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Groups quasi-isometric to symmetric spaces

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We determine the structure of finitely generated groups which are quasi-isometric to nonpositively curved symmetric spaces, allowing Euclidean de Rham factors. If X is a symmetric space of noncompact type (i.e. it has no Euclidean de Rham factor), and Γ is a finitely generated group quasi-isometric to the product $\mathbb{E}^k \times X$, then there is an exact sequence $1 \to H \to \Gamma \to L \to 1$ where H contains a finite index copy of $\mathbb{Z}^k$ and L is a uniform lattice in the isometry group of X.

1. Introduction.

If X is a symmetric space with no Euclidean de Rham factor, then any finitely generated group Γ quasi-isometric to X is a finite extension of a uniform lattice in $Isom(X)$. This result is a direct corollary of the main results of [KlLe97b] together with earlier work in the rank 1 cases [Tuk88, Gro81a, Hin90, Pan89, Ga92, CJ94], and was first announced in June 1994 at MSRI, and in [KlLe97a]. This result does not extend to symmetric spaces with a nontrivial Euclidean factor: it was observed by Epstein, Gersten, and Mess that any extension of a Fuchsian group by $\mathbb{Z}$ is quasi-isometric to $\mathbb{H}^2 \times \mathbb{R}$, and such extensions are typically not finite extensions of lattices in $Isom(\mathbb{H}^2 \times \mathbb{R})$. In this paper we treat the case of groups quasi-isometric to symmetric spaces with a Euclidean de Rham factor.

Theorem 1.1. *Let X be a symmetric space of noncompact type, and let Nil be a simply connected nilpotent Lie group equipped with a left-invariant Riemannian metric. Suppose Γ is a finitely generated group quasi-isometric to $Nil \times X$ (endowed with the product metric). Then there is an exact sequence*

$$1 \longrightarrow H \longrightarrow \Gamma \stackrel{p}{\longrightarrow} L \longrightarrow 1 \tag{1.2}$$

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where H is a finitely generated group quasi-isometric to Nil and L is a uniform lattice in the isometry group of X, and this sequence is unique up to isomorphism. Furthermore, given any quasi-isometry $\Gamma \xrightarrow{\phi} Nil \times X$, there is a quasi-isometry $L \xrightarrow{\bar{\phi}} X$ so that the diagram

$$\begin{array}{ccc} \Gamma & \xrightarrow{p} & L \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\bar{\phi}} \\ Nil \times X & \xrightarrow{\pi_2} & X \end{array} \tag{1.3}$$

commutes up to bounded error. In particular, H is undistorted[3] *in Γ.*

When Nil is the trivial group then Γ is a finite extension of a uniform lattice in $Isom(X)$, and when $Nil \simeq \mathbb{R}^k$ then H is virtually abelian of rank k by [Gro81b, Pan83]. The case when X is the hyperbolic plane and $Nil \simeq \mathbb{R}$ is due to Rieffel [Rie93].

We further refine Theorem 1.1 when $Nil \simeq \mathbb{R}^n$.

Theorem 1.4. *Let X be as in Theorem 1.1. Then any finitely generated group Γ quasi-isometric to $\mathbb{R}^n \times X$ contains a finite index subgroup $\Gamma_1 \subset \Gamma$ which is a central extension of the form*

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma_1 \longrightarrow L_1 \longrightarrow 1 \tag{1.5}$$

where L_1 is a finite extension of a lattice in $Isom(X)$.

In general, one cannot arrange that the group L_1 is a lattice in $Isom(X)$ rather than a finite extension of a lattice. Examples of Raghunathan [Rag84] show that this is impossible in general even when $n = 0$.

Theorem 1.4 raises the question of which central extensions (1.5) are quasi-isometric to $\mathbb{E}^n \times X$. Theorem 1.8 below gives a homological answer to this.

Definition 1.6. An extension $1 \to K \to G \xrightarrow{p} Q \to 1$ of finitely generated groups is *quasi-isometrically trivial* if there is a quasi-isometry $G \xrightarrow{\phi} K \times Q$ so that the diagram

$$\begin{array}{ccc} G & \xrightarrow{p} & Q \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle id_Q} \\ K \times Q & \xrightarrow{\pi_2} & Q \end{array} \tag{1.7}$$

[3] The inclusion of H in Γ is biLipschitz with respect to the word metrics.

commutes up to bounded error.

The central extension (1.5) is quasi-isometrically trivial by the second part of Theorem 1.1. The next result gives a general characterisation of quasi-isometrically trivial extensions.

Theorem 1.8. *(See section 7 for the definition of L^∞ cochains for CW complexes.) Let*

$$1 \to \mathbb{Z}^n \to G \to Q \to 1 \tag{1.9}$$

be a central extension of finitely generated groups, and let $\alpha \in H^2(Q;\mathbb{Z}^n)$ be the associated cohomology class. Let K be a CW-complex with finite 1-skeleton which is an Eilenberg-Maclane space for Q, and identify α with a class in $H^2(K;\mathbb{Z}^n) \simeq H^2(Q;\mathbb{Z}^n)$. Then the extension (1.9) is quasi-isometrically trivial iff the pullback of α to $H^2(\tilde{K};\mathbb{Z}^n)$ is in the image of $H^2_{L^\infty}(\tilde{K};\mathbb{Z}^n) \xrightarrow{\delta} H^2(\tilde{K},\mathbb{Z}^n)$, where $\tilde{K}$ denotes the universal cover of K.

Remarks. Using bounded cohomology instead of L^∞ cohomology, Gersten [Ger92] gave a sufficient condition for a central extension by $\mathbb{Z}$ to be quasi-isometric to a trivial extension. In [ReNe97, Section 4] the authors give another cohomological characterization of quasi-isometrically trivial central extensions.

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2. Preliminaries.

In this section we recall some basic definitions and notation. See [Gro93] for more discussion and background.

Definition 2.1. A map $f : X \longrightarrow Y$ between metric spaces is an (L, A) *quasi-isometry* if for every $x_1, x_2 \in X$

$$L^{-1}d(x_1, x_2) + A \leq d(x_1, x_2) \leq Ld(x_1, x_2) + A,$$

and for every $y \in Y$ we have $d(y, f(X)) < A$. Two quasi-isometries $f_1, f_2 : X \longrightarrow Y$ are *equivalent* if $d(f_1, f_2) < \infty$.

If Γ is a finitely generated group, then any two word metrics on Γ are biLipschitz to one another by $id_\Gamma : \Gamma \to \Gamma$. We will implicitly endow our finitely generated groups with word metrics.

Definition 2.2. An (L, A)*-quasi-action* of a group Γ on a metric space Z is a map $\rho : \Gamma \times Z \to Z$ so that $\rho(\gamma, \cdot) : Z \to Z$ is an (L, A) quasi-isometry for every $\gamma \in \Gamma$, $d(\rho(\gamma_1, \rho(\gamma_2, z)), \rho(\gamma_1\gamma_2, z)) < A$ for every $\gamma_1, \gamma_2 \in \Gamma$, $z \in Z$, and $d(\rho(e, z), z) < A$ for every $z \in Z$.

