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so(3)–Topological Quantum Field Theory

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We give the choice of basic data for $so(3)$ topological quantum theory and show that it satisfies the Moore-Seiberg-Walker equations.

1. Introduction.

The Alexander polynomial has a dual nature. Extrinsically, it can easily be computed from the placement of a knot in space using skein relations. Intrinsically, it can be computed in a cut and paste fashion via homology theory. The power of the Alexander polynomial as a tool for studying knots derives from this connection between the extrinsic and intrinsic structure of a knot. The current study of state sum invariants of three-manifolds was initiated with the discovery of the Jones polynomial [J]. Shortly after its introduction the need for a cut and paste theory to explain the Jones polynomial became understood. Witten [Wi] realized such a theory using ideas from quantum field theory. The cut and paste theory he developed is called topological quantum field theory. Witten's construction of topological quantum field theory rested on deep physical intuitions that had yet to be completely justified from a mathematical viewpoint. In addition to explaining the Jones polynomial Witten's theory produced a whole new realm of three-manifold invariants.

Reshetikhin and Turaev [R-T] have given a rigorous construction of a topological quantum field theory based on the representation theory of an algebra $U_q(sl_2)$. The representation theory of the algebra $U_q(sl_2)$ was largely worked out by Kirilov and Reshetikhin [K-R]. Along another path Kohno [K-1], [K-2] worked out a construction using the holonomy of the Knizhnik-Zamoldchikov connection. Finally a skein theoretic approach based on the Kauffman bracket was worked out by Lickorish [L] and by Blanchet, Habbegger, Masbaum and Vogel [B-H-M-V]. (See also Kauffman and Lins [K-L].)

[1]Authors thank Tomek Bartoszyński for drawing the pictures.

Many arguments in three-manifold topology proceed by cutting a three-manifold open along a surface with boundary. Walker, basing his work [Wa] on Moore and Seiberg [M-S], carefully described what is needed to construct a topological quantum field theory where you can glue together along surfaces with boundary. This paper is a worked example of his theory. By discarding half of the representations that are normally used we get examples of topological quantum field theories with corners.

Basic data for a topological quantum field theory is a list of vector spaces, pairings, and linear mappings from which any other vector space, pairing or linear mapping in the topological quantum field theory can be derived. This collection must satisfy the Moore-Seiberg equations in order to actually be the basic data of some topological quantum field theory. A *modular* Hopf algebra is a Hopf algebra with a list of representations and some intertwiners that satisfy the axioms of [R-T]. Walker shows that, given a modular Hopf algebra, one can easily construct basic data for a modular topological quantum field theory. The goal of this paper is to work out the basic data of these theories to the point where, if interested, one could begin with the formulas and a reference to basic hypergeometric functions (e.g. [G-R]) and start working examples. The half of the representations that we retain correspond to representations of $SO(3)$, hence the title of the paper. The invariants for closed three-manifolds from this theory differ from the invariants of [R-T] by conjugation and by a product with a function of the signature of a four manifold that the three-manifold bounds.

Section 2 of this paper reviews the definitions from [Wa]. In section 3 we give our choices of basic data, using the notation from [K-M] for dealing with $U_q(sl_2)$ when q is a root of unity . In section 4, using more common imagery in the manner of [K-R], [K-2], [R-T], we prove that our choice of basic data satisfies the Moore-Seiberg equations. This is in lieu of presenting our basic data in the different form as used by Walker and then quoting his theorem. Many maps in a topological quantum field theory constructed by the Hopf algebra approach are isomorphisms corresponding to duality pairings. Our proof that the Moore-Seiberg equations are satisfied translates them into statements about pairings, whereas Walker uses tangle descriptions of the maps. Finally, in section 5 we derive numerical values for the linear maps in our basic data.

The computations in this section are based on the standard techniques for computing the Clebsch-Gordan coefficients and the $6j$-symbols. There are two pitfalls occuring in the literature that we avoid. Some formulas for Clebsch-Gordan coefficients turn out to involve division by zero when you substitute numbers for letters. In some derivations the adjoint of the action of $U_q(sl_2)$ on the tensor product of representations is taken incorrectly. Since we are working at roots of unity, we are dealing with cases where the standard presentation of the techniques is not valid. This requires more care in defining the Clebsch-Gordan coefficients and more careful attention to the steps in the derivation.

We also have an innovation resulting in more satisfying computations. In the classical example, the representations of $U(sl_2)$ are made into Hilbert spaces, so that the adjoint of operators on the representations induces an involution on $U(sl_2)$ that is independent of the representation. Thus the representations form a category of star representations. In addition to allowing the answer to be normalized via the trace norm on intertwiners, it has the advantage that the adjoint of an intertwiner is an intertwiner. The algebras $U_q(sl_2)$ converge to $U(sl_2)$ as q goes to 1. Hence it is traditional to make the representations of $U_q(sl_2)$ into Hilbert spaces so that as q goes to 1 the Hilbert spaces converge to the classical examples. When q is not real, the problem is that the representations of $U_q(sl_2)$ no longer form a category of star representations. This results from noncocommutativity of the comultiplication. One solution to this problem is given by Durhus [D]. If one decides to extend Hilbert pairings to tensor products in a nonstandard way, then one can still have the adjoint of an intertwiner be an intertwiner. However, in our resolution we define a nondegenerate sesquilinear pairing on our representations that is not positive definite. The adjoint of operators on any representation in our category induces the same involution on $U_q(sl_2)$. Hence the adjoint of an intertwiner is an intertwiner. We are amazed that the trace norm induced by our pairing on intertwiners is positive definite! In fact, we obtain the same normalization as the one given by making the representations into Hilbert spaces, but the adjoint of an intertwiner is an intertwiner. Our answers differ from the answers of [K-R] only by signs since we use [K-M]'s choice of operators for the associated tangle functors rather than [K-R]'s choice. This gives us some convenient formulas in the associated tangle functors. For instance, we have a very simple derivation of a formula relating the S matrix to the F matrix that was originally derived with a great deal of effort in [L-Y].

The last section consists of a list of formulas that summarize our findings for the impatient who just want to work some examples.

2. What is Topological Quantum Field Theory?.

Our treatment of topological quantum field theory is taken from Walker [Wa]. We repeat his definitions for completeness of exposition.

A topological quantum field theory consists of

- a modular functor **V** from the category of labeled extended surfaces to the category of vector spaces and morphisms;
- a partition function **Z** from the set of extended 3-manifolds which assigns to a manifold M an element $Z(M)$ of a vector space associated with its boundary, i.e. $Z(M) \in V(\partial M)$.

These must satisfy some axioms that describe their behavior under gluing, taking disjoint sums and changing orientation, together with naturality and mapping cylinder axioms and several dimension axioms.

All considered manifolds are assumed to be piecewise-linear, compact and oriented.

Each boundary component of an *extended surface* has a fixed parameterization by a standard circle. In addition, an extended surface Y is equipped with maximal isotropic (Lagrangian) subspace L of $H_1(Y)$. A subspace is *isotropic* if the intersection form restricted to that subspace is zero. This subspace L corresponds to choosing a system of disjoint curves which cut Y into a collection of disks, annuli and pants.

A *labeled, extended surface* has an element of a finite label set $\mathcal{L}$ assigned to each boundary component. A morphism of labeled extended surfaces is an ordered pair consisting of a homeomorphism that preserves boundary parameterizations and labels, and an integer. These morphisms are composed by composing the homeomorphisms and adding the integers along with a correction term coming from Wall's non-additivity function [Wall], and the Langrangian subspaces. More specifically, if

$$(f, m) : (Y_1, L_1) \to (Y_2, L_2) \quad \text{and} \quad (g, n) : (Y_2, L_2) \to (Y_3, L_3)$$

then

$$(g, n) \circ (f, m) = (g \circ f, m + n + c(L_3, g(L_2), f(g(L_1))))$$

where c is Wall's non-additivity function. For the precise definition of c see [Wall].

In general, the label set should be equipped with an involution $a \leftrightarrow \hat{a}$ and a distinguished "trivial label", mapped to itself by this involution. In our case the involution is trivial, so we will ignore it. Our notation differs from Walker's, since we use 0 for the "trivial label", which he denotes by 1.

An *extended 3-manifold* consists of a triple (M, L, n), where M is a 3-manifold, L is a maximal isotropic subspace of $H_1(\partial M)$ and n is an integer. There is a procedure for gluing two extended 3-manifolds together along subsurfaces of their boundary that involves the non-additivity function of Wall [Wa]. Since we will not be using the gluing procedure in this paper, we do not describe it.

Let (Y, l) denote an extended surface Y with labeling l, where l is a function from the set of boundary components of Y to $\mathcal{L}$.

The modular functor $\mathbf{V}$ and partition function $\mathbf{Z}$ have to satisfy the following axioms (taken from [Wa]).

1. **Disjoint Union Axiom for V**

$$V(Y_1 \sqcup Y_2, l_1 \sqcup l_2) = V(Y_1, l_1) \otimes V(Y_2, l_2)$$

These identifications have to be associative and compatible with the action of the mapping class groupoids.

2. **Gluing Axiom for V**

Let C and C' be disjoint closed components of the boundary of Y, and let ϕ and ϕ' denote their parameterizations. If $g : C \to C'$ is a homeomorphism such that $(\phi')^{-1}g\phi$ acts as complex conjugation, denote by Y_g the surface Y glued by g. Then

$$V(Y_g, l) = \bigoplus_{x \in \mathcal{L}} V(Y, l, x, x),$$

where l is a labeling of Y_g, and x is a label assigned to both C and C'.

3. **Duality Axiom**

$$V(-Y, l) = V(Y, l)^*$$

Here $V(Y, l)^*$ denotes the space of complex linear maps $V(Y, l) \to \mathbb{C}$.

The identifications $V(Y) = V(-Y)^*$ and $V(-Y) = V(Y)^*$ are mutually adjoint. Furthermore there is a function $S : \mathcal{L} \to \mathbb{C}$ so that if

$$\bigoplus_x \alpha_x \epsilon \bigoplus_{x \in \mathcal{L}} V(Y, l, x, x) \quad \text{and} \quad \bigoplus_x \beta_x \epsilon \bigoplus_{x \in \mathcal{L}} V(-Y, l, x, x)$$

then the pairing on the glued surface is given by

$$\bigoplus_x \beta_x(\bigoplus_x \alpha_x) = \sum_x S(x)\beta_x(\alpha_x).$$

It is traditional to call this pairing the Kronecker pairing. We will use the symbol $\langle \cdot, \cdot \rangle_{k,\Sigma_{g,n}}$ to denote this pairing on a surface of genus g with n boundary components.

4. **Empty Space Axiom**

$$V(\emptyset) = \mathbb{C}$$

5. **Disk Axiom**

$$V(D, a) \simeq \begin{cases} \mathbb{C}, & a = 0 \\ 0, & a \neq 0. \end{cases}$$

D denotes an extended disk.

6. **Annulus Axiom**

$$V(A, a, b) \simeq \begin{cases} \mathbb{C}, & a = b \\ 0, & a \neq b. \end{cases}$$

A denotes an extended annulus.

7. **Naturality Axiom**

Let M_1 and M_2 be two extended 3-manifolds and let $f : M_1 \to M_2$ be an orientation preserving homeomorphism which maps the Lagrangian subspaces to each other. Furthermore assume that the integer part of the extended 3-manifolds is the same. The naturality axiom states:

$$V(f \mid \partial M_1) Z(M_1) = Z(M_2).$$

8. **Gluing Axiom for Z**

Let M be an extended 3-manifold, and $Y_1, Y_2 \subset \partial M$ disjoint extended surfaces of genus g with n boundary components. Let $f : Y_1 \to Y_2$ be an extended morphism, and let M_f denote the extended 3-manifold in which Y_i's are glued together by f.

$$Z(M_f) = \bigoplus_l \sum_j \langle V(f)\alpha_l^j, \beta_l^j \rangle_{k,\Sigma_{g,n}} \gamma_{ll}^j,$$

where α, β and γ are described below, and the pairing is defined by Axiom 3.

By Axiom 2

$$V(\partial M) = \bigoplus_{l_1,l_2} V(Y_1, l_1) \otimes V(Y_2, l_2) \otimes V(\partial M \backslash (Y_1 \cup Y_2), (l_1, l_2)),$$

where l_i 's are all possible labelings of ∂Y_i. Thus $Z(M)$ can be expressed as

$$Z(M) = \bigoplus_{l_1,l_2} \sum_j \alpha_{l_1}^j \otimes \beta_{l_2}^j \otimes \gamma_{l_1 l_2}^j,$$

with $\alpha_{l_1}^j, \beta_{l_2}^j, \gamma_{l_1 l_2}^j$ from the respective factors of the tensor products in the sum above. By Axiom 2

$$V(\partial M_f) = \bigoplus_l V(\partial M \backslash (Y_1 \cup Y_2), (l, l)).$$

9. **Mapping Cylinder Axiom**

Let I_{id} be the mapping cylinder of the identity on Y.

$$Z(I_{id}) = \bigoplus_{l \in \mathcal{L}(Y)} id_l$$

Note that by Axioms 2 and 3 we have

$$V(\partial I_{id}) = \bigoplus_{l \in \mathcal{L}(Y)} V(Y, l) \otimes V(Y, l)^*$$

By id_l we mean the identity in $V(Y, l) \otimes V(Y, l)^*$.

A functor V satisfying Axioms 1-6 is called a *modular functor*, and Z is called a *partition function.*

Notice that for a planar surface S the intersection form is trivial. Hence, the only Lagrangian subspace of $H_1(S)$ is all of $H_1(S)$. Hence we supress the Lagrangian subspace in the notation for the extended disk, annulus or pair of pants.

Every surface can be cut open into a disjoint union of disks, annuli and pairs of pants (thrice punctured spheres). Thus to describe a topological quantum theory it is enough to assign vector spaces to these simple surfaces. The axioms (in particular 2 and 1) will determine $\mathbf{V}$ for all other surfaces.

The operators corresponding to the generators of the mapping class groups of the above surfaces also need to be defined.

Since any given surface can be sliced up in several different ways, we need to identify the corresponding vector spaces.

Summarizing, the *basic data* for a topological quantum field theory consists of

- assignation of vector spaces to:
  - extended disk D,
  - extended annulus A ,
  - extended pair of pants P;
- choice of certain basis elements (denoted by β_0, β_{aa} and β_a^{cb}) in these vector spaces;
- definition of standard orientation reversing maps on D, A and P. We will denote each of these maps by ψ.

  ψ induces identifications of the corresponding vector spaces with their duals and defines a pairing

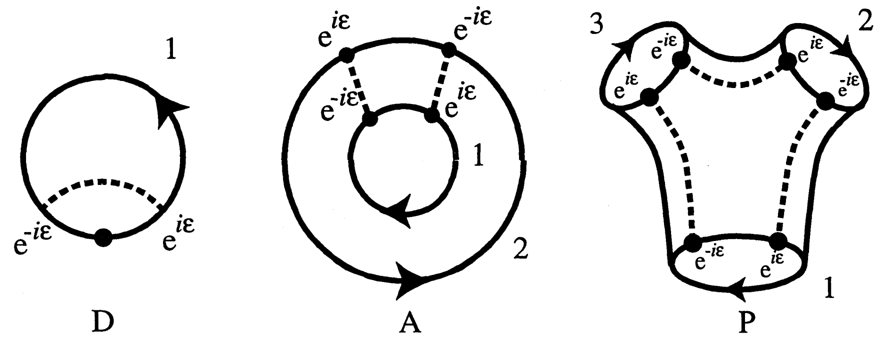
$$\langle x, y\rangle = \psi(x)(y).$$

- description of operators corresponding to the maps T, R, B described below (generators of mapping class groups of A and P);
- description of isomorphisms F and S described further below (corresponding to some decompositions of an annulus, torus, once punctured torus and four times punctured sphere).

This basic data will uniquely determine a topological quantum field theory if and only if it will satisfy the 14 relations stated at the end of this section (see Theorem 2.1).

We will number the boundary components of the surfaces D, A, and P. Recall that each boundary component has a fixed parameterization by a standard circle. Equip each surface with *seams*, i.e. disjoint, properly embedded arcs. A seam on a disk joins the point $e^{-i\epsilon}$ with the point $e^{i\epsilon}$, for

Figure 1: Disk, Annulus and Pair of Pants, with numbered boundary components and seams.



some fixed $0 < \epsilon < \pi$. Seams on an annulus or on a pair of pants join the point $e^{-i\epsilon}$ on the j'th boundary component with the point $e^{i\epsilon}$ on the $j+1$'st boundary component (see figure 1).

Notice, that for any two pairs of pants with numbered boundary components and seams, there is a unique (up to isotopy) orientation preserving homeomorphism between them which preserves the numberings of the boundary components and the seams; similarly for annuli or disks.

Recall that the vector space assigned to the labeled, extended disk $(D, 0)$ is denoted by $V(D, 0)$ (see axiom 5), the vector space assigned to an annulus (A, a, a) is denoted by $V(A, a, a)$ (see axiom 6). Similarly denote the vector space assigned to a pair of pants (P, a, b, c) by $V(P, a, b, c)$.

The standard orientation reversing maps ψ on D, A, and P (see figure 2) induce linear isomorphisms identifying vector spaces $V(D, 0)$, $V(A, a, a)$ and $V(P, a, b, c)$ respectively, with their duals.

The mapping class group of the extended disk is generated by the (identity, 1). Recall that a morphism of an extended surface is an ordered pair consisting of a homeomorphism and an integer. The isomorphism of vector spaces induced by the morphism of the extended disk to itself consisting of the identity map and the integer 1 will be denoted by C.

The mapping class group of an extended annulus is generated by the morphism consisting of the identity map and the integer 1 (the induced isomorphism of vector spaces is also denoted by C), and by maps T , and R, with integer parts equal to 0, where T and R are pictured in figure 3.

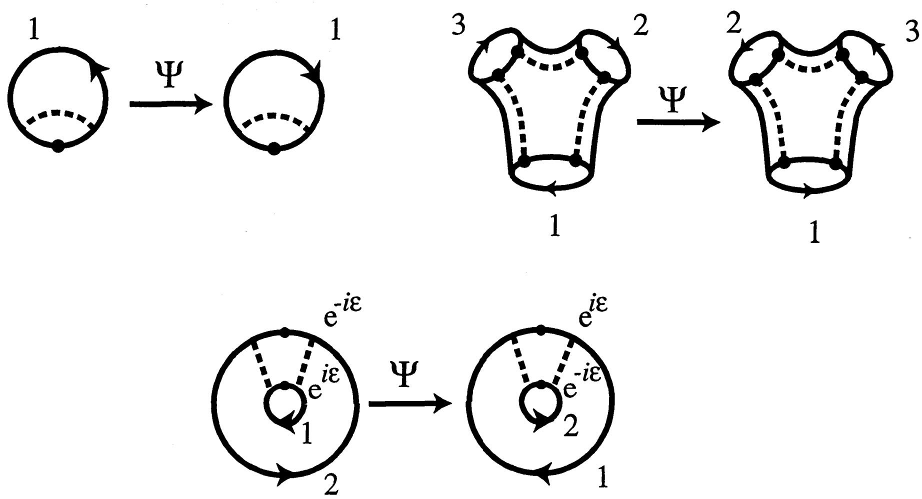


Figure 2: Maps ψ on D, A and P

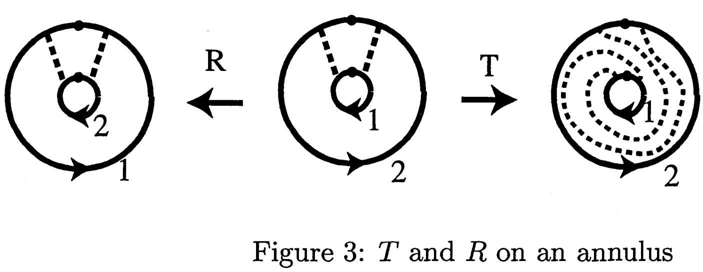


Figure 3: T and R on an annulus

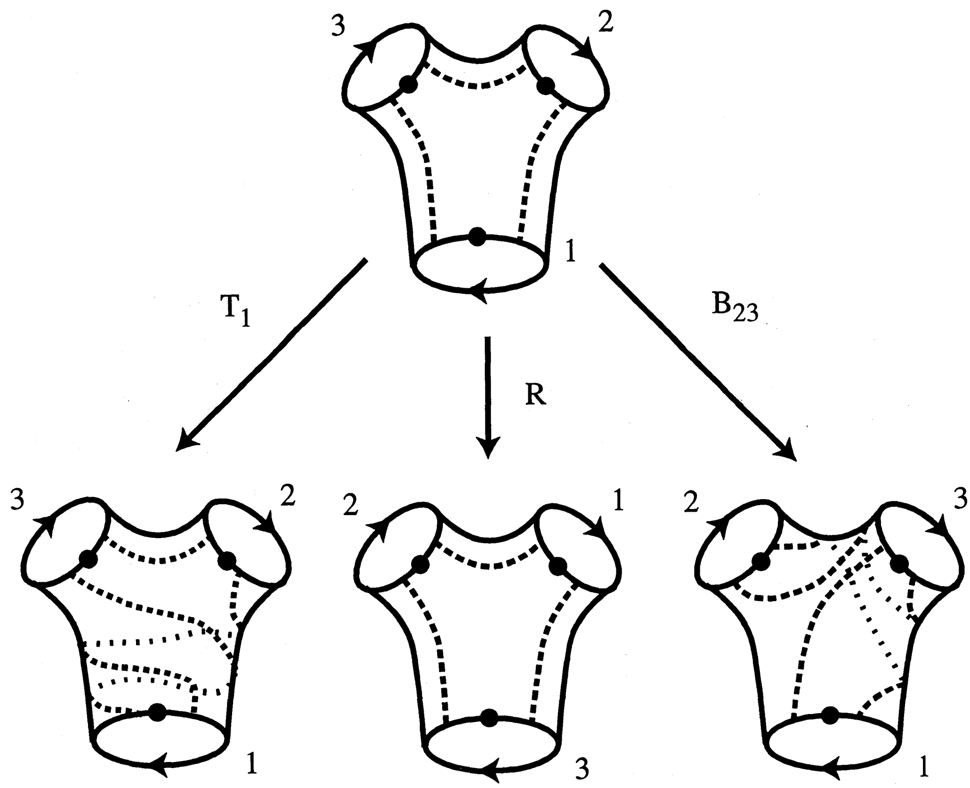


Figure 4: T_1, R and B_{23}

The mapping class group of an extended pair of pants is generated by the pair $(id, 1)$ (denoted again by C) and by maps T_1, R and B_{23} pictured in figure 4, with integer parts equal to 0 . We will also use maps B_{12}, B_{31}, T_2, and T_3, defined by:

(2.1)

$$B_{12} = R^{-1}B_{23}R, \quad B_{13} = RB_{23}R^{-1}, \quad T_2 = RT_1R^{-1}, \quad T_3 = R^{-1}T_1R.$$

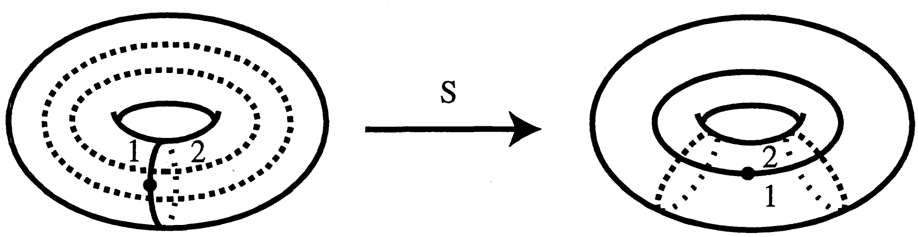
The integer parts of the corresponding morphisms of extended surfaces are equal to 0.

In order to identify vector spaces assigned to a surface via different ways of slicing it, it is enough to describe the isomorphisms S, S_a and F. The integer parts of the morphisms are 0.

