COMMUNICATIONS IN
ANALYSIS AND GEOMETRY
Volume 14, Number 1, 1-24, 2006

Degeneration of Kähler–Einstein Manifolds II: the Toroidal case

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In this paper, we prove that the Kähler–Einstein metrics for a toroidal canonical degeneration family of Kähler manifolds with ample canonical bundles Gromov–Hausdorff converge to the complete Kähler–Einstein metric on the smooth part of the central fiber when the base locus of the degeneration family is empty. We also prove the incompleteness of the Weil–Peterson metric in this case.

1. Introduction.

This paper is a sequel to [8]. In algebraic geometry, when discussing the compactification of the moduli space of complex manifold X with ample canonical bundle K_X, it is necessary to consider holomorphic degeneration family $\pi : \mathcal{X} \to B$, where $X_t = \pi^{-1}(t)$ are smooth for $t \neq 0$, $\mathcal{X}$ and X_0 are $\mathbb{Q}$-Gorenstein, such that the canonical bundle of X_t for $t \neq 0$ and the dualizing sheaf of X_0 are ample. We will call such degeneration *canonical degeneration*. We are interested in studying the degeneration behavior of the family of Kähler–Einstein metrics g_t on X_t when t approaches 0. Following his seminal proof of Calabi conjecture [13], Yau [11] initiated the program of studying the application of Kähler–Einstein metrics to algebraic geometry with the belief that the behavior of Kähler–Einstein metrics should reflect the topological, geometric and algebraic structure of the underlying complex algebraic manifolds. According to this philosophy, one would expect the metric degeneration of the Kähler–Einstein manifolds to be closely related to the algebraic degeneration of the underlying algebraic manifolds. In [9], Tian made the first important contribution along this direction. He proved that the Kähler–Einstein metrics on X_t converge to the complete Cheng-Yau Kähler–Einstein metric on the smooth part of X_0 in the sense of Cheeger–Gromov, when $\mathcal{X}$ is smooth and the central fibre X_0 is the union of smooth normal crossing divisors $D_1, \cdots, D_L$, with a technical restriction that no

[1]Partially supported by NSF Grant DMS-0104150.

three divisors have common intersection. Following the general framework in Tian's paper ([9]), [4] and later [8] studied the general normal crossing case and removed the technical restriction in [9].

In this paper, we generalize the result in [8] to the case when the central fibre X_0 is a union of toroidal orbifolds that results from the so-called toroidal canonical degeneration of smooth X_t (see Section 2 for definitions). The total space $\mathcal{X}$ for this kind of degeneration will be toroidal and generally not smooth. Please note that a toroidal canonical degeneration, where X_0 is not normal crossing in $\mathcal{X}$, cannot be reduced to a normal crossing canonical degeneration. The normal crossing case is a very special case of toroidal canonical degeneration. For an algebraic curve, a toroidal canonical degeneration is equivalent to Deligne–Mumford stable degeneration into stable curves.

In this paper, we always require an algebraic variety X to possess a (set-theoretical) *canonical* (Whitney) *stratification* $X = \bigcup_{p\in\Sigma} D_p$ by smooth algebraic strata. By "canonical", we mean that any other (Whitney) stratification $X = \bigcup_{p'\in\Sigma'} D'_{p'}$ by smooth strata is a refinement of the canonical (Whitney) stratification. More precisely, we have $D'_{p'} \subset D_p$ when $D'_{p'} \cap D_p \neq \emptyset$. For example, the toroidal varieties defined in Section 2 satisfy such requirement.

The degeneration family is called *base point free* if each smooth strata of X_0 is inside a smooth strata of $\mathcal{X}$. The smoothness condition of $\mathcal{X}$ when X_0 is normal crossing is equivalent to requiring the degeneration family to be base point free. In some sense, toroidal canonical degenerations that we consider in this paper are generic base point free canonical degenerations. (Toroidal canonical degenerations and related concepts and constructions are discussed in Section 2.)

Our first main theorem (proved in Section 5) is the following.

Theorem 1.1. *Let $\pi : \mathcal{X} \to B$ be a toroidal canonical degeneration of Kähler–Einstein manifolds $\{X_t, g_{E,t}\}$ with $\mathrm{Ric}(g_{E,t}) = -g_{E,t}$. Then, the Kähler–Einstein metrics $g_{E,t}$ on X_t converge in the sense of Cheeger–Gromov to a complete Cheng–Yau Kähler–Einstein metric $g_{E,0}$ on the smooth part of the canonical limit X'_0 (which is a finite cover of the central fibre X_0).*

To prove this theorem, we follow the three steps outlined in [8]. The first step is to construct certain smooth family of background Kähler metrics $\hat{g}_t$

on X_t and their Kähler potential volume forms $\hat{V}_t$. The second step is to construct a smooth family of approximate Kähler metrics g_t with Kähler form $\omega_t = \frac{i}{2\pi}\partial\bar{\partial}\log V_t$, where $V_t = h\hat{V}_t$ (h is a function on $\mathcal{X}$) satisfies a certain uniform estimate independent of t. The third step is to use Monge–Ampère estimate of Aubin [1] and Yau [13] to derive a uniform estimate (independent of t) for the smooth family of Kähler–Einstein metrics $g_{E,t}$, starting with the smooth family of approximate Kähler metric g_t, which is enough to ensure the Gromov–Hausdorff convergence of the family to the unique complete Kähler–Einstein metric $g_{E,0} = \{g_{0,i}\}_{i=1}^l$ on the smooth part of X_0. The first and the third steps are carried out in the very brief Sections 3 and 5 and are virtually the same as in the normal crossing case [8]. The second step, carried out in Section 4, is more involved than the simple global construction in [8].

The following similar but much more non-trivial (comparing to [8]) estimate of the Weil–Peterson metric near the degeneration, which implies the incompleteness of the Weil–Peterson metric, is worked out in Section 6.

Theorem 1.2. *The restriction of the Weil–Peterson metric on the moduli space of complex structures to the toroidal canonical degeneration* $\pi : \mathcal{X} \to B$ *is bounded from above by a constant multiple of* $\frac{dt \wedge d\bar{t}}{|\log|t||^3|t|^2}$. *In particular, Weil–Peterson metric is incomplete at* $t = 0$.

Note on notation: We say $A \sim B$ if there exist constants $C_1, C_2 > 0$ such that $C_1 B \leq A \leq C_2 B$.

2. Toroidal canonical degeneration.

In this section, we introduce the concepts of toric degeneration and toroidal degeneration, and discuss the details of relevant stratification structures and the construction of compatible partition functions that we need for the construction of approximate metrics in Section 4.

2.1. Toric degenerations.

(Unless specified otherwise, the notations in this subsection will not be carried over to other parts of this paper.)

Let us first introduce the basic notions in toric geometry. An $(n+1)$-dimensional affine toric variety A_{σ_0} is determined by a strongly convex $(n+1)$-dimensional integral polyhedral cone σ_0 in a rank $n+1$ lattice $\tilde{M}$.

Let $\sigma_0(k)$ denote the set of k-dimensional subfaces of σ_0. Then, $\sigma_0(n)$ corresponds to toric Weil divisors $\{D_i\}_{i\in\sigma_0(n)}$ in A_{σ_0}, and $\sigma_0(1)$ corresponds to toric Cartier divisors $\{(f_i)\}_{i\in\sigma_0(1)}$ in A_{σ_0}. For $i\in\sigma_0(1)$,

$$(f_i)=\sum_{j\in\sigma_0(n)} a_{ij}D_j,$$

where a_{ij} is the natural pairing of the primitive elements in $i\in\sigma_0(1)$ and $j\in\sigma_0^\vee(1)\cong\sigma_0(n)$. $\sigma_0^\vee$ denotes the dual cone of σ_0.

A toric map $\pi:\mathcal{X}\to B\cong\mathbb{C}$ is called a *toric degeneration*, if $\mathcal{X}=A_{\sigma_0}$ is an affine toric variety such that $\mathcal{X}\setminus X_0$ is the big open torus. Consequently, X_t for $t\neq 0$ are codimension one subtori in $\mathcal{X}\setminus X_0$. A toric degeneration is determined by a strongly convex integral polyhedral cone $\sigma_0\subset\tilde{M}$ with a marked primitive element t in the interior of the cone σ_0. Under such notation, the central fibre is

$$X_0=D=\bigcup_{i\in\sigma_0(n)} D_i.$$

Let $M=\tilde{M}/\mathbb{Z}\{t\}$. Since t is in the interior of σ_0, the projection of σ_0 to M determines a complete fan Σ on M. Splittings $\tilde{M}\cong M\times\mathbb{Z}\{t\}$ can be parameterized (non-canonically) by $\mathbb{Z}$-valued linear functions on M. Each such splitting realizes σ_0 as a $\mathbb{Q}$-valued function w_{σ_0} on the lattice M. In such a way, σ_0 can be understood as an equivalence class $[w_{\sigma_0}]$ (modulo $\mathbb{Z}$-valued linear functions) of convex piecewise linear $\mathbb{Q}$-valued functions on the lattice M that are compatible with a complete fan Σ in M. Let $\Sigma(k)$ denote the set of k-dimensional cones in Σ. Naturally, $\Sigma(k)\cong\sigma_0(k)$ for $1\leq k\leq n$. We will use $\tilde{\sigma}\in\sigma_0(k)$ to denote the cone corresponding to $\sigma\in\Sigma(k)$.

