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# PERTURBING AWAY HIGHER DIMENSIONAL SINGULARITIES FROM AREA MINIMIZING HYPERSURFACES

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## 1. Introduction

Area minimizing cones are the simplest examples of area minimizing varieties in Euclidean spaces (or indeed any Riemannian manifold) which have singularities. Minimizing cones also occur as tangent cones at points of an arbitrary minimizing submanifold. At smooth points of the submanifold the tangent cone is unique and is simply a linear subspace, but at singular points its uniqueness is generally unknown and its structure might be rather complicated. Still, it is hoped that a reasonable understanding of the singular set of an arbitrary minimizing submanifold can be reached through analysis of its tangent cones. Conversely, many new examples of minimizing submanifolds with singularities have been constructed as perturbations of minimizing cones.

The only case where this program has been carried out in any degree of completeness is when the singularities are isolated. The results here are quite satisfactory: there is a unique tangent cone at any isolated singular point of a minimizing submanifold (provided at least one tangent cone at this point is of multiplicity one and has an isolated singularity), and the submanifold can be written as a graph over its tangent cone near this singular point in a well-understood manner, cf. [S1], [S3], and [A-S]. Conversely, given a minimal cone with an isolated singularity (necessarily at its vertex) and with boundary a smooth submanifold of the unit sphere, a finite codimensional family of small perturbations of this boundary span nearby minimal submanifolds with

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the same singularity [C-H-S]. If the cone is strictly minimizing then these perturbations are minimizing [H-S2]. On the other hand, (codimension one) one sided perturbations of the boundary span minimizing submanifolds with no singularities [H-S2], cf. also [Mc]. Finally, submanifolds with many isolated singularities can be constructed by bridging these examples together [Sm1], [Sm2].

Very little is known when the singularities are not isolated. F. Almgren proved the existence of a measure-theoretic stratification for the singular set of a mod 2 minimizer, and recently L. Simon established finiteness of the measure of the top-dimensional stratum, amongst other results, [S6]. Some examples of minimizing submanifolds with nonisolated singularities are created by taking products of minimizing cones with isolated singularities with other manifolds. There are also codimension one minimal examples with singularities invariant with respect to a compact Lie group [Sm3]. Since algebraic varieties are minimizing, many other singular examples are known, but since we shall be concerned primarily with hypersurfaces below, these examples are irrelevant then. A direct generalization of the techniques of [C-H-S] to construct perturbations of these product examples seems, for certain technical reasons, quite difficult. In this paper we address the question of how to perturb away singularities from these minimizing products.

More specifically, let $C \subset \mathbb{R}^{n+1}$ be a strictly minimizing hypercone with an isolated singularity (necessarily at its vertex) and $C_1$ its truncation to the unit ball. Denote its boundary link by $\Sigma \subset S^n$. The notion of strict minimization was introduced in [H-S2], and is intended to rule out cases which are only 'borderline' minimizing. We shall introduce one of the equivalent definitions for it in the next section. Let $M^k$ be an arbitrary compact $k$-dimensional Riemannian manifold. Then $T_0 \equiv C_1 \times M$ is minimizing in $\mathbb{R}^{n+1} \times M$, and has boundary $\Sigma \times M$ and singular set $\{0\} \times M$. As discussed in [H-S2], $\Sigma$ is orientable and connected, so it separates $S^n$ into two components, $U^{\pm}$. Submanifolds $Z$ of dimension $n+k-1$ in $S^n \times M$ which are $\mathcal{C}^{1,\alpha}$ close to $\Sigma \times M$ can be parametrized by $\mathcal{C}^{1,\alpha}$ functions $\psi$ as described in the next section. A one sided perturbation of the boundary is one contained either in $U^+ \times M$ or $U^- \times M$. For any choice of boundary there is a unique minimizing current $T$

in $\mathbb{R}^{n+1} \times M$ near to $C_1 \times M$. We shall study minimizers spanning one sided boundary perturbations; for definiteness, these boundaries will lie in $U^+ \times M$. For any $\epsilon > 0$ let $U^+_\epsilon$ denote the $\epsilon$-tubular neighbourhood of $\Sigma$ in $U^+$. The goal of this paper is to prove the

**Theorem.** *Given any $\delta$ with $0 < \delta < 1$ there exists an $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$ if $Z$ is a boundary perturbation contained in $(U^+_\epsilon \times M)\backslash(U^+_{\delta\epsilon} \times M)$, then the minimizer $T$ spanning $Z$ is a smooth hypersurface.*

There is a more accurate statement of this theorem in the next section. This result is the first one concerning general perturbations of minimizing currents with nonisolated singularities. The hypotheses here are somewhat more complicated than those of the analogous result in [H-S2] because of the requirement of $\delta$-pinching of the boundary values, which we call *strongly one sided* below. This requirement seems necessary to our proof, but its necessity for the result is unclear. Unfortunately it seems to preclude a generalization such as [Mc] to this setting. The proof of this theorem proceeds similarly in broad outline to the one in [H-S2], but differs markedly in several important aspects. We use in a crucial way the foliation of $\mathbb{R}^{n+1}\backslash C$ by homothetically related smooth minimizing hypersurfaces constructed in [H-S2]. The difficulties all occur in trying to control the 'horizontal' behaviour of dilates of putative counterexamples of the theorem, where by horizontal we mean in directions parallel to the singular set. In the linear analysis, these difficulties are manifested by the fact that the Jacobi operator is of 'edge' type, as in [M1], rather than of the simpler conic type as in [H-S2].

One use for results such as the one above and its precursor in [H-S2] is in clarifying the issue of whether singularities of minimizing varieties are 'necessary.' Specifically, suppose $X$ is a variety which either minimizes area within a homology class of a Riemannian manifold and has no boundary, or minimizes area amongst competing currents sharing the same boundary, and which has singularities. Sometimes these singularities are necessary for topological reasons: either the homology class has no smooth representatives or the boundary is not nullcobordant. But if this is not the case (e.g. for the homological case in codimension one) then it is reasonable to ask whether for small generic perturbations of the background metric the corresponding minimizer is smooth. The perturbation result of [H-S2] after which our result is modelled implies this fact for seven dimensional strictly minimizing cones in $\mathbb{R}^8$ with six dimensional boundary in $S^7$, although this implication is not stated there. Less obvious is the generic regularity of seven dimensional minimizing hypersurfaces in homology classes, but this is proved in [Sm4]. The restriction to dimension seven is important here, for as is well known, seven dimensional minimizing hypercurrents can have only isolated singularities. It is hoped that the result here will have a similar application, although the pinching hypothesis will presumably need to be removed for this.

In §2 below the proof is given assuming a basic estimate for solutions of the linear Jacobi equation on the product $C \times \mathbb{R}^k$ (which is the space obtained as a limit of dilations of $C \times M$). The relevant linear theory is discussed in §3.

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## 2. Strongly One Sided Perturbations

**Minimizing Cones.** We first need some preliminary results on area minimizing hypercones, which for the most part are taken from [H-S2]. Let $C$ be a regular, $n$-dimensional, area minimizing cone in $\mathbb{R}^{n+1}$. That is, $C$ is a multiplicity one cone described by $C = \{z = r\theta : 0 \leq r < \infty,\ \theta \in \Sigma\}$, where $\Sigma$ is a smooth minimal submanifold of the unit $n$-sphere, and $C$ is area minimizing in the sense of currents (see [S1], [S2]). Thus, the singular set of $C$ equals the origin, and by the standard regularity theory for area minimizing codimension one currents, we must have $n \geq 7$. We will use polar coordinates $(r, \theta)$ on $C$, where $r$ is Euclidean distance to the origin, and $\theta$ is a local coordinate on $\Sigma$. We also use the notation $C_\rho$ to denote the truncated cone $\{r\theta : 0 < r \leq \rho\}$. Now $C$ divides $\mathbb{R}^{n+1}$ into two disjoint components, $E^\pm$. According to §2 of [H-S2], there exist smooth area minimizing hypersurfaces $V^\pm \subset E^\pm$, with $\operatorname{dist}\{V^\pm, 0\} = 1$. Furthermore, any area-minimizing $n$-dimensional current contained in $E^+$ or $E^-$ must be a homothety of $V^+$ or $V^-$; these homothetic images will be denoted $V^{\pm}_{\lambda} = \{\lambda x : x \in V\}$ for $\lambda > 0$. Further properties of $V^{\pm}$ will be described shortly.

