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Donaldson–Thomas invariants of certain Calabi–Yau 3-folds

WEI-PING LI AND ZHENBO QIN

We compute the Donaldson–Thomas invariants for two types of Calabi–Yau 3-folds. These invariants are associated to the moduli spaces of rank-2 Gieseker semistable sheaves. None of the sheaves are locally free, and their double duals are locally free stable sheaves investigated earlier in [12, 25, 33]. We show that these Gieseker moduli spaces are isomorphic to some Quot-schemes. We prove a formula for Behrend's ν-functions when torus actions present with positive dimensional fixed point sets, and use it to obtain the generating series of the relevant Donaldson–Thomas invariants in terms of the McMahon function.

1. Introduction

The Donaldson–Thomas invariants of a Calabi–Yau 3-fold Y essentially count the number of stable sheaves on Y. It attracts intensive activities recently due to the conjectural relations with Gromov–Witten invariants proposed by Maulik, Nekrasov, Okounkov and Pandharipande. The moduli space of stable sheaves in [28] consists of ideal sheaves defining one-dimensional closed subschemes of Y with some 0-dimensional components and some embedded points. The Donaldson–Thomas invariants associated to the 0-dimensional closed subschemes of Y have been determined in [4, 21, 23]. Curves on Y may also be related to rank-2 vector bundles via the Serre construction. In fact, the first example of Donaldson–Thomas invariants in [12, 33] counts certain stable rank-2 sheaves on the Calabi–Yau 3-fold Y, which is the smooth intersection of a quartic hypersurface and a quadric hypersurface in $\mathbb{P}^5$. In [25], we studied some moduli spaces of rank-2 stable sheaves on a Calabi–Yau hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$ for $n \geq 2$ and computed the corresponding Donaldson–Thomas invariants of the 3-fold Y, when $n = 2$. The idea in [12, 33] and [25] is to give a complete description of the moduli spaces. Let L (resp. $\mathbf{c}_0$) be the ample line bundle (resp. total Chern class) considered in [12, 33] or [25]. Then the coarse moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_0)$ of rank-2 Gieseker L-semistable sheaves with total Chern classes $\mathbf{c}_0$

consists of two smooth points for the case in [12, 33] and is a projective space for the case in [25]. It turns out that in both cases, all the sheaves in $\overline{\mathfrak{M}}_L(\mathbf{c}_0)$ are stable and locally free. Here $\mathbf{c}_0$ is chosen as in Theorem A and Theorem B. For a general total Chern class $\mathbf{c}_0$, a full description of the moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_0)$ is yet to be done.

In this paper, we compute the Donaldson–Thomas invariant, denoted by $\lambda(L, \mathbf{c}_m)$, associated to the Gieseker moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_m)$ where

$$\mathbf{c}_m = \mathbf{c}_0 - m[y_0]$$

and $y_0 \in Y$ is a fixed point. Again, an important ingredient is to understand $\overline{\mathfrak{M}}_L(\mathbf{c}_m)$. We show that whenever $\overline{\mathfrak{M}}_L(\mathbf{c}_m)$ with $m \neq 0$ is not empty, none of the sheaves $E \in \overline{\mathfrak{M}}_L(\mathbf{c}_m)$ are locally free but their double duals E^{**} are locally free and contained in the moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_0)$ considered in [12, 25, 33].

More precisely, for the pair $(L, \mathbf{c}_0)$ from [12, 33], let the two smooth points in $\overline{\mathfrak{M}}_L(\mathbf{c}_0)$ be represented by the rank-2 stable bundles $E_{0,1}$ and $E_{0,2}$. If $\overline{\mathfrak{M}}_L(\mathbf{c}_m) \neq \emptyset$, then m is even and non-negative. When $m \geq 0$, the moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_{2m})$ is isomorphic to the disjoint union of the Quot-schemes $\mathrm{Quot}^m_{E_{0,1}}$ and $\mathrm{Quot}^m_{E_{0,2}}$.

Theorem A. *Let Q_0 be a smooth quadric in $\mathbb{P}^5$, H be a hyperplane in $\mathbb{P}^5$, P be a plane on Q_0, and Y be the Calabi–Yau 3-fold which is the smooth intersection of a quartic hypersurface and Q_0 in $\mathbb{P}^5$. Let the total Chern class be*

$$\mathbf{c}_0 = 1 + H|_Y + P|_Y.$$

Let $\chi(Y)$ be the Euler characteristic of Y, and $M(q) = \prod_{m=1}^{+\infty} \frac{1}{(1-q^m)^m}$ be the McMahon function. Then,

$$\sum_{m \in \mathbb{Z}} \lambda(L, \mathbf{c}_m)\, q^m = 2 \cdot M(q^2)^{2\,\chi(Y)}.$$

Next, let Y be the Calabi–Yau 3-fold considered in [25], i.e., Y is a smooth Calabi–Yau hypersurface of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. Let $L_r^Y = L = \pi_2^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^2}(r)|_Y$ be a $\mathbb{Q}$-line bundle on Y where π_i is the ith projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. Let

$$2(2-\epsilon_2)/(2+\epsilon_1) < r < 2(2-\epsilon_2)/\epsilon_1$$

where $\epsilon_1, \epsilon_2 = 0, 1$ appear in the definition of the total Chern class $\mathbf{c}_m$ in (1.1). In [25], we proved that $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$ is isomorphic to a projective space and consists of stable bundles. Let $\mathcal{E}_0$ be a universal vector bundle over $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times Y$, and let

$$\mathrm{Quot}^m_{\mathcal{E}_0/} = \mathrm{Quot}^m_{\mathcal{E}_0/\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)\times Y/\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)}$$

be the relative Quot-scheme. If $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m) \neq \emptyset$, then m is even and non-negative. When $m \geq 0$, the moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_{2m})$ is isomorphic to $\mathrm{Quot}^m_{\mathcal{E}_0/}$.

Theorem B. *Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ be a generic smooth Calabi–Yau hypersurface. Let $\epsilon_1, \epsilon_2 = 0, 1$, and $k = (1+\epsilon_1)(4-\epsilon_2)(3-\epsilon_2)/2 - 1$. Let $\pi : Y \to \mathbb{P}^1 \times \mathbb{P}^2$ be the restriction to Y of the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ to the product of the last two factors. Fix a point $y_0 \in Y$, and define in $A^*(Y)$ the class*

$$\mathbf{c}_m = -m[y_0] + \big(1 + \pi^*(-1,1)\big) \cdot \big(1 + \pi^*(\epsilon_1 + 1, \epsilon_2 - 1)\big) \tag{1.1}$$

where (a,b) denotes the divisor $a(\{p\} \times \mathbb{P}^2) + b(\mathbb{P}^1 \times H)$ for a line H in $\mathbb{P}^2$.

(i) *If $0 < r < 2(2-\epsilon_2)/(2+\epsilon_1)$, then $\lambda(L_r^Y, \mathbf{c}_m) = 0$ for all $m \in \mathbb{Z}$.*

(ii) *If $2(2-\epsilon_2)/(2+\epsilon_1) < r < 2(2-\epsilon_2)/\epsilon_1$, then*

$$\sum_{m \in \mathbb{Z}} \lambda(L_r^Y, \mathbf{c}_m) q^m = (-1)^k \cdot (k+1) \cdot M(q^2)^{2\chi(Y)}.$$

In fact, if $0 < r < 2(2-\epsilon_2)/(2+\epsilon_1)$, then the moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_m)$ is empty for all $m \in \mathbb{Z}$. Therefore $r = 2(2-\epsilon_2)/(2+\epsilon_1)$ may be regarded as a wall, and the open intervals $(0, 2(2-\epsilon_2)/(2+\epsilon_1))$ and $(2(2-\epsilon_2)/(2+\epsilon_1), 2(2-\epsilon_2)/\epsilon_1)$ may be regarded as two chambers. From this point of view, Theorem B (ii) provides a wall-crossing formula for the Donaldson–Thomas invariants. Due to their connections with Hall algebras, wall-crossing formulas for Donaldson–Thomas invariants have been investigated intensively in the past few years (see [7, 17–20, 34] and the references there). A concrete wall-crossing formula was obtained in [19] under certain conditions which are not satisfied in our present situation. Our results might shed some light on the general properties of these wall-crossing formulas.

In addition to understanding the moduli spaces of Gieseker semistable sheaves, another essential ingredient in the proofs of Theorem A and Theorem B is the following result concerning Behrend's ν-function when a torus action exists.

Theorem C. *Assume that $\mathbb{T} = \mathbb{C}^*$ acts on a complex scheme X which admits a symmetric obstruction theory compatible with the $\mathbb{T}$-action. Let $P \in X^{\mathbb{T}}$. Then,*

(i) *$X^{\mathbb{T}}$ admits a symmetric obstruction theory;*

(ii) *$\nu_X(P) = (-1)^{\dim T_P X - \dim T_P(X^{\mathbb{T}})} \cdot \nu_{X^{\mathbb{T}}}(P)$, where $T_P X$ denotes the Zariski tangent space of X at P and ν_X denotes Behrend's ν-function for X.*

When $P \in X^{\mathbb{T}}$ is an isolated $\mathbb{T}$-fixed point, Theorem C (ii) has been proved in [4]. It also follows easily when X is locally the critical scheme of a regular function f on a smooth scheme M, i.e., $X = Z(df)$ locally. This is due to the fact that, in this case, $\nu_X(P)$ can be computed via the Euler characteristic of the Milnor fiber obtained from f. In [5, 18], it is shown that the moduli spaces of Gieseker stable sheaves on a Calabi–Yau 3-fold are locally critical schemes. Therefore the results from [5, 18] are sufficient for the computation of Donaldson–Thomas invariants. However, in [29], an example where a scheme X admitting a symmetric obstruction theory is not locally critical has been constructed (see [30] for the correction). Hence Theorem C could be useful to a general X with just a symmetric obstruction theory.

By the results in [2], the Donaldson–Thomas invariant $\lambda(L, \mathbf{c}_m)$ coincides with the weighted Euler characteristic $\widetilde{\chi}\big(\overline{\mathfrak{M}}_L(\mathbf{c}_m)\big)$ of the moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_m)$ (see (2.1) for the t definition of $\widetilde{\chi}(\cdot)$). By standard techniques, the computation of $\widetilde{\chi}\big(\overline{\mathfrak{M}}_L(\mathbf{c}_m)\big)$ reduces to the relevant punctual Quot-schemes. It turns out that these punctual Quot-schemes admit $\mathbb{T}$-actions. The $\mathbb{T}$-fixed loci are the unions of certain products of the punctual Hilbert schemes of 0-dimensional closed subschemes on Y. A combination of Theorem C and the results in [4, 21, 23] regarding punctual Hilbert schemes yield Theorem A and Theorem B.

In [28], the Donaldson–Thomas invariants are defined via the moduli space of ideal sheaves I_Z where the dimensions of the closed subschemes Z are equal to one. Most of the computational results in the literature are concentrated on this type of invariants. Via the Serre construction, curves on Y correspond to rank-2 vector bundles on Y. With this point of view, Theorem A is comparable to the Donaldson–Thomas invariants corresponding to super-rigid curves considered in [3]. An irreducible super-rigid curve C in Y has normal bundle $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ and thus can not deform; in our case, $E_{0,1}$ and $E_{0,2}$ have no deformation either, and thus are similar to super-rigid curves. Theorem B is comparable to the Donaldson–Thomas invariants corresponding to the moduli spaces of ideal sheaves I_Z where the topological invariant $[Z]$ of Z is fixed in $H_2(Y, \mathbb{Z})$ but the curve component of Z has a positive dimensional moduli.

We remark that recently Stoppa [32] and Toda [35] worked on D0–D6 states counting which is similar to this paper. For example, sheaves in [32] are isomorphic to the trivial vector bundle of some rank outside a finite set of points. Thus the invariants counted there are higher rank generalizations of degree zero Donaldson–Thomas invariants for ideal sheaves defined in [28]. While their work deal with sheaves with vanishing first and second Chern classes on arbitrary Calabi–Yau three-folds, sheaves in this paper have non-zero first and second Chern classes on some special Calabi–Yau three-folds. So our paper studies the generalization of the Donaldson–Thomas invariants for ideal sheaves in [28] with non-trivial contributions from curve components. While the methods used in [32, 35] include powerful wall-crossing techniques developed in [18, 20] and Bridgeland stability conditions, this paper uses complete descriptions of moduli spaces to carry out the computations.

The paper is organized as follows. In Section 2, we prove Theorem C. In Section 3, we review virtual Hodge polynomials, and compute the Euler characteristics of Grothendieck Quot-schemes. The results are of independent interest, and will be used in Section 4. In Section 4, we verify Theorems A and B.

2. Behrend's ν-functions for schemes with $\mathbb{C}^*$-actions

For a complex scheme X, an invariant ν_X of X was introduced in [2]. The invariant ν_X is an integer-valued constructible function defined over X. Following [2], *the weighted Euler characteristic* of X is defined to be

$$\widetilde{\chi}(X) = \chi(X, \nu_X) := \sum_{n \in \mathbb{Z}} n \cdot \chi\big(\{x \in X | \nu_X(x) = n\}\big) \tag{2.1}$$

where $\chi(\cdot)$ denotes the usual Euler characteristic of topological spaces.

Recall from [4] that *a symmetric obstruction theory* for X is a triple (E, ϕ, θ) where $\phi : E \to L_X$ is a perfect obstruction theory for X and $\theta : E \to E^v[1]$ is a non-degenerate symmetric bilinear form. Let $\mathbb{T} = \mathbb{C}^*$ act on X, which admits a symmetric obstruction theory compatible with the $\mathbb{T}$-action. By [4], if $P \in X^{\mathbb{T}}$ is an isolated $\mathbb{T}$-fixed point of X, then

$$\nu_X(P) = (-1)^{\dim T_P X} \tag{2.2}$$

where $T_P X$ denotes the Zariski tangent space of X at P. However, for the cases considered in our paper, the fixed point sets $X^{\mathbb{T}}$ will be positive dimensional. Our goal is to prove Theorem C that generalizes (2.2) to the case when $P \in X^{\mathbb{T}}$ is not necessarily an isolated $\mathbb{T}$-fixed point of X. Theorem C reduces the computation of the ν-function of X to that of the fixed point set $X^{\mathbb{T}}$.

We begin with a few technical lemmas regarding equivariant Thom classes. Elementary properties about Thom classes can be found in [6], and basic materials on equivariant cohomology and equivariant Thom classes can be found in [1, 27, 31].