We will denote the self-map $\rho(\gamma, \cdot) : Z \to Z$ by $\rho(\gamma)$. ρ is *discrete* if for any point $z \in Z$ and any radius $R > 0$, the set of all $\gamma \in \Gamma$ such that $\rho(\gamma, z)$ is contained in the ball $B_R(z)$ is finite. ρ is *cobounded* if Z coincides with a finite tubular neighborhood of the "orbit" $\rho(\Gamma)z \subset Z$ for every z. If ρ is a discrete cobounded quasi-action of a finitely generated group Γ on a geodesic metric space Z, it follows easily that the map $\Gamma \to Z$ given by $\gamma \mapsto \rho(\gamma, z)$ is a quasi-isometry for every $z \in Z$.

Definition 2.3. Two quasi-actions ρ and ρ' are *equivalent* if there exists a constant D so that $d(\rho(\gamma), \rho'(\gamma)) < D$ for all $\gamma \in \Gamma$.

Definition 2.4. Let ρ and ρ' be a quasi-actions of Γ on Z and Z' respectively, and let $\phi : Z \to Z'$ be a quasi-isometry . Then ρ is *quasi-isometrically conjugate to ρ' via ϕ* if there is a D so that $d(\phi \circ \rho(\gamma), \rho'(\gamma) \circ \phi) < D$ for all $\gamma \in \Gamma$.

Lemma 2.5 (cf. [Gro87, 8.2.K]). *Let X be a Hadamard manifold of dimension ≥ 2 with sectional curvature $\leq K < 0$, and let $\partial_\infty X$ denote the geometric boundary of X with the cone topology. Recall that every quasi-isometry $\Phi : X \longrightarrow X$ induces a boundary homeomorphism $\partial_\infty\Phi : \partial_\infty X \to \partial_\infty X$.*

1. *If $\rho : \Gamma \times X \to X$ is a quasi-action on X, then ρ is discrete (respectively cobounded) iff $\partial_\infty \phi$ acts properly discontinuously (respectively cocompactly) on the space of distinct triples in $\partial_\infty X$.*

2. *Given (L, A) there is a D so that if ϕ_k, ψ are (L, A) quasi-isometries, then $\partial_\infty \phi_k$ converges uniformly to $\partial_\infty \psi$ iff $\limsup d(\phi_k x, \psi x) < D$ for every $x \in X$. In particular, if $\phi_1, \phi_2 : X \longrightarrow X$ are (L, A) quasi-isometries with the same boundary mappings, then $d(\phi_1, \phi_2) < D$.*

Proof. Let $\partial^3 X \subset \partial_\infty X \times \partial_\infty X \times \partial_\infty X$ denote the subspace of distinct triples. The uniform negative curvature of X implies that there is a D_0 depending only on K such that

(a) For every $x \in X$ there is a triple $(\xi_1, \xi_2, \xi_3) \in \partial^3 X$ such that $d(x, \overline{\xi_i \xi_j}) < D_0$ for every $1 \le i \ne j \le 3$, where $\overline{\xi_i \xi_j}$ denotes the geodesic with ideal endpoints ξ_i, ξ_j. Moreover for every C the set $\{(\xi_1, \xi_2, \xi_3) \mid d(x, \overline{\xi_i \xi_j}) < C \text{ for all } 1 \le i \ne j \le 3\}$ has compact closure in $\partial^3 X$.

and

(b) For every $(\xi_1, \xi_2, \xi_3) \in \partial^3 X$ there is a point $x \in X$ so that $d(x, \overline{\xi_i \xi_j}) < D_0$ for each $1 \le i \ne j \le 3$. And for every C there is a C' depending only on C and K so that $\{x \in X \mid d(x, \overline{\xi_i \xi_j}) < C \text{ for every } 1 \le i \ne j \le 3\}$ has diameter $< C'$.

easily from this. □

3. Projecting quasi-actions to the factors.

Let Nil and X be as in Theorem 1.1 and decompose X into irreducible factors:

$$X = \prod_{i=1}^{l} X_i \tag{3.1}$$

Suppose ρ is a quasi-action of the finitely generated group Γ on $Nil \times X$. We denote by $p : Nil \times X \to X$ the canonical projection. Theorem 1.1.2 from [KlLe97b][4] implies that every quasi-isometry of $Nil \times X$ respects the

[4] Although Theorem 1.1.2 is only formulated in the case that $Nil \simeq \mathbb{R}^n$, the same proof works in general provided one uses [Pan83] to conclude that all asymptotic cones of Nil are homeomorphic to $\mathbb{R}^k$ where $k = Dim(Nil)$.

fibering p and covers a product quasi-isometry $X \to X$, up to bounded error. Applying this theorem to $\rho(\gamma)$ for each γ, we construct quasi-actions ρ_i of Γ on X_i so that

$$d\left(p \circ \rho(\gamma), \prod_{i=1}^{k} \rho_i(\gamma) \circ p\right) < D$$

for all $\gamma \in \Gamma$ and some positive constant D.

4. Straightening cocompact quasi-actions on irreducible symmetric spaces.

The following result is a direct consequence of [Pan89, Théorème 1] and [KlLe97b, Theorem 1.1.3].

Fact 4.1. *Let X be an irreducible symmetric space other than a real or complex hyperbolic space. Then every quasi-action on X is equivalent to an isometric action.*

Proof. Let ρ be a quasi-action of a group Γ on X. By the results just cited, there is an isometry $\bar{\rho}(\gamma)$ at finite distance from the quasi-isometry $\rho(\gamma)$ for every $\gamma \in \Gamma$. This isometry is unique and its distance from $\rho(\gamma)$ is uniformly bounded[5] in terms of the constants of the quasi-action. So $\bar{\rho}$ is an isometric action equivalent to ρ. □

We recall that the real and complex hyperbolic spaces of all dimensions admit quasi-isometries which are not equivalent to isometries [Pan89].

Fact 4.2. *Any cobounded quasi-action ρ on a real or complex hyperbolic space of dimension > 2 is quasi-isometrically conjugate to an isometric action.*

This result is due to Sullivan in the $\mathbb{H}^3$ case, and to [Gro81a, Tuk86] in the real-hyperbolic case. Using Pansu's theory of Carnot differentiability one can carry out Tukia's arguments for all rank-one symmetric spaces other than hyperbolic plane, cf. [Pan89, sec. 11]. Another proof for the complex-hyperbolic case can be found in [Chow96].

[5] The uniformity in the rank one case follows from Lemma 2.5.

Fact 4.3. *Let ρ be a cobounded quasi-action of a group Γ on $\mathbb{H}^2$. Then ρ is quasi-isometrically conjugate to a cocompact isometric action of Γ on $\mathbb{H}^2$.*

Proof. We recall that every quasi-isometry $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ induces a quasi-symmetric homeomorphism $\partial_\infty\phi : \partial_\infty\mathbb{H}^2 \to \partial_\infty\mathbb{H}^2$, see [TuVa82]; moreover the quasi-symmetry constant of $\partial_\infty\phi$ can be estimated in terms of the quasi-isometry constants of ϕ. Since equivalent quasi-isometries yield the same boundary homeomorphism, every quasi-action ρ on $\mathbb{H}^2$ induces a genuine action $\partial_\infty\rho$ on $\partial_\infty\mathbb{H}^2$ by uniformly quasi-symmetric homeomorphisms.

