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Multiplier ideal sheaves and the Kähler–Ricci flow on toric Fano manifolds with large symmetry

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The purpose of this paper is to calculate the support of the multiplier ideal subschemes derived from the Kähler–Ricci flow on certain toric Fano manifolds with large symmetry. The early idea of this paper has already been in the Appendix of [12].

1. Introduction

In [12], Futaki and Sano investigate the relationship between the multiplier ideal subvariety derived from the continuity method on toric Fano manifolds and Futaki invariant, and calculate the multiplier ideal subvariety on a simple example. On the other hand, the relationship between the multiplier ideal sheaves and the Kähler–Ricci flow has recently been studied. The first work on this topic is given by Phong–Sesum–Sturm [20]. They give a sufficient and necessary condition for the convergence of the Kähler–Ricci flow in the terms of the multiplier ideal sheaves. After [20], Rubinstein [21] proves that the Kähler–Ricci flow will induce a multiplier ideal sheaf satisfying the same properties as Nadel's multiplier ideal sheaves derived from the continuity method. The purpose of this paper is to calculate the multiplier ideal subvarieties from the Kähler–Ricci flow in the sense of [21] on certain toric Fano manifolds with large symmetry. Our method owes largely to the result about the convergence of the Kähler–Ricci flow on toric Fano manifolds due to Zhu [31].

Let (X, ω) be an n-dimensional Fano manifold with a Kähler form ω representing $c_1(X)$. The normalized Kähler–Ricci flow on X is defined by

$$\frac{d}{dt}\omega_t = -\mathrm{Ric}(\omega_t) + \omega_t, \tag{1.1}$$

where $t \in \mathbb{R}_{\geq 0}$, $\mathrm{Ric}(\omega_t)$ is the Ricci form of ω_t and $\omega_0 = \omega$. Since the flow (1.1) preserves the Kähler class, we can consider the corresponding equation to (1.1) with respect to Kähler potentials

$$\begin{cases} \dfrac{\partial \varphi_t}{\partial t} = \log \dfrac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} + \varphi_t - h_0, \\ \varphi_0 \equiv c_0, \end{cases} \tag{1.2}$$

where $\omega_t = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_t$, c_0 is a constant and h_0 is a real-valued function determined by

$$\mathrm{Ric}(\omega_0) - \omega_0 = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}h_0, \ \int_X \mathrm{e}^{h_0}\omega_0^n = V. \tag{1.3}$$

Here we denote the volume of X with respect to ω_0 by V. The existence of the solution of (1.2) for all $t > 0$ is proved by Cao [3] by following Yau's argument in [30]. Since there are some obstructions for the existence of Kähler–Einstein metrics on Fano manifolds [11, 15, 26], the Kähler–Ricci flow does not necessarily converge on Fano manifolds. Nadel [16] and Demailly and Kollár [7] prove that if X does not admit Kähler–Einstein metrics, then the failure of the closedness condition for the continuity method induces a multiplier ideal sheaf. (This fact can be extended in the cases of other canonical Kähler metrics such as Kähler–Ricci solitons [12] and Kähler–Einstein metrics in the sense of Mabuchi [22].) The analogous result for the Kähler–Ricci flow is proved recently by Rubinstein [21]. In this paper, we consider the multiplier ideal sheaves in the sense of [7]. Let ψ be an almost plurisubharmonic function (psh function) on X, i.e., ψ is written locally as a sum of a psh function and a smooth function. For ψ, we define the multiplier ideal sheaf $\mathcal{I}(\psi) \subset \mathcal{O}_X$ as follows; for every open subset $U \subset X$, the space $\Gamma(U, \mathcal{I}(\psi))$ of local sections of $\mathcal{I}(\psi)$ over U is given by

$$\Gamma(U, \mathcal{I}(\psi)) = \{f \in \mathcal{O}_X(U) \mid \int_U |f|^2 \, \mathrm{e}^{-\psi} d\nu < \infty\},$$

where f is a holomorphic function on U and $d\nu$ is a fixed volume form on X. Note that $\mathcal{I}(\psi)$ is a coherent ideal sheaf (cf.[7]) and invariant up to an additive constant. The result of [21] is as follows.

Theorem 1.1 [21]. *Let (X, ω) be an n-dimensional Fano manifold with a Kähler form ω in $c_1(X)$, and $G \subset Aut(X)$ be a compact subgroup of the group $Aut(X)$ of holomorphic automorphisms of X. Let $\gamma \in (n/(n+1), 1)$. Suppose that X does not admit Kähler–Einstein metrics. Then there is an initial condition $\varphi_0 \equiv c_0$ in (1.2) and a sequence $\{\varphi_{t_j}\}_{j \geq 0}$ such that $\varphi_{t_j} - \sup \varphi_{t_j}$*

converges to an almost psh function φ_∞ *in* L^1*-topology and the associated multiplier ideal sheaf* $\mathcal{I}(\gamma\varphi_\infty)$ *is* $G^{\mathbb{C}}$*-invariant and proper, i.e.,* $\mathcal{I}(\gamma\varphi_\infty)$ *is equal neither to* 0 *nor* $\mathcal{O}_X$*, where* $G^{\mathbb{C}}$ *is the complexification of* G*.*

Remark that in [21] the multiplier ideal sheaf is constructed from the sequence of $\{\varphi_t - \frac{1}{V}\int_X \varphi_t\omega^n\}_t$ instead of $\{\varphi_t - \sup\varphi_t\}_t$ but there is no difference between them due to a standard argument by the Green function, more precisely, there is a constant C such that $\sup\varphi_t - C \leq \int_X \varphi_t\omega^n \leq \sup\varphi_t$. For $\gamma \in (0,1)$, we denote the subscheme of X cut out by $\mathcal{I}(\gamma\varphi_\infty)$ by V_γ. In this paper, we call it the *KRF-multiplier ideal subscheme* (KRF-MIS) of exponent γ. We often abbreviate the subschemes cut out by (general) multiplier ideal sheaves to the MIS. In particular, for an almost psh function φ we call the subscheme cut out by $\mathcal{I}(\gamma\varphi)$ the MIS of exponent γ (with respect to φ). The exponent of the MIS is closely related to the complex singularity exponent, which is introduced by Demailly–Kollár [7]. The definition of the complex singularity exponent will be explained in Section 3. Here let us remark that the complex singularity exponent is a local version of a holomorphic invariant, which is called the α-invariant defined by Tian [24]. In fact, Theorem 1.1 in [21] is obtained by effectively proving that if $\alpha_G(X) > \frac{n}{n+1}$ then the Kähler–Ricci flow converges. Remark that $\alpha_G(X) \geq 1$ if there is no multiplier ideal sheaf $\mathcal{I}(\psi)$ such that there is a positive constant ε satisfying that $\mathcal{I}(\gamma\psi)$ is proper for $\gamma \in (1-\varepsilon, 1)$. See [2, 4, 5, 9, 13, 23, 25] for other works related to the α-invariant.

On the other hand, it has been conjectured that the existence of Kähler–Einstein metrics would be equivalent to certain stability of manifolds in the sense of Geometric Invariant Theory (cf.[8],[26]). In this viewpoint, we expect that the KRF-MIS would be related to stability of manifolds. Hence, concrete calculations of the KRF-MIS would be useful to verify the expectation. Our main result is as follows.

Theorem 1.2. *Let* X *be a toric Fano manifold in* $\mathcal{W}_1$*. Suppose that* X *does not admit Kähler–Einstein metrics. Let* v_{KRS} *be the holomorphic vector field for a Kähler–Ricci soliton on* X*. Suppose that the imaginary part of* v_{KRS} *generates a one-parameter subgroup of a compact subgroup* G *(defined in Section 2) of* $\mathrm{Aut}(X)$*. Let* $\{\sigma_t := \exp(tv_{\mathrm{KRS}})\}$ *and* $\gamma \in (0,1)$*. Then, there exists a subsequence of* $\{t_j\}$ *in Theorem 1.1 such that the support of the induced KRF-MIS of exponent* γ *is equal to the support of the MIS of exponent* γ *derived from a sequence of Kähler potentials of* $\{(\sigma_{t_j}^{-1})^*\omega\}$ *for any* G*-invariant Kähler form* ω*.*

The definitions of $\mathcal{W}_1$ and Kähler–Ricci solitons will be explained in Section 2. In [30], it is proved that a Kähler–Ricci soliton always exists on any toric Fano manifold.

Remark 1.1. The author expects that the restriction to $\mathcal{W}_1$ would be just a technical assumption and it would be removed.[1]

We can calculate the support of the KRF-MIS concretely by using Theorem 1.2 and a formula in Theorem 3.1 to compute the complex singularity exponent of the almost psh function induced from one-parameter subgroups of the torus action. For example,

Corollary 1.1. *Let X be the blow up of $\mathbb{CP}^2$ at p_1 and p_2. Let E_1 and E_2 be the exceptional divisors of the blow up, and E_0 be the proper transform of $\overline{p_1p_2}$ of the line passing through p_1 and p_2. Then, the support of the KRF-MIS on X of exponent γ is*

$$\begin{cases} \cup_{i=0}^2 E_i & \text{for } \gamma \in \left(\dfrac{1}{2}, 1\right), \\ E_0 & \text{for } \gamma \in \left(\dfrac{1}{3}, \dfrac{1}{2}\right). \end{cases}$$

The organization of this paper is as follows. In Section 2, we reduce the KRF-MIS to a simpler one by following the proof of [31]. In Section 3, we give the formula to calculate the complex singularity exponent of the associated almost psh function derived from one-parameter subgroups of the torus action and complete the proof of Theorem 1.2. In Section 4, we calculate examples of toric Fano n-folds ($n = 2, 3$) contained in $\mathcal{W}_1$ by using our results.