Let

$$S : \bigoplus_{x \in \mathcal{L}} V(A, x, x) \to \bigoplus_{y \in \mathcal{L}} V(A, y, y) \tag{2.2}$$

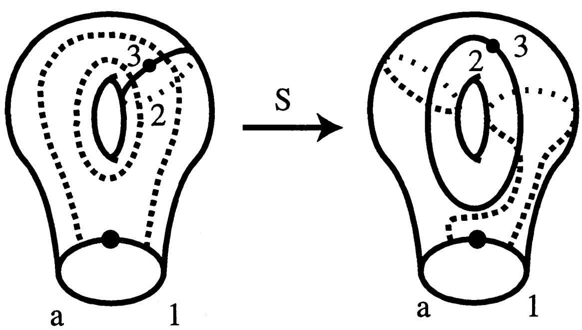
correspond to cutting an extended torus into an extended annulus in two different ways:



Let

$$S_a : \bigoplus_{x \in \mathcal{L}} V(P, a, x, x) \to \bigoplus_{y \in \mathcal{L}} V(P, a, y, y) \tag{2.3}$$

be the analogous map corresponding to cutting open a punctured torus into a pair of pants in two different ways:

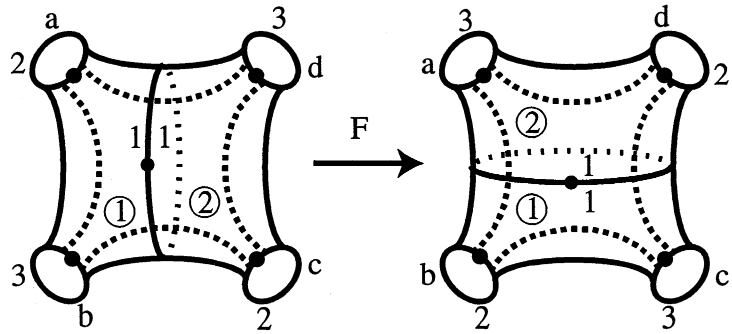


Let

(2.4)

$$F : \bigoplus_{x \in \mathcal{L}} V(P, x, a, b) \otimes V(P, x, c, d) \to \bigoplus_{y \in \mathcal{L}} V(P, y, b, c) \otimes V(P, y, d, a)$$

be the map corresponding to cutting a four times punctured sphere into two pairs of pants in two different ways:



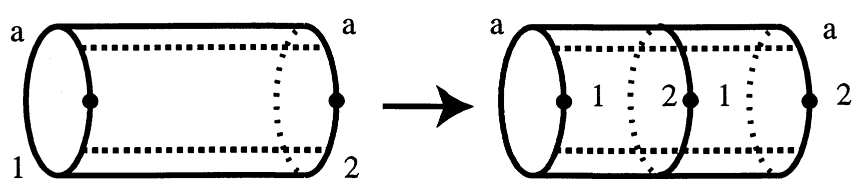
Denote basis elements of $V(D,0)$, $V(A,a,a)$ and $V(P,a,b,c)$ by β_0, β_{aa} and β_a^{cb} respectively. These have to be chosen in such a way that the following conditions are satisfied. First,

$$\langle \beta_0, \beta_0 \rangle = 1, \tag{2.5}$$

where pairing is defined by ψ.

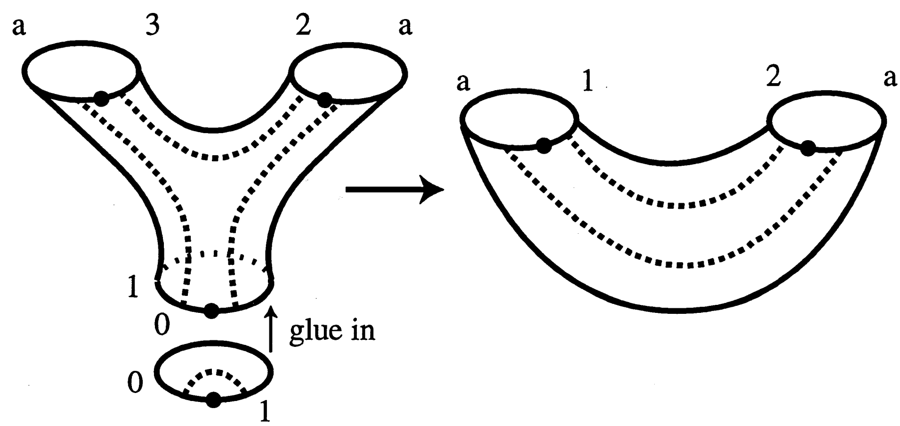
Second, if we cut an annulus into two annuli then the morphism $(id, 0)$ induces a map $V(A,a,a) \to V(A,a,a) \otimes V(A,a,a)$. We require that

$$\beta_{aa} \mapsto \beta_{aa} \otimes \beta_{aa} \tag{2.6}$$



And finally, if we glue together a disk and a pair of pants, obtaining an annulus, then the map corresponding to the identity map should have the property that

$$\beta_0 \otimes \beta_0^{aa} \mapsto \beta_{aa}. \tag{2.7}$$



The following theorem is quoted from [Wa].

Theorem 2.1 (Moore, Seiberg, Walker). *Basic data determines a modular functor if and only if, for all a, b, c, d $\in \mathcal{L}$:*

(i) $P^{(13)}R^{(2)}F^{(12)}R^{(2)}F^{(23)}R^{(2)}F^{(12)}R^{(2)}F^{(23)}R^{(2)}F^{(12)} = 1$;

(ii) $F\left(B_{23}^{(2)}\right)^{-1} F\left(B_{23}^{(2)}\right)^{-1} F\left(B_{23}^{(2)}\right)^{-1} T_2^{(2)} = 1$;

(iii) $\left(T_3^{(2)}\right)^{-1} T_1^{(2)} B_{23}^{(2)} F\left(B_{23}^{(1)}\right)^{-1}\left(B_{23}^{(2)}\right)^{-1} F(S^{(2)})^{-1}FR^{(2)}(R^{(1)})^{-1}FS^{(2)} = 1$;

(iv) $CB_{23}^{-1}T_3^2ST_3ST_3S = 1$; $CRTSTSTS = 1$;

(v) $F\left(R^{-1}(x) \otimes \beta_c^{0c}\right) = x \otimes \beta_a^{a0}$ *for all* $x \in V_{abc}$;

(vi) $S = \phi^{-1}S_1\phi$, *where* $\phi : \beta_{xx} \mapsto \beta_0^{xx}$;

(vii) $F^2P = 1$;

(viii) $T_3B_{23}^{-1}S^2 = 1$; $RS^2 = 1$;

(ix) $R(\beta_{aa}) = \beta_{aa}$;

(x) $R = \phi^{-1}\left(T_1^{-1}B_{12}\right)\phi$, *where* $\phi : \beta_{xx} \mapsto \beta_x^{0x}$;

(xi) $F_{abcd} = F_{cbad}^{\dagger}$;

(xii) $S_a = S_a^{\dagger}$;

(xiii) $\langle \beta_{aa}, \beta_{aa} \rangle = S(a)^{-1}$;

(xiv) $\langle\beta_0^{aa},\beta_0^{aa}\rangle = S(0)^{-1}S(a)^{-1}$.

The superscripted numbers in parentheses refer to the factors of a tensor product that the operator is acting on. For instance $P^{(13)}(\alpha\otimes\beta\otimes\gamma) = \gamma\otimes\beta\otimes\alpha$. Similarly $R^{(2)}(\alpha\otimes\beta\otimes\gamma) = \alpha\otimes R(\beta)\otimes\gamma)$. We will explicate the meaning of these equations more fully as we prove that our choice of basic data satisfies them.

3. The basic data.

In this section we recall the algebra A_r, $r > 1$, (see [K-M], [K-R] [R-T], [V-K]) and the associated tangle functor. We then give choices of the basic data we use to define $so(3)$-tqft. Finally, we work out some basic identities that we will use in section 4.

Recall from [K-M]; if $r > 1$ is an integer, then

$$q = e^{\frac{2\pi i}{r}}, \quad s = e^{\frac{\pi i}{r}} \text{ and } t = e^{\frac{\pi i}{2r}}. \tag{3.1}$$

If n is an integer, let

$$[n] = \frac{s^n - s^{-n}}{s - \bar{s}} = \frac{\sin\frac{\pi n}{r}}{\sin\frac{\pi}{r}}. \tag{3.2}$$

Clearly $[1] = 1$, $[-n] = -[n]$ and $[n-r] = -[n]$. By $[n]!$ we mean the function defined recursively by

$$[0]! = 1, \quad \text{and} \quad [n]! = [n]\cdot[n-1]!. \tag{3.3}$$

Finally the quantized binomial coefficient is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}. \tag{3.4}$$

There is a a "Pascal triangle" for $\begin{bmatrix} n \\ k \end{bmatrix}$, which was explained to us by R. Gelca [G]. Specifically, for $n \geq 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix} s^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} s^{k-n}. \tag{3.5}$$

As a corollary Gelca gets, for $n \geq 1$,

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} s^{k(n-1)} = 0. \tag{3.6}$$

To see this, note that

$$\sum_{k=0}^{n}(-1)^k \begin{bmatrix} n \\ k \end{bmatrix} s^{k(n-1)} = \sum_k (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} s^{kn} + \sum_k (-1)^k \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} s^{(k-1)n} =$$

$$= \sum_k (-1)^k \begin{bmatrix} n-1 \\ k \end{bmatrix} s^{kn} + \sum_k (-1)^{k+1} \begin{bmatrix} n-1 \\ k \end{bmatrix} s^{kn} = 0$$

The algebra $\mathcal{A}_r$ is the associative algebra over $\mathbb{C}$, with unit, generated by X, Y, K and $\bar{K}$, with relations

(3.7)
$$K\bar{K} = \bar{K}K = 1, \quad K^{4r} = 1, \quad X^r = Y^r = 0, \quad KX = sXK, \quad KY = \bar{s}YK,$$
$$XY - YX = \frac{K^2 - \bar{K}^2}{s - \bar{s}}.$$

In fact, $\mathcal{A}_r$ is a Hopf algebra. The counit $\epsilon : \mathcal{A}_r \to \mathbb{C}$ is the morphism satisfying

(3.8) $$\epsilon(X) = \epsilon(Y) = 0 \text{ and } \epsilon(K) = 1.$$

The antipode is the antimorphism defined by

(3.9) $$S(X) = -sX, \quad S(Y) = -\bar{s}Y, \text{ and } S(K) = \bar{K}.$$

The comultiplication is the morphism $\Delta : \mathcal{A}_r \to \mathcal{A}_r \otimes \mathcal{A}_r$ with

(3.10)
$$\Delta(X) = X \otimes K + \bar{K} \otimes X, \quad \Delta(Y) = Y \otimes K + \bar{K} \otimes Y, \quad \Delta(K) = K \otimes K.$$

As in [K-M] we use ϵ, S and Δ to define the trivial representation, the dual representation and the tensor product of two representations.

We will use the following representations of $\mathcal{A}_r$. Let m be an integer with $0 \leq m < \frac{r-1}{2}$. The vector space $\underline{m}$ has dimension $2m+1$. It is spanned by the vectors $\{e_i\}$, where i runs from $-m$ to m. The action of $\mathcal{A}_r$ on $\underline{m}$ is induced by

(3.11) $$Xe_i = [m+i+1]\, e_{i+1} \text{ for } i < m \text{ and } Xe_m = 0$$

$$Ye_i = [m-i+1]\, e_{i-1} \text{ when } i > -m \text{ and } Ye_{-m} = 0$$

$$Ke_i = s^i e_i.$$

There is an $\mathcal{A}_r$-linear isomorphism from $\underline{m}^*$ to $\underline{m}$. Let $\{e^i\}$ be the basis for $\underline{m}^*$ that is dual to $\{e_i\}$. Define

$$D(e^i) = \begin{bmatrix} 2m \\ m-i \end{bmatrix}^{-1} (-s)^i e_{-i}. \tag{3.12}$$

The morphism D allows us to define a nondegenerate sesquilinear form

$$(\ ,\ ) : \underline{m} \otimes \underline{m} \to \mathbb{C}.$$

If $v \in \underline{m}$ then $v = \sum_{i=-m}^{m} \alpha_i e_i$. The complex conjugate of v is $\bar{v} = \sum_{i=-m}^{m} \bar{\alpha}_i e_i$. We define

$$(w, v) = (-1)^m D^{-1}(w)(\bar{v}). \tag{3.13}$$

It is easy to see that (w, v) is the complex conjugate of (v, w). We extend $(\ ,\ )$ to tensor products of representations by

$$(w_1 \otimes w_2, v_1 \otimes v_2) = (w_1, v_1)(w_2, v_2). \tag{3.14}$$

This allows us to define the adjoint of any $\mathcal{A}_r$-linear morphism between tensor products of the representation $\underline{m}$. Specifically, if α is a morphism, α^* is the unique morphism satisfying

$$(\alpha(v), w) = (v, \alpha^*(w)) \quad \text{for all } v \text{ and } w. \tag{3.15}$$

We say that an $\mathcal{A}_r$-linear morphism α is *unitary* if

$$(\alpha(v), \alpha(w)) = (v, w) \quad \text{for all } v \text{ and } w. \tag{3.16}$$

Notice that $(\ ,\ )$ is not hermitian. Hence the appellation "unitary" is not completely standard.

Proposition 3.1. *The morphism $\check{R} : \underline{m} \otimes \underline{m'} \to \underline{m'} \otimes \underline{m}$ given by (see [K-M])*

$$\check{R}(e_i \otimes e_j) = \tag{3.17}$$
$$= \sum_{\substack{n \geq 0 \\ i+n \leq m \\ j-n \geq -m'}} \frac{(s-\bar{s})^n}{[n]!} \frac{[m+i+n]!}{[m+i]!} \frac{[m'-j+n]!}{[m'-j]!} t^{4ij-2n(i-j)-n(n+1)} e_{j-n} \otimes e_{i+n}$$

is unitary.

PROOF We need to check that

$$\big(\check{R}(e_i \otimes e_j), \check{R}(e_k \otimes e_l)\big) = (e_i \otimes e_j, e_k \otimes e_l).$$

Since $\check{R}(e_i \otimes e_j)$ is a linear combination of terms in $e_{j-n} \otimes e_{i+n}$, it suffices to check the formula for $e_k \otimes e_l$ with $k + l = -i - j$. Hence we may assume that $e_k \otimes e_l = e_{-i-z} \otimes e_{-j+z}$.

If $z \leq 0$ there is at most one nonzero term in the evaluation of the pairing. Hence the assertion is easy to check.

Assume then that $z > 0$. In this case

$$\big(\check{R}(e_i \otimes e_j), \check{R}(e_{-i-z} \otimes e_{-j+z})\big) =$$

$$= \Bigg(\sum_{n\geq 0} \frac{(s-\bar{s})^n}{[n]!} \frac{[m+i+n]!}{[m+i]!} \frac{[m'-j+n]!}{[m'-j]!} t^{4ij-2n(i-j)-n(n+1)} e_{j-n} \otimes e_{i+n}\,,$$

$$\sum_{p\geq 0} \frac{(s-\bar{s})^p}{[p]!} \frac{[m-i-z+p]!}{[m-i-z]!} \frac{[m+j-z+p]!}{[m+j-z]!} t^{4(-i-z)(-j+z)-2p(-i+j-2z)-p(p+1)}$$

$$e_{-j+z-p} \otimes e_{-i-z+p}\Bigg)$$

$$= (-1)^{m+m'} \Bigg((D^{-1} \otimes D^{-1}) \Bigg(\sum_{n\geq 0} \frac{(s-\bar{s})^n}{[n]!} \frac{[m+i+n]!}{[m+i]!} \frac{[m'-j+n]!}{[m'-j]!}$$

$$t^{4ij-2n(i-j)-n(n+1)} e_{j-n} \otimes e_{i+n} \Bigg)\Bigg)$$

$$\Bigg(\sum_{p\geq 0} \frac{(-1)^p (s-\bar{s})^p}{[p]!} \frac{[m-i-z+p]!}{[m-i-z]!} \frac{[m'+j-z+p]!}{[m'+j-z]!} \cdot$$

$$t^{-4(-i-z)(-j+z)+2p(-i+j-2z)+p(p+1)} e_{-j+z-p} \otimes e_{-i-z+p} \Bigg).$$

Since

$$D^{-1} \otimes D^{-1}(e_{j-n} \otimes e_{i+n}) = \begin{bmatrix} 2m' \\ m'-j+n \end{bmatrix} \begin{bmatrix} 2m \\ m-i-n \end{bmatrix} (-s)^{i+j} e^{n-j} \otimes e^{-n-i}$$

the only nonzero terms occur when $p+n=z$. This yields:

$$\big(\check{R}(e_i \otimes e_j), \check{R}(e_{-i-z} \otimes e_{-j+z})\big) =$$

$$(-1)^{m+m'} \sum_{n=0}^{z} \frac{(-1)^{z-n}}{[n]!\,[z-n]!} s^{n(z-1)} \frac{[2m]!\,[2m']!}{[m+i]!\,[m'-j]!\,[m-i-z]!\,[m'+j-z]!}$$

$$(-s)^{i+j}(s-\bar{s})^z t^{z^2+2z(i-j)+z} =$$

$$\frac{(-1)^{m+m'}[2m]!\,[2m']!(-s)^{i+j}(s-\bar{s})^z t^{z^2+2z(i-j)+z}}{[m+i]!\,[m'-j]!\,[m-i-z]!\,[m'+j-z]!\,[z]!} \sum_{n=0}^{z}(-1)^{z-n}\begin{bmatrix} n \\ z \end{bmatrix} s^{n(z-1)} = 0.$$

The last sum is equal to zero by our earlier discussion of quantized binomial coefficients (equation 3.6).

Of particular interest to us are the spaces of $\mathcal{A}_r$-linear maps $\alpha : \underline{m} \otimes \underline{n} \to \underline{p}$, which we denote by V_p^{mn}, and the spaces of the $\mathcal{A}_r$-linear maps $\beta : \underline{p} \to \underline{m} \otimes \underline{n}$, which we denote by V_{mn}^p.

Using (,) (equation 3.13) to define the adjoint, we get the trace pairing $\langle\ ,\ \rangle_t : V_p^{mn} \otimes V_p^{mn} \to \mathbb{C}$. We define

$$\langle \alpha, \beta \rangle_t \cdot 1_{\underline{p}} = \alpha \circ \beta^*, \tag{3.18}$$

where $1_{\underline{p}}$ is the identity map on $\underline{p}$. Schur's lemma assures us that the formula for $\langle\ ,\ \rangle_t$ makes sense.

Let

$$X = \sqrt{\sum_m [2m+1]^2} = \frac{\sqrt{r}}{2\sin\frac{\pi}{r}}. \tag{3.19}$$

We define the Kronecker pairing $\langle\ ,\ \rangle_k : V_{mn}^p \otimes V_p^{mn} \to \mathbb{C}$ by

$$\langle \alpha, \beta \rangle_k 1_{\underline{p}} = X^2 \beta \circ \alpha. \tag{3.20}$$

Notice that

$$\langle \alpha, \beta \rangle_k = X^2 \langle \beta, \alpha^* \rangle_t. \tag{3.21}$$

The Kronecker pairing identifies V_{mn}^p with $(V_p^{mn})^*$.

The algebra $\mathcal{A}_r$ is not semisimple, but in the eyes of the $\underline{m}$, tensor products of the $\underline{m}$ act as if they are semisimple.

We say that a *representation* V *is bad* if for every $\mathcal{A}_r$-linear map $\alpha : V \to V$ the operator $K^2\alpha$ has trace 0. (i.e. quantum trace of α, $tr_q(\alpha) = 0$.)

It is shown in [R-T] that

$$\underline{m} \otimes \underline{n} \cong \oplus_q \underline{q} \oplus B, \tag{3.22}$$

where

$$|m-n| \leq q \leq \min\{m+n, r-2-m-n\},$$

and B is bad. The summands are uniquely determined.

Let $\beta_q^{mn} : \underline{m} \otimes \underline{n} \to \underline{q}$ be an $\mathcal{A}_r$-linear map with $\langle \beta_q^{mn}, \beta_q^{mn} \rangle_t = 1$. Then the morphism

$$(\beta_q^{mn})^* \circ \beta_q^{mn} : \underline{m} \otimes \underline{n} \to \underline{m} \otimes \underline{n}$$

is an idempotent and its image is the summand of $\underline{m} \otimes \underline{n}$ that is isomorphic to $\underline{q}$. Notice that $\underline{0}$ is a summand of $\underline{m} \otimes \underline{n}$ if and only if $m = n$.

We have

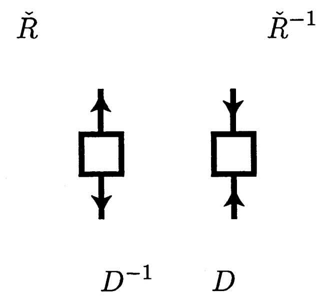
(3.23)

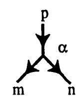
$$V_q^{mn} \cong \begin{cases} \mathbb{C} & \text{if } |m-n| \leq q \leq \min\{m+n, r-m-n-2\} \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

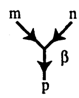
We use the tangle functor associated to $\mathcal{A}_r$ as described in [K-M], [R-T] and [Wa]. The discussion here is not sufficient to learn from. The reader who has not seen this before is referred to [K-M]. Our goal is to recall what we need in order to construct the basic data. We make the assignments below.

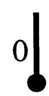
$1 \mapsto \sum e_j \otimes e^j \qquad\qquad 1 \mapsto \sum s^{-2j} e^j \otimes e_j$

$x \otimes f \mapsto f(K^2 x) \qquad\qquad f \otimes x \mapsto f(x)$

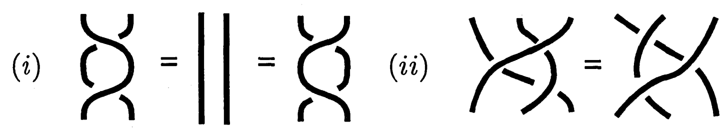


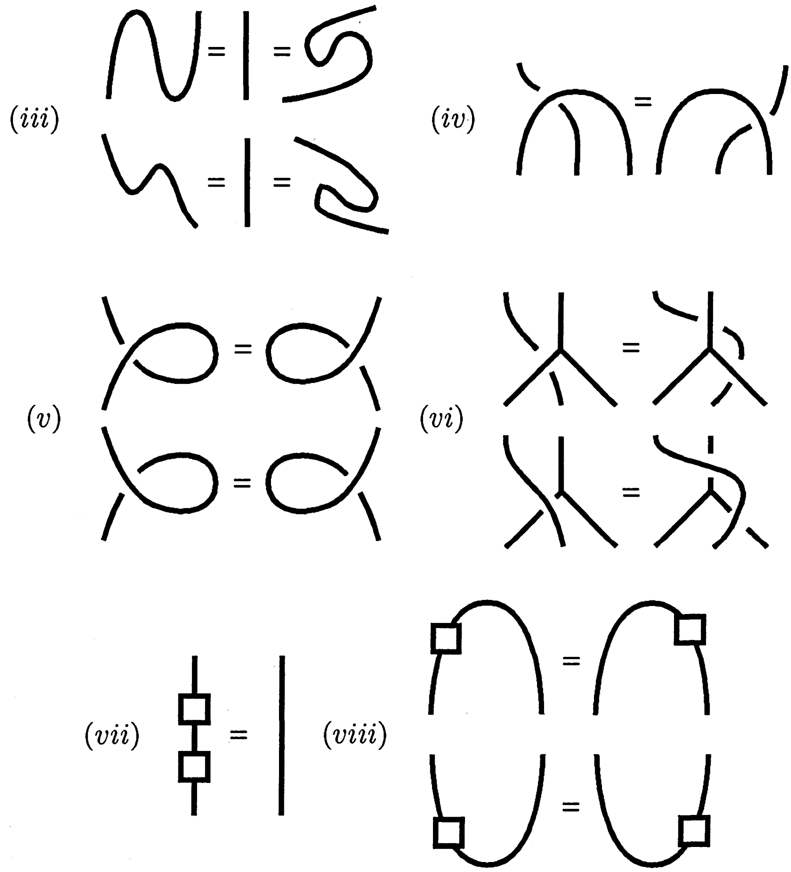
If $\alpha \in V_p^{mn}$ then the morphism is represented by  .

