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# Manifolds with weighted Poincaré inequality and uniqueness of minimal hypersurfaces

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In this paper, we obtain results on rigidity of complete Riemannian manifolds with weighted Poincaré inequality. As an application, we prove that if $M$ is a complete $\frac{n-2}{n}$-stable minimal hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 3$ and has bounded norm of the second fundamental form, then $M$ must either have only one end or be a catenoid.

## 1. Introduction

In this paper, we will discuss complete Riemannian manifolds with weighted Poincaré inequality and minimal hypersurfaces with $\delta$-stability in the Euclidian space $\mathbb{R}^{n+1}$ with $n \geq 3$. We first recall some backgrounds.

Let $M$ be an $n$-dimensional Riemannian manifold. Given a Schrödinger operator $L = \Delta + q(x)$ on $M$, we consider the eigenvalue problem on a compact subdomain $D \subset M$:

$$\begin{cases} Lf + \lambda f = 0, & \text{in } D; \\ f|_{\partial D} = 0. \end{cases}$$

It has discrete spectrum and the number of negative eigenvalues is finite. The (Morse) index of L on $M$ is defined as the supremum, over compact domains of $M$, of the number of negative eigenvalues (counted with multiplicity) of $L$ with Dirichlet boundary condition.

If $M$ is a complete connected immersed minimal hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$ and if $L$ is the Jacobi operator $L = \Delta + |A|^2$, where $|A|$ denotes the norm of the second fundamental form of $M$, then the index of $L$ is said to be the (Morse) index of $M$. $M$ is said to be stable if the index of $M$ is 0, which is equivalent to say that, for all compactly supported piecewise smooth function $f \in C_o^\infty(M)$,

$$\int_M \left(|\nabla f|^2 - |A|^2 f^2\right) \geq 0. \tag{1.1}$$

It is known that a complete stable minimal surface in $\mathbb{R}^3$ is plane, which was proved by do Carmo and Peng and Fischer-Cobrie and Schoen independently [6, 7]; and that only index one complete minimal surfaces in $\mathbb{R}^3$ are the catenoid and Enneper surface, which was proved by Lopez and Ros [9].

While it is unknown that a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ is a hyperplane when $n \leq 7$, Cao *et al.* [1] proved that a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ must have only one end for all dimension $n \geq 3$. Tam and Zhou [16] recently showed that an ($n$-dimensional) catenoid in the Euclidean space $\mathbb{R}^{n+1}$ with $n \geq 3$ has index one (see the definition of $n$-dimensional catenoid in $\mathbb{R}^{n+1}$ in [5], also in [16]).

Now let us assume $L = \Delta + \delta|A|^2$ on minimal hypersurface $M$ in $\mathbb{R}^{n+1}$ for some number $0 < \delta \leq 1$ . We may similarly define that $M$ is *$\delta$-stable* if

$$\int_M \left(|\nabla f|^2 - \delta|A|^2 f^2\right) \geq 0 \tag{1.2}$$

for all $f \in C_o^\infty(M)$.

Obviously, given $\delta_1 > \delta_2$, $\delta_1$-stable implies $\delta_2$-stable. So $M$ is stable implies that $M$ is $\delta$-stable for all $0 < \delta \leq 1$. Hyperplane is $\delta$-stable for all $0 < \delta \leq 1$.

There are some work on $\delta$-stable minimal hypersurfacs. Kawai [8] proved a $\delta$-stable, $\delta > \frac{1}{8}$ complete minimal surface in $\mathbb{R}^3$ must be plane. Recently, Meeks *et al.* [14] showed that any complete embedded $\delta$-stable minimal surface in $\mathbb{R}^3$ with finite genus is flat. In the case of higher dimension $n \geq 3$, we have, directly from the argument in [1], that the result of Cao, Shen and Zhu also holds for $\frac{n-1}{n}$-stable. Recently, Tam and Zhou [16] showed that a catenoid in $\mathbb{R}^{n+1}$ is $\frac{n-2}{n}$-stable. Also they proved that if $M$ is an $\frac{n-2}{n}$-stable complete immersed minimal hypersurface in $\mathbb{R}^{n+1}$ and if

$$\lim_{R \to +\infty} \frac{1}{R^2} \int_{B(2R)\setminus B(R)} |A|^{2(n-2)/n} = 0,$$

then $M$ is either a hyperplane or a catenoid.