2. Convergence of the Kähler–Ricci flow to Kähler–Ricci solitons on toric Fano manifolds

Through this section and the next section, we prove Theorem 1.2 by following the proof of [31].

First, we explain the setup concerning toric Fano manifolds briefly. (Consult [18] and [10] for general information of toric geometry.) A toric variety

[1] The restriction to $\mathcal{W}_1$ does not imply the non-existence of Kähler–Einstein metrics. In fact, examples of Kähler–Einstein toric Fano manifolds in $\mathcal{W}_1$ are found (cf.[17, 19]).

X is an algebraic variety with an effective action of $T_{\mathbb{C}} := (\mathbb{C}^*)^n$, where $\dim_{\mathbb{C}} X = n$. Let $T_{\mathbb{R}} := (S^1)^n$ be the real torus in $T_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{R}}$ be the associated Lie algebra. Let $N_{\mathbb{R}} := J\mathfrak{t}_{\mathbb{R}} \simeq \mathbb{R}^n$ where J is the complex structure of $T_{\mathbb{C}}$. Let $M_{\mathbb{R}}$ be the dual space $\mathrm{Hom}(N_{\mathbb{R}}, \mathbb{R}) \simeq \mathbb{R}^n$ of $N_{\mathbb{R}}$. Denoting the group of algebraic characters of $T_{\mathbb{C}}$ by M, then $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. Let P^* be the image of X under the moment map, which is a convex reflexive polytope. Let $\mathrm{L}P^* = \{p^{(i)}\}_{1 \le i \le m}$ be the set of all lattice points contained in P^*. Let P be the dual polytope defined by

$$P = \{x \in N_{\mathbb{R}} \mid \langle p, x \rangle \le 1 \ \text{ for any vertex } p \text{ of } P^*\},$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing on $M_{\mathbb{R}} \times N_{\mathbb{R}}$. The polytope P is often called the Fano polytope of X.

Let $\mathcal{N}(T_{\mathbb{C}})$ be the normalizer of $T_{\mathbb{C}}$ in $\mathrm{Aut}(X)$. Then the Weyl group $\mathcal{W}(X) := \mathcal{N}(T_{\mathbb{C}})/T_{\mathbb{C}}$ of $\mathrm{Aut}(X)$ with respect to $T_{\mathbb{C}}$ is equal to the finite subgroup of $\mathrm{GL}(n, \mathbb{Z})$ consisting of all elements, which preserve P where $N \simeq \mathbb{Z}^n$ is the dual of M (Proposition 3.1 in [2]). We define G in Theorem 1.2 by the compact subgroup of $\mathrm{Aut}(X)$ generated by $T_{\mathbb{R}}$ and $\mathcal{W}(X)$. Let $N_{\mathbb{R}}^{\mathcal{W}(X)} := \{x \in N_{\mathbb{R}} \mid x^g = x \ \text{ for all } g \in \mathcal{W}(X)\}$. Then, the class of toric Fano manifolds, which we shall consider is

$$\mathcal{W}_1 := \{X : \ \text{toric Fano manifold with } \dim N_{\mathbb{R}}^{\mathcal{W}(X)} = 1\}.$$

As an initial Kähler form ω_0 on X, we take a standard metric determined by the moment polytope P^* as follows. Let $(\frac{1}{2}x_1 + \sqrt{-1}\theta_1, \ldots, \frac{1}{2}x_n + \sqrt{-1}\theta_n)$ be an affine logarithm coordinates on $T_{\mathbb{C}} = T_{\mathbb{R}} \times N_{\mathbb{R}}$, i.e., $t_i = \exp(\frac{1}{2}x_i + \sqrt{-1}\theta_i)$ where $t = (t_1, \ldots, t_n) \in T_{\mathbb{C}}$. By the invariance, we define $\omega_0 := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_0$ on a dense orbit of the action of $T_{\mathbb{C}}$, where u_0 is also a convex function on $N_{\mathbb{R}}$ defined by

$$u_0(x) := \log\left(\sum_{i=1}^{m} \mathrm{e}^{\langle p^{(i)}, x \rangle}\right) \tag{2.1}$$

and $x = (x_1, \ldots, x_n) \in N_{\mathbb{R}}$. It is known that ω_0 can be extended to a well-defined Kähler form on X. Obviously ω_0 and u_0 are $\mathcal{W}(X)$-invariant.

Now, we begin the proof of Theorem 1.2. The proof is based on the following result concerning the convergence of the Kähler–Ricci flow. Recall that a pair $(v_{\mathrm{KRS}}, \omega_{\mathrm{KRS}})$ of a holomorphic vector field and a Kähler form on a Fano manifold is called a Kähler–Ricci soliton if

$$\mathrm{Ric}(\omega_{\mathrm{KRS}}) - \omega_{\mathrm{KRS}} = \mathcal{L}_{v_{\mathrm{KRS}}}\omega_{\mathrm{KRS}},$$

where $\mathcal{L}_v$ is the Lie derivative along v. Let $\mathrm{Aut}_r(X)$ be the reductive part of $\mathrm{Aut}(X)$ and K be a maximal compact subgroup of $\mathrm{Aut}_r(X)$. Note that $\mathrm{Aut}_r(X)$ is the complexification of K. From the uniqueness of Kähler–Ricci solitons proved by Tian–Zhu in [27], we may assume that a Kähler–Ricci soliton $(v_{\mathrm{KRS}}, \omega_{\mathrm{KRS}})$ is K-invariant and the imaginary part of v_{KRS} generates a one-parameter subgroup $K_{v_{\mathrm{KRS}}}$ of K. For a holomorphic vector field v, let F_v be the holomorphic invariant of Tian–Zhu [27] , which is a generalization of Futaki invariant. Then v_{KRS} satisfies that $F_{v_{\mathrm{KRS}}}$ vanishes on $\mathrm{Aut}_r(X)$.

Theorem 2.1 Zhu, [31]. *On a toric Fano manifold X, the normalized Kähler–Ricci flow with any $T_{\mathbb{R}}$-invariant Kähler metric converges to a Kähler–Ricci soliton ω_{KRS} in the sense of Cheeger–Gromov.*

This is a toric version of [28]. Theorem 2.1 indicates that the asymptotic behavior of ω_t in (1.1) is similar to $(\sigma_t^{-1})^*\omega_{\mathrm{KRS}}$ where $\sigma_t = \exp(tv_{\mathrm{KRS}})$. Indeed, $\{(\sigma_t^{-1})^*\omega_{\mathrm{KRS}}\}_t$ satisfies equation (1.1). The key of the proof of Theorem 1.2 is to estimate the difference between ω_t in (1.1) and $(\sigma_t^{-1})^*\omega_{\mathrm{KRS}}$. So, we recall a part of [31] needed for our proof.

Let us consider the equation of (1.2) with the initial metric ω_0 defined by using u_0 in (2.1) and $c_0 = 0$. Remark that the assumption of c_0 is different from the initial constant in [21], but we shall see in the proof of Lemma 2.1 that this difference does not affect the KRF-MIS. Since all objects concerning (1.2) are $T_{\mathbb{R}}$-invariant, we can reduce (1.2) to a real Monge–Ampère equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \log\det(u_{ij}) + u, \\ u(0,\cdot) = u_0, \end{cases} \tag{2.2}$$

where $u_t = u(t,\cdot) = u_0 + \varphi_t$ on $N_{\mathbb{R}}$. Here we reduce the potential function φ_t of ω_t to a function on $N_{\mathbb{R}}$ by the invariance under $T_{\mathbb{R}}$-action. To avoid the complication of symbols, we denote it by the same φ_t. Note that φ_t on $N_{\mathbb{R}}$ is normalized by requiring that the image of the gradient map of u_t in $M_{\mathbb{R}}$ is equal to P^*. For each solution u_t of (2.2), let x_t be the minimal point of u_t. The assumption that X is contained in $\mathcal{W}_1$ makes the behavior of $\{x_t\}_t$ simple. Let β_{KRS} be the vector in $N_{\mathbb{R}}$, which induces the holomorphic vector field v_{KRS} of the Kähler–Ricci soliton. More precisely, if $v^\sharp_{\mathrm{KRS}}$ is the real vector field induced by β_{KRS} then $v_{\mathrm{KRS}} = \frac{1}{2}(v^\sharp_{\mathrm{KRS}} - \sqrt{-1}(Jv^\sharp_{\mathrm{KRS}}))$. Since β_{KRS} is $\mathcal{W}(X)$-invariant and $X \in \mathcal{W}_1$, the line $\{s\beta_{\mathrm{KRS}} \mid s \in \mathbb{R}\}$ is equal to the fixed subspace of $N_{\mathbb{R}}$ under the action of $\mathcal{W}(X)$. Since u_t is also $\mathcal{W}(X)$-invariant, $\{x_t\}_t$ is contained in the line $\{s\beta_{\mathrm{KRS}} \mid s \in \mathbb{R}\}$, that is to say, for each t there is a constant $s_t \in \mathbb{R}$ such that $x_t = s_t\beta_{\mathrm{KRS}}$. Remark that $|ds_t/dt|$ is uniformly bounded for all t, which is due to Lemma 4.6 [6] and Lemma 4.1 [31].

