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The mean value theorem and basic properties of the obstacle problem for divergence form elliptic operators

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In 1963, Littman *et al.* proved a mean value theorem for elliptic operators in divergence form with bounded measurable coefficients. In the Fermi lectures in 1998, Caffarelli stated a much simpler mean value theorem for the same situation, but did not include the details of the proof. We show all of the nontrivial details needed to prove the formula stated by Caffarelli, and in the course of showing these details we establish some of the basic facts about the obstacle problem for general elliptic divergence form operators, in particular, we show a basic quadratic nondegeneracy property.

1. Introduction

Based on the ubiquitous nature of the mean value theorem in problems involving the Laplacian, it is clear that an analogous formula for a general divergence form elliptic operator would necessarily be very useful. In [14], Littman *et al.* stated a mean value theorem for a general divergence form operator, L. If μ is a nonnegative measure on Ω and u is the solution to

$$\begin{aligned} Lu = \mu \quad & \text{in} \;\; \Omega \\ 0 \quad & \text{on} \;\; \partial\Omega, \end{aligned} \tag{1.1}$$

and $G(x,y)$ is the Green's function for L on Ω, then Equation (8.3) in their paper states that $u(y)$ is equal to

$$\lim_{a\to\infty} \frac{1}{2a} \int_{a \le G \le 3a} u(x) a^{ij}(x) D_{x_i} G(x,y) D_{x_j} G(x,y) \, dx \tag{1.2}$$

almost everywhere, and this limit is nondecreasing. The pointwise definition of u given by this equation is necessarily lower semicontinuous. There are a few reasons why this formula is not as nice as the basic mean value formulas for Laplace's equation. First, it is a weighted average and not a simple average. Second, it is not an average over a ball or something which is even homeomorphic to a ball. Third, it requires knowledge of derivatives of the Green's function.

A simpler formula was stated by Caffarelli in [4, 5]. That formula provides an increasing family of sets, $D_R(x_0)$, which are each comparable to B_R and such that for a supersolution to $Lu = 0$ the average

$$\frac{1}{|D_R(x_0)|} \int_{D_R(x_0)} u(x)\, dx$$

is nondecreasing as $R \to 0$. On the other hand, Caffarelli did not provide any details about showing the existence of an important test function used in the proof of this result, and showing the existence of this function turns out to be nontrivial. This paper grew out of an effort to prove rigorously all of the details of the mean value theorem that Caffarelli asserted in [4, 5].

In order to get the existence of the key test function, one must be able to solve the variational inequality or obstacle type problem:

$$D_i a^{ij} D_j V_R = \frac{1}{R^n} \chi_{\{V_R > 0\}} - \delta_{x_0}, \tag{1.3}$$

where δ_{x_0} denotes the Dirac mass at x_0. In [5], the book by Kinderlehrer and Stampacchia is cited (see [12]) for the mean value theorem. Although many of the techniques in that book are used in the current work, an exact theorem to give the existence of a solution to Equation (1.3) was not found in [12] by either author of this paper or by Kinderlehrer [11]. The authors of this work were also unable to find a suitable theorem in other standard sources for the obstacle problem. (See [7, 16].) Indeed, we believe that without the nondegeneracy theorem stated in this paper, there is a gap in the proof.

To understand the difficulty inherent in proving a nondegeneracy theorem in the divergence form case, it helps to review the proof of nondegeneracy for the Laplacian and/or in the nondivergence form case. (See [1, 3, 4].) In those cases, good use is made of the barrier function $|x - x_0|^2$. The relevant properties are that this function is nonnegative and vanishing at x_0, it grows quadratically, and most of all, for a nondivergence form elliptic operator L, there exists a constant $\gamma > 0$, such that $L(|x - x_0|^2) \geq \gamma$. On the other hand, when L is a divergence form operator with only bounded measurable coefficients, it is clear that $L(|x - x_0|^2)$ does not make sense in general.

Now we give an outline of the paper. In Section 2, we almost get the existence of a solution to a partial differential equation (PDE) formulation of the obstacle problem. In Section 3, we first show the basic quadratic regularity and nondegeneracy result for our functions which are only “almost” solutions, and then we use these results to show that our “almost” solutions are true solutions. In Section 4, we get existence and uniqueness of solutions of a variational formulation of the obstacle problem, and then show that the two formulations are equivalent. In Section 5, we show the existence of a function which we then use in Section 6 to prove the mean value theorem stated in [4, 5], and give some corollaries.

Throughout the paper, we assume that $a^{ij}(x)$ are bounded, symmetric and uniformly elliptic, and we define the divergence form elliptic operator

$$L := D_j\, a^{ij}(x) D_i, \tag{1.4}$$

or, in other words, for a function $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ we say “$Lu = f$ in Ω” if for any $\phi \in W_0^{1,2}(\Omega)$ we have

$$-\int_\Omega a^{ij}(x) D_i u D_j \phi = \int_\Omega f\phi. \tag{1.5}$$

(Notice that with our sign conventions we can have $L = \Delta$ but not $L = -\Delta$.) With our operator L we let $G(x, y)$ denote the Green’s function for all of $\mathbb{R}^n$ and observe that the existence of G is guaranteed by the work of Littman *et al*. (See [14].)

The results in this paper are used in a forthcoming sequel where we establish some weak regularity results for the free boundary in the case where the coefficients are assumed to belong to the space of vanishing mean oscillation. The methods of that paper rely on stability, flatness and compactness arguments. (See [2].) In the case where the coefficients are assumed to be Lipschitz continuous, recent work of Focardi *et al.* establishes stronger regularity results of the free boundary. The methods of that work have a more “energetic” flavor: they generalize some important monotonicity formulas, and use these formulas along with the epiperimetric inequality due to Weiss and a generalization of Rellich and Něcas’ identity to prove their regularity results. (See [6].)