In this paper, we prove that if an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ with $n \geq 3$ and the norm of its second fundamental form satisfies some growth condition, then it either has only one end or is a catenoid. More precisely, we show

**Theorem 1.1.** *Let $M$ be an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}$ for $n \geq 3$ and the norm of its second fundamental form satisfies*

$$\begin{aligned} &\lim_{R\to+\infty} \sup_{B(R)} |A|/R^{(n-3)/2} = 0 \quad \text{for} \quad n \geq 4; \\ &\lim_{R\to+\infty} \sup_{B(R)} |A|/\ln R = 0 \quad \text{for} \quad n = 3, \end{aligned} \tag{1.3}$$

*then $M$ either has only one end or is a catenoid.*

From Theorem 1.1, we have the following result:

**Corollary 1.1.** *Let $M$ be an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, with at least two ends. If it has bounded norm of the second fundamental form, then $M$ must be a catenoid.*

Our results for minimal hypersurfaces rely on the study on complete manifolds with weighted Poincaré inequality which is of independent interest.

Let $M$ be a complete Riemannian manifold. Recall from [13] that a complete Riemannian manifold $(M, ds^2)$ is said to satisfy a weighted Poincaré inequality with non-negative weight function $\rho$ if the inequality

$$\int_M |\nabla f|^2 \geq \int_M \rho f^2$$

holds for all compactly supported piecewise smooth function $f \in C_o^{+\infty}(M)$.

Further, $M$ is said to satisfy property $(\mathcal{P}_\rho)$ for non-zero non-negative weight function $\rho(x)$ if,

(1) $M$ satisfies a weighted Poincaré inequality with $\rho$; and

(2) the conformal metric $\rho ds^2$ is complete.

In [13], Li and Wang studied complete manifolds satisfying property $(\mathcal{P}_\rho)$ and obtained many theorems on rigidity. Later the first author [3] discussed complete manifolds with Poincaré inequality and obtain results on the uniqueness of ends which can be applied to study stable minimal hypersurfaces in a Riemannian manifold. In this paper, we generalize one result of Li and Wang [13, Theorem 5.2] to the following:

**Theorem 1.2.** *Let $M$ be a complete $n$-dimensional $(n \geq 3)$ Riemannian manifold with property $(\mathcal{P}_\rho)$ for some non-zero weight function $\rho$. Suppose* *the Ricci curvature of $M$ has the lower bound*

$$\mathrm{Ric}_M(x) \geq -(n-1)\tau(x), \quad x \in M,$$

*where $\tau(x)$ satisfies Poincaré inequality*

$$\int_M |\nabla f|^2 \geq (n-2)\int_M \tau f^2, \text{ for all } f \in C_o^{+\infty}(M).$$

*If $\rho$ and $\tau$ satisfy the growth condition*

$$\begin{aligned} &\liminf_{R\to+\infty} S(R)e^{-((n-3)/(n-2))R} = 0 \quad \text{for} \quad n \geq 4, \\ &\liminf_{R\to+\infty} S(R)R^{-1} = 0 \quad \text{for} \quad n = 3, \end{aligned} \tag{1.4}$$

*where*

$$S(R) = \sup_{x\in B_\rho(R)} (\sqrt{\rho(x)}, \sqrt{\tau(x)}),$$

*then either*

(1) *$M$ has only one non-parabolic end*; *or*

(2) *$M$ has two non-parabolic ends and is given by $M = \mathbb{R} \times N$ with the warped product metric*

$$ds_M^2 = dt^2 + \eta^2(t)ds_N^2,$$

*for some positive function $\eta(t)$ and some compact manifold $N$. Moreover, $\tau(t)$ is a function of $t$ alone satisfying*

$$(n-2)\eta''\eta^{-1} = \tau.$$

If we choose $\tau = \frac{1}{n-2}\rho$ in Theorem 1.2, it is just Theorem 5.2 of [13]. In the case of minimal hypersurfaces, we could not find any weight function $\rho$ in a Poincaré inequality, which satisfies both the completeness of the metric $\rho ds^2$ and the lower bound estimate of Ricci curvature of $M$. Hence, we could not apply the theorem of Li and Wang. Instead, our Theorem 1.2 is suitable to our minimal case (see Theorem 1.1).

The work of Li and Wang on complete manifolds satisfying weighted Poincaré inequality is a generalization of their one on complete manifolds with positive spectrum [11, 12]; see [13] and the references therein). Let $\lambda_1(M)$ be the largest lower bound of the spectrum of the Laplacian with respect to the metric of $M$. Theorem 1.2 implies the following result.