Let ρ_t be a holomorphic transformation, which induces the shift transformation on $N_{\mathbb{R}}$ defined by $x \mapsto x + s_t\beta_{\mathrm{KRS}}$ for each t. Let $\tilde{\varphi}_t$ be a Kähler potential defined by

$$\rho_t^*\omega_{\varphi_t} = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\tilde{\varphi}_t, \tag{2.3}$$

which is what we desire. Remark that $\tilde{\varphi}_t$ is equal to $u_t(\cdot + s_t\beta_{\mathrm{KRS}}) - u_0(\cdot)$ up to constant. The ambiguity of an additive constant in (2.3) is removed by requiring

$$\frac{\partial\tilde{\varphi}_t}{\partial t} = \log\frac{\det(g_{i\bar{j}} + \tilde{\varphi}_{i\bar{j}})}{\det(g_{i\bar{j}})} + \tilde{v}_t(\tilde{\varphi}_t) + \tilde{\varphi}_t - \tilde{h}_0 + \theta_{\tilde{v}_t} \tag{2.4}$$

on X, where

$$\tilde{v}_t := \frac{dx_t}{dt} = \beta_{\mathrm{KRS}}\cdot\frac{ds_t}{dt},$$

$\theta_{\tilde{v}_t} := \tilde{v}_t(u_0)$. The function $\tilde{h}_0$ is the renormalized function of h_0 as follows. Let V be the volume of X with respect to ω_0. Let $\theta_{v_{\mathrm{KRS}}} = v_{\mathrm{KRS}}(u_0)$ and $\theta_{\tilde{v}_t} := \tilde{v}_t(u_0)$. Let φ'_t be the Kähler potential defined by

$$\sigma_t^*\omega_{\varphi_t} = \omega_0 + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi'_t$$

and

$$\frac{\partial\varphi'_t}{\partial t} = \log\frac{\det(g_{i\bar{j}} + (\varphi'_t)_{i\bar{j}})}{\det(g_{i\bar{j}})} + v_{\mathrm{KRS}}(\varphi'_t) + \varphi'_t - h_0 + \theta_{v_{\mathrm{KRS}}}.$$

Then $\tilde{h}_0$ satisfies

$$\frac{1}{V}\int_X(\tilde{h}_0 - \theta_{v_{\mathrm{KRS}}})\omega_0^n = -\frac{1}{V}\int_0^\infty\int_X \|\bar{\partial}\frac{\partial\varphi'_t}{\partial t}\|^2\exp(\theta_{v_{\mathrm{KRS}}} + v_{\mathrm{KRS}}(\varphi'_t) - t)\\ \wedge(\sigma_t^*\omega_{\varphi_t})^n\wedge dt$$

as Proposition 3.2 [31]. Then, it is proved

Proposition 2.1 [31]. *The family $\{\omega_{\tilde{\varphi}_t}\}_t$ converges to a Kähler–Ricci soliton associated to v_{KRS} and $\tilde{v}_t$ converges to v_{KRS} as t goes to infinity.*

Remark that $\frac{ds_t}{dt} \to 1$ as $t \to \infty$, because $\tilde{v}_t$ converges to v_{KRS}. Therefore we can conclude the following lemma. Let ψ_t be the function defined by

$$(\rho_t^{-1})^*\omega = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi_t,\ \sup\psi_t = 0, \tag{2.5}$$

where ω is a G-invariant Kähler form. By replacing the sequence $\{t_j\}$ in Theorem 1.1 by an appropriate subsequence if necessary, we have the limit of $\{\psi_{t_j}\}$ in L^1-topology. We denote it by ψ_∞. Remark that we replace φ_∞ in Theorem 1.1 by the one with respect to the above new sequence $\{t_j\}$.

Lemma 2.1. *$\mathcal{I}(\gamma\varphi_\infty)$ and $\mathcal{I}(\gamma\psi_\infty)$ coincide for any $\gamma \in (0, 1)$.*

Proof. First, we shall see that the difference of the choice of initial constant c_0 does not matter when we consider the KRF-MIS in the sense of [21]. Let $\varphi_{1,t}$ and $\varphi_{2,t}$ be the solutions of (1.2) with different initial constants c_1 and c_2. Then, $\varphi_{2,t} = \varphi_{1,t} - (c_1 - c_2)\,\mathrm{e}^t$. Since $\varphi_{1,t} - \sup\varphi_{1,t}$ is equal to $\varphi_{2,t} - \sup\varphi_{2,t}$ for each t, their MIS coincide.

The KRF-MIS is independent of the choice of ω in (2.5). Take any G-invariant Kähler form ω. As seen in the above argument, we find that ω_{φ_t} is equal to $(\rho_t^{-1})^*\omega_{\tilde{\varphi}_t}$ for each t. Let $\phi_t \in C^\infty(X)$ be the discrepancy function defined by $\omega_{\tilde{\varphi}_t} - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi_t$ and $\sup\phi_t = 0$. Since $\omega_{\tilde{\varphi}_t}$ converges in C^∞-sense, $\|\phi_t\|_{C^0}$ is uniformly bounded. Then

$$\psi_t = (\varphi_t - (\rho_t^{-1})^*\phi_t) - \sup(\varphi_t - (\rho_t^{-1})^*\phi_t).$$

Since $\|(\rho_t^{-1})^*\phi_t\|_{C^0}$ is also uniformly bounded, the MIS of $\{\psi_{t_j}\}$ and of $\{\varphi_{t_j} - \sup\varphi_{t_j}\}$ coincide. The proof is completed. □

3. Complex singularity exponents of multiplier ideal sheaves on toric Fano manifolds

In this section, we prove the following lemma to finish the proof of Theorem 1.2. Let $\overline{\varphi_t}$ be the Kähler potential defined by

$$(\sigma_t^{-1})^*\omega = \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\overline{\varphi}_t,\ \sup\overline{\varphi}_t = 0,$$

where ω is a G-invariant Kähler form. By replacing $\{t_j\}$ in Theorem 1.1 again by its appropriate subsequence if necessary, we have the limit of $\{\overline{\varphi}_{t_j}\}$ in L^1-topology. We denote it by $\overline{\varphi}_\infty$. Remark that we replace φ_∞ again by the one with respect to the new subsequence of $\{t_j\}$.

Lemma 3.1. *For any $\gamma \in (0,1)$, the support of the MIS of exponent γ with respect to ψ_∞ is equal to the one with respect to $\overline{\varphi}_\infty$.*

To prove Lemma 3.1, we shall give a formula to calculate the complex singularity exponent of ψ_∞ with respect to each face of the polytope P^*. Furthermore, we give a way to determine the support of the KRF-MIS. We should note that the assumption on $\mathcal{W}_1$ is not needed for the argument in this section.

First, let us recall the complex singularity exponent. Let X be a complex manifold and φ be an almost psh function on X. Let $K \subset X$ be a compact subset of X. The complex singularity exponent $c_K(\varphi)$ of φ on K is defined by

$$c_K(\varphi) := \sup\{c \geq 0; \exp(-c\varphi) \text{ is } L^1 \text{ on a neighborhood of } K\}.$$

From its definition, $c_{\{p\}}(\varphi)$ is strictly less than some positive constant γ if and only if the local section 1_{U_p} of $\mathcal{O}_X(U_p)$ is not contained in $\Gamma(U_p, \mathcal{I}(\gamma\varphi))$ for any open neighborhood U_p at p, i.e., p is contained in the support of the subscheme cut out by $\mathcal{I}(\gamma\varphi)$. That is to say, the support of the MIS of exponent γ with respect to φ is equal to $\{p \in X \mid c_{\{p\}}(\varphi) < \gamma\}$.

From now on, let X be a toric Fano manifold. Let ψ_t and ψ_∞ be as before. We take the standard Kähler form ω_0 using by (2.1) as ω in (2.5). For a point $y \in P^*$, we denote the complex singularity exponent of ψ_∞ on $\mu^{-1}(y)$ by $c_{\{y\}}(\psi_\infty)$ where μ is the moment map with respect to ω_0. This notation makes sense. In fact, $\mu^{-1}(y)$ is contained in the support of the MIS of exponent γ with respect to ψ_∞ if and only if $c_{\{y\}}(\psi_\infty) < \gamma$, because the MIS on a toric manifold is $T_{\mathbb{R}}$-invariant and

$$c_{\{y\}}(\psi_\infty) = \inf_{p \in \mu^{-1}(y)} c_{\{p\}}(\psi_\infty). \tag{3.1}$$

It is easy to check as follows. It is trivial that $c_{\{y\}}(\psi_\infty) \leq \inf_{p\in\mu^{-1}(y)} c_{\{p\}}(\psi_\infty)$ from the definition. For any $c < \inf_{p\in\mu^{-1}(y)} c_{\{p\}}(\psi_\infty)$, there is an open covering $\cup_{p\in\mu^{-1}(y)} U_p$ of $\mu^{-1}(y)$ such that U_p is an open neighborhood at p and $\mathrm{e}^{-c\psi_\infty}$ is integrable over U_p. Since $\mu^{-1}(y)$ is compact, we find that $\mathrm{e}^{-c\psi_\infty}$ is integrable over $\cup_{p\in\mu^{-1}(y)} U_p$, i.e., $c \leq c_{\{y\}}(\psi_\infty)$. Hence (3.1) is proved.