Let $\bar{\Gamma}$ be the quotient of Γ by the kernel of the action $\partial_\infty\rho$, and let $\pi : \Gamma \to \bar{\Gamma}$ be the canonical epimorphism. If two elements $\gamma_1, \gamma_2 \in \Gamma$ have the same boundary map then $d(\rho(\gamma_1), \rho(\gamma_2))$ is uniformly bounded by Lemma 2.5. Hence we may obtain a quasi-action $\bar{\rho}$ of $\bar{\Gamma}$ on $\mathbb{H}^2$ by choosing $\gamma \in \pi^{-1}(\bar{\gamma})$ for each $\bar{\gamma} \in \bar{\Gamma}$, and setting $\bar{\rho}(\bar{\gamma}) = \rho(\gamma)$. If $\bar{\tau}$ is an isometric action of $\bar{\Gamma}$ on $\mathbb{H}^2$ and $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ quasi-isometrically conjugates $\bar{\rho}$ into $\bar{\tau}$, then ϕ will quasi-isometrically conjugate ρ into the isometric action $\tau : \Gamma \times \mathbb{H}^2 \to \mathbb{H}^2$ given by $\tau(\gamma) = \bar{\tau}(\pi(\gamma))$. Hence it suffices to treat the case when $\bar{\Gamma} = \Gamma$, and so we will assume that $\partial_\infty\rho$ is an effective action.

Lemma 4.4. *The quasi-action ρ is discrete if and only if the action $\partial_\infty\rho$ on $\partial_\infty\mathbb{H}^2$ is discrete in the compact-open topology.*

Proof. Suppose $\partial_\infty\rho$ is discrete, and let (γ_i) be a sequence in Γ so that $\rho(\gamma_i)$ maps a point $p \in \mathbb{H}^2$ into a fixed ball $B_R(p)$. Then by a selection argument we may assume – after passing to a subsequence if necessary – that there is a quasi-isometry $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ so that for every $q \in \mathbb{H}^2$ we have $\limsup_i d(\rho(\gamma_i)(q), \phi(q)) < D$ for some D. Hence the boundary maps $\partial_\infty\rho(\gamma_i)$ converge to $\partial_\infty\phi$, and so the sequence $\partial_\infty\rho(\gamma_i)$ is eventually constant. Since ρ is effective we conclude that γ_i is eventually constant. Therefore ρ is a discrete quasi-action.

If ρ is a discrete quasi-action on $\mathbb{H}^2$, then $\partial_\infty\rho$ is discrete by Lemma 2.5.
□

Proof of 4.3 *continued.*

Case 1: $\partial_\infty\rho$ is discrete. In this case, ρ is a discrete convergence group action (Lemma 2.5) and by the work of [CJ94, Ga92], there is a discrete isometric action τ of Γ on $\mathbb{H}^2$ so that $\partial_\infty\rho$ is topologically conjugate to $\partial_\infty\tau$.

Since ρ is cobounded, $\partial_\infty \rho$ acts cocompactly on the set of distinct triples of points in $\partial_\infty \mathbb{H}^2$ (lemma 2.5); therefore $\partial_\infty \tau$ also acts cocompactly on the space of triples and so τ is a discrete, cocompact, isometric action of Γ on $\mathbb{H}^2$. We now have two discrete, cobounded, quasi-actions of Γ on $\mathbb{H}^2$, so they are quasi-isometrically conjugate by some quasi-isometry $\psi : \mathbb{H}^2 \to \mathbb{H}^2$.

Case 2: $\partial_\infty \rho$ is nondiscrete. By [Hin90, Theorem 4], $\partial_\infty \rho$ is quasi-symmetrically conjugate to $\partial_\infty \tau$, where τ is an isometric action on $\mathbb{H}^2$. The conjugating quasi-symmetric homeomorphism is the boundary of a quasi-isometry $\psi : \mathbb{H}^2 \to \mathbb{H}^2$, [TuVa82], which quasi-isometrically conjugates $\partial_\infty \rho$ into the isometric action action τ. Applying Lemma 2.5 again, we conclude that τ is cocompact. $\square$

Section 3, and facts 4.1, 4.2 and 4.3 imply:

Corollary 4.5. *Let X be a symmetric space of noncompact type without Euclidean factor. Then any cobounded quasi-action on X is quasi-isometrically conjugate to a cocompact isometric action on X.*

5. A Growth estimate for small elements in nondiscrete cocompact subgroups of $Isom(X)$.

5.1. Parabolic isometries of symmetric spaces.

Let X be a symmetric space of noncompact type, and let $G = Isom(X)$.

Recall that the displacement function of an isometry g is the convex function $\delta_g : X \to \mathbb{R}$ defined by the formula $\delta_g(x) := d(gx, x)$. An isometry $g \in G$ is *semisimple* if its displacement function δ_g attains its infimum and *parabolic* otherwise.

Lemma 5.1. *Let $A \subset G$ be a finitely generated abelian group all of whose nontrivial elements are parabolic. Then A has a fixed point at infinity.*

Proof. Recall that the nearest point projection to a closed convex subset is well-defined and distance non-increasing. This implies that if C is a non-empty A-invariant closed convex set, then for all displacement functions δ_a, $a \in A$, we have $\inf \delta_a = \inf \delta_a|_C$. Hence for all $n \in \mathbb{N}$, the intersection of the sublevel sets $\{p \mid \delta_{a_i}(p) \leq \inf \delta_{a_i} + 1/n\}$ is non-empty and contains a point p_n. We have $\delta_{a_i}(p_n) \to \inf \delta_{a_i}$ for all a_i, and since the isometries a_i

are parabolic the sequence $\{p_n\}$ subconverges to an ideal boundary point $\xi \in \partial_\infty X$. It follows that the a_i fix ξ. □

Lemma 5.2. *Let $a_1, \ldots, a_k \in Isom(X)$ be commuting parabolic isometries. Then there is a sequence of isometries $\{g_n\} \subset G$ so that for every i the sequence $g_n a_i g_n^{-1}$ subconverges to a semisimple isometry $\bar{a}_i$.*

Proof. From the proof of the previous lemma, there is a sequence of points $\{p_n\} \subset X$ converging to an ideal point ξ so that $\delta_{a_i}(p_n) \to \inf \delta_{a_i}$ for all a_i. Pick isometries $g_n \in G$ such that $g_n \cdot p_n = p_0$. The conjugates $g_n a_i g_n^{-1}$ have the same infimum displacement as a_i. Since

$$\delta_{g_n a_i g_n^{-1}}(p_0) = \delta_{a_i}(p_n) \to \inf \delta_{a_i} \quad ,$$

the $g_n a_i g_n^{-1}$ subconverge to a semisimple isometry. □

We call an isometry $g \neq e$ *purely parabolic*[6] if the identity is the only semisimple element in $\overline{Ad_G(G) \cdot g}$.