**Corollary 1.2.** *Let $M$ be a complete $n$-dimensional $(n \geq 3)$ Riemannian manifold with positive spectrum (i.e., $\lambda_1(M) > 0$). Suppose the Ricci curvature of $M$ has the lower bound*

$$\mathrm{Ric}_M(x) \geq -(n-1)\tau(x), \quad x \in M,$$

*where $\tau(x)$ satisfies Poincaré inequality*

$$\int_M |\nabla f|^2 \geq (n-2)\int_M \tau f^2, \text{ for all } f \in C_o^{+\infty}(M).$$

*If $\tau$ satisfies the growth condition*

$$\liminf_{R\to+\infty}\left(\sup_{x\in B(R)} \tau(x)\right)e^{-(2(n-3)/(n-2))R} = 0 \quad \text{for} \quad n \geq 4,$$

$$\liminf_{R\to+\infty}\left(\sup_{x\in B(R)} \tau(x)\right)R^{-2} = 0 \quad \text{for} \quad n = 3, \tag{1.5}$$

*then either*

(1) *$M$ has only one non-parabolic end; or*

(2) *$M$ has two non-parabolic ends and is given by $M = \mathbb{R} \times N$ with the warped product metric*

$$ds_M^2 = dt^2 + \eta^2(t)ds_N^2,$$

*for some positive function $\eta(t)$ and some compact manifold $N$. Moreover, $\tau(t)$ is a function of $t$ alone satisfying*

$$(n-2)\eta''\eta^{-1} = \tau.$$

This corollary generalizes Theorem 2.1 in [11] (just choose $\tau(x) = \frac{\lambda_1(M)}{n-2}$ and use the fact a non-parabolic end with $\lambda_1(M) > 0$ has infinite volume).

Throughout this paper, all manifolds are assumed to be oriented.

## 2. Rigidity of complete manifolds

In this section, we will consider the structure of a complete manifold $M$ with property $(\mathcal{P}_\rho)$. Since we follow the argument of Li and Wang [13, Theorem 5.2] with some changes of techniques in the proof of our Theorem 1.2, we recommend [13] as a complement when necessary.

Let $d(x,y)$ and $d_\rho(x,y)$ denote the distances between $x$ and $y$ with respect to $ds^2$ and $\rho^2 ds^2$, respectively; $B(x,R)=\{y\in M; d(x,y)<R\}$ and $B_\rho(x,R)=\{y\in M; d_\rho(x,y)<R\}$. For a fixed point $p\in M$, we denote $r(x)$ and $r_\rho(x)$ the distance functions with respect to metric $ds^2$ and conformal metric $\rho ds^2$ from $p$, respectively; $B(R)=\{x\in M; r(x)<R\}$ and $B_\rho(R)=\{x\in M; r_\rho(x)<R\}$.

We need the following construction of harmonic functions [13, §5].

Suppose $M$ has at least two non-parabolic ends $E_1$ and $E_2$. A theory of Li and Tam [10] asserts that one can get a non-constant bounded harmonic function $f$ with finite Dirichlet integral by taking a convergent subsequence of the harmonic functions $f_R$ as $R\to+\infty$, satisfying

$$\Delta f_R = 0 \qquad \text{on } B(R),$$

with boundary conditions

$$\begin{cases} f_R = 1, & \text{on } \partial B(R)\cap E_1;\\ f_R = 0, & \text{on } \partial B(R)\setminus E_1. \end{cases}$$

It follows from the maximum principle that $0\le f_R\le 1$ for all $R$ and hence $0\le f\le 1$.

Now we prove Theorem 1.2.

*Proof.* If $M$ has at least two non-parabolic ends, then there exists a bounded harmonic function $f$ with finite Dirichlet integral constructed as above. We may assume that $\inf f=0$ and $\sup f=1$.

Then the Bochner formula and the lower bound of the Ricci curvature imply (cf. [13, Lemm 4.1])

$$\Delta|\nabla f| \ge -(n-1)\tau|\nabla f| + \frac{1}{n-1}\frac{|\nabla|\nabla f||^2}{|\nabla f|}. \tag{2.1}$$

Set $\alpha=\frac{n-2}{n-1}$ and $g=|\nabla f|^\alpha$. Equation (2.1) implies

$$\begin{aligned} \Delta g &= \alpha(\alpha-1)|\nabla f|^{\alpha-2}|\nabla|\nabla f||^2 + \alpha|\nabla f|^{\alpha-1}\Delta|\nabla f| \\ &\ge -(n-2)\tau g. \end{aligned} \tag{2.2}$$

We will show inequality (2.2) is actually an equality. For any non-negative compactly supported piecewise smooth function $\phi$ on $M$, we have

$$\begin{aligned}\int_M &\phi^2 g(\Delta g + (n-2)\tau g)\\ &= -2\int_M \phi g\langle \nabla g, \nabla\phi\rangle - \int_M \phi^2|\nabla g|^2 + \int_M (n-2)\tau(\phi g)^2\\ &\le -2\int_M \phi g\langle \nabla g, \nabla\phi\rangle - \int_M \phi^2|\nabla g|^2 + \int_M |\nabla(\phi g)|^2\\ &= \int_M |\nabla\phi|^2|\nabla f|^{2(n-2)/(n-1)} = \int_M |\nabla\phi|^2 g^2.\end{aligned} \tag{2.3}$$

The inequality in (2.3) holds since $\tau$ satisfies Poincaré inequality.