For each face δ^* of P^*, let us calculate $c_{\{y\}}(\psi_\infty)$ where y is a point in the relative interior of δ^*. In order to do it, we shall choose a reference point in the interior of δ^* as follows. Let δ^* be an $(n-l-1)$-dimensional face of P^*. Let $\tilde{\mu}$ be the G-equivariant moment map from $N_{\mathbb{R}}$ to $M_{\mathbb{R}}$ with respect to ω_0 defined by

$$\tilde{\mu}(x) := \left(\frac{\partial u_0}{\partial x_1}(x), \dots, \frac{\partial u_0}{\partial x_n}(x)\right),$$

where u_0 is defined by (2.1). Remark that the image of $\tilde{\mu}$ is equal to the interior of P^*. From the duality between P and P^*, for δ^* there is a unique l-dimensional face δ of P. Let $\{q^{(i)}\}_{i=1,\dots,l+1}$ be the set of vertices of δ. For $a_i \in \mathbb{R}_{>0}$ satisfying $\sum_{i=1}^{l+1} a_i = 1$, we put $x^{(a)} := a_1 q^{(1)} + \cdots + a_{l+1} q^{(l+1)}$. Obviously $x^{(a)}$ is contained in the relative interior of δ. Then

$$\begin{aligned}
\frac{\partial u_0}{\partial x_j}(sx^{(a)}) &= \frac{1}{\sum_{i=1}^m \mathrm{e}^{\langle p^{(i)}, sx^{(a)}\rangle}} \left\{ \sum_{i=1}^m p_j^{(i)} \, \mathrm{e}^{\langle p^{(i)}, sx^{(a)}\rangle} \right\} \\
&= \frac{1}{\left(\sum_{i_\alpha \in A} \mathrm{e}^s\right) + o(\mathrm{e}^s)} \left\{ \mathrm{e}^s \big(\sum_{i_\alpha \in A} p_j^{(i_\alpha)} \big) + o(\mathrm{e}^s) \right\} \\
&\to \frac{\sum_{i_\alpha \in A} p_j^{(i_\alpha)}}{\sharp A}
\end{aligned} \tag{3.2}$$

as $s \to \infty$, where A is a subset of $\{1, \dots, m\}$ such that $i_\alpha \in A$ if and only if $p^{(i_\alpha)}$ is contained in $\cap_{i=1}^{l+1} H_i$, where $H_i := \{y \in M_{\mathbb{R}} \mid \langle y, q^{(i)} \rangle = 1\}$. In the above $f(s) \in o(\mathrm{e}^{cs})$ means $\lim_{s\to\infty} f(s)\,\mathrm{e}^{-cs} = 0$ and $\sharp A$ denotes the number of integers in A. Equation (3.2) means that the point

$$p^{(\delta^*)} := \lim_{s\to\infty} \tilde{\mu}(sx^{(a)}) \tag{3.3}$$

is independent of the choice of a vector a and is contained in the relative interior of the face δ^*. In fact, $\{p^{(i_\alpha)}\}_{i_\alpha \in A}$ is the set of all integral points on δ^* and $p^{(\delta^*)}$ is the average of them. So, in order to determine whether $\mu^{-1}(\delta^*)$ is contained in the MIS of exponent γ or not, it is sufficient to determine whether $c_{\{p^{(\delta^*)}\}}(\psi_\infty)$ is strictly smaller than γ or not. In fact, the $T_{\mathbb{C}}$-invariance of the MIS implies that if $p^{(\delta^*)}$ is contained in the MIS then δ^* is also contained in it.

Next, we shall give a formula to calculate $c_{\{p^{(\delta^*)}\}}(\psi_\infty)$ for each face δ^* of P^*. Let $\mathrm{L}P^*_{\max}$ be the set of all integral points p contained in P^* satisfying

$$\langle p, -\beta_{\mathrm{KRS}} \rangle = \max_{p^{(i)} \in \mathrm{L}P^*} \langle p^{(i)}, -\beta_{\mathrm{KRS}} \rangle. \tag{3.4}$$

Let $u_0'(t,x)$ be a convex function on $N_{\mathbb{R}}$ defined by

$$u_0'(t,x) := \log\left(\sum_{i=1}^{m} \mathrm{e}^{\langle p^{(i)}, x - s_t\beta_{\mathrm{KRS}}\rangle}\right) - s_t \max_{i=1,\dots,m}\langle p^{(i)}, -\beta_{\mathrm{KRS}}\rangle.$$

Then, $(\rho_t^{-1})^*\omega_0 = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}u_0'(t,x)$. We find

$$u_0'(t,x) - u_0(x) = \log\left(\frac{\sum_{i=1}^{m} \mathrm{e}^{\langle p^{(i)},x\rangle + s_t(\langle p^{(i)},-\beta_{\mathrm{KRS}}\rangle - \max_j\langle p^{(j)},-\beta_{\mathrm{KRS}}\rangle)}}{\sum_{i=1}^{m} \mathrm{e}^{\langle p^{(i)},x\rangle}}\right) \le 0 \tag{3.5}$$

for all $x \in N_{\mathbb{R}}$ and all $t \in \mathbb{R}_{\ge 0}$, and on the other hand we also find

$$\begin{aligned} &uw_0'(t,-s\beta_{\mathrm{KRS}}) - u_0(-s\beta_{\mathrm{KRS}}) \\ &= \log\left(\frac{\sum_{i=1}^{m} \mathrm{e}^{\langle p^{(i)},-s\beta_{\mathrm{KRS}} - s_t\beta_{\mathrm{KRS}}\rangle - s_t\max(\langle p^{(i)},-\beta_{\mathrm{KRS}}\rangle)}}{\sum_{i=1}^{m} \mathrm{e}^{\langle p^{(i)},-s\beta_{\mathrm{KRS}}\rangle}}\right) \\ &\ge \log\left(\frac{(\sharp \mathrm{L}P^*_{\max})\cdot \mathrm{e}^{s\max(\langle p^{(i)},-\beta_{\mathrm{KRS}}\rangle)}}{\sum_{i=1}^{m} \mathrm{e}^{\langle p^{(i)},-s\beta_{\mathrm{KRS}}\rangle}}\right) \\ &\to 0 \end{aligned} \tag{3.6}$$

as $s \to \infty$. From (3.5) and (3.6), we find

$$\lim_{s\to\infty}(u_0'(t,-s\beta_{\mathrm{KRS}}) - u_0(-s\beta_{\mathrm{KRS}})) = 0. \tag{3.7}$$

From (3.5) and (3.7) we find $\sup_{x\in N_{\mathbb{R}}}(u_0'(t,x) - u_0(x)) = 0$, that is to say, $\psi_t(x) = u_0'(t,x) - u_0(x)$.

Theorem 3.1. *Let δ^* be an $(n-l-1)$-dimensional face of P^*. Let δ is the l-dimensional face of P associated with δ^*. Then, we have the following two possibilities;*

(i) *If for any $x \in \delta$ there is an integral point $p_x \in \mathrm{L}P^*_{\max}$ such that $\langle p_x, x\rangle \ge 0$, then $c_{\{p^{(\delta^*)}\}}(\psi_\infty) \ge 1$. In particular $\mu^{-1}(\delta^*)$ is not contained in the support of the MIS of $\mathcal{I}(\gamma\psi_\infty)$ for any $\gamma < 1$.*

(ii) *Suppose that there is a point $x \in \delta$ such that*

$$\langle p, x\rangle < 0 \ \textit{for any}\ p \in \mathrm{L}P^*_{\max}. \tag{3.8}$$

Let $p_{\max} \in \mathrm{L}P^*_{\max}$ *and* $x^{(0)}$ *be a point in* δ *such that*

$$\langle p_{\max}, x^{(0)} \rangle = \min_{x} \max_{p \in \mathrm{L}P^*_{\max}} \langle p, x \rangle, \tag{3.9}$$

where x *runs over* $\{x \in \delta \mid x \text{ satisfies (3.8)}\}$*. Then, we have*

$$c_{\{p^{(\delta^*)}\}}(\psi_\infty) = \frac{1}{1 - \langle p_{\max}, x^{(0)} \rangle} < 1.$$

In particular, $\mu^{-1}(\delta^*)$ *is contained in the support of the MIS of* $\mathcal{I}(\gamma\psi_\infty)$ *for any* $\gamma \in (c_{\{p^{(\delta^*)}\}}(\psi_\infty), 1)$*.*

Proof. First, we shall show the case (i). For any $x \in \delta$, the assumption implies

$$u_0'(t, sx) \geq s\langle p_x, x \rangle \geq 0 \tag{3.10}$$

for all $s \geq 0$. Since there exists a constant C independent of t and x such that

$$|u_0'(t, y) - u_0'(t, x)| \leq C, \quad \text{for any } y \in B_1(x),$$

where $B_1(x) = \{y \in N_{\mathbb{R}} \mid |x - y| < 1\}$, equation (3.10) implies that $u_0'(t, \cdot)$ is bounded uniformly from below over

$$\tilde{U} := \{s_1 x + s_2 \eta \in N_{\mathbb{R}} \mid x \in \delta,\ \eta \in N_{\mathbb{R}}, |\eta| = 1,\ s_i \in \mathbb{R}_{\geq 0}, |s_2| < 1\}.$$

Let $U_{p^{(\delta^*)}} \subset X$ be the interior of $\mu^{-1}(\overline{\tilde{\mu}(\tilde{U})})$, where $\overline{\tilde{\mu}(\tilde{U})}$ denotes the closure of $\tilde{\mu}(\tilde{U})$. Then, from (3.2), we find that $U_{p^{(\delta^*)}}$ is an open neighborhood around $\mu^{-1}(p^{(\delta^*)})$. Then, for $c \geq 0$,

$$\begin{aligned}
\int_{U_{p^{(\delta^*)}}} \mathrm{e}^{-c\psi_t} \omega_0^n &\leq C \int_{\tilde{U}} \mathrm{e}^{-c\psi_t - u_0} dx && (3.11)\\
&= C \int_{\tilde{U}} \mathrm{e}^{-cu_0'(t,x) + (-1+c)u_0(x)} dx && (3.12)\\
&\leq C \int_{\tilde{U}} \mathrm{e}^{(-1+c)u_0(x)} dx \\
&\leq C \Big(\int_{s=0}^{\infty} \mathrm{e}^{(-1+c)s} ds \Big)^{l+1}. && (3.13)
\end{aligned}$$

In (3.11), we use the fact (cf. Lemma 4.3 in [23]) that $\mathrm{e}^{u_0}\det\left(\frac{\partial^2 u_0}{\partial x_i \partial x_j}\right)$ is bounded. From (3.13), we find that $\int_{U_{p^{(\delta^*)}}} \mathrm{e}^{-c\psi_t}\omega_0^n$ is bounded if $0 \le c < 1$. Hence we find that $c_{\{p^{(\delta^*)}\}}(\psi_\infty) \ge 1$.