5.2. The growth estimate.

Proposition 5.3. *Let X be a symmetric space of noncompact type with no Euclidean de Rham factors. Let $\Gamma \subset G = Isom(X)$ be a finitely generated, nondiscrete, cocompact subgroup. Let $U \subset Isom(X)$ be a neighborhood of the identity, and set*

$$f(k) := \#\{g \in \Gamma : |g|_\Gamma < k,\ g \in U\},$$

where $|\cdot|_\Gamma$ denotes a word norm on Γ. Then f grows faster than any polynomial, i.e. for every $d > 0$ $\limsup_{k\to\infty} \frac{f(k)}{k^d} = \infty$.

Proof. Let $\bar{\Gamma}$ be the closure of Γ in G with respect to the Hausdorff topology, and let $\bar{\Gamma}^o$ be the identity component of $\bar{\Gamma}$.
Case 1: $\bar{\Gamma}^o$ is nilpotent. Let A be the last non-trivial subgroup in the derived series of $\bar{\Gamma}^o$. Then $A \subset \bar{\Gamma}$ is a connected abelian subgroup of positive dimension, A is normal in $\bar{\Gamma}$, and $\Gamma \cap A$ is dense in A.

Lemma 5.4. *For every $\delta \in (0,1)$ there is a $\gamma \in \Gamma$ such that all eigenvalues of the automorphism $Ad_G(\gamma)\big|_A : A \to A$ have absolute value $< \delta$.*

[6] This is a geometric way of defining unipotent isometries.

Proof. See section 5.1 for terminology.

Step 1: A contains no semisimple isometries other than e. Otherwise we can consider the intersection C of the minimum sets for the displacement functions δ_a where a runs through all semisimple elements in A. C is a nonempty convex subset of X which splits metrically as $C \cong \mathbb{E}^k \times Y$. The flats $\mathbb{E}^k \times \{y\}$ are the minimal flats preserved by all semisimple elements in A. Since Γ normalises A it follows that C is Γ-invariant. The cocompactness of Γ implies that $C = X$ and $k = 0$ because X has no Euclidean factor. This means that the semisimple elements in A fix all points, a contradiction.

Step 2: All non-trivial isometries in A are purely parabolic. If $a \in A$, $a \neq e$, is not purely parabolic then there is a sequence of isometries g_n so that $g_n a g_n^{-1}$ converges to a semisimple isometry $\bar{a} \neq e$. We can uniformly approximate the g_n by elements in Γ, i.e. there exist $\gamma_n \in \Gamma$ and a bounded sequence $k_n \in G$ subconverging to $k \in G$ so that $\gamma_n = k_n g_n$. Then $\gamma_n a \gamma_n^{-1} = k_n g_n a g_n^{-1} k_n^{-1}$ subconverges to the non-trivial semisimple element $k\bar{a}k^{-1}$. This contradicts step 1.

Step 3: Pick a basis $\{a_1, \ldots, a_k\}$ for $A \simeq \mathbb{R}^k$. By Lemma 5.2 there exist elements $g_n \in G$ so that $g_n a_i g_n^{-1} \to e$ for all a_i. We approximate the g_n as above by γ_n so that the sequence $\gamma_n g_n^{-1}$ is bounded. Then $\gamma_n a_i \gamma_n^{-1} \to e$ for all a_i. The lemma follows by setting $\gamma = \gamma_n$ for sufficiently large n. □

Proof of case 1 continued. By Lemma 5.4, there is a $\gamma \in \Gamma$, $\gamma \neq e$, and a norm $\|\cdot\|_A$ on A such that for all $a \in A$ we have

$$\|\gamma a \gamma^{-1}\|_A < \frac{1}{2}\|a\|_A.$$

Consider a neighborhood U of e in G. Let $r > 0$ be small enough so that $\{a \in A : \|a\|_A < r\} \subset U$ and pick $\alpha \in \Gamma \cap A$ with $\|\alpha\|_A < r/2$. Then the elements

$$\gamma_{\epsilon_0 \ldots \epsilon_{n-1}} = \alpha^{\epsilon_0} \cdot (\gamma\alpha\gamma^{-1})^{\epsilon_1} \cdot \cdots \cdot (\gamma^{n-1}\alpha\gamma^{1-n})^{\epsilon_{n-1}}$$

for $\epsilon_i \in \{0, 1\}$ are 2^n pairwise distinct elements contained in $\Gamma \cap U$ with word norm $|\gamma_{\epsilon_0 \ldots \epsilon_{n-1}}|_\Gamma < n^2(|\alpha|_\Gamma + |\gamma|_\Gamma)$. This implies superpolynomial growth of f.

Case 2: $\bar{\Gamma}^o$ is not nilpotent. Define an increasing sequence (the upper central series) of nilpotent Lie subgroups $Z_i \subset \bar{\Gamma}^o$ inductively as follows: Set $Z_0 = \{e\}$ and let Z_{i+1} be the inverse image in $\bar{\Gamma}^o$ of the center in $\bar{\Gamma}^o/Z_i$. The dimension of Z_i stabilizes and we choose k so that $\dim Z_k$ is maximal. Then the center of $\bar{\Gamma}/Z_k$ is discrete and, since $\bar{\Gamma}^o$ is not nilpotent, we have $\dim Z_k < \dim \bar{\Gamma}$. Proposition 5.3 now follows by applying the next lemma with $H = \bar{\Gamma}$ and $H_1 = Z_k$. □

Lemma 5.5. *Let H be a Lie group, let $H_1 \lhd H$ be a closed normal subgroup so that $\bar{H} := H/H_1$ is a positive dimensional Lie group with discrete center, and suppose $\Gamma \subset H$ is a dense, finitely generated subgroup. If U is any neighborhood of e in H, then the function $f(k) := \#\{g \in \Gamma : |g|_\Gamma \leq k,\ g \in U\}$ grows superpolynomially.*

Proof. The idea of the proof is to use the contracting property of commutators to produce a sequence $\{\alpha_k\}$ in $H \cap \Gamma$ which converges exponentially to the identity. The word norm $|\alpha_k|_\Gamma$ grows exponentially with k, but the number of elements of $\langle \alpha_1, \dots, \alpha_k \rangle$ in U also grows exponentially with k; by comparing growth exponents we find that f grows superpolynomially.

Fix $M \in \mathbb{N}$, a positive real number $\epsilon < 1/3$ and some left-invariant Riemannian metric on H. Since the differential of the commutator map $(h, h') \mapsto [h, h']$ vanishes at (e, e) we can find a neighborhood V of e in H such that:

$$h, h' \in V \quad \Longrightarrow \quad [h, h'] \in V \quad \text{and} \quad d([h, h'], e) < \frac{1}{4M} d(h, e) \tag{5.6}$$

Since the differential of the k-th power $h \mapsto h^k$ at e is $k \cdot id_{T_eH}$ for all $k \in \mathbb{Z}$, we can furthermore achieve that, whenever $1 \leq k, k' \leq M$ and $h, h^k, h^{k'} \in V$, then

$$d(h^k, h^{k'}) \geq (|k - k'| - \epsilon) \cdot d(h, e) \tag{5.7}$$

By our assumption, there exist finitely many elements $\gamma_1, \dots, \gamma_m \in \Gamma \cap V$ such that the centralizers $Z_{\bar{H}}(\bar{\gamma}_j)$ of their images in $\bar{H}$ have discrete intersubsection. We construct an infinite sequence of elements $\alpha_i \in (\Gamma \cap V) \setminus H_1$ by picking $\alpha_0 \in V$ arbitrarily and setting $\alpha_{i+1} = [\alpha_i, \gamma_{j(i)}] \notin H_1$ for suitably chosen $1 \leq j(i) \leq m$. Then

$$0 < d(\alpha_{i+1}, e) < \frac{1}{4M} d(\alpha_i, e) \tag{5.8}$$

by (5.6).