For each $\sigma\in\Sigma$, there is an affine variety $A_\sigma=\mathrm{Spec}(\mathbb{C}[\sigma])$. For $\sigma,\sigma'\in\Sigma$ satisfying $\sigma\subset\sigma'$, there is a natural semi-group morphism $\sigma'\to\sigma$ that restricts to identity map on $\sigma\subset\sigma'$ and restricts to zero map on $\sigma'\setminus\sigma$, which induces the map $h_{\sigma\sigma'}:A_\sigma\to A_{\sigma'}$. Using $\{h_{\sigma\sigma'}\}_{\sigma,\sigma'\in\Sigma}$, we may glue the affine pieces $\{A_\sigma\}_{\sigma\in\Sigma}$ into the singular variety X_Σ. We have the following natural canonical (Whitney) stratification

$$X_\Sigma=\bigcup_{\sigma\in\Sigma} T_\sigma,\ \text{ where } T_\sigma=(\mathrm{Span}_{\mathbb{Z}}\sigma)^\vee\otimes_{\mathbb{Z}}\mathbb{C}^*=(M^\vee/\sigma^\perp)\otimes_{\mathbb{Z}}\mathbb{C}^*. \tag{2.1}$$

In such a way, Σ determines a singular variety X_Σ that is a mirror dual to the usual toric variety P_Σ in certain sense.

For $\sigma\in\Sigma$, the natural injection $\tilde{\sigma}\hookrightarrow\sigma$ over $\mathbb{Z}$ induces a cover map $p_\sigma:A_\sigma\to A_{\tilde{\sigma}}$ and subsequently, $q_\sigma=h_{\tilde{\sigma}\sigma_0}\circ p_\sigma:A_\sigma\to A_{\sigma_0}=\mathcal{X}$. It is easy to check that p_σ, q_σ for $\sigma \in \Sigma$ glue together to form the maps $p_\Sigma : X_\Sigma \to X_0$, $q_\Sigma : X_\Sigma \to \mathcal{X}$.

Recall that a complex torus has a canonical *toric holomorphic volume form*, and consequently, a canonical real *toric volume form*. Via this toric holomorphic volume form on the complex torus $\mathcal{X} \setminus X_0$, the dualizing sheaf $K_\mathcal{X}$ can be naturally identified with $\mathcal{O}_\mathcal{X}(-D)$. We call π *simple* when each divisor D_i is of multiplicity one under π. Then, the Cartier divisor $(t) = D$, and the dualizing sheaf $K_\mathcal{X}$ is a line bundle.

Proposition 2.1. *A toric degeneration $\pi : \mathcal{X} \to B \cong \mathbb{C}$ is simple if and only if $[w_{\sigma_0}]$ is $\mathbb{Z}$-valued on M if and only if $q_\Sigma : X_\Sigma \to \mathcal{X}$ is an imbedding (or equivalently, $p_\Sigma : X_\Sigma \to X_0$ is an isomorphism).*

Proof. For $\tilde{\sigma} \in \sigma_0(n)$, it is straightforward to check that the multiplicity of t along $D_{\tilde{\sigma}}$ is $|(\mathrm{Span}_\mathbb{Z}\sigma)/(\mathrm{Span}_\mathbb{Z}\tilde{\sigma})|$. Consequently, π is simple if and only if for each $\sigma \in \Sigma(n)$, the natural injection $\tilde{\sigma} \hookrightarrow \sigma$ over $\mathbb{Z}$ is bijection (which amounts to that $[w_{\sigma_0}]$ is $\mathbb{Z}$-valued on σ) if and only if $p_\sigma : A_\sigma \to A_{\tilde{\sigma}}$ is an isomorphism for each $\sigma \in \Sigma(n)$. These local results together imply the proposition. □

Proposition 2.2. *For a toric degeneration $\pi : \mathcal{X} \to B \cong \mathbb{C}$, let d be the smallest positive integer so that $d[w_{\sigma_0}]$ is $\mathbb{Z}$-valued. Then, the canonical d'-fold base extension $\pi' : \mathcal{X}' \to B'$ is a simple toric degeneration if and only if $d|d'$.*

Proof. It is easy to see that the canonical d'-fold base extension $\pi' : \mathcal{X}' \to B'$ is determined by $d'\sigma_0 \subset \tilde{M}$. Since $[w_{d'\sigma_0}] = d'[w_{\sigma_0}]$ is $\mathbb{Z}$-valued if and only if $d|d'$, by Proposition 2.1, we get the desired conclusion. □

Remark: Propositions 2.1 and 2.2 are well known. (A special case of Proposition 2.2, where Σ is simplicial fan, was proved and used in the proof of the semistable reduction theorem [5] by Mumford and Kun.) We provide simple proofs of them here for the convenience of the readers.

$\Sigma(1)$ can be equivalently interpreted as the set of primitive generating elements of 1-dimensional cones in Σ. The piecewise linear function w_{σ_0} is determined by $\{w_m\}_{m\in\Sigma(1)}$, with $w_m = w_{\sigma_0}(m) \in \mathbb{Q}$ for $m \in \Sigma(1)$. The toric degeneration family can be equivalently characterized by the following family of toric immersions:

$$i_t : N_{\mathbb{C}^*} \to \mathbb{C}^{|\Sigma(1)|}$$

defined as $\{t^{w_m} z^m\}_{m\in\Sigma(1)}$, where $N = M^\vee$ and $N_{\mathbb{C}^*} = (N \otimes_{\mathbb{Z}} \mathbb{C})/N$. We are also interested in generalized toric degenerations, where $w_m \in \mathbb{R}$ are not necessarily rational.

Example: The simplest toric degenerations that are not normal crossing are:
(1) $X_t = \{z \in \mathbb{C}^4 | z_1 z_2 = z_3 z_4 = t\}$ (product of normal crossing degenerations).
(2) $X_t = \{z \in \mathbb{C}^4 | z_1 z_2 = t,\ z_3 z_4 = t z_1\}$. □

Remark: *A priori*, the piecewise linear function f generated by $\{w_m\}_{m\in\Sigma(1)}$ need not be convex. Then, we may take the largest convex function $\tilde{f} \leq f$. The piecewise linear convex function $\tilde{f}$ will be generated by $\{w_m\}_{m\in\tilde{\Sigma}(1)}$, where $\tilde{\Sigma}(1)$ is a subset of $\Sigma(1)$. There is a natural projection $P : \mathbb{C}^{|\Sigma(1)|} \to \mathbb{C}^{|\tilde{\Sigma}(1)|}$. It is easy to check that P induces an equivalence between the toric degeneration families determined by toric embeddings i_t and $\tilde{i}_t = P \circ i_t$. Therefore, we only need to consider the case when f is convex. For f *generic*, the fan it determines is a *simplicial* fan. Namely, the toric divisors are all *toric orbifolds.*

2.2. Toroidal degenerations.

A holomorphic degeneration $\pi : \mathcal{X} \to B = \{t \in \mathbb{C} : |t| < 1\}$ is called a *toroidal degeneration* if it is locally toric. Let

$$X_0 = \bigcup_{k=0}^{n} X_0^{(k)} = D = \bigcup_{p\in\Sigma} D_p,\ X_0^{(k)} = \bigcup_{p\in\Sigma(k)} D_p,\ \Sigma = \bigcup_{k=0}^{n} \Sigma(k) \tag{2.2}$$

be the canonical stratification for X_0, with $\{D_p\}_{p\in\Sigma}$ parameterizing all the strata and $X_0^{(k)}$ denoting the union of all k-dimensional strata. π is called *simple* if each divisor $\bar{D}_p$ is of multiplicity 1 under π for $p \in \Sigma(n)$. Propositions 2.1 and 2.2 imply the following generalization to toroidal case.

Proposition 2.3. *For a toroidal degeneration $\pi : \mathcal{X} \to B$, there exists an integer $d > 0$ such that the canonical d'-fold base extension $\pi' : \mathcal{X}' \to B'$ is a simple toroidal degeneration if and only if $d|d'$. X_0' (which will be called the* canonical limit*) is independent of d' satisfying $d|d'$ with the natural finite cover map $X_0' \to X_0$.*

Proof. Since d'-fold base extension is canonical and local, the $d > 0$ here can be taken to be the lowest common multiple of the d's specified in Proposition 2.2 for all local toric models. When $d|d'$, namely π' is simple, Proposition 2.1 implies that X'_0 restricted to each local toric model can be identified with X_Σ in Proposition 2.1, therefore is canonical and independent of d'. □

Through Proposition 2.3, all discussions for toroidal degeneration can be reduced to discussions for simple toroidal degeneration via base extension. For this reason, we will always assume that π is simple. Consequently, the dualizing sheaf $K_{\mathcal{X}}$ is a line bundle. For such generic degeneration π, the Weil divisors $\bar{D}_p$ for $p \in \Sigma(n)$ are *toroidal orbifolds*. Without loss of generality and for simplicity of notations, we will also assume that each $\bar{D}_p$ does not self-intersect.