Next we shall prove the case (ii). Before proving it, remark that the existence of the points $p_{\max}$ and $x^{(0)}$ in (3.9) is assured. In fact a function $x \mapsto \max_{p\in \mathrm{L}P^*_{\max}}\langle p, x\rangle$ is continuous on a compact set $\{x \in \delta \mid \langle p, x\rangle \le 0 \text{ for all } p \in \mathrm{L}P^*_{\max}\}$ and it is

$$\begin{cases} \text{equal to zero} & \text{if} \quad \langle p, x\rangle = 0 \text{ for some } p \in \mathrm{L}P^*_{\max}, \\ \text{strictly less than zero} & \text{if} \quad \langle p, x\rangle < 0 \text{ for all } p \in \mathrm{L}P^*_{\max}. \end{cases}$$

These mean that the minimal point $x^{(0)}$ of the above function is contained in $\{x \in \delta \mid \langle p, x\rangle < 0 \text{ for all } p \in \mathrm{L}P^*_{\max}\}$. Let us begin to prove (ii). The definition (3.9) implies that for all $x \in \delta$

$$u_0'(t, sx) \ge s \max_{p\in \mathrm{L}P^*_{\max}} \langle p, x\rangle \ge s\langle p_{\max}, x^{(0)}\rangle. \tag{3.14}$$

Since for any $x \in \delta$ there is a vertex p of P^* such that $\langle x, p\rangle = 1$, then we have

$$u_0(sx) \ge s. \tag{3.15}$$

As (3.12), for $0 \le c < 1$, (3.14) and (3.15) imply

$$\begin{aligned} \int_{U_{p^{(\delta^*)}}} \mathrm{e}^{-c\psi_t}\omega_0^n &\le C \int_{\tilde U} \mathrm{e}^{-cu_0'(t,x)+(-1+c)u_0(x)}dx \\ &\le C\left(\int_{s=0}^{\infty} \mathrm{e}^{s\{-1+c(1-\langle p_{\max},x^{(0)}\rangle)\}}ds\right)^{l+1}. \end{aligned} \tag{3.16}$$

From (3.16) we find that

$$c_{\{p^{(\delta^*)}\}}(\psi_\infty) \ge \frac{1}{1-\langle p_{\max}, x^{(0)}\rangle}. \tag{3.17}$$

Next we shall prove $c_{\{p^{(\delta^*)}\}}(\psi_\infty) \le \frac{1}{1-\langle p_{\max},x^{(0)}\rangle}$. For each $p^{(i)} \in \mathrm{L}P^*$, let

$$A_i(s) := \langle p^{(i)}, sx^{(0)} - s_t\beta_{\mathrm{KRS}}\rangle - s_t\langle p_{\max}, -\beta_{\mathrm{KRS}}\rangle.$$

Then by (3.4), for all $i = 1, \dots, m$, we have

$$A_i(s) = s(\langle p_{\max}, x^{(0)}\rangle + \langle p^{(i)} - p_{\max}, x^{(0)}\rangle) - s_t(\langle p^{(i)}, \beta_{\mathrm{KRS}}\rangle - \langle p_{\max}, \beta_{\mathrm{KRS}}\rangle) \leq s(\langle p_{\max}, x^{(0)}\rangle + \langle p^{(i)} - p_{\max}, x^{(0)}\rangle). \tag{3.18}$$

Let us consider the following two possibilities separately.

$$\begin{cases} \text{(a) } \langle p^{(i)} - p_{\max}, x^{(0)}\rangle \leq 0, \\ \text{(b) } \langle p^{(i)} - p_{\max}, x^{(0)}\rangle > 0. \end{cases}$$

In the case (a), we have

$$A_i(s) \leq s\langle p_{\max}, x^{(0)}\rangle, \quad \text{for all } s \geq 0. \tag{3.19}$$

Let us consider the case (b). If $p^{(i)}$ would be contained in $\mathrm{L}P^*_{\max}$, (3.9) induces that

$$\langle p_{\max}, x^{(0)}\rangle = \max_{p \in \mathrm{L}P^*_{\max}} \langle p, x^{(0)}\rangle \geq \langle p^{(i)}, x^{(0)}\rangle.$$

This is a contradiction. Therefore, we have $p^{(i)} \notin \mathrm{L}P^*_{\max}$. This and (3.4) imply that $\langle p^{(i)}, \beta_{\mathrm{KRS}}\rangle - \langle p_{\max}, \beta_{\mathrm{KRS}}\rangle$ is *strictly* bigger than zero. Then,

$$s\langle p^{(i)} - p_{\max}, x^{(0)}\rangle - s_t(\langle p^{(i)}, \beta_{\mathrm{KRS}}\rangle - \langle p_{\max}, \beta_{\mathrm{KRS}}\rangle) \leq 0$$

for all $s \in [0, s_t T'_i]$. Here

$$T'_i := \frac{\langle (p^{(i)} - p_{\max}), \beta_{\mathrm{KRS}}\rangle}{\langle (p^{(i)} - p_{\max}), x^{(0)}\rangle}.$$

Then, (3.18) implies

$$A_i(s) \leq s\langle p_{\max}, x^{(0)}\rangle \tag{3.20}$$

for all $s \in [0, s_t T'_i]$. Let $T' := \min\{T'_i \mid i = 1, \dots, m\} > 0$. This constant depends only on β_{KRS} and independent of s and i. Therefore, (3.19) and (3.20) imply that for all $i = 1, \dots, m$

$$A_i(s) \leq s\langle p_{\max}, x^{(0)}\rangle, \quad \text{for all } s \in [0, s_t T']. \tag{3.21}$$

Let $\tilde{U}_\varepsilon := \{x \in N_{\mathbb{R}} \mid |x - sx^{(0)}| < \varepsilon,\ s \geq \frac{1}{\varepsilon}\}$. For any open neighborhood U' of $\mu^{-1}(p^{(\delta^*)})$, there is a sufficiently small constant $\varepsilon > 0$ such that $\tilde{\mu}(\tilde{U}_\varepsilon) \subset$

$\mu(U')$. In fact, for the point $x^{(0)}$ in (3.9) we have

$$\begin{aligned}\frac{\partial u_0}{\partial x_j}(sx^{(0)}+\eta) &= \frac{1}{\sum_{i=1}^m \mathrm{e}^{\langle p^{(i)}, sx^{(0)+\eta}\rangle}}\Big\{\sum_{i=1}^m p_j^{(i)}\, \mathrm{e}^{\langle p^{(i)}, sx^{(0)+\eta}\rangle}\Big\}\\ &\to \frac{\sum_{i_\alpha\in A} \mathrm{e}^{\langle p^{(i_\alpha)},\eta\rangle} p_j^{(i_\alpha)}}{\sum_{i_\alpha\in A} \mathrm{e}^{\langle p^{(i_\alpha)},\eta\rangle}}\end{aligned}$$

as $s\to\infty$, where A is the subset of $\{1,\dots,m\}$ defined by (3.2). Since A is independent of η, there is a positive constant C independent of ε and η such that

$$\begin{aligned}(3.22)\qquad & |\lim_{s\to\infty}\tilde\mu(sx^{(0)}+\eta)-p^{(\delta^*)}|^2\\ &= \sum_{j=1}^n \left|\frac{\sum_{i_\alpha\in A} \mathrm{e}^{\langle p^{(i_\alpha)},\eta\rangle} p_j^{(i_\alpha)}}{\sum_{k_\alpha\in A} \mathrm{e}^{\langle p^{(k_\alpha)},\eta\rangle}} - \frac{\sum_{i_\alpha\in A} p_j^{(i_\alpha)}}{\sharp A}\right|^2\\ &= C\sum_{j=1}^n \frac{\left|\sum_{i_\alpha\in A}\big(\sum_{k_\alpha\in A}(\mathrm{e}^{\langle p^{(i_\alpha)},\eta\rangle}-\mathrm{e}^{\langle p^{(k_\alpha)},\eta\rangle})\big)p_j^{(i_\alpha)}\right|^2}{\sharp A\big\{\sum_{k_\alpha\in A}\mathrm{e}^{\langle p^{(k_\alpha)},\eta\rangle}\big\}}\\ &\le C\varepsilon^2,\end{aligned}$$

for any sufficiently small $\varepsilon>0$ and any $\eta\in N_{\mathbb{R}}$ with $|\eta|<\varepsilon$. The inequality (3.22) follows from that for sufficiently small ε there exists positive constants c and C such that

$$c<\sum_{k_\alpha\in A}\mathrm{e}^{\langle p^{(k_\alpha)},\eta\rangle},\quad |\mathrm{e}^{\langle p^{(i_\alpha)},\eta\rangle}-\mathrm{e}^{\langle p^{(k_\alpha)},\eta\rangle}|^2\le C|\eta|^2\le C\varepsilon^2$$

for any i_α, $k_\alpha\in A$. Then, we find that there is a positive constant C independent of s and η such that

$$(3.23)\qquad |\tilde\mu(sx^{(0)}+\eta)-\tilde\mu(sx^{(0)})|\le C\varepsilon,$$

for all $s\in\mathbb{R}_{\ge 0}$ and any η with $|\eta|<\varepsilon$. This implies that $\tilde\mu(\tilde U_\varepsilon)\subset\mu(U')$ for any sufficiently small ε. Remark that $\tilde\mu(\tilde U_\varepsilon)$ is not necessarily a neighborhood of $p^{(\delta^*)}$. (For instance, when δ^* is a 0-dimensional face, $\tilde\mu(sx^{(0)}+\eta)$ goes to the point $p^{(\delta^*)}$ for any η, because $\sharp A=1$.) There is a positive constant C_ε depending only on ε such that, for any $x\in\tilde U_\varepsilon$ with $|x-sx^{(0)}|<\varepsilon$,

$$(3.24)\qquad u_0'(t,x)\le u_0'(t,sx^{(0)})+C_\varepsilon=\log\Big(\sum_i \exp A_i(s)\Big)+C_\varepsilon.$$