Let $z_1, \ldots, z_n, \tilde{z}_1, \ldots, \tilde{z}_n$ be the coordinates of $\mathbb{A}^n \times \mathbb{A}^n$. For $t \in S^1$ and $1 \le i \le n$, let $t(z_i) = t^{r_i} z_i$ and $t(\tilde{z}_i) = t^{-r_i} \tilde{z}_i$, where $r_i \in \mathbb{Z}$. Let $r_i = 0$ for $1 \le i \le m$ and $r_i \ne 0$ for $m+1 \le i \le n$. Define the subsets Δ_n and F of $\mathbb{A}^n \times \mathbb{A}^n$ by

$$\begin{aligned} \Delta_n &= \{\tilde{z}_1 = \overline{z}_1, \ldots, \tilde{z}_n = \overline{z}_n\}, \\ F &= \{z_{m+1} = \cdots = z_n = \tilde{z}_{m+1} = \cdots = \tilde{z}_n = 0\}. \end{aligned} \tag{2.3}$$

Then, Δ_n is S^1-invariant, and $F = (\mathbb{A}^n \times \mathbb{A}^n)^{S^1}$. We orient Δ_n so that the natural map $\mathbb{A}^n \to \Delta_n$ defined by

$$(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_n; \overline{z}_1, \ldots, \overline{z}_n) \tag{2.4}$$

is orientation-preserving. Let $\Delta_m = \Delta_n \cap F$. The intersection of Δ_n and F along Δ_m is not transversal. However, a direct computation shows that $(T_{\Delta_n}|_{\Delta_m}) \cap (T_F|_{\Delta_m}) = T_{\Delta_m}$. Consider the following commutative diagram of maps:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & N_{\Delta_m \subset F} & \to & N_{\Delta_n \subset \mathbb{A}^n \times \mathbb{A}^n}|_{\Delta_m} & \to & M & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & T_F|_{\Delta_m} & \to & T_{\mathbb{A}^n \times \mathbb{A}^n}|_{\Delta_m} & \to & N_{F \subset \mathbb{A}^n \times \mathbb{A}^n}|_{\Delta_m} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & T_{\Delta_m} & \to & T_{\Delta_n}|_{\Delta_m} & \to & N_{\Delta_m \subset \Delta_n} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{2.5}$$

Note that M is a rank-$(2n-2m)$ real vector bundle over Δ_m.

Let N_n denote $N_{\Delta_n \subset \mathbb{A}^n \times \mathbb{A}^n}$ for simplicity. We use

$$\xi = \big((z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n), (w_1, \ldots, w_n; \tilde{w}_1, \ldots \tilde{w}_n)\big)$$

to represent a point in $T_{\mathbb{A}^n \times \mathbb{A}^n} = (\mathbb{A}^n \times \mathbb{A}^n) \times (\mathbb{A}^n \times \mathbb{A}^n)$. Define a map $\varphi\colon T_{\mathbb{A}^n \times \mathbb{A}^n} \to (\mathbb{A}^n \times \mathbb{A}^n) \times \mathbb{A}^n$,

$$\varphi(\xi) = \big((z_1, \ldots, z_n; \tilde{z}_1, \ldots, \tilde{z}_n), (\tilde{w}_1 - \overline{w}_1, \ldots \tilde{w}_n - \overline{w}_n)\big).$$

φ is a homomorphism of real vector bundles on $\mathbb{A}^n \times \mathbb{A}^n$. It is easy to see that $\varphi|_{\Delta_n}(T_{\Delta_n}) = 0$. Thus we get an induced isomorphism

$$\varphi_n \colon = \varphi|_{\Delta_n} \colon N_n = \frac{T_{\mathbb{A}^n \times \mathbb{A}^n}|_{\Delta_n}}{T_{\Delta_n}} \longrightarrow \Delta_n \times \mathbb{A}^n. \tag{2.6}$$

Similarly, let $N_m = N_{\Delta_m \subset F}$. It is a subbundle of $N_n|_{\Delta_m}$. Define the standard projection to the last $(n-m)$ factors:

$$\begin{aligned} \mathrm{pr}\colon \Delta_n \times \mathbb{A}_n &\longrightarrow \Delta_n \times \mathbb{A}^{n-m}, \\ \big(\mathrm{pt}, (a_1, \ldots, a_n)\big) &\longrightarrow \big(\mathrm{pt}, (a_{m+1}, \ldots, a_n)\big). \end{aligned}$$

Then $(\mathrm{pr} \circ \varphi_n)|_{\Delta_m}$ maps N_m to zero. Thus, we get an induced isomorphism of real vector bundles, denoted by ψ_m, i.e.,

$$\psi_m \colon M = \frac{N_n|_{\Delta_m}}{N_m} \longrightarrow \Delta_m \times \mathbb{A}^{n-m}. \tag{2.7}$$

Also the following exact sequence splits:

$$0 \longrightarrow N_m \longrightarrow N_n|_{\Delta_m} \longrightarrow M \longrightarrow 0. \tag{2.8}$$

If we introduce an S^1-action on $\mathbb{A}^n$ in $N_n \cong \Delta_n \times \mathbb{A}^n$ via

$$t \cdot (a_1, \ldots, a_n) = (t^{-r_1} a_1, \ldots, t^{-r_n} a_n),$$

then the map φ_n in (2.6) is S^1-equivariant. In fact, fiberwise, for any point $\mathrm{pt} \in \Delta_n$, we have

$$\begin{aligned} &\varphi_n|_{\mathrm{pt}}\big(t \cdot [(w_1, \ldots, w_n; \tilde{w}_1, \ldots, \tilde{w}_n)]\big) \\ &\quad = \varphi_n|_{\mathrm{pt}}\big([(t^{r_1} w_1, \ldots, t^{r_n} w_n; t^{-r_1}\tilde{w}_1, \ldots, t^{-r_n}\tilde{w}_n)]\big) \\ &\quad = (t^{-r_1}\tilde{w}_1 - \overline{t^{r_1} w_1}, \ldots, t^{-r_n}\tilde{w}_n - \overline{t^{r_n} w_n}) \\ &\quad = \big(t^{-r_1}(\tilde{w}_1 - \overline{w}_1), \ldots, t^{-r_n}(\tilde{w}_n - \overline{w}_n)\big) \\ &\quad = t \cdot \varphi_n|_{\mathrm{pt}}\big([(w_1, \ldots, w_n; \tilde{w}_1, \ldots, \tilde{w}_n)]\big). \end{aligned}$$

Similarly, if we define an S^1-action on $\mathbb{A}^{n-m}$ in $M \cong \Delta_m \times \mathbb{A}^{n-m}$ via

$$t \cdot (a_{m+1}, \ldots, a_n) = (t^{-r_{m+1}} a_{m+1}, \ldots, t^{-r_n} a_n), \tag{2.9}$$

then we get an S^1-action on M such that the map $N_n|_{\Delta_m} \to M$ in the exact sequence (2.8) is S^1-equivariant, and the exact sequence (2.8) splits S^1-equivariantly.

Let ω_{Δ_n} (resp. $\omega_{\Delta_n}^{S^1}$) be the Thom class (resp. equivariant Thom class) of the vector bundle $N_{\Delta_n \subset \mathbb{A}^n \times \mathbb{A}^n}$ in the cohomology $H^*(\mathbb{A}^n \times \mathbb{A}^n, \mathbb{A}^n \times \mathbb{A}^n - \Delta_n)$ (resp. $H^*_{S^1}(\mathbb{A}^n \times \mathbb{A}^n, \mathbb{A}^n \times \mathbb{A}^n - \Delta_n)$). Similarly, let ω_{Δ_m} (resp. $\omega_{\Delta_m}^{S^1}$) be the Thom class (resp. equivariant Thom class) of the vector bundle $N_{\Delta_m \subset F}$ in the cohomology $H^*(F, F - \Delta_m)$ (resp. $H^*_{S^1}(F, F - \Delta_m)$). Let $\omega_M^{S^1}$ be the equivariant Thom class of the vector bundle M in the equivariant cohomology $H^*_{S^1}(M)$. Let $\chi^{S^1}(M)$ be the equivariant Euler class of M.

Lemma 2.1. *Let $i : F \hookrightarrow \mathbb{A}^n \times \mathbb{A}^n$ be the inclusion map, and let $\tau : N_{\Delta_m \subset F} \to \Delta_m$ be the natural projection. Then, in the equivariant cohomology $H^*_{S^1}(F, F - \Delta_m)$, we have*

$$i^* \omega_{\Delta_n}^{S^1} = \omega_{\Delta_m}^{S^1} \cup \tau^* \chi^{S^1}(M).$$

Proof. The map i can be regarded as a map of pairs, still denoted by i,

$$(F, F - \Delta_m) \to (\mathbb{A}^n \times \mathbb{A}^n, \mathbb{A}^n \times \mathbb{A}^n - \Delta_n).$$

Since the exact sequence (2.8) splits S^1-equivariantly, let π_1 and π_2 be the natural projections of $N_{\Delta_m \subset F} \oplus M$. Then, we have a commutative diagram:

$$\begin{array}{cccc} N_{\Delta_n \subset \mathbb{A}^n \times \mathbb{A}^n}|_{\Delta_m} \cong & N_{\Delta_m \subset F} \oplus M & \xrightarrow{\pi_2} & M \\ & \downarrow \pi_1 & & \downarrow \\ & N_{\Delta_m \subset F} & \xrightarrow{\tau} & \Delta_m. \end{array}$$

Thus we have

$$\omega_{\Delta_n}^{S^1}|_{(N_{\Delta_n \subset \mathbb{A}^n \times \mathbb{A}^n}|_{\Delta_m})} = \pi_1^* \omega_{\Delta_m}^{S^1} \cup \pi_2^* \omega_M^{S^1}. \tag{2.10}$$

where $\pi_2^* \omega_M^{S^1}$ is the equivariant Thom class of $\tau^* M$ over $N_{\Delta_m \subset F}$. Since $N_{\Delta_m \subset F} \oplus M$ can be identified with $\tau^* M$ and $N_{\Delta_m \subset F} \hookrightarrow N_{\Delta_m \subset F} \oplus M = \tau^* M$ can be regarded as the zero section of the bundle $\tau^* M$ over $N_{\Delta_m \subset F}$, pulling back (2.10) to $N_{\Delta_m \subset F}$, we have $\omega_{\Delta_n}^{S^1}|_{(N_{\Delta_m \subset F})} = \omega_{\Delta_m}^{S^1} \cup \tau^* \chi^{S^1}(M)$. Here we have used the fact that if s is the zero section of a vector bundle

$E \to Y$ and ω^{S^1} is the equivariant Thom class of E, then $s^*\omega^{S^1}$ is the equivariant Euler class $\chi^{S^1}(E)$ of E. Since Δ_m is a real linear subspace of F, $N_{\Delta_m \subset F} = F$. Therefore, we conclude that $i^*\omega^{S^1}_{\Delta_n} = \omega^{S^1}_{\Delta_m} \cup \tau^*\chi^{S^1}(M)$. □

Lemma 2.2. *In the S^1-equivariant cohomology $H^*_{S^1}(F, F - \Delta_m)$, we have*

$$i^*\omega^{S^1}_{\Delta_n} = (-1)^{n-m}(r_{m+1}\dots r_n)t^{n-m}\ \omega^{S^1}_{\Delta_m}.$$

Proof. In view of Lemma 2.1, it remains to prove that

$$\chi^{S^1}(M) = (-1)^{n-m}(r_{m+1}\cdots r_n)t^{n-m}. \tag{2.11}$$

Firstly, let us prove that the map φ_n in (2.6) is orientation preserving. Let $P \in \mathbb{A}^n \times \mathbb{A}^n$ be the origin. Since

$$T_P\Delta_n = \{(z_1, \dots, z_n; \bar{z}_1, \dots, \bar{z}_n)\} \subset T_P(\mathbb{A}^n \times \mathbb{A}^n),$$

as an $\mathbb{R}$-subspace of $T_P(\mathbb{A}^n \times \mathbb{A}^n) \cong \mathbb{A}^n \times \mathbb{A}^n$, $T_P\Delta_n$ has an ordered basis

$$\vec{u}_i = \vec{\epsilon}_i + (-1)^{i+1}\vec{v}_i, \quad 1 \le i \le 2n$$

where $\{\vec{\epsilon}_1, \dots, \vec{\epsilon}_{2n}\}$ (resp. $\{\vec{v}_1, \dots, \vec{v}_{2n}\}$) is the standard basis of the subspace $\mathbb{R}^{2n} \times \{0\} = \mathbb{A}^n \times \{0\}$ (resp. $\{0\} \times \mathbb{R}^{2n} = \{0\} \times \mathbb{A}^n$) of $T_P(\mathbb{A}^n \times \mathbb{A}^n)$.

Recall that the orientation of Δ_n is such that the map (2.4) is orientation preserving. Thus the ordered basis $\{\vec{u}_1, \dots, \vec{u}_{2n}\}$ is the orientation of $T_P\Delta_n$.

Now $\{\varphi_n(\vec{v}_1), \dots, \varphi_n(\vec{v}_{2n})\}$ is the standard basis of $\mathbb{R}^{2n} = \mathbb{A}^n$. Since

$$\{\vec{u}_1, \dots, \vec{u}_{2n}, \vec{v}_1, \dots, \vec{v}_{2n}\}$$

agrees with the orientation of $T_P(\mathbb{A}^n \times \mathbb{A}^n)$, φ_n is orientation-preserving.

Similarly, ψ_m defined in (2.7) is also orientation preserving. Recall the S^1-action on M defined in (2.9). Thus, we have

$$\chi^{S^1}(M) = (-r_{m+1})t \cdots (-r_n)t = (-1)^{n-m}(r_{m+1}\cdots r_n)t^{n-m}.$$

This completes the proof of (2.11). □

We continue with the S^1-action on $\mathbb{A}^n \times \mathbb{A}^n$ defined earlier. Fix holomorphic functions $f_1, \dots, f_n$ in the variables $z_1, \dots, z_n$ such that the degree of each f_i with respect to the S^1-action is $-r_i$. Then $f_i(z_1, \dots, z_m, 0, \dots, 0) = 0$ for all $m+1 \le i \le n$. For $1 \le i \le m$, let $\tilde{f}_i(z_1, \dots, z_m) = f_i(z_1, \dots, z_m, 0, \dots, 0)$. Then, $\tilde{f}_1, \dots, \tilde{f}_m$ are holomorphic in $z_1, \dots, z_m$. Let $\Gamma \subset \mathbb{A}^n \times \mathbb{A}^n$

be defined by the equations $\tilde{z}_1 = f_1, \ldots, \tilde{z}_n = f_n$. Then $\Gamma \cap F$ is given by $\{\tilde{z}_1 = \tilde{f}_1, \ldots, \tilde{z}_m = \tilde{f}_m\} \subset F$.