If $\beta \in V_{mn}^p$ then it is represented by  . These coupons are rigid, and there is a preferred side that must remain up in our tangles.

It should be remarked that we allow valence one vertices when the edge containing the vertex is labeled with 0. This is because the representation $\underline{0}$ is isomorphic to the representation induced on $\mathbb{C}$ by ϵ (the counit). Let $\beta_0 : \underline{0} \to \mathbb{C}$ be the $\mathcal{A}_r$-linear map that takes λe_0 to λ. Then  in a diagram corresponds to β_0^* and  corresponds to β_0.

If we change a diagram either by an isotopy of the plane that does not rotate our coupons, or by any of the moves below, then we do not change the operator represented by a tangle via the tangle functor. (Add your own arrows.)



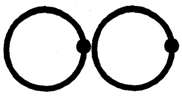
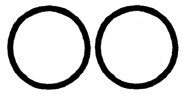


Remark 1. There are analogues of the representations $\underline{m}$ where m is a half integer. Relation (viii) does not hold for these.

In the language of [R-T] we are working with isotopy classes of oriented, labeled, homogeneous, directed ribbon tangles with coupons in $\mathbb{R}^2 \times I$. *Oriented* means that there are arrows, *labeled* means that each edge is labeled with an $\underline{m}$, *ribbon* means that the edges are actually embedded "strips" $I \times I$ and $S^1 \times I$. (The strip is the one that is parallel to the plane of the paper.) Finally, *homogeneous* means that the coupons and the strips have a preferred side which is up.

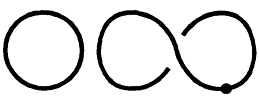
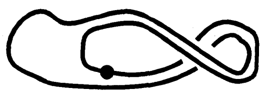
When a simple closed curve is labeled with a dot instead of a representation, this means: take the weighted sum of the morphisms obtained by putting each label $\{0, 1, 2, \ldots\}$ on that component, where the weight assigned to the morphism with label m is $\frac{[2m+1]}{X}$.

Example 1.

 $= \sum_{m,n} \frac{[2m+1][2n+1]}{X^2}$ 
m n

$$= \sum_{m,n} \frac{[2m+1]^2[2n+1]^2}{X^2} = \frac{X^4}{X^2} = X^2.$$

Since the morphism corresponds to a tangle in a three manifold obtained by doing surgery along the dotted components, we may also do handle slides over dotted components and not change the value of the invariant.

Example 2.  = 
m m

We are now ready to present our choices of the basic data. We will omit the subscripts $\Sigma_{g,n}$ in the notation for the Kronecker pairing when its domain is unambiguous from the context.

Let r be an odd integer greater than 1. The set of labels $\mathcal{L}$ is the set of integers i with

$$0 \leq i < \frac{r-1}{2}. \tag{3.24}$$

The involution $\check{}: \mathcal{L} \to \mathcal{L}$ required in [Wa] is trivial so we omit it from our notation. The preferred "trivial label" is 0. (Walker calls this 1.)

To start with, $C =$ , that is C is a multiplication by the number associated to this tangle.

The vector space $V(D,0)$ is the space of $\mathcal{A}_r$-linear maps from $\underline{0}$ to $\mathbb{C}$, where the action on $\mathbb{C}$ is given by the counit $\epsilon : \mathcal{A}_r \to \mathbb{C}$.

The vector space $V(-D,0)$ is the space of $\mathcal{A}_r$-morphisms from $\mathbb{C}$ to $\underline{0}$.

The Kronecker pairing $\langle\ ,\ \rangle_k : V(-D,0) \otimes V(D,0) \to \mathbb{C}$ is

$$\langle \alpha, \beta \rangle_k = \beta \circ \alpha(1). \tag{3.25}$$

Let $\beta_0 : \underline{0} \to \mathbb{C}$ be the map

$$\beta_0(\lambda e_0) = \lambda. \tag{3.26}$$

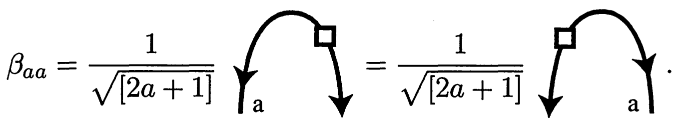
This is the same map we called β_0 earlier.

The vector space $V(A,a,a)$ will be the space of $\mathcal{A}_r$-linear maps from $\underline{a}\otimes\underline{a}$ to $\mathbb{C}$. Its dual $V(-A,a,a)$ is the space of $\mathcal{A}_r$-linear maps from $\mathbb{C}$ to $\underline{a}\otimes\underline{a}$. The Kronecker pairing $\langle\ ,\ \rangle_k : V(-A,a,a)\otimes V(A,a,a)\to\mathbb{C}$ is given by

(3.27) $$\langle\alpha,\beta\rangle_k = X\cdot\beta\circ\alpha(1).$$

Let

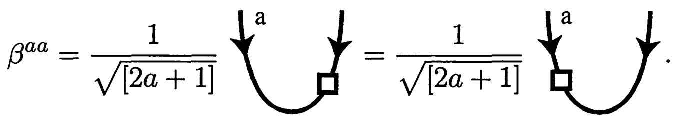
(3.28)



Any element of $V(A,a,a)$ can be written as $\lambda\beta_{aa}$ for some $\lambda\in\mathbb{C}$.

Let

(3.29)

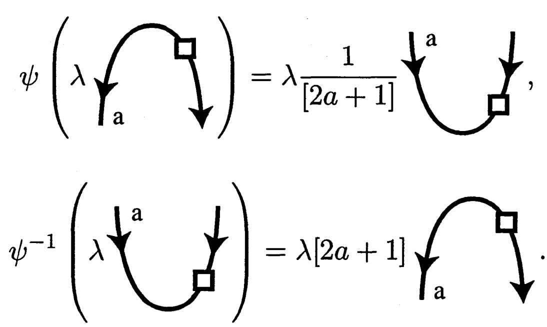


Notice that β_{aa} is the adjoint of β^{aa} with respect to the trace pairing.

Let $R : V(A,a,a)\to V(A,a,a)$ be the identity.

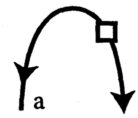
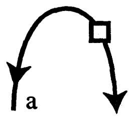
Let $\psi : V(A,a,a)\to V(-A,a,a)$ be:

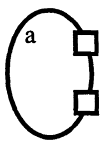
(3.30)



Define $\langle\ ,\ \rangle : V(A,a,a)\otimes V(A,a,a)\to\mathbb{C}$ by

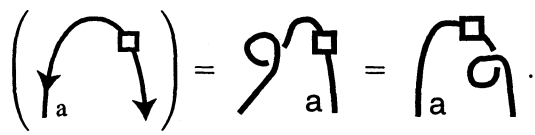
(3.31) $$\langle\alpha,\beta\rangle = \langle\psi(\alpha),\beta\rangle_k = X\langle\beta,\psi(\alpha)^*\rangle_t.$$

It is easy to check that $\langle\ ,\ \rangle$ is symmetric. If $\alpha = \lambda$ , $\beta =$ μ  then

$$\langle \alpha, \beta \rangle = \frac{\lambda\mu}{[2a+1]} X$$  $$= \langle \beta, \alpha \rangle.$$

It is worthy to note that each β_{aa} has norm 1 with respect to the trace pairing. In fact all our basis elements will have this property.

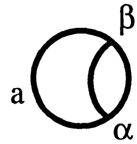
Finally, the Dehn twist $T_a : V(A, a, a) \to V(A, a, a)$ is given by

(3.32) T_a 

Next we describe the basic data associated to the pair of pants. The vector space $V(P, a, b, c)$ will be V_a^{cb}, while $V(-P, a, b, c) = V_{cb}^a$. The Kronecker pairing is, for $\alpha \in V_{cb}^a$, $\beta \in V_a^{cb}$, defined by

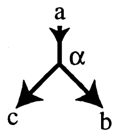
$$\langle \alpha, \beta \rangle_k \cdot 1_{\underline{a}} = X^2 \cdot \beta \circ \alpha. \tag{3.33}$$

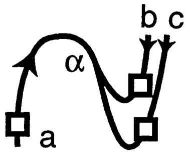
So that

$$\langle \alpha, \beta \rangle_k \cdot 1_{\underline{a}} = X^2$$  $$, \quad \frac{X^2}{[2a+1]}$$  $$= X^2 \langle \beta, \alpha^* \rangle_t.$$

Let $\psi : V(P, a, b, c) \to V(-P, a, c, b)$ be

(3.34)

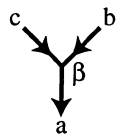
$\psi\Bigg($ 

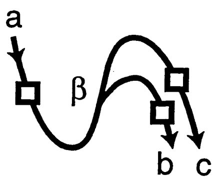
$\Bigg) = \dfrac{1}{\sqrt{[2a+1][2b+1][2c+1]}}$ 

.

Notice that $\psi^{-1} : V(-P, a, b, c) \to V(P, a, c, b)$ is given by:

(3.35)

$\psi^{-1}\Bigg($ 

$\Bigg) = \sqrt{[2a+1][2b+1][2c+1]}$ 

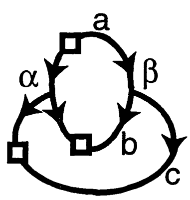
.

The associated pairing

$$\langle\ ,\ \rangle : V_a^{cb} \otimes V_a^{bc} \to \mathbb{C}$$

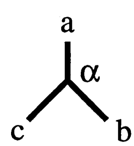
is given by

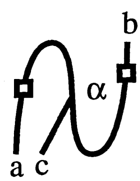
(3.36)

$\langle \alpha, \beta \rangle = \langle \psi(\alpha), \beta \rangle_k = X^2 \langle \beta, \psi(\alpha)^* \rangle_t =$ 

$\cdot \dfrac{X^2}{[2a+1]^{3/2}\sqrt{[2b+1][2c+1]}}.$

The rotation $R : V(P, a, b, c) \to V(P, b, c, a)$ is given by

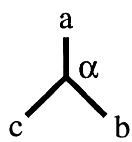
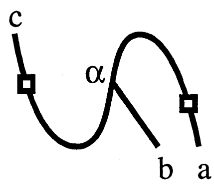
(3.37) $R\Bigg($ 

$\Bigg) = \dfrac{\sqrt{[2b+1]}}{\sqrt{[2a+1]}}$ 

.

In diagrams we will use $\tilde{\alpha}$ to denote $R(\alpha)$. The rotation in the other direction is

(3.38)

$L\Bigg($  $\Bigg) = \dfrac{\sqrt{[2c+1]}}{\sqrt{[2a+1]}}$  .

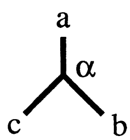
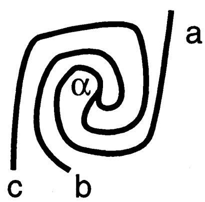
In diagrams we denote $L(\alpha)$ by $\underset{\sim}{\alpha}$.

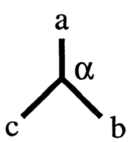
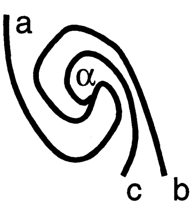
There could be some notational confusion, because we have used R and ψ to denote several different maps. However, the maps with the same names have different domains and ranges, so it is always possible to divine what is meant from the context. We will also start repressing some of the boxes corresponding to D and D^{-1}.

It is easy to see that:

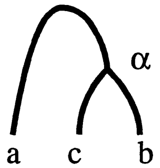
(3.39) $$R^3 = Id, \quad RL = LR = Id \quad \text{and} \quad R^* = L.$$

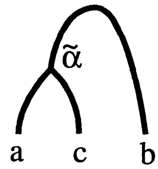
The identities $R^3 = Id$ and $L^3 = Id$ will be most useful. Pictorially:

Proposition 3.2.  =  ,

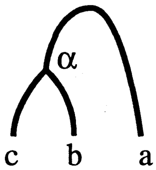
 =  .

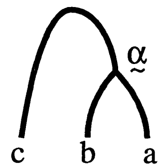
Next we develop two formulas for dealing with R and L in diagrams.

Proposition 3.3. *i)* 

$= \dfrac{\sqrt{[2a+1]}}{\sqrt{[2b+1]}}$ 

;

ii) 

$= \dfrac{\sqrt{[2a+1]}}{\sqrt{[2c+1]}}$ 

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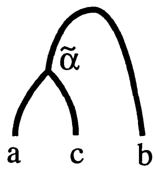
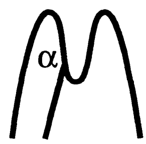
Proof

$\dfrac{\sqrt{[2a+1]}}{\sqrt{[2b+1]}}$ 

$= \dfrac{\sqrt{[2a+1]}}{\sqrt{[2b+1]}} \dfrac{\sqrt{[2b+1]}}{\sqrt{[2a+1]}}$ 

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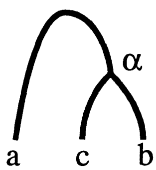
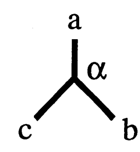
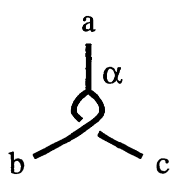


.

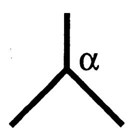
Formula ii) follows from a similar computation.
Next the map $B_{23} : V_a^{cb} \to V_a^{bc}$ is given by

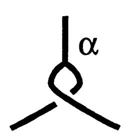
(3.40) $B_{23}\Bigg($ 

$\Bigg) =$ 

,

and

(3.41) $B_{23}^{-1}\Bigg($ 

$\Bigg) =$ 

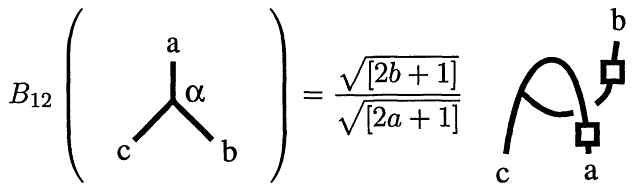
.

Also, it is easy to see from Proposition 3.1 that

(3.42) $$B_{23}^{*} = B_{23}^{-1}.$$

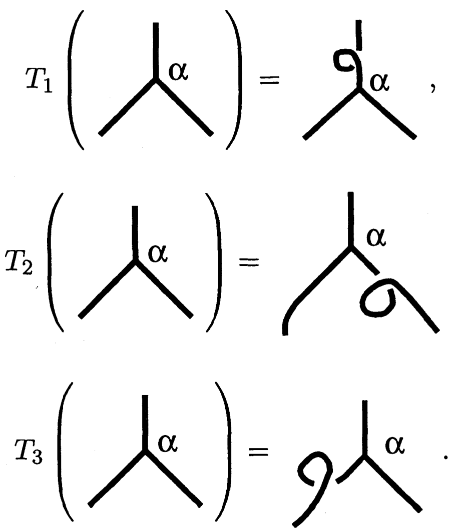
Recall that $B_{12} : V_a^{cb} \to V_b^{ca}$ is given by $B_{12} = R^{-1} B_{23} R$. it follows that:

(3.43)



Then there are the Dehn twists about the boundary components of P:

(3.44)



Next there are the basis elements β_a^{cb}. The adjoint of β_a^{cb} with respect to the trace norm is $\beta_{cb}^a : \underline{a} \to \underline{c} \otimes \underline{b}$. The latter map is easier to describe. Let

(3.45)
$$\nu_a^{c,b} = \sqrt{\frac{[c+b-a]![2a+1]!}{[2b]![2c]![a-b+c]![a+b-c]![a+b+c+1]!}}.$$

The map β_{cb}^a is the unique intertwiner with

(3.46)
$$\beta_{cb}^a(e_a) = (-1)^{a(a+b)} t^{b(b+1)-a(a+1)-c(c+1)} \nu_a^{c,b} \sum_{i+j=a} (-1)^i s^{i(a+1)} [c+i]![b+j]! e_i \otimes e_j.$$

We define (for $i+j=a$):

(3.47)
$$\beta_a^{cb}(e_i\otimes e_j)=(-1)^{a(a+b)+i}\frac{[2c]![2b]!}{[c-i]![b-j]!}s^{i(a+1)}\nu_a^{c,b}t^{b(b+1)-a(a+1)-c(c+1)}e_a.$$

It will become evident from computations in section 5 that β_a^{cb} is unitary. Specializing, we get:

(3.48)
$$\beta_0^{aa}(e_{-i}\otimes e_i)=\frac{1}{\sqrt{[2a+1]}}D^{-1}(e_i)(e_{-i}).$$

Also

(3.49)
$$\beta_a^{0a}(e_0\otimes e_a)=e_a,$$

and

(3.50)
$$\beta_a^{a0}(e_a\otimes e_0)=e_a.$$

The map that gives the isomorphism between $V(P,0,a,a)\otimes V(D,0)$ and $V(A,a,a)$, corresponding to gluing a disk into the first boundary component of P, is defined by:

(3.51)
$$(\beta_0^{aa}\otimes\beta_0)\mapsto\beta_{aa}.$$

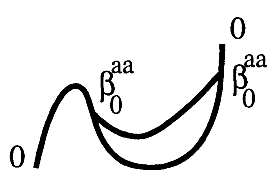
Notice that

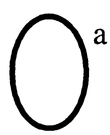
(3.52)
$$\psi(\beta_0^{aa}\otimes\beta_0)=\psi(\beta_0^{aa})\otimes\psi(\beta_0),$$

and

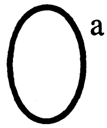
(3.53)
$$\langle\beta_0^{aa}\otimes\beta_0,\beta_0^{aa}\otimes\beta_0\rangle_{k,A}\cdot 1_{\underline{0}}=\langle\psi(\beta_0^{aa})\otimes\psi(\beta_0),\beta_0^{aa}\otimes\beta_0\rangle_{k,A}\cdot 1_{\underline{0}}=$$

$$=\frac{1}{X}\langle\psi(\beta_0^{aa}),\beta_0^{aa}\rangle_{k,P}\cdot 1_{\underline{0}}=\frac{1}{X}\frac{X^2}{[2a+1]}\cdot 1_{\underline{0}}=$$

$$=\frac{X}{[2a+1]}$$ 

$$=\frac{X}{[2a+1]^2}$$ 

$$\cdot 1_{\underline{0}},$$

$$\frac{X}{[2a+1]^2}$$ 

$$= \frac{X}{[2a+1]} = \langle \beta_{aa}, \beta_{aa} \rangle_{k,A}.$$

We use the subscripted A and P to indicate whether the pairing is taken on an annulus or on a pair of pants.

Next we must consider the basic data corresponding to F. Consider V_i^{jkl}, i.e. the space of A_r-linear maps from $\underline{j} \otimes \underline{k} \otimes \underline{l}$ to $\underline{i}$. There is a trace pairing $\langle\ ,\ \rangle_t : V_i^{jkl} \otimes V_i^{jkl} \to \mathbb{C}$ which is defined, as usual, by

(3.54) $$\langle \alpha, \beta \rangle_t 1_{\underline{i}} = \alpha \circ \beta^*.$$

Notice that $\bigoplus_p V_i^{jp} \otimes V_p^{kl}$ and $\bigoplus_q V_i^{ql} \otimes V_q^{jk}$ are innerproduct spaces via the direct sum of the trace pairings on $V_i^{jp} \otimes V_p^{kl}$ and $V_i^{ql} \otimes V_q^{jk}$, that is

(3.55) $$\langle \alpha \otimes \beta, \delta \otimes \gamma \rangle_t = \langle \alpha, \delta \rangle_t \langle \beta, \gamma \rangle_t.$$

Let $\Theta : \bigoplus_p V_i^{jp} \otimes V_p^{kl} \to V_i^{jkl}$ be given by

(3.56) $$\Theta(\alpha \otimes \beta) = \alpha(1_{\underline{j}} \otimes \beta).$$

Let $\eta : \bigoplus_q V_i^{ql} \otimes V_q^{jk} \to V_i^{jkl}$ be defined by

(3.57) $$\eta(\delta \otimes \gamma) = \delta \circ (\gamma \otimes 1_{\underline{l}}).$$

Let $N \begin{bmatrix} j & k \\ i & l \end{bmatrix} : \bigoplus_p V_i^{jp} \otimes V_p^{kl} \to \bigoplus_q V_i^{ql} \otimes V_q^{jk}$ be given by

(3.58) $$N \begin{bmatrix} j & k \\ i & l \end{bmatrix} = \eta^* \circ \Theta.$$

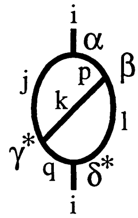
By $N_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix}$ we mean the projection onto the summand $V_i^{ql} \otimes V_q^{jk}$ of the restriction of $N \begin{bmatrix} j & k \\ i & l \end{bmatrix}$ to the summand $V_i^{jp} \otimes V_p^{kl}$. In symbols:

(3.59)
$$\left\langle N_{pq} \begin{bmatrix} j & k \\ i & l \end{bmatrix} (\alpha \otimes \beta), \delta \otimes \gamma \right\rangle_t = \langle \alpha(1_j \otimes \beta), \delta(\gamma \otimes 1_{\underline{l}}) \rangle_t.$$

In pictures

(3.60)

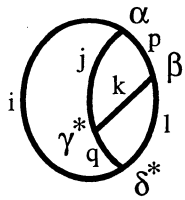
$$\left\langle N_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\alpha\otimes\beta),\delta\otimes\gamma\right\rangle_t \cdot 1_{\underline{i}} =$$



or

(3.61)

$$\left\langle N_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\alpha\otimes\beta),\delta\otimes\gamma\right\rangle_t = \frac{1}{[2i+1]}$$



.

Now we describe the pentagon identity for N (see [D-J-N], [K-R], [V-K]). For arbitrary $i, j, k, l, p, q, m, n, r \in \mathcal{L}$ we define $N^{(1,2)}_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} : V_i^{jp}\otimes V_p^{kl}\otimes V_r^{mn} \to V_i^{ql}\otimes V_q^{jk}\otimes V_r^{mn}$ by

(3.62) $$N^{(1,2)}_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = N_{pq}\begin{bmatrix} j & k \\ i & l \end{bmatrix}\otimes 1.$$

Let $P^{(23)} : \bigoplus_i V_i\otimes W_i\otimes U_i \to \bigoplus_i V_i\otimes U_i\otimes W_i$ be given by

(3.63) $$P^{(23)}(\alpha\otimes\beta\otimes\gamma) = \alpha\otimes\gamma\otimes\beta,$$

and $P^{(12)} : \bigoplus_i V_i\otimes W_i\otimes U_i \to \bigoplus_i W_i\otimes V_i\otimes U_i$ by

(3.64) $$P^{(12)}(\alpha\otimes\beta\otimes\gamma) = \beta\otimes\alpha\otimes\gamma,$$

then

(3.65) $$N^{(1,3)}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = P^{(23)}N^{(1,2)}\begin{bmatrix} j & k \\ i & l \end{bmatrix}P^{(23)},$$

$$N^{(2,3)}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = P^{(12)}P^{(23)}N^{(1,2)}\begin{bmatrix} j & k \\ i & l \end{bmatrix} P^{(23)}P^{(12)} = 1 \otimes N \begin{bmatrix} j & k \\ i & l \end{bmatrix}.$$

Then by [D-J-N], [K-R] or [V-K]

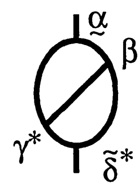
$$N^{(2,3)}\begin{bmatrix} n & j \\ r & k \end{bmatrix} \circ N^{(1,2)}\begin{bmatrix} n & q \\ u & l \end{bmatrix} \circ N^{(2,3)}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = \tag{3.66}$$

$$= P^{(23)}N^{(1,3)}\begin{bmatrix} m & k \\ n & l \end{bmatrix} N^{(1,2)}\begin{bmatrix} n & j \\ u & p \end{bmatrix}.$$

The map $F\begin{bmatrix} j & k \\ i & l \end{bmatrix} : \bigoplus_p V_p^{ij} \otimes V_p^{kl} \to \bigoplus_q V_q^{li} \otimes V_q^{jk}$ is obtained by precomposing and postcomposing N with $L \otimes Id$. Specifically,

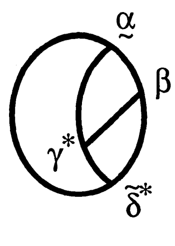
$$F\begin{bmatrix} j & k \\ i & l \end{bmatrix} = (L \otimes Id) \circ N \begin{bmatrix} j & k \\ i & l \end{bmatrix} \circ (L \otimes Id). \tag{3.67}$$

It can be described most simply in terms of the trace pairing:

(3.68) $\left\langle F\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\alpha \otimes \beta), \delta \otimes \gamma \right\rangle_t \cdot 1_{\underline{i}} =$ 

or

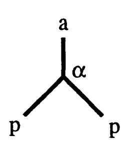
(3.69)

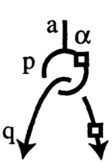
$\left\langle F\begin{bmatrix} j & k \\ i & l \end{bmatrix}(\alpha \otimes \beta), \delta \otimes \gamma \right\rangle_t = \dfrac{1}{[2i+1]}$ 

.