We now discuss the Jacobi operator on $C$. First, let $n_C$ denote the unit normal of $C$ pointing into $E^{+}$. Then the Jacobi operator $\mathcal{L}$ is the second variation of the area operator, or equivalently, the first derivative of the mean curvature,

$$\mathcal{L}u = \frac{d}{dt}H(tu)|_{t=0}, \quad u \in C^2_0(C),$$

where $H(v)$ is the mean curvature of the variety described by $x \to x + v(x)n_C(x)$, $x \in C$. It is standard that $\mathcal{L}$ has the form

$$\mathcal{L} = \Delta_C + |A_C|^2 = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}L_\theta,$$

where $\Delta_C$ and $|A_C|^2$ are the Laplacian and the norm squared of the second fundamental form on $C$, respectively, and $L_\theta = \Delta_\theta + q(\theta)$ with $\Delta_\theta$ the Laplacian on $\Sigma$ and $q$ the restriction of $|A_C|^2$ to $\Sigma$ ($q$ differs from $|A_\Sigma|^2$ by a constant). $L_\theta$ is a self adjoint elliptic operator on $\Sigma$; let $\lambda_1$ and $\phi_1$ be its first eigenvalue and eigenfunction, so that $L_\theta\phi_1 = -\lambda_1\phi_1$. Now, set

$$\gamma_1^{\pm} = \frac{2-n}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda_1}. \tag{2.1}$$

These numbers, as well as the ones obtained by replacing $\lambda_1$ by the higher eigenvalues of $L_\theta$, are fundamental in the analysis of $L_C$, as described in the next section. Since $C$ is area minimizing, it is stable (the second variation of area for compactly supported variations is non-negative), which implies that $\lambda_1 \geq -\frac{(n-2)^2}{4}$ (see [C-H-S]); hence the numbers $\gamma_1^{\pm}$ are real. $C$ is *strictly* stable if we have strict inequality above, in which case $\gamma_1^{-} < \gamma_1^{+}$. Regarding the area minimizers $V^{\pm}$, it was shown in §2 of [H-S2] using the Allard regularity theorem and an anaysis of the minimal surface equation on $C$ that $V^{\pm}$ is the graph over $C$ near infinity of a function with a certain rate of decay; that is

$$V^{\pm} \cap (\mathbb{R}^{n+1}\backslash B_{R_0}(0)) = \{x + v^{\pm}(x)n_C(x) : x \in C, |x| \geq R_0\}$$

for some $R_0 > 0$ and where $v^{\pm} \in \mathcal{C}^\infty(C \cap (\mathbb{R}^{n+1}\backslash B_{R_0}(0)))$. Furthermore, $v^{\pm}$ can be decomposed as

$$v^{\pm} = (c_1\log(r) + c_2)r^{\gamma_1^{+}}\phi_1 + O(r^{\gamma_1^{+}-\beta}) \qquad \text{if } \gamma_1^{+} = \gamma_1^{-}, \tag{2.2}$$

where $\beta > 0$ and either $c_1 \neq 0$ or $c_2 \neq 0$, while if $\gamma_1^- < \gamma_1^+$ then

$$(2.3) \qquad v^{\pm} = \begin{cases} \text{a)} \quad c_1 r^{\gamma_1^-} \phi_1 + O(r^{\gamma_1^- - \beta}) \\ \text{or} \\ \text{b)} \quad c_1 r^{\gamma_1^+} \phi_1 + O(r^{\gamma_1^+ - \beta}) \end{cases}$$

where $c_1 \neq 0$ and $\beta > 0$. Evidently $c_1 > 0$ for $v^+$ and $c_1 < 0$ for $v^-$, since $V^{\pm} \subset E^{\pm}$ and $n_C$ was chosen to point into $E^+$. We also use the notation $v_\lambda^{\pm}$ for the corresponding function for $V_\lambda^{\pm}$, so that $V_\lambda^{\pm} \cap (\mathbb{R}^{n+1} \backslash B_{\lambda R_0}(0)) = \{x + v_\lambda^{\pm}(x) n_C(x) : x \in C, |x| \geq \lambda R_0\}$. Of course $v_\lambda^{\pm}(\lambda x) = \lambda v^{\pm}(x)$.

Finally, $C$ is said to be *strictly minimizing* if there is a constant $\mu > 0$ such that

$$(2.4) \qquad \text{Area}(C_1) \leq \text{Area}(S) - \mu \varepsilon^n$$

whenever $\varepsilon > 0$ and $S$ is an integer multiplicity current with $\operatorname{spt} S \subset \mathbb{R}^{n+1} \backslash B_\varepsilon(0)$, and $\partial S = \partial C_1$ (technically, Area$(S)$ should be the mass of $S$). Section 3 of [H-S2] contains several equivalent characterizations of strict minimization. The characterization useful here is that if $C$ is strictly minimizing, then $V^{\pm}$ has the slower of the possible rates of decay at infinity, that is, (2.3b) holds when $C$ is strictly stable and strictly minimizing, while if $C$ is only stable then $c_1 \neq 0$ in (2.2).

**Perturbing Away Singularities.** We first describe the area minimizing surfaces that will be perturbed. Let $C$ be an area minimizing hypercone as described above. The proof of the main theorem requires that $C$ be strictly minimizing, and so we shall assume this (although it would be nice to remove this hypothesis). We shall also assume that $C$ is strictly stable. This hypothesis is not necessary, although it clarifies the exposition; we indicate later how the proof can be modified to allow for non-strictly stable cones. Thus, alternative (2.3b) holds. It should be pointed out that all *known* area minimizing hypercones are both strictly minimizing and strictly stable (see the discussion in section 6 of [S5]). Now, let $(M, g)$ be a smooth compact $k$-dimensional Riemannian manifold, and define $T_0 = C_1 \times M$. We view $T_0$ as a singular hypercurrent in $\mathbb{R}^{n+1} \times M$ equipped with the product metric. Thus

$\partial T_0 = \partial C_1 \times M$, and the singular set of $T_0$ is $\operatorname{sing}(T_0) = \{0\} \times M$. Since $C_1$ is area minimizing, $T_0$ is certainly area minimizing in $\mathbb{R}^{n+1} \times M$. Similarly $V_\lambda^\pm \times M$ is also area minimizing. We will consider one sided perturbations of $\partial T_0$, and for convenience these will be perturbations into $E^+ \times M$, although of course an analogous theorem holds for perturbations into $E^- \times M$. To simplify notation we now let $E$, $V_\lambda$ and $v_\lambda$ denote $E^+$, $V_\lambda^+$ and $v_\lambda^+$, respectively. The unit normal of $T_0$ pointing into $E \times M$ will be called simply $n$. If $W_1$ and $W_2$ are closed sets contained in $\bar{E} \times \mathbb{R}^k$, each dividing $\bar{E} \times \mathbb{R}^k$ into at most two connected components, then the notation $W_1 \leq W_2$ will mean that $W_2$ is contained in the component of $\bar{E} \times \mathbb{R}^k \backslash W_1$ not bordering on $C \times \mathbb{R}^k$ (that is $W_1$ lies between $C \times \mathbb{R}^k$ and $W_2$). We will also use the same notation when $M$ replaces $\mathbb{R}^k$, and strict inequality will be used if $W_1 \cap W_2 = \emptyset$.

For $\psi \in \mathcal{C}^{1,\alpha}(\partial T_0)$ with sufficiently small $\mathcal{C}^{1,\alpha}$ norm, we let $Z_\psi$ denote the perturbed boundary $\{x + \psi(x)n(x) : x \in \partial T_0\}$. The boundary perturbation is one sided if $\psi \geq 0$. We will show that area minimizers whose boundaries are *strongly* one sided small perturbations of $\partial T_0$ are smooth. By strongly one sided, we mean that there is a fixed bound to the ratio of the maximum of $\psi$ to the minimum.