2. The PDE obstacle problem with a gap

We wish to establish the existence of weak solutions to an obstacle type problem which we now describe. We assume that we are given

$$f, a^{ij} \in L^\infty(B_1) \quad \text{and} \quad g \in W^{1,2}(B_1) \cap L^\infty(B_1), \tag{2.1}$$

which satisfy

$$
(2.2)\qquad
\begin{aligned}
&0 < \bar{\lambda} \leq f \leq \bar{\Lambda},\\
&a^{ij} \equiv a^{ji},\\
&0 < \lambda|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \ \text{ for all } \xi \in \mathbb{R}^n,\ \xi \neq 0, \ \text{ and}\\
&g \not\equiv 0 \text{ on } \partial B_1,\ g \geq 0.
\end{aligned}
$$

We want to find a nonnegative function $w \in W^{1,2}(B_1)$, which is a weak solution of

$$
(2.3)\qquad
\begin{aligned}
Lw &= \chi_{\{w>0\}} f \quad \text{in } B_1,\\
w &= g \quad \text{on } \partial B_1.
\end{aligned}
$$

In this section, we will content ourselves to produce a nonnegative function $w \in W^{1,2}(B_1)$ which is a weak solution of

$$
(2.4)\qquad
\begin{aligned}
Lw &= h \quad \text{in } B_1,\\
w &= g \quad \text{on } \partial B_1,
\end{aligned}
$$

where we know that h is a nonnegative function satisfying

$$
(2.5)\qquad
\begin{aligned}
h(x) &= 0, && \text{for } x \in \{w = 0\}^{\mathrm{o}},\\
h(x) &= f(x), && \text{for } x \in \{w > 0\}^{\mathrm{o}},\\
h(x) &\leq \bar{\Lambda}, && \text{for } x \in \partial\{w = 0\} \cup \partial\{w > 0\},
\end{aligned}
$$

where for any set $S \subset \mathbb{R}^n$, we use S^{o} to denote its interior. Thus h agrees with $\chi_{\{w>0\}} f$ everywhere except possibly the free boundary. (The "gap" mentioned in the title to this section is the fact that we would not know that $h = \chi_{\{w>0\}} f$ *a.e.* until we show that the free boundary (that is $\partial\{w = 0\} \cup \partial\{w > 0\}$) has measure zero.) We will show such a w exists by obtaining it as a limit of functions w_s which are solutions to the semilinear PDE:

$$
(2.6)\qquad
\begin{aligned}
Lw &= \Phi_s(w) f \quad \text{in } B_1\\
w &= g \quad \text{on } \partial B_1,
\end{aligned}
$$

where for $s > 0$, $\Phi_s(x) := \Phi_1(x/s)$ and $\Phi_1(x)$ is a function which satisfies

(1) $\Phi_1 \in C^\infty(\mathbb{R})$,

(2) $0 \leq \Phi_1 \leq 1$,

(3) $\Phi_1 \equiv 0$ for $x < 0$, $\quad \Phi_1 \equiv 1$ for $x > 1$ and

(4) $\Phi_1'(x) \geq 0$ for all x.

The function Φ_s has a derivative which is supported in the interval $[0, s]$ and notice that for a fixed x, $\Phi_s(x)$ is a nonincreasing function of s.

If we let H denote the standard Heaviside function, but make the convention that $H(0) := 0$, then we can rewrite the PDE in Equation (2.4) as

$$Lw = H(w)f$$

to see that it is formally the limit of the PDEs in Equation (2.6). We also define

$$\Phi_{-s}(x) := \Phi_s(x + s),$$

so that we will be able to "surround" our solutions to our obstacle problem with solutions to our semilinear PDEs.

The following theorem seems like it should be stated somewhere, but without further smoothness assumptions on the a^{ij} we could not find it within [8], [10] or [13]. The proof is a fairly standard application of the method of continuity, so we will only sketch it.

Theorem 2.1 (Existence of solutions to a semilinear PDE)**.** *Given the assumptions above, for any $s \in [-1, 1] \setminus \{0\}$, there exists a w_s which satisfies Equation (2.6).*

Proof. We provide only a sketch. Fix $s \in [-1, 1] \setminus \{0\}$. Let T be the set of $t \in [0, 1]$ such that there is a unique solution to the problem

$$\begin{aligned} Lw &= t\Phi_s(w)f \quad \text{in } B_1 \\ w &= g \quad \text{on } \partial B_1. \end{aligned} \tag{2.7}$$

We know immediately that T is nonempty by observing that Theorem 8.3 of [8] shows us that $0 \in T$. Now we need to show that T is both open and closed.

As in [14] we let $\tau^{1,2}$ denote the Hilbert space formed as the quotient space $W^{1,2}(B_1)/W_0^{1,2}(B_1)$ and then we define the Hilbert space

$$H := W_0^{1,2}(B_1)^* \oplus \tau^{1,2}, \tag{2.8}$$

where $W_0^{1,2}(B_1)^*$ denotes the dual space to $W_0^{1,2}(B_1)$. Next we define the nonlinear operator $L^t : W^{1,2}(B_1) \to H$. For a function $w \in W^{1,2}(B_1)$, we set

$$L^t(w) = \ell^t(w) \oplus \mathcal{R}(w), \tag{2.9}$$

where $\mathcal{R}(w)$ is simply the restriction from w to its boundary values in $\tau^{1,2}$, and for any $\phi \in W_0^{1,2}(B_1)$ we let

$$[\ell^t(w)](\phi) := \int_{B_1} \left(a^{ij}(x) D_i w D_j \phi + t\Phi_s(w) f \phi \right) \, dx. \tag{2.10}$$

In order to show that T is open, we need the implicit function theorem in Hilbert space. In order to use that theorem, we need to show that the Gateaux derivative of L^t is invertible. The relevant part of that computation is simply the observation that the Gateaux derivative of ℓ^t, which we denote by $D\ell^t$, is invertible. Letting $v \in W^{1,2}(B_1)$ we have

$$\left[[D\ell^t(w)](\phi) \right](v) = \int_{B_1} \left(a^{ij}(x) D_i v D_j \phi + t\Phi_s'(w) f v \phi \right) \, dx. \tag{2.11}$$

The function $d(x) := t\Phi_s'(w(x)) f(x)$ is a nonnegative bounded function of x and so we can apply Theorem 8.3 of [8] again in order to verify that L^t is invertible.

In order to show that T is closed, we let $t_n \to \tilde{t}$, and assume that $\{t_n\} \subset T$. We let w_n solve

$$\begin{aligned} Lw &= t_n \Phi_s(w) f \quad \text{in } B_1 \\ w &= g \quad \text{on } \partial B_1, \end{aligned} \tag{2.12}$$

and observe that the right-hand side of our PDE is bounded by $\bar{\Lambda}$. Knowing this information we can use Corollary 8.7 of [8] to conclude $||w_n||_{W^{1,2}(B_1)} \leq C$, and we can use the theorems of De Giorgi, Nash and Moser to conclude that for any $r < 1$, we have $||w_n||_{C^\alpha(\overline{B_r})} \leq C$. Elementary functional analysis allows us to conclude that a subsequence of our w_n will converge weakly in $W^{1,2}(B_r)$ and strongly in $C^{\alpha/2}(\overline{B_r})$ to a function $\tilde{w}$. Using a simple diagonalization argument, we can show that $\tilde{w}$ satisfies

$$\begin{aligned} Lw &= \tilde{t} \Phi_s(w) f \quad \text{in } B_1 \\ w &= g \quad \text{on } \partial B_1, \end{aligned} \tag{2.13}$$

and this fact show us that $\tilde{t} \in T$. □

We will also need the following comparison results.