Finally, we introduce $S_a : \bigoplus_p V_a^{pp} \to \bigoplus_q V_a^{qq}$ by

(3.70)

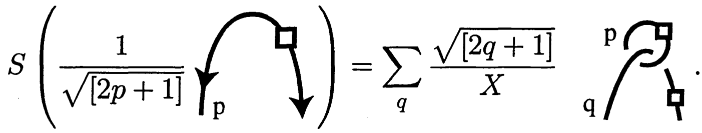
S_a (

) $= \sum_q \dfrac{\sqrt{[2p+1][2q+1]}}{X}$ 

.

Also,

(3.71)



This completes our presentation of the basic data. We will now discuss some identities satisfied by this basic data. These will be useful in proving that our choice of basic data satisfies the Moore-Seiberg-Walker equations (theorem 2.1).

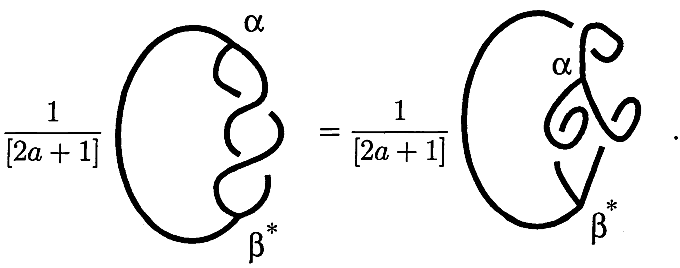
Proposition 3.4. $B_{23}^2 = T_1^{-1}T_2T_3$.

PROOF

The statement is true if for every α, $\beta \in V_a^{cb}$ we have

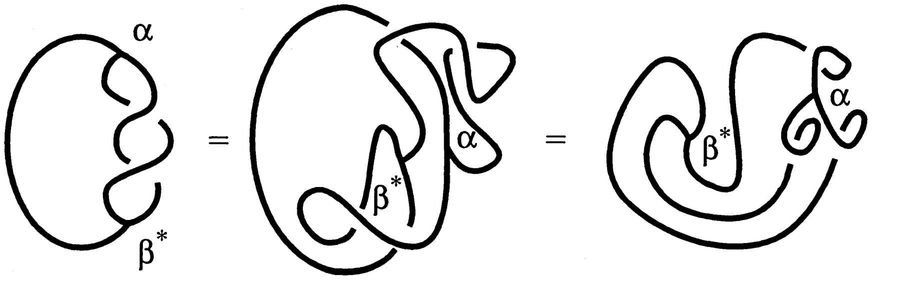
$$\langle B_{23}^2(\alpha), \beta\rangle_t = \langle T_1^{-1}T_2T_3(\alpha), \beta\rangle_t. \tag{3.72}$$

We will prove this by proceeding from the left hand side of the equation to the right hand side. In diagrams the equation (3.72) says

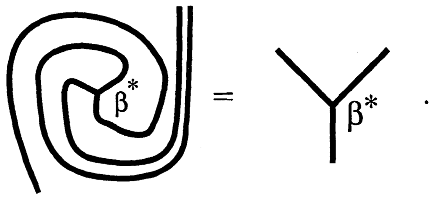


But

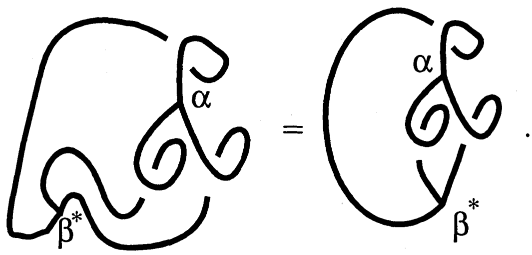
(3.73)



Recall from Proposition 3.2 that



Hence the last term in the equation (3.73) above is equal to



This proves the desired result.

Proposition 3.5. *The pairing $\langle\ ,\ \rangle$ is symmetric.*

Proof

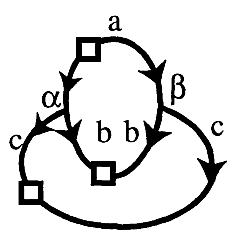
We are to show that if $\alpha \in V_a^{cb}$ and $\beta \in V_a^{bc}$ then

(3.74) $$\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle.$$

In pictures

(3.75)

$$\frac{X^2}{[2a+1]^{3/2}\sqrt{[2b+1][2c+1]}} \cdot$$



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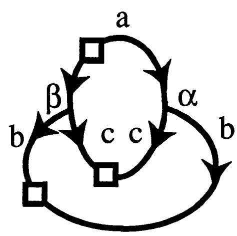
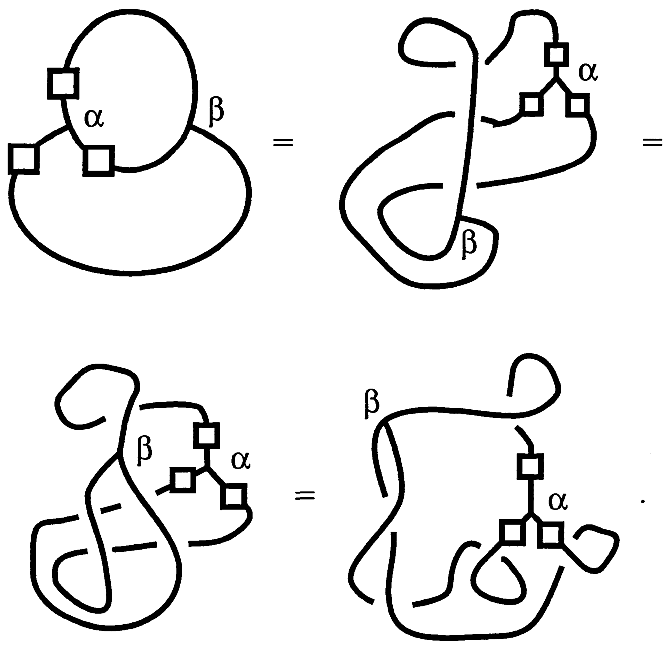
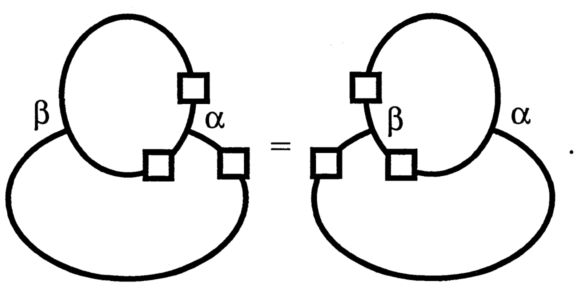


$$\cdot \frac{X^2}{[2a+1]^{3/2}\sqrt{[2b+1][2c+1]}}.$$

Once again, we start on the left hand side and work to the right hand side.



By Proposition 3.4 this reduces to:



Suppose V is a vector space and $\langle\ ,\ \rangle$ is a hermitian pairing on V. Let ξ_i be an orthonormal basis for V. Let $\vec{v}$, $\vec{w} \in V$ and suppose that $\vec{w} = \sum \lambda_i \xi_i$. Then

$$\langle \vec{v}, \vec{w} \rangle = \left\langle \vec{v}, \sum \lambda_i \xi_i \right\rangle = \sum \bar{\lambda}_i \langle \vec{v}, \xi_i \rangle.$$

If $L : U \to V$ and $M : V \to W$, and all the spaces are hermitian then

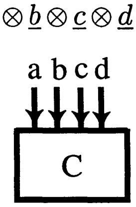
$$\langle M \circ L(\vec{u}), \vec{w} \rangle = \langle L(\vec{u}), M^*(\vec{w}) \rangle. \tag{3.76}$$

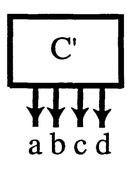
Since

$$M^*(\vec{w}) = \sum \langle M^*(\vec{w}), \xi_i \rangle \xi_i, \tag{3.77}$$

we have

$$\begin{aligned} \langle L(\vec{u}), M^*(\vec{w}) \rangle &= \sum_i \langle L(\vec{u}), \xi_i \rangle \overline{\langle M^*(\vec{w}), \xi_i \rangle} = \\ &= \sum_i \langle L(\vec{u}), \xi_i \rangle \langle \xi_i, M^*(\vec{w}) \rangle = \sum_i \langle L(\vec{u}), \xi_i \rangle \langle M(\xi_i), \vec{w} \rangle. \end{aligned} \tag{3.78}$$

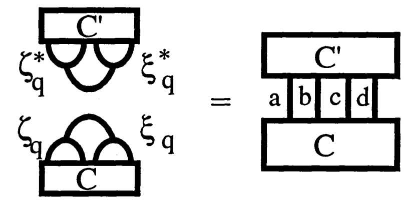
Suppose now that ξ_q is an element of V_q^{ab} so that when $V_q^{ab} \neq \{0\}$ then $\xi_q \circ \xi_q{}^* = 1_{\underline{q}}$, and let $\zeta_q \in V_q^{cd}$ so that when $V_q \neq \{0\}$, $\zeta_q \circ \zeta_q^* = 1_q$. Then notice that $\phi_q = \dfrac{1}{\sqrt{[2q+1]}} \cdot \beta_{qq} \circ (\xi_q \otimes \zeta_q) : \underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d} \to \mathbb{C}$ has the property that $\phi_q \phi_q^* : \mathbb{C} \to \mathbb{C}$ is the identity. Hence $\phi_q^* \circ \phi_q : \underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d} \to \underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$ is an idempotent. Further $\sum_q \phi_q^* \circ \phi_q$ is an idempotent, and its image is the trivial part of $\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$. That is any morphism $\mathbb{C} \to \underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$ has its image in $im\left(\sum_q \phi_q^* \circ \phi_q\right)$. From this we conclude that if 

is any morphism from $\mathbb{C}$ to $\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$ and 

is any morphism from

$\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$ to $\mathbb{C}$ then

(3.79) $$\sum_q \frac{1}{[2q+1]}$$



Similarly

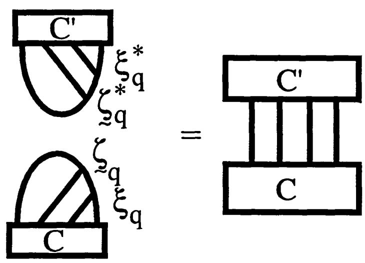
(3.80) $$\psi_q = \frac{1}{\sqrt{[2a+1]}} \cdot \beta_{aa} \circ \left(\zeta_q \circ (1_b \otimes \xi_q)\right)$$

has

$$\psi_q \circ \psi_q^* = 1_{\mathbb{C}}$$

and $\sum_q \psi_q^* \circ \psi_q$ is an idempotent on $\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$ whose image is the trivial part of $\underline{a} \otimes \underline{b} \otimes \underline{c} \otimes \underline{d}$. Hence

(3.81) $$\sum_q \frac{1}{[2q+1]}$$



We record these facts in the following proposition.

Proposition 3.6. *Let all the quantities be as in the above discussion.*

a)
$$\langle M \circ L(\vec{v}), \vec{w} \rangle = \sum_i \langle L(\vec{v}), \xi_i \rangle \langle M(\xi_i), \vec{w} \rangle$$

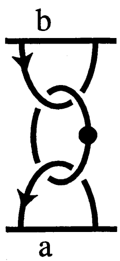
b) Equation 3.79 *is satisfied.*

c) Equation 3.81 *is satisfied.*

The following proposition can be found in [Wa] and in [T-We].

If $f : \underline{a} \otimes \underline{b} \to \underline{a} \otimes \underline{b}$ is $\mathcal{A}_r$-linear, f_G will mean the restriction of f to the good part of $\underline{a} \otimes \underline{b}$.

Proposition 3.7. *Let*

$t_{ab} =$  .

Then

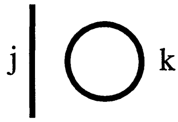
$$[t_{ab}]_G = \begin{cases} 0 & a \neq b \\ \dfrac{X}{[2a+1]} \cdot Id_G & a = b \end{cases}$$

where $Id : a \otimes a^* \longrightarrow a \otimes a^*$.

The following formulas are quoted from [K-M].

Proposition 3.8. *[K-M]*

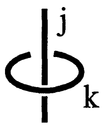
a.

 $= [2k+1]$ 

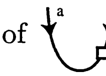
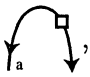
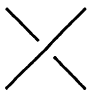
b.

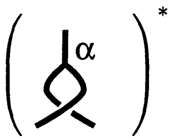
 $= q^{j(j+1)}$ 

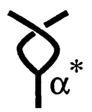
c.



$= \dfrac{[(2j+1)(2k+1)]}{[2j+1]}$ 

Finally a little discussion of how to pictorially represent the adjoint of an operator with respect to the trace norm. Recall that the adjoint reverses the order of composition of operators. Further, the adjoint of  is , and the adjoint of  is . Hence to take the adjoint of the diagram turn it upside down, change the crossings and take the adjoints of all the coupons.

Example 3. 

$=$ 

4. The Moore-Seiberg-Walker Equations.

In this section we use the properties of tangle functors to show that our choice of basic data satisfies the Moore-Seiberg-Walker Equations (theorem 2.1). We confirm the equations in the order (*ix*), (*x*), (*xi*), (*xii*), (*xiii*), (*xiv*), (*vii*), (*i*), (*ii*), (*iii*), (*iv*), (*v*), (*vi*), and (*viii*).

Equation (ix)

Equation (*ix*) states that

$$R(\beta_{aa}) = \beta_{aa}. \tag{4.1}$$

This is obvious, since we defined R on the annulus to be the identity.

Equation (x)

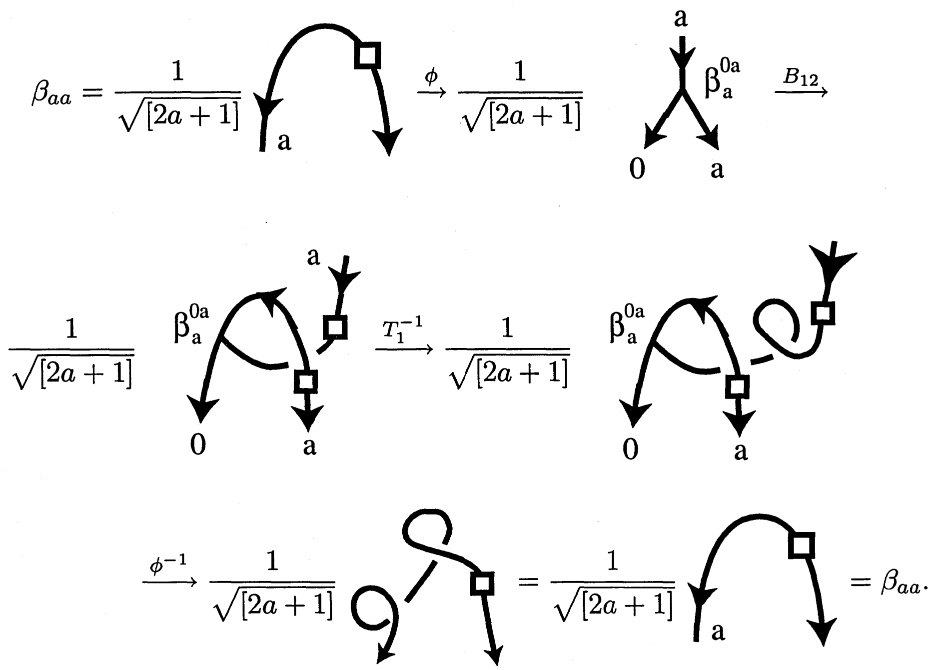
Equation (x) says:

$$R=\phi^{-1}\left(T_1^{-1}B_{12}\right)\phi, \tag{4.2}$$

where $\phi : V(A,a,a) \to V(P,a,a,0)$ is the map that takes β_{aa} to β_a^{0a}. Since R on the annulus is the identity, we need to check that

$$\beta_{aa}=\phi^{-1}\left(T_1^{-1}B_{12}\right)\phi\beta_{aa}. \tag{4.3}$$

Let's see:



Equation (xi)

We use a superscripted † to represent the adjoint with respect to pairings on tensor products of $V(P,a,b,c)$'s induced by $\langle\ ,\ \rangle$. Equation (xi) states

$$F=F^{\dagger}. \tag{4.4}$$

In terms of pairings it says

$$\langle F(\alpha\otimes\beta),\delta\otimes\gamma\rangle=\langle\alpha\otimes\beta,F(\delta\otimes\gamma)\rangle \tag{4.5}$$

where $\alpha \otimes \beta \in V_p^{ab} \otimes V_p^{cd}$, $\delta \otimes \gamma \in V_q^{ad} \otimes V_q^{cb}$. By the symmetry of $\langle\ ,\ \rangle$ (proposition 3.5) this is the same as

$$\langle \delta \otimes \gamma, F(\alpha \otimes \beta) \rangle = \langle \alpha \otimes \beta, F(\delta \otimes \gamma) \rangle.$$

By the definition of $\langle\ ,\ \rangle$ this equation becomes

(4.6)
$$\langle \psi \otimes \psi(\delta \otimes \gamma), F(\alpha \otimes \beta) \rangle_{k,\Sigma_{0,4}} = \langle \psi \otimes \psi(\alpha \otimes \beta), F(\delta \otimes \gamma) \rangle_{k,\Sigma_{0,4}}.$$

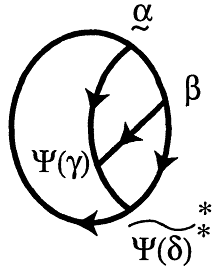
The Kronecker pairing on a surface, that is obtained by identifying two pairs of pants along boundary components, is the sum, over each label on the identified components, of the extension of the Kronecker pairing to tensor products, multiplied by $\frac{[2a+1]}{X}$. Hence, by the relation of the Kronecker pairing to the trace pairing, formula (4.6) becomes

(4.7)
$$\frac{X^2 \cdot [2q+1]}{X} \langle F(\alpha \otimes \beta), \psi(\delta)^* \otimes \psi(\gamma)^* \rangle_t = \frac{X^2 \cdot [2p+1]}{X} \langle F(\delta \otimes \gamma), \psi^*(\alpha) \otimes \psi^*(\beta) \rangle.$$

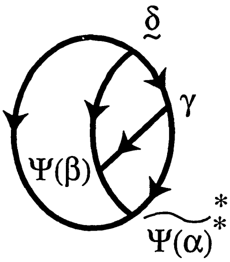
Expanding these as diagrams and using the defining formula for F we get

(4.8)

$\frac{[2q+1]}{[2a+1]}$



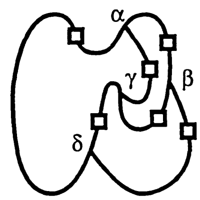
$= \frac{[2p+1]}{[2a+1]}$

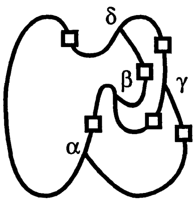


.

In order to affirm the verity of this formula we expand the coupons on the left hand side followed by the right hand side. The hardest coupons to expand are the ones corresponding to $\widetilde{\psi(\delta)^*}^*$ and $\widetilde{\psi(\alpha)^*}^*$. We use the identity

(4.9)
$$\widetilde{n^*}^* = \underset{\sim}{n}.$$

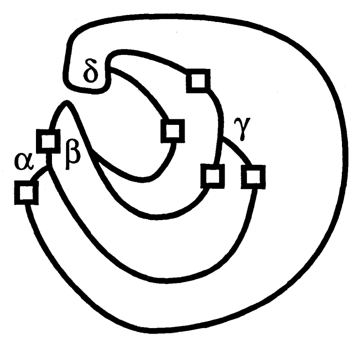
The two sides of equation (4.8) can be expressed as the multiples of the following diagrams: 

for the left hand side, and 

for the right hand side. The coefficient is in both cases equal to

$$\frac{1}{\sqrt{[2a+1][2b+1][2c+1][2d+1][2p+1][2q+1]}}.$$

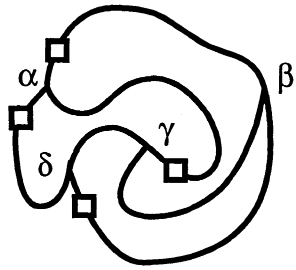
Since the coefficients are the same we just need to check that the diagrams represent the same morphism under the tangle functor. The first step is to replace the arc joining δ to α in the last diagram by an arc running around the other side of the diagram. This move can be achieved by sliding the arc across the top and canceling two nugatory crossings with opposite signs. So the right hand side becomes

(4.10) 

.

Now using the identities from proposition 3.2 on δ and γ, and performing an isotopy we get

(4.11)



.

This is the left hand side.

Equation (xii)

Equation (xii) states

$$S = S^{\dagger}. \tag{4.12}$$

By the definition of $S^{\dagger}$, the equation says that for all $\alpha \in V_a^{pp}$ and $\beta \in V_a^{qq}$,

$$\langle S(\alpha), \beta\rangle = \langle \alpha, S(\beta)\rangle. \tag{4.13}$$

By the symmetry of $\langle\ ,\ \rangle$ this means

$$\langle S(\alpha), \beta\rangle = \langle S(\beta), \alpha\rangle. \tag{4.14}$$

By the definition of $\langle\ ,\ \rangle$ the last equation is equivalent to

$$\langle \psi(\beta), S(\alpha)\rangle_{k,\Sigma_{1,1}} = \langle \psi(\alpha), S(\beta)\rangle_{k,\Sigma_{1,1}}. \tag{4.15}$$

Recall that the subscript $(k, \Sigma_{1,1})$ indicates that we are considering Kronecker pairing on a surface $\Sigma_{1,1}$ of genus one with one boundary component.

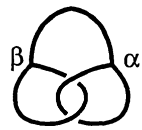
Remembering that $\Sigma_{1,1}$ is obtained by gluing two boundary components of a pair of pants P together, and using the relation between $\langle\ ,\ \rangle_k$ and $\langle\ ,\ \rangle_t$, (equation 3.21) we get

$$\frac{[2q+1]}{X}\langle \psi(\beta), S(\alpha)\rangle_{k,P} = \langle \psi(\alpha), S(\beta)\rangle_{k,P}\cdot\frac{[2p+1]}{X}. \tag{4.16}$$

Hence

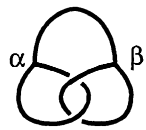
$$X[2q+1]\langle S(\alpha), \psi(\beta)^*\rangle_t = X[2p+1]\langle S(\beta), \psi(\alpha)^*\rangle_t. \tag{4.17}$$

Expanding as diagrams, on the left hand side we get

(4.18) $$\frac{\sqrt{[2p+1][2q+1]}}{[2a+1]^{3/2}}$$ 

,

and the right hand side is

(4.19) $$\frac{\sqrt{[2p+1][2q+1]}}{[2a+1]^{2/3}}$$ 

.

The same sequence of moves as used to show the symmetry of $\langle\ ,\ \rangle$ in proposition 3.5 applies to show that the left hand side equals the right hand side.