On the other hand,

$$u_0(sx) \leq s + \log m, \tag{3.25}$$

where $x \in \delta$ and m is the number of lattice points in P^*. From (3.21), (3.24) and (3.25) we find that for $0 \leq c < 1$ and a fixed sufficiently small ε,

$$\begin{aligned}\int_{U'} \mathrm{e}^{-c\psi_t}\omega_0^n &\geq C\int_{\mu^{-1}(\tilde{\mu}(\tilde{U}_\varepsilon))} \mathrm{e}^{-c\psi_t}\omega_0^n \\ &\geq C\int_{\tilde{U}_\varepsilon} \mathrm{e}^{-cu_0'(t,x)+(-1+c)u_0}dx \geq C\int_{\frac{1}{\varepsilon}}^{s_tT'} \mathrm{e}^{-c\max_i A_i(s)+(-1+c)s}ds \\ &\geq C\int_{\frac{1}{\varepsilon}}^{s_tT'} \mathrm{e}^{s\{c(1-\langle p_{\max},x^{(0)}\rangle)-1\}}ds. \end{aligned} \tag{3.26}$$

If $c \geq \frac{1}{1-\langle p_{\max},x^{(0)}\rangle}$, the RHS of (3.26) goes to $+\infty$ as $t \to \infty$, because s_t goes to $+\infty$. The semi-continuity of the complex singularity exponent ([7]) implies that

$$c_{\{p^{(\delta^*)}\}}(\psi_\infty) \leq \frac{1}{1-\langle p_{\max},x^{(0)}\rangle}. \tag{3.27}$$

Hence we get the desired equation from (3.17) and (3.27). The proof is completed. □

Remark 3.1. Theorem 3.1 is kind of a local version of Song's formula [23] of the α-invariant on toric Fano manifolds.

Then, Lemma 3.1 is a corollary of Theorem 3.1.

Proof of Lemma 3.1. The behavior of s_t does not affect Theorem 3.1 and its proof. Hence, the support of the MIS of $\mathcal{I}(\gamma\psi_\infty)$ and of $\mathcal{I}(\gamma\overline{\varphi}_\infty)$ coincide. □

Let $\varepsilon > 0$ be a sufficiently small constant. Theorem 3.1 gives us a way to determine the support of the MIS of exponent γ from any one-parameter subgroup of $\mathrm{Aut}(X)$ for $\gamma \in (1-\varepsilon, 1)$ as follows. Before describing this, let us introduce some terminologies. Let σ_t be a one-parameter subgroup of the holomorphic vector field v_ζ which is associated with a vector $\zeta \in N_\mathbb{R}$. Let us consider the MIS coming from $\{(\sigma_t^{-1})^*\omega_0\}_t$ as before. For distinct points $x^{(1)}$ and $x^{(2)}$ on ∂P, we define that $x^{(1)} \sim x^{(2)}$ if and only if $x^{(1)}$ and $x^{(2)}$ are contained in a common $(n-1)$-dimensional facet of P. For a point $x \in \partial P$, we define the star set of x by

$$st(x) := \{y \in \partial P \mid y \sim x\}.$$

Let $x(-\zeta) \in \partial P$ be a point, which is the intersection between ∂P and the half line $\{-s\zeta \in N_{\mathbb{R}} \mid s \geq 0\}$. Then, $st(x(-\zeta))$ is a union of $(n-1)$-dimensional facets $\{\delta_k\}_{k=1,\ldots,k_\zeta}$ of P. For each δ_k, there corresponds to a hyperplane $\{x \in N_{\mathbb{R}} \mid H_k(x) = 1\}$ in $N_{\mathbb{R}}$, which contains δ_k. Then, the star set $st(x(-\zeta))$ divides $N_{\mathbb{R}}$ into two. This means that $N_{\mathbb{R}}$ is divided into $N_{\mathbb{R}}^{\leq} := \{x \mid H_k(x) \leq 1 \text{ for all } k\}$ and its complement. Then, by translating $N_{\mathbb{R}}^{\leq}$ along the line $\{-s\zeta \in N_{\mathbb{R}} \mid s \in \mathbb{R}\}$ so that the origin is contained in its boundary, we define

$$\widetilde{st(x(-\zeta))} := \{x \in N_{\mathbb{R}} \mid H_k(x) \leq 0, \text{ for all } k\}.$$

Corollary 3.1. *Let X be a toric Fano manifold. Let σ_t be a one-parameter subgroup of the holomorphic vector field v_ζ which is associated with a vector $\zeta \in N_{\mathbb{R}}$. Suppose $\gamma \in (1-\varepsilon, 1)$ where ε is a sufficiently small positive constant. Let δ^* be an $(n-l-1)$-dimensional face of P^* and δ be its associated l-dimensional face. Then, δ^* is contained in the image of the support of the MIS of exponent γ from $\{(\sigma_t^{-1})^*\omega_0\}_t$ under the moment map μ if and only if $\delta \cap int(\widetilde{st(x(-\zeta))}) \neq \emptyset$, where $int(\widetilde{st(x(-\zeta))})$ is the interior of $\widetilde{st(x(-\zeta))}$.*

Proof. From the duality of P and P^*, we find that for each H_k there is a point $p^{(k)} \in \mathrm{L}P^*_{\max}$ corresponding to it and, that $st(x(-\zeta))$ is equal to

$$\{x \in N_{\mathbb{R}} \mid \langle p^{(k)}, x\rangle = 1, \quad \text{for all } 1 \leq k \leq k_\zeta\}.$$

Hence we find that $int(\widetilde{st(x(-\zeta))})$ is equal to

$$\{x \in N_{\mathbb{R}} \mid \langle p^{(k)}, x\rangle < 0, \quad \text{for all } 1 \leq k \leq k_\zeta\}.$$

If $\delta \cap \mathrm{int}(\widetilde{st(x(-\zeta))}) \neq \emptyset$, then Theorem 3.1 (ii) implies $c_{(p^{\delta^*})}(\psi_\infty) < 1 - \varepsilon(\delta)$ for some $\varepsilon(\delta) > 0$ which might depend on δ. By taking a sufficiently small ε, we get that $c_{(p^{\delta^*})}(\psi_\infty) < 1 - \varepsilon$ if $\delta \cap \mathrm{int}(\widetilde{st(x(-\zeta))}) \neq \emptyset$, because the number of faces in P is finite. On the other hand, if $\delta \cap int(\widetilde{st(x(-\zeta))}) = \emptyset$, then Theorem 3.1 (i) implies $c_{(p^{\delta^*})}(\psi_\infty) \geq 1 \geq 1 - \varepsilon$. This completes the proof. $\square$

4. Examples

In this section, we shall calculate several examples that are contained in $\mathcal{W}_1$. Let v_{KRS} be the holomorphic vector field of Kähler–Ricci soliton, which is contained in the reductive part $\mathfrak{h}_r(X)$ of the Lie algebra $\mathfrak{h}(X)$ consisting of all holomorphic vector fields on X. Since manifolds are contained in $\mathcal{W}_1$, we can determine the vector β_{KRS} in $N_{\mathbb{R}}$ which induces v_{KRS} by calculating the sign of its Futaki invariant. Futaki invariant [11] is a Lie character of $\mathfrak{h}(X)$, defined by

$$F(v) := \int_X v h_g \omega_g^n.$$

It is proved that F is independent of the choice of g. Let $(v_{\mathrm{KRS}}, \omega_{\mathrm{KRS}})$ be a Kähler–Ricci soliton. It is not difficult to see

$$F(v_{\mathrm{KRS}}) > 0. \tag{4.1}$$

Then, in order to determine β_{KRS} under the assumption that X is contained in $\mathcal{W}_1$, it is sufficient to calculate the sign of Futaki invariant of the holomorphic vector field coming from a vector in $N_{\mathbb{R}}$, which is invariant under $\mathcal{W}(X)$. To calculate it, we shall use the following result;

Theorem 4.1 (Mabuchi [14]). *Let $\mathbb{F} := (F(t_1\frac{\partial}{\partial t_1}), \ldots, F(t_n\frac{\partial}{\partial t_n})) \in \mathbb{R}^n$. Remark that $t_i\frac{\partial}{\partial t_i}$ is a $T_{\mathbb{C}}$-invariant holomorphic vector field on $T_{\mathbb{C}}$, which can be extended on X. Let $b(P^*) \in M_{\mathbb{R}}$ be the barycenter of P^*, i.e.,*

$$\frac{1}{\int_{P^*} dy}\left(\int_{P^*} y_1 dy, \ldots, \int_{P^*} y_n\, dy\right),$$

where $dy = dy_1 \wedge \cdots \wedge dy_n$. Then $\mathbb{F}$ is equal to $-b(P^*)$.

