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Higher cohomology triples and holomorphic extensions

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We introduce equations for special metrics, and notions of stability for some new types of augmented holomorphic bundles. These new examples include holomorphic extensions, and in this case we prove a Hitchin–Kobayashi correspondence between a certain deformation of the Hermitian–Einstein equations and our definition of stability for an extension.

1. Introduction.

There are three natural moduli spaces associated to a smooth complex bundle over a Kähler manifold; one algebraic, one complex analytic, and one symplectic. The first is the moduli space of slope stable holomorphic structures on E, and is constructed by Geometric Invariant Theory. The second, the moduli space of Hermitian–Einstein connections, is constructed by gauge theory and deformation theory. For the third, one uses the symplectic structure induced on the space of unitary connections and considers the moment map for the action of the unitary gauge group. The symplectic moduli space is then the Marsden–Weinstein quotient of the zero level of the moment map by the action of the unitary gauge group.

In fact, it is well known that these three quotients can all be identified, and this is referred to as the Hitchin–Kobayashi correspondence. Furthermore, this triad of descriptions has, in recent years, been found to be a common feature in an ever expanding range of situations. In most of these, the moduli spaces are for augmented bundles of one kind or another, i.e. for objects consisting of one or more holomorphic bundle together with prescribed holomorphic sections. A summary of such results can be found in [BDGW].

In this paper we discuss some extensions of these ideas in two directions that have not hitherto been pursued. This involves consideration of an interesting class of equations which includes deformations of the Hermitian–Einstein equations as well as certain generalizations of the equations known as the vortex equations. It also requires the introduction of a new concepts of stability for various augmented bundles.

In the one class of examples that we discuss, the starting point is the observation that symplectic reduction can be carried out more generally than simply at the 0-level set. In particular, symplectic quotients can be constructed from the inverse images of coadjoint orbits in the dual of the Lie algebra of the unitary gauge group. It is natural to look for a description of such reduced spaces as complex quotients and to try to find an algebraic characterization of these quotients as moduli spaces.

The simplest example of such a generalization can be described as follows. Suppose that $E = E_1 \oplus E_2$. Fix a smooth metric K on E such that the above splitting of E is an orthogonal decomposition. Let $T_{\tau_1,\tau_2} \in \mathfrak{G}$ be the global gauge transformation given by

$$T_{\tau_1,\tau_2} = \begin{pmatrix} i\tau_1 \mathbf{I}_1 & 0 \\ 0 & i\tau_2 \mathbf{I}_2 \end{pmatrix}$$

with respect to the given splitting of E, and let $\mathcal{O}(\tau_1, \tau_2)$ be the coadjoint orbit of T_{τ_1,τ_2}. Pursuing this example, we find interesting "deformations" of both the Hermitian–Einstein equations and the notion of bundle stability. Furthermore, these are naturally interpreted in terms of holomorphic extensions.

The second type of structure we consider is a natural generalization of the triples described in [BGP]. In [BGP] we described objects consisting of two holomorphic bundles, $\mathcal{E}_1$ and $\mathcal{E}_2$ plus a map between them, i.e. a section $\Phi \in H^0(X, Hom(\mathcal{E}_2, \mathcal{E}_1))$. In the generalization we have in mind, we take Φ in $H^p(X, Hom(\mathcal{E}_2, \mathcal{E}_1))$, for any p. We call such objects *p-cohomology triples.* Apart from their interest as natural generalizations of the original triples, such objects (with $p = 2$) have been encountered in the work of Pidstrigach and Tyurin ([PT]), and more recently in connection with the Seiberg–Witten invariants for Kähler surfaces (cf. [W]). For the case $p = 0$, we described in [BGP] what the natural notion of stability is, and what the corresponding equations for special metrics look like. In this paper we discuss how these can be modified to describe the more general situation.

The case of $p = 1$ is of particular interest, since elements in

$$H^1(X, Hom(\mathcal{E}_2, \mathcal{E}_1))$$

can be interpreted as extension classes. This leads to interesting relations between the two kinds of situations described above. We describe in some detail how these points of view compare. We also relate these to yet another description of holomorphic extensions, namely one in terms of the bundles $\mathcal{E}$, $\mathcal{E}_2$ plus surjective maps $\pi : \mathcal{E} \longrightarrow \mathcal{E}_2$. Such objects, which describe extensions of $\mathcal{E}_2$ by the kernel of the map, can be thought of as a special type of $p = 0$ triples. More specifically, they correspond to such triples in which the map between the bundles is surjective. We thus discuss the relation between such surjective ($p = 0$) triples, 1-cohomology triples and extensions.

Remark. The result given in Theorem 3.9 has been proved independently by Daskalopoulos, Uhlenbeck and Wentworth [DUW]. With stability defined as in in Definition 3.4, they have gone on to give analytic as well as invariant theory constructions of the moduli spaces of stable extensions.

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2. Cohomology triples.

Let (X, ω) be a compact Kähler manifold of dimension n, and fix two smooth complex bundles $E_i \longrightarrow X$, $i = 1, 2$. Denote their ranks and degrees by d_i and r_i, where by the degree we mean, in general, $\int_X c_1(E_1) \wedge \omega^{n-1}$. In order to simplify certain formulae, we assume that the volume of X is normalised to 2π.

As defined in [GP] and [BGP], a holomorphic triple based on E_1 and E_2 consists of holomorphic structures (given by $\overline{\partial}$-operators $\overline{\partial}_1$ and $\overline{\partial}_2$) on these bundles plus a holomorphic section of $Hom(E_2, E_1)$.

There are two distinct ways in which one might want to generalize this to allow form-valued augmentations.

1. In the first, which we call *p-cocycle triples*, one replaces holomorphic sections in $\Omega^0(Hom(E_2, E_1))$ by *holomorphic sections in* $\Omega^{0,p}(Hom(E_2, E_1))$.

2. In the second, which we call *p-cohomology triples*, the augmentation is considered to be the class in $H^{0,p}(Hom(E_2, E_1))$ represented by a holomorphic section in $\Omega^{0,p}(Hom(E_2, E_1))$.

As will be seen (cf. Section 2.2), there are compelling reasons for regarding the second approach as the "correct" one. Nevertheless, at least in the case where X is a Riemann surface, there are interesting features of both types of augmentation. In the case that $\mathcal{E}_2$ is fixed to be the structure sheaf, the resulting objects may be considered as p-cocycle and p-cohomology pairs.

2.1. The basics.

Set

$$\chi^{(p)} = \mathcal{C}_1 \times \mathcal{C}_2 \times \Omega^{0,p}(Hom(E_2, E_1)) \ , \tag{2.1.1}$$

where $\mathcal{C}_i$ denotes the space of holomorphic structures (or equivalently, the space of $\overline{\partial}$-operators) on E_i.

Definition 2.1. We can define the space of all *p-cocycle triples* on (E_1, E_2) by the holomorphic subspace

$$\mathcal{Z}^{(p)} = \{(\overline{\partial}_1, \overline{\partial}_2, \phi) \in \chi^{(p)} \ : \ \overline{\partial}_{1,2}(\phi) = 0\} \ , \tag{2.1.2}$$

where $\overline{\partial}_{1,2}(\phi) = \overline{\partial}_1 \circ \phi - \phi \circ \overline{\partial}_2$.

We can define an equivalence relation on $\mathcal{Z}^{(p)}$ by

$$(\overline{\partial}_1, \overline{\partial}_2, \phi) \sim (\overline{\partial}_1, \overline{\partial}_2, \phi + \overline{\partial}_{1,2}(\alpha)) \ ,$$

for any $\alpha \in \Omega^{0,p-1}(Hom(E_2, E_1))$. The p-cohomology triples are described by the equivalence classes in $\mathcal{Z}^{(p)}/\sim$. Notice that these equivalence classes correspond to orbits of the additive group $\Omega^{p-1}_{1,2} := \Omega^{0,p-1}(Hom(E_2, E_1))$ under the action

$$\alpha \circ (\overline{\partial}_1, \overline{\partial}_2, \phi) = (\overline{\partial}_1, \overline{\partial}_2, \phi + \overline{\partial}_{1,2}(\alpha)) \ .$$

Definition 2.2. The space of all *p-cohomology triples* on (E_1, E_2) is defined by

$$\mathcal{H}^{(p)} = \mathcal{Z}^{(p)}/\Omega^{p-1}_{1,2} \ . \tag{2.1.3}$$

Definition/Lemma 2.3. The complex gauge group $\mathfrak{G}_{\mathbb{C}} = \mathfrak{G}_{\mathbb{C}}^{(1)} \times \mathfrak{G}_{\mathbb{C}}^{(2)}$ acts on both $\mathcal{Z}^{(p)}$ and $\mathcal{H}^{(p)}$. In both cases, the $\mathfrak{G}_{\mathbb{C}}$-orbits correspond to isomorphism classes, with the notion of isomorphism defined in the obvious way. Thus the "moduli spaces" of isomorphism classes of p-cocycle (resp. p-cohomology) triples corresponds to the orbit space $\mathcal{Z}^{(p)}/\mathfrak{G}_{\mathbb{C}}$ (resp. $\mathcal{H}^{(p)}/\mathfrak{G}_{\mathbb{C}}$).

It is important to observe that in the double quotient

$$\mathcal{H}^{(p)}/\mathfrak{G}_{\mathbb{C}} = (\mathcal{Z}^{(p)}/\Omega_{1,2}^{p-1})/\mathfrak{G}_{\mathbb{C}} \ ,$$

the order of the quotient operations cannot be reversed. Not only do the actions of $\Omega_{1,2}^{p-1}$ and $\mathfrak{G}_{\mathbb{C}}$ fail to commute, but $\Omega_{1,2}^{p-1}$ does not act on $\mathcal{Z}^{(p)}/\mathfrak{G}_{\mathbb{C}}$ in any obvious way. Nevertheless, the quotient $\mathcal{H}^{(p)}/\mathfrak{G}_{\mathbb{C}}$ can be described as a quotient of $\mathcal{Z}^{(p)}$, namely as the quotient by the group action of the semidirect product $\Omega_{1,2}^{p-1} \ltimes \mathfrak{G}_{\mathbb{C}}$.

Definition/Lemma 2.4. We can identify

$$(\mathcal{Z}^{(p)}/\Omega_{1,2}^{p-1})/\mathfrak{G}_{\mathbb{C}} = \mathcal{H}^{(p)}/\Omega_{1,2}^{p-1} \ltimes \mathfrak{G}_{\mathbb{C}}$$

where the group structure on the semidirect product is defined by

$$(\alpha, g_1, g_2)(\alpha', g_1', g_2') = ({g_1'}^{-1}\alpha g_2' + \alpha', g_1 g_1', g_2 g_2') \ , \tag{2.1.4}$$

and the action on $\mathcal{Z}^{(p)}$ is

$$(\alpha, g_1, g_2)(\overline{\partial}_1, \overline{\partial}_2, \phi) = (g_1(\overline{\partial}_1), g_2(\overline{\partial}_2), g_1(\phi + \overline{\partial}_{1,2}(\alpha))g_2^{-1}) \ . \tag{2.1.5}$$

2.2. Stability with parameters.

As usual, one cannot expect the orbit spaces $\mathcal{Z}^{(p)}/\mathfrak{G}_{\mathbb{C}}$ or $\mathcal{H}^{(p)}/\mathfrak{G}_{\mathbb{C}}$ to yield well behaved moduli spaces without restricting to suitably defined spaces of "stable" orbits. The definition of stability that we propose for cohomology triples is a reasonably straightforward extensions of the stability defined for triples in [BGP]. Since this definition is in terms of a condition on subtriples, we need to specify precisely what we mean by the subobjects of cohomology triples.

Let $\mathcal{E}_1 = (E_1, \overline{\partial}_1)$ and $\mathcal{E}_2 = (E_2, \overline{\partial}_2)$ holomorphic vector bundles on X and $\Phi \in H^p(Hom(\mathcal{E}_2, \mathcal{E}_1))$. To define the subobjects of the cohomology triple $T = (\mathcal{E}_1, \mathcal{E}_2, \Phi)$ we need to determine the category to which T belongs.

The subobjects of T will be then certain objects in this category—the ones for which there is an injective morphism to T.

The category we need to consider is the category of "Ext^p" triples. Its elements consist of triples $(\mathcal{F}_1, \mathcal{F}_2, \Psi)$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are coherent sheaves on X and Ψ is an element of $\mathrm{Ext}^p(\mathcal{F}_2, \mathcal{F}_1)$.

Recall that $\mathrm{Ext}^0(\mathcal{F}_2, \mathcal{F}_1) = \mathrm{Hom}(\mathcal{F}_2, \mathcal{F}_1) \cong H^0(Hom(\mathcal{F}_2, \mathcal{F}_1))$, and if $\mathcal{F}_2$ is locally free:

1) $Hom(\mathcal{F}_2, \mathcal{F}_1) \cong \mathcal{F}_1 \otimes \mathcal{F}_2^*$

2) $\mathrm{Ext}^p(\mathcal{F}_2, \mathcal{F}_1) \cong H^p(\mathcal{F}_1 \otimes \mathcal{F}_2^*)$.

Let $T = (\mathcal{F}_1, \mathcal{F}_2, \Psi)$ and $T' = (\mathcal{F}_1', \mathcal{F}_2', \Psi')$ be two Ext^p triples. A morphism $T' \longrightarrow T$ consists of morphisms $f_1 : \mathcal{F}_1' \longrightarrow \mathcal{F}_1$ and $f_2 : \mathcal{F}_2' \longrightarrow \mathcal{F}_2$ such that under the induced maps

$$\mathrm{Ext}^p(\mathcal{F}_2', \mathcal{F}_1') \xrightarrow{f_{1*}} \mathrm{Ext}^p(\mathcal{F}_2', \mathcal{F}_1) \xleftarrow{f_2^*} \mathrm{Ext}^p(\mathcal{F}_2, \mathcal{F}_1)$$

one has that

$$f_{1*}(\Psi') = f_2^*(\Psi). \tag{2.2.1}$$

Note that when $p = 0$, this is equivalent to having the following commutative diagram

$$\begin{array}{ccc} \mathcal{F}_2 & \xrightarrow{\Psi} & \mathcal{F}_1 \\ \uparrow{\scriptstyle f_2} & & \uparrow{\scriptstyle f_1} \\ \mathcal{F}_2' & \xrightarrow{\Psi'} & \mathcal{F}_1'. \end{array}$$

Definition 2.5. Let $T = (\mathcal{F}_1, \mathcal{F}_2, \Psi)$ be an Ext^p triple. A *subobject* T' of T consists of an Ext^p triple $(\mathcal{F}_1', \mathcal{F}_2', \Psi')$ such that one has injections $i_1 : \mathcal{F}_1' \hookrightarrow \mathcal{F}_1$ and $i_2 : \mathcal{F}_2' \hookrightarrow \mathcal{F}_2$, which induce a morphism from T' to T, i.e. $i_{1*}(\Psi') = i_2^*(\Psi)$.