Lemma 2.3. *In the S^1-equivariant cohomology $H^*_{S^1}(F, F - F \cap \Gamma)$, we have*

$$i^* \omega_\Gamma^{S^1} = (-1)^{n-m} (r_{m+1} \cdots r_n) t^{n-m} \, \omega_{\Gamma \cap F}^{S^1}$$

where $\omega_\Gamma^{S^1}$ is the equivariant Thom class of the normal bundle of Γ in $\mathbb{A}^n \times \mathbb{A}^n$ in the equivariant cohomology $H^*_{S^1}(\mathbb{A}^n \times \mathbb{A}^n, \mathbb{A}^n \times \mathbb{A}^n - \Gamma)$, and $\omega_{\Gamma \cap F}^{S^1}$ is the equivariant Thom class of the normal bundle of $\Gamma \cap F$ in F in the equivariant cohomology $H^*_{S^1}(F, F - F \cap \Gamma)$.

Proof. The map i can be regarded as the map of pairs $i\colon (F, F - F \cap \Gamma) \to (\mathbb{A}^n \times \mathbb{A}^n, \mathbb{A}^n \times \mathbb{A}^n - \Gamma)$.

First of all, let $\{\vec{\xi}_1, \ldots, \vec{\xi}_n\}$ (resp. $\{\vec{\zeta}_1, \ldots, \vec{\zeta}_n\}$) be the standard complex basis of the subspace $\mathbb{A}^n \times \{0\}$ (resp. $\{0\} \times \mathbb{A}^n$) of $\mathbb{A}^n \times \mathbb{A}^n$. Then the tangent space of Γ is spanned by the ordered basis

$$\vec{\xi}_i + \sum_{j=1}^n \frac{\partial f_j}{\partial z_i} \vec{\zeta}_j, \quad 1 \le i \le n. \tag{2.12}$$

Thus, the normal vector bundle $N_{\Gamma \subset \mathbb{A}^n \times \mathbb{A}^n}$ is spanned by

$$\sum_{j=1}^n \frac{\partial f_i}{\partial z_j} \vec{\xi}_j - \vec{\zeta}_i, \quad 1 \le i \le n.$$

Using these, one checks that $(T_\Gamma|_{\Gamma \cap F}) \cap (T_F|_{\Gamma \cap F}) = T_{\Gamma \cap F}$ and that

$$\widetilde{M} := \big(N_{\Gamma \subset \mathbb{A}^n \times \mathbb{A}^n}|_{\Gamma \cap F}\big) / N_{\Gamma \cap F \subset F}$$

is a complex bundle. By the same arguments as in the proofs of Lemmas 2.1 and 2.2, $i^* \omega_\Gamma^{S^1} = \omega_{\Gamma \cap F}^{S^1} \cup \chi^{S^1}(\widetilde{M})$. Hence, it remains to prove that

$$\chi^{S^1}(\widetilde{M}) = (-1)^{n-m} (r_{m+1} \cdots r_n) t^{n-m}.$$

Next, note from the assumptions about the functions $f_1, \ldots, f_m$, $f_{m+1}, \ldots, f_n$ that the restriction $N_{\Gamma \subset \mathbb{A}^n \times \mathbb{A}^n}|_{\Gamma \cap F}$ is spanned by $\{\vec{w}_1, \ldots, \vec{w}_n\}$

where

$$\vec{w}_i = \begin{cases} \sum_{j=1}^{m} \dfrac{\partial \tilde{f}_i}{\partial z_j}\, \vec{\xi}_j - \vec{\zeta}_i, & \text{if } 1 \le i \le m, \\ \sum_{j=m+1}^{n} \dfrac{\partial f_i}{\partial z_j}\, \vec{\xi}_j - \vec{\zeta}_i, & \text{if } m+1 \le i \le n. \end{cases}$$

In addition, since the degree of f_{m+1} with respect to the S^1-action is $-r_{m+1}$, for each component $\partial f_{m+1}/\partial z_j$ in $\vec{w}_{m+1}$, we have either $\partial f_{m+1}/\partial z_j = 0$ or $r_j = -r_{m+1}$. So $t \cdot \vec{w}_{m+1} = t^{-r_{m+1}} \vec{w}_{m+1}$ under the S^1-action on $\mathbb{A}^n \times \mathbb{A}^n$, where we regard $\vec{w}_{m+1} \in \mathbb{A}^n \times \mathbb{A}^n$ with $z_{m+1} = \cdots = z_n = 0$ and with $z_1, \ldots, z_m$ fixed. Similarly,

$$t \cdot \vec{w}_j = t^{-r_j} \vec{w}_j, \quad \text{for } j \ge m+1.$$

Since the normal bundle $N_{\Gamma \cap F \subset F}$ is spanned by $\vec{w}_1, \ldots, \vec{w}_m$, the bundle $\widetilde{M}$ is spanned by $\vec{w}_{m+1}, \ldots, \vec{w}_n$, i.e., the bundle $\widetilde{M}$ is S^1-equivariantly isomorphic to the trivial bundle $(\Gamma \cap F) \times \mathbb{A}^{n-m}$ where S^1 acts on $\mathbb{A}^{n-m}$ by

$$t \cdot (b_{m+1}, \ldots, b_n) = (t^{-r_{m+1}} b_{m+1}, \ldots, t^{-r_n} b_n).$$

Therefore, $\chi^{S^1}(\widetilde{M}) = (-r_{m+1})t \cdots (-r_n)t = (-1)^{n-m}(r_{m+1} \ldots r_n)t^{n-m}$. □

Theorem 2.4. *Assume that $\mathbb{T}$ acts on a complex scheme X and that X admits a symmetric obstruction theory compatible with the $\mathbb{T}$-action. Let $P \in X^{\mathbb{T}}$. Then,*

(i) *$X^{\mathbb{T}}$ admits a symmetric obstruction theory;*

(ii) *$\nu_X(P) = (-1)^{\dim T_P X - \dim T_P(X^{\mathbb{T}})} \cdot \nu_{X^{\mathbb{T}}}(P)$.*

Proof. Part (i) follows from the proof of a similar statement about equivariant obstruction theory in [15] and the definition of equivariant symmetric obstruction theory. In the following, we verify (ii). As in Subsection 3.2 of [4], we may assume that $X = Z(\omega) \subset \mathbb{A}^n$ and $P \in X$ is the origin of $\mathbb{A}^n$. Here $n = \dim T_P X$ and ω is a $\mathbb{T}$-invariant almost closed 1-form on $\mathbb{A}^n$. Let $z_1, \ldots, z_n$ be the coordinates of $\mathbb{A}^n$, and $z_1, \ldots, z_n, \tilde{z}_1, \ldots, \tilde{z}_n$ be the coordinates of $\Omega_{\mathbb{A}^n} = \mathbb{A}^n \times \mathbb{A}^n$. Let $\omega = \sum_{i=1}^n f_i dz_i$ where the functions f_i are holomorphic in $z_1, \ldots, z_n$.

For $t \in \mathbb{T}$ and $1 \le i \le n$, let $t(z_i) = t^{r_i} z_i$ where $r_i \in \mathbb{Z}$. Then the degrees of both $\tilde{z}_i$ and f_i with respect to the $\mathbb{T}$-action are equal to $-r_i$. Assume that $r_i = 0$ for $1 \le i \le m$ and $r_i \ne 0$ for $m+1 \le i \le n$. Then $f_i(z_1, \ldots, z_m, 0, \ldots, 0) = 0$ for all $m+1 \le i \le n$. For $1 \le i \le m$, let $\tilde{f}_i(z_1, \ldots, z_m) =$

$f_i(z_1, \dots, z_m, 0, \dots, 0)$. Since $X = Z(\omega) = \{f_1 = 0, \dots, f_n = 0\}$ and $(\mathbb{A}^n)^{\mathbb{T}} = \{z_{m+1} = 0, \dots, z_n = 0\}$,

$$\begin{aligned} X^{\mathbb{T}} = X \cap (\mathbb{A}^n)^{\mathbb{T}} &= \{f_1 = 0, \dots,\ f_n = 0,\ z_{m+1} = 0, \dots,\ z_n = 0\} \\ &= \{\tilde{f}_1 = 0, \dots,\ \tilde{f}_m = 0,\ z_{m+1} = 0, \dots,\ z_n = 0\}. \end{aligned}$$

So $X^{\mathbb{T}} = Z(\omega^{\mathbb{T}}) \subset \mathbb{A}^m$ where $\omega^{\mathbb{T}} = \sum_{i=1}^m \tilde{f}_i dz_i$. By [15], $\dim T_P(X^{\mathbb{T}}) = m$.

Next, let $\mathcal{C} \hookrightarrow \Omega_{\mathbb{A}^n}$ be the embedding of the normal cone $\mathcal{C}_{X/\mathbb{A}^n}$ into $\Omega_{\mathbb{A}^n}$ given by ω. Let $\Delta_n \subset \Omega_{\mathbb{A}^n} = \mathbb{A}^n \times \mathbb{A}^n$ be defined as in (2.3). For $\eta \in \mathbb{C} - \{0\}$, let Γ_η be the graph of the section $1/\eta \cdot \omega$ of $\Omega_{\mathbb{A}^n}$, i.e., Γ_η is the subspace of $\Omega_{\mathbb{A}^n}$ defined by the equations $\eta \tilde{z}_i = f_i$, $1 \le i \le n$. Again, orient Γ_η so that the map $\mathbb{A}^n \to \Gamma_\eta$ is orientation-preserving. From the proof of Proposition 4.22 in [2], we see that P is an isolated point of the intersection $\mathcal{C} \cap \Delta_n$, and $\nu_X(P) = I_{\{P\}}([\mathcal{C}], [\Delta_n])$, where $I_{\{P\}}(\cdot, \cdot)$ denotes the intersection number at the point P. Moreover, we have $\lim_{\eta \to 0} [\Gamma_\eta] = [\mathcal{C}]$. Since $\mathcal{C}$ is the specialization of Γ_η, for small η, $\Gamma_\eta \cap \Delta_n$ will be a set of finitely many isolated points. Write $\Gamma_\eta \cap \Delta_n = \{P_i(\eta)\}_i$. So for a sufficiently small $\eta \ne 0$, by the specialization property of localized intersection theory (see [13] or [9] Section 2.6.30), we conclude that

$$\nu_X(P) = \sum_i I_{\{P_i(\eta)\}}([\Gamma_\eta], [\Delta_n]), \tag{2.13}$$

which is the degree of $[\Gamma_\eta] \cdot [\Delta_n]$ where $[\Gamma_\eta] \cdot [\Delta_n]$ is a class in the Borel–Moore homology $H_0^{\mathrm{BM}}(\Gamma_\eta \cap \Delta_n)$ with the canonical degree map $\deg\colon H_0^{\mathrm{BM}}(\Gamma_\eta \cap \Delta_n) \to \mathbb{Z}$.

For simplicity, let $\eta = 1$ and $\Gamma = \Gamma_1$.

Recall that ω_Γ is in $H^*(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Gamma)$ which is Poincaré dual to the homology class $[\Gamma]$ in the Borel–Moore homology $H_*^{\mathrm{BM}}(\Gamma)$ (see [9] for properties of Borel–Moore homology used here), that ω_{Δ_n} is in $H^*(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Delta_n)$, which is Poincaré dual to the homology class $[\Delta_n]$ in the Borel–Moore homology $H_*^{\mathrm{BM}}(\Delta_n)$, and that we have the localized intersection pairing on Borel–Moore homology:

$$H_*^{\mathrm{BM}}(\Gamma) \times H_*^{\mathrm{BM}}(\Delta_n) \to H_0^{\mathrm{BM}}(\Gamma \cap \Delta_n),$$

which is dual to the cup product in cohomology,

$$H^*(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Gamma) \times H^*(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Delta_n) \to H^{\mathrm{top}}(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Gamma \cap \Delta_n).$$

Thus, $\omega_\Gamma \cup \omega_{\Delta_n}$ is a cohomology class in $H^{\mathrm{top}}(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Gamma \cap \Delta_n)$, which is Poincaré dual to the class $[\Gamma_\eta] \cdot [\Delta_n]$ in the homology group $H_0^{\mathrm{BM}}(\Gamma \cap \Delta_n)$

under the map:

$$D\colon H^{\mathrm{top}}(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Gamma \cap \Delta_n) \to H_0^{\mathrm{BM}}(\Gamma \cap \Delta_n).$$

If we write $\int_{\Omega_{\mathbb{A}^n}} = \deg \circ D$, then we have

$$\nu_X(P) = \int_{\Omega_{\mathbb{A}^n}} \omega_\Gamma \cup \omega_{\Delta_n}. \tag{2.14}$$

The $\mathbb{T}$-action on $\Omega_{\mathbb{A}^n}$ induces an S^1-action on $\Omega_{\mathbb{A}^n}$, and both Γ and Δ_n are S^1-invariant. Computing the right-hand side of (2.14) in the equivariant cohomology $H^*_{S^1}(\Omega_{\mathbb{A}^n})$, we get

$$\nu_X(P) = \int_{\Omega_{\mathbb{A}^n}} \omega_\Gamma^{S^1} \cup \omega_{\Delta_n}^{S^1} = \iota_{n!}\big(\omega_\Gamma^{S^1} \cup \omega_{\Delta_n}^{S^1}\big), \tag{2.15}$$

where $\omega_\Gamma^{S^1} \cup \omega_{\Delta_n}^{S^1} \in H^*_{S^1}(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Gamma \cap \Delta_n)$, and $\iota_n : \Omega_{\mathbb{A}^n} \to p$ is the trivial map from $\Omega_{\mathbb{A}^n}$ to the point p.