Proposition 2.2 (Basic comparisons)**.** *Under the assumptions of the previous theorem and letting w_s denote the solution to Equation (2.6), we have the following comparison results:*

(1) $s > 0 \;\Rightarrow w_s \geq 0$;

(2) $s < 0 \;\Rightarrow w_s \geq s$;

(3) $t < s \;\Rightarrow w_t \geq w_s$;

(4) $t < 0 < s \;\Rightarrow w_s \leq w_t + s - t$; *and*

(5) *For a fixed* $s \in [-1, 1] \setminus \{0\}$ *the solution,* w_s *is unique.*

Proof. All five statements are proved in very similar ways, and their proofs are fairly standard, but for the convenience of the reader, we will prove the fourth statement. We assume that it is false, and we let

$$\Omega^- := \{w_s - w_t > s - t\}. \tag{2.14}$$

Obviously $w_s - w_t = s - t$ on $\partial\Omega^-$. Next, observe that by the second statement we know that Ω^- is a subset of $\{w_s > s\}$. Thus, within Ω^- we have $L(w_s - w_t) = 1 - \Phi_t(w_t) \geq 0$ and so if Ω^- is not empty, then we contradict the weak maximum principle. □

We are now ready to give our existence theorem for our "problem with the gap."

Theorem 2.3 (Existence theorem)**.** *Given the assumptions above, there exists a pair* (w, h)*, such that* $w \geq 0$ *satisfies Equation (2.4) with an* $h \geq 0$*, which satisfies Equation (2.5).*

Proof. Using the last proposition, we can find a sequence $s_n \to 0$, and a function w such that (with w_n used as an abbreviation for w_{s_n}) we have strong convergence of the w_n to w in $C^\alpha(\overline{B_r})$ for any $r < 1$ and weak convergence of the w_n to w in $W^{1,2}(B_1)$. Elementary functional analysis allows us to conclude that the functions $\chi_{\{w_n>0\}} f$ converge weak-$*$ in $L^\infty(B_1)$ to a function h which automatically satisfies $0 \leq h \leq \bar{\Lambda}$. By looking at the equations satisfied by the w_n's and using the convergences, it then follows very easily that the function w satisfies Equation(2.4) but it remains to verify that the function h is equal to $\chi_{\{w>0\}} f$ away from the free boundary.

Since the limit is continuous, the set $\{w > 0\}$ is already open, and by the uniform convergence of the w_n's we can say that on any set of the form $\{w > \gamma\}$ (where $\gamma > 0$) we will have $\Phi_{s_n}(w_n) \equiv 1$ once n is sufficiently large. Thus, we must have $h = f$ on this set. On the other hand, in the interior of the set $\{w = 0\}$ we have $\nabla w \equiv 0$, and so it is clear that in that set $h \equiv 0$ *a.e.* □

3. Regularity, nondegeneracy and closing the gap

Now, we begin with a pair (w, h) like the pair given by Theorem (2.3), except that we do not insist that it have any particular boundary data on ∂B_1. In other words, in this section w will always satisfy

$$L(w) = h \quad \text{in } B_1 \tag{3.1}$$

for a function h which satisfies Equation (2.5). In addition, we will assume Equations (2.1) and (2.2) hold. By the end of this section, we will know that the set $\partial\{w = 0\}$ has Lebesgue measure zero and so w actually satisfies:

$$L(w) = \chi_{\{w>0\}} f \quad \text{in } B_1, \tag{3.2}$$

which will allow us to forget about h afterward. Before we eliminate h, we have two main results: first, w enjoys a parabolic bound from above at any free boundary point, and second, w has a quadratic nondegenerate growth from such points. It turns out that these properties are already enough to ensure that the free boundary has measure zero.

Lemma 3.1. *Assume that w satisfies everything described above, but in addition, assume that $w(0) = 0$. Then there exists a $\tilde{C}$, such that*

$$\| w \|_{L^\infty(B_{1/2})} \leq \tilde{C}. \tag{3.3}$$

Proof. Let u solve the following PDE:

$$\begin{cases} Lu = h & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{3.4}$$

Then Theorem 8.16 of [8] gives

$$\| u \|_{L^\infty(B_1)} \leq C_1. \tag{3.5}$$

Now, consider the solution to

$$\begin{cases} Lv = 0 & \text{in } B_1 \\ v = w & \text{on } \partial B_1. \end{cases} \tag{3.6}$$

Notice that $u(x) + v(x) = w(x)$, and in particular $0 = w(0) = u(0) + v(0)$. Then by the Weak Maximum Principle and the Harnack Inequality, we have

$$\sup_{B_{1/2}} |v| = \sup_{B_{1/2}} v \leq C_2 \inf_{B_{1/2}} v \leq C_2 v(0) \leq C_2(-u(0)) \leq C_2 \cdot C_1. \tag{3.7}$$

Therefore

$$\| w \|_{L^\infty(B_{1/2})} \leq C \tag{3.8}$$

□

Theorem 3.2 (Optimal regularity). *If* $0 \in \partial\{w > 0\}$, *then for any* $x \in B_{1/2}$ *we have*

$$w(x) \leq 4\tilde{C}|x|^2, \tag{3.9}$$

where $\tilde{C}$ *is the same constant as in the statement of Lemma (3.1).*

Proof. By the previous lemma, we know $\| w \|_{L^\infty(B_{1/2})} \leq \tilde{C}$. Notice that for any $\gamma > 1$,

$$u_\gamma(x) := \gamma^2 w\left(\frac{x}{\gamma}\right) \tag{3.10}$$

is also a solution to the same type of problem on B_1, but with a new operator $\tilde{L}$, and with a new function $\tilde{f}$ multiplying the characteristic function on the right-hand side. On the other hand, the new operator has the same ellipticity as the old operator, and the new function $\tilde{f}$ has the same bounds that f had. Suppose there exist some point $x_1 \in B_{1/2}$ such that

$$w(x_1) > 4\tilde{C}|x_1|^2. \tag{3.11}$$

Then since $\frac{1}{2|x_1|} > 1$ and since $\frac{x_1}{2|x_1|} \in \partial B_{\frac{1}{2}}$, we have

$$u_{\left(\frac{1}{2|x_1|}\right)}\left(\frac{x_1}{2|x_1|}\right) = \frac{1}{4|x_1|^2} w(x_1) > \tilde{C}, \tag{3.12}$$

which contradicts Lemma (3.1). □

Now we turn to the nondegeneracy statement. The first thing we need is a variant of the following result from [14].