Notice that if $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ is a cohomology triple, (where $\mathcal{E}_1$ and $\mathcal{E}_2$ are locally free and $\Phi \in H^p(Hom(\mathcal{E}_2, \mathcal{E}_1)))$, a subobject *need not be* a cohomology triple. In general it will be only an Ext^p triple, i.e. $\mathcal{E}_1' \hookrightarrow \mathcal{E}_1$ and $\mathcal{E}_2' \hookrightarrow \mathcal{E}_2$ are not necessarily locally free and $\Phi' \in \mathrm{Ext}^p(\mathcal{E}_2', \mathcal{E}_1')$ (If $\mathcal{E}_2'$ is not locally free this cannot be identified with $H^p(Hom(\mathcal{E}_2', \mathcal{E}_1'))$.

The definition of stability is given in terms of *defect functions* (Alastair King's terminology) for pairs of bundles:

Definition 2.6. Let (E_1, E_2) be a pair of bundles of degree d_1 and d_2, and rank r_1 and r_2. Fix real numbers $\{a_1, a_2, \tau_1, \tau_2\}$, and define $\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1, \mathcal{E}_2)$ by

$$\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1, \mathcal{E}_2) = a_1 d_1 + a_2 d_2 - \tau_1 r_1 - \tau_2 r_2 \ . \tag{2.2.2}$$

Of course this definition makes also sense if $\mathcal{E}_1$ and $\mathcal{E}_2$ are coherent sheaves not necessarily locally free.

Definition 2.7. Let $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ be a p-cohomology triple based on the smooth bundles (E_1, E_2). Fix real numbers $\{a_1, a_2, \tau_1, \tau_2\}$ with a_1 and a_2 non-negative, and such that

$$a_1 d_1 + a_2 d_2 - \tau_1 r_1 - \tau_2 r_2 = 0 \ ,$$

i.e. such that $\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1, \mathcal{E}_2) = 0$. We say that the triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ is $(a_1, a_2, \tau_1, \tau_2)$-*stable* if

$$\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', \mathcal{E}_2') < 0$$

for all non-trivial subobjects $(\mathcal{E}_1', \mathcal{E}_2', \Phi')$.

Remarks.

1. As usual, to study stability questions it suffices to consider *saturated* subobjects of $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ (we are assuming that $\mathcal{E}_1$ and $\mathcal{E}_2$ are torsion free). These are subobjects $(\mathcal{E}_1', \mathcal{E}_2', \Phi')$ for which the inclusions $\mathcal{E}_1' \hookrightarrow \mathcal{E}_1$ and $\mathcal{E}_2' \hookrightarrow \mathcal{E}_2$ are saturated, i.e. $\mathcal{E}_1/\mathcal{E}_1'$ and $\mathcal{E}_2/\mathcal{E}_2'$ are torsion free.

2. As in the case of *cohomology* triples, before we can define stability for a *cocycle* triple, we must first establish what the legitimate subobjects are. We immediately run into difficulty when we consider what the appropriate category for such objects should be. Denote the objects in the category by $(\mathcal{F}_1, \mathcal{F}_2, \phi)$. The problem is that we do not want to require that $\mathcal{F}_1$ and $\mathcal{F}_2$ be locally free sheaves. This means we need to have a replacement for $\Omega^{0,p}(Hom(\mathcal{F}_2, \mathcal{F}_1))$ in the case that $\mathcal{F}_1$ and $\mathcal{F}_2$ are *not* locally free. This is one of the reasons that cohomology triples are to be preferred to their cocycle cousins .

   Notice that when X is a Riemann surface, these difficulties do not arise, and sensible definitions can be given. Subobjects are defined to be cocycle triples $(\mathcal{F}_1', \mathcal{F}_2', \phi')$ with injections $i_1 : \mathcal{F}_1' \longrightarrow \mathcal{F}_1$ and $i_2 : \mathcal{F}_2' \longrightarrow \mathcal{F}_2$ such that

   $$i_1 \circ \phi' = \phi' \circ i_2 \ .$$

Stability with respect to parameters $\{a_1, a_2, \tau_1, \tau_2\}$ is then defined exactly as for cohomology triples.

3. Finally, suppose that $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ is a 0-cohomology triple, i.e. a triple with $\phi \in H^0(Hom(E_2, E_1))$. We recover the old definition of τ-stability given in [BGP] by taking $\{a_1, a_2, \tau_1, \tau_2\} = \{1, 1, \tau, \tau'\}$. The definition above is thus a generalization of τ-stability.

The parameter space for the parameters in the definition of stability can be described as follows. Let $\mathbf{Par} \subset \mathbb{R}^4$ be the subspace

(2.2.3)

$$\mathbf{Par} = \{(a_1, a_2, \tau_1, \tau_2) \mid a_1 \geq 0 \ , \ a_2 \geq 0 \ , \ a_1 d_1 + a_2 d_2 - \tau_1 r_1 - \tau_2 r_2 = 0 \ \} \ .$$

Notice that the definition of $(a_1, a_2, \tau_1, \tau_2)$-stability is insensitive to an overall scaling of $(a_1, a_2, \tau_1, \tau_2)$ by a positive scale factor. The effective parameter space is thus $\mathbf{Par}/\mathbb{R}^+$. The "geography" of this parameter space is an interesting issue, which we will return to in a later paper. There are however, a few features which are immediately apparent.

The first feature comes from the fact that (at least for the case when X is algebraic), the degrees and ranks of subobjects may be assumed to be integers, i.e. to lie in a discrete subset of $\mathbb{R}$. It follows immediately that

Lemma 2.8. *The parameter space* $\mathbf{Par}/\mathbb{R}^+$ *is partitioned into chambers. The walls are determined by the choices of* $(a_1, a_2, \tau_1, \tau_2)$ *at which the relation* $\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', \mathcal{E}_2') = 0$ *is numerically possible. Within a fixed chamber the definition of* $(a_1, a_2, \tau_1, \tau_2)$*-stability is independent of the values of* $(a_1, a_2, \tau_1, \tau_2)$.

The next result identifies a special region within $\mathbf{Par}/\mathbb{R}^+$.

Proposition 2.9. *(1) Suppose that* $a_1 > 0$. *Then the space of* $(a_1, a_2, \tau_1, \tau_2)$*-stable objects is empty unless* $\tau_1/a_1 > \mu(\mathcal{E}_1)$.

(2) There are positive numbers ϵ_1, ϵ_2 *such that the following is true: Let* $(a_1, a_2, \tau_1, \tau_2)$ *be any point in* $\mathbf{Par}$ *such that*

$$a_1 a_2 \neq 0 \ ,$$
$$0 < \frac{\tau_1}{a_1} - \mu(E_1) < \epsilon_1 \ ,$$
$$\frac{\tau_1}{a_1} - \mu(E_1) < (\frac{a_2}{a_1})\epsilon_2 \ .$$

Then in any $(a_1, a_2, \tau_1, \tau_2)$*-stable object, say* $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$, *the bundles* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are semistable. Conversely, if* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are stable bundles, then all cohomology triples* $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ *are* $(a_1, a_2, \tau_1, \tau_2)$*-stable.*

Proof. Both parts of the proposition use the following observations. Let $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ be any p-cohomology triple. For any subsheaf $\mathcal{E}_1'$ of $\mathcal{E}_1$ we can construct the subtriple $(\mathcal{E}_1', 0, 0)$. Furthermore, for any subsheaf $\mathcal{E}_2' \hookrightarrow \mathcal{E}_2$ we can construct the subtriple $(\mathcal{E}_1, \mathcal{E}_2', \Phi')$ by taking $\Phi' = i_2^*(\Phi)$. Notice that

$$\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', 0) = a_1 r_1'(\mu(\mathcal{E}_1') - \tau_1/a_1) \ . \tag{2.2.4}$$

Part (1) follows immediately from this. For part (2), we observe that if $a_1 \neq 0$, then we can write (2.2.4) as

$$\frac{\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', 0)}{a_1} = r_1'(\mu(\mathcal{E}_1') - \mu(\mathcal{E}_1)) + r_1'(\mu(\mathcal{E}_1) - \tau_1/a_1) \ . \tag{2.2.5}$$

It follows that if $|\mu(\mathcal{E}_1) - \tau_1/a_1|$ is sufficiently small, and if $\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', 0) < 0$, then $\mu(\mathcal{E}_1') - \mu(\mathcal{E}_1) \leq 0$. We now consider the subobjects coming from subsheaves of $\mathcal{E}_2$. For these,we get

$$\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1, \mathcal{E}_2') = a_1 d_1 + a_2 d_2' - r_1\tau_1 - r_2'\tau_2 \ . \tag{2.2.6}$$

Using the constraint equation $a_1 d_1 + a_2 d_2 - \tau_1 r_1 - \tau_2 r_2 = 0$, this can be written as

(2.2.7)
$$\frac{\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1, \mathcal{E}_2')}{a_2} = r_2'(\mu(\mathcal{E}_2') - \mu(\mathcal{E}_2)) - r_1(\frac{r_2 - r_2'}{r_2})(\frac{\tau_1/a_1 - \mu(\mathcal{E}_1)}{a_2/a_1}) \ .$$

Thus if $\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1, \mathcal{E}_2') < 0$, then

$$\mu(\mathcal{E}_2') - \mu(\mathcal{E}_2) < (\frac{r_1}{r_2'})(\frac{r_2 - r_2'}{r_2})(\frac{\tau_1/a_1 - \mu(\mathcal{E}_1)}{a_2/a_1}) \ .$$

Since $(\frac{r_1}{r_2'})(\frac{r_2-r_2'}{r_2})$ is bounded above, it follows that $\mu(\mathcal{E}_2') - \mu(\mathcal{E}_2) \leq 0$ if $\frac{\tau_1/a_1-\mu(\mathcal{E}_1)}{a_2/a_1}$ is sufficiently small. This completes the proof of the first claim in (2). The second claim also follows from the identities (2.2.5) and (2.2.7), which show that for any subtriple $(\mathcal{E}_1, \mathcal{E}_2', \Phi')$ we have

$$\begin{aligned}
&\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', \mathcal{E}_2') = \theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1, \mathcal{E}_2') + \theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', 0) - a_1 r_1(\mu(\mathcal{E}_1) - \tau_1/a_1) \\
&= a_2 r_2'(\mu(\mathcal{E}_2') - \mu(\mathcal{E}_2)) + a_1 r_1'(\mu(\mathcal{E}_1') - \mu(\mathcal{E}_1)) + a_1(r_1 - r_1')(\tau_1/a_1 - \mu(\mathcal{E}_1)) \\
&- a_2 r_1(\frac{r_2 - r_2'}{r_2})(\frac{\tau_1/a_1 - \mu(\mathcal{E}_1)}{a_2/a_1}) \ .
\end{aligned}$$

□

2.3. Comparison of cocycle and cohomology.

In the case that X is a Riemann surface, the following comparison between cohomology and cocycle triples makes sense. Let $\pi : \mathcal{Z}^{(p)} \longrightarrow \mathcal{H}^{(p)} = \mathcal{Z}^{(p)}/\Omega^{p-1}_{1,2}$ denote the projection map. Let $(a_1, a_2, \tau_1, \tau_2)$ be any set of real numbers satisfying the constraint

$$a_1 d_1 + a_2 d_2 - \tau_1 r_1 - \tau_2 r_2 = 0 \ .$$

Proposition 2.10. *Suppose that X is a Riemann surface. For any cohomology triple $(\overline{\partial}_1, \overline{\partial}_2, \Phi) \in \mathcal{H}^{(p)}$, the following are equivalent:*

1. $(\overline{\partial}_1, \overline{\partial}_2, \Phi) \in \mathcal{H}^{(p)}$ *is $(a_1, a_2, \tau_1, \tau_2)$-stable,*
2. *all cocycle triples $(\overline{\partial}_1, \overline{\partial}_2, \phi) \in \pi^{-1}(\overline{\partial}_1, \overline{\partial}_2, \Phi)$ are $(a_1, a_2, \tau_1, \tau_2)$-stable,*
3. *given any $(\overline{\partial}_1, \overline{\partial}_2, \phi) \in \pi^{-1}(\overline{\partial}_1, \overline{\partial}_2, \Phi)$, every cocycle triple on the $\Omega^{p-1}_{1,2}$-orbit through $(\overline{\partial}_1, \overline{\partial}_2, \phi)$ is $(a_1, a_2, \tau_1, \tau_2)$-stable.*

Proof. Statements (2) and (3) are obviously equivalent. We thus need only prove that (2) or (3) is equivalent to (1). To do so, we need to compare the definitions of stability for a cocycle triple and for a cohomology triple. In both cases, the definition is given in terms of the values of $\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}'_1, \mathcal{E}'_2)$, where $\mathcal{E}'_1$ and $\mathcal{E}'_2$ are the subbundles in either a cocycle subtriple, $(\overline{\partial}'_1, \overline{\partial}'_2, \phi')$, or a cohomology subtriple, $(\overline{\partial}'_1, \overline{\partial}'_2, \Phi')$. Notice that neither the ϕ' nor the Φ' affect the value of $\theta_{a_1,a_2,\tau_1,\tau_2}$ —their only role is to determine *on which pairs* $(\mathcal{E}'_1, \mathcal{E}'_2)$ the function must be evaluated. The proof thus consists essentially of a comparison of the *subobjects* of cocycle triples and of cohomology triples.

Suppose first that $(\overline{\partial}_1, \overline{\partial}_2, \Phi) \in \mathcal{H}^{(p)}$ is $(a_1, a_2, \tau_1, \tau_2)$-stable. Let $(\overline{\partial}_1, \overline{\partial}_2, \phi)$ be any cocycle triple in $\pi^{-1}(\overline{\partial}_1, \overline{\partial}_2, \Phi)$, and let $(\overline{\partial}'_1, \overline{\partial}'_2, \phi')$ be a cocycle subtriple. Then ϕ' defines a cohomology class, Φ', in $H^1(X, Hom(\mathcal{E}'_2, \mathcal{E}'_1))$, and $(\overline{\partial}'_1, \overline{\partial}'_2, \Phi')$ is clearly a cohomology subtriple of $(\overline{\partial}_1, \overline{\partial}_2, \Phi)$. Thus, by the stability of $(\overline{\partial}_1, \overline{\partial}_2, \Phi)$,

$$\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}'_1, \mathcal{E}'_2) < 0 \ ,$$

i.e. $(\overline{\partial}_1, \overline{\partial}_2, \phi)$ is $(a_1, a_2, \tau_1, \tau_2)$-stable.