**Theorem 2.5.** *Given $\delta \in (0,1)$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, if $\psi \in \mathcal{C}^{1,\alpha}(\partial T_0)$ with $\delta\varepsilon \leq \psi \leq \varepsilon$ and $|\psi|_{\mathcal{C}^{1,\alpha}} \leq C(M,\Sigma)$, then there is a* smooth *area minimizing hypersurface $T$ with $\partial T = Z_\psi$.*

*Proof.* The proof will be by contradiction. If the theorem were false, then for a fixed $\delta \in (0,1)$, there would be a sequence $\varepsilon_j \downarrow 0$, and $\mathcal{C}^2$ functions $\psi_j$, with $\delta\varepsilon_j \leq \psi_j \leq \varepsilon_j$, and area minimizing currents $T_j$, $\partial T_j = Z_{\psi_j}$, such that $\operatorname{sing}(T_j) \neq \emptyset$. Note that $T_j \subset E \times M$, and that $\operatorname{area}(T_j) \leq K$ for some constant $K$ depending only on $C$ and $M$. By the compactness property of minimizing currents, there is an area minimizing current $T$ such that (after passing to a subsequence) $T_j \rightharpoonup T$ in the sense of currents. From standard geometric measure theory, $\operatorname{spt} T_j \to \operatorname{spt} T$ in Hausdorff distance. Since $\partial T = \partial T_0$ and $T_0$ is the unique area minimizer with boundary $\partial C \times M$, we must have $T = T_0$. From the Allard regularity theorem, $T_j$ converges locally in $\mathcal{C}^2$ to $T_0$ away from $\operatorname{sing}(T_0)$. Furthermore, from [H-S1] sing $T_j$ is uniformly bounded away from $\operatorname{spt}\partial T_j$. Now choose $x_j \in \operatorname{sing}(T_j)$ such that $x_j$ minimizes distance to $\operatorname{sing}(T_0)$ amongst all points in $\operatorname{sing}(T_j)$, and let $d_j = \operatorname{dist}(x_j, \operatorname{sing}(T_0))$. Also define $z_j \in C$, $y_j \in M$ by $x_j = (z_j, y_j)$. Then, from the discussion above, upon passing to a subsequence again, $x_j \to x_0 = (0, y_0) \in \operatorname{sing} T_0$. Note that

$$V_{c\delta\varepsilon_j} \times M \leq \operatorname{spt} T_j \leq V_{c\varepsilon_j} \times M$$

for some constant $c$ independent of $j$.

Let $\mathcal{N}$ be a neighborhood of $x_0$ in $\mathbb{R}^{n+1} \times M$, with coordinates $\{x^1, \dots, x^{n+k+1}\}$, where $x^{n+2}, \dots, x^{n+k+1}$ are normal coordinates about $y_0$ in $M$. Fix $\rho_0 > 0$ sufficiently small so that $B_{\rho_0}(y_j) \subset \mathcal{N}$ for all $j$. Now let $\tau_j : B_\rho(y_j) \to B_{d_j^{-1}\rho}(y_j)$ be the homothetic expansion about $(0, y_j)$ by the factor $d_j^{-1}$, i.e. $\tau_j((z,y)) = d_j^{-1}(z, y - y_j)$. Define the current $S_j$ by $S_j = (\tau_j)_\#(T_j \cap B_\rho(y_j))$ (so that $\operatorname{spt} S_j = \tau_j(\operatorname{spt} T_j \cap B_\rho(y_j))$. We consider $S_j$ as a hypercurrent in $\mathbb{R}^{n+1} \times \mathbb{R}^k$, endowed with the metric $\tau_j^*(h)$ where $h$ is the product metric on $\mathbb{R}^{n+1} \times M$; it is clearly (locally) area minimizing. From the discussion above it follows that

$$V_{c\delta} \times M \leq \operatorname{spt} S_j \leq V_c \times M. \tag{2.6}$$

Since $\tau_j^*(h)$ converges to the Euclidean metric $dx^2$ in $C^3$ on compact subsets of $\mathbb{R}^{n+1} \times \mathbb{R}^k$, it follows from the compactness theorems of geometric measure theory that $S_j \rightharpoonup S$ (in the sense of currents) to an area minimizing hypercurrent $S \subset \mathbb{R}^{n+1} \times \mathbb{R}^k$, and from the remark above, $\operatorname{spt} S \subset E \times \mathbb{R}^k$, and

$$V_{c\delta} \times \mathbb{R}^k \leq \operatorname{spt}(S) \leq V_c \times \mathbb{R}^k. \tag{2.7}$$

Also, since $S_j$ has a singularity at $\tau_j(x_j)$, and $\tau_j(x_j) \to (z_0, 0)$ for some $z_0 \in \mathbb{R}^{n+1}$ with $|z_0| = 1$ (again, after passing to a subsequence if necessary), it follows from the Allard regularity theorem that

$$(z_0, 0) \in \operatorname{sing}(S) \text{ so that } \operatorname{sing}(S) \neq \emptyset. \tag{2.8}$$

We will show that $S = V_\lambda \times \mathbb{R}^k$ for some $\lambda$, contradicting (2.8). First of all, from (2.7) it is clear that $C \times \mathbb{R}^k$ is the *unique* tangent cone at infinity for $S$ (that is, for any sequence $\{t_i\}$ with $t_i \downarrow 0$, we have $t_i S \rightharpoonup C \times \mathbb{R}^k$). It then follows from the Allard regularity theorem and standard elliptic theory that $S$ is the graph of a function over $C \times \mathbb{R}^k$ in the following sense. There exists a map $\rho : \mathbb{R}^k \to \mathbb{R}$, $\rho > 0$ with $\rho(y)/|y| \to 0$ as $|y| \to \infty$, and a $C^\infty$ function $u$ such that

$$\text{(2.9)}\quad \begin{aligned} S \cap \{x = (z, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^k : |z| \geq \rho(y)\} \\ = \{x + u(x)n(x) : x \in C \times \mathbb{R}^k, \text{ with } |z| \geq \rho(y)\}\end{aligned}$$

where

$$\text{(2.10)}\qquad \frac{u(x)}{|x|} + |\nabla u(x)| + |\nabla^2 u(x)||x| \to 0, \quad \text{as } |x| \to 0.$$

Of course, because of the sandwiching (2.7), $u$ actually tends to zero as fast as $r^{\gamma_1^+}$ by virtue of (2.3). Now we claim that (2.9) and (2.10) actually hold in a cylinder $\{x = (z, y) \in C \times \mathbb{R}^k : r = |z| \geq R_1\}$ for some $R_1 > 0$. For if not then there would exist a sequence $\{(z_j, y_j)\} \subset C \times \mathbb{R}^k$ with $|z_j| = R_j \to +\infty$ and $|y_j| \to +\infty$, and such that $S$ is not the graph of a function over $B_\sigma(z_j, y_j) \cap C \times \mathbb{R}^k$ for any $\sigma > 0$. Let $\sigma_j : \mathbb{R}^{n+1} \times \mathbb{R}^k \to \mathbb{R}^{n+1} \times \mathbb{R}^k$ be the map $\sigma_j(z, y) = R_j^{-1}(z, y - y_j)$, and let $\hat{S}_j = (\sigma_j)_\#(S)$. Then of course $\hat{S}_j$ is area minimizing and

$$\text{(2.11)}\qquad C \times \mathbb{R}^k \leq \operatorname{spt}(\hat{S}_j) \leq V_c \times \mathbb{R}^k.$$