Choose a suitable tubular neighborhood $\tilde{U}_p$ of $\bar{D}_p$ for each $p \in \Sigma$ such that for any $p_1, p_2 \in \Sigma$, we have

$$\tilde{U}_{p_1} \cap \tilde{U}_{p_2} \subset \bigcup_{q \in \Sigma, D_q \in \bar{D}_{p_1} \cap \bar{D}_{p_2}} \tilde{U}_q.$$

For each $p \in \Sigma$, we can construct a tubular neighborhood U_p of D_p as $\tilde{U}_p$ minus the union of divisors $\bar{D}_q$ for $q \in \Sigma(n)$ satisfying $D_p \not\subset \bar{D}_q$. We will also need $U_p^0 \subset U_p$ defined as $\tilde{U}_p$ minus the union of (slightly shrunk) $\overline{\tilde{U}_q}$ for $q \in \Sigma(n)$ satisfying $D_p \not\subset \bar{D}_q$. Let $D_p^0 = D_p \cap U_p^0$. Since

$$\bigcup_{p \in \Sigma} U_p = \bigcup_{p \in \Sigma} U_p^0$$

forms a neighborhood of X_0 that contains X_t for t small, many of our discussions on X_t can be reduced locally to either $U_p \cap X_t$ or $U_p^0 \cap X_t$ for $p \in \Sigma$. Notice that for any $p_1, p_2 \in \Sigma$, we also have

$$U_{p_1} \cap U_{p_2} \subset \bigcup_{q \in \Sigma, D_q \in \bar{D}_{p_1} \cap \bar{D}_{p_2}} U_q, \quad U_{p_1}^0 \cap U_{p_2}^0 \subset \bigcup_{q \in \Sigma, D_q \in \bar{D}_{p_1} \cap \bar{D}_{p_2}} U_q^0.$$

Locally, $U_p = A_p \times D_p$ and $U_p^0 = A_p \times D_p^0$. A_p is a neighborhood of the origin of the affine toric local model determined by the fan Σ_p and the integral convex function $\{w_m\}_{m \in \Sigma_p(1)}$ (notation as in 2.1). Let $|p| := \dim D_p$ and $l = n - |p|$. $\Sigma_p(l)$ (which can be naturally identified with a subset of $\Sigma(n)$) corresponds to toroidal Weil divisors $\{D_q \cap U_p\}_{q \in \Sigma_p(l) \subset \Sigma(n)}$ in U_p. $\Sigma_p(1)$ corresponds to toroidal Cartier divisors $\{(s_m)\}_{m \in \Sigma_p(1)}$ in U_p containing D_p. We may choose local coordinate $(t, z, \tilde{z})$ for U_p, $z = (z_1, \cdots, z_l)$,

$\tilde{z} = (z_{l+1}, \cdots, z_n)$, so that $s_m = t^{w_m} z^m$, (t, z) and $\tilde{z}$ form coordinates for A_p and D_p. $(z, \tilde{z})$ can be considered as coordinate for $X_t \cap U_p$. For $m \in \Sigma_p(1)$, s_m can be viewed as a section of a line bundle on U_p that defines the Cartier divisor. One can choose a Hermitian metric $\|\cdot\|_m$ on the line bundle over U_p^0 such that $\|s_m\|_m \leq 1$ and $\|s_m\|_m = 1$ outside a small neighborhood of the Cartier divisor (s_m). More precisely, we require that $\|s_m\|_m = 1$ on U_q^0 for $q \in \Sigma$ when s_m is non-vanishing on D_q.

For $p, q \in \Sigma$ satisfying $D_q \subset \bar{D}_p$, Cartier divisors in U_p can be naturally extended to certain $\mathbb{Q}$-Cartier divisors in U_q that can be expressed by the natural injective map $e_{pq} : \Sigma_p(1) \to \Sigma_q(1)$. By suitably adjusting the Hermitian metric of the line bundle, for $m \in \Sigma_p(1)$, we may assume that $\|s_m\|_m = \|s_{e_{pq}(m)}\|_{e_{pq}(m)}$ in the common domain $U_p^0 \cap U_q^0$. It is easy to check that $e_{pq'} = e_{qq'} \circ e_{pq}$ for $q' \in \Sigma$ satisfying $D_{q'} \subset \bar{D}_q$. Therefore, the Cartier divisor (s_m) in U_p^0 for $m \in \Sigma_p(1)$ naturally extends to the $\mathbb{Q}$-Cartier divisor (still denoted by (s_m)) in $\tilde{U}_p$. $\|s_m\|_m$ for $m \in \Sigma_p(1)$ can similarly be extended from U_p^0 to $\tilde{U}_p$.

Let $\Sigma^p(1)$ denote the set of $q \in \Sigma(|p| + 1)$ satisfying $D_q \subset \bar{D}_p$. For $q \in \Sigma^p(1)$, D_p can be naturally identified with an element $[D_p] \in \Sigma_q(|q| - |p|) = \Sigma_q(1)$, which can also be viewed as a Cartier divisor s_q in $\tilde{U}_p$ supported in $\tilde{U}_p \setminus U_p$. $\Sigma_p^p = \Sigma_p(1) \cup \Sigma^p(1)$ (resp. $\Sigma^p(1)$) can be characterized as the set of Cartier divisors on $\tilde{U}_p$ whose defining functions are not identically zero (resp. nowhere zero) on $X_0 \cap U_p$.

A (*holomorphic*) *volume form* on $U_p \setminus D$ is called *toroidal* if its pullback to the local toric model differs from the standard toric (holomorphic) volume form by a bounded nowhere zero (holomorphic) factor on U_p. By examining the holomorphic toric volume form, it is easy to see that a holomorphic toroidal volume form on $\mathcal{X} \setminus D$ can be naturally identified with a nowhere zero holomorphic section of $K_{\mathcal{X}}(D)$, or in another word, a meromorphic section of $K_{\mathcal{X}}$ with a pole of order 1 along D.

2.3. Partition functions.

Let $\mu(x)$ be a smooth increasing function on $\mathbb{R}$ with bounded derivatives satisfying $\mu(x) = 0$ for $x \leq 0$ and $\mu(x) = 1$ for $x \geq 1$. Let $\min'(x_1, \cdots, x_l)$ be a smooth function with bounded derivatives that coincide with $\min(x_1, \cdots, x_l)$ when $\min\limits_{i \neq j}(|x_i - x_j|) \geq 1$. (In another word, $\min'(x_1, \cdots, x_l)$ is a smoothing of $\min(x_1, \cdots, x_l)$ with bounded derivatives.)

For each $p \in \Sigma$ and $\eta > 0$ large, we may define the smooth function

$$\tilde{\mu}_p = \mu\left(\frac{1}{\log(\tau/\eta^2)}\min'\left(\{\log(a_m/\eta)\}_{m\in\Sigma_p(1)}, \{\log(\tau/a_m\eta)\}_{m\in\Sigma^p(1)}\right)\right),$$

where $\tau = -\log|t|^2$ and $a_m = \eta - \log\|s_m\|_m^2$. These will give us the partition functions $\{\mu_p\}_{p\in\Sigma}$, where $\mu_p = \tilde{\mu}_p\left(\sum_{p\in\Sigma}\tilde{\mu}_p\right)^{-1}$. We generally have $D_p^0 \subset \operatorname{supp}(\mu_p) \subset U_p$. The condition on $\|\cdot\|_m$ implies that

$$U_p^0 \cap \operatorname{supp}(\mu_q) = \emptyset \quad \text{when} \quad D_p \not\subset \bar{D}_q. \tag{2.3}$$