Conversely, suppose that all cocycle triples, $(\overline{\partial}_1, \overline{\partial}_2, \phi)$, in $\pi^{-1}(\overline{\partial}_1, \overline{\partial}_2, \Phi)$ are $(a_1, a_2, \tau_1, \tau_2)$-stable. Let $(\overline{\partial}'_1, \overline{\partial}'_2, \Phi')$ be any cohomology subtriple. Fix any representatives ϕ' (resp. ϕ) of Φ' (resp.Φ). Then the condition $i_{1*}(\Phi') =$

$i_2^*(\Phi)$ (cf. Definition 2.5) implies that $i_1 \circ \phi' = \phi \circ i_2 + (\overline{\partial}_1 \circ \alpha' - \alpha' \circ \overline{\partial}_2')$, for some $\alpha' \in \Omega^0(X, Hom(E_2', E_1))$. Let $\alpha \in \Omega^0(X, Hom(E_2, E_1))$ be any element such that $\alpha \circ i_2 = \alpha'$. Then, since i_2 is a holomorphic map, we get

$$\begin{aligned} \overline{\partial}_1 \circ \alpha' - \alpha' \circ \overline{\partial}_2' &= \overline{\partial}_1 \circ \alpha \circ i_2 - \alpha \circ i_2 \circ \overline{\partial}_2' \\ &= \overline{\partial}_1 \circ \alpha \circ i_2 - \alpha \circ \overline{\partial}_2 \circ i_2 \\ &= \overline{\partial}_{1,2}(\alpha) \circ i_2. \end{aligned}$$

That is, $i_1 \circ \phi' = (\phi + \overline{\partial}_{1,2}(\alpha)) \circ i_2$, and hence $(\overline{\partial}_1', \overline{\partial}_2', \phi')$ is a cocycle subtriple of $(\overline{\partial}_1, \overline{\partial}_2, \phi + \overline{\partial}_{1,2}(\alpha))$. Since $\pi(\overline{\partial}_1, \overline{\partial}_2, \phi + \overline{\partial}_{1,2}(\alpha)) = (\overline{\partial}_1, \overline{\partial}_2, \Phi)$, it now follows from the stability of this cocycle triple that $\theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', \mathcal{E}_2') < 0$, i.e. $(\overline{\partial}_1, \overline{\partial}_2, \Phi)$ is $(a_1, a_2, \tau_1, \tau_2)$-stable. □

2.4. Metric equations.

In this section we describe the metric equations corresponding to the above definitions of stability. Recall that for a triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ with $\Phi \in H^0(X, Hom(E_2, E_1))$, there is a Hitchin–Kobayashi correspondence between stability (as defined in [BGP]) and metrics satisfying the coupled vortex equations. As equations for metrics H_1 and H_2 on E_1 and E_2, these are

$$i\Lambda F_{H_1} + \Phi\Phi^* = \tau_1 \mathbf{I}_1 \ , \tag{2.4.1a}$$

$$i\Lambda F_{H_2} - \Phi^*\Phi = \tau_2 \mathbf{I}_2 \ , \tag{2.4.1b}$$

where Φ^* denotes the adjoint with respect to the metrics H_1 and H_2. To obtain the analogous equations corresponding to $(a_1, a_2, \tau_1, \tau_2)$-stability of a p-cohomology triple, we need the following operations on form-valued sections of bundles over Kähler manifolds (cf. [W]).

$$\wedge : \Omega^{p,q}(X, E) \times \Omega^{k,l}(X, E^*) \longrightarrow \Omega^{p+k,q+l}(X, \mathbb{C}) \ , \tag{2.4.2}$$

$$\begin{aligned} \circ : \Omega^{p,q}(X, Hom(E_1, E_2)) \times \Omega^{k,l}(X, Hom(E_2, E_1)) \\ \longrightarrow \Omega^{p+k,q+l}(X, Hom(E_1, E_1)) \ , \end{aligned} \tag{2.4.3}$$

$$\overline{*}_E : \Omega^{p,q}(X, E) \longrightarrow \Omega^{n-p,n-q}(X, E^*) \ . \tag{2.4.4}$$

These are defined such that for $\phi_i \in \Omega^{p,q}(X,E)$,

$$\phi_1 \wedge \bar{*}_E \phi_2 = (\phi_1, \phi_2)\frac{\omega^n}{n!} \tag{2.4.5}$$

where ω is the Kähler form, and (ϕ_1, ϕ_2) is the inner product coming from the metric on E and the metric on forms of type (p,q). Also, for $\phi_i \in \Omega^{p,q}(X, Hom(E_1, E_2))$, we have

$$\phi_1 \wedge \bar{*}_E \phi_2 = \mathrm{Tr}(\phi_1 \circ \bar{*}_E \phi_2) \tag{2.4.6}$$

Definition 2.11. Given a p-cohomology triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$, and real parameters $(a_1, a_2, \tau_1, \tau_2)$ we define the following equations for metrics on E_1 and E_2 and a representative $\phi \in \Omega^{0,p}(X, Hom(E_2, E_1))$ of the cohomology class Φ:

$$i\Lambda a_1 F_{H_1} + \Lambda^n(\phi \circ \bar{*}_E \phi) = \tau_1 \mathbf{I}_1 \ , \tag{2.4.7a}$$

$$i\Lambda a_2 F_{H_2} - (-1)^p \Lambda^n(\bar{*}_E \phi \circ \phi) = \tau_2 \mathbf{I}_2 \ , \tag{2.4.7b}$$

$$\overline{\partial}^*_{1,2}(\phi) = 0 \ . \tag{2.4.7c}$$

Remarks 2.12.
2.12.1 The sign of the terms involving ϕ are chosen such that $\mathrm{Tr}(\Lambda^n(\phi \circ \bar{*}_E \phi))$ and $\mathrm{Tr}((-1)^p \Lambda^n(\bar{*}_E \phi \circ \phi))$ are positive. This will be important in section 2.6.
2.12.2 The coefficients a_1 and a_2 will be assumed non-negative, and the parameters $(a_1, a_2, \tau_1, \tau_2)$ must satisfy the constraint

$$a_1 d_1 + a_2 d_2 - \tau_1 r_1 - \tau_2 r_2 = 0 \ . \tag{2.4.8}$$

2.12.3 The coupled vortex equations given in [BGP] correspond to the case $p = 0$ and $a_1 = a_2 = 1$. There is however no good reason to single out these special values for a_1, a_2. This is most clearly seen in the symplectic interpretation of the equations, and will be discussed in the next section. We remark in passing that there is no need to add scale factors to the terms involving ϕ since these can be absorbed in ϕ, or by a rescaling of the metrics.

2.5. Moment maps.

If we fix metrics K_1 and K_2 on E_1 and E_2, we can reduce the gauge groups to the real unitary groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$. In addition, $\mathcal{C}_i$, $i = 1,2$ and $\Omega^{0,p}(X, Hom(E_2, E_1))$ acquire symplectic structures in the usual way. We denote these by ω_1, ω_2, and $\omega_{(0,p)}$ respectively. A symplectic form on

$$\chi^{(p)} = \mathcal{C}_1 \times \mathcal{C}_2 \times \Omega^{0,p}(Hom(E_2, E_1)) \ ,$$

can be produced by taking the sum $\omega_1 + \omega_2 + \omega_{(0,p)}$. This is, however, merely one possibility; given any real positive numbers a_1 and a_2, we can form a symplectic structures on $\chi^{(p)}$ by defining

$$\omega_{a_1,a_2} = a_1\omega_1 + a_2\omega_2 + \omega_{(0,p)} \ . \tag{2.5.1}$$

Lemma 2.13. *The group $\mathfrak{G}_1 \times \mathfrak{G}_2$ acts symplectically on (χ, ω_{a_1,a_2}), and has a moment map*

$$\Psi_{a_1,a_2} : \chi \longrightarrow \mathfrak{g}_1 \times \mathfrak{g}_2$$

given by
(2.5.2)

$$\Psi_{a_1,a_2}(\overline{\partial}_1, \overline{\partial}_2, \phi) = (a_1 \Lambda F_{K_1} - i\Lambda^n(\phi \circ \overline{*}_E \phi), a_2 \Lambda F_{K_2} - i(-1)^p \Lambda^n(\overline{*}_E \phi \circ \phi) \ .$$

Proof. Exactly the same as for the $p = 0$ case. The sign factor $(-1)^p$ comes from interchanging the order in a wedge product forms of type $(0, p)$ and $(n, n-p)$. □

With $\mathcal{Z}^{(p)}$ as in Definition 2.1, define $\mathcal{H}^{eq}_{a_1,a_2,\tau_1,\tau_2} \subset \mathcal{Z}^{(p)}$ to be the subset on which solutions to equations (2.4.7a,b) can be found. Also define $\mathcal{H}ar^{(p)} \subset \mathcal{Z}^{(p)}$ to be the subset on which solutions to equation (2.4.7c) can be found. Then we have

Proposition 2.14. *There is a bijective correspondence*

$$\mathcal{H}^{eq}_{a_1,a_2,\tau_1,\tau_2} / \mathfrak{G}^{(1)}_{\mathbb{C}} \times \mathfrak{G}^{(2)}_{\mathbb{C}} \longleftrightarrow (\Psi^{-1}_{a_1,a_2}(-i\tau_1, -i\tau_2) \cap \mathcal{H}ar^{(p)}) / \mathfrak{G}_1 \times \mathfrak{G}_2 \ .$$

Unfortunately, there does not seem to be a way to realize the harmonicity condition as a moment map condition.

2.6. Hitchin–Kobayashi correspondence.

In this section we show how our stability conditions follow as a consequence from the existence of solutions to the appropriate metric equations.

Lemma 2.15. *Let $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ be a p-cocycle triple (so $\phi \in \Omega^{0,p}(Hom(\mathcal{E}_2, \mathcal{E}_1))$ and $\overline{\partial}(\phi) = 0$). Let $(a_1, a_2, \tau_1, \tau_2)$ be any set of real numbers with $a_i \geq 0$ for $i = 1, 2$, and such that*

$$a_1 d_1 + a_2 d_2 - \tau_1 r_1 - \tau_2 r_2 = 0 \ .$$

Suppose there are bundle metrics H_1 and H_2 which satisfy the coupled equations (2.4.7a,b), i.e.

$$i\Lambda a_1 F_{H_1} + \Lambda^n(\phi \circ \overline{*}_E \phi) = \tau_1 \mathbf{I}_1 \ ,$$

$$i\Lambda a_2 F_{H_2} - (-1)^p \Lambda^n(\overline{*}_E \phi \circ \phi) = \tau_2 \mathbf{I}_2 \ .$$

Let $(\mathcal{E}_1', \mathcal{E}_2', \phi')$ be a locally free subtriple, i.e. suppose that $i_1 : \mathcal{E}_1' \hookrightarrow \mathcal{E}_1$ is a subbundle of $\mathcal{E}_1$, $i_2 : \mathcal{E}_2' \hookrightarrow \mathcal{E}_2$ is a subbundle of $\mathcal{E}_2$, and $\phi' \in \Omega^{0,1}(Hom(\mathcal{E}_2', \mathcal{E}_1'))$ satisfies the condition $i_1 \circ \phi' = \phi \circ i_2$.

Then

$$\theta_{a_1, a_2, \tau_1, \tau_2}(\mathcal{E}_1', \mathcal{E}_2') \leq 0 \ ,$$

with equality if and only if $(\mathcal{E}_1, \mathcal{E}_2, \phi)$ splits with $(\mathcal{E}_1', \mathcal{E}_2', \phi')$ as a direct summand.

Proof. Using the metrics H_1 and H_2 we can make orthogonal decompositions $\mathcal{E}_1 = \mathcal{E}_1' \oplus (\mathcal{E}_1/\mathcal{E}_1')$ and $\mathcal{E}_2 = \mathcal{E}_2' \oplus (\mathcal{E}_2/\mathcal{E}_2')$. With respect to these decompositions we can write ϕ as

$$\phi = \begin{pmatrix} \phi'' & \phi'^{\perp} \\ \phi^{\perp'} & \phi^{\perp\perp} \end{pmatrix} \ . \tag{2.6.1}$$

In view of the condition $i_1 \circ \phi' = \phi \circ i_2$, it follows that in (2.6.1) we have $\phi'' = \phi'$ and $\phi^{\perp'} = 0$, i.e.

$$\phi = \begin{pmatrix} \phi' & \phi'^{\perp} \\ 0 & \phi^{\perp\perp} \end{pmatrix} \ . \tag{2.6.2}$$

The conclusion now follows precisely as in the case of ordinary triples. More specifically, after writing the curvature terms with respect to the above orthogonal decompositions of the bundles, the equations (2.4.7a,b) yield the following:

$$i a_1 \Lambda F_{H_1}' + i a_1 \Lambda \Pi_1 + \Lambda^n(\phi'' \circ \overline{*}_E \phi'') + \Lambda^n(\phi'^{\perp} \wedge \overline{*}_E \phi'^{\perp}) = \tau_1 \mathbf{I}_1' \ , \tag{2.6.3a}$$

$$i a_2 \Lambda F_{H_2}' + i a_2 \Lambda \Pi_2 - (-1)^p \Lambda^n(\phi'' \wedge \overline{*}_E \phi'') = \tau_2 \mathbf{I}_2' \ . \tag{2.6.3b}$$

We can take the trace of these equations, and use the fact that for any section $\psi \in \Omega^{0,p}(Hom(\mathcal{E}_2, \mathcal{E}_1))$ we have

$$\mathrm{Tr}((-1)^p \Lambda^n \bar{*}_E \psi \wedge \psi) = \mathrm{Tr}(\Lambda^n \psi \wedge \bar{*}_E \psi) = |\psi|^2 \ . \tag{2.6.4}$$

This gives

$$a_1 d_1' + a_2 d_2' + a_1 \mathrm{Tr}(i\Lambda\Pi_1) + a_2 \mathrm{Tr}(i\Lambda\Pi_2) + |\phi'^{\perp}|^2 = \tau_1 r_1' + \tau_2 r_2' \ . \tag{2.6.5}$$

The conclusion follows directly from this, since both $\mathrm{Tr}(i\Lambda\Pi_1)$ and $\mathrm{Tr}(i\Lambda\Pi_2)$ are non-negative. □

We now consider p-cohomology triples over Riemann surfaces.