Lemma 3.3 (Corollary 7.1 of [14])**.** *Suppose μ is a nonnegative measure supported in C which we assume is a compact subset of B_1. Suppose L and $\tilde{L}$ are divergence form elliptic operators exactly of the type considered in this work, and assume that their constants of ellipticity are all contained in the interval of positive numbers: $[\bar{\lambda}, \bar{\Lambda}]$. If*

$$\begin{aligned} Lu = \tilde{L}\tilde{u} = \mu \quad & \textit{in} \;\; B_1 \\ u = \tilde{u} = 0 \quad & \textit{on} \;\; \partial B_1, \end{aligned} \tag{3.13}$$

then there exists a constant $K = K(n, C, \bar{\lambda}, \bar{\Lambda})$, such that for all $x \in C$ we have

$$K^{-1}u(x) \leq \tilde{u}(x) \leq Ku(x).$$

We need to do away with the restriction that μ be supported on a compact subset of B_1, but we can restrict our attention to much simpler nonnegative measures. In fact, the following lemma is good enough for our purposes.

Lemma 3.4. *Assume that L and $\tilde{L}$ are taken exactly as in Lemma (3.3), and assume*

$$\begin{aligned} Lw = \tilde{L}\tilde{w} = 1 \quad & \text{in} \;\; B_1 \\ w = \tilde{w} = 0 \quad & \text{on} \;\; \partial B_1. \end{aligned} \tag{3.14}$$

Then there exists a positive constant $C_0 = C_0(n, \bar{\lambda}, \bar{\Lambda})$, such that for all $x \in B_{1/4}$ we have

$$C_0^{-1}w(x) \leq \tilde{w}(x) \leq C_0 w(x). \tag{3.15}$$

Proof. Without loss of generality we can assume that $\tilde{L}$ is the Laplacian, and we can also replace the assumption $Lw = \Delta\tilde{w} = 1$ with the assumption $Lw = \Delta\tilde{w} = -1$, so that w and $\tilde{w}$ are positive functions. In fact, $\tilde{w}(x) = \Theta(x)$ where we define

$$\Theta(x) := \frac{1 - |x|^2}{2n}.$$

It will be convenient to define the following positive universal constants:

$$\theta_1 := \int_{B_1} |\nabla\Theta|^2 \qquad \text{and} \qquad \theta_2 := \int_{B_{1/2}} \Theta. \tag{3.16}$$

Let u solve

$$Lu = -\chi_{\{B_{1/2}\}} \quad \text{in} \quad B_1 \\ u = 0 \quad \text{on} \quad \partial B_1 \tag{3.17}$$

and let v solve

$$Lv = -1 + \chi_{\{B_{1/2}\}} \quad \text{in} \quad B_1 \\ v = 0 \quad \text{on} \quad \partial B_1. \tag{3.18}$$

By the strong maximum principle, both u and v are positive in B_1, and since $w = u + v$ in B_1, we have $w > u$ in B_1. By Theorem 8.18 of [8]

$$\left(\frac{1}{4}\right)^{-n} ||u||_{L^1(B_{1/2})} \leq C \inf_{B_{1/4}} u. \tag{3.19}$$

By basic facts from the Calculus of Variations, u is characterized as the unique minimizer of the functional

$$J(\phi; r) := \int_{B_1} \nabla\phi A(x) \nabla\phi - 2 \int_{B_r} \phi, \tag{3.20}$$

when r is taken to be $1/2$. (We are letting $A(x)$ be the matrix of coefficients for the operator L.) Now we observe that for any $t > 0$, we have

$$\begin{aligned} J(t\Theta; 1/2) &= t^2 \int_{B_1} \nabla\Theta A(x) \nabla\Theta - 2t \int_{B_{1/2}} \Theta \\ &\leq t^2 \Lambda\theta_1 - 2t\theta_2. \end{aligned}$$

(Recall that θ_1 and θ_2 are the positive universal constants defined in Equation (3.16) above.) Now by taking

$$t := \frac{\theta_2}{\Lambda\theta_1},$$

we can conclude

$$J(u; 1/2) \leq J(t\Theta; 1/2) \leq -\frac{{\theta_2}^2}{\Lambda\theta_1} =: -C_1 < 0. \tag{3.21}$$

Now since

$$J(u; 1/2) \geq -2 \int_{B_1/2} u = -2||u||_{L^1(B_{1/2})},$$

we can conclude that

$$||u||_{L^1(B_{1/2})} \geq C_1/2,$$

which can be combined with Equation (3.19) to get

$$\inf_{B_{1/4}} w \geq \inf_{B_{1/4}} u \geq C, \tag{3.22}$$

which is half of what we need.

On the other hand, by Theorem 8.17 of [8] we know

$$\sup_{B_{1/2}} w \leq C(||w||_{L^2(B_1)} + 1). \tag{3.23}$$

Using the fact that w is the unique minimizer of $J(\cdot; 1)$ and reasoning in a fashion almost identical to what we did above we get

$$\begin{aligned} 0 &\geq J(w;1) \\ &\geq \lambda \int_{B_1} |\nabla w|^2 - 2\int_{B_1} w \\ &= \lambda ||\nabla w||^2_{L^2(B_1)} - 2||w||_{L^1(B_1)} \\ &\geq C\lambda ||w||^2_{L^2(B_1)} - 2||w||_{L^1(B_1)} \qquad \text{by Poincaré's inequality} \\ &\geq C\lambda ||w||^2_{L^2(B_1)} - 2(||w||_{L^2(B_1)} + |B_1|), \end{aligned}$$

which forces $||w||_{L^2(B_1)} \leq C_0$ for some universal C_0. Combining this equation with Equation (3.23) gives us what we need. □

Lemma 3.5. *Let W satisfy the following:*

$$\bar{\lambda} \leq L(W) \leq \bar{\Lambda} \quad \text{in } B_r \qquad \text{and } W \geq 0, \tag{3.24}$$

then there exists a positive constant, C, such that

$$\sup_{\partial B_r} W \geq W(0) + Cr^2. \tag{3.25}$$

Proof. Let u solve

$$L(u) = 0 \quad \text{in } B_r \qquad \text{and } u = W \quad \text{on} \quad \partial B_r. \tag{3.26}$$

Then the Weak Maximum Principle gives

$$\sup_{\partial B_r} u \geq u(0). \tag{3.27}$$

Let v solve

$$L(v) = L(W) \quad \text{in } B_r \quad \text{and } v = 0 \quad \text{on} \quad \partial B_r. \tag{3.28}$$