The minus sign of $b(P^*)$ above comes from that our choice of affine logarithmic coordinates has the opposite sign to the one in [14]. Combining (4.1) and Theorem 4.1 we find

$$\langle b(P^*), \beta_{\mathrm{KRS}} \rangle < 0. \tag{4.2}$$

4.1. Toric Fano 2-folds

There are five types of toric Fano 2-folds; $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the blow–up of $\mathbb{CP}^2$ at k points, where $k = 1, 2, 3$. Kähler–Einstein manifolds among them are $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the blow up of $\mathbb{CP}^2$ at 3 points. Meanwhile the blow up of $\mathbb{CP}^2$ at k points ($k = 1, 2$) does not admit Kähler–Einstein metrics and it is contained in $\mathcal{W}_1$. So we can apply our results to them. First, let us consider the blow up of $\mathbb{CP}^2$ at one point.

Example 4.1. The support of the KRF-MIS on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$ of exponent γ is the exceptional divisor for all $\gamma \in (\frac{1}{2}, 1)$.

Proof. The polytope in $N_{\mathbb{R}}$ whose vertices are

$$(-1,-1), (-1,0), (0,-1), (1,1),$$

corresponds to the Fano polytope P of $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$. Then, $N_{\mathbb{R}}^{\mathcal{W}(X)}$ is the one-dimensional subspace of $N_{\mathbb{R}}$ generated by a vector $(-1,-1)$ and β_{KRS} is proportional to $(-1,-1)$. Since the vertices of the polytope P^* are

$$(-1,0), (0,-1), (2,-1), (-1,2),$$

it is easy to see that $\langle b(P^*), (1,1) \rangle > 0$. Then, (4.2) implies that $\beta_{\mathrm{KRS}} = \beta(-1,-1)$ where $\beta > 0$. Also we find that

$$st(\widetilde{x(-\beta_{\mathrm{KRS}})}) = \{x = (x_1, x_2) \mid x_1 - 2x_2 \geq 0, 2x_1 - x_2 \leq 0\}.$$

The vertex of P contained in $int(st(\widetilde{x(-\beta_{\mathrm{KRS}})}))$ is $(-1,-1)$ which represents the exceptional divisor. Then, Corollary 3.1 implies that the support of the KRF-MIS of exponent γ is the exceptional divisor where γ is strictly smaller than 1 and sufficiently close to 1. $\mathrm{L}P^*_{\max}$ is

$$\{(2,-1), (-1,2)\}.$$

For the facet δ^* of P^* associated with the vertex $(-1,-1)$ of P,

$$\langle p_{\max}, x^{(0)} \rangle = \langle (2,-1), (-1,-1) \rangle = \langle (-1,2), (-1,-1) \rangle = -1.$$

Hence,

$$c_{\{p^{(\delta^*)}\}}(\psi_\infty) = c_{\{(-\frac{1}{2}, -\frac{1}{2})\}}(\psi_\infty) = \frac{1}{2}.$$

Therefore, the proof is completed. □

Next, let us consider the blow up of $\mathbb{CP}^2$ at p_1 and p_2. Let E_1 and E_2 be the exceptional divisors of the blow up. In X, there is another (-1)-curve denoted by E_0, which intersects with E_1 and E_2 Remark that E_0 is the proper transform of $\overline{p_1 p_2}$ of the line passing through p_1 and p_2. Then,

Example 4.2. The support of the KRF-MIS on $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}^2}$ of exponent γ is

$$\begin{cases} \cup_{i=0}^{2} E_i, & \text{for } \gamma \in (\frac{1}{2}, 1), \\ E_0, & \text{for } \gamma \in (\frac{1}{3}, \frac{1}{2}). \end{cases}$$

Proof. The polytope in $N_{\mathbb{R}}$ whose vertices are

$$(-1,0),(0,-1),(1,0),(1,1),(0,1),$$

corresponds to the Fano polytope P of $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}^2}$. Then, $N_{\mathbb{R}}^{\mathcal{W}(X)}$ is the one-dimensional subspace of $N_{\mathbb{R}}$ generated by a vector $(1,1)$ and β_{KRS} is proportional to $(1,1)$. Since the vertices of the polytope P^* are

$$(-1,-1),(-1,1),(0,1),(1,0),(1,-1),$$

we find that $\langle b(P^*),(1,1)\rangle < 0$. Then, (4.2) implies that $\beta_{\mathrm{KRS}} = \beta(1,1)$ where $\beta > 0$. Also we find that

$$st(\widetilde{x(-\beta_{\mathrm{KRS}})}) = \{x = (x_1,x_2) \mid x_1 + x_2 \geq 0\}.$$

The vertices of P contained in $\mathrm{int}(st(\widetilde{x(-\beta_{\mathrm{KRS}})}))$ are $(1,0),(0,1)$, which represent the exceptional divisors E_1 and E_2, and $(1,1)$, which represents the proper transform E_0. Then, Corollary 3.1 implies that the support of the KRF-MIS of exponent γ is the sum of E_0, E_1 and E_2 where γ is strictly smaller than 1 and sufficiently close to 1. $\mathrm{L}P^*_{\max}$ is

$$\{(-1,-1)\}.$$

For the facets $\eta_i^*, (i=1,2)$ of P^* associated with the vertices $(1,0)$ and $(0,1)$ of P, respectively,

$$\langle p_{\max}, x^{(0)}\rangle = \langle(-1,-1),(1,0)\rangle = \langle(-1,-1),(0,1)\rangle = -1.$$

Hence,

$$c_{\{p^{(\eta_1^*)}\}}(\psi_\infty) = c_{\{(1,-\frac{1}{2})\}}(\psi_\infty) = \tfrac{1}{2}.$$

Also $c_{\{p^{(\eta_2^*)}\}}(\psi_\infty) = \frac{1}{2}$. For the facet δ^* associated with the vertex $(1,1)$ of P,

$$\langle p_{\max}, x^{(0)}\rangle = \langle(-1,-1),(1,1)\rangle = -2.$$

Hence,

$$c_{\{p^{(\delta^*)}\}}(\psi_\infty) = c_{\{(\frac{1}{2},\frac{1}{2})\}}(\psi_\infty) = \tfrac{1}{3}.$$

Therefore, the proof is completed. □

4.2. Toric Fano 3-folds

Toric Fano 3-folds are classified completely (Remark 2.5.10 in [1]). According to the classification, there are eighteen types of toric Fano 3-folds. Five of them are Kähler–Einstein manifolds, and eight of them are contained in $\mathcal{W}_1$ and do not admit Kähler–Einstein metrics. (As for the classification of Kähler–Einstein toric 3-folds, see [14].) In this section, we give the list of the support of the KRF-MIS for the eight examples, computed by applying our method. The data of the list consists of (a) the vertices of the Fano polytope P denoted by $\mathrm{Ver}P$ and the vertices of the moment polytope P^* denoted by $\mathrm{Ver}P^*$, (b) the vector β_{KRS} and (c) the vertices of contained in $int(st(\widetilde{x(-\beta_{\mathrm{KRS}}}))$ denoted by $\widetilde{\mathrm{Ver}P} = \{q^{(i)}\}_i$ and the complex singularity exponent corresponding to $q^{(i)}$, which is denoted by c_i.

Example 4.3. Let X_k be the blow up of $\mathbb{CP}^n$ along $\mathbb{CP}^k$, where $0 \leq k \leq n-2$. The support of the KRF-MIS of complex singular exponent γ is the exceptional divisor for $\gamma \in (\frac{1}{k+2}, 1)$.

(a)

$$\mathrm{Ver}P = \underbrace{\begin{pmatrix} a_1 & -1 & 0 & \cdots & 0 & 1 \\ a_2 & 0 & -1 & & 0 & 1 \\ \vdots & \vdots & 0 & \ddots & 0 & \vdots \\ a_n & 0 & 0 & \cdots & -1 & 1 \end{pmatrix}}_{n+2},$$

where $a_i = -1$ for $1 \leq i \leq n-k$, otherwise $a_i = 0$. $\mathrm{Ver}P^*$ consists of

$$\{p^{(i)} = (b_1^{(i)}, \dots, b_n^{(i)}) \mid b_i^{(i)} = n,\ b_j^{(i)} = -1 \text{ for } j \neq i\}_{1 \leq i \leq n-k}$$

and

$$\{p^{(i)} \in \mathrm{Ver}P^* \mid \langle p^{(i)}, q^{(1)} \rangle = 1\}_{n-k+1 \leq i \leq \sharp \mathrm{Ver}P^*},$$

where $q^{(1)} = (a_1, \dots, a_n)$.

(b) $\beta_{\mathrm{KRS}} = (a_1, \dots, a_n)$.

(c) $\widetilde{\mathrm{Ver}P} = \{q^{(1)}\}$ with

$$c_1 = \frac{1}{1 - \langle p^{(i)}, q^{(1)} \rangle} = \frac{1}{k+2} \quad \text{(for any } 1 \leq i \leq n-k).$$

The vertex $q^{(1)}$ represents the exceptional divisor.

Remark that $\mathcal{B}_2 = \mathbb{P}_{\mathbb{CP}^2}(\mathcal{O} \oplus \mathcal{O}(1))$ and $\mathcal{B}_3 = \mathbb{P}_{\mathbb{CP}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ in [1] are contained in Example 4.3 as the case for ($n = 3$, $k = 0$) and ($n = 3$, $k = 1$) respectively.