Equation (xiii)

Equation *xiii* says

(4.20) $$\langle \beta_{aa}, \beta_{aa} \rangle = \frac{X}{[2a+1]}.$$

Expanding the left hand side,

(4.21)

$$\langle \beta_{aa}, \beta_{aa} \rangle = \langle \psi(\beta_{aa}), \beta_{aa} \rangle_k = X \cdot \beta_{aa} \circ \psi(\beta_{aa}) = \frac{X}{[2a+1]^2}$$ 

$$= \frac{X}{[2a+1]}.$$

Equation (xiv)

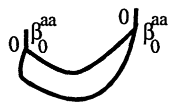
Similarly, equation *xiv* states

$$\langle \beta_0^{aa}, \beta_0^{aa} \rangle = \frac{X^2}{[2a+1]}. \tag{4.22}$$

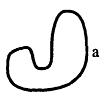
Expanding we get

$$\langle \beta_0^{aa}, \beta_0^{aa} \rangle \cdot 1_{\underline{0}} = \langle \psi(\beta_0^{aa}), \beta_0^{aa} \rangle_k \cdot 1_{\underline{0}} = X^2 \beta_0^{aa} \circ \psi(\beta_0^{aa}) 1_{\underline{0}} = \tag{4.23}$$

$$= \frac{X^2}{[2a+1]}$$



$$= \frac{X^2}{[2a+1]^2}$$



$$\cdot 1_{\underline{0}} = \frac{X^2}{[2a+1]} \cdot 1_{\underline{0}}.$$

Equation (vii)

Equation *vii* is the statement that

$$F^2 = P. \tag{4.24}$$

Let $P : V_p^{ab} \otimes V_p^{cd} \to V_p^{cd} \otimes V_p^{ab}$ be the map

$$P(\alpha \otimes \beta) = \beta \otimes \alpha. \tag{4.25}$$

In order to show (4.24) we will convert it to a statement about the trace pairing. Let $\alpha \otimes \beta \in V_p^{ab} \otimes V_p^{cd}$ and $\delta \otimes \gamma \in V_q^{ab} \otimes V_q^{cd}$ be arbitrary. $F^2 = P$ is equivalent to:

$$\langle F^2(\alpha \otimes \beta), \gamma \otimes \delta \rangle_t = \delta_{pq} \langle \alpha, \delta \rangle_t \langle \beta, \gamma \rangle_t. \tag{4.26}$$

Let $\zeta_r \otimes \xi_r \in V_r^{da} \otimes V_r^{bc}$ be orthonormal in the sense that $\zeta_r \circ \zeta_r^* = 1_r$ and $\xi_r \circ \xi_r^* = 1_r$. Then $\sum_r \underset{\sim}{\zeta_r} \otimes \xi_r \in V_d^{ar} \otimes V_r^{bc}$. Further $\sum_r \underset{\sim}{\zeta_r} \circ (1_a \otimes \xi_r) : \underline{a} \otimes \underline{b} \otimes \underline{c} \to \underline{d}$ has the property that

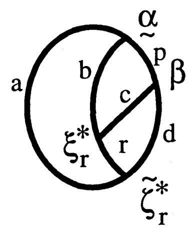
$$\sum_r \left(\underset{\sim}{\zeta_r} \circ (1_a \otimes \xi_r)\right)^* \circ \sum_r \underset{\sim}{\zeta_r} \circ (1_a \otimes \xi_r) : \underline{a} \otimes \underline{b} \otimes \underline{c} \to \underline{a} \otimes \underline{b} \otimes \underline{c} \tag{4.27}$$

is an idempotent whose image contains all copies of $\underline{d}$ in $\underline{a} \otimes \underline{b} \otimes \underline{c}$. Hence it acts as the identity on that subspace. Expanding $\langle F^2(\alpha \otimes \beta), \gamma \otimes \delta \rangle_t$ we get

(4.28) $\langle F^2(\alpha \otimes \beta), \gamma \otimes \delta \rangle_t = \langle F(\alpha \otimes \beta), F^*(\gamma \otimes \delta) \rangle_t =$

$$\sum_r \langle F(\alpha \otimes \beta), \zeta_r \otimes \xi_r \rangle_t \langle \zeta_r \otimes \xi_r, F^*(\gamma \otimes \delta) \rangle_t = \sum_r \langle F(\alpha \otimes \beta), \zeta_r \otimes \xi_r \rangle_t \langle F(\zeta_r \otimes \xi_r), \gamma \otimes \delta \rangle_t =$$

$$= \sum_r \frac{1}{[2a+1][2d+1]}$$



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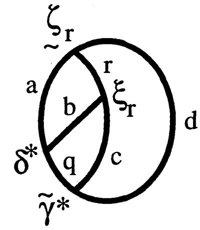
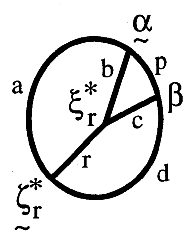


=

$$= \sum_r \frac{1}{\sqrt{[2a+1]}[2d+1]^{3/2}}$$



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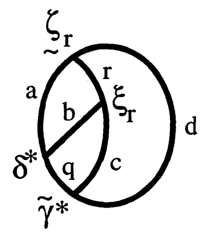
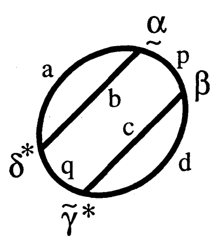


.

By proposition 3.6 we can splice these together and lose the sum to get:

(4.29) $$\frac{1}{\sqrt{[2a+1][2d+1]}}$$

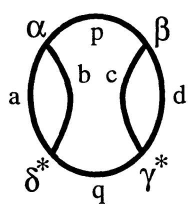


.

Expanding the coupons yields:

(4.30)

$$\frac{1}{\sqrt{[2p+1][2q+1]}}$$



$= \delta_{pq} \langle \alpha, \delta \rangle_t \langle \beta, \gamma \rangle_t.$

Equation (i)

Equation i says:

(4.31)
$$P^{(1,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)} = 1.$$

This occurs on a tensor product of three vector spaces. $P^{(1,3)}$ is the permutation of the first and third factors, $F^{(1,2)}$ and $F^{(2,3)}$ are F acting in the first and second, and second and third factors respectively. Recall from previous section (formula 3.66) that

$$N^{(2,3)}N^{(1,2)}N^{(2,3)} = P^{(2,3)}N^{(1,3)}N^{(1,2)} \tag{4.32}$$

Recalling (see formula 3.67)

$$F = L^{(1)} \circ N \circ L^{(1)} \tag{4.33}$$

we turn all our N's into F's.

(4.34)
$$R^{(2)}F^{(2,3)}R^{(2)}R^{(1)}F^{(1,2)}R^{(1)}R^{(2)}F^{(2,3)}R^{(2)} = P^{(2,3)}R^{(1)}F^{(1,3)}{R^{(1)}}^2F^{(1,2)}R^{(1)}.$$

Since $R^{(1)}$ acts in the first factor only, we can commute the two $R^{(1)}$'s on the left hand side to the front and back. Further we can bring the first $R^{(1)}$ on the right to the front. We can then cancel to get

(4.35)
$$R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}R^{(2)} = P^{(2,3)}F^{(1,3)}{R^{(1)}}^2F^{(1,2)}.$$

Recalling that $F^2 = P$ (formula 4.24) we get

(4.36)
$$R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}R^{(2)} = P^{(2,3)}P^{(1,3)}{F^{(1,3)}}^{-1}{R^{(1)}}^2P^{(1,2)}{F^{(1,2)}}^{-1}.$$

Now commute the $P^{(1,2)}$ on the right hand side forward using

(4.37)
$$R^{(1)}P^{(1,2)} = P^{(1,2)}R^{(2)}, \quad {F^{(1,3)}}^{-1}P^{(1,2)} = P^{(1,2)}{F^{(2,3)}}^{-1}$$

to get

(4.38)
$$R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}R^{(2)} = P^{(2,3)}P^{(1,3)}P^{(1,2)}{F^{(2,3)}}^{-1}{R^{(2)}}^2{F^{(1,2)}}^{-1}.$$

Now

(4.39)
$$P^{(2,3)}P^{(1,3)}P^{(1,2)} = P^{(1,3)}$$

so:

(4.40)
$$R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}R^{(2)} = P^{(1,3)}F^{(2,3)^{-1}}R^{(2)^2}F^{(1,2)^{-1}}.$$

Putting everything on the right hand side and using $R^3 = 1$ yields:

(4.41)
$$R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}P^{(1,3)} = 1.$$

We can now commute $P^{(1,3)}$ to the front. One consequence of $F^2 = P$ is that $F = PFP$, so

(4.42)
$$P^{(1,3)}R^{(2)}F^{(1,2)}F^{(2,3)}R^{(2)}F^{(1,2)}R^{(2)}F^{(2,3)}R^{(2)}F^{(1,2)} = 1,$$

which is equation i.

Equation (ii)

This says

(4.43)
$$\left(F \circ B_{23}^{(2)^{-1}}\right)^3 T_2^{(2)} = 1.$$

Let $\alpha \otimes \beta \in V_p^{ba} \otimes V_p^{dc}$, $\delta \otimes \gamma \in V_s^{ba} \otimes V_s^{dc}$ be arbitrary. Equation (4.43) is equivalent to

(4.44)
$$\left\langle \left(F \circ B_{23}^{(2)^{-1}}\right)^3 T_2^{(2)}(\alpha \otimes \beta), \delta \otimes \gamma \right\rangle_t = \langle \alpha, \delta \rangle_t \langle \beta, \gamma \rangle_t \delta_{ps}.$$

Choose $\phi_q \otimes \psi_q \in V_q^{db} \otimes V_q^{ac}$, $\zeta_r \otimes \xi_r \in V_r^{ad} \otimes V_r^{bc}$ so that when the vector spaces are nonzero,

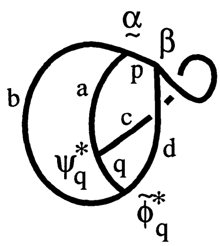
(4.45)
$$\phi_q \circ \phi_q^* = 1_q, \quad \psi_q \circ \psi_q^* = 1_q, \quad \zeta_r \circ \zeta_r^* = 1_r, \quad \text{and} \quad \xi_r \circ \xi_r^* = 1_r.$$

Using proposition 3.6 to expand the left hand side we get:

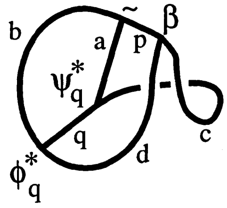
(4.46)
$$\left\langle \left(F \circ B_{23}^{(2)^{-1}}\right)^3 T_2^{(2)}(\alpha \otimes \beta), \delta \otimes \gamma \right\rangle_t = \sum_{q,r} \left\langle F \circ B_{23}^{(2)^{-1}} \circ T_2^{(2)}(\alpha \otimes \beta), \phi_q \otimes \psi_q \right\rangle_t$$
$$\cdot \left\langle F \circ B_{23}^{(2)^{-1}}(\phi_q \otimes \psi_q), \zeta_r \otimes \xi_r \right\rangle_t \cdot \left\langle F \circ B_{23}^{(2)^{-1}}(\zeta_r \otimes \xi_r), \delta \otimes \gamma \right\rangle_t .$$

We now display the three factors in the sum as diagrams. We suppress the arrows and  in the diagrams, as their locations can easily be deduced.

(4.47)

$$\left\langle F \circ B_{23}^{(2)^{-1}} \circ T_2^{(2)}(\alpha \otimes \beta), \phi_q \otimes \psi_q \right\rangle_t = \frac{1}{[2b+1]}$$ 

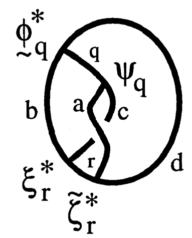
By proposition 3.3 this is equal to

(4.48) $$\frac{1}{\sqrt{[2b+1][2d+1]}}$$  .

Next,

(4.49)

$$\left\langle F\circ B_{23}^{(2)^{-1}}(\phi_q\otimes\psi_q),\zeta_r\otimes\xi_r\right\rangle_t=\frac{1}{[2d+1]}$$

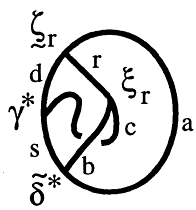


.

Finally,

(4.50)

$$\left\langle F\circ B_{23}^{(2)^{-1}}(\zeta_r\otimes\xi_r),\delta\otimes\gamma\right\rangle_t=\frac{1}{[2a+1]}$$

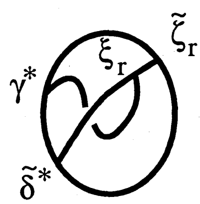


,

which by proposition 3.3 is

(4.51)

$$\frac{1}{\sqrt{[2a+1][2d+1]}}$$

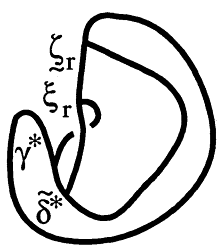


.

Isotoping the strand joining γ^* and $\tilde{\zeta}_r$, and canceling the nugatory crossings yields:

(4.52)

$$\frac{1}{\sqrt{[2a+1][2d+1]}}$$

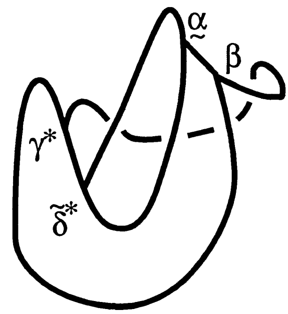


.

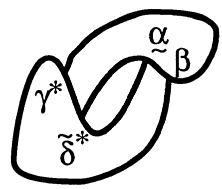
Two applications of proposition 3.6 allow us to join these diagrams and lose the sums, so:

(4.53)

$$\left\langle \left(F\circ B_{23}^{(2)^{-1}}\right)^3 T_2^{(2)}(\alpha\otimes\beta),\delta\otimes\gamma\right\rangle_t = \frac{1}{\sqrt{[2b+1][2a+1]}}$$



$$= \frac{1}{\sqrt{[2b+1][2a+1]}}$$

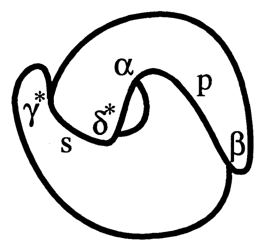


.

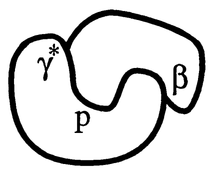
Using proposition 3.3 to slide $\tilde{\delta}^*$ and $\underset{\sim}{\alpha}$ past a local minimum and a local maximum respectively, we get the left hand side to be equal to:

(4.54)

$$\frac{1}{\sqrt{[2p+1][2s+1]}}$$



$$= \frac{1}{[2p+1]}\delta_{ps}\langle\alpha,\delta\rangle_t$$



.

Four applications of proposition 3.3 and some isotopies yield that formula (4.54) is equal to

(4.55) $$\delta_{ps}\langle\alpha,\delta\rangle_t\langle\beta,\gamma\rangle_t.$$

Equation (iii)

We are now ready to begin our assault on equation *iii*:

(4.56)

$$\left(T_3^{(2)^{-1}}T_1^{(2)}B_{23}^{(2)}FB_{23}^{(1)^{-1}}B_{23}^{(2)^{-1}}F\right)\left(S^{(2)^{-1}}FR^{(2)}R^{(1)^{-1}}FS^{(2)}\right)=1.$$

Moving half of it to the right hand side yields

(4.57)
$$T_3^{(2)^{-1}} T_1^{(2)} B_{23}^{(2)} F B_{23}^{(1)^{-1}} B_{23}^{(2)^{-1}} F = S^{(2)^{-1}} F^{-1} R^{(1)} R^{(2)^{-1}} F^{-1} S^{(2)}.$$

Using $F^{-1} = FP = PF$ we get:

(4.58)
$$T_3^{(2)^{-1}} T_1^{(2)} B_{23}^{(2)} F B_{23}^{(1)^{-1}} B_{23}^{(2)^{-1}} F = S^{(2)^{-1}} F P R^{(1)} R^{(2)^{-1}} P F S^{(2)}.$$

Using $PR^{(1)}R^{(2)^{-1}} = R^{(2)}R^{(1)^{-1}}P$ and $P^2 = 1$ we finally have:

(4.59)
$$T_3^{(2)^{-1}} T_1^{(2)} B_{23}^{(2)} F B_{23}^{(1)^{-1}} B_{23}^{(2)^{-1}} F = S^{(2)^{-1}} F R^{(2)} R^{(1)^{-1}} F S^{(2)}.$$

We will expand the left and right hand sides of equation (4.59) to see that they are the same. Once again we suppress arrows and boxes. First lets trace the domains, left hand side first.

(4.60)
$$\alpha \otimes \beta \in V_p^{ab} \otimes V_p^{cc} \xrightarrow{F} \bigoplus_q V_q^{ca} \otimes V_q^{bc} \xrightarrow{B_{23}^{(1)^{-1}} B_{23}^{(2)^{-1}}} \bigoplus_q V_q^{ac} \otimes V_q^{cb} \xrightarrow{F}$$
$$\bigoplus_e V_e^{ba} \otimes V_e^{cc} \xrightarrow{T_3^{(2)^{-1}} T_1^{(2)} B_{23}^{(2)}} \bigoplus_e V_e^{ba} \otimes V_e^{cc}$$

Let $\delta \otimes \gamma \in V_e^{ba} \otimes V_e^{rr}$ and let $\zeta_q^{ac} \otimes \zeta_q^{cb} \in V_q^{ac} \otimes V_q^{cb}$ be orthonormal. Notice that if LHS denotes the left hand side of (4.59) then it is clear that

$$\langle LHS(\alpha \otimes \beta), \delta \otimes \gamma \rangle = 0 \quad \text{unless} \quad \gamma \in V_e^{cc}. \tag{4.61}$$

Now the right hand side:

(4.62)
$$\alpha \otimes \beta \in V_p^{ab} \otimes V_p^{cc} \xrightarrow{S^{(2)}} \bigoplus_d V_p^{ab} \otimes V_p^{dd} \xrightarrow{F} \bigoplus_{d,q} V_q^{da} \otimes V_q^{bd} \xrightarrow{R^{(2)} R^{(1)^{-1}}}$$
$$\bigoplus_{d,q} V_d^{aq} \otimes V_d^{qb} \xrightarrow{F} \bigoplus_{e,q} V_e^{ba} \otimes V_e^{qq} \xrightarrow{S^{(2)^{-1}}} \bigoplus_{e,r} V_e^{ba} \otimes V_e^{rr}.$$

Denote by RHS the right hand side of equation (4.59). One of the things we will need to see is that

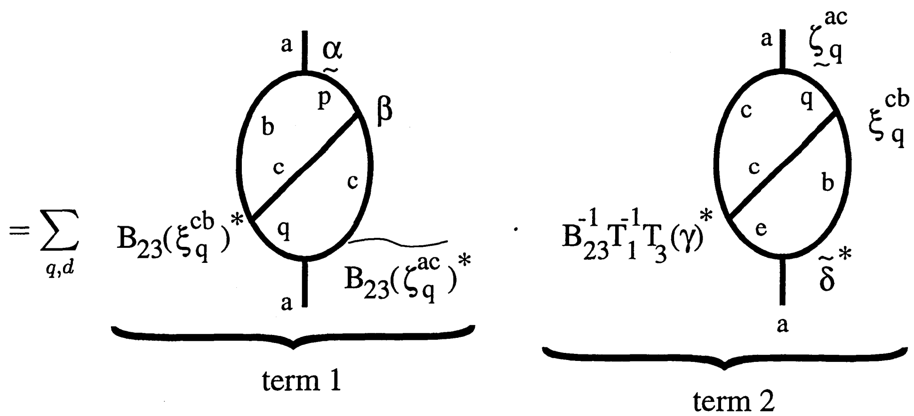
$$\langle RHS(\alpha \otimes \beta), \delta \otimes \gamma \rangle = 0 \quad \text{unless} \quad \gamma \in V_e^{cc}. \tag{4.63}$$

We choose $\phi_d^{aq} \otimes \psi_d^{qb} \in V_d^{aq} \otimes V_d^{qb}$ to be unitary.

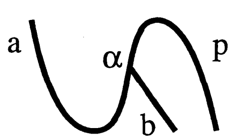
We begin by expanding $\langle LHS(\alpha \otimes \beta), \delta \otimes \gamma \rangle$.

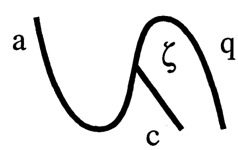
$$\left\langle T_3^{(2)^{-1}} T_1^{(2)} B_{23}^{(2)} F B_{23}^{(1)^{-1}} B_{23}^{(2)^{-1}} F(\alpha, \beta), \delta \otimes \gamma \right\rangle = \tag{4.64}$$

$$\sum_{q,d} \left\langle F(\alpha \otimes \beta, B_{23}\left(\zeta_q^{ac}\right) \otimes B_{23}\left(\xi_q^{cb}\right)\right\rangle \cdot \left\langle F\left(\zeta_q^{ac} \otimes \xi_q^{cb}\right), \delta \otimes B_{23}^{-1} T_1^{-1} T_3(\gamma)\right\rangle =$$

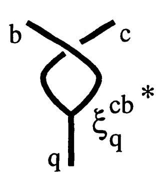


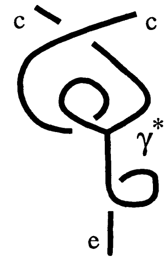
We need to expand some coupons:

$\underset{\sim}{\alpha} = \dfrac{\sqrt{[2a+1]}}{\sqrt{[2p+1]}}$ 

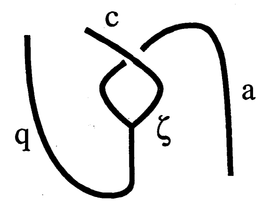
, $\underset{\sim}{\zeta}_q^{ac} = \dfrac{\sqrt{[2a+1]}}{\sqrt{[2q+1]}}$ 

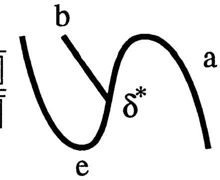
,

$B_{23}\left(\xi_q^{cb}\right)^* =$ 

, $B_{23}^{-1} T_1^{-1} T_3(\gamma)^* =$ 

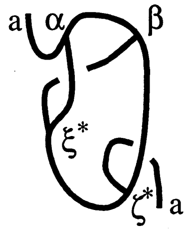
,

$\widetilde{B_{23}\left(\zeta_q^{ac}\right)}^{*} = \frac{\sqrt{[2a+1]}}{\sqrt{[2q+1]}}$ 

, $\tilde{\delta}^{*} = \frac{\sqrt{[2a+1]}}{\sqrt{[2e+1]}}$ 

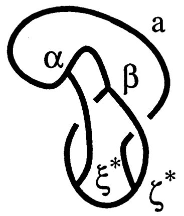
.

Expanding the coupons in term 1 yields:

(4.65) $$\frac{[2a+1]}{\sqrt{[2p+1][2q+1]}}$$ 

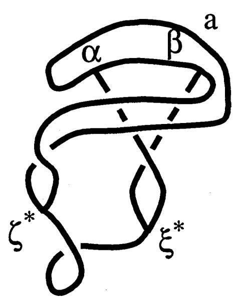
.

Closing we get

(4.66) $$\frac{1}{\sqrt{[2p+1][2q+1]}}$$ 

.

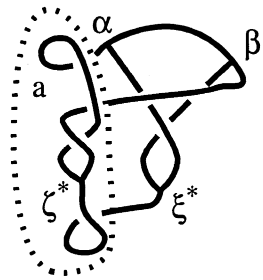
Now we slide ζ^* over ξ^*.

(4.67) $$\frac{1}{\sqrt{[2p+1][2q+1]}}$$ 

Next we slide strand labeled with a over.