Similarly, regarding $\Omega_{\mathbb{A}^m} \subset \Omega_{\mathbb{A}^n}$ and letting $\Gamma^{\mathbb{T}} = \Gamma \cap \Omega_{\mathbb{A}^m}$ be defined by the equations $\tilde{z}_i = \tilde{f}_i$, $1 \le i \le m$, we have

$$\nu_{X^{\mathbb{T}}}(P) = \iota_{m!}\big(\omega_{\Gamma^{\mathbb{T}}}^{S^1} \cup \omega_{\Delta_m}^{S^1}\big). \tag{2.16}$$

Let $i : F := \Omega_{\mathbb{A}^m} \hookrightarrow \Omega_{\mathbb{A}^n}$ be the inclusion map. By the same proof for the localization theorem in [1], we have the isomorphism of localized equivariant cohomologies, with $\mathbb{C}[t] = H^*_{S^1}(\mathrm{pt})$,

$$H^*_{S^1}(\Omega_{\mathbb{A}^n}, \Omega_{\mathbb{A}^n} - \Gamma \cap \Delta_n) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \cong H^*_{S^1}(F, F - \Gamma \cap \Delta_m) \otimes_{\mathbb{C}[t]} \mathbb{C}(t).$$

Using the localization theorem and computing in the localized equivariant cohomology, we have

$$\omega_\Gamma^{S^1} \cup \omega_{\Delta_n}^{S^1} = i_! \left(\frac{i^*(\omega_\Gamma^{S^1} \cup \omega_{\Delta_n}^{S^1})}{\chi^{S^1}(N_{\Omega_{\mathbb{A}^m} \subset \Omega_{\mathbb{A}^n}})} \right) = i_! \left(\frac{i^*\omega_\Gamma^{S^1} \cup i^*\omega_{\Delta_n}^{S^1}}{(-1)^{n-m}(r_{m+1}^2 \cdots r_n^2)t^{2n-2m}} \right).$$

Therefore, we conclude from Lemmas 2.2 and 2.3 that

$$\omega_\Gamma^{S^1} \cup \omega_{\Delta_n}^{S^1} = (-1)^{n-m} \cdot i_!\big(\omega_{\Gamma^{\mathbb{T}}}^{S^1} \cup \omega_{\Delta_m}^{S^1}\big).$$

In view of (2.15) and (2.16), we obtain $\nu_X(P) = (-1)^{n-m} \cdot \nu_{X^{\mathbb{T}}}(P)$. □

3. Euler characteristics of Grothendieck Quot-schemes

In this section, we will compute the Euler characteristics of the Quot scheme $\mathrm{Quot}^m_{\mathcal{O}_Y^{\oplus r}}$ where Y is a smooth projective variety. The results will be used in the next section for the computation of Behrend's ν-function.

3.1. Virtual Hodge polynomials and Euler characteristics

First of all, let Y be a reduced complex scheme (not necessarily projective, irreducible or smooth). Mixed Hodge structures are defined on the cohomology $H^k_c(Y, \mathbb{Q})$ with compact support (see [10, 11]). The mixed Hodge structures coincide with the classical one if Y is projective and smooth. For each pair of integers (m, n), define the virtual Hodge number

$$e^{m,n}(Y) = \sum_k (-1)^k h^{m,n}(H^k_c(Y, \mathbb{Q})).$$

Then the *virtual Hodge polynomial* of Y is defined to be

$$e(Y; s, t) = \sum_{m,n} e^{m,n}(Y) s^m t^n. \tag{3.1}$$

Next, for an arbitrary complex scheme Y, we put

$$e(Y; s, t) = e(Y_{\mathrm{red}}; s, t) \tag{3.2}$$

following [8]. By (3.2) and the results in [8, 10, 14] for reduced complex schemes, we see that virtual Hodge polynomials satisfy the following properties:

(i) When Y is projective and smooth, $e(Y; s, t)$ is the usual Hodge polynomial of Y. For a general complex scheme Y, we have

$$e(Y; 1, 1) = \chi(Y). \tag{3.3}$$

(ii) If $Y = \coprod_{i=1}^n Y_i$ is a finite disjoint union of locally closed subsets, then

$$e(Y; s, t) = \sum_{i=1}^n e(Y_i; s, t). \tag{3.4}$$

(iii) If $f : Y \to Y'$ is a Zariski-locally trivial bundle with fiber F, then

$$e(Y; s, t) = e(Y'; s, t) \cdot e(F; s, t). \tag{3.5}$$

(iv) If $f : Y \to Y'$ is a bijective morphism, then

$$e(Y; s, t) = e(Y'; s, t). \tag{3.6}$$

Let $\mathbb{T} = \mathbb{C}^*$. By the Theorem 4.1 in [22], if Y admits a $\mathbb{T}$-action, then

$$\chi(Y) = \chi\big(Y^{\mathbb{T}}\big). \tag{3.7}$$

3.2. Euler characteristics of Grothendieck Quot-schemes

Let Y be a projective scheme over a base Noetherian scheme B, and let $\mathcal{V}$ be a sheaf on Y flat over B. Let $\mathrm{Quot}^m_{\mathcal{V}/Y/B}$ be the (relative) Grothendieck Quot-scheme parameterizing all the surjections $\mathcal{V}|_{Y_b} \to Q \to 0$, modulo automorphisms of Q, with $b \in B$ such that the quotients Q are torsion sheaves supported at finitely many points and $h^0(Y_b, Q) = m$. When $B = \mathrm{Spec}(\mathbb{C})$, we put $\mathrm{Quot}^m_{\mathcal{V}} = \mathrm{Quot}^m_{\mathcal{V}/Y/\mathrm{Spec}(\mathbb{C})}$. An element in $\mathrm{Quot}^m_{\mathcal{V}}$ can also be regarded as a subsheaf $E \subset \mathcal{V}$ such that the quotient $\mathcal{V}/E$ is supported at finitely many points with $h^0(Y, \mathcal{V}/E) = m$.

By the Lemma 5.2 (ii) in [24], if $\mathcal{V}$ is a locally free rank-r sheaf on Y, then

$$e\big(\mathrm{Quot}^m_{\mathcal{V}/Y/B}; s, t\big) = e\big(\mathrm{Quot}^m_{\mathcal{O}_Y^{\oplus r}/Y/B}; s, t\big). \tag{3.8}$$

In the remaining of this section, let Y be smooth. For a fixed point $y \in Y$, let

$$\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(Y, y) \subset \mathrm{Quot}^m_{\mathcal{O}_Y^{\oplus r}}$$

be the *punctual* Quot-scheme consisting of all the surjections $\mathcal{O}_Y^{\oplus r} \to Q \to 0$ such that $\mathrm{Supp}(Q) = \{y\}$ and $h^0(Y, Q) = m$. Let $n = \dim Y$. Then,

$$\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(Y, y) \cong \mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O) \tag{3.9}$$

where O denotes the origin in $\mathbb{C}^n$. Also, when $r = 1$, $\mathrm{Quot}^m_{\mathcal{O}_Y}$ is the Hilbert scheme $\mathrm{Hilb}^m(Y)$, and $\mathrm{Quot}^m_{\mathcal{O}}(Y, y)$ is the punctual Hilbert scheme $\mathrm{Hilb}^m(Y, y)$.

Lemma 3.1. *Let Y be smooth, and let O be the origin in $\mathbb{C}^n$. Then,*

$$\sum_{m=0}^{+\infty} \chi(\mathrm{Quot}^m_{\mathcal{O}_Y^{\oplus r}}) q^m = \left(\sum_{m=0}^{+\infty} \chi\big(\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)\big) q^m \right)^{\chi(Y)}. \tag{3.10}$$

Proof. There exist unique rational numbers $Q_{n,r;k,\ell,m}$ such that

$$\sum_{m=0}^{+\infty} e\big(\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O); s, t\big) q^m = \prod_{k=1}^{+\infty} \prod_{\ell,m=0}^{+\infty} \left(\frac{1}{1 - q^k s^\ell t^m} \right)^{Q_{n,r;k,\ell,m}} \tag{3.11}$$

as elements in $\mathbb{Q}[s,t][[q]]$. Define $\mathfrak{Q}_{n,r}(q,s,t) \in \mathbb{Q}[s,t][[q]]$ to be the power series:

$$\mathfrak{Q}_{n,r}(q,s,t) = \sum_{k=1}^{+\infty} \left(\sum_{\ell,m=0}^{+\infty} Q_{n,r;k,\ell,m} s^\ell t^m \right) q^k. \tag{3.12}$$

Using the arguments similar to those in Section 6 of [26], we conclude that

$$\sum_{m=0}^{+\infty} e\big(\mathrm{Quot}^m_{\mathcal{O}_Y^{\oplus r}}; s, t\big) q^m = \exp\left(\sum_{m=1}^{+\infty} \frac{1}{m} e(Y; s^m, t^m) \mathfrak{Q}_{n,r}(q^m, s^m, t^m) \right). \tag{3.13}$$

In particular, setting $s = t = 1$ and using (3.3), we obtain (3.10). □

Next, we use a torus action to compute $\chi\big(\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)\big)$. Let

$$\mathbf{T}_0 = (\mathbb{C}^*)^{n+r}. \tag{3.14}$$

Then $\mathbf{T}_0$ acts on $\mathrm{Quot}^m_{\mathcal{O}_{\mathbb{C}^n}^{\oplus r}}$ as follows. On one hand, the n-dimensional torus $\mathbf{T}_1 := (\mathbb{C}^*)^n$ acts on $\mathbb{C}^n$. This induces a $\mathbf{T}_1$-action on $\mathrm{Quot}^m_{\mathcal{O}_{\mathbb{C}^n}^{\oplus r}}$. On the other hand, $\mathbf{T}_2 := (\mathbb{C}^*)^r$ acts on $\mathcal{O}_{\mathbb{C}^n}^{\oplus r}$ via $(\mathbb{C}^*)^r \subset \mathrm{Aut}\big(\mathcal{O}_{\mathbb{C}^n}^{\oplus r}\big)$. This induces a $\mathbf{T}_2$-action on $\mathrm{Quot}^m_{\mathcal{O}_{\mathbb{C}^n}^{\oplus r}}$. So $\mathbf{T}_0 = \mathbf{T}_1 \times \mathbf{T}_2$ acts on $\mathrm{Quot}^m_{\mathcal{O}_{\mathbb{C}^n}^{\oplus r}}$. Note that $\mathbf{T}_0$ preserves $\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)$. Therefore, we obtain a $\mathbf{T}_0$-action on $\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)$. More precisely, let

$$R = \mathbb{C}[z_1, \ldots, z_n]$$

be the affine coordinate ring of $\mathbb{C}^n$. Denote the elements in $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_0$ by

$$\mathbf{t}_1 = (t_{11}, \ldots, t_{1n}), \quad \mathbf{t}_2 = (t_{21}, \ldots, t_{2r}), \quad \mathbf{t}_0 = (\mathbf{t}_1, \mathbf{t}_2) \tag{3.15}$$

respectively. Then, the element $\mathbf{t}_1 \in \mathbf{T}_1$ acts on the ring R by

$$\mathbf{t}_1(z_1^{i_1} \cdots z_n^{i_n}) = (t_{11}z_1)^{i_1} \cdots (t_{1n}z_n)^{i_n},$$

and the element $\mathbf{t}_0 = (\mathbf{t}_1, \mathbf{t}_2) \in \mathbf{T}_0$ acts on the module R^r by

$$\mathbf{t}_0(f_1, \ldots, f_r) = \big(t_{21} \cdot \mathbf{t}_1(f_1), \ \ldots, \ t_{2r} \cdot \mathbf{t}_1(f_r)\big). \tag{3.16}$$

Lemma 3.2. *Let $E \in \mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)$. Let $R^{(i)}$ be the ith component of R^r, and let $I_{Z_i} = E \cap R^{(i)} \subset R^{(i)} \cong R$. Then, $E \in \mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)^{\mathbf{T}_0}$ if and only if*

$$E = I_{Z_1} \oplus \cdots \oplus I_{Z_r}, \tag{3.17}$$

$Z_i \in \mathrm{Hilb}^{\ell(Z_i)}(\mathbb{C}^n, O)^{\mathbf{T}_1}$ for all $1 \le i \le r$, and $\ell(Z_1) + \cdots + \ell(Z_r) = m$.

Proof. It is clear that if E is of the form (3.17), then $E \in \mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)^{\mathbf{T}_0}$. Conversely, let $E \in \mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)^{\mathbf{T}_0}$. Since both E and $R^{(i)}$ are $\mathbf{T}_0$-invariant, I_{Z_i} is $\mathbf{T}_0$-invariant. So I_{Z_i} is $\mathbf{T}_1$-invariant, and $Z_i \in \mathrm{Hilb}^{\ell(Z_i)}$ $(\mathbb{C}^n, O)^{\mathbf{T}_1}$ which consists of finitely many points. Let $\mathbf{1}$ denote the identity element in $\mathbf{T}_1$. Since E is invariant by $\{\mathbf{1}\} \times \mathbf{T}_2 \subset \mathbf{T}_0$, we see that E is the span of elements of the form $f\mathbf{e}_i \in E$ where $f \in R$ and $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ is the standard basis of $\mathbb{C}^r$. In particular,

$$E = I_{Z_1} + \cdots + I_{Z_r}.$$

So $E = I_{Z_1} \oplus \cdots \oplus I_{Z_r}$. Finally, $\ell(Z_1) + \cdots + \ell(Z_r) = m$ since $\dim R^r/E = m$. ☐

Definition 3.3. Let $n \ge 2$ and $m \ge 0$. *An n-dimensional partition* of m is an array $(m_{i_1,\ldots,i_{n-1}})_{i_1,\ldots,i_{n-1}}$ of non-negative integers $m_{i_1,\ldots,i_{n-1}}$ indexed by the tuples

$$(i_1, \ldots, i_{n-1}) \in (\mathbb{Z}_{\ge 0})^{n-1} \tag{3.18}$$

such that $m_{i_1,\ldots,i_{n-1}} \ge m_{j_1,\ldots,j_{n-1}}$ whenever $i_1 \le j_1, \ldots, i_{n-1} \le j_{n-1}$, and that

$$\sum_{i_1,\ldots,i_{n-1}} m_{i_1,\ldots,i_{n-1}} = m. \tag{3.19}$$

Theorem 3.4. *Let Y be a smooth projective variety of dimension n. Then,*

$$\sum_{m=0}^{+\infty} \chi(\mathrm{Quot}^m_{\mathcal{O}_Y^{\oplus r}}) q^m = \left(\sum_{m=0}^{+\infty} P_n(m) q^m \right)^{r \cdot \chi(Y)}, \tag{3.20}$$

where $P_n(m)$ denotes the number of n-dimensional partitions of m.