Furthermore, $\hat{S}_j$ is not the graph of a function over $B_\sigma(R_j^{-1} z_j, 0) \cap (C \times \mathbb{R}^k)$ for any $\sigma$. After passing to a subsequence we have $\hat{S}_j \rightharpoonup \hat{S}_0$, for some area minimizing current $\hat{S}_0$, and $\sigma_j(z_j, y_j) \to (z_0, 0) \in \operatorname{spt}(\hat{S}_0)$, with $|z_0| = 1$. Evidently from (2.7) $\hat{S}_0 = C \times \mathbb{R}^k$. But then by Allard's regularity theorem again there is a $\sigma > 0$ sufficiently small, such that $\operatorname{spt}(\hat{S}_j)$ is the graph of a smooth function over $B_\sigma(z_0, 0) \cap (C \times \mathbb{R}^k)$ for $j$ large enough (since $(z_0, 0)$ is in the regular set of $C \times \mathbb{R}^k$) contradicting our assumption. Thus, we must have

$$\text{(2.12)}\quad S \cap \{(z, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^k : |z| \geq R_1\} = \{x + u(x)n(x) : x \in C \times \mathbb{R}^k : r(x) \geq R_1\},$$

for some $R_1 > 0$ and some smooth function $u > 0$ satisfying

$$\text{(2.13)}\qquad c_2^{-1} r^{\gamma_1^+} \leq u(x) \leq c_2 r^{\gamma_1^+}, \quad r(x) \geq R_1$$

for some constant $c_2$; this inequality follows from (2.3b) and (2.7). It now follows from standard elliptic theory and the fact that $u$ satisfies the minimal surface equation on $C \times \mathbb{R}^k \backslash \{r < R_1\}$ that there are constants $c_3(m)$, $m = 1, \ldots,$ independent of the multi-index $\beta$, such that

$$|\nabla_C^m \partial_y^\beta u(x)| \leq c_3(m) r^{\gamma_1^+ - m} \quad \text{for } r \geq 2R_1, \tag{2.14}$$

where $\nabla_C$ is the gradient operator on $C$ and, as usual $y \in \mathbb{R}^k$.

The minimal surface operator $H$ on $C \times \mathbb{R}^k$ has the form

$$H(w) = \mathcal{L}w + \mathcal{Q}(w)$$

where $\mathcal{L} = \Delta_{C \times \mathbb{R}^k} + |A_{C \times \mathbb{R}^k}|^2$ now is the Jacobi operator on $C \times \mathbb{R}^k$, and $\mathcal{Q}$ is a nonlinear term satisfying

$$|\mathcal{Q}(w)| \leq c_4(r^{-3}|w|^2 + r^{-1}|\nabla w|^2 + r^{-1}|w||\nabla^2 w| + |\nabla w|^2|\nabla^2 w|), \tag{2.15}$$

for $w \in C^2(\{C \times \mathbb{R}^k : r > 2R_1\})$ provided $|r\nabla w|$ is sufficiently small. The estimate for $\mathcal{Q}$ essentially follows from the proof of Lemma 5 in §3 of [Sm1] (see also [K]). We thus have $\mathcal{L}u = F$ for $r \geq 2R_1$, where $u, F \in \mathcal{C}^\infty(\{C \times \mathbb{R}^k : r > 2R_1\})$, with $u$ satisfying (2.13) and (2.14), and $|F| \leq c_4 r^{2\gamma_1^+ - 3}$. §3 below is devoted to an analysis of the asymptotic behavior of such solutions as $r \to \infty$, and the main result, Proposition (3.9), states that $u$ has a decomposition:

$$u(x) = c_5 r^{\gamma_1^+} \phi_1(\theta) + O(r^{\gamma_1^+ - \mu}) \quad \text{for } r \geq R_2, \tag{2.16}$$

for some constants $\mu > 0$, $c_5 > 0$, and $R_2 \geq 2R_1$. We emphasize that the second term on the right is bounded by $r^{\gamma_1^+ - \mu}$ *independently* of $y \in \mathbb{R}^k$.

Define $\underline{\lambda} = \sup\{\lambda : V_\lambda \times \mathbb{R}^k \leq \operatorname{spt}(S)\}$, and $\overline{\lambda} = \inf\{\lambda : \operatorname{spt}(S) \leq V_\lambda \times \mathbb{R}^k\}$. Then we have

$$V_{\underline{\lambda}} \times \mathbb{R}^k \leq \operatorname{spt}(S) \leq V_{\overline{\lambda}} \times \mathbb{R}^k, \tag{2.17}$$

so that

$$v_{\underline{\lambda}} \leq u \leq v_{\overline{\lambda}} \quad \text{for } r \geq R_2. \tag{2.18}$$

Let $c_{\underline{\lambda}}$ and $c_{\overline{\lambda}}$ be the corresponding constants as in (2.3b), so that

$$\begin{aligned} v_{\underline{\lambda}} &= c_{\underline{\lambda}} r^{\gamma_1^+} \phi_1 + O(r^{\gamma_1^+ - \beta}) \\ v_{\overline{\lambda}} &= c_{\overline{\lambda}} r^{\gamma_1^+} \phi_1 + O(r^{\gamma_1^+ - \beta}). \end{aligned} \tag{2.19}$$

From (2.17) and (2.18) it follows that

$$c_{\underline{\lambda}} \le c_5 \le c_{\overline{\lambda}}. \tag{2.20}$$

To finish the proof, it suffices to show that these are equalities, for then we would have $V_{\underline{\lambda}} = V_{\overline{\lambda}}$; together with (2.17) this implies that $S = V_{\underline{\lambda}} \times \mathbb{R}^k$ and so $\operatorname{sing}(S) = \emptyset$, contradicting (2.8).

Suppose that $c_{\underline{\lambda}} < c_5$. Then, for any sequence $\{\lambda_j\}$, with $\lambda_j \downarrow \underline{\lambda}$ there exists, by definition of $\underline{\lambda}$,

$$(z_j, y_j) \in (V_{\lambda_j} \times \mathbb{R}^k) \cap \operatorname{spt}(S)$$

where clearly we can assume that $|y_j| \to \infty$. As above, we translate $S$ back by $y_j$. Define $\tau_j : \mathbb{R}^{n+1+k} \to \mathbb{R}^{n+1+k}$ by $\tau_j(z,y) = (z, y - y_j)$, and set $\bar{S}_j = (\tau_j)_\#(S)$. Thus $\bar{S}_j$ is an area minimizing current with $V_{\underline{\lambda}} \times \mathbb{R}^k < \operatorname{spt}(\bar{S}_j) < V_{\overline{\lambda}} \times \mathbb{R}^k$ and $\bar{S}_j$ has the asymptotic expansion (2.16) (with a different $u$ but same constant $c_5$). As in previous arguments, after taking a subsequence, there is an area minimizing current $\bar{S}$ such that $\operatorname{spt} \bar{S}_j \to \operatorname{spt} \bar{S}$ locally in Hausdorff distance, and smoothly near regular points of $\bar{S}$. Clearly $\bar{S}$ also satisfies (2.16) with the same constant $c_5$, and $V_{\underline{\lambda}} \times \mathbb{R}^k \le \operatorname{spt}(\bar{S})$. We have arranged that $(z_j, 0) \in \operatorname{spt}(\bar{S}_j) \cap (V_{\lambda_j} \times \mathbb{R}^k)$. Furthermore, because of the different rates of growth as $r \to \infty$, $\sup |z_j| < \infty$. Thus (after taking a subsequence if necessary) $z_j \to z_0$, and we would have

$$\operatorname{spt}(\bar{S}) \cap (V_{\underline{\lambda}} \times \mathbb{R}^k) \neq \emptyset \quad \text{and} \quad V_{\underline{\lambda}} \times \mathbb{R}^k \le \operatorname{spt}(\bar{S})$$

which implies that $\bar{S} = V_{\underline{\lambda}} \times \mathbb{R}^k$ by [S4], contradicting the hypothesis that $c_{\underline{\lambda}} \neq c_5$. Thus we must have $c_{\underline{\lambda}} = c_5$, and an identical argument shows that $c_{\overline{\lambda}} = c_5$. This completes the proof of the theorem assuming that $C$ is strictly stable. When $C$ is only assumed to be stable, so that (2.2) holds with $c_1 \neq 0$, then there is an analogue of (2.16). This states that if $\mathcal{L}u = F$ where $|F| \le c_4 r^{-n-1} (\log r)^2$, then

$$u = c_5 r^{\frac{2-n}{2}} \log r + O(r^{\frac{2-n}{2}}). \tag{2.21}$$

Since the functions determining the $V_\lambda$ have the same asymptotics, the rest of the proof in this section proceeds in exactly the same way. □