Sublemma 5.9. *Pick $n_0 \in \mathbb{N}$. The M^n elements*

$$\gamma_{\epsilon_1 \dots \epsilon_n} = \alpha_{n_0+1}^{\epsilon_1} \cdots \alpha_{n_0+n}^{\epsilon_n} \qquad \epsilon_i \in \{0, \dots, M-1\} \tag{5.10}$$

are distinct.

Proof. Assume that $\gamma_{\epsilon_1\ldots\epsilon_n} = \gamma_{\epsilon'_1\ldots\epsilon'_n}$, $\epsilon_l \neq \epsilon'_l$ and $\epsilon_i = \epsilon'_i$ for all $i < l$. Then

$$\alpha_{n_0+l}^{\epsilon_l-\epsilon'_l} = \left(\alpha_{n_0+l+1}^{\epsilon'_{l+1}} \cdots \alpha_{n_0+n}^{\epsilon'_n}\right)\left(\alpha_{n_0+l+1}^{\epsilon_{l+1}} \cdots \alpha_{n_0+n}^{\epsilon_n}\right)^{-1}.$$

By (5.8) and the triangle inequality

$$d\left(\left(\alpha_{n_0+l+1}^{\epsilon'_{l+1}} \cdots \alpha_{n_0+n}^{\epsilon'_n}\right)\left(\alpha_{n_0+l+1}^{\epsilon_{l+1}} \cdots \alpha_{n_0+n}^{\epsilon_n}\right)^{-1}, e\right)$$
$$< 2M \sum_{j=1}^{\infty} \frac{1}{(4M)^j} d\left(\alpha_{n_0+l}, e\right) \leq \frac{2}{3} d\left(\alpha_{n_0+l}, e\right).$$

On the other hand, by (5.7) we have

$$d\left(\alpha_{n_0+l}^{\epsilon_l-\epsilon'_l}, e\right) \geq (1-\epsilon) d\left(\alpha_{n_0+l}, e\right) > \frac{2}{3} d\left(\alpha_{n_0+l}, e\right)$$

which is a contradiction. □

To complete the proof of the lemma, we observe that the elements (5.10) have word norm $|\gamma_{\epsilon_1\ldots\epsilon_n}|_\Gamma \leq const(n_0) \cdot 3^n$ and are contained in U if n_0 is sufficiently large. This shows that $f(k)$ grows polynomially of order at least $\frac{log(M)}{log(3)}$ for all M, hence the claim. □

6. Proof Theorem 1.1.

Let $\rho_0 : \Gamma \times \Gamma \to \Gamma$ be the isometric action of Γ on itself by left translation, and let $\phi : \Gamma \to Nil \times X$ be a quasi-isometry. Then there is a quasi-action ρ of Γ on $Nil \times X$ such that ϕ quasi-isometrically conjugates ρ_0 into ρ. According to section 3, ρ projects (up to bounded error) to a cobounded quasi-action $\bar{\rho}$ of Γ on X. $\bar{\rho}$ is quasi-isometrically conjugate to a cocompact isometric action $\hat{\rho}$, cf. Corollary 4.5. Pick $x \in X$, $y \in Nil \times \{x\}$, and $R > 0$. Since the quasi-action ρ covers $\bar{\rho}$, we know that for all $\gamma \in \Gamma$ with $\hat{\rho}(\gamma) \cdot x \in B_R(x)$, the distance $d(\rho(\gamma) \cdot y, Nil \times \{x\})$ is uniformly bounded. The map $\Gamma \to Nil \times X$ given by $\gamma \mapsto \rho(\gamma) \cdot y$ being a quasi-isometry, we conclude that the function

$$(6.1) \qquad N(k) := \#\{\gamma \in \Gamma \mid |\gamma|_\Gamma < k, \, \hat{\rho}(\gamma) \cdot x \in B_R(x)\}$$

grows at most as fast as the volume of balls in Nil, i.e. it is $< Ck^d$ for some C, $d \in \mathbb{R}$. Proposition 5.3 implies that $L := \hat{\rho}(\Gamma)$ is a discrete subgroup in

$Isom(X)$ and hence a uniform lattice. The kernel H of the action $\hat{\rho}$ is then a finitely generated group quasi-isometric to the fiber Nil, since it clearly (quasi)-acts discretely and coboundedly on the fiber.

To see that the sequence (1.2) is unique up to isomorphism, let

$$1 \to H' \to \Gamma \xrightarrow{p'} L' \to 1$$

be an exact sequence with $L' \subset Isom(X)$ a uniform lattice and H' a group quasi-isometric to Nil. Then by [Gro81b, Pan83] H' is a virtually nilpotent group. Now if $\Gamma \xrightarrow{f} \Gamma$ is an isomorphism then $p'(H) \subset L'$ is a normal, finitely generated, virtually nilpotent subgroup; it follows that $p'(f(H))$ is trivial. Similarly $p(f^{-1}(H'))$ is trivial and we conclude that f induces an isomorphism of the two exact sequences.

We now prove the last statement of Theorem 1.1. When we restrict $\bar{\rho}$ to H we get a quasi-action which is equivalent to the trivial action of H on X. Hence $\bar{\rho}$ induces a quasi-action η of $L = \Gamma/H$ on X, which is discrete and cobounded. The action η_0 of L on itself by left translations is also discrete and cobounded, so $g \mapsto \eta(g)(\pi_2(\phi(e)))$ defines a quasi-isometry $L \xrightarrow{\bar{\phi}} X$. It follows that the diagram

(6.2)
$$\begin{array}{ccc} \Gamma & \xrightarrow{p} & L \\ \phi\downarrow & & \bar{\phi}\downarrow \\ Nil \times X & \xrightarrow{\pi_2} & X \end{array}$$

commutes up to bounded error since ϕ quasi-isometrically conjugates ρ_0 into ρ, ρ projects to $\bar{\rho}$, and $d(\bar{\rho}(\gamma H), \eta(\gamma H))$ is uniformly bounded (independent of γ). □

7. Proof of Theorem 1.4.

Sketch of proof. If Γ is quasi-isometric to $\mathbb{R}^n \times X$ where X is a symmetric space with no Euclidean de Rham factor, then by Theorem 1.1, Γ fits into an exact sequence (1.2) where H is an undistorted virtually $\mathbb{Z}^n$ subgroup. We will use the undistortedness of H to pass to a finite index subgroup of Γ which is a central extension, cf. [Ger91].

If S is a subset of a group G, we will use the notation $Z(S, G)$ to denote the centralizer of S in G, and $Z(G)$ to denote the center of G.