Proof. We conclude from formula (3.7) and Lemma 3.2 that

$$\begin{aligned} \sum_{m=0}^{+\infty} \chi\big(\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)\big) q^m &= \sum_{m=0}^{+\infty} \chi\big(\mathrm{Quot}^m_{\mathcal{O}^{\oplus r}}(\mathbb{C}^n, O)^{\mathbf{T}_0}\big) q^m \\ &= \left(\sum_{m=0}^{+\infty} \chi\big(\mathrm{Hilb}^m(\mathbb{C}^n, O)^{\mathbf{T}_1}\big) q^m \right)^r \\ &= \left(\sum_{m=0}^{+\infty} \chi\big(\mathrm{Hilb}^m(\mathbb{C}^n, O)\big) q^m \right)^r. \end{aligned}$$

The Euler characteristic $\chi\big(\mathrm{Hilb}^m(\mathbb{C}^n, O)\big)$ is given by the formula:

$$\sum_{m=0}^{+\infty} \chi\big(\mathrm{Hilb}^m(\mathbb{C}^n, O)\big) q^m = \sum_{m=0}^{+\infty} P_n(m) q^m \tag{3.21}$$

(see Proposition 5.1 in [8]). Now (3.20) follows from Lemma 3.1. □

The generating series for $P_3(m), m \geq 0$ is given by the McMahon function:

$$\sum_{m=0}^{+\infty} P_3(m) q^m = M(q) := \prod_{m=1}^{+\infty} \frac{1}{(1-q^m)^m}. \tag{3.22}$$

Corollary 3.5. *Let Y be a smooth projective complex 3-fold. Then,*

$$\sum_{m=0}^{+\infty} \chi(\mathrm{Quot}^m_{\mathcal{O}_Y^{\oplus r}}) q^m = M(q)^{r \cdot \chi(Y)}. \tag{3.23}$$

Proof. Follows immediately from Theorem 3.4 and (3.22). □

4. Donaldson–Thomas invariants of certain Calabi–Yau 3-folds

In this section, we compute the Donaldson–Thomas invariants for two types of Calabi–Yau 3-folds. The first type comes from [12, 33] and is studied in Section 4.2. The second type comes from [25] and is studied in Section 4.3.

4.1. Donaldson–Thomas invariants and weighted Euler characteristics

Let L be an ample line bundle on a smooth projective variety Y of dimension n, and V be a rank-r torsion-free sheaf on Y. We say that V is (*slope*) *L-stable* if

$$\frac{c_1(F)\cdot c_1(L)^{n-1}}{\mathrm{rank}(F)} < \frac{c_1(V)\cdot c_1(L)^{n-1}}{r}$$

for any proper subsheaf F of V, and V is *Gieseker L-stable* if

$$\frac{\chi(F\otimes L^{\otimes k})}{\mathrm{rank}(F)} < \frac{\chi(V\otimes L^{\otimes k})}{r}, \quad k \gg 0$$

for any proper subsheaf F of V. Similarly, we define *L-semistability* and *Gieseker L-semistability* by replacing the above strict inequalities $<$ by inequalities $\leq$. For a class c in the Chow group $A^*(Y)$, let $\mathfrak{M}_L(c)$ be the moduli space of L-stable rank-2 bundles with total Chern class c, and let $\overline{\mathfrak{M}}_L(c)$ be the moduli space of Gieseker L-semistable rank-2 torsion-free sheaves with total Chern class c.

Next, let (Y,L) be a polarized smooth Calabi–Yau 3-fold. Assume that all the rank-2 torsion-free sheaves in $\overline{\mathfrak{M}}_L(c)$ are actually Gieseker L-stable. By the Definition 3.54 and Corollary 3.39 in [33], *the Donaldson–Thomas invariant*

$$\lambda(L,c)\in\mathbb{Z} \tag{4.1}$$

(also known as *the homolomorphic Casson invariant*) can be defined via the moduli space $\overline{\mathfrak{M}}_L(c)$. By the Proposition 1.26 in [4], $\overline{\mathfrak{M}}_L(c)$ admits a symmetric obstruction theory. It follows from the Theorem 4.18 in [2] that

$$\lambda(L,c) = \widetilde{\chi}\big(\overline{\mathfrak{M}}_L(c)\big). \tag{4.2}$$

Note that if $\overline{\mathfrak{M}}_L(c)$ is further assumed to be smooth, then

$$\lambda(L,c) = (-1)^{\dim \overline{\mathfrak{M}}_L(c)}\cdot \chi\big(\overline{\mathfrak{M}}_L(c)\big). \tag{4.3}$$

4.2. Donaldson–Thomas invariants, I

Let Q_0 be a smooth quadric in $\mathbb{P}^5$. Identifying Q_0 with the Grassmaniann $G(2,4)$, we obtain universal rank-2 bundles B_1 and B_2 sitting in the exact sequence:

$$0 \to (B_1)^* \to (\mathcal{O}_{Q_0})^{\oplus 4} \to B_2 \to 0. \tag{4.4}$$

The Chern classes of B_1 and B_2 are the same, and

$$c_1(B_i) = H|_{Q_0}, \quad c_2(B_i) = P \tag{4.5}$$

where H is a hyperplane in $\mathbb{P}^5$, and P is a plane contained in Q_0.

Next, let Y be a smooth quartic hypersurface in Q_0. Then, Y is a smooth Calabi–Yau 3-fold with $H_1(Y,\mathbb{Z}) = 0$, and the pull-back $\mathrm{Pic}(\mathbb{P}^5) \to \mathrm{Pic}(Y)$ is an isomorphism. Let

$$E_{0,i} = B_i|_Y$$

for $i = 1, 2$, and let $L = \mathcal{O}_{\mathbb{P}^5}(1)|_Y$. Fix a point $y_0 \in Y$. For $m \in \mathbb{Z}$, define

$$\mathbf{c}_m = -m[y_0] + \big(1 + H|_Y + P|_Y\big) \in A^*(Y). \tag{4.6}$$

By Theorem 3.55 in [33], the moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_0)$ is smooth and

$$\overline{\mathfrak{M}}_L(\mathbf{c}_0) = \big\{E_{0,1},\ E_{0,2}\big\}. \tag{4.7}$$

Moreover, since $\mathrm{Pic}(Y) = \mathbb{Z}L$, both $E_{0,1}$ and $E_{0,2}$ are L slope-stable. It follows that $\lambda(L, \mathbf{c}_0) = 2$.

Lemma 4.1. *Let $m \ge 0$. Then $\overline{\mathfrak{M}}_L(\mathbf{c}_{2m})$ is isomorphic to* $\mathrm{Quot}^m_{E_{0,1}} \coprod \mathrm{Quot}^m_{E_{0,2}}$.

Proof. Let $E \in \mathrm{Quot}^m_{E_{0,i}}$ with $i = 1$ or 2. Then, we have an exact sequence:

$$0 \to E \to E_{0,i} \to Q \to 0 \tag{4.8}$$

where Q is supported at finitely many points and $h^0(Y, Q) = m$. Note that

$$c(E) = c(E_{0,i})/c(Q) = \mathbf{c}_0/(1 + 2m[y_0]) = -2m[y_0] + \mathbf{c}_0 = \mathbf{c}_{2m}.$$

Also, E is L slope-stable since $E_{0,i}$ is L slope-stable. Hence

$$E \in \overline{\mathfrak{M}}_L(\mathbf{c}_{2m}). \tag{4.9}$$

Conversely, let $E \in \overline{\mathfrak{M}}_L(\mathbf{c}_{2m})$. The same argument in the proof of Theorem 3.55 in [33], which uses only the first and second Chern classes of E, shows that $E^{**} \cong E_{0,i}$ where $i = 1$ or 2. Calculating the Chern classes from the canonical exact sequence $0 \to E \to E^{**} \to Q \to 0$, we get $c(Q) = 1 + 2m[y_0]$. So Q is supported at finitely many points with $h^0(Y, Q) = m$, and

$$E \in \mathrm{Quot}^m_{E_{0,i}}. \tag{4.10}$$

It is well known that the Grothendieck Quot-schemes are fine moduli spaces. So over $\mathrm{Quot}^m_{E_{0,i}} \times Y$, there exists universal exact sequence

$$0 \to \mathcal{E}_{m,i} \to \rho_2^* E_{0,i} \to \mathcal{Q}_{m,i} \to 0$$

where ρ_2 is the second projection of $\mathrm{Quot}^m_{E_{0,i}} \times Y$. By (4.9), the sheaf

$$\mathcal{E}_{m,1} \coprod \mathcal{E}_{m,2} \tag{4.11}$$

over $\mathrm{Quot}^m_{E_{0,1}} \coprod \mathrm{Quot}^m_{E_{0,2}}$ parameterizes a flat family of Gieseker L-stable rank-2 sheaves with Chern class $\mathbf{c}_{2m}$. To show that (4.11) is universal, let $\mathcal{E}$ be a flat family of Gieseker L-semistable rank-2 sheaves with Chern class $\mathbf{c}_{2m}$ parameterized by T, and let $p_T : T \times Y \to T$ be the projection. By (4.10) and the universal property of Quot-schemes, there exist a morphism $\psi : T \to \mathrm{Quot}^m_{E_{0,1}} \coprod \mathrm{Quot}^m_{E_{0,2}}$ and a line bundle G on T such that

$$\mathcal{E} \otimes p_T^* G = (\psi \times \mathrm{Id}_Y)^* (\mathcal{E}_{m,1} \coprod \mathcal{E}_{m,2}).$$

Therefore, (4.11) is a universal family. □

Remark 4.2. From the proof of Lemma 4.1, we see that if the moduli space $\overline{\mathfrak{M}}_L(\mathbf{c}_m)$ is not empty, then m must be even and non-negative.

Proposition 4.3. *Let Y be a smooth quartic hypersurface in the quadric Q_0, and let $\mathbf{c}_m = -m\,[y_0] + \big(1 + H|_Y + P|_Y\big) \in A^*(Y)$. Then,*

$$\sum_{m \in \mathbb{Z}} \chi\big(\overline{\mathfrak{M}}_L(\mathbf{c}_m)\big)\, q^m = 2 \cdot M(q^2)^{2\chi(Y)}. \tag{4.12}$$

Proof. By Remark 4.2, $\overline{\mathfrak{M}}_L(\mathbf{c}_m) = \emptyset$ if $m < 0$ or m is odd. So

$$\sum_{m \in \mathbb{Z}} \chi\big(\overline{\mathfrak{M}}_L(\mathbf{c}_m)\big)\, q^m = \sum_{m=0}^{+\infty} \chi\big(\overline{\mathfrak{M}}_L(\mathbf{c}_{2m})\big)\, q^{2m}. \tag{4.13}$$

By Lemma 4.1 and (3.8), $e\big(\overline{\mathfrak{M}}_L(\mathbf{c}_{2m}); s, t\big) = 2 \cdot e\big(\operatorname{Quot}^m_{\mathcal{O}_Y^{\oplus 2}}; s, t\big)$. Setting $s = t = 1$ and using (3.3), we conclude that $\chi\big(\overline{\mathfrak{M}}_L(\mathbf{c}_{2m})\big) = 2 \cdot \chi\big(\operatorname{Quot}^m_{\mathcal{O}_Y^{\oplus 2}}\big)$. Now our formula (4.12) follows immediately from (4.13) and Corollary 3.5. □

Let $\widetilde{F}_m \subset \operatorname{Quot}^m_{E_{0,1}}$ be the punctual Quot-scheme defined by:

$$\widetilde{F}_m = \{E \in \operatorname{Quot}^m_{E_{0,1}} \,|\, E_{0,1}/E \text{ is supported at } y_0\}. \tag{4.14}$$

Fix a Zariski open neighborhood Y_0 of the point $y_0 \in Y$ such that $E_{0,1}|_{Y_0} \cong \mathcal{O}_{Y_0}^{\oplus 2}$, and define the open subset $\operatorname{Quot}^m_{E_{0,1}}(Y_0) \subset \operatorname{Quot}^m_{E_{0,1}}$ by

$$\operatorname{Quot}^m_{E_{0,1}}(Y_0) = \{E \in \operatorname{Quot}^m_{E_{0,1}} \,|\, E_{0,1}/E \text{ is supported in } Y_0\}.$$

Then, $\widetilde{F}_m \subset \operatorname{Quot}^m_{E_{0,1}}(Y_0) \cong \operatorname{Quot}^m_{\mathcal{O}_{Y_0}^{\oplus 2}}$. Consider the embedding $\mathbb{T} = \mathbb{C}^* \hookrightarrow \mathbb{T}_2 = (\mathbb{C}^*)^2 \subset \operatorname{Aut}(\mathcal{O}_{Y_0}^{\oplus 2})$ via $t \mapsto (1, t)$ and the induced $\mathbb{T}$-action on $\operatorname{Quot}^m_{\mathcal{O}_{Y_0}^{\oplus 2}}$. Arguments similar to those in the proof of Lemma 3.2 show that $E \in \big(\operatorname{Quot}^m_{\mathcal{O}_{Y_0}^{\oplus 2}}\big)^{\mathbb{T}}$ if and only if $E = I_{Z_1} \oplus I_{Z_2} \subset \mathcal{O}_{Y_0} \oplus \mathcal{O}_{Y_0}$ where Z_1 and Z_2 are 0-dimensional closed subschemes of Y_0 with $\ell(Z_1) + \ell(Z_2) = m$. Thus, we obtain

$$\big(\operatorname{Quot}^m_{\mathcal{O}_{Y_0}^{\oplus 2}}\big)^{\mathbb{T}} \cong \coprod_{i=0}^{m} \operatorname{Hilb}^i(Y_0) \times \operatorname{Hilb}^{m-i}(Y_0). \tag{4.15}$$

Lemma 4.4. *Let Z_1 and Z_2 be 0-dimensional closed subschemes of Y. Then,*

$$\dim \operatorname{Hom}(I_{Z_1}, \mathcal{O}_{Z_2}) + \dim \operatorname{Hom}(I_{Z_2}, \mathcal{O}_{Z_1}) \equiv \ell(Z_1) + \ell(Z_2) \pmod 2. \tag{4.16}$$

Proof. We have $H^2(Y, \mathcal{O}_Y) \cong H^1(Y, \mathcal{O}_Y)^* = 0$. Taking cohomology from the exact sequence $0 \to I_{Z_2} \to \mathcal{O}_Y \to \mathcal{O}_{Z_2} \to 0$, we get $H^2(Y, I_{Z_2}) \cong$

$H^1(Y, \mathcal{O}_{Z_2}) = 0$. Thus,

$$\mathrm{Ext}^1(I_{Z_2}, \mathcal{O}_Y) \cong H^2(Y, I_{Z_2})^* = 0.$$

Applying $\mathrm{Hom}(I_{Z_2}, \cdot)$ to the exact sequence $0 \to I_{Z_1} \to \mathcal{O}_Y \to \mathcal{O}_{Z_1} \to 0$, we obtain the following exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}(I_{Z_2}, I_{Z_1}) &\to \mathrm{Hom}(I_{Z_2}, \mathcal{O}_Y) \to \mathrm{Hom}(I_{Z_2}, \mathcal{O}_{Z_1}) \\ &\to \mathrm{Ext}^1(I_{Z_2}, I_{Z_1}) \to \mathrm{Ext}^1(I_{Z_2}, \mathcal{O}_Y). \end{aligned}$$