Choose $\phi = \psi\chi$, where $\psi$ and $\chi$ will be chosen later and $\psi$ denotes a piecewise smooth compactly supported function on $M$. Then

$$\begin{aligned}\int_M |\nabla\phi|^2 g^2 \le{}& 2\int_M |\nabla\psi|^2\chi^2|\nabla f|^{2(n-2)/(n-1)}\\ &+ 2\int_M |\nabla\chi|^2\psi^2|\nabla f|^{2(n-2)/(n-1)}.\end{aligned} \tag{2.4}$$

We first consider the case of $n \ge 4$. For $R > 1$, we let $\psi(x)$ be a function depending on the $\rho$-distance:

$$\psi(x) = \begin{cases} 1 & \text{on } B_\rho(R-1),\\ R - r_\rho & \text{on } B_\rho(R)\setminus B_\rho(R-1),\\ 0 & \text{on } M\setminus B_\rho(R).\end{cases}$$

For $\sigma \in (0,1)$ and $\epsilon \in (0, \frac{1}{2})$, we define $\chi$ on the level sets of $f$:

$$\chi(x) = \begin{cases} 0 & \text{on } \mathcal{L}(0,\sigma\epsilon)\cup\mathcal{L}(1-\sigma\epsilon,1),\\ (\epsilon-\sigma\epsilon)^{-1}(f-\sigma\epsilon) & \text{on } \mathcal{L}(\sigma\epsilon,\epsilon)\cap(M\setminus E_1),\\ (\epsilon-\sigma\epsilon)^{-1}(1-\sigma\epsilon-f) & \text{on } \mathcal{L}(1-\epsilon,1-\sigma\epsilon)\cap E_1,\\ 1 & \text{otherwise,}\end{cases}$$

where we denote the set $\mathcal{L}(a,b) = \{x \in M | a < f(x) < b\}$.

Denote the set

$$\Omega = E_1 \cap (B_\rho(R)\setminus B_\rho(R-1)) \cap (\mathcal{L}(\sigma\epsilon, 1-\sigma\epsilon)).$$

Recall the growth estimate for $|\nabla f|$ [13, Corollary 2.3]:

$$\int_{B_\rho(R+1)\setminus B_\rho(R)} |\nabla f|^2 \leq Ce^{-2R},$$

and the decay estimate for $f$ [13, (2.10)]:

$$\int_{E_1\cap B_\rho(R+1)\setminus E_1\cap B_\rho(R)} \rho(1-f)^2 \leq Ce^{-2R}.$$

We have

$$\left(\int_\Omega |\nabla f|^2\right)^{(n-2)/(n-1)} \leq Ce^{-(2(n-2)/(n-1))R} \tag{2.5}$$

and with notation $S(R)$ as in the statement of theorem,

$$\begin{aligned}\int_\Omega \rho^{n-1} &\leq (S(R))^{2(n-2)} \int_\Omega \rho \\ &\leq (\sigma\epsilon)^{-2}(S(R))^{2(n-2)} \int_\Omega \rho(1-f)^2 \\ &\leq C(S(R))^{2(n-2)}(\sigma\epsilon)^{-2}e^{-2R}.\end{aligned} \tag{2.6}$$

Hence, by $|\nabla r_\rho|(x) = \rho(x)$, (2.5) and (2.6), we have

$$\begin{aligned}&\int_{E_1} |\nabla\psi|^2\chi^2|\nabla f|^{2(n-2)/(n-1)} \\ &\leq \int_\Omega \rho|\nabla f|^{2(n-2)/(n-1)} \\ &\leq \left(\int_\Omega |\nabla f|^2\right)^{(n-2)/(n-1)} \left(\int_\Omega \rho^{n-1}\right)^{1/(n-1)} \\ &\leq C(\sigma\epsilon)^{-2/(n-1)}(S(R))^{2(n-2)/(n-1)}e^{-2((n-2)/(n-1))R-(2/(n-1))R} \\ &\leq C(\sigma\epsilon)^{-2/(n-1)}(S(R))^{2(n-2)/(n-1)}e^{-2R}.\end{aligned} \tag{2.7}$$

Note the assumption that the Ricci curvature of $M$ is bounded from below by $-(n-1)\tau(x)$. Then the local gradient estimate of Cheng and Yau [2] (cf. [12]) implies that there exists a constant $C_n$ depending on $n$ such that

$$|\nabla f|(x) \leq C_n \left(\sup_{y\in B(x,R)} \sqrt{\tau(y)} + R^{-1}\right) |f(x)|, \quad x \in M,$$

for all $R > 0$.