## 3. Analysis of the Linearized Equation

As demonstrated in the last section, the proof of the main theorem may be reduced to establishing Proposition (3.7) below. Let $H$ denote the minimal surface operator for graphs over the minimizing wedge $W = C \times \mathbb{R}^k$. Also let $u$ denote the function parametrizing the minimizing hypersurface $T$ constructed in §2 as a graph over $W_{r_0} = W \cap \{r \geq r_0\}$. As indicated earlier, the behaviour of the linearization for $H$ is regulated by its indicial roots

$$\gamma_j^{\pm} = \frac{2-n}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j}, \tag{3.1}$$

where $\{\lambda_j, \phi_j(\theta)\}$ are the eigenvalues and eigenfunctions for the operator $L_\theta = \Delta_\Sigma + q$ on $\Sigma$. Our convention is that $\lambda_j \to \infty$. Also, by the variational characterization of $\lambda_1$ and the fact that $q \geq 0$, $\lambda_1 < 0$. Hence, assuming also that $C$ is strictly stable,

$$0 > \gamma_1^+ > \frac{2-n}{2} > \gamma_1^-. \tag{3.2}$$

By the geometric constraints of the last section,

$$0 < u \leq C r^{\gamma_1^+}. \tag{3.3}$$

Define the spaces

$$\mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma) = \{\psi(y,\theta) : |\partial_y^\alpha \partial_\theta^\beta \psi| \leq C_{\alpha\beta} \ \forall\ \alpha, \beta \text{ and } (y,\theta) \in \mathbb{R}^k \times \Sigma\} \tag{3.4}$$

and

$$\begin{aligned} S^\ell([r_0,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma)) = \\ \{v : |\partial_r^j \partial_y^\alpha \partial_\theta^\beta v| \leq C_{j,\alpha,\beta} r^{\ell - j} \ \forall\ j, \alpha, \beta \text{ and } \ r \geq r_0,\ (y,\theta) \in \mathbb{R}^k \times \Sigma\}. \end{aligned} \tag{3.5}$$

The (modified) minimal surface operator $r^2 H$ on $W$ is dilation invariant in $(r, y)$ and translation invariant in $y$, so straightforward scaling arguments as in [H-S2] combined with the estimate (3.3) yield the proof of

**Lemma 3.6.** *If $u$ is a solution of $H(u) = 0$ and also satisfies the estimate (3.3), then $u \in S^{\gamma_1^+}([r_0,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$.*

Our goal in this section is to establish the following

**Proposition 3.7.** *Let $u$ be the solution of $H(u) = 0$ on $W_{r_0}$ obtained in the last section. Then there exists a constant $A > 0$ such that*

$$u = Ar^{\gamma_1^+} + v, \qquad v \in S^q([r_0, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$$

*where* $q = \max\{\gamma_1^-, 2\gamma_1^+ - 1\} < \gamma_1^+$.

For convenience, set

$$p = 2\gamma_1^+ - 1,$$

and also assume that $r_0 = 1$. (3.2) implies that $p < \gamma_1^+$.

The first step in the proof of this Proposition is to rewrite the equation $H(u) = 0$ as $Lu = f$, where $L$ is the linearization of $r^2 H$ about the zero solution, and $f$ is the quadratic remainder. This was discussed already in §2 (but note that $L$ here is $r^2$ times the Jacobi operator on $C \times \mathbb{R}^k$ and $f = r^2 F$). Hence, as explained there,

$$|f| \leq Cr^{2\gamma_1^+ - 1} = Cr^p. \tag{3.8}$$

The same scaling arguments as before now imply that $f \in S^p([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$. We restate Proposition (3.7) as

**Proposition 3.9.** *Let $Lu = f$, where $L$ is the linearization of $r^2 H$, the function $u \in S^{\gamma_1^+}([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ and $f \in S^p([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$. Then there exists a constant $A$ such that $u = Ar^{\gamma_1^+} + v$, where $v \in S^q([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ with $q$ as above.*

The linearization $L$ may be written explicitly in terms of the coordinates $(r, y, \theta)$ as

$$L = r^2 \partial_r^2 + (n-1) r \partial_r + L_\theta + r^2 \Delta_y. \tag{3.10}$$

We shall reduce the proof of Proposition (3.9) to the following two results.

**Lemma 3.11.** *There exists a unique Poisson operator for the operator $L$ on the set $W_1$ which satisfies*

$$P : \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma) \longrightarrow S^{\gamma_1^-}([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma)). \tag{3.12}$$

**Lemma 3.13.** *Given $f \in S^p([1,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$, there exists a unique solution $w \in S^q([1,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ of the equation $Lw = f$ with boundary value $w(1, y, \theta) = \psi(y, \theta) \in \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma)$ prescribed.*

The proof of (3.13) uses (3.11) in a standard way, namely first some solution of the equation $Lw = f$ in $S^q([1,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ is found, and then (3.11) is used to correct its boundary values.

Assuming these we now give the

*Proof of Proposition* 3.7. Use Lemma 3.13 to find $w \in S^q([1,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ with $U = u - w$ vanishing at $r = 1$. Thus $LU = 0$, $U \in S^q([1,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ and $U(1, y, \theta) = 0$. First decompose $U$ according to the eigenfunctions of $L_\theta$: $U = \sum U_j(r, y)\phi_j(\theta)$. Each eigencomponent $U_j \in S^q(\mathbb{R}^k \times \Sigma)$, so in particular is bounded in $\mathcal{C}^\infty$ in $y$ for each $r \geq 1$. Also, $U_j(1, y) = 0$. Now take the Fourier transform in $y$ to produce a sequence of bounded measures $\hat{U}_j(r, \eta)$. Using that $U_j(r, y) \in \mathcal{C}_b^\infty(\mathbb{R}^k)$ for $y$ fixed, we see that each $\hat{U}_j(r, \eta)$ is rapidly decreasing (in the weak sense). The $\hat{U}_j(1, \eta)$ all vanish.

As functions of $r$ the $\hat{U}_j$ solve the equations $\hat{L}_j \hat{U}_j = 0$ where

$$\hat{L}_j = r^2 \partial_r^2 + (n-1) r \partial_r - \lambda_j - r^2 |\eta|^2. \tag{3.14}$$

Since the $\hat{L}_j$ are ODEs, we can write down all solutions to the equation $\hat{L}_j h = 0$ in terms of the basic solutions

$$\begin{aligned} h_j^+(r, \eta) &= r^{\frac{2-n}{2}} I_{\nu_j}(r|\eta|)/I_{\nu_j}(|\eta|) \text{ and} \\ h_j^-(r, \eta) &= r^{\frac{2-n}{2}} K_{\nu_j}(r|\eta|)/K_{\nu_j}(|\eta|). \end{aligned} \tag{3.15}$$

Here $\nu_j = \sqrt{(n-2)^2/4 + \lambda_j}$ and the functions $I_\nu(t)$ and $K_\nu(t)$ are Bessel functions of imaginary argument. Those of their properties that we shall use are described, for example, in [L]. In particular, we record their asymptotics:

$$I_\nu(t) \sim \begin{cases} e^t/\sqrt{t} & \text{as } t \to \infty \\ t^\nu & \text{as } t \to 0 \end{cases} \tag{3.16}$$

and

$$K_\nu(t) \sim \begin{cases} e^{-t}/\sqrt{t} & \text{as } t \to \infty \\ t^{-\nu} & \text{as } t \to 0, \end{cases} \tag{3.17}$$

at least up to constant factors. We have chosen the normalizations of the $h_j^\pm(r,\eta)$ so that

$$h_j^\pm(1,\eta) = 1, \text{ and } h_j^\pm(r,0) = r^{\gamma_j^\pm}. \tag{3.18}$$

Notice also that for any $\eta$, $h_j^\pm \longrightarrow r^{\gamma_j^\pm}$ as $r \longrightarrow 0$.