Since $\mathrm{Ext}^1(I_{Z_2}, \mathcal{O}_Y) = 0$ and $\mathrm{Hom}(I_{Z_2}, \mathcal{O}_Y) \cong \mathbb{C}$, we conclude that

$$\dim \mathrm{Hom}(I_{Z_2}, \mathcal{O}_{Z_1}) = -\dim \mathrm{Hom}(I_{Z_2}, I_{Z_1}) + \dim \mathrm{Ext}^1(I_{Z_2}, I_{Z_1}) + 1.$$

By symmetry, we have a similar formula for $\dim \mathrm{Hom}(I_{Z_1}, \mathcal{O}_{Z_2})$. Therefore,

$$\begin{aligned} &\dim \mathrm{Hom}(I_{Z_2}, \mathcal{O}_{Z_1}) + \dim \mathrm{Hom}(I_{Z_1}, \mathcal{O}_{Z_2}) \\ &\quad = -\dim \mathrm{Hom}(I_{Z_2}, I_{Z_1}) + \dim \mathrm{Ext}^1(I_{Z_2}, I_{Z_1}) + 2 \\ &\qquad - \dim \mathrm{Hom}(I_{Z_1}, I_{Z_2}) + \dim \mathrm{Ext}^1(I_{Z_1}, I_{Z_2}) \\ &\quad \equiv -\dim \mathrm{Hom}(I_{Z_2}, I_{Z_1}) + \dim \mathrm{Ext}^1(I_{Z_2}, I_{Z_1}) \\ &\qquad + \dim \mathrm{Ext}^3(I_{Z_2}, I_{Z_1}) - \dim \mathrm{Ext}^2(I_{Z_2}, I_{Z_1}) \pmod 2 \\ &\quad \equiv -\chi(I_{Z_2}, I_{Z_1}) \pmod 2. \end{aligned} \tag{4.17}$$

Since $c_3(I_{Z_i}) = -c_3(\mathcal{O}_{Z_i}) = -2[Z_i]$, the Hirzebruch–Riemann–Roch formula gives

$$\chi(I_{Z_2}, I_{Z_1}) = \int_Y \mathrm{ch}(I_{Z_2})^* \cdot \mathrm{ch}(I_{Z_1}) \cdot \mathrm{td}(T_Y) = -\ell(Z_1) + \ell(Z_2).$$

Combining this with (4.17) yields the desired formula (4.16). □

For simplicity, regard $\mathrm{Quot}^m_{E_{0,1}}(Y_0) = \mathrm{Quot}^m_{\mathcal{O}_{Y_0}^{\oplus 2}}$. The symmetric obstruction theory on $\mathrm{Quot}^m_{E_{0,1}}$ restricts to a symmetric obstruction theory on $\mathrm{Quot}^m_{E_{0,1}}(Y_0) = \mathrm{Quot}^m_{\mathcal{O}_{Y_0}^{\oplus 2}}$. This symmetric obstruction theory on $\mathrm{Quot}^m_{E_{0,1}}(Y_0) = \mathrm{Quot}^m_{\mathcal{O}_{Y_0}^{\oplus 2}}$ is $\mathbb{T}$-equivariant since the construction of the symmetric obstruction theory is stable under base change (see [4, 33]) and our Gieseker moduli space is fine. Let $\widetilde{\nu}_m$ be the restriction of Behrend's function $\nu_{\mathrm{Quot}^m_{E_{0,1}}}$ to $\widetilde{F}_m$. Then $\nu_{\mathrm{Quot}^m_{\mathcal{O}_{Y_0}^{\oplus 2}}}|_{\widetilde{F}_m} = \widetilde{\nu}_m$.

Lemma 4.5. $\chi(\widetilde{F}_m, \widetilde{\nu}_m) = \chi(\widetilde{F}_m)$.

Proof. Note that $\widetilde{F}_m \subset \mathrm{Quot}^m_{\mathcal{O}^{\oplus 2}_{Y_0}}$ is $\mathbb{T}$-invariant. For each $n \in \mathbb{Z}$, the subset

$$\{E \in \widetilde{F}_m - (\widetilde{F}_m)^{\mathbb{T}} | \widetilde{\nu}_m(E) = n\}$$

is $\mathbb{T}$-invariant and does not contain any fixed point. By (3.7),

$$\chi\big(\{E \in \widetilde{F}_m - (\widetilde{F}_m)^{\mathbb{T}} | \widetilde{\nu}_m(E) = n\}\big) = 0.$$

By definition, $\chi(\widetilde{F}_m, \widetilde{\nu}_m) = \sum_{n \in \mathbb{Z}} n \cdot \chi\big(\{E \in \widetilde{F}_m | \widetilde{\nu}_m(E) = n\}\big)$. Thus,

$$\chi(\widetilde{F}_m, \widetilde{\nu}_m) = \sum_{n \in \mathbb{Z}} n \cdot \chi\big(\{E \in (\widetilde{F}_m)^{\mathbb{T}} | \widetilde{\nu}_m(E) = n\}\big).$$

In view of (4.15), $E \in (\widetilde{F}_m)^{\mathbb{T}}$ if and only if $E = I_{Z_1} \oplus I_{Z_2} \subset \mathcal{O}_{Y_0} \oplus \mathcal{O}_{Y_0}$ where $Z_1 \in \mathrm{Hilb}^i(Y_0, y_0)$ and $Z_2 \in \mathrm{Hilb}^{m-i}(Y_0, y_0)$ for some integer i satisfying $0 \le i \le m$. In this case, we obtain from Theorem 2.4 (ii) that

$$\begin{aligned}
\widetilde{\nu}_m(E) &= \nu_{\mathrm{Quot}^m_{\mathcal{O}^{\oplus 2}_Y}}(E) = \nu_{\mathrm{Quot}^m_{\mathcal{O}^{\oplus 2}_{Y_0}}}(E) \\
&= (-1)^a \cdot \nu_{\mathrm{Hilb}^i(Y_0) \times \mathrm{Hilb}^{m-i}(Y_0)}(Z_1, Z_2) \\
&= (-1)^a \cdot \nu_{\mathrm{Hilb}^i(Y) \times \mathrm{Hilb}^{m-i}(Y)}(Z_1, Z_2),
\end{aligned}$$

where a is the difference between the dimensions of the Zariski tangent spaces:

$$\begin{aligned}
a &= \dim T_E \,\mathrm{Quot}^m_{\mathcal{O}^{\oplus 2}_{Y_0}} - \big(\dim T_{Z_1} \mathrm{Hilb}^i(Y_0) + \dim T_{Z_2} \mathrm{Hilb}^{m-i}(Y_0)\big) \\
&= \dim \mathrm{Hom}(I_{Z_1} \oplus I_{Z_2}, \mathcal{O}_{Z_1} \oplus \mathcal{O}_{Z_2}) - \sum_{k=1}^{2} \dim \mathrm{Hom}(I_{Z_k}, \mathcal{O}_{Z_k}) \\
&\equiv \ell(Z_1) + \ell(Z_2) \pmod 2 \\
&\equiv m \pmod 2,
\end{aligned}$$

by Lemma 4.4. Therefore, $\widetilde{\nu}_m(E) = (-1)^m \cdot \nu_{\mathrm{Hilb}^i(Y)\times \mathrm{Hilb}^{m-i}(Y)}(Z_1, Z_2)$ and

$$\begin{aligned}
&\chi(\widetilde{F}_m, \widetilde{\nu}_m) \\
&\quad = \sum_{n\in\mathbb{Z}} n \cdot \sum_{i=0}^{m} \chi\big(\{(Z_1, Z_2) \in \mathrm{Hilb}^i(Y, y_0) \\
&\qquad \times \mathrm{Hilb}^{m-i}(Y, y_0) | \nu(Z_1, Z_2) = (-1)^m n\}\big) \\
&\quad = (-1)^m \sum_{i=0}^{m} \sum_{n\in\mathbb{Z}} n \cdot \chi\big(\{(Z_1, Z_2) \in \mathrm{Hilb}^i(Y, y_0) \\
&\qquad \times \mathrm{Hilb}^{m-i}(Y, y_0) | \nu(Z_1, Z_2) = n\}\big) \\
&\quad = (-1)^m \sum_{i=0}^{m} \chi(\mathrm{Hilb}^i(Y, y_0) \times \mathrm{Hilb}^{m-i}(Y, y_0), \nu) \\
&\quad = (-1)^m \sum_{i=0}^{m} \chi(\mathrm{Hilb}^i(Y, y_0), \nu_{\mathrm{Hilb}^i(Y)}) \cdot \chi(\mathrm{Hilb}^{m-i}(Y, y_0), \nu_{\mathrm{Hilb}^{m-i}(Y)}),
\end{aligned}$$

where ν denotes $\nu_{\mathrm{Hilb}^i(Y)\times \mathrm{Hilb}^{m-i}(Y)}$. By the Corollary 4.3 in [4],

$$\chi(\mathrm{Hilb}^i(Y, y_0), \nu_{\mathrm{Hilb}^i(Y)}) = (-1)^i \chi(\mathrm{Hilb}^i(Y, y_0)).$$

Combining this with $\chi((\widetilde{F}_m)^{\mathbb{T}}) = \chi(\widetilde{F}_m)$, we conclude that

$$\chi(\widetilde{F}_m, \widetilde{\nu}_m) = \sum_{i=0}^{m} \chi(\mathrm{Hilb}^i(Y, y_0)) \cdot \chi(\mathrm{Hilb}^{m-i}(Y, y_0)) = \chi((\widetilde{F}_m)^{\mathbb{T}}) = \chi(\widetilde{F}_m).$$

□

Theorem 4.6. *Let Y be a smooth quartic hypersurface in the quadric Q_0, and let $\mathbf{c}_m = -m\,[y_0] + \big(1 + H|_Y + P|_Y\big) \in A^*(Y)$. Then,*

$$\sum_{m\in\mathbb{Z}} \lambda(L, \mathbf{c}_m)\, q^m = 2 \cdot M(q^2)^{2\,\chi(Y)}. \tag{4.18}$$

Proof. By Remark 4.2, $\overline{\mathfrak{M}}_L(\mathbf{c}_m) = \emptyset$ if $m < 0$ or m is odd. By (4.2),

$$\sum_{m\in\mathbb{Z}} \lambda(L, \mathbf{c}_m)\, q^m = \sum_{m=0}^{+\infty} \lambda(L, \mathbf{c}_{2m})\, q^{2m} = \sum_{m=0}^{+\infty} \widetilde{\chi}\big(\overline{\mathfrak{M}}_L(\mathbf{c}_{2m})\big)\, q^{2m}. \tag{4.19}$$

Adopting the proof of the Theorem 4.11 in [4], we conclude that

$$\widetilde{\chi}\big(\operatorname{Quot}^m_{E_{0,1}}\big) = \sum_{\alpha \vdash n} |G_\alpha| \cdot \chi\big(Y_0^{\ell(\alpha)}\big) \cdot \prod_i \chi(\widetilde{F}_{\alpha_i}, \widetilde{\nu}_{\alpha_i}),$$
$$\chi\big(\operatorname{Quot}^m_{E_{0,1}}\big) = \sum_{\alpha \vdash n} |G_\alpha| \cdot \chi\big(Y_0^{\ell(\alpha)}\big) \cdot \prod_i \chi(\widetilde{F}_{\alpha_i}).$$

Here, for a partition α of n, G_α denotes the automorphism group of α, $\ell(\alpha)$ denotes the length of α, and $Y_0^{\ell(\alpha)}$ denotes the open subset of the product $Y^{\ell(\alpha)}$ consisting of $\ell(\alpha)$-tuples with pairwise distinct entries. By Lemma 4.5,

$$\widetilde{\chi}\big(\operatorname{Quot}^m_{E_{0,1}}\big) = \chi\big(\operatorname{Quot}^m_{E_{0,1}}\big).$$

Similarly, $\widetilde{\chi}\big(\operatorname{Quot}^m_{E_{0,2}}\big) = \chi\big(\operatorname{Quot}^m_{E_{0,2}}\big)$. By (4.19) and Lemma 4.1,

$$\sum_{m \in \mathbb{Z}} \lambda(L, \mathbf{c}_m)\, q^m = \sum_{m=0}^{+\infty} \chi\big(\overline{\mathfrak{M}}_L(\mathbf{c}_{2m})\big)\, q^{2m} = \sum_{m \in \mathbb{Z}} \chi\big(\overline{\mathfrak{M}}_L(\mathbf{c}_m)\big)\, q^m.$$

Finally, we obtain $\sum_{m \in \mathbb{Z}} \lambda(L, \mathbf{c}_m)\, q^m = 2 \cdot M(q^2)^{2\chi(Y)}$ from Proposition 4.3. □

4.3. Donaldson–Thomas invariants, II

Let $n \geq 2$ and $X = \mathbb{P}^1 \times \mathbb{P}^n$. Let p be a point in $\mathbb{P}^1$, and H be a hyperplane in $\mathbb{P}^n$. For simplicity, denote the divisor $a(\{p\} \times \mathbb{P}^n) + b(\mathbb{P}^1 \times H)$ by (a, b). When a and b are rational numbers, (a, b) is a $\mathbb{Q}$-divisor and $\mathcal{O}_X(a, b)$ is a $\mathbb{Q}$-line bundle. The divisor $(1, r)$ is ample if and only if $r > 0$. Put

$$L_r = \mathcal{O}_X(1, r). \tag{4.20}$$

Let Y be a generic divisor of type $(2, 2, n+1)$ in the product

$$Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n.$$

Then Y is a smooth Calabi–Yau $(n+1)$-fold. By the Lefschetz hyperplane theorem,

$$\operatorname{Pic}(Y) \cong \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n).$$

Let π_i be the projection from $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$ to the ith factor, and let

$$\pi = (\pi_2 \times \pi_3)|_Y : \ Y \to X = \mathbb{P}^1 \times \mathbb{P}^n. \tag{4.21}$$

Put $\mathcal{O}_Y(a,b,c) = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(b) \otimes \pi_3^*\mathcal{O}_{\mathbb{P}^n}(c)|_Y$, and

$$L_r^Y = \mathcal{O}_Y(0,1,r) = \pi^* L_r. \tag{4.22}$$

Then the projection $\pi : Y \to X = \mathbb{P}^1 \times \mathbb{P}^n$ is a ramified double covering with the ramification locus $B \subset X$ being a smooth divisor of type $(4, 2n+2)$. In particular,

$$\pi_*\mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{O}_X(-2, -n-1). \tag{4.23}$$

By the projection formula, if $b < (n+1)$, then we obtain

$$\begin{aligned} H^1(Y, \pi^*\mathcal{O}_X(a,b)) &\cong H^1(X, \mathcal{O}_X(a,b) \otimes \pi_*\mathcal{O}_Y) \\ &\cong H^1(X, \mathcal{O}_X(a,b) \oplus \mathcal{O}_X(a-2, b-n-1)) \\ &\cong H^1(X, \mathcal{O}_X(a,b)). \end{aligned} \tag{4.24}$$

Fix $\epsilon_1, \epsilon_2 = 0, 1$, and fix a point $y_0 \in Y$. For $m \in \mathbb{Z}$, define

$$\begin{aligned} \mathbf{c}_m &= -m[y_0] + \big(1 + \pi^*(-1,1)\big) \cdot \big(1 + \pi^*(\epsilon_1 + 1, \epsilon_2 - 1)\big) \\ &= \big(1 - m[y_0]\big) \cdot \big(1 + \pi^*(-1,1)\big) \cdot \big(1 + \pi^*(\epsilon_1 + 1, \epsilon_2 - 1)\big) \in A^*(Y). \end{aligned} \tag{4.25}$$

Our first goal is to study the Gieseker moduli space $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m)$. When $m = 0$, the structure of the moduli space $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$ has been determined in [25] (for convenience, we adopt the convention that $e/0 = +\infty$ when $e > 0$):

Lemma 4.7. (Theorem 4.6 in [25]) *Let* $k = (1+\epsilon_1)\binom{n+2-\epsilon_2}{n} - 1$.