Set $\bar\rho(x) = \frac{1}{2}\rho(x) + \frac{1}{2}(n-2)\tau(x), x \in M$. Then $\sqrt{\tau} \le \sqrt{\frac{2}{n-2}\bar\rho}$ and

$$|\nabla f|(x) \le C\left(\sup_{y\in B(x,R)} \sqrt{\bar\rho}(y) + R^{-1}\right)|f(x)|. \tag{2.8}$$

Fix $x \in M$ and consider the function $\eta(R) = \sqrt{2}R - (\sup_{B(x,R)}\sqrt{\bar\rho})^{-1}$. Observe that $\eta(R)$ tends to $+\infty$ as $R \to \infty$ and tends to a negative number as $R \to 0$. There exists a $R_0$ depending on $x$ such that $\sqrt{2}R_0 = (\sup_{B(x,R_0)} \sqrt{\bar\rho})^{-1}$. Hence,

$$|\nabla f|(x) \le C\left(\sup_{B(x,R_0)} \sqrt{\bar\rho}\right)|f(x)|. \tag{2.9}$$

For any $y \in B(x,R_0)$, let $\gamma(s), s \in [0,l]$ be a minimizing geodesic connecting $x$ and $y$ with respect to the background metric $ds^2$, where $s$ is the arc-length of $\gamma$ in $ds^2$. The distance $d_\rho(x,y)$ with respect to $\rho ds^2$ satisfies

$$\begin{aligned} d_\rho(x,y) &\le \int_0^l \sqrt{\rho(\gamma(s))}ds \\ &\le \int_0^l \sqrt{2}\sqrt{\bar\rho(\gamma(s))}ds \\ &\le \left(\sup_{B(x,R_0)} \sqrt{\bar\rho}\right)(\sqrt{2}R_0) = 1. \end{aligned} \tag{2.10}$$

This implies $B(x,R_0) \subset B_\rho(x,1)$. Hence,

$$|\nabla f|(x) \le C\left(\sup_{B_\rho(x,1)} \sqrt{\bar\rho}\right)|f(x)|, \quad x \in M. \tag{2.11}$$

Similarly, we have

$$|\nabla f|(x) \le C\left(\sup_{B_\rho(x,1)} \sqrt{\bar\rho}\right)|1-f(x)|, \quad x \in M. \tag{2.12}$$

On $E_1$, we have

$$\begin{aligned}
&\int_{E_1} |\nabla\chi|^2\psi^2|\nabla f|^{2(n-2)/(n-1)}\\
&\le C((1-\sigma)\epsilon)^{-2}\int_{\mathcal{L}(1-\epsilon,1-\sigma\epsilon)\cap E_1\cap B_\rho(R)} |\nabla f|^{2(n-2)/(n-1)+2}\\
&\le C(S(R+1))^{2(n-2)/(n-1)}((1-\sigma)\epsilon)^{-2}\\
&\quad\times\int_{\mathcal{L}(1-\epsilon,1-\sigma\epsilon)\cap E_1\cap B_\rho(R)} |\nabla f|^2(1-f)^{2(n-2)/(n-1)}.
\end{aligned} \tag{2.13}$$

Note that [13, Lemma 5.1] asserts that the integral of $|\nabla f|$ on the level set $l(t)=\{x\in M|f(x)=t\}$, $0\le t\le 1$, is independent of $t$. Using this conclusion and the co-area formula and [13, Lemma 5.1], we have

$$\begin{aligned}
&\int_{\mathcal{L}(1-\epsilon,1-\sigma\epsilon)\cap E_1\cap B_\rho(R)} |\nabla f|^2(1-f)^{2(n-2)/(n-1)}\\
&\quad\le\int_{1-\epsilon}^{1-\sigma\epsilon}(1-t)^{2(n-2)/(n-1)}\int_{l(t)\cap E_1\cap B_\rho(R)}|\nabla f|dA\,dt\\
&\quad\le C\int_{l(b)}|\nabla f|dA\int_{1-\epsilon}^{1-\sigma\epsilon}(1-t)^{2(n-2)/(n-1)}\,dt\\
&\quad= C\int_{l(b)}|\nabla f|dA\cdot(1-\sigma^{2(n-2)/(n-1)+1})\epsilon^{2(n-2)/(n-1)+1}.
\end{aligned} \tag{2.14}$$