Proof of Theorem 1.4. By Theorem 1.1 we get an exact sequence

$$1 \longrightarrow H \longrightarrow \Gamma \xrightarrow{p} L \longrightarrow 1$$

where H is a finitely generated group quasi-isometric to $\mathbb{Z}^n$, and $L \subset Isom(X)$ is a uniform lattice. Applying the second part of the theorem we can get a quasi-isometry $\Gamma \xrightarrow{f} \mathbb{Z}^n \times L$ so that

$$\begin{array}{ccc} \Gamma & \xrightarrow{p} & L \\ f\downarrow & & \downarrow id \\ \mathbb{Z}^n \times L & \xrightarrow{\pi_2} & L \end{array} \tag{7.1}$$

commutes up to bounded error. Clearly $f(H) \subset \mathbb{Z}^n \times L$ has finite Hausdorff distance from $\mathbb{Z}^n \times \{e\} \subset \mathbb{Z}^n \times L$, so H is undistorted[7] in Γ. By [Gro81b, Pan83] H contains a finite index copy of $\mathbb{Z}^n$.

Next we will identify a finite index abelian subgroup of H which is normal in Γ. Let T be the subgroup of "translations" in H, i.e.

$$T = \{h \in H \mid [H : Z(h, H)] < \infty\}. \tag{7.2}$$

Clearly T is a characteristic subgroup of H, and has finite index in H; in particular T is finitely generated. Note that $Z(T)$, the center of T, has finite index in T since if $T = \langle t_1, \dots, t_k \rangle$, then $Z(T) = \cap_i Z(t_i, T)$ is a finite intersection of finite index subgroups of T. Hence $Z(T)$ is a finitely generated abelian group of the form $\mathbb{Z}^n \oplus A$ where A is a finite abelian group. Note $Z(T)$ is normal in Γ since it is characteristic in H, and H is normal in Γ.

Lemma 7.3. *The centralizer of $Z(T)$ in Γ, $Z(Z(T), \Gamma)$, has finite index in Γ.*

The proof uses properties of translation numbers, see [Gro81a, pp. 189-191]. The paper [Ger91] uses a similar setup.

Definition 7.4. Let G be a finitely generated group, and let $|\cdot|_G$ be a word norm on G. Then the *translation length* of $g \in G$ is

$$\delta_G(g) := \lim_{k \to \infty} \frac{|g^k|_G}{k}.$$

The limit exists since $k \mapsto |g^k|_G$ is a subadditive function.

[7] A finitely generated subgroup of a finitely generated group is undistorted if the inclusion homomorphism is a quasi-isometric embedding.

The translation length is conjugacy invariant, vanishes on torsion elements, and changes by at most a bounded factor if one passes to a different word metric. If a homomorphism $\alpha : H \to G$ is a quasi-isometric embedding of finitely generated groups (i.e. $\exists C > 0$ such that $|\alpha(h)|_G \geq C|h|_H$ for all $h \in H$) then the pullback of δ_G to H agrees with δ_H up to a bounded factor.

Proof of Lemma 7.3. We know that $Z(T)$ is undistorted in Γ since $Z(T)$ has finite index in H and H is undistorted in Γ. Hence δ_Γ restricts to a function on $Z(T)$ which is equivalent to $\delta_{Z(T)}$. The latter function clearly factors through the homomorphism $Z(T) \to \mathbb{Z}^n$ whose kernel is the torsion subgroup $A \subset Z(T)$. Hence $\delta_{Z(T)} : Z(T) \to \mathbb{R}$ is a proper function on $Z(T)$ which is invariant under conjugacy by elements of Γ. If R is large enough that $K_R := \{g \in Z(T) \mid \delta_\Gamma(g) \leq R\}$ generates $Z(T)$, then any finite index subgroup of Γ centralizing K_R will centralize $Z(T)$, so $Z(Z(T), \Gamma)$ has finite index in Γ. □

Proof of Theorem 1.4 *concluded.* Let $\Gamma_1 := Z(Z(T), \Gamma)$, let $H_1 \subseteq Z(T) \subseteq \Gamma_1 \cap H$ be a finite index subgroup of $Z(T)$ isomorphic to $\mathbb{Z}^n$, and set $L_1 := \Gamma_1/H_1$. Then clearly L_1 is a finite extension of a uniform lattice in $Isom(X)$, and hence

$$1 \to H_1 \to \Gamma_1 \to L_1 \to 1$$

is an exact sequence as in (1.5). □

8. Geometry of central extensions by $\mathbb{Z}^n$.

The objective of this section is Proposition 8.3, which provides criteria for recognizing quasi-isometrically trivial central extensions.

Definition 8.1. Let X be a CW-complex. A cellular k-cochain $\alpha \in C^k(X; \mathbb{Z}^n)$ is *bounded* if its values on the k-cells of X are uniformly bounded. The collection of bounded cochains forms a subgroup[8] of $C^k(X; \mathbb{Z}^n)$ which will be denoted by $C^k_{L^\infty}(X; \mathbb{Z}^n)$.

Lemma 8.2. *Suppose $k > 0$, X is a CW-complex with finitely many $(k-1)$-cells, $\tilde{X} \xrightarrow{p} X$ is the universal covering, and $\alpha \in B^k(X; \mathbb{Z}^n)$ is a k-coboundary. Then $p^*\alpha \in Im(C^{k-1}_{L^\infty}(\tilde{X}; \mathbb{Z}^n) \xrightarrow{\delta} C^k(\tilde{X}; \mathbb{Z}^n))$.*

[8]Under appropriate finiteness conditions $C^*_{L^\infty}(X; \mathbb{Z}^n)$ will be a subcomplex of $C^*(X; \mathbb{Z}^n)$ and the L^∞ cohomology $H^*_{L^\infty}(X; \mathbb{Z}^n)$ will be well-defined.

Proof. If $\theta \in C^{k-1}(X;\mathbb{Z}^n)$ and $\alpha = \delta\theta$ then $p^*\alpha = p^*\delta\theta = \delta p^*\theta$ and $p^*\theta \in C^{k-1}_{L^\infty}(\tilde{X};\mathbb{Z}^n)$ since X has a finitely many $(k-1)$ cells. □

Let X be a CW complex with finitely many $(k-1)$-cells. By the Lemma, the subgroup

$$Z^k_{sp}(X;\mathbb{Z}^n) := \{\alpha \in Z^k(X;\mathbb{Z}^n) \mid p^*\alpha \in Im(C^{k-1}_{L^\infty}(\tilde{X};\mathbb{Z}^n) \xrightarrow{\delta} C^k(\tilde{X};\mathbb{Z}^n))\}$$

descends to a subgroup $H^k_{sp}(X;\mathbb{Z})$ of $H^k(X;\mathbb{Z}^n)$; we will refer elements of $H^k_{sp}(X;\mathbb{Z}^n)$ as *special cohomology classes*[9]. If $X \xrightarrow{f} X'$ is a continuous map from X to another CW complex with finitely many $(k-1)$-cells, we can homotope f to a cellular map, so we have an induced homomorphism

$$H^k_{sp}(X';\mathbb{Z}^n) \xrightarrow{f^*} H^k_{sp}(X;\mathbb{Z}^n).$$

When G is a finitely generated group, the *special cohomology group* $H^2_{sp}(G;\mathbb{Z}^n)$ of G is defined as follows: pick a $K(G,1)$ with finite 1-skeleton; the special cohomology group $H^2_{sp}(X;\mathbb{Z}^n) \subset H^2(X;\mathbb{Z}^n) \simeq H^2(G;\mathbb{Z}^n)$ defines a subgroup of $H^2(G;\mathbb{Z}^n)$ which is independent of the choice of X.