Now we use these functions to write $\hat{U}_j$ as

$$\hat{U}_j(r,\eta) = a_j(\eta)h_j^+(r,\eta) + b_j(\eta)h_j^-(\eta) \tag{3.19}$$

for some measures $a_j$ and $b_j$. However, the fact that $U$ is of polynomial growth (in fact, decaying) in $r$ implies that each $\hat{U}_j$ has the same property. This necessitates that $a_j(\eta)$ be supported at the origin. Since it is only a measure, it must be a constant multiple of the delta function. Next, since $\hat{U}_j(1,\eta) = 0$, it also follows that $b_j$ is supported at the origin, hence also a multiple of the delta function. In fact, these values must cancel, so we conclude upon taking inverse Fourier transforms that

$$U_j(r,y) = C_j(r^{\gamma_j^+} - r^{\gamma_j^-}) \tag{3.20}$$

for some constants $C_j$. Finally, because $U \in S^{\gamma_1^+}([1,\infty);\mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$, all but the first of the $C_j$ must vanish, so that $u = U + w = C_1 r^{\gamma_1^+} + v$, with $v \in S^q([1,\infty);\mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$, as desired. □

**The Poisson Operator.** In this subsection we construct the Poisson operator for $L$ and prove Lemma (3.11). The strategy is to construct Poisson operators $\hat{P}_j$ for each of the $\hat{L}_j$, transform these to Poisson operators $P_j$ for

$$L_j = r^2\partial_r^2 + (n-1)r\partial_r - \lambda_j + r^2\Delta_y, \tag{3.21}$$

and finally sum these to obtain $P$. This is a more straightforward procedure than the construction of Poisson operators in [M2] for operators such as $L$ where the initial data is along the hypersurface $\{r = 0\}$ where $L$ degenerates.

Since $\hat{L}_j$ is only a scalar ordinary differential operator, its (temperate) Poisson operator $\hat{P}_j$ is just the exponentially decreasing solution $h_j^-(r,\eta)$ from (3.15), for the unique temperate solution to the boundary problem

$$\hat{L}_j\hat{u}_j = 0, \quad \hat{u}_j(1,\eta) = \hat{\psi}_j(\eta) \tag{3.22}$$

is $\hat{u}_j = \hat{\psi}_j(r,\eta)h_j^-(r,\eta)$. Since $h_j^-$ is continuous in $\eta$, this makes sense even when $\hat{\psi}_j(r,\eta)$ is only a measure in this variable.

The Poisson operator $P_j(r,y)$ for $L_j$ is now obtained by taking the inverse Fourier transform of $\hat{P}_j$:

$$P_j(r,y) = (2\pi)^{-k} \int e^{iy\cdot\eta}\, h_j^-(r,\eta)\, d\eta. \tag{3.23}$$

Because $r \geq 1$, $h_j^-$ has a uniform rate of exponential decrease as a function of $|\eta|$, so this integral converges to a smooth function which is bounded on each line $r =$ constant. More is true: since $\hat{P}_j$ is smooth away from $\eta = 0$, is continuous and has a polyhomogeneous conormal singularity there commencing with the terms $c_0 + c_1|\eta| + \cdots$ where $c_0$, $c_1$ are constants ( $c_1$ depends on $j$ but $c_0$ does not), it also follows that for $r$ fixed, $P_j(r,y)$ is a classical symbol of order $-k-1$. It is still temperate in $r$. $P_j$ acts by convolution in the $y$ variable.

By construction, $P_j(1,y) = \delta(y-\tilde{y})$, so the unique temperate solution of $L_j u_j = 0$, $u_j(1,y) = \psi_j(y)$ for $\psi_j$ well enough behaved is given by

$$u_j(r,y) = \int P_j(r, y-\tilde{y})\psi_j(\tilde{y})d\tilde{y}. \tag{3.24}$$

The integrability of $P_j$ in $y$ implies that this integral makes sense and defines a solution of the problem even when $\psi_j(y)$ is only bounded, and in particular when $\psi_j \in \mathcal{C}_b^\infty(\mathbb{R}^k)$.

The next step is to show that $w_j \equiv P_j\psi_j$ is an element of $S^{\gamma_j^-}([1,\infty);\mathcal{C}_b^\infty(\mathbb{R}^k))$. The $\mathcal{C}^\infty$ boundedness in $y$ is quite easy, for observe that at least for $r > 1$ fixed, $|\partial_y^\alpha P_j| \leq C(1+|y|)^{-k-1-|\alpha|}$, so that the integration by parts in $\partial_y^\alpha P_j\psi_j = (-1)^{|\alpha|}P_j\partial_y^\alpha\psi_j$ is valid when $\psi_j \in \mathcal{C}_b^\infty(\mathbb{R}^k)$. Using the integrability of $P_j$ in $y$ again we see that $w_j(r,\cdot) \in \mathcal{C}_b^\infty(\mathbb{R}^k)$ for each $r \geq 1$.

To get estimates on the decay of $w_j$ as $r \to \infty$ we first note that $(r\partial_r)^j P_j$ is a kernel with the same regularity and growth as $P_j$. This means that $P_j\psi_j$ satisfies symbol estimates as $r \to \infty$, i.e. $w_j \in S^\ell([1,\infty);\mathcal{C}_b^\infty(\mathbb{R}^k))$ for some $\ell$. It remains then to establish that $\ell = \gamma_j^-$.

The precise decay in $r$ is easy to obtain because the $P_j$ (and later $P$ as well) are positive kernels. This follows directly from the maximum principle whenever $\lambda_j \geq 0$, i.e. for all but the first few values of $j$. To cover these cases as well, conjugate the operator $L_j$ by $r^{(2-n)/2}$. The resulting operator $L'_j$ is of the same form as $L_j$, but has zeroeth order term $-\lambda_j - (n-2)^2/4$ which is negative by the strict stability of $\Sigma$. The Poisson operator for $L'_j$ is therefore positive, and since $P_j$ is obtained by multiplying this operator by the appropriate power of $r$, it too is positive. The important point in this argument is that we could conjugate $L_j$ to obtain an operator with negative zeroeth order part using a real power of $r$ – the transformation would be true in general without assuming stability, but the correct power of $r$ would be imaginary, and positivity of the Poisson operator would be destroyed.

Now, to estimate $w_j$ as $r \to \infty$ it is legitimate to replace $\psi_j$ by its supremum. Thus it suffices to estimate the integral

$$\int P_j(r, y - \tilde{y})\, d\tilde{y} = \int P_j(r, y)\, dy.$$

The integral of any integrable function is given by the value of its Fourier transform at $\eta = 0$, and we already computed this in (3.19). It is $h_j^-(r, 0) = r^{\gamma_j^-}$. This proves that $w_j \in S^{\gamma_j^-}([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k))$.

The final step in the proof of Lemma (3.11) is to carry out the summation in $j$. This too turns out to be quite simple provided we use the smoothness of $\psi$ in $\theta$. For it is standard that by virtue of this smoothness, the supremum of any quantity $|\partial_y^\alpha \psi_j(y)|$ decreases rapidly in $j$. Furthermore, the suprema of any fixed derivative of the eigenfunctions $\phi_j(\theta)$ increase at most polynomially in $j$, so fixing any $N > 0$ there exists a constant $C_N$ (depending on $\psi$) so that $\sup |\psi_j(y)\phi_j(\theta)| \le C_N j^{-N}$, and similarly for any derivative. Inserting this into the Poisson kernel

$$P\psi(r, y, \theta) = \sum_j \int P_j(r, y - \tilde{y}) \psi_j(\tilde{y})\, d\tilde{y}\, \phi_j(\theta) \tag{3.25}$$

yields the estimate

$$|P\psi| \le C \sum_j j^{-N} r^{\gamma_j^-} \le C r^{\gamma_1^-} \sum j^{-N} \le C' r^{\gamma_1^-}.$$

Similar estimates hold for all derivatives. This concludes the proof of Lemma (3.11).