3. Construction of the background metric.

For construction in this section to work, it is necessary to assume that the dualizing line bundle $K_{\mathcal{X}}$ of the total space $\mathcal{X}$ exists and is ample, which is valid in our situation. (The construction in this section is partially inspired by our work [7] on Bergmann metrics.) Recall that $K_{\mathcal{X}/B} = K_{\mathcal{X}} \otimes K_B^{-1}$ and $K_{X_t} = K_{\mathcal{X}/B}|_{X_t} \cong K_{\mathcal{X}}|_{X_t}$. (The last equivalence is not canonical, depending on the trivialization $K_B \cong \mathcal{O}_B$. We will use dt to fix the trivialization of K_B.) Since K_{X_t} is ample for all t, certain multiple $K_{X_t}^m$ will be very ample for all t. Equivalently, $K_{\mathcal{X}}^m$ is very ample on $\mathcal{X}$. It is not hard to find sections $\{\Omega_k\}_{k=0}^{N_m}$ of $K_{\mathcal{X}}^m$ that determine an embedding $e : \mathcal{X} \to \mathbb{CP}^{N_m}$, such that $\{\Omega_{t,k}\}_{k=0}^{N_m}$ forms a basis of $H^0(K_{X_t})$ for all t, where $\Omega_{t,k} = (\Omega_k \otimes (dt)^{-m})|_{X_t}$. $\{\Omega_{t,k}\}_{k=0}^{N_m}$ will determine a family of embedding $e_t : X_t \to \mathbb{CP}^{N_m}$ such that $e_t = e|_{X_t}$. Choose the Fubini–Study metric ω_{FS} on $\mathbb{CP}^{N_m}$, and define

$$\hat{\omega} = \frac{1}{m}e^*\omega_{FS}, \quad \hat{\omega}_t = \hat{\omega}|_{X_t} = \frac{1}{m}e_t^*\omega_{FS}.$$

Since $K_{\mathcal{X}}^m$ is very ample on $\mathcal{X}$, $\hat{\omega}$ is a smooth metric on $\mathcal{X}$. The Kähler potential of $\hat{\omega}$ and $\hat{\omega}_t$ are the logarithm of the volume forms

$$\hat{V} = \left(\sum_{k=0}^{N_m}\Omega_k \otimes \bar{\Omega}_k\right)^{\frac{1}{m}}, \text{ and } \hat{V}_t = \left(\sum_{k=0}^{N_m}\Omega_{t,k} \otimes \bar{\Omega}_{t,k}\right)^{\frac{1}{m}} = \hat{V} \otimes (dt \otimes d\bar{t})^{-1}\Big|_{X_t}.$$

Since $K_{\mathcal{X}}^m$ is ample and therefore base point free, $\hat{V}$ is a non-degenerate smooth volume form on $\mathcal{X}$. Recall $(t) = D$. Hence, $\frac{\hat{V}}{|t|^2}$ is a toroidal volume form on $\mathcal{X}$. On the other hand, $\frac{dt}{t}$ is the standard toric holomorphic form on B. Therefore,

$$\hat{V}_t = \left(\sum_{k=0}^{N_m} \Omega_{t,k} \otimes \bar{\Omega}_{t,k}\right)^{\frac{1}{m}} = \hat{V} \otimes (dt \otimes d\bar{t})^{-1}\Big|_{X_t} = \frac{\hat{V}}{|t|^2} \otimes \left(\frac{dt}{t} \otimes \frac{d\bar{t}}{\bar{t}}\right)^{-1}\Bigg|_{X_t}$$

is also toroidal, namely

$$\hat{V}_t = \rho(t,z,\tilde{z}) \left(\prod_{j=1}^{l} \frac{dz_j d\bar{z}_j}{|z_j|^2}\right) \left(\prod_{j=l+1}^{n} dz_j d\bar{z}_j\right) \tag{3.1}$$

under the coordinate $(z, \tilde{z})$ for $X_t \cap U_p$, where $\rho(t,z,\tilde{z}) \sim 1$ is a smooth positive function on U_p.

Since $e : \mathcal{X} \to \mathbb{CP}^{N_m}$ is an embedding, locally in U_p, there exists a decomposition $e = \hat{e} \circ i_{\Sigma_p}$, where $i_{\Sigma_p} = (s_{\Sigma_p}, \tilde{z}) : \mathcal{X} \to \mathbb{C}^{|\Sigma_p(1)|+|p|}$ and $\hat{e} : \mathbb{C}^{|\Sigma_p(1)|+|p|} \to \mathbb{CP}^{N_m}$ are smooth embeddings and $s_{\Sigma_p} = (s_m)_{m \in \Sigma_p(1)}$. Therefore,

$$\hat{\omega} = \sum_{m,m' \in \Sigma_p(1)} g_{mm'}(s_{\Sigma_p}, \tilde{z}) ds_m d\bar{s}_{m'} + (\text{terms involving } d\tilde{z}, d\bar{\tilde{z}}). \tag{3.2}$$

4. Construction of the approximate metric.

The approximate metric is constructed by gluing together appropriate metrics on the neighborhood of each strata by the partition functions constructed in Section 2.3.

For $p \in \Sigma$ and $m \in \Sigma_p^p$, in U_p, define

$$h_p = \tau^{2(|\Sigma_p(1)|-l)} \prod_{m \in \Sigma_p^p} \frac{\eta^2}{a_m^2}, \quad a_m = \eta - \log \|s_m\|_m^2, \ \tau = -\log|t|^2.$$

On X_t, let $V_t = h\hat{V}_t$, where $\log h = \sum_{p \in \Sigma} \mu_p \log h_p$, and let

$$\omega_t = \frac{i}{2\pi}\partial\bar{\partial}\log V_t = \hat{\omega}_t + \frac{i}{2\pi}\partial\bar{\partial}\log h = \hat{\omega}_t + \gamma_t + \alpha_t,$$

where

$$\alpha_t = \sum_{p\in\Sigma} \mu_p \alpha_{t,p}, \quad \alpha_{t,p} = \frac{i}{\pi} \sum_{m\in\Sigma_p^p} \frac{1}{a_m^2} \partial a_m \bar{\partial} a_m,$$

$$\gamma_t = \sum_{p\in\Sigma} \mu_p \sum_{m\in\Sigma_p(1)} \frac{2}{a_m} \mathrm{Ric}(\|\cdot\|_m)$$
$$+ \frac{i}{2\pi} \sum_{p\in\Sigma} (\log h_p \partial\bar{\partial}\mu_p + \partial \log h_p \bar{\partial}\mu_p + \partial\mu_p \bar{\partial} \log h_p).$$

The main result of this section is the estimate (Propositions 4.6 and 4.7) on the approximate Kähler metric g_t with the Kähler form ω_t on X_t.

Since Σ_p is a simplicial fan, $\sigma \in \Sigma_p(l)$ naturally corresponds to a subset $S_\sigma \subset \Sigma_p(1)$ with l elements.

Proposition 4.1. *There exist $\lambda_1, \lambda_2 > 0$ such that $\log \|s_m\|_m^2 \geq \lambda_2 \log |t|^2$ on U_p^0 for all $m \in \Sigma_p(1)$. And for any $x \in U_p^0$, $S_x = \{m \in \Sigma_p(1) | \log \|s_m(x)\|^2 \geq \lambda_1 \log |t|^2\} \subset S_\sigma$ for some $\sigma \in \Sigma_p(l)$.*

Proof. Since $\log \|s_m\|_m^2 = \log |s_m|^2 + O(1)$, where $|s_m|$ is the absolute value of s_m viewed as monomial in the toric local model, it is sufficient to prove the proposition for $\log |s_m|^2$ in the place of $\log \|s_m\|_m^2$. For $m \in \Sigma_p(1)$, there exists $\sigma \in \Sigma_p(l)$ such that $-m$ belongs to the cone spanned by S_σ. Namely,

$$m = -\sum_{m'\in S_\sigma} b_{m'} m'$$

where $b_{m'} \geq 0$ for all $m' \in S_\sigma$. Therefore,

$$\log |s_m|^2 = w_m \log |t|^2 + \log |z^m|^2 = w_m \log |t|^2 - \sum_{m'\in S_\sigma} b_{m'} \log |z^{m'}|^2$$
$$= (w_m + \sum_{m'\in S_\sigma} b_{m'} w_{m'}) \log |t|^2 - \sum_{m'\in S_\sigma} b_{m'} \log |s_{m'}|^2$$
$$\geq (w_m + \sum_{m'\in S_\sigma} b_{m'} w_{m'}) \log |t|^2.$$

We may take λ_2 to be the maximum of such $(w_m + \sum_{m'\in S_\sigma} b_{m'} w_{m'})$.