Theorem 2.16. *Let $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ be a p-cohomology triple over a Riemann surface X (so $\Phi \in H^p(Hom(\mathcal{E}_2, \mathcal{E}_1))$). Let $(a_1, a_2, \tau_1, \tau_2)$ be any set of real numbers with $a_i \geq 0$ and satisfying the constraint*

$$a_1 d_1 + a_2 d_2 - \tau_1 r_1 - \tau_2 r_2 = 0 \ .$$

Suppose there is a representative $\phi \in \Omega^{0,p}(Hom(\mathcal{E}_2, \mathcal{E}_1))$ for Φ, and bundle metrics H_1 and H_2, which satisfy the coupled equations (2.4.7a-c). Then $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ is $(a_1, a_2, \tau_1, \tau_2)$-stable.

Proof. Since X is a Riemann surface, all saturated subtriples are locally free. Let $(\mathcal{E}_1', \mathcal{E}_2', \Phi')$ be any such subtriple. To prove the Theorem we need to show that

$$\theta_{a_1, a_2, \tau_1, \tau_2}(\mathcal{E}_1', \mathcal{E}_2') < 0 \ .$$

Notice that by equation (2.4.7c), ϕ is the harmonic representative of Φ with respect to the metrics H_1 and H_2. Using the induced metrics on $\mathcal{E}_1'$ and $\mathcal{E}_2'$, take the harmonic representative, ϕ', of Φ'. We claim that $(\mathcal{E}_1', \mathcal{E}_2', \phi')$ is a subtriple of the 1-cocycle triple $(\mathcal{E}_1, \mathcal{E}_2, \phi)$, i.e. we claim that $\phi \circ i_2 = i_1 \circ \phi'$ where $i_1 : \mathcal{E}_1' \hookrightarrow \mathcal{E}_1$ and $i_2 : \mathcal{E}_2' \hookrightarrow \mathcal{E}_2$ are the inclusions. This will prove the proposition, since then by Lemma 2.12, we have $\theta_{a_1, a_2, \tau_1, \tau_2}(\mathcal{E}_1', \mathcal{E}_2') < 0$.

We now prove our claim. From the very definition of the maps induced by i_1 and i_2 in cohomology, we get that $\phi \circ i_2 = i_1 \circ \phi' + \overline{\partial}(\alpha)$, where $\alpha \in \Omega^0(Hom(\mathcal{E}_2', \mathcal{E}_1))$ and $\overline{\partial}$ denotes the operator induced by $\overline{\partial}_2'$ and $\overline{\partial}_1$. With respect to the orthogonal decompositions $\mathcal{E}_1 = \mathcal{E}_1' \oplus (\mathcal{E}_1/\mathcal{E}_1')$ and $\mathcal{E}_2 = \mathcal{E}_2' \oplus (\mathcal{E}_2/\mathcal{E}_2')$ we can thus write

$$\phi = \begin{pmatrix} \phi' + \beta'' & \phi'^{\perp} \\ \beta^{\perp'} & \phi^{\perp\perp} \end{pmatrix} ,$$

where $\overline{\partial}(\alpha) = \beta'' + \beta^{\perp'}$. But harmonic representative are norm minimizing. Thus $||\phi' + \beta''||^2 \geq ||\phi'||^2$, and therefore

$$||\phi||^2 \geq \left\| \begin{matrix} \phi' & \phi'^{\perp} \\ 0 & \phi^{\perp\perp} \end{matrix} \right\|^2 .$$

Thus we get $||\phi||^2 \geq ||\phi - \overline{\partial}(\alpha)||^2$, which is a contradiction unless $\overline{\partial}(\alpha) = 0$. □

3. Extensions.

The case of 1-cohomology triples deserves special attention because of the fact that a 1-cohomology class $\Phi \in H^1(Hom(\mathcal{E}_2, \mathcal{E}_1))$ can be interpreted as an extension class for extensions of $\mathcal{E}_2$ by $\mathcal{E}_1$. This can be exploited to study moduli space questions for the set of all such extensions, i.e. for the set of all short exact sequences

$$(e) \qquad 0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0 ,$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ have fixed underlying smooth bundles (denoted by E_1 and E_2 respectively).

Such extensions can also be considered from the point of view of the bundle $\mathcal{E}$. This leads to a metric problem and definition of stability that appear somewhat different to the ones considered in the previous section. In this section we discuss such an approach. In the next section we indicate the relationship between the two approaches.

Let us begin therefore with a compact Kähler manifold X, and a holomorphic bundle $\mathcal{E} \longrightarrow X$ given as an extensions of bundles as in (e).

3.1. Stability.

To formulate the stability condition, we consider extensions as objects in the category of short exact sequences of coherent sheaves of the form

$$(f) \qquad 0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0 .$$

Definition 3.1. A morphism of two extensions is defined by the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_2 & \longrightarrow & 0 \\ & & {\scriptstyle f_1}\uparrow & & {\scriptstyle F}\uparrow & & \uparrow{\scriptstyle f_2} & & \\ 0 & \longrightarrow & \mathcal{F}_1' & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}_2' & \longrightarrow & 0. \end{array} \tag{3.1.1}$$

A subobject of (f) consists then of an extension

$$(f') \qquad 0 \longrightarrow \mathcal{F}_1' \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F}_2' \longrightarrow 0,$$

and injective maps i_1, i_2, i such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}_2 & \longrightarrow & 0 \\ & & {\scriptstyle i_1}\uparrow & & {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i_2} & & \\ 0 & \longrightarrow & \mathcal{F}_1' & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F}_2' & \longrightarrow & 0. \end{array} \tag{3.1.2}$$

The extension (f') will be called a *subextension* of (f).

Lemma 3.2. *Let us consider the extension (f). Any subsheaf $i : \mathcal{F}' \hookrightarrow \mathcal{F}$ defines a subextension of (f).*

Proof. Let $g : \mathcal{F}' \longrightarrow \mathcal{F}_2$ be the map obtained by composing i with the surjection $\mathcal{F} \longrightarrow \mathcal{F}_2$. Then

$$0 \longrightarrow \operatorname{Ker} g \longrightarrow \mathcal{F}' \longrightarrow \operatorname{Im} g \longrightarrow 0$$

is the desired subextension. □

There is hence a one-to-one correspondence between subsheaves of $\mathcal{F}$ and subextensions of (f).

Definition 3.3. Let e be the extension

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

and e' the subextension

$$0 \longrightarrow \mathcal{E}_1' \longrightarrow \mathcal{E}_1' \longrightarrow \mathcal{E}_2' \longrightarrow 0.$$

For $\alpha \in \mathbb{R}$ we define the α-slope of e' as

$$\mu_\alpha(e') = \mu(\mathcal{E}') + \alpha \, \frac{\operatorname{rank} \mathcal{E}_2'}{\operatorname{rank} \mathcal{E}'}. \tag{3.1.3}$$

Definition 3.4. The extension e is said to be α-stable (resp. semistable) if and only if for every subextension $e' \subset e$ (resp. $e' \subseteq e$)

$$\mu_\alpha(e') < \mu_\alpha(e) \quad (\text{resp. } \leq). \tag{3.1.4}$$

Remark 3.5. If $\alpha = 0$, then α-stability is equivalent to ordinary stability.

Proposition 3.6. *Let e be α-stable, then*

$$\alpha > \mu(\mathcal{E}_1) - \mu(\mathcal{E}_2). \tag{3.1.5}$$

Conversely, there is some $\epsilon > 0$ such that for α in the interval

$$(\mu(E_1) - \mu(E_2), \mu(E_1) - \mu(E_2) + \epsilon) \ ,$$

the following is satisfied:
(1) If e is α-stable, then $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable.
(2) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are stable then e is α-stable.

Proof. For the first statement it suffices to apply the numerical stability condition to the trivial subextension

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_1 \longrightarrow 0 \longrightarrow 0.$$

The proof of the second statement is essentially identical to that of Proposition 2.9. In fact, in view of the results of section 4.2, this result can be treated as a special case of Proposition 2.9, corresponding to the case $p = 1, a_1 = a_2 = 1$. We can also give a direct proof which depends on an examination of the α-stability condition for special sub-objects. In this case the subobjects are subextensions with either $\mathcal{E}_2' = 0$ or $\mathcal{E}_1' = \mathcal{E}_1$. □

3.2. Metric equations.

Given an extension

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0 \ , \tag{e}$$

the natural metric problem is to look for a metric H on $\mathcal{E}$ satisfying the equation

$$i\Lambda F_H = \begin{pmatrix} \tau_1 \mathbf{I}_1 & 0 \\ 0 & \tau_2 \mathbf{I}_2 \end{pmatrix} . \tag{3.2.1}$$

Here τ_1 and τ_2 are real numbers an $\mathbf{I}_1$ and $\mathbf{I}_2$ are the identity endomorphisms in E_1 and E_2 respectively. We can make sense of the right hand side since the metric H on $\mathcal{E}$ gives a C^∞ splitting of (e), i.e. an identification of the smooth underlying bundle to $\mathcal{E}$ with $E_1 \oplus E_2$.
Remark. If $\tau_1 = \tau_2 = \lambda$, equation (3.2.1) reduces to the Hermitian–Einstein equation.

Proposition 3.7. *If H satisfies (3.2.1), then the parameters τ_1 and τ_2 are related by*

$$r_1\tau_1 + r_2\tau_2 = d_1 + d_2, \tag{3.2.2}$$

where $r_1 = \operatorname{rank}\mathcal{E}_1$, $r_2 = \operatorname{rank}\mathcal{E}_2$, $d_1 = \deg \mathcal{E}_1$ *and* $d_2 = \deg \mathcal{E}_2$.

Proof. This is easily proved by taking the trace in both sides of (3.2.1) and integrating. □

3.3. Hitchin–Kobayashi correspondence.

We first prove that α-stability is a necessary condition for existence of solutions to the equation (3.2.1).

Proposition 3.8. *Let e be the extension of vector bundles*

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0, \tag{e}$$

and let τ_1 and τ_2 satisfy (3.2.2). Set $\alpha = \tau_1 - \tau_2$. If $\mathcal{E}$ is indecomposable and admits a metric H satisfying the metric equation for extensions (3.2.1), then e is α-stable.

Proof. We need to show that $\mu_\alpha(e') < \mu_\alpha(e)$ for every subextension e'

$$0 \longrightarrow \mathcal{E}_1' \longrightarrow \mathcal{E}_1' \longrightarrow \mathcal{E}_2' \longrightarrow 0\ .$$

The proof is a minor modification of the analogous result for the ordinary Hitchin–Kobayashi correspondence. Consider first the locally free subextensions, i.e. the e' in which $\mathcal{E}_1'$ and $\mathcal{E}_2'$ are locally free. Denote the underlying smooth bundle for $\mathcal{E}'$ by E', and let $E^\perp$ be its orthogonal complement with respect to H. Then with respect to the smooth orthogonal splitting $E = E' \oplus E^{\perp}$, we get the block diagonal decomposition

$$\sqrt{-1}\Lambda F_H = \begin{pmatrix} \sqrt{-1}\Lambda F' + \Pi' & * \\ * & \sqrt{-1}\Lambda F^{\perp} - \Pi^{\perp} \end{pmatrix} \tag{3.3.1}$$

where $\Lambda F'$ and $\Lambda F^{\perp}$ are the induced metric connections on $\mathcal{E}'$ and $\mathcal{E}^{\perp}$ respectively, and Π', $\Pi^{\perp}$ are positive definite endomorphisms coming from the second fundamental form for the inclusion of E' in E. With respect to this splitting of E, the endomorphism on the right hand side of the metric equation is no longer diagonal, but has the form

$$\begin{pmatrix} T' & * \\ * & T^{\perp} \end{pmatrix} = A \begin{pmatrix} \tau_1 \mathbf{I}_1 & 0 \\ 0 & \tau_2 \mathbf{I}_2 \end{pmatrix} A^{-1} , \tag{3.3.2}$$

where the matrix A gives the transformation from the frame $E_1 \oplus E_2$ to $E' \oplus E^{\perp}$. If we make the further orthogonal decompositions of E_1 and E_2 into components in E' and $E^{\perp}$, then

$$E_1 \oplus E_2 = E_1' \oplus E_1^{\perp} \oplus E_2' \oplus E_2^{\perp} ,$$

and

$$E' \oplus E^{\perp} = E_1' \oplus E_2' \oplus E_1^{\perp} \oplus E_2^{\perp} .$$

With respect to these frames, the transformation A is represented by

$$\begin{pmatrix} \mathbf{I}_1' & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_2' & 0 \\ 0 & \mathbf{I}_1^{\perp} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_2^{\perp} \end{pmatrix} . \tag{3.3.3}$$

In fact, all we need is the trace of T'. It follows by a straightforward linear algebra computation that

$$\mathrm{Tr}(T') = r_1'\tau_1 + r_2'\tau_2 ,$$

where $r_1' = \operatorname{rank} \mathcal{E}_1'$ and $r_2' = \operatorname{rank} \mathcal{E}_2'$. We apply this to the condition

$$\sqrt{-1}\Lambda F' + \Pi' = T' ,$$

which can be extracted from the full metric equation. After taking the trace and integrating over X, we thus get

$$\int_X \mathrm{Tr}(\sqrt{-1}\Lambda F') + \int_X \mathrm{Tr}(\Pi') = r_1'\tau_1 + r_2'\tau_2 . \tag{3.3.4}$$

Using the Chern-Weil formula for $\deg(\mathcal{E}')$, and the positivity of Π', we obtain

$$\deg(\mathcal{E}') \leq r'_1\tau_1 + r'_2\tau_2 \ , \tag{3.3.5}$$

with equality if and only if $\Pi' = 0$, i.e. if and only if $\mathcal{E}$ splits. If $\alpha = \tau_1 - \tau_2$, and $\mu_\alpha(\mathcal{E}')$ is as in Definition 3.3, then (3.3.5) is equivalent to $\mu_\alpha(e') < \mu_\alpha(e)$. This proves the result for locally free subobjects. If e' is not locally free, then there is a subvariety $\Sigma \subset X$ of codimension at most two, such that $\mathcal{E}'|_{X-\Sigma}$ is locally free. We can thus apply the above arguments over $X - \Sigma$. This is good enough, because of the size of the codimension of Σ. □

We now prove that α-stability is a sufficient condition for existence of special metrics in the sense defined by equations (3.2.1). That is, we prove

Theorem 3.9. *Suppose that $\alpha < 0$ and*

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

is an α-stable extension. Let τ_1 and τ_2 be such that $\alpha = \tau_1 - \tau_2$ and $\deg(E) = r_1\tau_1 + r_2\tau_2$. *Then there is a metric H on $\mathcal{E}$ satisfying the equation (3.2.1), i.e*

$$i\Lambda F_H = \begin{pmatrix} \tau_1 \mathbf{I}_1 & 0 \\ 0 & \tau_2 \mathbf{I}_2 \end{pmatrix}.$$

The proof is an adaptation of the methods used in [Do1] (also [S] and [UY]) in proving the Hitchin–Kobayashi correspondence for ordinary stable bundles. As shown by Donaldson, the Hermitian–Einstein equation is the equation satisfied by the critical points of a certain functional defined on the space of Hermitian metrics on $\mathcal{E}$. We shall modify this functional to show that our equations appear in the same way.