Example 4.4. Let $\mathcal{B}_1$ be $\mathbb{P}_{\mathbb{CP}^2}(\mathcal{O} \oplus \mathcal{O}(2))$. The support of the KRF-MIS on $\mathcal{B}_1$ of exponent γ is S_∞ for $\gamma \in (\frac{1}{2}, 1)$. Here S_∞ is the divisor defined by a section $(0, \sigma)$ of $\mathcal{O} \oplus \mathcal{O}(2)$ over $\mathbb{CP}^2$. More precisely, S_∞ is the closure of

$$\{[0; \sigma(p)] \in \mathcal{B}_1 \mid \sigma(p) \neq 0\}.$$

Remark that it is not an exceptional divisor.

(a)

$$\mathrm{Ver}P = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, \quad \mathrm{Ver}P^* = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & -1 & 0 & 2 & -3 & 2 \\ 0 & 0 & -1 & 2 & 2 & -3 \end{pmatrix}.$$

(b) $\beta_{\mathrm{KRS}} = (1, 0, 0)$.

(c) $\widetilde{\mathrm{Ver}P} = \{q^{(1)} = (1, 0, 0)\}$ with $c_1 = \frac{1}{2}$, which represents S_∞.

Example 4.5. Let $\mathcal{C}_1$ be $\mathbb{P}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(\mathcal{O} \oplus \mathcal{O}(1, 1))$. The support of the KRF-MIS on $\mathcal{C}_1$ of exponent γ is S_∞ for $\gamma \in (\frac{1}{2}, 1)$. Here S_∞ is the divisor defined by a section $(0, \sigma_1 \otimes \sigma_2)$ of $\mathcal{O} \oplus \mathcal{O}(1, 1)$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$ and σ_i is the pull-back of the section of $\mathcal{O}_{\mathbb{CP}^1}(1)$ with respect to the ith projection $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$. Remark that S_∞ is not an exceptional divisor.

(a)

$$\mathrm{Ver}P = \begin{pmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 1 & -1 & 1 & 1 & 0 & 0 \end{pmatrix},$$
$$\mathrm{Ver}P^* = \begin{pmatrix} 0 & 0 & -1 & -1 & 2 & 2 & -1 & -1 \\ 0 & -1 & 0 & -1 & -1 & 2 & 2 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

(b) $\beta_{\mathrm{KRS}} = (0, 0, 1)$.

(c) $\widetilde{\mathrm{Ver}P} = \{q^{(1)} = (0, 0, 1)\}$ with $c_1 = \frac{1}{2}$, which represents S_∞.

Example 4.6. Let $\mathcal{C}_4$ be $(\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}) \times \mathbb{CP}^1$, which is the blow up of $\mathbb{CP}^2 \times \mathbb{CP}^1$ along $\{\text{point}\} \times \mathbb{CP}^1$. The support of the KRF-MIS on $\mathcal{C}_4$ of exponent γ is the exceptional divisor of the blow up for $\gamma \in (\frac{1}{2}, 1)$.

(a)

$$\mathrm{Ver}P = \begin{pmatrix} 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{Ver}P^* = \begin{pmatrix} 0 & -1 & 2 & -1 & 0 & -1 & 2 & -1 \\ -1 & 2 & -1 & 0 & -1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

(b) $\beta_{\mathrm{KRS}} = (-1, -1, 0)$.

(c) $\widetilde{\mathrm{Ver}P} = \{q^{(1)} = (-1, -1, 0)\}$ with $c_1 = \frac{1}{2}$, which represents the exceptional divisor.

Next, we consider a $(\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2})$-bundle $\mathcal{E}_1$ over $\mathbb{CP}^1$. This manifold is derived as follows. Let $\tilde{E}_0$ be its exceptional divisor of the blow up $\pi : \mathcal{B}_3 \to \mathbb{CP}^3$ along a curve

$$F_0 := \{[z_0; z_1 : 0 : 0] \in \mathbb{CP}^3 \mid z_i \in \mathbb{C}\} \simeq \mathbb{CP}^1.$$

Let $\tilde{F}_1$ and $\tilde{F}_2$ are the two ($T_{\mathbb{C}}$-fixed) curves, which are reduced to F_0 under π. Then $\mathcal{E}_1$ is constructed from the blow up of $\mathcal{B}_3$ along $\tilde{F}_1$ and $\tilde{F}_2$. Let $\tilde{\tilde{E}}_0$ be the proper transform of $\tilde{E}_0$ and $\cup_{i=1,2}\tilde{\tilde{E}}_i$ be the exceptional divisors with respect to the blow up of $\mathcal{B}_3$. Remark that $\tilde{\tilde{E}}_0$ is not exceptional in $\mathcal{E}_1$.

Example 4.7. Let $\mathcal{E}_1$ be a $(\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2})$-bundle over $\mathbb{CP}^1$ defined as above. The support of the KRF-MIS on $\mathcal{E}_1$ of exponent γ is

$$\begin{cases} \tilde{\tilde{E}}_0 \cup (\cup_{i=1,2}\tilde{\tilde{E}}_i), & \text{for } \gamma \in (\frac{1}{2}, 1), \\ \tilde{\tilde{E}}_0, & \text{for } \gamma \in (\frac{1}{3}, \frac{1}{2}). \end{cases}$$

(a)

$$\mathrm{Ver}P = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$\mathrm{Ver}P^* = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 1 & 1 & -1 & -1 & 0 \\ 0 & 1 & 3 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

(b) $\beta_{\mathrm{KRS}} = (1, 1, 0)$.

(c) $\widetilde{\mathrm{Ver}P} = \{q^{(1)}, q^{(2)}, q^{(3)}\} = \{(1,1,0),(0,1,0),(1,0,0)\}$, where the vertices $\{q^{(1)}\}$ and $\{q^{(2)}, q^{(3)}\}$ represent $\tilde{\tilde{E}}_0$ and $\{\tilde{\tilde{E}}_1, \tilde{\tilde{E}}_2\}$, respectively. Their complex singularity exponents are given by $c_1 = \frac{1}{3}$ and $c_2 = c_3 = \frac{1}{2}$.

Example 4.8. Let $\mathcal{E}_3$ be $(\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}^2}) \times \mathbb{CP}^1$, which is the blow–up of $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ along $\{p_1\} \times \{p_2\} \times \mathbb{CP}^1$. The support of the KRF-MIS on $\mathcal{E}_3$ of exponent γ is

$$\begin{cases} \cup_{i=0}^2 E_i, & \text{for } \gamma \in (\frac{1}{2}, 1), \\ E_0, & \text{for } \gamma \in (\frac{1}{3}, \frac{1}{2}). \end{cases}$$

Here E_0 denotes the exceptional divisor of the blow–up and E_1 (resp. E_2) denotes the proper transform of $\mathbb{CP}^1 \times \{p_2\} \times \mathbb{CP}^1$ (resp. $\{p_1\} \times \mathbb{CP}^1 \times \mathbb{CP}^1$)

(a)

$$\mathrm{Ver}P = \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$\mathrm{Ver}P^* = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 1 & 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

(b) $\beta_{\mathrm{KRS}} = (1,1,0)$.

(c) $\widetilde{\mathrm{Ver}P} = \{q^{(1)}, q^{(2)}, q^{(3)}\} = \{(1,1,0),(0,1,0),(1,0,0)\}$. The vertices $\{q^{(1)}\}$ and $\{q^{(2)}, q^{(3)}\}$ represent E_0 and $\{E_1, E_2\}$ respectively. Their complex singularity exponents are given by $c_1 = \frac{1}{3}$ and $c_2 = c_3 = \frac{1}{2}$.

Finally let us consider a $(\mathbb{CP}^2 \sharp 3\overline{\mathbb{CP}^2})$-bundle $\mathcal{F}_2$ over $\mathbb{CP}^1$. Let $\tilde{E}_0$, $\tilde{\tilde{E}}_i$ $(i = 0,1,2)$, F_0, and $\tilde{F}_i$ $(i = 1,2)$ be as in Example 4.7. Let $\tilde{\pi} : \mathcal{E}_1 \to \mathcal{B}_3$ be the blow up of $\mathcal{B}_3$ along $\tilde{F}_1$ and $\tilde{F}_2$. Let F_3 be the $\mathbb{CP}^1$ in $\mathbb{CP}^3$ defined by

$$F_3 := \{[0:0:z_3;z_4] \mid z_i \in \mathbb{C}\}.$$

The manifold $\mathcal{F}_2$ is constructed from the blow-up of $\mathcal{E}_1$ along the curve $\tilde{\pi}^{-1}(\pi^{-1}(F_3))$. Let $\tilde{\tilde{\tilde{E}}}_0$ be the proper transform of $\tilde{\tilde{E}}_0$ with respect to the blow up of $\mathcal{E}_1$ along the curve $\tilde{\pi}^{-1}(\pi^{-1}(F_3))$. Remark that $\tilde{\tilde{\tilde{E}}}_0$ is not an exceptional divisor.

Example 4.9. Let $\mathcal{F}_2$ be a $(\mathbb{CP}^2\sharp 3\overline{\mathbb{CP}^2})$-bundle over $\mathbb{CP}^1$ defined as above. The support of the KRF-MIS on $\mathcal{F}_2$ of exponent γ is $\tilde{\tilde{\tilde{E}}}_0$ for $\gamma \in (\frac{1}{2}, 1)$.

(a)

$$\mathrm{Ver}P = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$\mathrm{Ver}P^* = \begin{pmatrix} 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.$$

(b) $\beta_{\mathrm{KRS}} = (1, 0, 0)$.

(c) $\widetilde{\mathrm{Ver}P} = \{q^{(1)} = (1, 0, 0)\}$ with $c_1 = \frac{1}{2}$, which represents $\tilde{\tilde{\tilde{E}}}_0$.

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