Notice that $v_0(x) := \frac{|x|^2 - r^2}{2n}$ solves

$$\Delta(v_0) = 1 \quad \text{in } B_r \quad \text{and } v_0 = 0 \quad \text{on} \quad \partial B_r. \tag{3.29}$$

By Lemma (3.4) above, there exist constants C_1, C_2, such that $C_1 v_0(x) \leq v(x) \leq C_2 v_0(x)$ in $B_{r/4}$. In particular,

$$-v(0) \geq C_2 \frac{r^2}{2n}. \tag{3.30}$$

By the definitions of u and v, we know $W = u + v$, therefore by Equations (3.27) and (3.30) we have

$$\sup_{\partial B_r} W(x) = \sup_{\partial B_r} u(x) \geq u(0) = W(0) - v(0) \geq W(0) + C_2 \frac{r^2}{2n}. \tag{3.31}$$

□

Lemma 3.6. *Take w as above, and assume that $w(0) = \gamma > 0$. Then $w > 0$ in a ball B_{δ_0} where $\delta_0 = C_0\sqrt{\gamma}$*

Proof. By Theorem (3.2), we know that if $w(x_0) = 0$, then

$$\gamma = |w(x_0) - w(0)| \leq C|x_0|^2, \tag{3.32}$$

which implies $|x_0| \geq C\sqrt{\gamma}$. □

Lemma 3.7 (Nondegenerate increase on a polygonal curve)**.** *Let w be exactly as above except that we assume that everything is satisfied in B_2 instead of B_1. Suppose again that $w(0) = \gamma > 0$, but now we may require γ to be sufficiently small. Then there exists a positive constant, C, such that*

$$\sup_{B_1} w(x) \geq C + \gamma. \tag{3.33}$$

Proof. We can assume without loss of generality that there exists a $y \in B_{1/3}$ such that $w(y) = 0$. Otherwise we can apply the maximum principle along with Lemma (3.5) to get

$$\sup_{B_1} w(x) \geq \sup_{B_{1/3}} w(x) \geq \gamma + C, \tag{3.34}$$

and we would already be done.

By Lemmas (3.5) and (3.6), there exist $x_1 \in \partial B_{\delta_0}$, such that

$$w(x_1) \geq w(0) + C\frac{\delta_0^2}{2n} = (1 + C_1)\gamma. \tag{3.35}$$

For this x_1 and $B_{\delta_1}(x_1)$ where $\delta_1 = C_0\sqrt{w(x_1)}$, Lemma (3.6) guarantees the existence of an $x_2 \in \partial B_{\delta_1}(x_1)$, such that

$$w(x_2) \geq (1 + C_1)w(x_1) \geq (1 + C_1)^2\gamma. \tag{3.36}$$

Repeating the steps we can get finite sequences $\{x_i\}$ and $\{\delta_i\}$ with $x_0 = 0$ such that

$$w(x_i) \geq (1 + C_1)^i\gamma \quad \text{and} \quad \delta_i = |x_{i+1} - x_i| = C_0\sqrt{w(x_i)}. \tag{3.37}$$

Observe that as long as $x_i \in B_{1/3}$, because of the existence of $y \in B_{1/3}$ where $w(y) = 0$ we know that $\delta_i \leq 2/3$, and so x_{i+1} is still in B_1. Pick N to be the smallest number which satisfies the following inequality:

$$\Sigma_{i=0}^{N}\delta_i = \Sigma_{i=0}^{N} C_0\sqrt{\gamma}(1 + C_1)^{\frac{i}{2}} \geq \frac{1}{3}, \tag{3.38}$$

that is,

$$N \geq \frac{2\ln\left[\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}} + 1\right]}{\ln(1 + C_1)} - 1. \tag{3.39}$$

Plugging this into Equation (3.37) gives

$$\begin{aligned} w(x_N) &\geq \gamma(1+C_1)^{\frac{2\ln\left[\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}}+1\right]}{\ln(1+C_1)}-1} \\ &= \frac{\gamma}{1+C_1}\left(\frac{(1+C_1)^{\frac{1}{2}}-1}{3C_0\sqrt{\gamma}}+1\right)^2 \\ &= (\tilde{C}_0+\tilde{C}_1\sqrt{\gamma})^2 \\ &\geq C_2(1+\gamma), \end{aligned}$$

where the last inequality is guaranteed by the fact that we allow γ to be sufficiently small. □

Lemma 3.8. *Take w as above, but assume that $0 \in \overline{\{w > 0\}}$. Then*

$$\sup_{\partial B_1} w(x) \geq C. \tag{3.40}$$

Proof. By applying the maximum principle and the previous lemma this lemma is immediate. □

Theorem 3.9 (Nondegeneracy)**.** *With $C = C(n, \lambda, \Lambda, \bar{\lambda}, \bar{\Lambda}) > 0$ exactly as in the previous lemma, and if $0 \in \overline{\{w > 0\}}$, then for any $r \leq 1$ we have*

$$\sup_{x \in B_r} w(x) \geq Cr^2. \tag{3.41}$$

Proof. Assume that there exists some $r_0 \leq 1$, such that

$$\sup_{x \in B_{r_0}} w(x) = C_1 r_0^2 < C{r_0}^2. \tag{3.42}$$

Notice that for $\gamma \leq 1$,

$$u_\gamma(x) := \frac{w(\gamma x)}{\gamma^2} \tag{3.43}$$

is also a solution to the same type of problem with a new operator $\tilde{L}$ and new function $\tilde{h}$ defined in B_1, but the new operator has the same ellipticity as the old operator, and the new $\tilde{h}$ has the same bounds and properties that

h had. Now in particular for $u_{r_0}(x) = \frac{w(r_0 x)}{{r_0}^2}$, we have for any $x \in B_1$

$$u_{r_0}(x) = \frac{w(r_0 x)}{{r_0}^2} \leq \frac{1}{r_0^2} \sup_{x \in B_{r_0}} w(x) = C_1 < C, \tag{3.44}$$

which contradicts the previous lemma. □

Corollary 3.10 (Free boundary has zero measure). *The Lebesgue measure of the set*

$$\partial\{w = 0\}$$

is zero.

Proof. The idea here is to use nondegeneracy together with regularity to show that contained in any ball centered on the free boundary, there has to be a proportional subball where w is strictly positive. From this fact it follows that the free boundary cannot have any Lebesgue points. Since the argument is essentially identical to the proof within Lemma 5.1 of [3] that $\mathcal{P}$ has measure zero, we will omit it. □

Remark 3.11 (Porosity). In fact, more can be said from the same argument. Indeed, it shows that the free boundary is strongly porous and therefore has a Hausdorff dimension strictly less than n. (See [15] for definitions of porosity and other relevant theorems and references.)