Take a subset $S' \subset S$ such that S' span a simplicial cone and $S' \not\subset S_\sigma$ for any $\sigma \in \Sigma_p(l)$. There exists a linear function v_m on M such that $v_m =$

$\frac{\log|s_m|^2}{\log|t|^2} \leq \lambda_1$ for $m \in S'$ and $|v_m| \leq C\lambda_1$ for $m \in \Sigma_p(1) \setminus S'$. Then,

$$w'_m = \frac{\log|s_m|^2}{\log|t|^2} - v_m = w_m + \frac{\log|z^m|^2}{\log|t|^2} - v_m$$

is an adjustment of w_m by a linear function on M, such that $w'_m = 0$ for $m \in S'$. Since $S' \not\subset S_\sigma$ for any $\sigma \in \Sigma_p(l)$. The strict convexity of $\{w_m\}_{m\in\Sigma_p(1)}$ implies that there exists an $m' \in \Sigma_p(1) \setminus S'$ such that $w'_{m'} < 0$ is the smallest. Take λ_3 to be the maximum of such $w'_{m'} < 0$ for all possible S', which have only finite many possibilities. Then, $\lambda_3 < 0$ and

$$\frac{\log|s_{m'}|^2}{\log|t|^2} = w'_{m'} + v_{m'} \leq \lambda_3 + C\lambda_1.$$

We may take $\lambda_1 > 0$ to be small so that $\lambda_3 + C\lambda_1 < 0$. Then, $|s_{m'}|^2$ has to be big, contradicting the fact that $|s_{m'}|^2$ is small in U_p. Therefore, $S \subset S_\sigma$ for some $\sigma \in \Sigma_p(l)$. □

Lemma 4.2.

$$\gamma_t = O(\hat{\omega}_t/\eta) + O((\hat{\omega}_t + \alpha_t)/\log\tau), \quad \text{where} \quad \tau = -\log|t|^2.$$

Proof. In the argument of this paper, we will always first fix $\eta > 0$ large and then take τ large according to the fixed η. By our construction, $a_m \geq \eta$ is large. Hence,

$$\sum_{p\in\Sigma} \mu_p \sum_{m\in\Sigma_p(1)} \frac{2}{a_m}\mathrm{Ric}(\|\cdot\|_m) = O(\hat{\omega}_t/\eta).$$

For any $x \in X_t$, there exist a $q \in \Sigma$ such that $x \in X_t \cap U_q^0$. Since

$$\sum_{p\in\Sigma} \mu_p = 1, \quad \sum_{p\in\Sigma} \partial\bar{\partial}\mu_p = 0.$$

We have

$$\sum_{p\in\Sigma} \log h_p \partial\bar{\partial}\mu_p = \sum_{p\in\Sigma} (\log h_p - \log h_q)\partial\bar{\partial}\mu_p.$$

Since $U_q^0 \cap \mathrm{supp}(\mu_p) = \emptyset$ when $D_q \not\subset \bar{D}_p$ according to (2.3), we may consider only those $p \in \Sigma$ satisfying $D_q \subset \bar{D}_p$. Then, there are the natural inclusions $\Sigma^q(1) \subset \Sigma^p(1)$, $\Sigma_p(1) \subset \Sigma_q(1)$ and the Cartier divisors in $\Sigma^p(1) \setminus \Sigma^q(1)$

vanishing along D_q can be naturally identified with a subset of $\Sigma_p(1)\setminus\Sigma_q(1)$. Under such identifications $\Sigma_q^q\cap\Sigma_p^p$ is defined. For any $m\in\Sigma_p^p\setminus\Sigma_q^q$, $(s_m)\cap\bar{D}_q=\emptyset$. Consequently, $\|s_m\|_m^2=1$ and $a_m=\eta$ on $\tilde{U}_q$ for $m\in\Sigma_p^p\setminus\Sigma_q^q$. Hence,

$$\log h_p-\log h_q=2\sum_{m\in\Sigma_q^q\setminus\Sigma_p^p}\log\frac{a_m}{\tau\eta}$$

is bounded on $\mathrm{supp}(\mu_p)\cap U_q\subset U_p\cap U_q$. From the explicit expressions of μ_p and h_p, it is straightforward to check that $\partial\bar{\partial}\mu_p=O(1/\log\tau)$, $\partial\log h_p=O(1)$ and $\partial\mu_p=O(1/\log\tau)$ with respect to the Hermitian metric $\hat{\omega}_t+\alpha_t$. (Such kind of verification is more carefully done in the proof of Proposition 4.5 using (4.1).) Consequently, $\gamma_t=O(\hat{\omega}_t/\eta)+O((\hat{\omega}_t+\alpha_t)/\log\tau)$. □

For $\sigma\in\Sigma_p(l)$, let $A_\sigma(x)=\min\limits_{m\in\Sigma_p(1)\setminus S_\sigma}a_m(x)$ and

$$U^0_{p\sigma}=U_{p\sigma}\cap U^0_p,\ U_{p\sigma}=\{x\in U_p|A_\sigma(x)\geq A_{\sigma'}(x)\text{ for }\sigma'\in\Sigma_p(l)\}.$$

Then, the Proposition 4.1 implies that $A_\sigma(x)\geq\lambda_1\tau>0$ for $x\in U^0_{p\sigma}$ and

Proposition 4.3. *For t small enough, we have*

$$X_t\cap U^0_p=\bigcup_{\sigma\in\Sigma_p(l)}X_t\cap U^0_{p\sigma},$$

and $a_m^2\sim(\log|t|^2)^2$ in $U^0_{p\sigma}$ for $m\in\Sigma(1)\setminus S_\sigma$.

For $S_\sigma=\{m_1,\cdots,m_l\}$, on $U_{p\sigma}$, we may choose coordinate $z=\{z_k\}_{k=1}^l=\{s_{m_k}\}_{k=1}^l$. By adjusting the convex function $w=\{w_m\}_{m\in\Sigma_p(1)}$ by linear function, we may assume that $w_m=0$ for $m\in S_\sigma$ and $w_m>0$ for $m\in\Sigma_p(1)\setminus S_\sigma$. Then, we have $s_m=t^{w_m}z^m$, where $m=\{m^k\}_{k=1}^l$ also denotes the coordinate of m with respect to the basis $\{m_k\}_{k=1}^l$. It is easy to see that this coordinate z is a special case of the toroidal coordinate z defined in Section 2. Let

$$\alpha_{p\sigma}=\frac{i}{\pi}\sum_{m\in S_\sigma}\frac{1}{a_m^2}\partial a_m\bar{\partial}a_m,\ \alpha_{t,p\sigma}=\alpha_{p\sigma}|_{X_t}.$$

Proposition 4.4.

$$\alpha_{p\sigma}\leq\alpha\leq C(w)\alpha_{p\sigma}$$

along z direction in $U^0_{p\sigma}$. Consequently,

$$C_1\left(\prod_{m\in S_\sigma}\frac{1}{a_m^2}\right)\hat{V}_t\leq\omega_t^n\leq C_2\left(\prod_{m\in S_\sigma}\frac{1}{a_m^2}\right)\hat{V}_t,\quad\text{in}\quad U^0_{p\sigma}\cap X_t.$$

Proof. By the definition of $U^0_{p\sigma}$, clearly $\alpha_{p\sigma} \leq \alpha \leq C(w)\alpha_{p\sigma}$ along z direction in $U^0_{p\sigma}$. Therefore, $\omega_t \sim \hat{\omega}_t + \alpha_t \sim \hat{\omega}_t + \alpha_{t,p\sigma}$ according to Lemma 4.2. Since $\alpha^{l+1}_{t,p\sigma} = 0$. In $X_t \cap U^0_{p\sigma}$, we have

$$\omega_t^n \sim (\hat{\omega}_t + \alpha_{t,p\sigma})^n \sim \hat{\omega}_t^{n-l} \wedge \alpha^l_{t,p\sigma}.$$

According to formula (3.1),

$$\hat{V}_t = \rho(z)\left(\prod_{j=1}^{l} \frac{dz_j d\bar{z}_j}{|z_j|^2}\right)\left(\prod_{j=l+1}^{n} dz_j d\bar{z}_j\right).$$

Hence,

$$\hat{\omega}_t^{n-l} \wedge \alpha^l_{t,p\sigma} \sim \left(\prod_{m\in S_j} \frac{\partial a_m \bar{\partial} a_m}{a_m^2}\right)\left(\prod_{j=l+1}^{n} dz_j d\bar{z}_j\right) \sim \left(\prod_{m\in S_p} \frac{1}{a_m^2}\right)\hat{V}_t.$$

□

Notice that $V_t = h\hat{V}_t$ is the Kähler potential of ω_t. Assume

$$e^{-\phi_t} = \frac{\omega_t^n}{V_t}.$$

Proposition 4.5. *$|\phi_t|$ is bounded independent of t.*

Proof. According to Proposition 4.3, it is sufficient to verify in each $U^0_{p\sigma} \cap X_t$ for $p \in \Sigma$ and $\sigma \in \Sigma_p(l)$. Proposition 4.4 implies that

$$\frac{\omega_t^n}{V_t} \sim \eta^{2|\Sigma_p(1)|} \sim 1 \ \text{ in } U^0_{p\sigma} \cap X_t.$$

Therefore, $|\phi_t|$ is bounded independent of t. □

Let g_t denote the Kähler metric corresponding to the Kähler form ω_t, then we have

Proposition 4.6. *The curvature of g_t and its derivatives are all uniformly bounded with respect to t.*

Proof. On a Riemannian manifold (M,g), we call a basis $\{v_i\}$ *proper* if the corresponding metric matrix satisfies $C_1(\delta_{ij}) \leq (g_{ij}) \leq C_2(\delta_{ij})$ for $C_1, C_2 > 0$. To verify that the curvature of the Riemannian metric g and all its covariant derivatives are bounded, it is sufficient to find a proper basis $\{v_i\}$ satisfying that the coefficients of $[v_i, v_j]$ with respect to the basis $\{v_i\}$ and all their derivatives with respect to $\{v_i\}$ are bounded, such that g_{ij} and all their derivatives with respect to $\{v_i\}$ are bounded.