Substitute (2.14) into (2.13). Then

$$\begin{aligned}
\int_{E_1}|\nabla\chi|^2\psi^2|\nabla f|^{2(n-2)/(n-1)} &\le C(S(R+1))^{2(n-2)/(n-1)}(1-\sigma)^{-2}\\
&\quad\times(1-\sigma^{2(n-2)/(n-1)+1})\epsilon^{(n-3)/(n-1)}.
\end{aligned} \tag{2.15}$$

Setting $\sigma=\frac{1}{2}$, we have

$$\begin{aligned}
\int_{E_1}|\nabla\phi|^2|\nabla f|^{2(n-2)/(n-1)} &\le C(S(R+1))^{2(n-2)/(n-1)}\\
&\quad\times(e^{-2R}\epsilon^{-2/(n-1)}+\epsilon^{(n-3)/(n-1)}).
\end{aligned}$$

Let us choose $\epsilon = e^{-2R}$. Then

(2.16)
$$\int_{E_1} |\nabla\phi|^2 |\nabla f|^{2(n-2)/(n-1)} \leq C(S(R+1))^{2(n-2)/(n-1)} e^{-2((n-3)/(n-1))R}.$$

Using $f$ instead of $1-f$, similar to the above argument, we have that on $M \backslash E_1$,

(2.17)
$$\int_{M\backslash E_1} |\nabla\phi|^2 |\nabla f|^{2(n-2)/(n-1)} \leq C(S(R+1))^{2(n-2)/(n-1)} e^{-2((n-3)/(n-1))R}.$$

Hence,

(2.18)
$$\int_{M} |\nabla\phi|^2 |\nabla f|^{2(n-2)/(n-1)} \leq C(S(R+1))^{2(n-2)/(n-1)} e^{-2((n-3)/(n-1))R}.$$

Let $R \to +\infty$. By the assumption on $S(R)$, the left in (2.18) is identically zero. By (2.3), we conclude that (2.2) is actually an equality and hence the improved Bochner inequality (2.1) must be an equality. Note that [13, Lemma 4.1] asserts that if equality in inequality (2.1) holds, the metric of $M$ must be a warped product as described in the theorem. We obtain the conclusion of theorem in the case of $n \geq 4$.

In the case of $n = 3$, we may choose $\psi$ as above and $\chi$ to be

$$\chi(x) = \begin{cases} 0 & \text{on } \mathcal{L}(0,\sigma\epsilon) \cup \mathcal{L}(1-\sigma\epsilon, 1), \\ (-\log\sigma)^{-1}(\log f - \log(\sigma\epsilon)) & \text{on } \mathcal{L}(\sigma\epsilon,\epsilon) \cap (M \setminus E_1), \\ (-\log\sigma)^{-1}(\log(1-f) - \log(1-\sigma\epsilon)) & \text{on } \mathcal{L}(1-\epsilon, 1-\sigma\epsilon) \cap E_1, \\ 1 & \text{otherwise.} \end{cases}$$

By an argument similar to the above one for $n \geq 4$ (combining with the corresponding estimates for $n = 3$ in [13, Theorem 5.2]), we have the estimate

$$\int_M |\nabla\phi|^2 |\nabla f|^{2(n-2)/(n-1)} \leq CS(R+1)(\sigma^{-1}\epsilon^{-1}e^{-2R} + (-\log\sigma)^{-1}). \tag{2.19}$$

Choose $\sigma = \epsilon = e^{-Rq(R)}$ with $q(R) = \sqrt{\frac{S(R+1)}{R}}$. Then using the argument in [13], we have the right side of (2.19) tends to zero as $R \to +\infty$. We conclude that (2.2) is actually an equality and hence the theorem holds for $n = 3$. □

## 3. Application to minimal hypersurfaces

Let $M^n$ be a complete minimal hypersurface in $\mathbb{R}^{n+1}$ for $n \geq 3$. We first give some examples of the metric $\rho ds^2$ such that $M$ satisfies property $(\mathcal{P}_\rho)$.

**Example 3.1.** Let $\bar{d}(x,y), x, y \in \mathbb{R}^{n+1}$, denote the distance between $x$ and $y$ in $\mathbb{R}^{n+1}$. Denote by $\bar{r}(x), x \in M$, the extrinsic distance function $\bar{d}(x,o)$ from a fixed point $o \in \mathbb{R}^{n+1}$ ($o$ may or may not be in $M$). It is known that

$$\Delta \bar{r} \geq (n-1)\bar{r}^{-1},$$

where $\Delta$ is the Laplacian on $M$.