Corollary 3.12 (Removing the "Gap"). *The existence, uniqueness, regularity and nondegeneracy theorems from this section and the previous section all hold whenever*

$$L(w) = h$$

is replaced by

$$L(w) = \chi_{\{w>0\}} f.$$

4. Equivalence of the obstacle problems

There are two main points to this section. First, we deal with the comparatively simple task of getting existence, uniqueness and continuity of certain minimizers to our functionals in the relevant sets. Second, and more importantly we show that the minimizer is the solution of an obstacle problem of the type studied in the previous two sections. We start with some definitions and terminology.

We continue to assume that a^{ij} is strictly and uniformly elliptic and we keep L defined exactly as above. We let $G(x,y)$ denote the Green's function for L for all of $\mathbb{R}^n$ and observe that the existence of G is guaranteed by the work of Littman *et al.* (See [14].)

Let

$$\begin{aligned} C_{sm,r} &:= \min_{x\in\partial B_r} G(x,0), \\ C_{\text{big},r} &:= \max_{x\in\partial B_r} G(x,0), \\ G_{sm,r}(x) &:= \min\{G(x,0), C_{sm,r}\} \end{aligned}$$

and observe that $G_{sm,r} \in W^{1,2}(B_M)$ by results from [14] combined with the Cacciopoli Energy Estimate. We also know that there is an $\alpha \in (0,1)$ such that $G_{sm,r} \in C^{0,\alpha}(\overline{B_M})$ by the De Giorgi–Nash–Moser theorem. (See [8] or [10] for example.) For M large enough to guarantee that $G_{sm}(x) := G_{sm,1}(x) \equiv G(x,0)$ on ∂B_M, we define:

$$H_{M,G} := \{w \in W^{1,2}(B_M) \ : \ w - G_{sm} \in W_0^{1,2}(B_M) \ \}$$

and

$$K_{M,G} := \{ \ w \in H_{M,G} \ : \ w(x) \leq G(x,0) \text{ for all } x \in B_M \ \}.$$

(The existence of such an M follows from [14], and henceforth any constant M will be large enough so that $G_{sm,1}(x) \equiv G(x,0)$ on ∂B_M.)

Define:

$$\begin{aligned} \Phi_\epsilon(t) &:= \begin{cases} 0 & \text{for } t \geq 0, \\ -\epsilon^{-1}t & \text{for } t \leq 0, \end{cases} \\ J(w,\Omega) &:= \int_\Omega (a^{ij}D_iwD_jw - 2R^{-n}w), \quad \text{and} \\ J_\epsilon(w,\Omega) &:= \int_\Omega (a^{ij}D_iwD_jw - 2R^{-n}w + 2\Phi_\epsilon(G-w)). \end{aligned}$$

Theorem 4.1 (Existence and uniqueness)**.**

$$\begin{aligned} \textit{Let } \ell_0 &:= \inf_{w\in K_{M,G}} J(w,B_M) \quad \textit{and} \\ \textit{let } \ell_\epsilon &:= \inf_{w\in H_{M,G}} J_\epsilon(w,B_M). \end{aligned}$$

Then there exists a unique $w_0 \in K_{M,G}$ such that $J(w_0,B_M) = \ell_0$, and there exists a unique $w_\epsilon \in H_{M,G}$ such that $J_\epsilon(w_\epsilon,B_M) = \ell_\epsilon$.

Proof. Both of these results follow by a straightforward application of the direct method of the Calculus of Variations. □

Remark 4.2. Notice that we cannot simply minimize either of our functionals on all of $\mathbb{R}^n$ instead of B_M as the Green's function is not integrable at infinity. Indeed, if we replace B_M with $\mathbb{R}^n$ then

$$\ell_0 = \ell_\epsilon = -\infty$$

and so there are many technical problems.

Theorem 4.3 (Continuity). *For any $\epsilon > 0$, the function w_ϵ is continuous on $\overline{B_M}$.*

See Chapter 7 of [9].

Lemma 4.4. *There exists $\epsilon > 0$, $C < \infty$, such that $w_0 \leq C$ in B_ϵ.*

Proof. Let $\bar{w}$ minimize $J(w, B_M)$ among functions $w \in H_{M,G}$. Then we have

$$w_0 \leq \bar{w}.$$

Set $b := C_{\text{big},M} = \max_{\partial B_M} G(x, 0)$, and let w_b minimize $J(w, B_M)$ among $w \in W^{1,2}(B_M)$ with

$$w - b \in W_0^{1,2}(B_M).$$

Then by the weak maximum principle, we have

$$\bar{w} \leq w_b.$$

Next define $\ell(x)$ by

$$\ell(x) := b + R^{-n}\left(\frac{M^2 - |x|^2}{4n}\right) \leq b + \frac{R^{-n}M^2}{4n} < \infty. \tag{4.1}$$

With this definition, we can observe that ℓ satisfies

$$\Delta\ell = -\frac{R^{-n}}{2}, \text{ in } B_M \quad \text{and}$$
$$\ell \equiv b := \max_{\partial B_M} G \text{ on } \partial B_M.$$

Now let $\widetilde{\alpha}$ be $b + \frac{R^{-n}M^2}{4n}$. By Corollary 7.1 in [14] applied to $w_b - b$ and $\ell - b$, we have

$$w_b \leq b + K(\ell - b) \leq b + K\widetilde{\alpha} < \infty.$$

Chaining everything together gives us

$$w_0 \leq b + K\widetilde{\alpha} < \infty.$$

□

Lemma 4.5. *If* $0 < \epsilon_1 \leq \epsilon_2$, *then*

$$w_{\epsilon_1} \leq w_{\epsilon_2}.$$

Proof. Assume $0 < \epsilon_1 \leq \epsilon_2$, and assume that

$$\Omega_1 := \{w_{\epsilon_1} > w_{\epsilon_2}\}$$

is not empty. Since $w_{\epsilon_1} = w_{\epsilon_2}$ on ∂B_M, since $\Omega_1 \subset B_M$, and since w_{ϵ_1} and w_{ϵ_2} are continuous functions, we know that $w_{\epsilon_1} = w_{\epsilon_2}$ on $\partial\Omega_1$. Then it is clear that among functions with the same data on $\partial\Omega_1$, w_{ϵ_1} and w_{ϵ_2} are minimizers of $J_{\epsilon_1}(\cdot, \Omega_1)$ and $J_{\epsilon_2}(\cdot, \Omega_1)$ respectively. Since we will restrict our attention to Ω_1 for the rest of this proof, we will use $J_\epsilon(w)$ to denote $J_\epsilon(w, \Omega_1)$.