According to Proposition 4.3, it is sufficient to verify in each $U^0_{p\sigma} \cap X_t$. Let $W_j = a_{m_j} z_j \frac{\partial}{\partial z_j}$ for $1 \leq j \leq l$ and $W_j = \frac{\partial}{\partial z_j}$ for $l+1 \leq j \leq n$. According to Proposition 4.4, it is straightforward to check that the basis $\{W_j, \bar{W}_j\}_{j=1}^n$ is proper in $U^0_{p\sigma} \cap X_t$. Namely, $C_1(\delta_{ij}) \leq (g_{i\bar{j}}) \leq C_2(\delta_{ij})$ for some $C_1, C_2 > 0$, where $(g_{i\bar{j}})$ denotes the metric matrix with respect to the basis $\{W_j, \bar{W}_j\}_{j=1}^n$. (For the upper bound estimate, we need $\frac{a_{m_j}}{a_m}$ to be bounded for $1 \leq j \leq l$ and $m \in \Sigma_I(1) \setminus S_i$, which is due to our restriction to $U^0_{p\sigma}$.)

For $1 \leq j \leq l$, $\|s_{m_j}\|^2_{m_j} = \rho_j |z_j|^2$.

$$W_k(a_{m_j}) = \frac{W_k(\|s_{m_j}\|^2_{m_j})}{\|s_{m_j}\|^2_{m_j}} = \frac{W_k(\rho_j)}{\rho_j} + \frac{W_i(|z_j|^2)}{|z_j|^2}.$$

$$W_k(a_{m_j}) = a_{m_k}\left(z_k \frac{\partial \log \rho_j}{\partial z_k} + \delta_{kj}\right) \quad \text{for } 1 \leq k \leq l.$$

$$W_k(a_{m_j}) = \frac{\partial \log \rho_j}{\partial z_k} \quad \text{for } l+1 \leq k \leq n.$$

The functions

$$\begin{aligned} &\frac{1}{a_{m_j}},\ s_m P(a),\ \bar{s}_m P(a),\ z_j P(a_{m_j}),\ \bar{z}_j P(a_{m_j}), \\ &\frac{a_{m_j}}{a_m},\ \frac{\log|t|^2}{a_m},\ \frac{a_m}{\log|t|^2},\quad \text{for } m \in \Sigma_p(1) \setminus S_\sigma,\ 1 \leq j \leq l. \end{aligned} \tag{4.1}$$

are all bounded in $U^0_{p\sigma} \cap X_t$, where $P(a)$ is a polynomial on $(\{a_{m_j}\}_{j=1}^l, \log t)$ and $P(a_{m_j})$ is a polynomial on a_{m_j}. The above computations imply that the derivatives of functions in (4.1) with respect to $\{W_j, \bar{W}_j\}_{j=1}^n$ are smooth functions of terms in (4.1) and other smooth bounded terms. Therefore, they are bounded.

It is straightforward to check that $g_{i\bar{j}}$ and the coefficients of $[W_j, W_k]$, $[W_j, \bar{W}_k]$, $[\bar{W}_j, \bar{W}_k]$ with respect to the basis $\{W_j, \bar{W}_j\}_{j=1}^n$ are all smooth functions of terms in (4.1) and other bounded smooth terms. Consequently, any derivatives of theirs with respect to $\{W_j, \bar{W}_j\}_{j=1}^n$ are also smooth functions of terms in (4.1) and other bounded terms, therefore, are all bounded. □

Proposition 4.7. *For any k, $\|\phi_t\|_{C^k,g_t}$ is uniformly bounded with respect to t.*

Proof. Similar to the proof of the previous proposition, in $U^0_{p\sigma} \cap X_t$, it is straightforward to check according to Proposition 4.5 and the explicit expression of ϕ_t that ϕ_t is a bounded smooth function of terms in (4.1) and other smooth bounded terms. Consequently, all multi-derivatives of ϕ_t with respect to $\{W_j, \bar{W}_j\}_{j=1}^n$ are smooth functions of terms in (4.1) and other smooth bounded terms. Therefore, they are bounded. □

5. Construction of Kähler–Einstein metric via complex Monge–Ampère.

In this section, we will use the same notions as in the previous sections. In [9], using the Monge–Ampère estimate of Aubin and Yau, Tian essentially proved the following.

Theorem 5.1. (*Tian*) *Assume that ϕ_t, the curvature of g_t and their multi-derivatives are all bounded uniformly independent of t, then the Kähler–Einstein metric $g_{E,t}$ on X_t will converge to the complete Cheng–Yau Kähler–Einstein metric $g_{E,0}$ on $X_0\backslash\mathrm{Sing}(X_0)$ in the sense of Cheeger–Gromov: there are an exhaustion of compact subsets $F_\beta \subset X_0 \setminus \mathrm{Sing}(X_0)$ and diffeomorphisms $\psi_{\beta,t}$ from F_β into X_t satisfying:*

(1) $X_t \setminus \bigcup_{\beta=1}^{\infty} \psi_{\beta,t}(F_\beta)$ consists of finite union of submanifolds of real codimension 1;

(2) for each fixed β, $\psi^*_{\beta,t} g_{E,t}$ converge to $g_{E,0}$ on F_β in C^k-topology on the space of Riemannian metrics as t goes to 0 for any k.

Proof of Theorem 1.1. Proposition 2.3 reduces the theorem to the case that $\pi : \mathcal{X} \to B$ is simple, which is a direct corollary of Theorem 5.1 and Propositions 4.5, 4.6, and 4.7. □

It is easy to see that our construction actually implies the following asymptotic description of the family of Kähler–Einstein metrics.

Theorem 5.2. *Kähler–Einstein metric $g_{E,t}$ on X_t is uniformly quasi-isometric to the explicit approximate metric g_t. More precisely, there exist constants $C_1, C_2 > 0$ independent of t such that $C_1 g_t \leq g_{E,t} \leq C_2 g_t$.*

Proof. The uniform C^0-estimate of the complex Monge–Ampère equations implies that $C_1\omega_t^n \leq \omega_{E,t}^n \leq C_2\omega_t^n$ for some $C_1, C_2 > 0$. The uniform C^2-estimate of the complex Monge–Ampère equations implies that $\mathrm{Tr}_{g_t} g_{E,t}$ is uniformly bounded from above. Combining these two estimates, we get our conclusion. □

6. Weil–Peterson metric near degeneration.

In this section, we will start with the discussion of the toric case, which is of independent interest and the estimate is more precise. Then, we will proceed to the global toroidal case.

6.1. The Toric case.

Note: The notations in this subsection are the same as in Subsection 2.1. Unless specified otherwise, the notations in this section will not be carried over to other parts of this paper.

Example: Consider a toric degeneration $\pi : \mathcal{X} \to B \cong \mathbb{C}$ determined by a complete fan Σ in M and an integral piecewise linear convex function determined by $\{w_m\}_{m\in\Sigma(1)}$. For $i \in \Sigma(n)$, assume $w_m = 0$ for $m \in S_i$ and $w_m > 0$ for $m \in \Sigma(1) \setminus S_i$. With $S_i = \{m_1, \cdots, m_n\}$ and toric coordinate $z_j = s_{m_j}$ for $1 \leq j \leq n$, we have

$$\omega = \frac{i}{\pi}\sum_{j=1}^{n} \frac{dz_j \wedge d\bar{z}_j}{|z_j|^2(\log|z_j|^2)^2} + \frac{i}{\pi}\sum_{m\in\Sigma(1)\setminus S_i} \frac{ds_m \wedge d\bar{s}_m}{|s_m|^2(\log|s_m|^2)^2}.$$

Let

$$W = \frac{\nabla \log t}{|\nabla \log t|^2},$$

then $\bar{\partial}W|_{X_t}$ is a natural representative of Kodaira–Spencer deformation class in the Dolbeault cohomology $H^1(T_{X_t})$. W can also be determined by the conditions $\pi_* W = t\frac{d}{dt}$ and $i(W)\omega|_{X_t} = 0$ for all t. Let $a_j = \log|z_j|^2$ and $a_m = \log|s_m|^2$ for $m \in \Sigma(1) \setminus S_i$. We will use $\rho = 1 + O(a_j/a_m)$ to denote a bounded smooth function on a_j/a_m for $1 \leq j \leq n, m \in \Sigma(1) \setminus S_i$. (Here,

$O(a_j/a_m)$ is a shorthand for $O(a_j/a_m, 1 \leq j \leq n, m \in \Sigma(1) \setminus S_i)$.) It is straightforward to derive that

$$\omega_t = \omega|_{X_t} = \frac{i}{\pi}\sum_{j,k=1}^{n} g_{j\bar{k}}\frac{\partial a_j}{a_j} \wedge \frac{\bar{\partial} a_k}{a_k},$$

$$g_{j\bar{k}} = \delta_{jk} + a_j a_k O\left(\frac{1}{a_m^2}\right), \quad g^{j\bar{k}} = \delta_{jk} + a_j a_k O\left(\frac{1}{a_m^2}\right),$$

$$\omega_t^n = n!\left(\frac{i}{\pi}\right)^n \rho \prod_{j=1}^{n} \frac{dz_j \wedge d\bar{z}_j}{a_j^2|z_j|^2} = n!\left(\frac{1}{\pi}\right)^n \rho \prod_{j=1}^{n} \frac{da_j \wedge d\theta_j}{a_j^2}.$$