For any $\phi \in C_o^\infty(M)$,

$$\begin{aligned}(n-1)\int_M \bar{r}^{-2}\phi^2 &\leq \int_M \bar{r}^{-1}\phi^2\Delta\bar{r} \\ &= -2\int_M \bar{r}^{-1}\phi\langle\nabla\phi, \nabla\bar{r}\rangle + \int_M \bar{r}^{-2}\phi^2|\nabla\bar{r}|^2 \\ &\leq 2\int_M \bar{r}^{-1}\phi|\nabla\phi| + \int_M \bar{r}^{-2}\phi^2.\end{aligned}$$

$$\begin{aligned}(n-2)\int_M \bar{r}^{-2}\phi^2 &\leq 2\int_M \bar{r}^{-1}\phi|\nabla\phi| \\ &\leq 2\left(\int_M \bar{r}^{-2}\phi^2\right)^{1/2}\left(\int_M |\nabla\phi|^2\right)^{1/2}\end{aligned}$$

Hence,

$$\int_M |\nabla\phi|^2 \geq \frac{(n-2)^2}{4}\int_M \bar{r}^{-2}\phi^2 \quad \text{for all} \quad \phi \in C_o^{+\infty}(M). \tag{3.1}$$

Let $\rho(x) = \frac{(n-2)^2}{4}\bar{r}^{-2}(x), x \in M$. Inequality (3.1) asserts the Poincaré inequality holds with weight function $\rho$.

Further the metric $\rho ds^2$ is complete. Indeed, take a fixed point $p \in M$ with $p \neq o$. Let $r(x), x \in M$, denote the intrinsic distance from $p$. Note that $\bar{r}(x) \leq \bar{d}(o,p) + \bar{d}(x,p) \leq r_0 + r(x)$, where $r_0 = \bar{d}(o,p) > 0$. Then $\bar{r}^{-2}(x) > (r_0 + r(x))^{-2}$. It is known that the metric $(r_0 + r(x))^{-2}ds^2$ is complete. Hence $\rho ds^2$ is complete.

Thus we obtain that $M$ has property $(\mathcal{P}_\rho)$ for $\rho$.

**Example 3.2.** Using smoothing technique, we may modify $\rho = \frac{(n-2)^2}{4}\bar{r}^{-2}$ in Example 3.1 to get a bounded smooth positive function $\rho_1(x) = \rho_1(\bar{r}(x))$, $x \in M$, such that $M$ has property $(\mathcal{P}_\rho)$ for $\rho_1$.

Indeed, let positive number $0 < b \le r_0$ be fixed, we can choose number $a, 0 < a < b$, such that function $\zeta(\bar{r}) = \frac{(n-2)^2}{4}(\bar{r}^{-2} - e^{-1/(\bar{r}-b)^2})$ is strictly decreasing in $(a, b)$ as $\bar{r}$ tends increasingly to $b$ and construct the smooth $\rho_1$

$$\rho_1(\bar{r}(x)) = \begin{cases} h(\bar{r}) & \text{for } \bar{r}(x) \le a, \\ \zeta(\bar{r}) & \text{for } a < \bar{r}(x) < b, \\ \rho(\bar{r}) & \text{for } \bar{r}(x) \ge b, \end{cases}$$

where $h(\bar{r})$ is chosen to be bounded and to satisfy $\rho(\bar{r}) \ge h(\bar{r}) \ge \rho(b)$ for $\bar{r} \le a$.

Observe that $\rho_1 \le \rho$. Hence the Poincaré inequality holds for $\rho_1$. Moreover, $\rho_1 ds^2$ is complete since $\rho_1(x) \ge \frac{(n-2)^2}{4}(r_0 + r(x))^{-2}$. In fact, for $\bar{r}(x) \ge b$, $\rho_1 = \rho$. Note that for $\bar{r}(x) < b$, $\rho_1(\bar{r}(x)) \ge \rho(b)$ and $0 < b \le r_0$. Hence, $\rho_1(\bar{r}(x)) \ge \frac{(n-2)^2}{4}(r_0 + r(x))^{-2}$ for $\bar{r}(x) < b$.

**Example 3.3.** Under the above notations, choose $\rho_2(x) = \frac{(n-2)^2}{4}(r_0 + r(x))^{-2}, x \in M$. Since $\rho_2 \le \rho$, Poincaré inequality holds with weight function $\rho_2$. By the completeness of the metric $\rho_2 ds^2$, we know $M$ has property $(\mathcal{P}_\rho)$ for $\rho_2$.