$J_{\epsilon_2}(w_{\epsilon_2}) \leq J_{\epsilon_2}(w_{\epsilon_1})$ implies

$$\begin{aligned}\int_{\Omega_1} a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2} + 2\Phi_{\epsilon_2}(G - w_{\epsilon_2}) \\ \leq \int_{\Omega_1} a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1} + 2\Phi_{\epsilon_2}(G - w_{\epsilon_1}),\end{aligned}$$

and by rearranging this inequality we get

$$\begin{aligned}\int_{\Omega_1} (a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2}) - \int_{\Omega_1} (a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1}) \\ \leq \int_{\Omega_1} 2\Phi_{\epsilon_2}(G - w_{\epsilon_1}) - 2\Phi_{\epsilon_2}(G - w_{\epsilon_2}).\end{aligned}$$

Therefore,

$$\begin{aligned} J_{\epsilon_1}(w_{\epsilon_2}) - J_{\epsilon_1}(w_{\epsilon_1}) &= \int_{\Omega_1} a^{ij} D_i w_{\epsilon_2} D_j w_{\epsilon_2} - 2R^{-n} w_{\epsilon_2} + 2\Phi_{\epsilon_1}(G - w_{\epsilon_2}) \\ &\quad - \int_{\Omega_1} a^{ij} D_i w_{\epsilon_1} D_j w_{\epsilon_1} - 2R^{-n} w_{\epsilon_1} + 2\Phi_{\epsilon_1}(G - w_{\epsilon_1}) \\ &\leq 2 \int_{\Omega_1} \Big[\Phi_{\epsilon_2}(G - w_{\epsilon_1}) - \Phi_{\epsilon_2}(G - w_{\epsilon_2}) \Big] \\ &\quad - 2 \int_{\Omega_1} \Big[\Phi_{\epsilon_1}(G - w_{\epsilon_1}) - \Phi_{\epsilon_1}(G - w_{\epsilon_2}) \Big] \\ &< 0, \end{aligned}$$

since $G - w_{\epsilon_1} < G - w_{\epsilon_2}$ in Ω_1 and Φ_{ϵ_1} decreases as fast or faster than Φ_{ϵ_2} decreases everywhere. This inequality contradicts the fact that w_{ϵ_1} is the minimizer of $J_{\epsilon_1}(w)$. Therefore, $w_{\epsilon_1} \leq w_{\epsilon_2}$ everywhere in Ω. □

Lemma 4.6. *$w_0 \leq w_\epsilon$ for every $\epsilon > 0$.*

Proof. Let $S := \{w_0 > w_\epsilon\}$ be a nonempty set, let $w_1 := \min\{w_0, w_\epsilon\}$, and let $w_2 := \max\{w_0, w_\epsilon\}$. It follows that $w_1 \leq G$ and both w_1 and w_2 belong to $W^{1,2}(B_M)$. Since $\Phi_\epsilon \geq 0$, we know that for any $\Omega \subset B_M$ we have

$$J(w, \Omega) \leq J_\epsilon(w, \Omega) \tag{4.2}$$

for any permissible w. We also know that since $w_0 \leq G$ we have

$$J(w_0, \Omega) = J_\epsilon(w_0, \Omega). \tag{4.3}$$

Now we estimate

$$\begin{aligned} J_\epsilon(w_1, B_M) &= J_\epsilon(w_1, S) + J_\epsilon(w_1, S^c) \\ &= J_\epsilon(w_\epsilon, S) + J_\epsilon(w_0, S^c) \\ &= J_\epsilon(w_\epsilon, B_M) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\ &\leq J_\epsilon(w_2, B_M) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\ &= J_\epsilon(w_0, S) + J_\epsilon(w_\epsilon, S^c) - J_\epsilon(w_\epsilon, S^c) + J_\epsilon(w_0, S^c) \\ &= J_\epsilon(w_0, S) + J_\epsilon(w_0, S^c) \\ &= J_\epsilon(w_0, B_M). \end{aligned}$$

Now by combining this inequality with Equations (4.2) and (4.3), we get

$$J(w_1, B_M) \leq J_\epsilon(w_1, B_M) \leq J_\epsilon(w_0, B_M) = J(w_0, B_M),$$

but if S is nonempty, then this inequality contradicts the fact that w_0 is the unique minimizer of J among functions in $K_{M,G}$. □

Now, since w_ϵ decreases as $\epsilon \to 0$, and since the w_ϵ's are bounded from below by w_0, there exists

$$\widetilde{w} = \lim_{\epsilon \to 0} w_\epsilon$$

and $w_0 \leq \widetilde{w}$.

Lemma 4.7. *With the definitions as above, $\tilde{w} \leq G$ almost everywhere.*

Proof. This fact is fairly obvious, and the proof is fairly straightforward, so we supply only a sketch.

Suppose not. Then there exists an $\alpha > 0$, such that

$$\tilde{S} := \{\tilde{w} - G \geq \alpha\}$$

has positive measure. On this set we automatically have $w_\epsilon - G \geq \alpha$. We compute $J_\epsilon(w_\epsilon, B_M)$ and send ϵ to zero. We will get $J_\epsilon(w_\epsilon, B_M) \to \infty$ which gives us a contradiction. □

Lemma 4.8. $\widetilde{w} = w_0$ *in* $W^{1,2}(B_M)$.

Proof. Since for any ϵ, w_ϵ is the minimizer of $J_\epsilon(w, B_M)$, we have

$$\begin{aligned} J_\epsilon(w_\epsilon, B_M) &\leq J_\epsilon(w_0, B_M) \\ &\leq \int_{B_M} a^{ij} D_i w_0 D_j w_0 - 2R^{-n} w_0 + 2\Phi_\epsilon(G - w_\epsilon), \end{aligned}$$

and after canceling the terms with Φ_ϵ we have

$$\int_{B_M} a^{ij} D_i w_\epsilon D_j w_\epsilon - 2R^{-n} w_\epsilon \leq \int_{B_M} a^{ij} D_i w_0 D_j w_0 - 2R^{-n} w_0.$$

Letting $\epsilon \to 0$ gives us

$$J(\tilde{w}, B_M) \leq J(w_0, B_M).$$

However, by Proposition (4.7), $\widetilde{w}$ is a permissible competitor for the problem $\inf_{w \in K_{M,G}} J(w, B_M)$, so we have

$$J(w_0, B_M) \leq J(\tilde{w}, B_M).$$

Therefore,

$$J(w_0, B_M) = J(\tilde{w}, B_M),$$

and then by uniqueness, $\widetilde{w} = w_0$. □

Let W solve

$$\begin{cases} L(w) = -\chi_{\{w<G\}} R^{-n} & \text{in } \; B_M \\ \\ w = G_{sm} & \text{on } \; \partial B_M. \end{cases} \tag{4.4}$$

The existence of such a W is guaranteed by combining Theorem (2.3) with Corollary (3.12). (Signs are reversed, so to be completely precise one must apply the theorems to the problem solved by $G - W$.)