Proposition 8.3. *Let*

$$1 \to \mathbb{Z}^n \xrightarrow{i} G \xrightarrow{p} Q \to 1 \tag{8.4}$$

be a central extension of finitely generated groups. Then the following are equivalent:

1. *The extension is quasi-isometrically trivial, i.e. there is a quasi-isometry $G \xrightarrow{f} \mathbb{Z}^n \times Q$ so that the diagram*

$$\begin{array}{ccc} G & \xrightarrow{p} & Q \\ f\downarrow & & \downarrow id \\ \mathbb{Z}^n \times Q & \xrightarrow{\pi_Q} & Q \end{array} \tag{8.5}$$

   commutes up to bounded error.

2. *There is a Lipschitz section $s : Q \to G$ of p.*

[9]It would be more descriptive to say that these classes "pullback to d(*bounded*)"; but we chose "special" for brevity.

3. *The cohomology class* $\alpha \in H^2(G;\mathbb{Z}^n)$ *associated with the extension (8.4) is a special cohomology class, i.e. if* K *is a* $K(G,1)$ *with finite 1-skeleton and* $c \in Z^2(K;\mathbb{Z}^n)$ *represents* α, *then* $p^*c \in Im(H^1_{L^\infty}(\tilde{K};\mathbb{Z}^n) \xrightarrow{\delta} H^1(\tilde{K};\mathbb{Z}^n))$.

Proof. ($1 \Longrightarrow 2$). Suppose f makes diagram (8.5) commute up to bounded error, and let f^{-1} be a quasi-inverse[10] for f. Define $s_0 : Q \to G$ to be the composition $Q \to \{e\}\times Q \to \mathbb{Z}^n \times Q \xrightarrow{f^{-1}} G$. The approximate commutativity of (8.5) implies that $d(p \circ s_0, id_Q) < \infty$. Define a section $s : Q \to G$ of p by letting $s(q)$ be a point in $p^{-1}(q)$ closest to $s_0(q)$, for all $q \in Q$. By Lemma 8.6 below, we have $d(s, s_0) < \infty$, and so s is Lipschitz since s_0 is Lipschitz and $d(q_1, q_2) \geq 1$ for distinct elements q_1, $q_2 \in Q$.

Lemma 8.6. *If* $H \lhd G$ *are finitely generated groups, we define a distance function* $d_{G/H}$ *on* G/H *by letting* $d_{G/H}(g_1H, g_2H)$ *be the distance between the subsets* g_1H, g_2H *of* G *with respect to a fixed word metric on* G. *Then the coset distance metric on* G/H *is equivalent*[11] *to any word metric on* G/H.

Proof. Let $\Sigma \subset G$ be a symmetric finite generating set, and let $\bar{\Sigma} \subset G/H$ be the image of Σ under $G \to G/H$. Then there is a canonical 1-Lipschitz map between the Cayley graphs $Cay(G,\Sigma)$ and $Cay(G/H,\bar{\Sigma})$. Paths in $Cay(G/H,\bar{\Sigma})$ can be lifted to paths in $Cay(G,\Sigma)$ of the same length which join the corresponding cosets of H. □

($2 \Longrightarrow 1$). If $s : Q \to G$ is a Lipschitz section of p, we may define a map $\pi_{\mathbb{Z}^n} : G \to \mathbb{Z}^n$ by the formula $\pi_{\mathbb{Z}^n}(g)s(p(g)) = g$, i.e. $\pi_{\mathbb{Z}^n}$ is the unique map $G \to \mathbb{Z}^n$ which sends $s(Q)$ to $e \in \mathbb{Z}^n$, and which is equivariant with respect to translation by elements of $\mathbb{Z}^n$.

Lemma 8.7. $\pi_{\mathbb{Z}^n}$ *is Lipschitz.*

Proof. Note that if $g_1, g_2 \in G$, $h \in \mathbb{Z}^n$, and $g_2 = g_1h$, then $\pi_{\mathbb{Z}^n}(g_2) = \pi_{\mathbb{Z}^n}(g_1)h$, so $d_{\mathbb{Z}^n}(\pi_{\mathbb{Z}^n}(g_1), \pi_{\mathbb{Z}^n}(g_2)) = d_{\mathbb{Z}^n}(e,h)$. The properness of the distance function $d_{\mathbb{Z}^n}(\cdot, e)$ implies that there is a function $\delta : \mathbb{N} \to \mathbb{N}$ so that

[10] $d(f^{-1} \circ f, id_G)$ and $d(f \circ f^{-1}, id_{\mathbb{Z}^n \times Q})$ are both finite.

[11] The two metrics have uniformly bounded ratio.

for all $h \in \mathbb{Z}^n$,

$$d_{\mathbb{Z}^n}(h,e) \le \delta(d_G(h,e)). \tag{8.8}$$

To prove Lemma 8.7, it suffices to find an L such that

$$d_{\mathbb{Z}^n}(\pi_{\mathbb{Z}^n}(g_1), \pi_{\mathbb{Z}^n}(g_2)) \le L$$

whenever $d_G(g_1, g_2) = 1$. Consider the unique $g_3 \in g_1\mathbb{Z}^n$ which satisfies $\pi_{\mathbb{Z}^n}(g_3) = \pi_{\mathbb{Z}^n}(g_2)$, i.e. $g_3 \in g_1\mathbb{Z}^n \cap (\pi_{\mathbb{Z}^n}(g_2)s(Q))$. Then $d_G(g_3, g_2) \le C$ for some constant C because the composition $s \circ p$ is Lipschitz. Applying triangle inequalities and (8.8), we get

$$\begin{aligned} d_{\mathbb{Z}^n}(\pi_{\mathbb{Z}^n}(g_1), \pi_{\mathbb{Z}^n}(g_2)) &= d_{\mathbb{Z}^n}(\pi_{\mathbb{Z}^n}(g_1), \pi_{\mathbb{Z}^n}(g_3)) \\ &\le \delta(d_G(g_1, g_3)) \le \delta(1 + C). \end{aligned}$$

□

To finish the proof that $(2 \Longrightarrow 1)$, note that we have a bijection $\hat{f} : \mathbb{Z}^n \times Q \to G$ given by $\hat{f}(h,q) = hs(q)$. $\hat{f}$ is clearly $Lip(s)$-Lipschitz in the Q direction. That $\hat{f}$ is Lipschitz in the $\mathbb{Z}^n$ direction follows from the fact that $\mathbb{Z}^n$ is a central subgroup of G:

$$\begin{aligned} d_G(\hat{f}(h_1,q), \hat{f}(h_2,q)) &= d_G(h_1 s(q), h_2 s(q)) \\ &= d_G(h_1 h_2^{-1}, e) \le d_{\mathbb{Z}^n}(h_1 h_2^{-1}, e) = d_{\mathbb{Z}^n}(h_1, h_2). \end{aligned}$$

Letting $f = \hat{f}^{-1}$, we see that $f = (\pi_{\mathbb{Z}^n}, p)$ is a biLipschitz bijection.