□

Lemma 6.1.

$$W = t\frac{\partial}{\partial t} - \sum_{j=1}^{n} \sum_{m \in \Sigma(1)\setminus S_i} w_m m^j \frac{a_j^2}{a_m^2}\rho z_j \frac{\partial}{\partial z_j}.$$

Proof. Since $\pi_* W = t\frac{d}{dt}$, we may assume that $W = t\frac{\partial}{\partial t} + \sum_{j=1}^{n} q_j z_j \frac{\partial}{\partial z_j}$. $i(W)\omega|_{X_t} = 0$ implies that

$$\sum_{m \in \Sigma(1)\setminus S_i} w_m \frac{d\bar{s}_m}{\bar{s}_m a_m^2} + \sum_{j=1}^{n} q_j \frac{d\bar{z}_j}{\bar{z}_j a_j^2} + \sum_{m \in \Sigma(1)\setminus S_i} \sum_{j=1}^{n} q_j m^j \frac{d\bar{s}_m}{\bar{s}_m a_m^2} = 0.$$

Consequently, $q_j = -\sum_{m \in \Sigma(1)\setminus S_i} w_m m^j \frac{a_j^2}{a_m^2}\rho$. □

Define $F = a = (a_1, \cdots, a_n) : \mathcal{X} \to \mathbb{R}^n$. Let $A_i(x) = \min_{m \in \Sigma(1)\setminus S_i} a_m(x)$. For $\eta > 0$, consider the domain $U_{i,\eta} = \{x \in U_\eta | A_i(x) \geq A_{i'}(x) \text{ for } i' \in \Sigma(n)\}$, where $U_\eta = \{x \in \mathcal{X} | a_m(x) \geq \eta \text{ for } m \in \Sigma(1)\}$. Notice that Proposition 4.1 implies that $A_i(x) \geq \lambda_1 \tau > 0$ for $x \in U_{i,\eta}$. It is easy to observe that there exist $c' > c > 0$ such that $[\eta, c\tau]^n \subset F(X_t \cap U_{i,\eta}) \subset [\eta, c'\tau]^n$. For ω_t and W as in the previous example, we have

Proposition 6.2. *There exists a constant $C_{i,\eta} \geq 0$, such that*

$$\frac{\displaystyle\int_{X_t \cap U_{i,\eta}} \|\bar{\partial} W\|^2 \omega_t^n}{\displaystyle\int_{X_t \cap U_{i,\eta}} \omega_t^n} = \frac{C_{i,\eta} + O(\tau^{-1}\log\tau)}{|\log|t|^2|^3}.$$

Proof. We may compute the volume of $X_t \cap U_{i,\eta}$.

$$\int_{X_t \cap U_{i,\eta}} \omega_t^n = n!2^n \int_{F(X_t \cap U_{i,\eta})} \rho \prod_{j=1}^n \frac{da_j}{a_j^2} = n!2^n \int_{[\eta, c\tau]^n} \rho \prod_{j=1}^n \frac{da_j}{a_j^2}(1+O(1/\tau))$$
$$= n!2^n \prod_{j=1}^n \left(\int_\eta^{c\tau} \frac{da_j}{a_j^2} \right) (1+O(\tau^{-1}\log\tau)) = \frac{n!2^n}{\eta^n}(1+O(\tau^{-1}\log\tau)).$$

Notice

$$\bar{\partial}\left(\frac{a_j^2}{a_m^2}\rho\right) = \rho\frac{2a_j}{a_m^2}\bar{\partial}a_j + \frac{a_j^2}{a_m^2}O\left(\frac{\bar{\partial}a_{j'}}{a_m}\right).$$

It is straightforward to compute

$$\left\| \sum_{m\in\Sigma(1)\setminus S_i} w_m m^j \bar{\partial}\left(\frac{a_j^2}{a_m^2}\rho\right) \right\|^2 = 4a_j^4\rho \left| \sum_{m\in\Sigma(1)\setminus S_i} \frac{w_m m^j}{a_m^2} \right|^2.$$

According to Lemma 6.1, we have

$$\bar{\partial}W = -\sum_{j=1}^n \sum_{m\in\Sigma(1)\setminus S_i} w_m m^j \bar{\partial}\left(\frac{a_j^2}{a_m^2}\rho\right) z_j \frac{\partial}{\partial z_j}$$
$$\|\bar{\partial}W\|^2 = \sum_{j=1}^n 4a_j^2\rho \left| \sum_{m\in\Sigma(1)\setminus S_i} \frac{w_m m^j}{a_m^2} \right|^2.$$
$$\int_{X_t \cap U_{i,\eta}} \|\bar{\partial}W\|^2 \omega_t^n = n!2^n \int_{F(X_t \cap U_{i,\eta})} \sum_{j=1}^n 4\rho \left| \sum_{m\in\Sigma(1)\setminus S_i} \frac{w_m m^j}{a_m^2} \right|^2 da_j \prod_{j'\neq j} \frac{da_{j'}}{a_{j'}^2}$$

For each j, let

$\tilde{U}^0_{ij,\eta} = \{a \in \mathbb{R}^n | \eta \leq a_j \leq c_j\tau,\ \eta \leq a_{j'} \leq c\tau,\ \text{for } j' \neq j\}$.
$\tilde{U}^1_{ij,\eta} = \{a \in F(X_t \cap U_{i,\eta}) | \eta \leq a_{j'} \leq c\tau,\ \text{for } j' \neq j\},\ \tilde{U}^2_{ij,\eta} = F(X_t \cap U_{i,\eta}) \setminus \tilde{U}^1_{ij,\eta}$.

It is straightforward to derive that

$$\begin{aligned}
&\int_{\tilde{U}^2_{ij,\eta}} 4\rho \left|\sum_{m\in\Sigma(1)\backslash S_i} \frac{w_m m^j}{a_m^2}\right|^2 da_j \prod_{j'\neq j} \frac{da_{j'}}{a_{j'}^2} \\
&= O\left(\frac{1}{\eta^{n-1}\tau^4}\right) \int_{\tilde{U}^1_{ij,\eta}} 4\rho \left|\sum_{m\in\Sigma(1)\backslash S_i} \frac{w_m m^j}{a_m^2}\right|^2 da_j \prod_{j'\neq j} \frac{da_{j'}}{a_{j'}^2} \\
&= \int_{\tilde{U}^1_{ij,\eta}} 4\rho_j(b_j) \left|\sum_{m\in\Sigma(1)\backslash S_i} \frac{w_m m^j}{(w_m\tau + m^j a_j)^2}\right|^2 da_j \prod_{j'\neq j} \frac{da_{j'}}{a_{j'}^2} + O\left(\frac{\log\tau}{\eta^{n-1}\tau^4}\right)
\end{aligned}$$

$$\begin{aligned}
&\left(\int_{\tilde{U}^1_{ij,\eta}} - \int_{\tilde{U}^0_{ij,\eta}}\right) 4\rho_j(b_j) \left|\sum_{m\in\Sigma(1)\backslash S_i} \frac{w_m m^j}{(w_m\tau + m^j a_j)^2}\right|^2 da_j \prod_{j'\neq j} \frac{da_{j'}}{a_{j'}^2} \\
&\quad = O\left(\frac{\log\tau}{\eta^{n-1}\tau^4}\right) \int_{\tilde{U}^0_{ij,\eta}} 4\rho_j(b_j) \left|\sum_{m\in\Sigma(1)\backslash S_i} \frac{w_m m^j}{(w_m\tau + m^j a_j)^2}\right|^2 da_j \prod_{j'\neq j} \frac{da_{j'}}{a_{j'}^2} \\
&\quad = \frac{4}{\eta^{n-1}|\log|t|^2|^3} \sum_{j=1}^n B_j \prod_{j'\neq j} \int_1^{+\infty} \frac{dx_{j'}}{x_{j'}^2} + O\left(\frac{\log\tau}{\eta^{n-1}\tau^4}\right),
\end{aligned}$$

where

$$B_j = \int_0^{c_j} \rho_j(b_j) \left|\sum_{m\in\Sigma(1)\backslash S_i} \frac{w_m m^j}{(w_m + m^j b_j)^2}\right|^2 db_j,$$

with $c_j = \frac{w_{\tilde{m}_j}}{1-\tilde{m}_j^j}$, $b_j = a_j/\tau$, $x_{j'} = a_{j'}/\eta$, and $\rho_j(b_j)$ is ρ replacing a_j/a_m by $b_j/(w_m + m^j b_j)$ and replacing $a_{j'}$ for $j' \neq j$ by zero. Combining all these estimates, we have

$$\int_{X_t\cap U_{i,\eta}} \|\bar{\partial} W\|^2 \omega_t^n = \frac{n! 2^{n+2}}{\eta^{n-1}} \frac{1}{|\log|t|^2|^3} \sum_{j=1}^n (B_j + O(\tau^{-1}\log\tau)),$$