- (i) *When* $0 < r < n(2-\epsilon_2)/(2+\epsilon_1)$, *the moduli space* $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$ *is empty;*
- (ii) *When* $n(2-\epsilon_2)/(2+\epsilon_1) < r < n(2-\epsilon_2)/\epsilon_1$, $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$ *is isomorphic to* $\mathbb{P}^k$ *and consists of all the bundles* E_0 *sitting in non-splitting extensions:*

$$0 \to \mathcal{O}_Y(0,-1,1) \to E_0 \to \mathcal{O}_Y(0, \epsilon_1 + 1, \epsilon_2 - 1) \to 0. \tag{4.26}$$

Moreover, all these rank-2 bundles E_0 *are (slope)* L_r^Y*-stable.*

Let $n(2-\epsilon_2)/(2+\epsilon_1) < r < n(2-\epsilon_2)/\epsilon_1$, and $E_0 \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$. Consider the Quot-scheme $\mathrm{Quot}^m_{E_0}$. Let $E \in \mathrm{Quot}^m_{E_0}$. Then we have an exact sequence:

$$0 \to E \to E_0 \to Q \to 0 \tag{4.27}$$

where Q is supported at finitely many points and $h^0(Y, Q) = m$. Note that

$$c(E) = c(E_0)/c(Q) = \mathbf{c}_0/(1 + 2m[y_0]) = -2m[y_0] + \mathbf{c}_0 = \mathbf{c}_{2m}.$$

Also, E is L_r^Y-stable since E_0 is L_r^Y-stable. Hence

$$E \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m}). \tag{4.28}$$

In the following, we show that the converse also holds, i.e., every element in $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m})$ is contained in $\mathrm{Quot}^m_{E_0}$ for some $E_0 \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$.

Lemma 4.8. *Let $n \ge 2$, $\epsilon_1, \epsilon_2 = 0, 1$ and $r < n(2-\epsilon_2)/\epsilon_1$. Let $E \in \overline{\mathfrak{M}}_{L_r^Y}$ $(\mathbf{c}_m)$.*

(i) *Then, $r \ge n(2-\epsilon_2)/(2+\epsilon_1)$ and E sits in an extension*

$$0 \to \mathcal{O}_Y(0, -1, 1) \otimes I_{Z_1} \to E \to \mathcal{O}_Y(0, \epsilon_1 + 1, \epsilon_2 - 1) \otimes I_{Z_2} \to 0 \tag{4.29}$$

for some 0-dimensional closed subschemes Z_1 and Z_2 of Y satisfying

$$m = 2\big(\ell(Z_1) + \ell(Z_2)\big);$$

(ii) *Moreover, if $r > n(2-\epsilon_2)/(2+\epsilon_1)$, then the above extension does not split.*

Proof. Our proof is slightly modified from the proof of Lemma 4.2 in [25] which handles the case $m = 0$. Since $c_1(E) = \pi^*(\epsilon_1, \epsilon_2)$ and

$$c_2(E) = \pi^*\big((2 + \epsilon_1 - \epsilon_2)[p \times H] - (1 - \epsilon_2)[\mathbb{P}^1 \times H^2]\big),$$

$(4c_2(E) - c_1(E)^2) \cdot c_1(L_{r_0}^Y)^{n-1} = 2(2-\epsilon_2)r_0^{n-2}[2(2+\epsilon_1)r_0 - (2-\epsilon_2)(n-1)]$. By the Bogomolov Inequality, E is $L_{r_0}^Y$-unstable if $0 < r_0 < (2-\epsilon_2)(n-1)/$

$(2(2+\epsilon_1))$. Fix such an r_0 with $r_0 < r$. Then there exists an exact sequence

$$0 \to \mathcal{O}_Y(a,b,c) \otimes I_{Z_1} \to E \to \mathcal{O}_Y(-a, \epsilon_1 - b, \epsilon_2 - c) \otimes I_{Z_2} \to 0 \tag{4.30}$$

such that $\mathcal{O}_Y(a,b,c) \otimes I_{Z_1}$ destabilizes E with respect to $L_{r_0}^Y$, where Z_1 and Z_2 are codimension at least two subschemes of Y. Therefore,

$$c_1(\mathcal{O}_Y(a,b,c)) \cdot c_1(L_{r_0}^Y)^n > c_1(E) \cdot c_1(L_{r_0}^Y)^n/2.$$

A straightforward computation shows that this can be simplified into

$$n[(2c - \epsilon_2) + (n+1)a] + (2a + 2b - \epsilon_1)r_0 > 0. \tag{4.31}$$

On the other hand, since E is L_r^Y-semistable, we must have

$$n[(2c - \epsilon_2) + (n+1)a] + (2a + 2b - \epsilon_1)r \leq 0. \tag{4.32}$$

Calculating the second Chern class from the exact sequence (4.30), we get

$$\mathcal{O}_Y(a,b,c) \cdot \mathcal{O}_Y(-a, \epsilon_1 - b, \epsilon_2 - c) \leq c_2(E) \tag{4.33}$$

since $c_2(I_{Z_1})$ and $c_2(I_{Z_2})$ are effective cycles. Regarding (4.33) as an inequality of cycles in Z and comparing the coefficients of $[p \times p \times H]$ and $[p \times \mathbb{P}^1 \times H^2]$ yield

$$[2a + (2b - \epsilon_1)](2c - \epsilon_2) + (n+1)a(2b - \epsilon_1) \geq -(\epsilon_1 + 2)(2 - \epsilon_2), \tag{4.34}$$

$$[(2c - \epsilon_2) + 2(n+1)a](2c - \epsilon_2) \geq (2 - \epsilon_2)^2. \tag{4.35}$$

Since $0 < r_0 < r$, we see from (4.31) and (4.32) that $(2c - \epsilon_2) + (n+1)a > 0$ and

$$(2a + 2b - \epsilon_1) < 0. \tag{4.36}$$

By (4.35), $(2c - \epsilon_2) + 2(n+1)a$ and $(2c - \epsilon_2)$ have the same sign, and so must be both positive. In particular, $c \geq 1$. By (4.34),

$$(n+1)a(2b - \epsilon_1) \geq -[2a + (2b - \epsilon_1)](2c - \epsilon_2) - (\epsilon_1 + 2)(2 - \epsilon_2). \tag{4.37}$$

In the following, we consider the cases $\epsilon_1 = 0$ and $\epsilon_1 = 1$ separately.

Assume $\epsilon_1 = 0$. Using (4.37) and (4.36), we obtain $(n+1)a(2b) \geq 0$. Together with (4.36) one more time, this implies either $a < 0$ and $b \leq 0$, or $a = 0$ and $b < 0$. If $a < 0$ and $b \leq 0$, then we see from (4.34) that

$$\begin{aligned}
-(2-\epsilon_2) &\leq (a+b)(2c-\epsilon_2) + (n+1)ab \\
&= a(2c-\epsilon_2) + [(2c-\epsilon_2) + (n+1)a]b \\
&\leq a(2c-\epsilon_2) \leq -(2c-\epsilon_2) \\
&\leq -(2-\epsilon_2).
\end{aligned}$$

So $a = -1$ and $c = 1$, contradicting to $(2c-\epsilon_2) + (n+1)a \geq 1$ and $n \geq 2$. If $a = 0$ and $b < 0$, then $b(2c-\epsilon_2) \geq -(2-\epsilon_2)$ by (4.34). Since $b(2c-\epsilon_2) \leq -(2c-\epsilon_2) \leq -(2-\epsilon_2)$, we must have $b = -1$ and $c = 1$. By (4.30), we obtain

$$\begin{aligned}
c(E) &= c(\mathcal{O}_Y(0,-1,1) \otimes I_{Z_1}) \cdot c(\mathcal{O}_Y(0,1,\epsilon_2-1) \otimes I_{Z_2}) \\
&= \frac{c(\mathcal{O}_Y(0,-1,1))}{c(\mathcal{O}_{Z_1}(0,-1,1))} \cdot \frac{c(\mathcal{O}_Y(0,1,\epsilon_2-1))}{c(\mathcal{O}_{Z_2}(0,1,\epsilon_2-1))}.
\end{aligned}$$

Since $c(E) = \mathbf{c}_m = (1 - m[y_0]) \cdot c(\mathcal{O}_Y(0,-1,1)) \cdot c(\mathcal{O}_Y(0,1,\epsilon_2-1))$, we get

$$c(\mathcal{O}_{Z_1}(0,-1,1)) \cdot c(\mathcal{O}_{Z_2}(0,1,\epsilon_2-1)) = \frac{1}{1-m[y_0]} = 1 + m[y_0]. \tag{4.38}$$

Thus Z_1 and Z_2 are 0-dimensional. Hence (4.30) becomes (4.29), and $m = 2\big(\ell(Z_1) + \ell(Z_2)\big)$. Note from (4.32) that $r \geq n(2-\epsilon_2)/2$. Moreover, if $r > n(2-\epsilon_2)/2$, then (4.29) does not split since $\mathcal{O}_Y(0,1,\epsilon_2-1) \otimes I_{Z_2}$ would destabilize E with respect to L_r^Y, contradicting to the assumption $E \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m)$.

Next, assume $\epsilon_1 = 1$. We see from (4.37) and (4.36) that

$$(n+1)a(2b-1) \geq -(2a+2b-1)(2c-\epsilon_2) - 3(2-\epsilon_2) \geq 1 - 6 = -5.$$

So $a(2b-1) \geq -1$ since $n \geq 2$. If $a(2b-1) = -1$, then we see from $2a + (2b-1) < 0$ that $a = -1$ and $b = 1$. By (4.37) again, we obtain $(2c-\epsilon_2) \leq 3(2-\epsilon_2) - (n+1) \leq (5-n)$ contradicting to $(2c-\epsilon_2) + 2(n+1)a \geq 1$ and $n \geq 2$. Therefore, we must have $a(2b-1) \geq 0$. Since $2a + (2b-1) < 0$, we conclude that either $a < 0$ and $(2b-1) \leq 0$, or $a = 0$ and $(2b-1) < 0$. As in the previous paragraph, we see that $a = 0$, $b = 0$ or -1. If $b = 0$, then we obtain from (4.32) that $r \geq n(2c-\epsilon_2) \geq n(2-\epsilon_2)$ contradicting to our assumption that $r < n(2-\epsilon_2)$. Therefore, $b = -1$. As in the previous paragraph again, we verify that $c = 1$, Z_1 and Z_2 are 0-dimensional, (4.30) becomes (4.29), $m = 2\big(\ell(Z_1) + \ell(Z_2)\big)$, and $r \ge n(2-\epsilon_2)/3$. Moreover, if $r > n(2-\epsilon_2)/3$, then (4.29) does not split. □

Remark 4.9. Let $n \ge 2$, and $\epsilon_1, \epsilon_2 = 0, 1$. By Lemma 4.8, the moduli space $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m)$ is empty if $0 < r < n(2-\epsilon_2)/(2+\epsilon_1)$. When

$$n(2-\epsilon_2)/(2+\epsilon_1) \le r < n(2-\epsilon_2)/\epsilon_1,$$

the moduli space $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m)$ is empty if $m < 0$ or m is odd.

Remark 4.10. As evidenced in [25], the critical value $r = n(2-\epsilon_2)/(2+\epsilon_1)$ is equivalent to saying that L_r^Y lies on a certain wall in the ample cone of Y.

Remark 4.11. Not every sheaf E sitting in a non-splitting extension (4.29) is contained in the moduli space $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m)$. The reason is that we might have

$$E^{**} \cong \mathcal{O}_Y(0,-1,1) \oplus \mathcal{O}_Y(0,\epsilon_1+1,\epsilon_2-1),$$

and then E will have a subsheaf $\mathcal{O}_Y(0,\epsilon_1+1,\epsilon_2-1) \otimes I_Z$ (for some 0-dimensional closed subscheme Z) destablizing E with respect to L_r^Y.