(4.68)

$$\frac{1}{\sqrt{[2p+1][2q+1]}}$$



Notice that the part of the tangle enclosed inside the dotted line is:

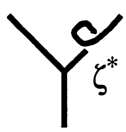
(4.69)

$$= \left(\bar{T}_3 T_1 B_{23}^2(\zeta)\right)^* = \left(T_2 \bar{T}_2 \bar{T}_3 T_1 B_{23}^2(\zeta)\right)^*$$

and since $\bar{T}_2 \bar{T}_3 T_1 B_{23}^2 = 1$ this is further equal to

(4.70)

$$(T_2(\zeta))^* =$$

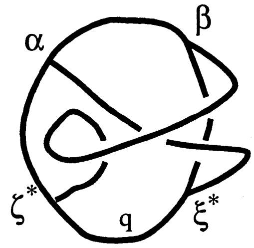


.

Substituting we get

(4.71)

$$\frac{1}{\sqrt{[2p+1][2q+1]}}$$



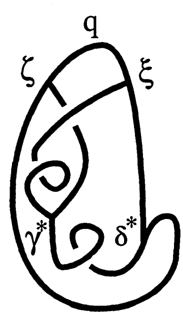
.

Now we unfold the second term:

(4.72) $$\frac{[2a+1]}{\sqrt{[2q+1][2e+1]}}$$ 

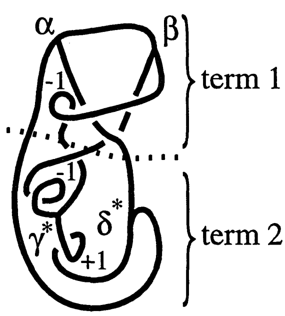
.

Closing it up we get

(4.73) $$\frac{1}{\sqrt{[2q+1][2e+1]}}$$ 

.

We are now ready to splice together (4.71) and (4.73). The factor $\frac{1}{\sqrt{[2q+1]}}$ in term (4.71) gets used to turn the minimum on q into a unitary projection. The factor $\frac{1}{\sqrt{[2q+1]}}$ in (4.73) gets used to turn the maximum on q into a unitary projection. We get:

(4.74) $$\frac{1}{\sqrt{[2p+1][2e+1]}}$$ 

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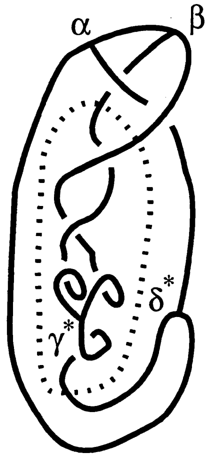
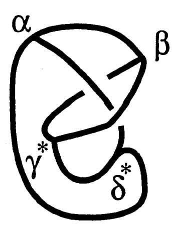
Drawing the Dehn twist closer to γ and pulling the right strand from β over the left yields:

(4.75) $$\frac{1}{\sqrt{[2p+1][2e+1]}}$$ 

.

The part enclosed by the dotted line is $\left(B_{23}^{-2}T_1^{-1}T_2T_3(\gamma)\right)^* = \gamma^*$. So the left hand side is equal to:

(4.76) $$\frac{1}{\sqrt{[2p+1][2e+1]}}$$ 

.

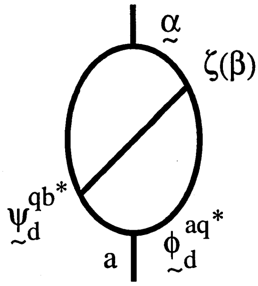
We now expand the right hand side of (4.59).

$$\left\langle S^{(2)^{-1}} F R^{(2)} R^{(1)^{-1}} F S^{(2)} (\alpha \otimes \beta), \delta \otimes \gamma \right\rangle =$$

$$= \sum_{q,d} \left\langle F\left(\alpha \otimes S(\beta)\right), \widetilde{\phi_d^{aq}} \otimes \underset{\sim}{\psi}{}_d^{qb} \right\rangle \cdot \left\langle F\left(\phi_d^{aq} \otimes \psi_d^{qb}\right), \delta \otimes \gamma \right\rangle =$$

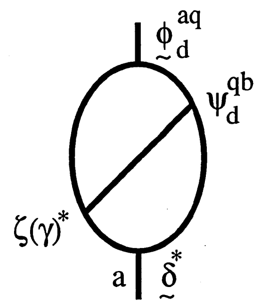
(4.77)

$$= \sum_{q,d}$$



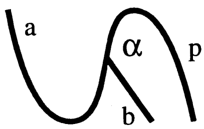
Term 1

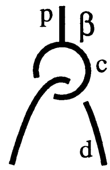
$\cdot$



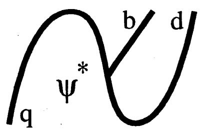
Term 2

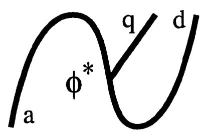
We need to expand coupons in (4.77).

$$\underset{\sim}{\alpha} = \frac{\sqrt{[2a+1]}}{\sqrt{[2p+1]}}$$ 

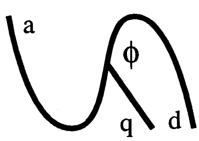
$$, \quad S(\beta) = \sum_{d} \frac{\sqrt{[2c+1][2d+1]}}{X}$$ 

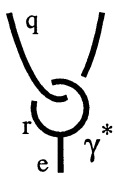
,

$$\underset{\sim}{\psi}{}_d^{qb*} = \frac{\sqrt{[2q+1]}}{\sqrt{[2d+1]}}$$ 

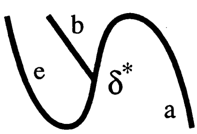
$$, \quad \underset{\sim}{\phi}{}_d^{aq*} = \frac{\sqrt{[2a+1]}}{\sqrt{[2d+1]}}$$ 

,

$$\phi_d^{aq} = \frac{\sqrt{[2a+1]}}{\sqrt{[2d+1]}}$$ 

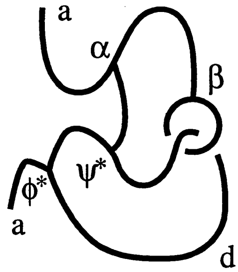
$$, \quad S(\gamma)^* = \sum_{q} \frac{\sqrt{[2r+1][2q+1]}}{X}$$ 

,

$$\tilde{\delta}^* = \frac{\sqrt{[2a+1]}}{\sqrt{[2e+1]}}$$ 

.

Expanding term 1 in (4.77):

(4.78) $$\frac{[2a+1]\sqrt{[2c+1][2q+1]}}{X\sqrt{[2d+1][2p+1]}}$$ 

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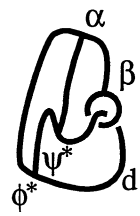
Closing up:

(4.79) $$\frac{\sqrt{[2c+1][2q+1]}}{X\sqrt{[2d+1][2p+1]}}$$ 

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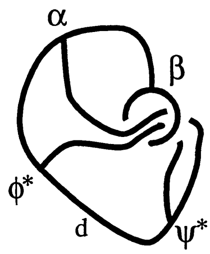
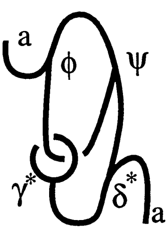
Finally we thread ψ^* through the ring:

(4.80) $$\frac{\sqrt{[2c+1][2q+1]}}{X\sqrt{[2d+1][2p+1]}}$$ 

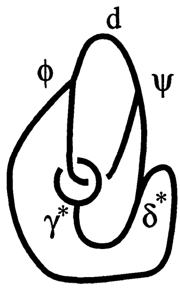
.

Expanding term 2 in (4.77):

(4.81) $$\frac{[2a+1]\sqrt{[2r+1][2q+1]}}{X\sqrt{[2d+1][2e+1]}}$$ 

.

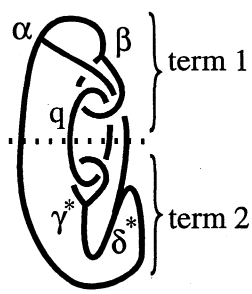
Closing up we get:

(4.82) $$\frac{\sqrt{[2r+1][2q+1]}}{X\sqrt{[2d+1][2e+1]}}$$ 

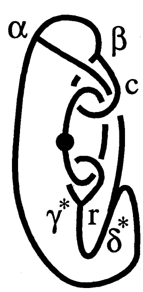
.

We now splice (4.80) and (4.82) together. Turning the local minimum and maximum on the two strands labeled with d into projections eats up the factor $\frac{1}{[2d+1]}$. The splice is just over d, with q fixed.

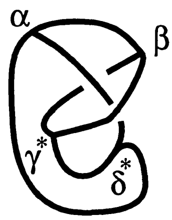
(4.83)

$$\sum_q \frac{[2q+1]\sqrt{[2c+1][2r+1]}}{X^2\sqrt{[2e+1][2p+1]}} \cdot$$ 

Now $\frac{2q+1}{X}$ and $\sum_q$ turn the simple closed curve labeled with q into surgery curve.

(4.84) $$\frac{\sqrt{[2c+1][2r+1]}}{X\sqrt{[2e+1][2p+1]}} \cdot$$ 

The surgery curve acts as 0 when $r \neq c$. When $r = c$ it acts as $\frac{X}{[2c+1]}$ followed by hooking two loops together. So the right hand side of (4.59) is finally equal to

(4.85) $$\frac{1}{\sqrt{[2e+1][2p+1]}}$$ 

.

A comparison of the diagrams (4.76) and (4.85) completes the proof.

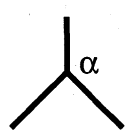
Equation (iv)

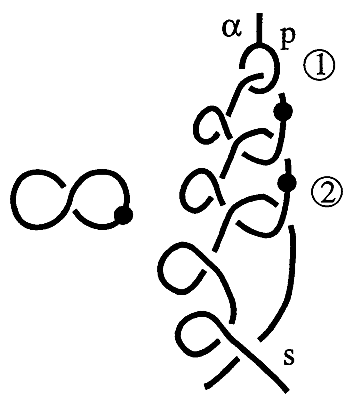
Equation (*iv*) is really two equations. However, the second one follows from the first and from equation (x) which we already established. The first one says:

(4.86) $$CB_{23}^{-1}T_3^2ST_3ST_3S = 1.$$

Let $\alpha \in V_a^{pp}$, then suppressing arrows and boxes:

(4.87)

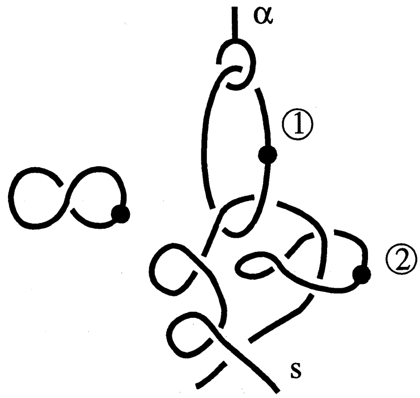
$$CB_{23}^{-1}T_3^2ST_3ST_3S \Bigg($$ 

$$\Bigg) = \sum_s \frac{\sqrt{[2p+1][2s+1]}}{X}$$ 

Sliding handle 1 over handle 2 yields:

(4.88)

$$\sum_s \frac{\sqrt{[2p+1][2s+1]}}{X}$$

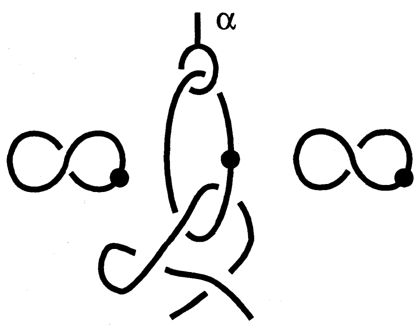


.

Sliding the strand labeled with s over handle 2 results in:

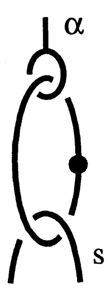
(4.89)

$$\sum_s \frac{\sqrt{[2p+1][2s+1]}}{X}$$



$=$

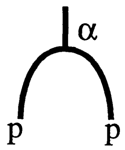
$$= \sum_s \frac{\sqrt{[2p+1][2s+1]}}{X}$$



.

By proposition 3.7 this is equal to

(4.90)



.

Equation (v)

Equation (v) says that for all $x \in V_a^{cb}$,

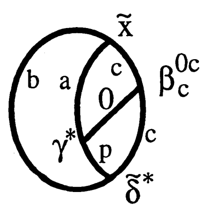
$$F\left(R^{-1}(x) \otimes \beta_c^{0c}\right) = x \otimes \beta_a^{a0}. \tag{4.91}$$

$R^{-1}(x) \in V_c^{ba}$ so $F\left(R^{-1}(x) \otimes \beta_c^{0c}\right) \in \bigoplus_p V_p^{cb} \otimes V_p^{a0}$, and equation (4.91) is equivalent to the statement that for all $\delta \otimes \gamma \in V_p^{cb} \otimes V_p^{a0}$

$$\left\langle F\left(R^{-1}(x) \otimes \beta_c^{0c}\right), \delta \otimes \gamma\right\rangle = \delta_{ap}\langle x, \delta\rangle_t \cdot \langle \beta_a^{a0}, \gamma\rangle_t. \tag{4.92}$$

Expanding the left hand side as a diagram we get:

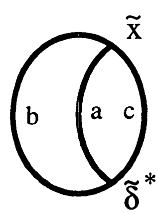
(4.93) $\dfrac{1}{[2b+1]} \cdot$



.

We may as well assume that $\gamma = k \cdot \beta_a^{a0}$, since $V_p^{a0} = 0$ unless $a = p$. We then use the fact that β_a^{a0} is just the deletion of an edge labeled with 0 to see (4.93) as:

(4.94) $\dfrac{1}{[2b+1]}\bar{k}\delta_{ap}$



$= \delta_{ap}\langle x, \delta\rangle_t \cdot \langle \beta_a^{a0}, \gamma\rangle_t.$

Equation (vi)

Equation (*vi*) is obvious from our choice of diagrams.

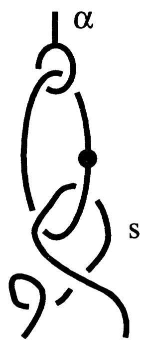
Equation (viii)

Equation (*viii*) consists of two equations. Once again, the second one follows from the first. The first equation states

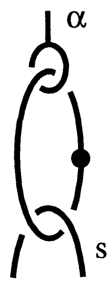
$$T_3 B_{23}^{-1} S^2 = 1. \tag{4.95}$$

We need to show that for any $\alpha \in V_a^{pp}$ we have $T_3 B_{23}^{-1} S^2(\alpha) = \alpha$. As a diagram

$$T_3 B_{23}^{-1} S^2(\alpha) = \sum_q \frac{\sqrt{[2p+1][2s+1]}}{X} \tag{4.96}$$



$$= \sum_q \frac{\sqrt{[2p+1][2s+1]}}{X}$$



which by proposition 3.7 equals α.

5. Numerical description of the basic data.

The goal of this section is to give a matrix description of the basic data. We begin with some facts about the basic hypergeometric functions. Next we prove that the trace norm is positive definite, and on the way give a method for computing it. We then choose $\beta^p_{m,n}$ and compute its coefficients in terms of $e_a \otimes e_b \otimes e^{a+b}$. These coefficients are commonly referred to as Clebsch-Gordan coefficients. Then we derive a formula for the $6j$-symbols from the Clebsch-Gordan coefficients. We work with this to produce a formula for the $6j$-symbols in terms of ${}_4\Phi_3$. We proceed to produce values of B_{23}, R, F and S_a in terms of this basis.The formulas for B_{12}, L and ψ are easily derivable from these.

Basic Hypergeometric Functions

For the complete treatment of basic hypergeometric functions we refer the reader to [V-K] or [G-R]. We state here some definitions and needed formulas. The results we cite are well known to the experts in the field.

Let

$$(5.1) \qquad \begin{array}{l} (b;q)_i = \prod_{k=1}^{i}(1-bq^{k-1}) \quad \text{if } i \neq 0 \\ (b;q)_0 = 1. \end{array}$$

Notice the following identities involving $(q^a;q)$.

Fact 10.

(i) $(q^a;q)_i = (\bar{s}-s)^i s^{ai+\frac{(i-1)i}{2}} \prod_{k=1}^{i}[a+k-1]$

(ii) $(q^a;q)_i = (-1)^i q^{ai+\frac{(i-1)i}{2}} \left(q^{-a-i+1};q\right)_i$

(iii) $(q^{a-u};q)_u = (-1)^u q^{(a-u)u+\frac{(u-1)u}{2}} \left(q^{-a+1};q\right)_u$

(iv) $(q^{a-u};q)_b = \dfrac{(q^a;q)_b(q^{a-u};q)_u}{(q^{a+b-u};q)_u} = q^{-bu}\dfrac{(q^{-a+1};q)_u(q^a;q)_b}{(q^{-a-b+1};q)_u}$

(v) $(q^a;q)_{b-x} = \dfrac{(q^a;q)_b}{(q^{a+b-x};q)_x}$

(vi) $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{\left(q^{n-k+1};q\right)_k}{(q;q)_k}$

Basic hypergeometric functions are defined by the following equality (see [G-R] or [V-K]). If $r = s+1$ then:

(5.2)

$$_r\phi_s(a_1,a_2,...,a_r;b_1,...,b_s;q;z) = \sum_{j=0}^{\infty} \frac{(a_1;q)_j...(a_r;q)_j}{(b_1;q)_j...(b_s;q)_j(q;q)_j} z^j.$$

If one of a_k is equal to q^{-m}, where m is a positive integer, then the series is finite and defines a polynomial in z. In our case all a_k and all b_k are powers of q, and we evaluate at $z = q^n$, where n is an integer. We adopt the following notation from [V-K]:

(5.3)
$$_r\Phi_s(\alpha_1, ..., \alpha_r; \beta_1, ..., \beta_s; q, z) =_r \phi_s(q^{\alpha_1}, ..., q^{\alpha_r}; q^{\beta_1}, ..., q^{\beta_s}; q; z).$$

We will be using

$$(5.4) \qquad {}_2\Phi_1(a, b; c; q, q^n) = \sum_{i=0}^{-b} \frac{(q^a; q)_i (q^b; q)_i}{(q^c; q)_i (q; q)_i} q^{ni},$$

and also $_3\Phi_2$ and $_4\Phi_3$. We will need the following q-analogue of Chu-Vandermonde convolution.

Proposition 5.1 ([G-R]). *If $-r < c \leq b \leq 0$ and $a > 0$ then*

$$_2\Phi_1(a, b; c; q, q) = \frac{(q^{c-a}; q)_{-b}}{(q^c; q)_{-b}} q^{-ab},$$

and

$$_2\Phi_1(a, b; c; q, q^{c-a-b}) = \frac{(q^{c-a}; q)_{-b}}{(q^c; q)_{-b}}.$$

Computing the Trace Pairing

Recall that V_p^{mn} is the space of $\mathcal{A}_r$-linear maps $\pi : \underline{m} \otimes \underline{n} \to \underline{p}$. On the way to coordinatizing V_p^{mn} we say that if V is any representation of $\mathcal{A}_r$ then

$$(5.5) \qquad Inv(V) = \{v \in V \mid Xv = 0, Yv = 0, Kv = v\}.$$

There is a natural identification of $Inv(\underline{p} \otimes \underline{n}^* \otimes \underline{m}^*)$ with V_p^{mn}. If

$$(5.6) \qquad \alpha = \sum \alpha_k^{i,j} e_k \otimes e^j \otimes e^i \in Inv(\underline{p} \otimes \underline{n}^* \otimes \underline{m}^*),$$

then

$$\alpha(v \otimes w) = \sum e^i(v) e^j(w) \alpha_k^{i,j} e_k.$$

Since $\alpha \in Inv(\underline{p} \otimes \underline{n}^* \otimes \underline{m}^*)$ we have that $K\alpha = \alpha$. This implies that if $i + j \neq k$ then $\alpha_k^{i,j} = 0$. Hence we can write

$$(5.7) \qquad \alpha = \sum_{i,j} \alpha^{i,j} e_{i+j} \otimes e^j \otimes e^i.$$

Similarly there is an identification between $Inv(\underline{m} \otimes \underline{n} \otimes \underline{p}^*)$ and $V^p_{mn} = Hom_{\mathcal{A}_r}(\underline{p}, \underline{m} \otimes \underline{n})$. We write

$$\beta = \sum_{i,j} \beta_{i,j} e_i \otimes e_j \otimes e^{i+j} \in Inv(\underline{m} \otimes \underline{n} \otimes \underline{p}^*) \tag{5.8}$$

and if $v \in \underline{p}$ then

$$\beta(v) = \sum_{i,j} e^{i+j}(v) e_i \otimes e_j.$$

Our next goal is to prove the following :

Theorem 5.2. *The trace pairing*

$$\langle \cdot, \cdot \rangle_t : V_p^{mn} \otimes V_p^{mn} \to \mathbb{C}$$

*defined by equation (*3.18*) is positive definite.*

We will get a formula for $\langle \alpha, \alpha \rangle_t$, given that $\alpha^*(e_p) = \sum \beta_{i,j} e_i \otimes e_j$, and use it to prove that the trace pairing is positive definite.

The first step is to find formulas for the adjoints of $\alpha \in V_p^{mn}$ and $\beta \in V^p_{mn}$.

Proposition 5.3. *If*

$$\alpha = \sum_{i,j} \alpha^{i,j} e_{i+j} \otimes e^j \otimes e^i$$

then

$$\alpha^* = (-1)^{p-m-n} \sum_{i,j} \frac{\begin{bmatrix} 2p \\ p-i-j \end{bmatrix}}{\begin{bmatrix} 2m \\ m-i \end{bmatrix} \begin{bmatrix} 2n \\ n-j \end{bmatrix}} \bar{\alpha}^{-i,-j}\; e_i \otimes e_j \otimes e^{i+j},$$

where $\bar{\alpha}^{-i,-j}$ denotes the complex conjugate of $\alpha^{-i,-j}$. Also if

$$\beta = \sum_{i,j} \beta_{i,j} e_i \otimes e_j \otimes e^{i+j}$$

then

$$\beta^* = (-1)^{m+n-p} \sum_{i,j} \frac{\begin{bmatrix} 2m \\ m-i \end{bmatrix} \begin{bmatrix} 2n \\ n-j \end{bmatrix}}{\begin{bmatrix} 2p \\ p-i-j \end{bmatrix}} \bar{\beta}_{-i,-j}\; e_{i+j} \otimes e^j \otimes e^i.$$

PROOF

Analyze the consequences of

$$(\alpha(e_i \otimes e_j), e_{-i-j}) = (e_i \otimes e_j, \alpha^*(e_{-i-j})) \tag{5.9}$$

and

$$(\beta(e_{i+j}), e_{-i} \otimes e_{-j}) = (e_{i+j}, \beta^* (e_{-i} \otimes e_{-j})) . \tag{5.10}$$

A direct effect of the formulas from Proposition 5.3 is that if $\alpha^* = \beta \in V^p_{mn}$ then

$$\begin{aligned} \langle \alpha, \alpha \rangle_t e_p &= \alpha \circ \alpha^*(e_p) = \beta^* \circ \beta(e_p) = \\ &= (-1)^{m+n-p} \sum_{i+j=p} \beta_{i,j} \bar{\beta}_{-i,-j} \begin{bmatrix} 2m \\ m-i \end{bmatrix} \begin{bmatrix} 2n \\ n-j \end{bmatrix} e_p. \end{aligned} \tag{5.11}$$

To get a formula for $\langle \alpha, \alpha \rangle_t$ in terms of $\beta(e_p)$ we need to establish a functional relationship between $\beta(e_p)$ and $\beta(e_{-p})$.