Just as in the case of the Hermitian–Einstein equation, we can separate out the trace and trace-free parts of the equation. We can fix the determinant of the metric on E to satisfy the trace part,

$$i\Lambda \operatorname{Tr}(F_H) = r_1\tau_1 + r_2\tau_2. \tag{3.3.6}$$

The problem then becomes one of finding a new metric with this same determinant, and which satisfies

$$i\Lambda F_H^0 = \begin{pmatrix} \frac{r_2}{r}\alpha \mathbf{I}_1 & 0 \\ 0 & -\frac{r_1}{r}\alpha \mathbf{I}_2 \end{pmatrix}, \tag{3.3.7}$$

where $\alpha = \tau_1 - \tau_2$.

Recall Donaldson's original functional to prove existence of solutions of the Hermitian–Einstein equation: Let $\mathcal{E}$ be a holomorphic vector bundle over a compact Kähler manifold (X, ω). Donaldson defined a functional $M(-,-)$ on pairs of Hermitian metrics on $\mathcal{E}$ using Bott–Chern secondary classes. Namely

$$M(H,K) = \int_X (R_2(H,K) - 2\lambda R_1(H,K)\omega) \wedge \omega^{n-1}, \tag{3.3.8}$$

where

$$R_1(H,K) = \log\det(K^{-1}H) = \mathrm{Tr}(\log K^{-1}H) \tag{3.3.9a}$$

(3.3.9b)

$$i\bar{\partial}\partial R_2(H,K) = (-\mathrm{Tr}(F_H^2)) - (-\mathrm{Tr}(F_K^2)),$$

$$\lambda = \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}\ . \tag{3.3.9c}$$

Now fix a smooth background metric K, with determinant satisfying (3.3.6). Let

$$S(K) = \{s \in \Omega^0(X, \operatorname{End} E) | s^{*K} = s\ ,\ \mathrm{Tr}(s) = 0\}\ . \tag{3.3.10}$$

Then any other metric with the same determinant as K can be described by Ke^s, with $s \in S(K)$. Fix an integer $p > 2n$, and define

$$\mathcal{M}et_2^p = \{H = Ke^s\ s \in L_2^p(S(K))\}\ . \tag{3.3.11}$$

Let $M : \mathcal{M}et(\mathcal{E}) \longrightarrow \mathbb{R}$ be given by $M(H) = M(K,H)$. The important property of M is that H is a critical point if and only if H satisfies the trace free part of the Hermitian–Einstein equations, i.e.

$$i\Lambda F_H^0 = 0.$$

Consider now the extension

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0\ . \tag{e}$$

Given a background metric K on $\mathcal{E}$ we can (smoothly) identify $\mathcal{E}_2$ with the orthogonal complement of $\mathcal{E}_1$ in $\mathcal{E}$, and in this way get metrics K_1 and K_2

on $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. Any other metric H can similarly be split into H_1 and H_2 (by using an H-orthogonal splitting of $\mathcal{E}$). Denote

$$M_D(H,K) = \int_X R_2(H,K) \wedge \omega^{n-1}. \tag{3.3.12}$$

Let τ_1 and τ_2 be real parameters. We shall consider the functional
(3.3.13)

$$M_{\tau_1,\tau_2}(H,K) = M_D(H,K) - 2\int_X (\tau_1 R_1(H_1,K_1) + \tau_2 R_1(H_2,K_2)) \wedge \omega^n.$$

Remark. If $\tau_1 = \tau_2 = \lambda$, then $M_{\tau_1,\tau_2}(H,K) = M(H,K)$, as can be easily seen from the following simple fact (see [Do1, Prop. 7. p. 10]).

Lemma 3.10. *Let H and K be Hermitian metrics on $\mathcal{E}$ and Let H_1, K_1 and H_2, K_2 the corresponding metrics induced on $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively, then*

$$R_1(H,K) = R_1(H_1,K_1) + R_1(H_2,K_2).$$

Notice that $R_1(H,K) = 0$ if the metrics have fixed determinant. We can thus simplify our definition to

$$M_{\tau_1,\tau_2}(H,K) = M_D(H,K) - 2(\tau_1 - \tau_2)\int_X R_1(H_1,K_1) \wedge \omega^n . \tag{3.3.14}$$

Let us fix K and define

$$M_{\tau_1,\tau_2}(H) = M_{\tau_1,\tau_2}(H,K). \tag{3.3.15}$$

Define $m^0 : \mathcal{M}et(\mathcal{E}) \longrightarrow \Omega^0(X, \operatorname{End} E)$ by

$$m^0(H) = \Lambda F^0_{\bar\partial_E,H} + \sqrt{-1}T^0_H , \tag{3.3.16}$$

where, with respect to the orthogonal splitting $E = E_1 \oplus E_2$ determined by H,

$$\begin{aligned} T^0_H &= \begin{pmatrix} \tau_1 \mathbf{I}_1 & 0 \\ 0 & \tau_2 \mathbf{I}_2 \end{pmatrix} - \operatorname{Tr}\begin{pmatrix} \tau_1 \mathbf{I}_1 & 0 \\ 0 & \tau_2 \mathbf{I}_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{r_2}{r}\alpha \mathbf{I}_1 & 0 \\ 0 & -\frac{r_1}{r}\alpha \mathbf{I}_2 \end{pmatrix}. \end{aligned}$$

The crucial properties of M_{τ_1,τ_2} are described in the next proposition.

Proposition 3.11. *(1) Given any three metrics H, K, J, we have*

$$M_{\tau_1,\tau_2}(H,K) + M_{\tau_1,\tau_2}(K,J) = M_{\tau_1,\tau_2}(H,J).$$

(2) If $H(t) = He^{ts}$ with $s \in S(H)$, then

$$\frac{d}{dt}M_{\tau_1,\tau_2}(H(t)) = 2i\int_X \operatorname{Tr}\left(sm^0(H(t))\right).$$

(3) If $s \in S(H)$ is given by $s = \begin{pmatrix} s_1 & u \\ u^* & s_2 \end{pmatrix}$ with respect to the orthogonal splitting $E = E_1 \oplus E_2$ determined by H, then

$$\begin{aligned}\frac{d^2}{dt^2}M_{\tau_1,\tau_2}(H(t))|_{t=0} &= 2i\int_X \operatorname{Tr}\left(s\frac{d}{dt}m^0(H(t))|_{t=0}\right)\\ &= \| D'_H(s) \|^2 - \alpha \| u \|^2\end{aligned}$$

Proof. (1) This follows immediately from the properties of the Bott-Chern classes.

(2) Chose a frame for E such that H can be written as

$$\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

that is a frame in which $\mathcal{E}_1$ and $\mathcal{E}_2$ are H-orthogonal. In terms of this frame we can write

$$s = \begin{pmatrix} s_1 & u \\ u^* & s_2 \end{pmatrix},$$

where $s_1 \in S(H_1)$, $s_2 \in S(H_2)$ and $u \in \operatorname{Hom}(E_2, E_1)$. We have to show that

$$\begin{aligned}\frac{d}{dt}M_{\tau_1,\tau_2}(H(t))|_{t=0} &= 2i\int_X \operatorname{Tr}\left(si\Lambda F_H^0 - \begin{pmatrix} \frac{r_2}{r}\alpha I_1 & 0 \\ 0 & -\frac{r_1}{r}\alpha I_2 \end{pmatrix}\right)\\ &= 2i\int_X \operatorname{Tr}\left(s\Lambda F^0_{H(t)}\right) - 2i\alpha\int_X \operatorname{Tr}(s_1).\end{aligned}$$

From [Do1] we know that $\frac{d}{dt}M_D(H(t)) = 2i\int_X \operatorname{Tr}\left(s\Lambda F^0_{H(t)}\right)$, so it remains to compute $\frac{d}{dt}R_1(H_1(t), H_1)|_{t=0}$. If we write

$$H(t) = He^{ts} = H\begin{pmatrix} h_1(t) & * \\ * & * \end{pmatrix},$$

with respect to the H-orthogonal frame, then
(3.3.17)

$$R_1(H_1(t), H_1) = \log \det h_1(t) = \log \det(1 + ts_1 + \frac{t^2}{2}(s_1^2 + uu^*) + O(t^3)) \ .$$

A straightforward computation yields the result

$$\frac{d}{dt} R_1(H_1(t), H_1)|_{t=0} = \mathrm{Tr}(s_1) \ .$$

(3) It follows from (3.3.17) that

$$\frac{d^2}{dt^2} R_1(H_1(t), H_1)|_{t=0} = \mathrm{Tr}(uu^*) = |u|^2 \ .$$

The result now follows from this, plus the fact that

$$\frac{d}{dt} M_D(H(t)) = \parallel D'_H(s) \parallel^2 \ .$$

□

Notice that as a consequence of (1) and (3) in Proposition 3.11 we get

Proposition 3.12. *Suppose that $\alpha < 0$ and (e) is an α-stable extension. Then*

1.
$$\frac{d^2}{dt^2} M_{\tau_1,\tau_2}(H(t)) > 0$$

2. $\mathrm{Ker}(L) = 0$, *where L is the operator on $L_2^p(S(H)$ defined by* $L(s) = \frac{d}{dt} m^0(H(t))|_{t=0}$.

Proof. Both of these statements follow from the fact that if s is as in (3) and $L(s) = 0$, then $\overline{\partial}_1(s_1) = \overline{\partial}_2(s_2) = u = 0$. The eiegenspaces of s thus split the extension (e) into a direct sum of extensions. This violates the stability criterion, since the α-slope inequality cannot be satisfied by both summands. □

The functional M_{τ_1,τ_2} thus has the convexity features we require. Furthermore,

Lemma 3.13. *Suppose that $\alpha < 0$ and let $H = Ke^s$ with $s \in L_2^p(S(K))$. Let $s = \begin{pmatrix} s_1 & u \\ u^* & s_2 \end{pmatrix}$ be the block decomposition of s with respect to the orthogonal splitting $E = E_1 \oplus E_2$ determined by K. Let $\Psi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function as in [B] (or [S]). Then*

$$M_{\tau_1,\tau_2}(H) = \sqrt{-1}\int_X \mathrm{Tr}(s\Lambda F_K) + \int_X (\Psi(s)\overline{\partial}_E s, \overline{\partial}_E s)_K - 2\alpha R_1(H_1, K_1)$$

(3.3.18)

$$\geq \sqrt{-1}\int_X \mathrm{Tr}(s\Lambda F_K) + \int_X (\Psi(s)\overline{\partial}_E s, \overline{\partial}_E s)_K - \alpha \int_X \mathrm{Tr}(s_1)$$

where the meaning of $\Psi(s)$ is as in [B] or [S].

Proof. The first line follows from the computations in [S] (or [Do1]). The second uses the convexity properties of the function $R_1(H(t)_1, K_1)$, and the fact that its first derivative at $t = 0$ is given by $\int_X (\mathrm{Tr}(s_1)$. □

This is slightly weaker than the analogous result for the original Donaldson functional, but is strong enough for our purposes.

The rest of the proof of Theorem 3.9 is precisely along the lines of the analogous result in [S]. We give here a sketch of the main ideas. Fix a real number B such that $\| m^0(K) \|_{L^p}^p \leq B$ (where $\| m^0(K) \|_{L^p}^p = \int_X |m^0(K)|_K^p dvol$). Define

$$\mathcal{M}et_2^p(B) = \{H \in \mathcal{M}et_2^p \mid \ \| m^0(H) \|_{L^p}^p \leq B \ \}.$$

We look for minima of $M_{\tau_1,\tau_2}(H)$ on $\mathcal{M}et_2^p(B)$. As in the case of the unmodified Donaldson functional, if the extension (e) is α-stable, then there are no extrema on the boundary of this constrained space, and the minima occur at solutions to the metric equation $m^0(H) = 0$.

To show that minima do occur, we need

Proposition 3.14. *Either (e) is α-stable or we can find positive constants C_1 and C_2 such that*

$$\mathrm{Sup}\,|s| < C_1 M_{\tau_1,\tau_2}(Ke^s) + C_2$$

for all $Ke^s \in \mathcal{M}et_2^p(B)$.

Sketch of Proof. As in the case of the unmodified Donaldson functional, one first shows that for metrics in the constrained set $\mathcal{M}et_2^p(B)$, the C^0 estimate given above is equivalent to a C^1 estimate of the same type. One then supposes that no such estimate holds. It follows that one may find a sequence $\{u_i\} \subset L_2^P(S(K)$ such that $\| u_i \|_{L^1} = 1$. This has a weakly convergent subsequence in $L_1^2(S(K)$, with non-trivial limit denoted by u_∞. One then shows that the eigenvalues of u_∞ are constant almost everywhere. This is done, as in [S], by making use of an estimate of the form:

Proposition 3.15. *Let $\mathcal{F} : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be any smooth positive function which satisfies $\mathcal{F}(x,y) \le 1/(x-y)$ whenever $x > y$. Then*
(3.3.18)
$$\sqrt{-1}\int_X \mathrm{Tr}(u_\infty \Lambda F_K) + \int_x (\mathcal{F}(u_\infty)\bar\partial_E u_\infty, \bar\partial_E u_\infty)_K - \alpha \int_x \mathrm{Tr}(u_{\infty,1}) \le 0 \ ,$$
where $u_\infty = \begin{pmatrix} u_{\infty,1} & * \\ * & * \end{pmatrix}$ *with respect to the splitting of E determined by K.*

Proof. This follows from the analysis in [S], plus the estimate given in Lemma 3.13. □

Since $\mathrm{Tr}(u_\infty) = 0$, there are at least two distinct eigenvalues. Let $\lambda_1 < \lambda_2, \dots, < \lambda_k$ denote the distinct eigenvalues. Setting $a_i = \lambda_{i+1} - \lambda_i$, one can thus define projections $\pi_i \in L_1^2(S(K))$ such that

$$u_\infty = \lambda_r \mathbf{I} - \sum_i^{k-1} a_i \pi_i. \tag{3.3.19}$$

By an important result of Uhlenbeck and Yau (cf. [UY]), the π_i define a filtration of $\mathcal{E}$ by reflexive subsheaves

$$\mathcal{E}_1 \subset \mathcal{E}_2 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}.$$

Each subsheaf $\mathcal{E}_j$ determines a subextension

$$0 \longrightarrow \mathcal{E}_{1,j} \longrightarrow \mathcal{E}_j \longrightarrow \mathcal{E}_{2,j} \longrightarrow 0 \ .$$

Now define the numerical quantity

$$Q = \lambda_k (r\mu(\mathcal{E}) - r_1\tau_1 - r_2\tau_2) - \sum_i^{k_1} a_i (r_i \mu(\mathcal{E}_i) - r_{1,i}\tau_1 - r_{2,i}\tau_2) \ , \tag{3.3.20}$$

where $\mu(\mathcal{E}_i)$ is the slope of $\mathcal{E}_j$, and $r_{a,i}$ is the rank of $\mathcal{E}_{a,i}$.