Lemma 4.12. *Let $n \ge 2$. Let $\epsilon_1, \epsilon_2 = 0, 1$, and $m \ge 0$. Assume that*

$$n(2-\epsilon_2)/(2+\epsilon_1) < r < n(2-\epsilon_2)/\epsilon_1.$$

(i) *If $E \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m})$, then $E^{**} \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$, $E \in \mathrm{Quot}^m_{E^{**}}$, and E is L_r^Y-stable;*

(ii) *We have $E \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m})$ if and only if $E \in \mathrm{Quot}^m_{E_0}$ for some rank-2 sheaf $E_0 \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$. Moreover, in this case, $E_0 \cong E^{**}$.*

Proof. Note that (ii) follows from (i) and (4.28). For (i), we see from Lemma 4.8 (i) that the double dual E^{**} sits in an exact sequence:

$$0 \to \mathcal{O}_Y(0,-1,1) \to E^{**} \to \mathcal{O}_Y(0,\epsilon_1+1,\epsilon_2-1) \to 0. \tag{4.39}$$

In addition, we have the canonical exact sequence:

$$0 \to E \to E^{**} \to Q \to 0 \tag{4.40}$$

where Q is supported at finitely many points. The exact sequence (4.39) does not split since otherwise, we see from (4.40) that E would have a subsheaf

$$\mathcal{O}_Y(0, \epsilon_1 + 1, \epsilon_2 - 1) \otimes I_Z$$

(for some 0-dimensional closed subscheme Z) destabilizing E with respect to L_r^Y. By Lemma 4.7 (ii), $E^{**} \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$. Calculating the Chern classes from (4.40), we conclude that $h^0(Y, Q) = m$. Therefore, $E \in \mathrm{Quot}^m_{E^{**}}$. By Lemma 4.7 (ii) again, E^{**} is (slope) L_r^Y-stable. Hence, the sheaf E is L_r^Y-stable as well. □

By Lemma 4.7 (ii) and a standard construction (see [16]), there exists a universal vector bundle $\mathcal{E}_0$ over $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times Y$ which sits in the exact sequence:

$$0 \to \rho_2^* \mathcal{O}_Y(0, -1, 1) \to \mathcal{E}_0 \to \rho_2^* \mathcal{O}_Y(0, \epsilon_1 + 1, \epsilon_2 - 1) \otimes \rho_1^* \mathcal{L} \to 0 \tag{4.41}$$

where ρ_1 and ρ_2 are the two natural projections of $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times Y$, and $\mathcal{L}$ stands for the line bundle $\mathcal{O}_{\mathbb{P}^k}(-1)$ on $\mathbb{P}^k \cong \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$. For simplicity, put

$$\mathrm{Quot}^m_{\mathcal{E}_0/} := \mathrm{Quot}^m_{\mathcal{E}_0/\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times Y/\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)}. \tag{4.42}$$

Note that the fiber of the natural morphism $\mathrm{Quot}^m_{\mathcal{E}_0/} \to \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$ at a point $E_0 \in \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$ is canonically identified with the Quot-scheme $\mathrm{Quot}^m_{E_0}$.

Proposition 4.13. *Let $n \geq 2$. Let $\epsilon_1, \epsilon_2 = 0, 1$, and $m \geq 0$. Assume that*

$$n(2 - \epsilon_2)/(2 + \epsilon_1) < r < n(2 - \epsilon_2)/\epsilon_1.$$

Then the moduli space $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m})$ is isomorphic to the Quot-scheme $\mathrm{Quot}^m_{\mathcal{E}_0/}$.

Proof. Follows immediately from Lemma 4.12 (ii), the universal property of Quot-schemes, and an argument similar to the proof of Lemma 4.1. □

Proposition 4.14. *Let $n \geq 2$. Let $\epsilon_1, \epsilon_2 = 0, 1$, $m \geq 0$, and*

$$k = (1 + \epsilon_1) \binom{n + 2 - \epsilon_2}{n} - 1.$$

Assume that $n(2 - \epsilon_2)/(2 + \epsilon_1) < r < n(2 - \epsilon_2)/\epsilon_1$. Then,

$$\sum_{m \in \mathbb{Z}} \chi\big(\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m)\big)\, q^m = (k + 1) \cdot \left(\sum_{m=0}^{+\infty} P_n(m) q^{2m} \right)^{2 \cdot \chi(Y)}. \tag{4.43}$$

Proof. Note from Remark 4.9 that $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m) = \emptyset$ if $m < 0$ or m is odd. So

$$\sum_{m\in\mathbb{Z}} \chi\big(\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m)\big)\, q^m = \sum_{m=0}^{+\infty} \chi\big(\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m})\big)\, q^{2m}. \tag{4.44}$$

By Proposition 4.13, $e\big(\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m}); s,t\big) = e\big(\operatorname{Quot}^m_{\mathcal{E}_0/}; s,t\big)$. Since the rank-2 universal sheaf $\mathcal{E}_0$ over $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times Y$ is locally free, we have

$$e\big(\operatorname{Quot}^m_{\mathcal{E}_0/}; s,t\big) = e\left(\operatorname{Quot}^m_{\mathcal{O}^{\oplus 2}_{\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)\times Y}/}; s,t\right)$$

by (3.8). Note that there exists a canonical isomorphism

$$\operatorname{Quot}^m_{\mathcal{O}^{\oplus 2}_{\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)\times Y}/} \cong \operatorname{Quot}^m_{\mathcal{O}_Y^{\oplus 2}} \times \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$$

by the universal property of Quot-schemes. It follows from (3.5) that

$$\begin{aligned} e\big(\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m}); s,t\big) &= e\left(\operatorname{Quot}^m_{\mathcal{O}_Y^{\oplus 2}} \times \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0); s,t\right) \\ &= e\left(\operatorname{Quot}^m_{\mathcal{O}_Y^{\oplus 2}}; s,t\right) \cdot e\left(\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0); s,t\right). \end{aligned}$$

Since $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \cong \mathbb{P}^k$ and $e(\mathbb{P}^k; s,t) = (1-(st)^{k+1})/(1-st)$, we get

$$e\big(\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_{2m}); s,t\big) = \frac{1-(st)^{k+1}}{1-st} \cdot e\left(\operatorname{Quot}^m_{\mathcal{O}_Y^{\oplus 2}}; s,t\right). \tag{4.45}$$

Setting $s = t = 1$, we see that (4.43) follows from (4.44), (3.3) and Theorem 3.4. □

Lemma 4.15. *Let $\mathcal{E}_0$ be the universal sheaf over $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times Y$ sitting in the exact sequence (4.41). Fix a point $y \in Y$. Let U be an open affine neighborhood of y such that $\mathcal{O}_Y(0,-1,1)|_U \cong \mathcal{O}_U$ and $\mathcal{O}_Y(0,\epsilon_1+1,\epsilon_2-1)|_U \cong \mathcal{O}_U$. Then,*

$$\mathcal{E}_0|_{\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)\times U} \cong \rho_1^*\left(\mathcal{O}_{\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)} \oplus \mathcal{L}\right)$$

where $\rho_1 : \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times U \to \overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)$ denotes the first projection.

Proof. Restricting the exact sequence (4.41) to $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times U$ yields

$$0 \to \mathcal{O} \to \mathcal{E}_0|_{\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)\times U} \to \rho_1^*\mathcal{L} \to 0$$

where $\mathcal{O}$ denotes $\mathcal{O}_{\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0)\times U}$. So it suffices to prove $\mathrm{Ext}^1(\rho_1^*\mathcal{L}, \mathcal{O}) = 0$, i.e.,

$$H^1(\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \times U, \rho_1^*\mathcal{L}^{-1}) = 0. \tag{4.46}$$

Recall that $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_0) \cong \mathbb{P}^k$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^k}(-1)$. Let $\tilde{\rho}_2 : \mathbb{P}^k \times U \to U$ be the second projection. By the Leray spectral sequence, we obtain an exact sequence

$$\begin{aligned} 0 &\to H^1(U, (\tilde{\rho}_2)_*\rho_1^*\mathcal{O}_{\mathbb{P}^k}(1)) \to H^1(\mathbb{P}^k \times U, \rho_1^*\mathcal{O}_{\mathbb{P}^k}(1)) \\ &\to H^0(U, R^1(\tilde{\rho}_2)_*\rho_1^*\mathcal{O}_{\mathbb{P}^k}(1)). \end{aligned}$$

Since U is affine, $H^1(U, (\tilde{\rho}_2)_*\rho_1^*\mathcal{O}_{\mathbb{P}^k}(1)) = 0$. Since $R^1(\tilde{\rho}_2)_*\rho_1^*\mathcal{O}_{\mathbb{P}^k}(1) = 0$, we conclude that $H^1(\mathbb{P}^k \times U, \rho_1^*\mathcal{O}_{\mathbb{P}^k}(1)) = 0$. This verifies (4.46). ☐

Now let $n = 2$. Then Y is a smooth Calabi–Yau 3-fold with $H_1(Y, \mathbb{Z}) = 0$. For the fixed point $y_0 \in Y$, let Y_0 be an open affine neighborhood of y_0 such that both $\mathcal{O}_Y(0, -1, 1)|_{Y_0}$ and $\mathcal{O}_Y(0, \epsilon_1 + 1, \epsilon_2 - 1)|_{Y_0}$ are trivial. Define

$$\begin{aligned} \widetilde{F}_m &= \{E \in \mathrm{Quot}^m_{\mathcal{E}_0/} \mid E^{**}/E \text{ is supported at } y_0\}, \\ \mathrm{Quot}^m_{\mathcal{E}_0/}(Y_0) &= \{E \in \mathrm{Quot}^m_{\mathcal{E}_0/} \mid E^{**}/E \text{ is supported in } Y_0\}. \end{aligned}$$

Then $\widetilde{F}_m \subset \mathrm{Quot}^m_{\mathcal{E}_0/}(Y_0) \cong \mathrm{Quot}^m_{\rho_1^*(\mathcal{O}_{\mathbb{P}^k}\oplus\mathcal{O}_{\mathbb{P}^k}(-1))/\mathbb{P}^k\times Y_0/\mathbb{P}^k}$ by Lemma 4.15. Consider the embedding $\mathbb{T} = \mathbb{C}^* \hookrightarrow \mathbb{T}_2 = (\mathbb{C}^*)^2 \subset \mathrm{Aut}\big(\mathcal{O}_{\mathbb{P}^k} \oplus \mathcal{O}_{\mathbb{P}^k}(-1)\big)$ via $t \mapsto (1, t)$ and the induced $\mathbb{T}$-action on $\mathrm{Quot}^m_{\rho_1^*(\mathcal{O}_{\mathbb{P}^k}\oplus\mathcal{O}_{\mathbb{P}^k}(-1))/\mathbb{P}^k\times Y_0/\mathbb{P}^k}$. As in Lemma 3.2 and (4.15),

$$\Big(\mathrm{Quot}^m_{\rho_1^*(\mathcal{O}_{\mathbb{P}^k}\oplus\mathcal{O}_{\mathbb{P}^k}(-1))/\mathbb{P}^k\times Y_0/\mathbb{P}^k}\Big)^{\mathbb{T}} \cong \coprod_{i=0}^{m} \mathbb{P}^k \times \mathrm{Hilb}^i(Y_0) \times \mathrm{Hilb}^{m-i}(Y_0). \tag{4.47}$$

An argument similar to the proof of Lemma 4.5 proves that

$$\chi(\widetilde{F}_m, \widetilde{\nu}_m) = (-1)^k \cdot \chi(\widetilde{F}_m) \tag{4.48}$$

where $\widetilde{\nu}_m$ is the restriction of Behrend's function $\nu_{\mathrm{Quot}^m_{\mathcal{E}_0/}}$ to $\widetilde{F}_m$.

Theorem 4.16. *Let $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ be a generic smooth Calabi–Yau hypersurface. Let $\epsilon_1, \epsilon_2 = 0, 1$, and $k = (1+\epsilon_1)(4-\epsilon_2)(3-\epsilon_2)/2 - 1$. Let $\pi : Y \to \mathbb{P}^1 \times \mathbb{P}^2$ be the restriction to Y of the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ to the product of the last two factors. Fix a point $y_0 \in Y$, and define in $A^*(Y)$ the class:*

$$\mathbf{c}_m = -m[y_0] + \big(1 + \pi^*(-1,1)\big) \cdot \big(1 + \pi^*(\epsilon_1 + 1, \epsilon_2 - 1)\big).$$

(i) *If $0 < r < 2(2-\epsilon_2)/(2+\epsilon_1)$, then $\lambda(L_r^Y, \mathbf{c}_m) = 0$ for all $m \in \mathbb{Z}$.*

(ii) *If $2(2-\epsilon_2)/(2+\epsilon_1) < r < 2(2-\epsilon_2)/\epsilon_1$, then*

$$\sum_{m\in\mathbb{Z}} \lambda(L_r^Y, \mathbf{c}_m) q^m = (-1)^k \cdot (k+1) \cdot M(q^2)^{2\chi(Y)}. \tag{4.49}$$

Proof. (i) In this case, $\overline{\mathfrak{M}}_{L_r^Y}(\mathbf{c}_m) = \emptyset$ by Remark 4.9. Hence $\lambda(L_r^Y, \mathbf{c}_m) = 0$.

(ii) By Remark 4.9, Proposition 4.13 and (4.2), we have

$$\sum_{m\in\mathbb{Z}} \lambda(L_r^Y, \mathbf{c}_m) q^m = \sum_{m=0}^{+\infty} \lambda(L_r^Y, \mathbf{c}_{2m}) q^{2m} = \sum_{m=0}^{+\infty} \widetilde{\chi}\big(\operatorname{Quot}^m_{\mathcal{E}_0/}\big)\, q^{2m}. \tag{4.50}$$

By Lemma 4.15, we can adopt the proof of the Theorem 4.11 in [4]. So

$$\widetilde{\chi}\big(\operatorname{Quot}^m_{\mathcal{E}_0/}\big) = \sum_{\alpha\vdash n} |G_\alpha| \cdot \chi\big(Y_0^{\ell(\alpha)}\big) \cdot \prod_i \chi(\widetilde{F}_{\alpha_i}, \widetilde{\nu}_{\alpha_i}),$$
$$\chi\big(\operatorname{Quot}^m_{\mathcal{E}_0/}\big) = \sum_{\alpha\vdash n} |G_\alpha| \cdot \chi\big(Y_0^{\ell(\alpha)}\big) \cdot \prod_i \chi(\widetilde{F}_{\alpha_i}).$$

By (4.48), $\widetilde{\chi}\big(\operatorname{Quot}^m_{\mathcal{E}_0/}\big) = (-1)^k \cdot \chi\big(\operatorname{Quot}^m_{\mathcal{E}_0/}\big)$. By (4.50) and Proposition 4.14,

$$\sum_{m\in\mathbb{Z}} \lambda(L_r^Y, \mathbf{c}_m) q^m = (-1)^k \cdot (k+1) \cdot M(q^2)^{2\chi(Y)}. \tag{4.51}$$

□

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Department of Mathematics
HKUST
Clear Water Bay
Kowloon
Hong Kong
Department of Mathematics
University of Missouri
Columbia
MO 65211
USA
E-mail address: mawpli@ust.hk
E-mail address: qinz@missouri.edu

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