**The Free Inverse.** As noted already, the proof of Lemma (3.13) requires only that we construct some solution $u \in S^q([1,\infty);\mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ of the equation $Lu = f$, for since $u(1,y,\theta) \in \mathcal{C}_b^\infty$ the Poisson operator from Lemma (3.11) can then be used to correct its boundary value to the prescribed one. So we shall construct the free inverse $G$ for $L$ on an appropriate space of functions over the whole wedge $W$. The solution $u$ is given by $u = Gf$. Of course, as it stands, this is not quite right since we need to extend $f$ to a function on all of $W$ rather than just $W_1$. To do this, choose an extension operator $E$ mapping smooth functions in $r \geq 1$ to functions on $W$ supported in $r \geq 1/2$ by Seeley's procedure. In fact, using the $\mathcal{C}^\infty$ boundedness of our data as $|y| \to \infty$ $E$ can be chosen as a continuous extension operator from functions $f$ in $S^p([1,\infty);\mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ on $W_1$ to functions in $S^p([1,\infty);\mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ on $W$ vanishing in $\{r \leq 1/2\}$. For convenience we still denote the extended function by $f$.

The free inverse $G$ corresponding to $L$ on the wedge $W$ is constructed following [M1], cf. also [M-S]. However, the dilation invariance of $L$ with respect to homotheties in $(r,y)$ means that we need only use the analysis of the model operators from [M1], which is considerably simpler than the general theory of edge operators. It is worth pointing out that the singularity of $L$ as $r \to \infty$ is irregular, unlike the case of the Jacobi operator on the cone $C$ as in [H-S2], so in principle the analysis could be much more complicated. However, the information we need is sufficiently basic that these difficulties do not appear here.

The construction of the inverse $G$ is quite straightforward, but we depart from [M1] in that we need somewhat different mapping properties. The starting point is the model Bessel operator

$$L_0 = t^2\partial_t^2 + (n-1)t\partial_t + L_\theta - t^2 \tag{3.26}$$

obtained from $L$ by first taking the Fourier transform in the $y$ variable, and then rescaling by the transformation $t = r|\eta|$. The eigenfunction decomposition for $L_\theta$ has not been used, for this would introduce substantial analytic difficulties of a sort not encountered for the Poisson operator in trying to reassemble the pieces. The spaces of functions on which we study the action of

$L_0$ are the weighted Sobolev spaces

$$(3.27) \qquad t^\delta \tilde{H}^\ell(W) = \{u : (t\partial_t)^j t^r \partial_\theta^\beta u \in t^\delta L^2 \text{ for } j + r + |\beta| \leq \ell\}$$

The reason that multiplication by $t$ is explicitly taken into account here is that it corresponds, under rescaling back to $r$ and $\eta$ and Fourier transform, to differentiation by $r\partial_y$. It suffices here to consider the mapping

$$(3.28) \qquad L_0 : t^{(3-n)/2}\tilde{H}^{\ell+2}(W) \longrightarrow t^{(3-n)/2}\tilde{H}^\ell(W)$$

which is evidently bounded and (at least formally) self-adjoint. Using the eigenfunction decomposition for $L_\theta$ it is easy to check that the map (3.28) has no nullspace or cokernel. Furthermore, a crude parametrix construction given in [M1] and [M-S] imply that its range is closed, so that (3.28) is in fact an isomorphism.

Let $G_0(t, \tilde{t}, \theta, \tilde{\theta})$ be the Schwartz kernel of the inverse of the map (3.28). Thus

$$(3.29) \qquad G_0 : t^{(3-n)/2}\tilde{H}^\ell \longrightarrow t^{(3-n)/2}\tilde{H}^{\ell+2}$$

is also an isomorphism. It is shown in [M1] that for $(t, \theta) \neq (\tilde{t}, \tilde{\theta})$ the map $G(t\lambda, \tilde{t}\lambda, \theta, \tilde{\theta})$ decreases rapidly as $\lambda \to \infty$ and has an asymptotic expansion in integer powers of $\lambda$ as $\lambda \to 0$ commencing with the term $F(t, \tilde{t}, \theta, \tilde{\theta})\lambda^{-1}$. This leading coefficient is denoted $N_f(G_0)$ and is easily seen to depend only on the ration $t/\tilde{t}$.

The operator $L$ itself is first taken to act on the spaces

$$(3.30) \quad r^{(3-n)/2}H_e^\ell \equiv \{u : (r\partial_r)^j (r\partial_y)^\alpha \partial_\theta^\beta u \in r^{(3-n)/2}L^2(dr\,dy\,d\theta) \text{ for } j + |\alpha| + |\beta| \leq \ell\}.$$

Thus

$$(3.31) \qquad L : r^{(3-n)/2}H_e^{\ell+2} \to r^{(3-n)/2}H_e^\ell$$

is bounded, (formally) self-adjoint and with trivial kernel and cokernel. An inverse to this map is provided by the operator with Schwartz kernel

$$(3.32) \qquad G(r, \tilde{r}, \theta, \tilde{\theta}, y, \tilde{y}) = \int e^{i(y-\tilde{y})\cdot\eta}\, G_0(r|\eta|, \tilde{r}|\eta|, \theta, \tilde{\theta})|\eta|\, d\eta.$$

For it may easily be checked that

$$LG = GL = \delta(r - \tilde{r})\delta(y - \tilde{y})\delta(\theta - \tilde{\theta}). \tag{3.33}$$

Notice that $G$ only depends on $y, \tilde{y}$ through the difference $y - \tilde{y}$. The extra factor of $|\eta|$ in the integral is included to insure this fact and also to make

$$G : r^{(3-n)/2} H_e^\ell \longrightarrow r^{(3-n)/2} H_e^{\ell+2} \tag{3.34}$$

bounded for any $\ell \in \mathbb{N}_0$. For this and the other facts recorded below concerning $G$ we refer, as usual, to [M1] for proofs.

It is immediate either from the definition (3.32) or the identity (3.33) that

$$G(\lambda r, \lambda \tilde{r}, \theta, \tilde{\theta}, \lambda y, \lambda \tilde{y}) = \lambda^{-k-1} G(r, \tilde{r}, \theta, \tilde{\theta}, y, \tilde{y}). \tag{3.35}$$

This means that $G$ is simply expressed in terms of polar coordinates

$$\rho = |(r, y - \tilde{y}, \tilde{r})|, \ \omega = (\omega_0, \omega', \omega_{k+1}) = (r, y - \tilde{y}, \tilde{r})/\rho. \tag{3.36}$$

In fact, (3.35) becomes

$$G(r, \tilde{r}, \theta, \tilde{\theta}, y - \tilde{y}) = \rho^{-k-1} N_f(G)(\omega, \theta, \tilde{\theta}). \tag{3.37}$$

Another crucial property, which is derived from an analogous property for $G_0$, is that $N_f(G)(\omega, \theta, \tilde{\theta})$ has asymptotic expansions with smooth coefficients in powers of $\omega_0$ or $\omega_{k+1}$ as either of these variables tends to zero, and has a product type expansion as both variables tend to zero. These expansions produce similar expansions for $G$ as either $r$ or $\tilde{r}$ tend to zero, and using the equation $LG = I$ these powers can be determined using only formal reasoning. For example, since near any point $(0, \theta, \tilde{\theta}, y - \tilde{y})$, $y \neq \tilde{y}$, the Schwartz kernel $G$ satisfies $LG = 0$, we see that

$$G(r, \tilde{r}, \theta, \tilde{\theta}, y - \tilde{y}) \sim \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} r^{\gamma_j^+ + \ell} G_\ell(\tilde{r}, \theta, \tilde{\theta}, y - \tilde{y}) \tag{3.38}$$

where the sequences stem from the basic solutions $r^{\gamma_j^+} \phi_j(\theta)$ of the equation $Lu = 0$. The $\gamma_j^+$ here are of course the indicial roots (3.1) and the $\phi_j$ are the eigenfunctions of $L_\theta$. The reason that only the $\gamma_j^+$, but not the $\gamma_j^-$ occur in this expansion is because of the choice of the particular weighted spaces $r^{(3-n)/2} H_e^\ell$ on which $G$ is bounded.