Using Lemma 3.15 and the fact that $u_\infty = \lambda_r \mathbf{I} - \sum_i^{k-1} a_i \pi_i$, one shows (by precisely the method in [S]) that $Q \leq 0$. On the other hand, τ_1 and τ_2 are related by $r\mu(\mathcal{E}) - r_1\tau_1 - r_2\tau_2 = 0$, and if (e) is α-stable, then

$$r_i\mu(\mathcal{E}_i) - r_{1,i}\tau_1 - r_{2,i}\tau_2 < 0$$

for all $i = 1, \ldots, k-1$. Thus $Q > 0$ if (e) is α-stable. This proves the proposition. $\square$

To complete the proof of the Hitchin–Kobayashi correspondence, it remains to show that the minimum of M_{τ_1,τ_2} is a smooth solution to the equation (3.2.1). This is done exactly as in [Do1] or [S].

3.4. An example.

As an example, we can consider the case where $E_1 = L_1$ and $E_2 = L_2$ are line bundles over a Riemann surface. We assume further that $d_1 < d_2$, where d_i denotes the degree of L_i. Let us denote such extensions by

$$(l) \qquad 0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0\ .$$

Since line bundles are automatically stable, Lemma 3.6(1) gives

Lemma 3.16. *Let L_1 and L_2 be as above. Then there is some $\epsilon > 0$ such that all extensions (l) as above are α-stable, for any α in the interval*

$$(d_1 - d_2, d_1 - d_2 + \epsilon)\ .$$

We can give a more detailed analysis. The main reason for this is that the possibilities for sub-extensions are so restricted; they all correspond to rank one (i.e. line-) subbundles of $\mathcal{E}$, and are of one of two types. The only possibilities are

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{L}_1 \longrightarrow 0 \longrightarrow 0\ , \tag{3.4.1}$$

or

$$0 \longrightarrow 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \longrightarrow 0\ . \tag{3.4.2}$$

Computing the α-slopes, we see that

$$\mu_\alpha(0, \mathcal{L}_1) = d_1\ , \tag{3.4.3}$$

$$\mu_\alpha(\mathcal{L}, 0) = d_{\mathcal{L}} + \alpha\ . \tag{3.4.4}$$

From this we see that if (l) is α-stable, then

$$0 < d_1 - d_2 < \alpha < d_1 + d_2 - 2d_{\mathcal{L}} \tag{3.4.5}$$

for all subbundles $\mathcal{L} \neq \mathcal{L}_1$. Now define

$$\operatorname{div}(\mathcal{E}) = \operatorname{Max}\{d_{\mathcal{L}} \mid d_{\mathcal{L}} \text{ is the degree of a line subbundle of } \mathcal{E}\} \ . \tag{3.4.6}$$

Lemma 3.17. *If*

$$0 < d_1 - d_2 < d_1 + d_2 - 2\operatorname{div}(\mathcal{E}) \ , \tag{3.4.7}$$

then (l) is α-stable for any α in the interval $(d_1 - d_2 \ , \ d_1 + d_2 - 2\operatorname{div}(\mathcal{E}))$.

Proof. Given any α such that $0 < d_1 - d_2 < \alpha < d_1 + d_2 - 2\operatorname{div}(\mathcal{E})$, we get

$$d_1 < \mu_\alpha(\mathcal{E}) = (d_1 + d_2)/2 + \alpha/2 \ ,$$

and

$$d_{\mathcal{L}} + \alpha < (d_1 + d_2)/2 + \alpha/2 \ .$$

By equations (3.4.3) and (3.4.4), and the above remarks concerning the possible subextensions of (l), this is all we need to check. □

Furthermore, the range for α is clearly partitioned into intervals of length 2, with the boundaries at the values $\{d_1 - d_2, d_1 - d_2 + 2, \ldots, d_2 - d_1 - 2, d_2 - d_1\}$.

Proposition 3.18. *Let L_1 and L_2 be as above, and let (l) denote an extension as above.*

1. *For α in the interval $(d_1 - d_2, d_1 - d_2 + 2)$, all non-trivial extensions (l) are α-stable.*
2. *Suppose that $\alpha_1 > \alpha_2 > d_1 - d_2$. If (l) is α_1-stable, then it is α_2-stable.*
3. *For $\alpha \geq -2$, if (l) is an α-stable extension, then $\mathcal{E}$ is a semistable bundle.*
4. *For $\alpha \geq 0$, if (l) is an α-stable extension, then $\mathcal{E}$ is a stable bundle.*
5. *If $\mathcal{E}$ is a stable (resp. semistable) bundle, then for any $d_1 - d_2 < \alpha \leq 0$ (resp. $d_1 - d_2 < \alpha < 0$), (l) is an α-stable extension.*

Proof. Part (1) follows from the fact that $\operatorname{div}(\mathcal{E}) \leq d_2$, with equality possible if and only if the extension is the trivial one (cf. [G]). Thus for any non-trivial extension, we have $d_1 + d_2 - 2\operatorname{div}(\mathcal{E}) > d_1 - d_2$. Now use Lemma 3.16. Part (2) follows from the observation that for any subextension of the type in (3.4.2), we have $\mu_\alpha(\mathcal{L}, 0) - \mu_\alpha(\mathcal{E}) = d_{\mathcal{L}} - \frac{d_1+d_2}{2} + \frac{\alpha}{2}$. Parts (3) and (4) both follows from the observation that if (l) is α-stable, then $\operatorname{div}(\mathcal{E}) < (d_1 + d_2)/2 - \alpha/2$. Part (5) follows from Lemma 3.16 and the fact that if $\mathcal{E}$ is stable (resp. semistable), then $\operatorname{div}(\mathcal{E}) < (d_1 + d_2)/2$ (resp. $\operatorname{div}(\mathcal{E}) \leq (d_1 + d_2)/2$. □

We thus get the following picture. Let

(3.4.8) $$\mathcal{E}xt(L_1, L_2) = \{ \text{ all extensions } 0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0\} \ ,$$

and let $\mathcal{E}xt^*(L_1, L_2) \subset \mathcal{E}xt(L_1, L_2)$ denote the *non-trivial* extensions. Given an integer k, define

(3.4.9)
$$\mathcal{E}xt_k(L_1, L_2) = \{(l) \in \ \mathcal{E}xt(L_1, L_2) \mid (l) \text{ is } \alpha\text{-stable, and } k < \alpha < k+2\}.$$

Set

(3.4.10) $$\mathcal{E}xt_-(L_1, L_2) = \begin{cases} \mathcal{E}xt_{-2}(L_1, L_2) & \text{if } (d_1 - d_2) \text{ is even} \\ \mathcal{E}xt_{-1}(L_1, L_2) & \text{if } (d_1 - d_2) \text{ is odd} \end{cases} ,$$

and

(3.4.11) $$\mathcal{E}xt_+(L_1, L_2) = \begin{cases} \mathcal{E}xt_0(L_1, L_2) & \text{if } (d_1 - d_2) \text{ is even} \\ \mathcal{E}xt_1(L_1, L_2) & \text{if } (d_1 - d_2) \text{ is odd} \end{cases} .$$

Also define

(3.4.12) $$\mathcal{E}xt_s(L_1, L_2) = \{(l) \in \mathcal{E}xt^*(L_1, L_2) \mid \mathcal{E} \text{ is a stable bundle}\} \ ,$$

(3.4.13) $$\mathcal{E}xt_{ss}(L_1, L_2) = \{(l) \in \mathcal{E}xt^*(L_1, L_2) \mid \mathcal{E} \text{ is a semistable bundle}\} \ .$$

Then we can summarize proposition 3.18 by the diagram

(3.4.14)
$$\begin{array}{ccccccc} \mathcal{E}xt_{d_1-d_2}(L_1, L_2) & \supset & \mathcal{E}xt_{d_1-d_2+2}(L_1, L_2) \supset \ldots & \supset & \mathcal{E}xt_-(L_1, L_2) & \supset & \mathcal{E}xt_+(L_1, L_2) \\ \Vert & & & & \Vert & & \Vert \\ \mathcal{E}xt^*(L_1, L_2) & & & & \mathcal{E}xt_{ss}(L_1, L_2) & \supset & \mathcal{E}xt_s(L_1, L_2). \end{array}$$

4. 1-Cohomology triples, extensions and surjective triples.

The correspondence with 1-cohomology triples is not the only way that extensions as in Section 3 are related to triples. Given an extension (e) as in §3, one can extract a (0-cohomology) triple $(\mathcal{E}_2, \mathcal{E}, \pi)$. Conversely, given a triple $(\mathcal{E}_2, \mathcal{E}, \pi)$ in which $E = E_1 \oplus E_2$ and π is surjective, we get an extension of $\mathcal{E}_2$ by $\mathcal{E}_1 = \mathrm{Ker}(\pi)$.

In this section we compare and relate notions of stability, moduli spaces, equations for special metric, etc. for

1. 1-cohomology triples on (E_1, E_2),
2. extensions on (E_1, E_2), and
3. surjective (0-cohomology) triples on (E_2, E).

4.1. Configuration spaces.

We begin with some definitions. Let E_1 and E_2 be (as usual) smooth bundles over X, and fix $E = E_1 \oplus E_2$ as a smooth bundle. Holomorphic bundles with these as their underlying smooth bundles will be denoted by $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}$ respectively.

Definition 4.1. 1. A *surjective triple* on (E_2, E) is a (0-cohomology) triple, $(\mathcal{E}_2, \mathcal{E}, \pi)$, in which $\pi : \mathcal{E} \longrightarrow \mathcal{E}_2$ is a surjective map. Set

$$\mathcal{H}_s(E_2, E) = \{(\mathcal{E}_2, \mathcal{E}, \pi) \ : \ \pi \text{ is surjective}\} \ .$$

2. Denote by $\mathcal{E}xt(E_1, E_2)$ the set of all holomorphic structures on E which can be described as extensions of $\mathcal{E}_2$ by $\mathcal{E}_1$, i.e.

$$\mathcal{E}xt(E_1, E_2) = \{0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0\} \ .$$

Recall also, from §2, that

$$\mathcal{H}^{(1)}(E_1, E_2) = \{(\mathcal{E}_1, \mathcal{E}_2, \Phi) \ |\Phi \in H^1(Hom(\mathcal{E}_2, \mathcal{E}_1)) \ \}$$

is the space of 1-cohomology pairs on (E_1, E_2).

On each of $\mathcal{H}_s(E_2, E)$, $\mathcal{E}xt(E_1, E_2)$, and $\mathcal{H}^{(1)}(E_1, E_2)$ there are natural equivalence relations.

Definition/Lemma 4.2. 1. In $\mathcal{H}^{(1)}(E_1, E_2)$, the equivalence relation is given by the action of the group $\mathfrak{G}_{\mathbb{C}}^{(1)} \times \mathfrak{G}_{\mathbb{C}}^{(2)}$.

2. In $\mathcal{E}xt(E_1, E_2)$ there are two equivalence relations to consider: We say that two extension $\mathcal{E}$ and $\mathcal{E}'$ in $\mathcal{E}xt(E_2, E_1)$are *weakly equivalent*, denoted by $\mathcal{E} \sim \mathcal{E}'$ if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[d, "g_1"'] & \mathcal{E} \arrow[r] \arrow[d, "g"'] & \mathcal{E}_2 \arrow[r] \arrow[d, "g_2"] & 0 \\ 0 \arrow[r] & \mathcal{E}_1' \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E}_2' \arrow[r] & 0 \end{tikzcd}$$

where g_1, g_2, and g are bundle automorphisms of the underlying smooth bundles.

We say that $\mathcal{E}$ and $\mathcal{E}'$ are *strongly equivalent*, denoted by $\mathcal{E} \approx \mathcal{E}'$ if $\mathcal{E}_i = \mathcal{E}_i'$ for $i = 1, 2$, and there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}_1 \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d, "g"'] & \mathcal{E}_2 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{E}_2 \arrow[r] & 0 \end{tikzcd}$$

where g is a bundle automorphism of E.

3. We can similarly define weak and strong equivalence for surjective triples: Let $(\mathcal{E}_2, \mathcal{E}, \pi)$ and $(\mathcal{E}_2', \mathcal{E}', \pi')$ be surjective triples in $\mathcal{H}_s(E_2, E)$. We say that $(\mathcal{E}_2, \mathcal{E}, \pi)$ and $(\mathcal{E}_2', \mathcal{E}', \pi')$ are *weakly equivalent*, denoted by $(\mathcal{E}_2, \mathcal{E}, \pi) \sim (\mathcal{E}_2', \mathcal{E}', \pi')$, if there is a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\pi"] \arrow[d, "g"'] & \mathcal{E}_2 \arrow[r] \arrow[d, "g_2"] & 0 \\ \mathcal{E}' \arrow[r, "\pi'"] & \mathcal{E}_2' \arrow[r] & 0 \end{tikzcd}$$

where g_2 and g are bundle automorphisms of the underlying smooth bundles.