Let $P \leq \underline{m} \otimes \underline{n}$, with P isomorphic to $\underline{p}$. The vector space P is spanned by the translates under Y of the kernel of X on $\langle e_a \otimes e_b \rangle_{a+b=p}$ (where the brackets denote the span of the list of vectors). Similarly, P is spanned by the translates under X of the kernel of the action of Y on $\langle e_{-a} \otimes e_{-b} \rangle_{a+b=p}$. Recall that

$$\begin{aligned} X(e_i \otimes e_j) &= (X \otimes K + \bar{K} \otimes X)(e_i \otimes e_j) = \\ &[m+i+1]s^j e_{i+1} \otimes e_j + [n+j+1]s^{-i} e_i \otimes e_{j+1}. \end{aligned} \tag{5.12}$$

Using (5.12) we find that if $\sum_{a+b=p} \gamma_a e_a \otimes e_b$ is in the kernel of X then

$$\gamma_a = -s^{p+1} \frac{[m+a]}{[n+b+1]} \gamma_{a-1}.$$

Hence the kernel of X on $\langle e_a \otimes e_b \rangle_{a+b=p}$ is spanned by

$$\sum_{a,b} (-1)^a s^{a(p+1)} [m+a]! [n+b]! e_a \otimes e_b. \tag{5.13}$$

Similarly the kernel of Y on $\langle e_{-a} \otimes e_{-b} \rangle_{a+b=p}$ is spanned by

$$\sum_{a,b} (-1)^a s^{-a(p+1)} [m+a]! [n+b]! e_{-a} \otimes e_{-b}. \tag{5.14}$$

Hence if $\alpha^* = \beta \in V^p_{mn}$ has

$$\beta(e_p) = \sum_{a,b} (-1)^a s^{a(p+1)} [m+a]![n+b]! e_a \otimes e_b \tag{5.15}$$

then, for some μ,

$$\beta(e_{-p}) = \mu \sum_{a,b} (-1)^{-a} s^{-a(p+1)} [m+a]![n+b]! e_{-a} \otimes e_{-b}. \tag{5.16}$$

In order to compute $\langle \alpha, \alpha \rangle_t$ we need to calculate μ. Since β is an intertwiner, we have

$$\begin{aligned} \frac{1}{[p+m-n]!} Y^{p+m-n} \beta(e_p) &= \beta\left(\frac{1}{[p+m-n]!} Y^{p+m-n} e_p\right) = \beta(e_{-m+n}) \\ &= \beta\left(\frac{1}{[p+n-m]!} X^{p+n-m} e_{-p}\right) = \frac{1}{[p+n-m]!} X^{p+n-m} \beta(e_{-p}). \end{aligned} \tag{5.17}$$

To find μ we only need to compare the coefficients of $e_{-m} \otimes e_n$ on the extreme left and right hand sides of the equation (5.17). We find that

$$\mu = (-1)^{m-n+p} s^{(p-n+m)(p+1)+(m-n)(m+n-p)} = (-1)^{m-n+p} s^{p(p+1)+m(m+1)-n(n+1)}. \tag{5.18}$$

Equipped with μ we compute the trace norm of $\alpha = \beta^*$, by plugging the appropriate quantities into our formula for $\langle \alpha, \alpha \rangle_t$, recognizing the sum as containing ${}_2\Phi_1(m+p, -n+1, p-m-n; -2n; q, q)$, and evaluating using proposition 5.1. We get

$$\langle \alpha, \alpha \rangle_t = \frac{[2m]![2n]![m+p-n]![n-m+p]![m+n+p+1]!}{[m+n-p]![2p+1]!} \tag{5.19}$$

Let us define

$$\nu_p^{m,n} = \sqrt{\frac{[m+n-p]![2p+1]!}{[2m]![2n]![m+p-n]![n-m+p]![m+n+p+1]!}} \tag{5.20}$$

and let

$$\beta^p_{m,n} = (-1)^{p(p+n)} \nu_p^{m,n} t^{n(n+1)-m(m+1)-p(p+1)} \beta, \tag{5.21}$$

and

$$\beta_p^{m,n} = (-1)^{p(p+n)}\nu_p^{m,n} t^{p(p+1)+m(m+1)-n(n+1)}\beta^*. \tag{5.22}$$

These coincide with (3.46) and (3.47).

Define the *Clebsch-Gordan coefficients* $C_{a,b}^{m,n,p}$ by

$$\beta^p_{m,n} = \sum_{a,b} C_{a,b}^{m,n,p} e_a \otimes e_b \otimes e^{a+b}. \tag{5.23}$$

With our choice of $\beta^p_{m,n}$ it is easy to see that

$$C_{-a,-b}^{m,n,p} = (-1)^{m-n+p}\bar{C}_{a,b}^{m,n,p}, \tag{5.24}$$

where the bar means the complex conjugate.

Since m, n, $p < \frac{r-1}{2}$, and $p+m+n+2 \leq r$, and $|m-n| \leq p \leq m+n$, all the quantities appearing in $\nu_p^{m,n}$ are positive. Thus the trace norm is positive definite and we proved Theorem 5.2.

Computing the Clebsch-Gordan Coefficients

Our next goal is to derive a complete formula for $\beta^p_{m,n}$. We follow the technique given in [V-K]. We start from

$$\beta^p_{m,n}(e_p) = (-1)^{p(p+n)}\nu_p^{m,n} t^{n(n+1)-m(m+1)-p(p+1)} \sum_{a+b=p} (-1)^a s^{a(p+1)}[m+a]![n+b]! e_a \otimes e_b. \tag{5.25}$$

Since $\beta^p_{m,n} : \underline{p} \to \underline{m} \otimes \underline{n}$ is an intertwiner we can use the action of Y to compute $\beta^p_{m,n}(e_c) = \beta^p_{m,n}\left(\frac{Y^{p-c}}{[p-c]!}e_p\right) = \frac{\Delta(Y)^{p-c}}{[p-c]!}\beta^p_{m,n}(e_p)$ for any $-p \leq c \leq p$.

After simplifying we obtain

$$C_{f,g}^{m,n,p} = (-1)^{p(p+n)}\nu_p^{m,n} t^{n(n+1)-m(m+1)-p(p+1)}[m+f]![n+g]! \sum_{j=0}^{p-f-g} (-1)^{p-g-j} s^{j(g-p+j)+(p-f-g-j)(g+j)+(p-g-j)(p+1)} \frac{\prod_{x=1}^{p-f-g-j}[m+f+x]\prod_{x=1}^{p-f-g-j}[m-f-x+1]\prod_{x=1}^{j}[n+g+x]\prod_{x=1}^{j}[n-g-x+1]}{[j]![p-f-g-j]!}. \tag{5.26}$$

Notice that we can replace m in (5.26) by any real number μ near m, and the formula still makes sense. Furthermore, the limit of the values of this formula as μ approaches m is equal to the value with m substituted in. This allows us to rewrite (5.26) as:

$$(5.27)\quad C^{m,n,p}_{f,g}(\mu) = \nu^{m,n}_{p} t^{n(n+1)-m(m+1)-p(p+1)}[m+f]![n+g]!$$
$$(-1)^{p(p+n)+p-g} s^{(p-f-g)g+(p-g)(p+1)} \frac{\prod_{x=1}^{p-f-g}[\mu+f+x]\prod_{x=1}^{p-f-g}[\mu-f-x+1]}{[p-f-g]!}$$
$${}_3\Phi_2\left(n+g+1, g-n, f+g-p; -\mu-p+g, \mu-p+g+1; q, q^{-p-f-g}\right).$$

Next we apply the following Sear's identity (see [G-R])

$$(5.28)\quad {}_3\Phi_2\left(-n, a, b; d, e; q, q^{d+e+n-a-b}\right) =$$
$$= \frac{(q^{e-a};q)_n}{(q^e;q)_n} \cdot {}_3\Phi_2\left(-n, a, d-b; d, a+1-n-e; q, q\right).$$

We make the following assignations in 5.28:

$$(5.29)\quad -n := f+g-p, \quad a := g-n, \quad b := n+g+1,$$
$$e := \mu-p+g+1, \quad d := -\mu-p+g.$$

After simplifying we take the limit as $\mu \to m$. The final answer is:

Proposition 5.4.

$$(5.30)\quad C^{m,n,p}_{f,g} = (-1)^{pn-g}\nu^{m,n}_{p} t^{n(n+1)-m(m+1)-p(p+1)} s^{(p-f-g)n+(p-g)(p+1)}$$
$$\cdot \frac{[m+p-g]![m+n-f-g]![n+g]!}{[p-f-g]![m+n-p]!}$$
$${}_3\Phi_2\left(f+g-p, g-n, -m-p-n-1; -m-p+g, f+g-m-n; q, q\right)$$

Computing the $6j$-symbols

The next piece of basic data we wish to describe is the map F. Once again we follow the technique given in [V-K]. Recall the map

$$N\begin{bmatrix} j & k \\ i & l \end{bmatrix} : \bigoplus_u V_i^{ju} \otimes V_u^{kl} \to \bigoplus_v V_i^{vl} \otimes V_v^{jk} \tag{5.31}$$

defined by requiring

$$\left\langle N\begin{bmatrix} j & k \\ i & l \end{bmatrix}\left(\beta_i^{j,u} \otimes \beta_u^{k,l}\right), \beta_i^{v,l} \otimes \beta_v^{j,k}\right\rangle_t 1_{\underline{i}} = \tag{5.32}$$

$$= \beta_i^{j,u} \circ \left(1_{\underline{j}} \otimes \beta_u^{k,l}\right)\left(\beta_{j,k}^{v} \otimes 1_{\underline{l}}\right)\beta_{v,l}^{i} = N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix} 1_{\underline{i}}$$

(see equations (3.58) and (3.59)).

The map $N\begin{bmatrix} j & k \\ i & l \end{bmatrix}$ is closely related to F and yet it is easier to compute. We will derive a formula for the coefficients $N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix}$ in terms of the Clebsch-Gordan coefficients.

Let

$$\mathcal{C} = \frac{(tK - \bar{t}\bar{K})^2}{(s - \bar{s})^2} + YX. \tag{5.33}$$

Notice that $\mathcal{C}$ is central in $\mathcal{A}_r$ and therefore acts as an intertwiner of any representation with itself. If the representation is irreducible, the intertwiner must be a scalar multiplication. Indeed, $\mathcal{C}$ acts on $\underline{i}$ as multiplication by $[i+\frac{1}{2}]^2$. In a tensor product $\underline{j}\otimes\underline{k}\otimes\underline{l}$, the eigenspace of $\mathcal{C} : \underline{j}\otimes\underline{k}\otimes\underline{l} \to \underline{j}\otimes\underline{k}\otimes\underline{l}$ corresponding to the eigenvalue $[i + \frac{1}{2}]^2$ is the unique maximal subspace of $\underline{j} \otimes \underline{k} \otimes \underline{l}$ that is isomorphic to a direct sum of copies of $\underline{i}$. Notice that if $v \in \underline{j} \otimes \underline{k} \otimes \underline{l}$ lives in a summand isomorphic to $\underline{i}$ then v lives in the eigenspace of $[i + \frac{1}{2}]^2$. Call that eigenspace I. We have two different bases for I: $\left\langle \xi_a^{v,l,i} \right\rangle_{v,a}$ and $\left\langle \zeta_a^{j,u,i} \right\rangle_{u,a}$, where

$$\xi_a^{v,l,i} = \left(\beta_{j,k}^{v} \otimes 1_{\underline{l}}\right) \circ \beta_{v,l}^{i}(e_a) \tag{5.34}$$

and

$$\zeta_a^{j,u,i} = \left(1_{\underline{j}} \otimes \beta_{k,l}^{u}\right) \circ \beta_{j,u}^{i}(e_a). \tag{5.35}$$

Notice that

$$(5.36)\quad \left\langle \xi_a^{v,l,i}, \zeta_{-a}^{j,u,i} \right\rangle_t = \left\langle \left(\beta_{j,k}^v \otimes 1_{\underline{l}}\right) \beta_{v,l}^i(e_a), \left(1_{\underline{j}} \otimes \beta_{k,l}^u\right) \beta_{j,u}^i(e_{-a}) \right\rangle_t$$

$$= \left\langle \beta_i^{j,u} \circ \left(1_{\underline{j}} \otimes \beta_u^{k,l}\right) \circ \left(\beta_{j,k}^v \otimes 1_{\underline{l}}\right) \left(\beta_i^{v,l}(e_a)\right), e_{-a} \right\rangle_t = N_{u,v} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \langle e_a, e_{-a} \rangle_t.$$

So

$$(5.37)\quad \xi_a^{v,l,i} = \sum_u N_{u,v} \begin{bmatrix} j & k \\ i & l \end{bmatrix} \zeta_a^{j,u,i}.$$

Consider now the following maps.

$$(5.38)\quad \sum_u \left(1_{\underline{j}} \otimes \beta_{k,l}^u\right) \circ \beta_{j,u}^i \circ \beta_i^{j,u} \circ \left(1_{\underline{j}} \otimes \beta_u^{k,l}\right) : \underline{j} \otimes \underline{k} \otimes \underline{l} \to \underline{j} \otimes \underline{k} \otimes \underline{l}$$

and

$$(5.39)\quad \sum_v \left(\beta_{j,k}^v \otimes 1_{\underline{l}}\right) \circ \beta_{v,l}^i \circ \beta_i^{v,l} \circ \left(\beta_v^{j,k} \otimes 1_{\underline{l}}\right) : \underline{j} \otimes \underline{k} \otimes \underline{l} \to \underline{j} \otimes \underline{k} \otimes \underline{l}.$$

Both maps are self adjoint idempotents with the same image I. Since the two maps are self adjoint, they have the same kernel $I^{\perp}$. This implies that they are equal. We call this map

$$(5.40)\quad \pi_I : \underline{j} \otimes \underline{k} \otimes \underline{l} \to \underline{j} \otimes \underline{k} \otimes \underline{l}.$$

We compute $\pi_I(e_a \otimes e_b \otimes e_c)$ using both formulas (5.38) and (5.39). Since π_I is an orthogonal projection, we have

$$(5.41)\quad \pi_I(e_a \otimes e_b \otimes e_c) = \sum_u \frac{\left\langle e_a \otimes e_b \otimes e_c, \zeta_{-a-b-c}^{j,u,i} \right\rangle_t}{\left\langle \zeta_{a+b+c}^{j,u,i}, \zeta_{-a-b-c}^{j,u,i} \right\rangle_t} \zeta_{a+b+c}^{j,u,i}$$

$$= \sum_v \frac{\left\langle e_a \otimes e_b \otimes e_c, \xi_{-a-b-c}^{v,l,i} \right\rangle_t}{\left\langle \xi_{a+b+c}^{v,l,i}, \xi_{-a-b-c}^{v,l,i} \right\rangle_t} \xi_{a+b+c}^{v,l,i}.$$

Since

$$(5.42)\quad \left\langle \zeta_{a+b+c}^{j,u,i}, \zeta_{-a-b-c}^{j,u,i} \right\rangle_t = \langle e_{a+b+c}, e_{a-b-c} \rangle_t = \left\langle \xi_{a+b+c}^{v,l,i}, \xi_{-a-b-c}^{v,l,i} \right\rangle_t$$

we can cancel these to get

(5.43)
$$\sum_u \left\langle e_a \otimes e_b \otimes e_c, \zeta^{j,u,i}_{-a-b-c}\right\rangle_t \zeta^{j,u,i}_{a+b+c} = \sum_v \left\langle e_a \otimes e_b \otimes e_c, \xi^{v,l,i}_{-a-b-c}\right\rangle_t \xi^{v,l,i}_{a+b+c}.$$

Next we use equation (5.37) to get

$$(5.44)\quad \sum_u \left\langle e_a \otimes e_b \otimes e_c, \zeta^{j,u,i}_{-a-b-c}\right\rangle_t \zeta^{j,u,i}_{a+b+c} =$$
$$\sum_{v,u} \left\langle e_a \otimes e_b \otimes e_c, \xi^{v,l,i}_{-a-b-c}\right\rangle_t N_{u,v} \left[\begin{array}{cc} j & k \\ i & l \end{array}\right] \zeta^{j,u,i}_{a+b+c}.$$

Finally, isolating the coefficients of $\zeta^{j,u,i}_{a+b+c}$, we have

(5.45)
$$\left\langle e_a \otimes e_b \otimes e_c, \zeta^{j,u,i}_{-a-b-c}\right\rangle_t = \sum_v \left\langle e_a \otimes e_b \otimes e_c, \xi^{v,l,i}_{-a-b-c}\right\rangle_t N_{u,v} \left[\begin{array}{cc} j & k \\ i & l \end{array}\right].$$

From the definition of the Clebsch-Gordan coefficients

$$(5.46)\qquad \xi^{v,l,i}_{-a-b-c} = \sum_{g+h+f=-a-b-c} C^{j,k,v}_{g,h} C^{v,l,i}_{g+h,f} e_g \otimes e_h \otimes e_f.$$

So

$$(5.47)\quad \left\langle e_a \otimes e_b \otimes e_c, \xi^{v,l,i}_{-a-b-c}\right\rangle_t =$$
$$= (-1)^{j+k+l} \left[\begin{array}{c} 2j \\ j-a \end{array}\right] \left[\begin{array}{c} 2k \\ k-b \end{array}\right] \left[\begin{array}{c} 2l \\ l-c \end{array}\right] (-s)^{a+b+c} \bar{C}^{j,k,v}_{-a,-b} \bar{C}^{v,l,i}_{-a-b,-c}.$$

Similarly

$$(5.48)\quad \zeta^{j,u,i}_{-a-b-c} = \left(1_{\underline{j}} \otimes \beta^u_{k,l}\right) \beta^i_{j,u}(e_{-a-b-c}) =$$
$$= \sum_{d+f+g=-a-b-c} C^{j,u,i}_{g,d+f} C^{k,l,u}_{d,f} e_g \otimes e_d \otimes e_f,$$

so

$$(5.49)\quad \left\langle e_a \otimes e_b \otimes e_c, \zeta^{j,u,i}_{-a-b-c}\right\rangle_t =$$
$$(-1)^{j+k+l} \left[\begin{array}{c} 2j \\ j-a \end{array}\right] \left[\begin{array}{c} 2k \\ k-b \end{array}\right] \left[\begin{array}{c} 2l \\ l-c \end{array}\right] (-s)^{a+b+c} \bar{C}^{j,u,i}_{-a,-b-c} \bar{C}^{k,l,u}_{-b,-c}.$$

Hence equation (5.45) becomes

(5.50)
$$\bar{C}^{j,u,i}_{-a,-b-c}\bar{C}^{k,l,u}_{-b,-c} = \sum_v C^{j,k,v}_{-a,-b}C^{v,l,i}_{-a-b,-c}N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix}.$$

Multiplying both sides of (5.50) by

$$(-1)^{j+k-h}\frac{\begin{bmatrix} 2j \\ j-a \end{bmatrix}\begin{bmatrix} 2k \\ k-b \end{bmatrix}}{\begin{bmatrix} 2h \\ h-a-b \end{bmatrix}}C^{j,k,h}_{a,b}$$

and summing over $a+b=$ constant we get

(5.51)
$$\sum_{a+b=const}(-1)^{j+k-h}\frac{\begin{bmatrix} 2j \\ j-a \end{bmatrix}\begin{bmatrix} 2k \\ k-b \end{bmatrix}}{\begin{bmatrix} 2h \\ h-a-b \end{bmatrix}}C^{j,k,h}_{a,b}\bar{C}^{j,u,i}_{-a,-b-c}\bar{C}^{k,l,u}_{-b,-c}$$

$$= \sum_{a+b=const}(-1)^{j+k-h}\frac{\begin{bmatrix} 2j \\ j-a \end{bmatrix}\begin{bmatrix} 2k \\ k-b \end{bmatrix}}{\begin{bmatrix} 2h \\ h-a-b \end{bmatrix}}C^{j,k,h}_{a,b}\bar{C}^{j,k,v}_{-a,-b}\bar{C}^{v,l,i}_{-a-b,-c}N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix}.$$

But since $\beta^{j,k}_h$ and $\beta^{j,k}_v$ have norm 1,

(5.52)
$$(-1)^{j+k-h}\sum_{a+b=d}C^{j,k,h}_{a,b}\bar{C}^{j,k,v}_{-a,-b}\frac{\begin{bmatrix} 2j \\ j-a \end{bmatrix}\begin{bmatrix} 2k \\ k-b \end{bmatrix}}{\begin{bmatrix} 2h \\ h-a-b \end{bmatrix}} = \delta_{h,v}$$

and we proved the following proposition.

Proposition 5.5.

(5.53)
$$\sum_{a+b=const}(-1)^{j+k-v}\frac{\begin{bmatrix} 2j \\ j-a \end{bmatrix}\begin{bmatrix} 2k \\ k-b \end{bmatrix}}{\begin{bmatrix} 2v \\ v-a-b \end{bmatrix}}C^{j,k,v}_{a,b}\bar{C}^{j,u,i}_{-a,-b-c}\bar{C}^{k,l,u}_{-b,-c} =$$

$$= \bar{C}^{v,l,i}_{-a-b,-c}N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix}$$

Recall (equation 5.24) that

$$C^{m,n,p}_{-a,-b} = (-1)^{m-n+p}\bar{C}^{m,n,p}_{a,b}.$$

Hence (5.53) becomes

$$(5.54)\quad (-1)^{l-v-i}C^{v,l,i}_{a+b,c}N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix} =$$

$$= (-1)^{l-k-j-i}\sum_{a+b=const}(-1)^{j+k-v}\frac{\begin{bmatrix} 2j \\ j-a \end{bmatrix}\begin{bmatrix} 2k \\ k-b \end{bmatrix}}{\begin{bmatrix} 2v \\ v-a-b \end{bmatrix}}C^{j,k,v}_{a,b}C^{k,l,u}_{b,c}C^{j,u,i}_{a,b+c}.$$

We specialize this formula by assuming that $a+b=v$ and $c=i-v$.

Proposition 5.6.

$$(5.55)\quad C^{v,l,i}_{v,i-v}N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix} =$$

$$= \sum_{a+b=v}\begin{bmatrix} 2j \\ j-a \end{bmatrix}\begin{bmatrix} 2k \\ k-b \end{bmatrix}C^{j,k,v}_{a,b}C^{j,u,i}_{a,b+i-v}C^{k,l,u}_{b,i-v}$$

We expand the right hand side (call it RHS) of (5.55) using equation (5.27) for $C^{k,l,u}_{b,i-v}$. We isolate terms to recognize ${}_2\Phi_1$ and simplify the answer using the identity from proposition 5.1. Then we use the identities from fact 10 to recognize a formula involving${}_4\Phi_3$.

$$(5.56)\quad RHS = (-1)^{vk+ui+ul}\nu^{j,k}_{v}\nu^{j,u}_{i}\nu^{k,l}_{u}t^{l(l+1)-v(v+1)-i(i+1)-2j(j+1)}[2j]![2k]!$$

$$\frac{[k+u-i+v]![l+i-v]![2j]![u+i-j]!}{[k+i-u-v]![u-i+j]!}\frac{\left(q^{-j-u-i-1};q\right)_{j+u-i}}{(q^{-2j};q)_{j+u-i}}$$

$$s^{(u-i+v)(u+1)+j(v+i+2)+(j+u-i)(i-v)}\;{}_4\Phi_3\left(l+i-v+1, i-l-v, i+j-u+1,\right.$$
$$\left. i-u-j; i-v-k-u, k+i-u-v+1, 2i+2; q, q\right).$$

Note that in case when $k+i-u-v<0$ we interpret $\frac{1}{[k+i-u-v]!}$ as $\prod_{x=1}^{u+v-i-k}[(\kappa-k)-x]$, where $\kappa\in R$ is close to k.

Now apply the following version of Sear's identity (see [G-R], equation 3.21):

(5.57) $${}_4\Phi_3\left(-n,a,b,c;d,e,a+b+c-n+1-d-e;q,q\right)=$$
$$\frac{(q^{e-a};q)_n\,(q^{d+e-b-c};q)_n}{(q^e;q)_n\,(q^{d+e-a-b-c};q)_n}\,{}_4\Phi_3\left(-n,a,d-b,d-c;d,d+e-b-c,a+1-n-e;q,q\right),$$

where we let:

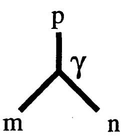
(5.58)
$$-n:=i-j-u,\quad a:=i-v-l,\quad b:=l+i-v+1,\quad c:=i+j-u+1,$$
$$d:=i-v-k-u,\quad e:=2i+1.$$

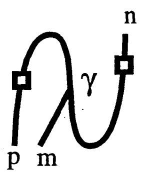
The final answer is:

Proposition 5.7.