$(2 \Longleftrightarrow 3)$. This follows from the obstruction theoretic interpretation of the characteristic class of the extension. Let K be a CW complex with finite 1-skeleton and one vertex, and which is an Eilenberg-Maclane space for Q. Let $P \to K$ be a principal T^n-bundle with characteristic class $[\alpha] \in H^2(K;\mathbb{Z}^n)$, so that the exact homotopy sequence $\pi_1(T^n) \to \pi_1(P) \to \pi_1(K)$ for the fibration $P \to K$ is isomorphic to (8.4). Let $\sigma : Skel_1(K) \to P$ be a section of P over the 1-skeleton of K. In the fiber over the point $Skel_0(K)$, choose a bouquet of n circles with vertex at $\sigma(Skel_0(K))$, which gives a standard basis for the fundamental group of the fiber. Let $M \subset P$ be the 1-complex consisting of the union of this bouquet of circles with the bouquet $\sigma(Skel_1(K)) \subset P$.

Let $\hat{P} \to \tilde{K}$ be the pullback of the bundle $P \to K$ under the covering projection $\tilde{K} \to K$, let $\hat{\sigma} : Skel_1(\tilde{K}) \to \hat{P}$ be the pullback of σ, and let

$\hat{M} \subset \hat{P}$ be the inverse image of M under the covering $\hat{P} \to P$. Finally, let $\tilde{P} \to \hat{P}$ be the universal covering, and let $\tilde{M} \subset \tilde{P}$ be the inverse image of $\hat{M}$ under $\tilde{P} \to \hat{P}$. Note that if we put path metrics on $Skel_1(\tilde{K})$ and $\tilde{M}$, then the projection map $Skel_0(\tilde{M}) \to Skel_0(\tilde{K})$ is naturally biLipschitz equivalent to $G \xrightarrow{p} Q$.

Now suppose 3 holds, and that $\alpha \in C^2_{L^\infty}(K;\mathbb{Z}^n) \subset C^2(K;\mathbb{Z}^n)$. We may assume that our section $\sigma : Skel_1(K) \to P$ was chosen so that the associated cellular obstruction cocycle is α. Then $\hat{\alpha}$, the image of α under the pullback $C^2_{L^\infty}(K;\mathbb{Z}^n) \to C^2_{L^\infty}(\tilde{K};\mathbb{Z}^n)$, is the obstruction cocycle for $\hat{\sigma} : Skel_1(\tilde{K}) \to \hat{P}$. By assumption, $\hat{\alpha} = \delta\theta$ for some $\theta \in C^1_{L^\infty}(\tilde{K};\mathbb{Z}^n)$. Hence we may modify $\hat{\sigma}$ using θ to get a new section $\hat{\sigma}_1 : Skel_1(\tilde{K}) \to \hat{P}$ with trivial obstruction cocycle. In particular, if $\tilde{P} \to \hat{P}$ is the universal covering map, then $\hat{\sigma}_1$ lifts to a section $\tilde{\sigma} : Skel_1(\tilde{K}) \to \tilde{P}$ of the $\mathbb{R}$-bundle $\tilde{P} \to \tilde{K}$. The fact that θ is an L^∞-cochain implies that $\tilde{\sigma}$ restricts to a 1-Lipschitz map from $Skel_0(\tilde{K})$ to $Skel_0(\tilde{M})$. Since the projection $Skel_0(\tilde{M}) \to Skel_0(\tilde{K})$ is biLipschitz equivalent to $G \to Q$, we get a Lipschitz section of p, so 2 holds.

Conversely, suppose 2 holds. Then we get a Lipschitz section $\tau : Skel_0(\tilde{K}) \to Skel_0(\tilde{M})$ of the projection $Skel_0(\tilde{M}) \to Skel_0(\tilde{K})$. We may extend τ to a section $\tilde{\sigma} : Skel_1(\tilde{K}) \to \tilde{P}$, and let $\hat{\sigma}_1 : Skel_1(\tilde{K}) \to \hat{P}$ be the composition of $\tilde{\sigma}$ with $\tilde{P} \to \hat{P}$. Notice that $\hat{\sigma}_1$ has trivial obstruction cocycle since it lifts to $\tilde{\sigma}$.

Lemma 8.9. *$\hat{\sigma}_1$ is obtained from $\hat{\sigma}$ by applying a bounded cochain $\theta \in C^1_{L^\infty}(\tilde{K};\mathbb{Z}^n)$.*

Proof. If e is a closed 1-cell in $Skel_1(\tilde{K})$, we want to show that the fixed endpoint homotopy classes of the two sections $\hat{\sigma}\big|_e : e \to \hat{P}$ and $\hat{\sigma}_1\big|_e : e \to \hat{P}$ (as maps into the inverse image of e in $\hat{P}$) agree up to bounded error. If $\gamma : [0,1] \to e$ is a characteristic map for e, lift the path $\hat{\sigma}\circ\gamma : [0,1] \to \hat{M} \subset \hat{P}$ to a path $\tilde{\gamma} : [0,1] \to \tilde{M} \subset \tilde{P}$ starting at $\tilde{\sigma}\circ\gamma(0)$. Then

$$\begin{aligned} d_{\tilde{M}}(\tilde{\gamma}(1), \tilde{\sigma}\circ\gamma(1)) &\leq d_{\tilde{M}}(\tilde{\gamma}(1), \tilde{\gamma}(0)) + d_{\tilde{M}}(\tilde{\gamma}(0), \tilde{\sigma}\circ\gamma(1)) \\ &= 1 + d_{\tilde{M}}(\tau(\gamma(0)), \tau(\gamma(1))) \\ &\leq 1 + L_\tau \end{aligned}$$

where L_τ is the Lipschitz constant of τ. But then $\tilde{\gamma}(1) = (\tilde{\sigma}\circ\gamma(1))h$ for some $h \in \mathbb{Z}^n$, and we can bound $d_{\mathbb{Z}^n}(h, e)$ by a constant C depending on L_τ, cf. (8.8). In other words, the fixed endpoint homotopy classes of $\hat{\sigma}\big|_e$ and

$\hat{\sigma}_1\big|_e$ (as maps from e to the inverse image of e in $\hat{P}$) differ by some $h \in \mathbb{Z}^n$ where $\|h\|_{\mathbb{Z}^n} < C$. □

If $\alpha \in C^2(K;\mathbb{Z}^n)$ is the obstruction cocycle for the section σ, then the pullback of α to $\hat{K}$ is the obstruction for $\hat{\sigma}$. As the obstruction for $\hat{\sigma}_1$ is 0, Lemma 8.9 gives

$$0 = \alpha + \delta\theta$$

for $\theta \in C^1_{L^\infty}(\tilde{K};\mathbb{Z}^n)$, so 3 holds. This completes the proof of Proposition 8.3. □

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