We may take $C_{i,\eta} = 4\eta\sum_{j=1}^n B_j$ for the proposition to hold. □

6.2. The Toroidal case.

With respect to the local Kähler metric $\omega_p = \hat{\omega} + \frac{i}{2\pi}\partial\bar{\partial}\log h_p$ and parameterizing function t on U_p, we can similarly define $W_{(p)} = \frac{\nabla \log t}{|\nabla \log t|^2}$. Let

$W = \sum_{p\in\Sigma} \mu_p W_{(p)}$. $\bar\partial W$ also represents the Kodaira–Spencer deformation class. We have

Proposition 6.3. *There exists a constant $C > 0$ independent of t such that*

$$\int_{X_t} \|\bar\partial W\|_{g_t}^2 \omega_t^n \le \frac{C}{|\log|t|^2|^3} \int_{X_t} \omega_t^n.$$

Proof. Locally in each $U^0_{q\sigma}$, we will use similar coordinate and proper basis $\{W_j, \bar W_j\}_{j=1}^n$ as in the proof of Proposition 4.6. Then the dual basis is $\{\beta_j, \bar\beta_j\}_{j=1}^n$, where $\beta_j = \frac{dz_j}{a_j z_j}$, $a_j = \log|z_j|^2$ for $1 \le j \le l$ and $\beta_i = dz_i$ for $l+1 \le j \le n$. Recall that $O(1)$ denotes a smooth function on terms in (4.1) and other smooth bounded terms. (Notice that here we assume $a_m = \log|s_m|^2$, which is slightly different from (4.1) and do not affect our arguments here. In this proof, we are using a_j to denote a_{m_j} and $O(a_j/a_m)$ as a shorthand for $O(a_j/a_m, 1 \le j \le n, m \in \Sigma_q(1) \setminus S_\sigma)$.) We will also use $O(1)$ to denote a tensor with $O(1)$ coefficients with respect to the proper and dual proper basis. It is easy to see that the action of the proper basis $\{W_j, \bar W_j\}_{j=1}^n$ will send $O(1)$ to $O(1)$, also $\bar\partial W_j = O(1)$. Under such notation, we have

$$\omega_{t,q} = \sum_{j,k=1}^n g_{j\bar k}\beta_j\bar\beta_k.$$

$$g_{j\bar k} = \delta_{jk}\left(1 + \frac{1}{a_k}O(1)\right) + a_j a_k O\left(\frac{1}{a_m^2}\right) + \frac{1}{a_j a_k}O(1), \text{ for } 1 \le j,k \le l.$$

$$g_{j\bar k} = \frac{1}{a_k}O(1),\ g^{j\bar k} = \frac{1}{a_k}O(1), \text{ for } 1 \le k \le l \text{ and } l+1 \le j \le n.$$

It is straightforward to derive that

$$i\left(t\frac{\partial}{\partial t}\right)\omega_q\bigg|_{X_t} = \sum_{j=1}^l \sum_{m\in\Sigma_q(1)\setminus S_\sigma} w_m m^j \frac{a_j}{a_m^2}\bar\beta_j + O\left(\frac{1}{a_m^2}\right),$$

$$W_{(q)} = t\frac{\partial}{\partial t} - \sum_{j=1}^l \sum_{m\in\Sigma_q(1)\setminus S_\sigma} w_m m^j \frac{a_j}{a_m^2}\rho W_j + O\left(\frac{1}{a_m^2}\right),$$

$$\bar\partial W_{(q)} = -\sum_{j=1}^l \sum_{m\in\Sigma_q(1)\setminus S_\sigma} w_m m^j \bar\partial\left(\frac{a_j^2}{a_m^2}\rho\right) z_j\frac{\partial}{\partial z_j} + O\left(\frac{1}{a_m^2}\right).$$

Applying Proposition 6.2, we can find $C > 0$ independent of t such that

$$\int_{U^0_{q\sigma}\cap X_t} \|\bar{\partial} W_{(q)}\|^2_{g_t}\omega_t^n \leq \frac{C}{|\log|t|^2|^3}\int_{U^0_{q\sigma}\cap X_t}\omega_t^n.$$

The rest of the proof closely resembles the proof of Lemma 4.2. For any $x \in X_t$, there exist a $q \in \Sigma$ such that $x \in X_t \cap U^0_q$. Since

$$\sum_{p\in\Sigma}\mu_p = 1, \quad \sum_{p\in\Sigma}\bar{\partial}\mu_p = 0.$$

We have

$$\sum_{p\in\Sigma}\bar{\partial}\mu_p W_{(p)} = \sum_{p\in\Sigma}\bar{\partial}\mu_p(W_{(p)} - W_{(q)}).$$

Since $U^0_q \cap \mathrm{supp}(\mu_p) = \emptyset$ when $D_q \not\subset \bar{D}_p$ according to (2.3), we may consider only those $p \in \Sigma$ satisfying $D_q \subset \bar{D}_p$. As in the proof of Lemma 4.2, for such $p, q \in \Sigma$, we can naturally define $\Sigma^q_q \cap \Sigma^p_p$. For any $m \in \Sigma^p_p \backslash \Sigma^q_q$, $(s_m) \cap \bar{D}_q = \emptyset$. Consequently, $\|s_m\|^2_m = 1$ and $a_m = \eta$ on $\tilde{U}_q$ for $m \in \Sigma^p_p \setminus \Sigma^q_q$. Hence

$$W_{(p)} - W_{(q)} = \sum_{j=1}^{l} O\left(\frac{a_j}{a_m^2}\right)W_j + O\left(\frac{1}{a_m^2}\right)$$
$$\bar{\partial}W_{(p)} = \sum_{j=1}^{l} O\left(\frac{a_j}{a_m^2}\right) + O\left(\frac{1}{a_m^2}\right)$$

on $\mathrm{supp}(\mu_p) \cap U_q \subset U_p \cap U_q$. From the explicit expressions of μ_p, it is straightforward to check that $\bar{\partial}\mu_p = O(1/\log\tau)$ with respect to the Hermitian metric ω_t. Consequently,

$$\int_{U^0_{q\sigma}\cap X_t}\left\|\sum_{p\in\Sigma}\bar{\partial}\mu_p W_{(p)}\right\|^2_{g_t}\omega_t^n \leq \frac{C}{\tau^3\log\tau}\int_{U^0_{q\sigma}\cap X_t}\omega_t^n.$$
$$\int_{U^0_{q\sigma}\cap X_t}\left\|\sum_{p\in\Sigma}\mu_p\bar{\partial} W_{(p)}\right\|^2_{g_t}\omega_t^n \leq \frac{C}{\tau^3}\int_{U^0_{q\sigma}\cap X_t}\omega_t^n.$$

Combine these estimates for all $\sigma \in \Sigma_q(l)$, $q \in \Sigma$ applying to

$$\bar{\partial}W = \sum_{p\in\Sigma}\mu_p\bar{\partial}W_{(p)} + \sum_{p\in\Sigma}\bar{\partial}\mu_p W_{(p)},$$

we will get the desired estimate. □

Remark: It is not hard to observe that the constant $C_{i,\eta} \geq 0$ in Proposition 6.2 is actually positive. With this observation and a bit more argument, one can show that the lower bound estimate in Proposition 6.3 (more precisely the estimate in Proposition 6.3 with the reversed inequality) is also true. Since such more precise estimates are not needed for arguments in this paper, we will omit them here.

Proof of Theorem 1.2. As pointed out in [9],

$$g_{WP}\left(\frac{d}{dt},\frac{d}{dt}\right)\Big|_{X_t} = \int_{X_t}\left\|H\left(\frac{d}{dt}\right)\right\|^2_{g_{E,t}}\omega^n_{E,t},$$

where $H\left(\frac{d}{dt}\right)$ denote the harmonic representative of the Kodaira–Spencer deformation class. As mentioned earlier, such class can also be represented by $\frac{\bar\partial W}{t}$. Applying Proposition 6.3 and Theorem 5.2, we have

$$\begin{aligned}\int_{X_t}\left\|H\left(\frac{d}{dt}\right)\right\|^2_{g_{E,t}}\omega^n_{E,t} &\leq \int_{X_t}\left\|\frac{\bar\partial W}{t}\right\|^2_{g_{E,t}}\omega^n_{E,t} \leq C\int_{X_t}\left\|\frac{\bar\partial W}{t}\right\|^2_{g_t}\omega^n_t \\ &\leq \frac{C}{|\log|t||^3|t|^2}.\end{aligned}$$

□

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Received August 29, 2003.