We say that $(\mathcal{E}_2, \mathcal{E}, \pi)$ and $(\mathcal{E}_2', \mathcal{E}', \pi')$ are *strongly equivalent*, denoted by $(\mathcal{E}_2, \mathcal{E}, \pi) \approx (\mathcal{E}_2', \mathcal{E}', \pi')$,if there is a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\pi"] \arrow[d, "g"'] & \mathcal{E}_2 \arrow[r] \arrow[d, equal] & 0 \\ \mathcal{E}' \arrow[r, "\pi'"] & \mathcal{E}_2' \arrow[r] & 0 \end{tikzcd}$$

where g is a bundle automorphism of E.

The relationships between these spaces can be seen as follows. With $\mathcal{Z}^{(1)}(E_1, E_2)$ as in (2.1.2), we have a map

$$f : \mathcal{Z}^{(1)}(E_1, E_2) \longrightarrow \mathcal{E}xt(E_1, E_2) \; . \tag{4.1.1}$$

Indeed, it is clear that given an element $(\overline{\partial}_1, \overline{\partial}_2, \phi) \in \mathcal{Z}^{(1)}(E_1, E_2)$ we can define a $\overline{\partial}$-operator on $E = E_1 \oplus E_2$ by

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_1 & \phi \\ 0 & \overline{\partial}_2 \end{pmatrix} \; .$$

This in turn defines an element in $\mathcal{E}xt(E_1, E_2)$. Conversely, given an element

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

in $\mathcal{E}xt(E_1, E_2)$, by choosing a metric on $\mathcal{E}$ we can identify the smooth underlying bundle to $\mathcal{E}$ with $E_1 \oplus E_2$, and in this way we can define an inverse to (4.1.1). This does however depends on the choice of the metric. In order to get a metric-independent map, we need to consider the image in $\mathcal{H}^{(1)}$, rather than in $\mathcal{Z}^{(1)}$. This is because two different metrics on $\mathcal{E}$ define second fundamental forms ϕ and ϕ' that are related by $\phi' = \phi + \overline{\partial}_{1,2}\alpha$ for $\alpha \in \Omega^0(\mathrm{Hom}(E_2, E_1))$. Moreover this map induces a bijection

$$\frac{\mathcal{E}xt(E_1, E_2)}{\approx} \longleftrightarrow \mathcal{H}^{(1)}(E_1, E_2) \; , \tag{4.1.2}$$

where $\approx$ denotes *strong* equivalence.

Similarly, by identifying $\mathcal{E}_1$ with $\mathrm{Ker}(\pi)$, we see that there is a bijective correspondence between extensions in $\mathcal{E}xt(E_1, E_2)$ and surjective triples in $\mathcal{H}_s(E_2, E)$. Furthermore, this correspondence holds at the level of weak or strong equivalence classes.

Let $\mathfrak{G}_{\mathbb{C}}^{(1)}$ and $\mathfrak{G}_{\mathbb{C}}^{(2)}$ be the complex gauge groups of E_1 and E_2 respectively. It is clear that these maps descend to the quotients and we obtain

Proposition 4.3. *There are one-to-one correspondences*

$$\frac{\mathcal{H}_s(E_2, E)}{\sim} \longleftrightarrow \frac{\mathcal{E}xt(E_1, E_2)}{\sim} \longleftrightarrow \frac{\mathcal{H}^{(1)}(E_1, E_2)}{\mathfrak{G}_{\mathbb{C}}^{(1)} \times \mathfrak{G}_{\mathbb{C}}^{(2)}} .$$

When we speak of a moduli space of extensions supported by the smooth bundles E_1 and E_2, it is the quotient $\frac{\mathcal{E}xt(E_1,E_2)}{\sim}$ that we have in mind. We will denote equivalence classes in each of these quotients by square brackets, thus for example,$[\mathcal{E}_1, \mathcal{E}_2, \Phi]$ is a class in $\frac{\mathcal{H}^{(1)}(E_1,E_2)}{\mathfrak{G}_{\mathbb{C}}^{(1)}\times\mathfrak{G}_{\mathbb{C}}^{(2)}}$.

4.2. Stability.

In view of the above bijections, it makes sense to compare the stability properties of the surjective triples, of the 1-cohomology triples, and of the extensions. This comparison is made considerably easier if we formulate the respective notions of stability in a uniform way. For this, we use the functions $\theta_{a_1,a_2,\tau_1,\tau_2}$ defined earlier.

Recall that for an ordinary triple, the definition of τ-stability in [BGP] is equivalent to $(1, 1, \tau, \tau')$-stability as defined in §2.2, with τ and τ' being related by $d_1 + d_2 = r_1\tau + r_2\tau'$. Similarly, taking the special values $\{a_1, a_2, \tau_1, \tau_2\} = \{1, 1, \tau, \tau'\}$ for a 1-cohomology triple, and defining $\alpha = \tau - \tau'$, we get

Definition/Lemma 4.4. The 1-cohomology triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ is said to be *α-stable* if for all subtriples, $(\mathcal{E}_1', \mathcal{E}_2', \Phi')$, we have

$$\mu_\alpha(\mathcal{E}_1', \mathcal{E}_2') < \mu_\alpha(\mathcal{E}_1, \mathcal{E}_2) ,$$

where

$$\mu_\alpha(\mathcal{E}_1', \mathcal{E}_2') = \mu(\mathcal{E}_1', \mathcal{E}_2') + \alpha\frac{r_2'}{r_1' + r_2'} . \tag{4.2.1}$$

This is equivalent to $(1, 1, \tau, \tau - \alpha)$-stability, as defined in Definition 2.7.

Now let $(\mathcal{E}_2, \mathcal{E}, \pi)$ be a surjective triple corresponding to the 1-cohomology triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$, i.e. $[\mathcal{E}_2, \mathcal{E}, \pi] = [\mathcal{E}_1, \mathcal{E}_2, \Phi]$ under the bijection in Proposition 4.3. If we compare the stability of $(\mathcal{E}_2, \mathcal{E}, \pi)$ and $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$, we find that we need to introduce a slightly restricted form of stability for the surjective triple. We will refer to this as *surjective stability*, with the precise definition as follows:

Definition 4.5. Given a surjective triple $(\mathcal{E}_2, \mathcal{E}, \pi)$, we say that a subtriple $(\mathcal{E}_2', \mathcal{E}', \pi')$ is a *surjective subtriple* if $\pi : \mathcal{E}' \longrightarrow \mathcal{E}_2'$ is surjective.

Fix real numbers $\{b_1, b_2, \sigma_1, \sigma_2\}$ such that

$$b_1 deg(\mathcal{E}_2) + b_2 deg(\mathcal{E}_1) - \sigma_1 rank(\mathcal{E}_2) - \sigma_2 rank(\mathcal{E}) = 0 ,$$

i.e. such that $\theta_{b_1,b_2,\sigma_1,\sigma_2}(\mathcal{E}_2,\mathcal{E}) = 0$. We say that the triple is $(b_1,b_2,\sigma_1,\sigma_2)$-*surjectively stable* if

$$\theta_{b_1,b_2,\sigma_1,\sigma_2}(\mathcal{E}_2',\mathcal{E}') < 0$$

for all surjective subtriples $(\mathcal{E}_2',\mathcal{E}',\pi')$.

Remark. In some cases surjective stability is equivalent to full stability. For example:

Proposition 4.6. *If* $b_1 = b_2$, *and* $\sigma_1 - \sigma_2 > 0$, *then* $(b_1,b_2,\sigma_1,\sigma_2)$-*surjective stability is equivalent to* $(b_1,b_2,\sigma_1,\sigma_2)$-*stability for a surjective triple.*

Proof. It is clear from the definitions that stability implies surjective stability. Conversely, suppose that $(\mathcal{E}_2,\mathcal{E},\pi)$ is a surjective triple which is not $\{b_1,b_2,\sigma_1,\sigma_2\}$-stable. Let $(\mathcal{E}_2',\mathcal{E}',\pi')$ be a destabilizing subtriple, i.e. suppose that $(\mathcal{E}_2',\mathcal{E}',\pi')$ is a subtriple (not necessarily a surjective subtriple), such that

$$\theta_{b_1,b_2,\sigma_1,\sigma_2}(\mathcal{E}_2',\mathcal{E}') \geq 0\ .$$

Suppose that $(\mathcal{E}_2',\mathcal{E}',\pi')$ is not a surjective subtriple. Let $\pi'(\mathcal{E}')$ be the image of the sheaf map, and denote by $\pi^{-1}(\mathcal{E}_2')$ the subsheaf of $\mathcal{E}$ defined by

$$0 \longrightarrow \mathrm{Ker}(\pi') \longrightarrow {\pi'}^{-1}(\mathcal{E}_2') \longrightarrow \mathcal{E}_2' \longrightarrow 0\ .$$

Then $(\pi'(\mathcal{E}'),\mathcal{E}',\pi')$ and $(\mathcal{E}_2',{\pi'}^{-1}(\mathcal{E}_2'),\pi')$ are both surjective subtriples of $(\mathcal{E}_2,\mathcal{E},\pi)$. We will show that if $\theta_{b_1,b_2,\sigma_1,\sigma_2}(\mathcal{E}_2',\mathcal{E}') \geq 0$, then at least one of these two surjective subtriples must likewise be destabilizing.

By their definition, the surjective subtriples lead to the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longrightarrow & {\pi'}^{-1}(\mathcal{E}_2')/\mathcal{E}' & \longrightarrow & \mathcal{E}_2'/\pi'(\mathcal{E}') & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Ker}(\pi') & \longrightarrow & {\pi'}^{-1}(\mathcal{E}_2') & \longrightarrow & \mathcal{E}_2' & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Ker}(\pi') & \longrightarrow & \mathcal{E}' & \longrightarrow & \pi'(\mathcal{E}') & \longrightarrow & 0.
\end{array}$$

It follows that

$$\begin{aligned}\deg({\pi'}^{-1}(\mathcal{E}_2')) - \deg(\mathcal{E}') &= \deg(\mathcal{E}_2') - \deg(\pi'(\mathcal{E}'))\\ \operatorname{rank}({\pi'}^{-1}(\mathcal{E}_2')) - \operatorname{rank}(\mathcal{E}') &= \operatorname{rank}(\mathcal{E}_2') - \operatorname{rank}(\pi'(\mathcal{E}')).\end{aligned}$$

(3.4.4)

Using this, a computation yields the relation

$$\begin{aligned}2\theta_{b_1,b_2,\sigma_1,\sigma_2}(\mathcal{E}_2',\mathcal{E}') = [\theta_{b_1,b_2,\sigma_1,\sigma_2}(\pi'(\mathcal{E}'),\mathcal{E}') + \Theta_{b_1,b_2,\sigma_1,\sigma_2}(\mathcal{E}_2',{\pi'}^{-1}(\mathcal{E}_2'))] +\\ + (a_1 - a_2)\Delta_d + (\tau_2 - \tau_1)\Delta_r \ ,\end{aligned}$$

where

$$\begin{aligned}\Delta_d &= \deg({\pi'}^{-1}(\mathcal{E}_2')) - \deg(\mathcal{E}') \ ,\\ \Delta_r &= \operatorname{rank}({\pi'}^{-1}(\mathcal{E}_2')) - \operatorname{rank}(\mathcal{E}') \ .\end{aligned}$$

The result follows from this, since $\Delta_r \geq 0$. □

Remark. In general, this relation between surjective stability and full stability does not seem to be true.

Proposition 4.7. *Let $(\mathcal{E}_2,\mathcal{E},\pi)$ and $(\mathcal{E}_1,\mathcal{E}_2,\Phi)$ be related by $[\mathcal{E}_1,\mathcal{E}_2,\Phi] = [\mathcal{E}_2,\mathcal{E},\pi]$ under the bijection of Proposition 4.2. Let $\{a_1,a_2,\tau_1,\tau_2\}$ be any set of real number such that $a_1d_1+a_2d_2-\tau_1r_1-\tau_2r_2 = 0$. Then the following are equivalent*

1. *The 1-cohomology triple $(\mathcal{E}_1,\mathcal{E}_2,\Phi)$ is (a_1,a_2,τ_1,τ_2)-stable,*
2. *The surjective triple $(\mathcal{E}_2,\mathcal{E},\pi)$ is $(a_2-a_1,a_1,\tau_2-\tau_1,\tau_1,)$-surjectively stable.*

Proof. The proof of this Proposition depends on the following lemma, which describes the relation between the subobjects of $(\mathcal{E}_2,\mathcal{E},\pi)$ and $(\mathcal{E}_1,\mathcal{E}_2,\Phi)$.

Lemma 4.8. *Let $(\mathcal{E}_2,\mathcal{E},\pi)$ and $(\mathcal{E}_1,\mathcal{E}_2,\Phi)$ be related by $(\mathcal{E}_1,\mathcal{E}_2,\Phi) = f(\mathcal{E}_2,\mathcal{E},\pi)$. Denote the sets of subobjects of $(\mathcal{E}_2,\mathcal{E},\pi)$ and $(\mathcal{E}_1,\mathcal{E}_2,\Phi)$ by $\mathcal{SUB}(\mathcal{E}_2,\mathcal{E},\pi)$ and $\mathcal{SUB}(\mathcal{E}_1,\mathcal{E}_2,\Phi)$ respectively. Then*

1. *There is a well defined map*

$$f : \mathcal{SUB}(\mathcal{E}_2,\mathcal{E},\pi) \longrightarrow \mathcal{SUB}(\mathcal{E}_1,\mathcal{E}_2,\Phi) \ ,$$

2. this map is surjective,

3. the function $\theta_{a_1,a_2,\tau_1,\tau_2}$ is constant on the fibers of this map.