There is a similar expansion for $G$ as $\tilde{r} \to 0$, but its powers are slightly different. To determine them it is simplest to use the self-adjointness of $L$ as in (3.31), i.e. with respect to the measure $r^{n-3} dr\, d\theta\, dy$. Since $G$ must also be self-adjoint with respect to this measure, it follows that

$$G(\tilde{r}, r, \theta, \tilde{\theta}, \tilde{y} - y) = r^{3-n} G(r, \tilde{r}, \theta, \tilde{\theta}, y - \tilde{y}) \tilde{r}^{n-3}. \tag{3.39}$$

Hence as $\tilde{r} \to 0$ (and in a bounded set where $r > 0$ and $|y - \tilde{y}| < \infty$), $G(r, \tilde{r}, \theta, \tilde{\theta}, y - \tilde{y})$ has an expansion involving the powers

$$\tilde{r}^{\gamma_j^+ + n - 3 + \ell}.$$

The smallest such power that can occur is

$$r \equiv \gamma_1^+ + n - 3 = \frac{n-2}{2} + \nu_1 - 1 \tag{3.40}$$

The quantity $\nu_1$ here is the usual square root as defined earlier.

One further comment about the expansions of $G$ as either $r$ or $\tilde{r}$ tend to zero is that they are controlled by the homogeneity (3.37) in conical regions of the form

$$0 < \tilde{r} \le C_1 r, \quad \text{and} \quad 0 < r \le C_2 \tilde{r}. \tag{3.41}$$

We finally return to our explicit problem, namely showing that the solution $w = Gf$ for $f \in S^p([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ supported in $r \ge 1/2$, is defined and an element of some $S^q([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$. The immediate problem is that it is not clear that $Gf$ even makes sense, for we have only determined that $G$ is defined and bounded on a certain $L^2$ space (3.34) which does not contain $S^p([1, \infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$. This is easily remedied, though, once we take the homogeneity (3.37) into account. Namely, the inner integral in

$$Gf(r, \theta, y) = \int \left\{ \int G(r, \tilde{r}, \theta, \tilde{\theta}, y - \tilde{y}) f(\tilde{r}, \theta, \tilde{y})\, d\theta\, d\tilde{y} \right\} d\tilde{r} \tag{3.42}$$

is convergent by virtue of the boundedness of $f$ and the fact that for $r, \tilde{r}$ fixed, $|G|$ is bounded by some power of $|y - \tilde{y}|^{-k-1}$ away from a neighbourhood of $y = \tilde{y}$ when $r = \tilde{r}, \theta = \tilde{\theta}$. The integral without the omission of this neighbourhood is also bounded by virtue of the usual boundedness of pseudodifferential operators.

It remains to check that the integral with respect to $\tilde{r}$ converges and produces a function with suitable decay in $r$ and with the requisite regularity. It is convenient at this point to split $G$ into a sum of two kernels $G = G_1 + G_2$ where the splitting is invariant with respect to dilations (hence corresponds to a splitting of $N_f(G)$). $G_1$ is taken to be supported in a neighbourhood of the diagonal and $G_2$ is supported outside a slightly smaller neighbourhood. Hence $G_1$ contains all the singularities of the kernel, while $G_2$ contains all asymptotic information.

The fact that $G_1 f$ is defined, smooth and decays like $r^p$ follows quite easily from the local boundedness of pseudodifferential operators on Sobolev spaces, the dilation invariance of $G_1$ and the specific decay properties of $f$. The more complete statement that $G_1 f \in S^p([1,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$ requires a standard argument involving the commutation of the vector fields $r\partial_r$, $\partial_y$ and $\partial_\theta$ through $G_1$ and is described in [M1]. Hence it suffices to focus attention on $G_2$.

Since $G_2$ has a smooth Schwartz kernel (away from the boundaries) and is dilation invariant, the regularity of $G_2 f$ again is immediate once we know this function is defined and has the correct decay as $r \to \infty$. Since we are only interested in this decay estimate, it now suffices to replace $f$ by the simple function $\tilde{r}^p$. Hence we are left with analyzing the integral

$$\int \left\{ \int G_2(r, \tilde{r}, \theta, \tilde{\theta}, y - \tilde{y})\, d\theta\, d\tilde{y} \right\} \tilde{r}^p\, d\tilde{r}. \tag{3.43}$$

The inner integral here may be regarded as a pushforward of the kernel $G_2$ with respect to the projection

$$\mathbb{R}^+ \times \mathbb{R}^+ \times \Sigma^2 \times \mathbb{R}^k \to \mathbb{R}^+ \times \mathbb{R}^+.$$

Denote this pushed forward kernel by $H(r, \tilde{r})$, and let $\rho = |(r, \tilde{r})|$, $\omega = \arctan(\tilde{r}/r)$ be the polar coordinates in the upper quadrant in $\mathbb{R}^2$. We have already observed that the pushforward defining $H$ is well-defined by virtue of the homogeneity (of degree $-k-1$) of $G_2$. If we change variables in the inner integral of (3.43) from $y - \tilde{y}$ to $(y - \tilde{y})/r$ and take this homogeneity into account, then the function $H$ can be seen to be homogeneous of degree $-1$ in $(r, \tilde{r})$, i.e.

$$H(r, \tilde{r}) = \rho^{-1} N_f(H)(\omega). \tag{3.44}$$

Now it is clear that since $p < 0$ the integral

$$\int_{\frac{1}{2}}^{\infty} H(r, \tilde{r})\, \tilde{r}^p \, d\tilde{r} \tag{3.45}$$

is at least convergent. At first glance, though, the resulting function of $r$ seems only to decay like $r^{-1}$, which is not good enough for our purposes. However, recall that in a sufficiently small angle $0 \le \omega \le \epsilon$ we may estimate $N_f(G_2)$ by $\omega^{\gamma_1^+ + n-3}$. Thus we split the integral (3.45) into two pieces and estimate $H$ by $r^{-1}(\tilde{r}/r)^{\gamma_1^+ n-3}$ in the first and simply by $r^{-1}$ in the second (we are using that $\tilde{r}/r$ is comparable to $\omega$ when $0 \le \omega \le \epsilon$). The resulting integrals can be evaluated explicitly. This gives, respectively

$$\begin{aligned}\int_{1/2}^{\epsilon r} r^{-1}(\tilde{r}/r)^{\gamma_1^+ + n-3} \tilde{r}^p \, d\tilde{r} &= \int_{1/2r}^{\epsilon} r^p s^{\gamma_1^+ n-3+p} \, ds \\ &= r^p(C + C' r^{-\gamma_1^+ + 2-n-p}) \\ &= C r^p + C' r^{\gamma_1^-}\end{aligned}$$

and

$$\int_{\epsilon r}^{\infty} \tilde{r}^{p-1} \, d\tilde{r} = C'' r^p.$$

The first integration here used the change of variables $s = \tilde{r}/r$ and in the second the homogeneity factor $1/r$ was replaced by $1/\tilde{r}$.

Since all terms here decay no slower than $r^q$, we have established at last that $Gf$ decays like $r^q$, and hence that $Gf \in S^q([1,\infty); \mathcal{C}_b^\infty(\mathbb{R}^k \times \Sigma))$. Continuity of the map $G$ is implicit in what we have done. This completes the proof of Lemma (3.13) and hence of the main theorem.

The results and proofs of this section generalize in a straightforward manner to produce the decomposition (2.21) when $\gamma_1^- = \gamma_1^+ = (2-n)/2$, and the details are left to the reader.

ADDED IN PROOF. Leon Simon has recently obtained very strong general results about the singular sets of harmonic maps in *Rectifiability of the singular set of energy minimizing maps*, to appear in Calc. of Variations and Partial Diff. Eqns.

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