**Theorem 3.1 (Theorem 1.1).** *Let $M$ be an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}, n \ge 3$, and the norm of its second fundamental form satisfies*

$$\lim_{R\to+\infty} \sup_{B(R)} |A|/R^{(n-3)/2} = 0, \quad \textit{for} \quad n \ge 4;$$
$$\lim_{R\to+\infty} \sup_{B(R)} |A|/\ln R = 0, \quad \textit{for} \quad n = 3, \tag{3.2}$$

*then $M$ either has one end or must be a catenoid.*

*Proof.* For any point $q \in M$ and any unit tangent vector $v \in T_qM$, we can choose an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ on $M$ at $q$ such that $e_1 = v$.

From the Gauss equation, we have

$$
\begin{aligned}
|A|^2 &\geq h_{11}^2 + \sum_{i=2}^{n} h_{ii}^2 + 2\sum_{i=1}^{n} h_{1i}^2 \\
&\geq h_{11}^2 + \frac{(\sum_{i=2}^{n} h_{ii})^2}{n-1} + 2\sum_{i=1}^{n} h_{1i}^2 \\
&\geq \frac{n}{n-1}\left(h_{11}^2 + \sum_{i=2}^{n} h_{1i}^2\right) \\
&\geq -\frac{n}{n-1}\mathrm{Ric}_M(e_1, e_1).
\end{aligned} \tag{3.3}
$$

Then $\mathrm{Ric}_M(v, v) \geq -\frac{n-1}{n}|A|^2$.

Let us choose $\tau = \frac{|A|^2}{n}$ and $\rho = \rho_1$ (or $\rho_2$) in Theorem 1.2. By the boundedness of $\rho_1$ (or $\rho_2$), the growth assumption (1.4) on $\rho$ is satisfied. Now we will assert that the growth assumption (1.5) on $\tau$ is also satisfied.

It can be verified directly that a minimizing geodesic starting from the fixed point $p$ with respect to $ds^2$ is also a minimizing geodesic starting from $p$ with respect to $\rho_2 ds^2$. Then by direct calculation, we have $B_{\rho_2}(\bar{R}) = B(R)$, where $\bar{R} = \frac{n-2}{2}\ln(1 + \frac{R}{r_0})$. Then for $n \geq 4$

$$
\lim_{\bar{R}\to+\infty} \sup_{B_{\rho_2}(\bar{R})} |A|e^{-(n-3)/(n-2)\bar{R}} = C \lim_{R\to+\infty} \sup_{B(R)} |A|R^{-(n-3)/2} = 0. \tag{3.4}
$$

For $n = 3$,

$$
\lim_{\bar{R}\to+\infty} \sup_{B_{\rho_2}(\bar{R})} |A|\bar{R}^{-1} = C \lim_{R\to+\infty} \sup_{B(R)} |A|(\ln R)^{-1} = 0. \tag{3.5}
$$

If $\rho = \rho_1$, by $\rho_1 \geq \rho_2$, $B_{\rho_1}(\bar{R}) \subseteq B_{\rho_2}(\bar{R})$ and hence the growth assumption on $\tau$ also holds for $\rho_1$.

Therefore, the conclusion of Theorem 1.2 is valid. Let us assume that $M$ has at least two ends. Since every end of a complete non-compact minimal hypersurface in $\mathbb{R}^{n+1}$ is non-parabolic ( [1], see its proof also in [4]), by Theorem 1.2, we know that $M$ has exactly two non-parabolic ends and $M = \mathbb{R} \times N$ with the warped product metric

$$
ds_M^2 = dt^2 + \eta^2(t) ds_N^2,
$$

for some compact manifold $N$ and some positive function $\eta(t)$. Moreover, $|A|$ is a function of $t$ alone satisfying

$$(n-2)\eta''\eta^{-1} = \frac{|A|}{n}.$$

Hence, $M$ has a rotationally symmetric metric. By a result of do Carmo and Dajczer [5, Corollary 4.4], it implies that every part of $M$ is a part of a catenoid. Hence $M$ is contained in a catenoid $\mathcal{C}$ by minimality of the immersion. Since $M$ is complete and the catenoid $\mathcal{C}$ is simply connected because $n \geq 3$, $M$ must be an embedded hypersurface, see [15, p. 330]. Hence $M$ is the catenoid. □

Theorem 3.1 implies directly that

**Corollary 3.1 (Corollary 1.1).** *Let $M$ be an $\frac{n-2}{n}$-stable complete minimal hypersurface in $\mathbb{R}^{n+1}, n \geq 3$, with at least two ends. If its second fundamental form is bounded, then $M$ must be a catenoid.*

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