Proof. (1) Let $(\mathcal{E}_2', \mathcal{E}', \pi')$ be a subtriple of $(\mathcal{E}_2, \mathcal{E}, \pi)$, and let Φ' be the extension class of the extension

$$0 \longrightarrow \mathrm{Ker}(\pi') \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E}_2' \longrightarrow 0 .$$

We need to check that $(\mathrm{Ker}(\pi'), \mathcal{E}_2', \Phi')$ is in $\mathcal{SUB}(\mathcal{E}_1, \mathcal{E}_2, \Phi)$. But this is an immediate consequence of the way in which subobjects are defined. We can thus define $f(\mathcal{E}_2', \mathcal{E}', \pi') = (\mathrm{Ker}(\pi'), \mathcal{E}_2', \Phi')$.
(2) If $(\mathcal{E}_1', \mathcal{E}_2', \Phi')$ is a subobject of $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$, then $\mathcal{E}_2'$ is a subbundle of $\mathcal{E}_2$. Furthermore, if $(\mathcal{E}_1, \mathcal{E}_2, \Phi) = f(\mathcal{E}_2, \mathcal{E}, \pi)$, then (again, by the defining properties of subobjects) $\mathcal{E}_2'$ can be lifted to a subbundle $\mathcal{E}' \subset \mathcal{E}$. Then, with $\pi' : \mathcal{E}' \longrightarrow \mathcal{E}_2'$ denoting the projection map, $(\mathcal{E}_2', \mathcal{E}', \pi')$ is in $\mathcal{SUB}(\mathcal{E}_2, \mathcal{E}, \pi)$. Thus the map f is surjective.
(3) This is clear since all subtriples $(\mathcal{E}_2', \mathcal{E}', \pi')$ in $f^{-1}(\mathcal{E}_1', \mathcal{E}_2', \Phi')$ have isomorphic underlying smooth bundles. $\square$

The proof of the proposition now follows from a straightforward computation. Given subobjects related by $(\mathcal{E}_2', \mathcal{E}', \pi') = f(\mathcal{E}_1', \mathcal{E}_2', \Phi')$, we get

$$\begin{aligned}
\Theta_{a_1,a_2,\tau_1,\tau_2}(\mathcal{E}_1', \mathcal{E}_2') &= a_1 d_1' + a_2 d_2' - \tau_1 r_1' - \tau_2 r_2' \\
&= a_1(d' - d_2') + a_2 d_2' - \tau_1(r' - r_2') - \tau_2 r_2' \\
&= a_1 d' + (a_2 - a_1) d_2' - \tau_1 r' - (\tau_2 - \tau_1) r_2' \\
&= \Theta_{a_2-a_1,a_1,\tau_2-\tau_1,\tau_1}(\mathcal{E}_2', \mathcal{E}').
\end{aligned}$$

$\square$

Given an extension

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0 , \tag{e}$$

the strong equivalence class of e can be identified, as we have seen above, with the 1-cohomology triple $T = (\mathcal{E}_1, \mathcal{E}_2, \Phi)$, where $\Phi \in H^1(\mathcal{E}_1 \otimes \mathcal{E}_2^*)$ is the class defined by e. The weak equivalence class, denoted by $[e]$ corresponds to the equivalence class of T in $\mathcal{H}^{(1)}(E_1, E_2)/\mathfrak{G}_{\mathbb{C}}^{(1)} \times \mathfrak{G}_{\mathbb{C}}^{(2)}$. By a comparison of the appropriate subobjects it is apparent that the stability notions we have defined for extensions is a property of weak equivalence classes. Similarly, stability of 1-cohomology triples is a property of equivalence classes under the action of $\mathfrak{G}_{\mathbb{C}}^{(1)} \times \mathfrak{G}_{\mathbb{C}}^{(2)}$. In other words

Lemma 4.9. *1. An extension e is α-stable if and only if every extension e' such that $[e] = [e']$ is α-stable.*

2. A 1-cohomology triple T is α-stable if and only if every triple T' such that $[T'] = [T]$ is α-stable.

Proposition 4.10. *Let $[e]$ be a class in $\mathcal{E}xt(E_1, E_2)/\sim$ and let $[T]$ be the corresponding equivalence class in $\mathcal{H}^{(1)}(E_1, E_2)/\mathfrak{G}_{\mathbb{C}}^{(1)} \times \mathfrak{G}_{\mathbb{C}}^{(2)}$. Then $[e]$ is α-stable if and only if $[T]$ is α-stable.*

Proof. Again, the proof depends on a comparison of subobjects. Suppose, for example, that for some $e \in [e]$ there is a subextension which violates the α-stability condition. But this subextension determines a sub-triple of $T(e)$, the 1-cohomology triple corresponding to e, and this subtriple violates the α-stability condition for $T(e)$. Conversely, suppose one is given a 1-cohomology triple T, and a subtriple T' which violates stability. This subtriple determines a subextension for some extension, say $e(T)$, in the class corresponding to T, and this subextension violates the α-stability condition for $e(T)$. □

Combining Definition/Lemma 4.4, Proposition 4.7, and Proposition 4.10, we thus get

Proposition 4.11. *Let the extension $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$, the surjective triple $(\mathcal{E}_2, \mathcal{E}, \pi)$, and the 1-cohomology triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ be related as described above. Then the following are equivalent*

1. *the extension $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$ is α-stable,*
2. *the 1-cohomology triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ is α-stable,*
3. *the 1-cohomology triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ is $(1, 1, \tau, \tau - \alpha)$-stable ,*
4. *the surjective triple $(\mathcal{E}_2, \mathcal{E}, \pi)$ is $(0, 1, -\alpha, \tau)$-stable.*

In (3) and (4), τ is determined by the relation $d_1 + d_2 = r_1\tau + r_2(\tau - \alpha)$.

4.3. Metric equations.

Corresponding to the comparison between the stability properties of surjective triples, 1-cohomology triples and extensions, there is an analogous comparison between the equations governing the metric problems in the three situations. In this section we spell out this equivalence of metric problems.

For the 1-cohomology triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$, the equations corresponding to $(a_1, a_2, \tau_1, \tau_2)$-stability are given by (2.4.7a-c), i.e.

$$i\Lambda a_1 F_{H_1} + \Lambda^n(\phi \circ \overline{*}_E \phi) = \tau_1 \mathbf{I}_1 \ ,$$

$$i\Lambda a_2 F_{H_2} - (-1)^p \Lambda^n(\overline{*}_E \phi \circ \phi) = \tau_2 \mathbf{I}_2 \ ,$$

$$\overline{\partial}^*_{1,2}(\phi) = 0 \ ,$$

where $\phi \in \Omega^{0,p}(X, Hom(E_2, E_1))$ is a representative of the cohomology class Φ.

The equations corresponding to, say, $(b_1, b_2, \sigma_1, \sigma_2)$- stability for a (surjective) triple $(\mathcal{E}_2, \mathcal{E}, \pi)$, come from (2.4.7) with $p = 0$. They are

$$b_1 i\Lambda F_{H_2} + \pi\pi^* = \sigma_1 \mathbf{I}_2 \ , \tag{4.3.1a}$$

$$b_2 i\Lambda F_H - \pi^*\pi = \sigma_2 \mathbf{I} \ . \tag{4.3.1b}$$

Proposition 4.11. *Let $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ be a 1-cohomology triple, and let $(\mathcal{E}_2, \mathcal{E}, \pi)$ be a corresponding surjective triple. Suppose that there are metrics H_2 and H on $(\mathcal{E}_2, \mathcal{E}, \pi)$ satisfying (4.3.1a,b).*

Then there are metrics H_1 and H_2 and a representative $\phi \in \Phi$ satisfying (2.4.7) with parameters $(b_2, b_2 + b_1, \sigma_2, \sigma_1 + \sigma_2)$.

Proof. We can use the metric on E to fix an orthogonal decomposition $E = E_1 \oplus E_2$. Let ϕ be the element in $\Omega^{0,p}(X, Hom(E_2, E_1))$ corresponding to the second fundamental form with respect to this metric. Then

$$F_H = \begin{pmatrix} F_{H_1} - \phi \wedge \phi^* & D'_{1,2}\phi \\ -\overline{\partial}_{1,2}\phi^* & F_{H_2} - \phi^* \wedge \phi \end{pmatrix} \tag{4.3.2}$$

where $D'_{1,2}$ is the holomorphic part of the metric connection on $Hom(E_2, E_1)$. Furthermore, in this frame, we get

$$\pi^*\pi = \begin{pmatrix} 0 & 0 \\ 0 & \pi\pi^* \end{pmatrix} \ .$$

Equation (4.3.1b) thus decomposes as

$$(4.3.3a)\qquad \overline{\partial}^*_{1,2}\phi = 0\ ,$$

$$(4.3.3b)\qquad i\Lambda b_2 F_{H_1} - ib_2\Lambda(\phi\wedge\phi^*) = \sigma_2\mathbf{I}_1\ ,$$

$$(4.3.3c)\qquad i\Lambda b_2 F_{H_2} - ib_2\Lambda(\phi^*\wedge\phi) = \sigma_2\mathbf{I}_2 + \pi\pi^*\ .$$

Since ϕ is in $\Omega^{0,1}(X, Hom(E_2, E_1))$, i.e. has form degree (0,1), we get

$$-i\Lambda(\phi\wedge\phi^*) = \frac{1}{n!}\Lambda^n(\phi\circ\overline{*}_E\phi) \text{ and } -i\Lambda(\phi^*\wedge\phi) = -\frac{1}{n!}\Lambda^n(\overline{*}_E\phi\circ\phi)\ .$$

Combining (4.3.3) with (4.3.1), we thus get

$$i\Lambda b_2 F_{H_1} + \frac{b_2}{n!}\Lambda^n(\phi\circ\overline{*}_E\phi) = \sigma_2\mathbf{I}_1\ ,$$

$$i\Lambda(b_2+b_1)F_{H_2} + \frac{b_2}{n!}\Lambda^n(\overline{*}_E\phi\circ\phi) = (\sigma_2+\sigma_1)\mathbf{I}_2\ ,$$

$$\overline{\partial}^*_{1,2}(\phi) = 0\ .$$

The factor $\frac{b_2}{n!}$ can be absorbed by rescaling the metric on E_1. We thus recover the 1-cohomology equations if we set $a_1 = b_2, a_2 = b_1 + b_2, \tau_1 = \sigma_2, \tau_2 = \sigma_1 + \sigma_2$. □

In the special case where $(b_1, b_2, \sigma_1, \sigma_2) = (0, 1, \tau, \tau')$, the correspondence between these equations and the deformation of the Hermitian–Einstein equation given in (3.2.1) can be seen as follows.

Notice first what happens to the surjective triples equations (4.3.1) in the special case where we take $(a_1, a_2, \tau_1, \tau_2) = (0, 1, -\alpha, \tau)$. Denoting the triple by $(\mathcal{E}_2, \mathcal{E}, \pi)$, the metric equations become

$$(4.3.4a)\qquad \pi\pi^* = -\alpha\mathbf{I}_2\ ,$$

$$(4.3.4b)\qquad i\Lambda F_H - \pi^*\pi = \tau\mathbf{I}\ .$$

The first of these equations says that $(-\alpha)^{-1}\pi^*$ is a left inverse of π, i.e. that $(-\alpha)^{-1}\pi^*$ splits the sequence

$$0 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0\ .$$

With respect to the smooth splitting $E = \mathrm{Ker}(\pi)\oplus\pi^*(E_2)$, the endomorphism $\pi^*\pi$ thus has the block decomposition

$$\pi^*\pi = \begin{pmatrix} 0 & 0 \\ 0 & -\alpha\mathbf{I}_2 \end{pmatrix}\ .$$

With τ' defined by $\alpha = \tau - \tau'$, the equation (4.3.4b) can then be rewritten as

$$i\Lambda F_H = \begin{pmatrix} \tau \mathbf{I}_1 & 0 \\ 0 & \tau' \mathbf{I}_2 \end{pmatrix} ,$$

which is precisely equation (3.2.1). On the other hand for the 1-cohomology triple $T = (\mathcal{E}_1, \mathcal{E}_2, \Phi)$, the equations become

$$\begin{aligned} \overline{\partial}^*_{1,2}\phi &= 0 \\ i\Lambda(F_{H_1} - \phi \wedge \phi^*) &= \tau_1 \mathbf{I}_1 \\ i\Lambda(F_{H_2} - \phi^* \wedge \phi &= \tau_2 \mathbf{I}_2 \end{aligned}$$

for a triple (H_1, H_2, ϕ) consisting of metrics on $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively, and $\phi \in \mathbf{O}(\Phi)$, where

$$\text{(4.3.5)} \qquad \mathbf{O}(\Phi) = \{\phi \in \Omega^{0,1}(\mathrm{Hom}(E_2, E_1)) \mid \overline{\partial}^*_{1,2}\phi = 0 \text{ and } [\phi] = \Phi\}.$$

The equivalence of these equations with (3.2.1) follows immediately from writing F_H as in (4.3.2), and from the fact that $i\Lambda\partial_{1,2} = \overline{\partial}^*_{1,2}$.

To have a complete equivalence between the solution of the two metric problems we need to prove the following.

Lemma 4.12. *There is a one-to-one correspondence*

$$\mathcal{M}et(\mathcal{E}) \longleftrightarrow \mathcal{M}et(\mathcal{E}_1) \times \mathcal{M}et(\mathcal{E}_2) \times \mathbf{O}(\Phi).$$

Proof. We have already mentioned above how from a metric H on $\mathcal{E}$ we obtain (H_1, H_2, ϕ). To prove the other direction we observe that giving a metric on $\mathcal{E}$ is equivalent to giving metrics on $\mathcal{E}_1$ and $\mathcal{E}_2$ and a C^∞-splitting of E. But there is a one-to-one correspondence between C^∞-splittings of E and elements of $\mathbf{O}(\Phi)$. These is clear since two different splittings $\gamma_1, \gamma_2 : \mathcal{E}_2 \longrightarrow \mathcal{E}$ differ by an element $\alpha \in \Omega^0(Hom(\mathcal{E}_2, \mathcal{E}_1))$, i.e. $\gamma_2 = \gamma_1 + j\alpha$, where j denotes the inclusion $\mathcal{E}_1 \longrightarrow \mathcal{E}$. The corresponding fundamental forms are related by $\phi_2 = \phi_1 + \overline{\partial}\alpha$. □

Summarizing the results for these special values of the parameters $(a_1, a_2, \tau_1, \tau_2)$, we get the following analog of Proposition 4.11:

Proposition 4.13. *Let the extension* $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0$, *the surjective triple* $(\mathcal{E}_2, \mathcal{E}, \pi)$, *and the 1-cohomology triple* $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ *be related as described above. Then the following are equivalent*

1. *The surjective triple* $(\mathcal{E}_2, \mathcal{E}, \pi)$ *admits a solution (i.e. metrics on* E *and* E_2*) to the equations (4.3.1) with* $b_1 = 0, b_2 = 1, \sigma_1 = -\alpha, \sigma_2 = \tau$.

2. *The 1-cohomology triple admits a solution (i.e. a representative of* Φ *and metrics on* E_1 *and* E_2*) to the equations (2.4.7) with* $a_1 = a_2 = 1, \tau_1 = \tau$*, and* $\tau_2 = \tau - \alpha$.

3. *The bundle* $\mathcal{E}$ *admits a solution (i.e. a metric on* E*) to the equation (3.2.1) with right hand side* $\begin{pmatrix} \tau\mathbf{I}_1 & 0 \\ 0 & (\tau-\alpha)\mathbf{I}_2 \end{pmatrix}$.

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