(5.59) $$N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix}=(-1)^{v(v+k)+l(u+i)+i(u+1)}\frac{\nu_v^{j,k}\nu_u^{k,l}}{\nu_i^{v,l}\nu_i^{u,j}}$$
$$\cdot\frac{[2k]![2i+1]![k+u+v-i]![j+k+l-i]![u+v+l+j+1]!}{[2u]![2v]![j+u+i+1]![k+l-u]![j+k-v]![i+l+v+1]!}$$
$${}_4\Phi_3\left(i-l-v,i-u-j,-k-l-u-1,-k-j-v-1;i-v-k-u,\right.$$
$$\left.-u-v-l-j-1,i-j-l-k;q,q\right).$$

Recall that $R:V_p^{mn}\to V_n^{pm}$ is given by

(5.60) $$R\left(\;\right)$$ 

$$=\frac{\sqrt{[2n+1]}}{\sqrt{[2p+1]}}$$ 

.

We wish to compute the matrix coefficient of R with respect to the bases $\beta_p^{m,n}$ and $\beta_n^{p,m}$. To do this we need to know some values of $\beta_p^{m,n}$ and $\beta_n^{p,m}$. Let $g^{a,b}$ be the coefficient of e_{a+b} in $\beta_p^{m,n}(e_a\otimes e_b)$, and let $h^{a,b}$ be the coefficient of e_{a+b} in $\beta_n^{p,m}(e_a\otimes e_b)$. We need to compute $g^{-m,m-p}$ and $h^{p,-m}$. The first follows directly from the formula for $\beta_{m,n}^p$ (see (5.25)) , the fact that $\beta_p^{m,n}=\beta_{m,n}^{p^*}$ and from proposition 5.3. In fact

(5.61)
$$g^{-m,m-p}=(-1)^{n-p+p(p+n)}\nu_p^{m,n}\frac{[2m]![2n]!}{[m+n-p]!}s^{-m(p+1)}t^{m(m+1)-n(n+1)+p(p+1)}.$$

To compute $h^{p,-m}$ first we note that

(5.62)
$$h^{m-n,-m} = (-1)^{p+n(n+m)}\nu_n^{p,m}\frac{[2p]![2m]!}{[p+m-n]!}s^{(m-n)(n+1)}t^{n(n+1)-m(m+1)+p(p+1)}.$$

Then observe that

(5.63)
$$[p+n-m]!s^{-m(p+n-m)}h^{m-n,-m} = \left(\beta_n^{p,m}\left(Y^{p+n-m}(e_p\otimes e_{-m})\right)\right)_{-n} =$$
$$= \left(Y^{p+n-m}\beta_n^{p,m}(e_p\otimes e_{-m})\right)_{-n} = \prod_{x=1}^{p+n-m}[n-p+m+x]h^{p,-m} = \frac{[2n]!}{[m+n-p]!}h^{p,-m},$$

where subscripted $-n$ denotes taking a coefficient of e_{-n}. Thus

$$(5.64)\quad h^{p,-m} = \frac{[m+n-p]![p+n-m]!}{[2n]!}s^{-m(p+n-m)}h^{m-n,-m} =$$
$$= (-1)^{p+n(n+m)}\nu_n^{p,m}\frac{[2p]![2m]!}{[p+m-n]!}\frac{[m+n-p]![p+n-m]!}{[2n]!}$$
$$s^{m(m+1)-n(n+1)-mp}t^{n(n+1)-m(m+1)+p(p+1)}.$$

Next we compute $R\left(\beta_p^{m,n}\right)(e_p\otimes e_{-m})$.

(5.65)
$$e_p\otimes e_{-m}\xrightarrow{D^{-1}\otimes Id}(-s)^pe^{-p}\otimes e_{-m}\xrightarrow{1\otimes N}\sum_k(-s)^pe^{-p}\otimes e_{-m}\otimes e_k\otimes e^k\xrightarrow{1\otimes\beta_p^{m,n}\otimes 1}$$
$$\sum_k(-s)^pg^{-m,k}e^{-p}\otimes e_{-m+k}\otimes e^k\xrightarrow{E\otimes 1}(-s)^pg^{-m,m-p}e^{m-p}\xrightarrow{D}$$
$$(-s)^pg^{-m,m-p}\begin{bmatrix}2n\\ n-m+p\end{bmatrix}^{-1}(-s)^{m-p}e_{p-m}\xrightarrow{\sqrt{\frac{[2n+1]}{[2p+1]}}}$$
$$(-s)^m\sqrt{\frac{[2n+1]}{[2p+1]}}\frac{[n-m+p]![n-p+m]!}{[2n]!}g^{-m,m-p}e_{p-m} =$$
$$= (-s)^m\sqrt{\frac{[2n+1]}{[2p+1]}}\frac{[n-m+p]![n-p+m]!}{[2n]!}(-1)^{n-p+p(p+n)}\nu_p^{m,n}\frac{[2m]![2n]!}{[m+n-p]!}$$
$$s^{-m(p+1)}t^{m(m+1)-n(n+1)+p(p+1)}e_{p-m} =$$
$$= (-1)^{m+n-p+p(p+n)}s^{-mp}t^{m(m+1)-n(n+1)+p(p+1)}[n+p-m]![2m]!\sqrt{\frac{[2n+1]}{[2p+1]}}\nu_p^{m,n}e_{p-m}$$

To find the matrix coefficient of R we divide the coefficient of e_{p-m} above by $h^{p,-m}$. This proves the following proposition.

Proposition 5.8. $R\left(\beta_p^{m,n}\right) = (-1)^{(m+p)(n+1)} \cdot \beta_n^{p,m}$.

From this we can derive the formula for L.

Corollary 5.9. $L\left(\beta_p^{m,n}\right) = (-1)^{(m+n)(p+1)} \beta_m^{n,p}$.

We defined the map $F\begin{bmatrix} j & k \\ i & l \end{bmatrix} : \bigoplus_p V_p^{ij} \otimes V_p^{kl} \to \bigoplus_q V_q^{li} \otimes V_q^{jk}$ as (see 3.67)

(5.66)
$$F\begin{bmatrix} j & k \\ i & l \end{bmatrix} = (L \otimes Id) \circ N \begin{bmatrix} j & k \\ i & l \end{bmatrix} \circ (L \otimes Id).$$

From Corollary 5.9 we immediately see that

(5.67)
$$F_{uv}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = (-1)^{(i+j)(u+1)+(v+l)(i+1)} N_{uv} \begin{bmatrix} j & k \\ i & l \end{bmatrix}.$$

The matrix coefficient of $B_{23} : V_p^{m,n} \longrightarrow V_p^{n,m}$ with respect to the bases $\beta_p^{m,n}$, $\beta_p^{n,m}$ is a straightforward computation. Recall

(5.68)
$$B_{23}(\gamma) =$$


So

(5.69)
$$B_{23}\left(\beta_p^{m,n}\right)(e_{m-p} \otimes e_{-m}) = (-1)^{n(p+1)} \nu_p^{m,n} \frac{[2m]![2n]!}{[m+n-p]!} q^{-m(m-p)} s^{-m(p+1)} \cdot t^{m(m+1)-n(n+1)+p(p+1)} e_{-p}$$

and

(5.70)
$$\beta_p^{n,m}(e_{m-p} \otimes e_{-m}) = (-1)^{n+p(p+m)} \nu_p^{n,m} \frac{[2m]![2n]!}{[m+n-p]!} s^{(m-p)(p+1)} t^{n(n+1)-m(m+1)+p(p+1)} e_{-p}.$$

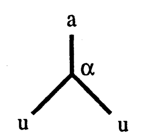
Taking the quotient of the coefficients and using the fact that $\nu_p^{m,n} = \nu_p^{n,m}$ we see the following proposition.

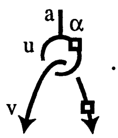
Proposition 5.10. $B_{23}\left(\beta_p^{m,n}\right) = (-1)^{p(m+n+p)} s^{p(p+1)-m(m+1)-n(n+1)} \cdot \beta_p^{n,m}$.

Recall the map $S_a : \bigoplus_u V_a^{uu} \to \bigoplus_v V_a^{vv}$ defined by equation (3.70):

(5.71)

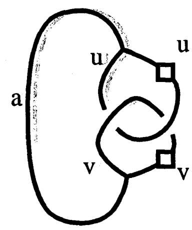
$$S_a\left(\quad \right) = \sum_q \frac{\sqrt{[2u+1][2v+1]}}{X} \quad .$$





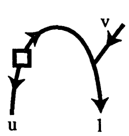
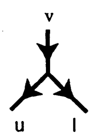
We need to compute the coefficient

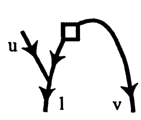
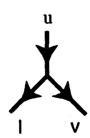
(5.72)
$$S_{uv}(a) = \frac{\sqrt{[2u+1][2v+1]}}{X[2a+1]} \quad .$$



To achieve this we will perform the sequence of changes on the tangle on the right hand side, and express it in terms of the coefficients of map N. We will use the following lemma in our computations.

Lemma 5.11.

(i) If  $= \tau_v^{u,l}$  *then* $\tau_v^{u,l} = \sqrt{\frac{[2l+1]}{[2v+1]}}$.

(ii) If  $= \sigma_u^{l,v}$  *then* $\sigma_u^{l,v} = (-1)^{(l+u)(v+1)} \sqrt{\frac{[2l+1]}{[2u+1]}}$.

PROOF To find τ we compare images of $e_{v-l} \otimes e_l$. On the left hand side:

$$e_{v-l} \otimes e_l \xrightarrow{D^{-1}\otimes 1} \begin{bmatrix} 2u \\ u-v+l \end{bmatrix} (-s)^{v-l} e^{l-v} \otimes e_l \xrightarrow{1\otimes \beta^l_{uv}} \tag{5.73}$$

$$\frac{[2u]!}{[u-v+l]![u+v-l]!}(-1)^{l(l+v)}(-s)^{v-l} t^{v(v+1)-u(u+1)-l(l+1)} \nu_l^{u,v} \sum_{i+j=l} (-1)^i s^{i(l+1)}$$

$$[u+i]![v+j]!\, e^{l-v} \otimes e_i \otimes e_j \xrightarrow{E\otimes 1} \frac{[2u]![2v]!}{[u+v-l]!}(-1)^{l(l+v)} \nu_l^{uv} s^{(l-v)l} t^{v(v+1)-u(u+1)-l(l+1)} e_v.$$

And on the right hand side

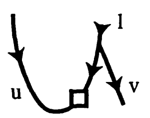
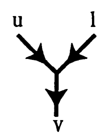
(5.74)

$$\beta_v^{ul}(e_{v-l} \otimes e_l) = \frac{[2u]![2l]!}{[u-v+l]!}(-1)^{v-l+v(v+l)} \nu_v^{ul} s^{(v-l)(v+1)} t^{l(l+1)-v(v+1)-u(u+1)}.$$

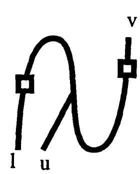
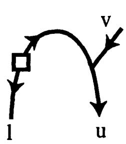
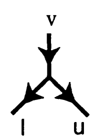
So

$$\tau_v^{u,l} = \sqrt{\frac{[2l+1]}{[2v+1]}}. \tag{5.75}$$

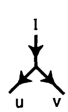
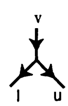
Taking adjoints we find the formula for $\bar{\tau} = \tau$:

(5.76)  $= \sqrt{\dfrac{[2l+1]}{[2v+1]}}$  .

To compute σ notice that

(5.77)  $= \bar{\sigma}$  $= \bar{\sigma}\tau$  .

Thus

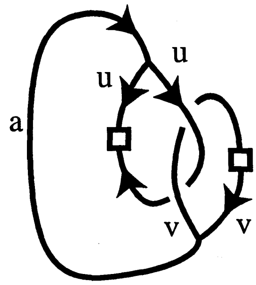
$\dfrac{\sqrt{[2l+1]}}{\sqrt{[2v+1]}} R \Big($  $\Big) = \bar{\sigma}\tau_v^{l,u}$  , so

(5.78)

$$\sigma = \bar{\sigma} = \frac{\sqrt{[2l+1]}}{\sqrt{[2v+1]}}(-1)^{(u+l)(v+1)}\frac{1}{\tau_v^{l,u}} = \frac{\sqrt{[2l+1]}}{\sqrt{[2v+1]}}(-1)^{u+v}(-1)^{(u+l)(v+1)}\frac{\sqrt{[2v+1]}}{\sqrt{[2u+1]}}.$$

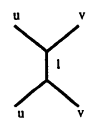
Equation (5.72) is equivalent to:

(5.79)

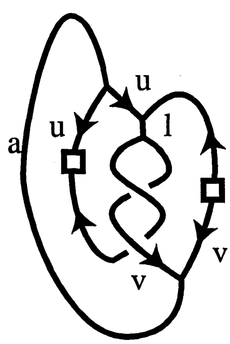
$$S_{uv}(a) = \frac{\sqrt{[2u+1][2v+1]}}{X[2a+1]}$$ 

.

Since our β's have norm 1 we can replace 

by $\sum_l$ 

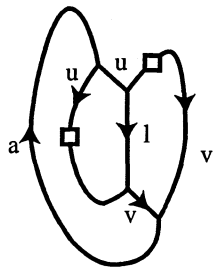
in any closed diagram. We obtain:

(5.80) $$\frac{\sqrt{[2u+1][2v+1]}}{X[2a+1]}\sum_l$$ 

.

Since $B_{23}^{-2}(\beta_l^{uv}) = s^{2u(u+1)+2v(v+1)-2l(l+1)}\beta_l^{uv}$ (by proposition 5.10) we get:

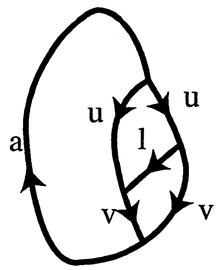
(5.81)

$$\frac{\sqrt{[2u+1][2v+1]}}{X[2a+1]}\sum_l s^{2u(u+1)+2v(v+1)-2l(l+1)}$$ 

.

By lemma 5.11 this is

(5.82)

$$\frac{\sqrt{[2u+1][2v+1]}}{X[2a+1]}\sum_l s^{2u(u+1)+2v(v+1)-2l(l+1)}\sigma_u^{l,v}\bar{\tau}_v^{u,l}$$ 

$$= \frac{\sqrt{[2u+1][2v+1]}}{X}\sum_l s^{2u(u+1)+2v(v+1)-2l(l+1)}\sigma_u^{l,v}\bar{\tau}_v^{u,l}N_{uv}\left[\begin{array}{cc} u & l \\ a & v \end{array}\right]$$

$$= \frac{\sqrt{[2u+1][2v+1]}}{X}\sum_l s^{2u(u+1)+2v(v+1)-2l(l+1)}\frac{\sqrt{[2l+1]}}{\sqrt{[2u+1]}} \cdot \frac{\sqrt{[2l+1]}}{\sqrt{[2v+1]}}(-1)^{(l+u)(v+1)}N_{uv}\left[\begin{array}{cc} u & l \\ a & v \end{array}\right].$$

We proved the following:

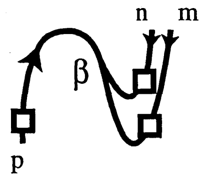
Proposition 5.12.

(5.83) $$S_{uv}(a) = (-1)^{u(v+1)}\frac{1}{X}q^{u(u+1)+v(v+1)} \sum_l (-1)^{l(v+1)}q^{-l(l+1)}[2l+1]N_{uv}\left[\begin{array}{cc} u & l \\ a & v \end{array}\right].$$

Lemma 5.11 is also useful in computing map $\psi : V_p^{mn} \to V_{nm}^p$ defined by equation (3.34):

(5.84)

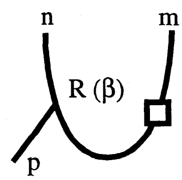
$$\psi\left(\beta_p^{m,n}\right) = \frac{1}{\sqrt{[2p+1][2m+1][2n+1]}}$$



Decomposing ψ in terms of R and using lemma 5.11 we get:

(5.85)

$$\psi\left(\beta_p^{m,n}\right) = \frac{1}{\sqrt{[2p+1][2m+1][2n+1]}} \frac{\sqrt{[2p+1]}}{\sqrt{[2n+1]}}$$



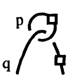
$$= \frac{\bar{\sigma}_p^{n,m}(-1)^{(m+p)(n+1)}}{[2n+1]\sqrt{[2m+1]}} \cdot \beta_{n,m}^p =$$

$$(-1)^{(m+p)(n+1)+(n+p)(m+1)} \frac{1}{[2n+1]\sqrt{[2m+1]}} \frac{\sqrt{[2n+1]}}{\sqrt{[2p+1]}} \beta_{n,m}^p =$$

$$(-1)^{(m+n)(p+1)} \frac{1}{\sqrt{[2m+1][2n+1][2p+1]}} \cdot \beta_{n,m}^p.$$

Finally recall the map $S : \bigoplus_{x\in\mathcal{L}} V(A,x,x) \to \bigoplus_{y\in\mathcal{L}} V(A,y,y)$ given by the equation (3.71)

(5.86) $$S\left(\frac{1}{\sqrt{[2p+1]}}\right.$$ 

$$\left.\right) = \sum_q \frac{\sqrt{[2q+1]}}{X}$$ 

From proposition 3.8 we immediately obtain:

(5.87) $$S\left(\beta_{pp}\right) = \sum_q \frac{[(2q+1)(2p+1)]}{X} \beta_{qq}.$$

6. Formulas.

We conclude with the list of formulas.

- Constant X:

$$X = \frac{\sqrt{r}}{2\sin\frac{\pi}{r}}$$

- Morphism C:

$$C =$$  $$= \frac{t^{-3}}{\sqrt{2}} e^{\frac{3\pi i}{4}}.$$

- Map S (equation 5.87):

$$S\left(\beta_{pp}\right) = \sum_{q} \frac{[(2q+1)(2p+1)]}{X} \beta_{qq}.$$

- Basis β^{p}_{mn} (equations 5.25, 5.23 and 5.4):

$$\beta^{p}_{m,n}(e_p) = (-1)^{p(p+n)} \nu_p^{m,n} t^{n(n+1)-m(m+1)-p(p+1)} \sum_{a+b=p} (-1)^a s^{a(p+1)} [m+a]![n+b]! e_a \otimes e_b$$

where $\nu_p^{m,n} = \sqrt{\dfrac{[m+n-p]![2p+1]!}{[2n]![2m]![p-n+m]![p+n-m]![p+n+m+1]!}}$.

If $\beta^{p}_{m,n} = \sum_{a,b} C^{m,n,p}_{a,b} e_a \otimes e_b \otimes e^{a+b}$ then:

$$C^{m,n,p}_{f,g} = \nu_p^{m,n} t^{n(n+1)-m(m+1)-p(p+1)} (-1)^{pn-g} s^{(p-f-g)n+(p-g)(p+1)}$$
$$\frac{[m+p-g]![m+n-f-g]![n+g]!}{[p-f-g]![m+n-p]!}$$
$${}_3\Phi_2(f+g-p, g-n, -m-p-n-1; -m-p+g, f+g-m-n; q, q).$$

- Map T (see proposition 3.8):

$$= q^{-j(j+1)}$$

.

- Map R (proposition 5.8):

$$R\left(\beta_p^{m,n}\right) = (-1)^{(m+p)(n+1)} \cdot \beta_n^{p,m}.$$

- Map L (corollary 5.9):

$$L\left(\beta_p^{m,n}\right) = (-1)^{(m+n)((p+1)}\beta_m^{n,p}.$$

- Map ψ (equation 5.85):

$$\psi\left(\beta_p^{m,n}\right) = (-1)^{(m+n)(p+1)}\frac{1}{\sqrt{[2m+1][2n+1][2p+1]}}\cdot\beta_{n,m}^{p}.$$

- Map B_{23} (proposition 5.10):

$$B_{23}\left(\beta_p^{m,n}\right) = (-1)^{p(m+n+p)}s^{p(p+1)-m(m+1)-n(n+1)}\cdot\beta_p^{n,m}.$$

- Map N (proposition 5.7):

$$N_{u,v}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = (-1)^{v(v+k)+l(u+i)+i(u+1)}\frac{\nu_v^{j,k}\nu_u^{k,l}}{\nu_i^{v,l}\nu_i^{u,j}}$$
$$\cdot\frac{[2k]![2i+1]![k+u+v-i]![j+k+l-i]![u+v+l+j+1]!}{[2u]![2v]![j+u+i+1]![k+l-u]![j+k-v]![i+l+v+1]!}$$
$${}_4\Phi_3\left(i-l-v, i-u-j, -k-l-u-1, -k-j-v-1; i-v-k-u,\right.$$
$$\left. -u-v-l-j-1, i-j-l-k; q, q\right).$$

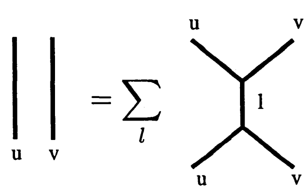
- Map F (equation 5.67):

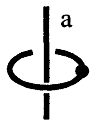
$$F_{uv}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = (-1)^{(i+j)(u+1)+(v+l)(i+1)}N_{uv}\begin{bmatrix} j & k \\ i & l \end{bmatrix}.$$

- Map S_a (proposition 5.12):

$$S_{uv}(a) = (-1)^v\frac{1}{X}q^{u(u+1)+v(v+1)}$$
$$\sum_l(-1)^{(u+l)(v+1)}q^{-l(l+1)}[2l+1]N_{uv}\begin{bmatrix} u & l \\ a & v \end{bmatrix}.$$

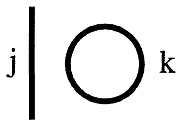
- Various tangle identities.





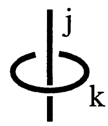
$$= \begin{cases} 0 & a \neq 0 \\ X & a = 0 \end{cases}$$

Proposition 3.8:



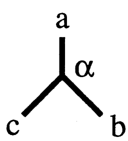
$= [2k{+}1]$

,



$= \dfrac{[(2j+1)(2k+1)]}{[2j+1]}$

Proposition 3.2:



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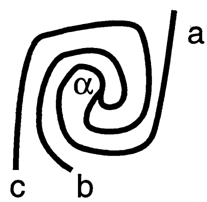

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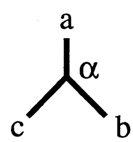

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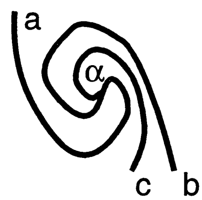

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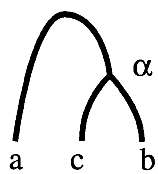
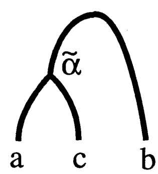
Proposition 3.3:



$= \dfrac{\sqrt{[2a+1]}}{\sqrt{[2b+1]}}$



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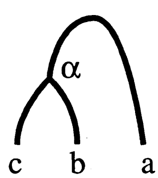
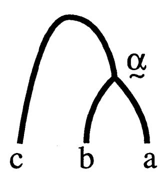


$= \dfrac{\sqrt{[2a+1]}}{\sqrt{[2c+1]}}$



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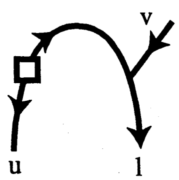
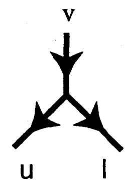
Lemma 5.11:



$$= \sqrt{\frac{[2l+1]}{[2v+1]}}$$



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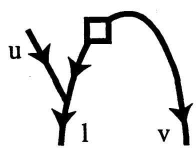
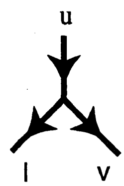


$$= (-1)^{(l+u)(v+1)} \sqrt{\frac{[2l+1]}{[2u+1]}}$$



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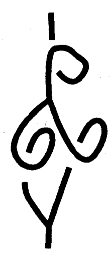
Proposition 3.4:

$=$



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