Lemma 4.9. *$W \leq G$ in B_M.*

Proof. Let $\Omega = \{W > G\}$ and $u := W - G$. Since G is infinite at 0, and since W is bounded, and both G and W are continuous, we know there exists an $\epsilon > 0$, such that $\Omega \cap B_\epsilon = \phi$. Then if $\Omega \neq \phi$, then u has a positive maximum in the interior of Ω. However, since $L(W) = L(G) = 0$ in Ω, we would get a contradiction from the weak maximum principle. Therefore, we have $W \leq G$ in B_M. □

Lemma 4.10. *$\tilde{w} \geq W$.*

Proof. It suffices to show $w_\epsilon \geq W$, for any ϵ. Suppose for the sake of obtaining a contradiction that there exists an $\epsilon > 0$ and a point x_0, where $w_\epsilon - W$ has a negative local minimum. So $w_\epsilon(x_0) < W(x_0) \leq G(x_0)$. Let $\Omega := \{w_\epsilon < W\}$ and observe that $w_\epsilon = W$ on $\partial\Omega$. Then x_0 is an interior point of Ω and

$$L(w_\epsilon) = -R^{-n} \;\text{ in }\; \Omega.$$

However,

$$L(W - w_\epsilon) \geq -R^{-n} + R^{-n} = 0 \;\text{ in }\; \Omega. \tag{4.5}$$

By the weak maximum principle, the minimum cannot be attained at an interior point, and so we have a contradiction. □

Lemma 4.11. *$w_0 = \tilde{w} = W$, and so w_0 and $\tilde{w}$ are continuous.*

Proof. We already showed that $w_0 = \tilde{w}$ in Lemma (4.8). By Lemma (4.10), in the set where $W = G$, we have

$$W = \tilde{w} = G. \tag{4.6}$$

Let $\Omega_1 := \{W < G\}$, it suffices to show $\tilde{w} = W$ in Ω_1. By definition of W, $L(W) = -R^{-n}$ in Ω_1.
Using the fact that w_0 is the minimizer, the standard argument in the calculus of variations leads to $L(w_0) \geq -R^{-n}$. Therefore,

$$L(\tilde{w} - W) = L(w_0 - W) \geq 0 \quad \text{in} \quad B_M. \tag{4.7}$$

Notice that on $\partial\Omega_1$, $W = \tilde{w} = G$. By weak maximum principle, we have

$$\tilde{w} = W \quad \text{in} \quad \Omega_1. \tag{4.8}$$

□

Using the last lemma along with our definition of W (see Equation (4.4)), we can now state the following theorem.

Theorem 4.12 (The PDE satisfied by w_0). *The minimizing function w_0 satisfies the following boundary value problem:*

$$\begin{cases} L(w_0) = -\chi_{\{w_0<G\}} R^{-n} & \text{in} \quad B_M \\ w_0 = G_{sm} & \text{on} \quad \partial B_M. \end{cases} \tag{4.9}$$

5. Minimizers become independent of M

At this point, we are no longer interested in the functions from the last section, with the exception of w_0. On the other hand, we now care about the dependence of w_0 on the radius of the ball on which it is a minimizer. Accordingly, we reintroduce the dependence of w_0 on M, and so we will let w_M be the minimizer of $J(w, B_M)$ within $K(M, G)$, and consider the behavior as $M \to \infty$. As we observed in Remark (4.2), it is not possible to start by minimizing our functional on all of $\mathbb{R}^n$, so we have to get the key function, "V_R," mentioned by Caffarelli on page 9 of [4] by taking a limit over increasing sets. Note that by Theorem (4.12) we know that w_M satisfies

$$\begin{cases} L(w_M) = -\chi_{\{G>w_M\}} R^{-n} & \text{in } B_M \\ \\ w_M = G_{sm} & \text{on } \partial B_M. \end{cases} \tag{5.1}$$

The theorem that we wish to prove in this section is the following.

Theorem 5.1 (Independence from M)**.** *There exists $M \in \mathbb{N}$, such that if $M_j > M$ for $j = 1, 2$, then*

$$w_{M_1} \equiv w_{M_2} \quad \text{within} \quad B_M$$

and

$$w_{M_1} \equiv w_{M_2} \equiv G \quad \text{within} \quad B_{M+1} \setminus B_M.$$

Furthermore, we can choose M such that $M < C(n, \lambda, \Lambda) \cdot R$.

This Theorem is an immediate consequence of the following Theorem.

Theorem 5.2 (Boundedness of the Noncontact Set)**.** *There exists a constant $C = C(n, \lambda, \Lambda)$, such that for any $M \in \mathbb{R}$*

$$\{w_M \neq G\} \subset B_{CR}. \tag{5.2}$$

Proof. First of all, if $M \leq CR$, then there is nothing to prove. For all $M > 1$ the function $W := G - w_M$ will satisfy

$$L(W) = R^{-n}\chi_{\{W>0\}}, \text{ and } 0 \leq W \leq G \text{ in } B_1^c. \tag{5.3}$$

If the conclusion to the theorem is false, then there exists a large M and a large C, such that

$$x_0 \in FB(W) \cap \{B_{M/2} \setminus B_{CR}\}.$$

Let $K := |x_0|/3$. By Theorem (3.9), we can then say that

$$\sup_{B_K(x_0)} W(x) \geq CR^{-n}K^2 > CK^{2-n} \geq \sup_{B_K(x_0)} G(x), \tag{5.4}$$

which gives us a contradiction since $W \leq G$ everywhere. Now note that in order to avoid the contradiction, we must have

$$CR^{-n}K^2 \leq CK^{2-n},$$

and this leads to

$$K \leq CR,$$

which means that $|x_0|$ must be less than CR. In other words, $FB(W) \subset B_{CR}$. □

At this point, we already know that when M is sufficiently large, the set $\{G > w_M\}$ is contained in B_{CR}. Then by uniqueness, the set will stay the same for any bigger M. Therefore, it makes sense to define w_R to be the solution of

$$Lw = -R^{-n}\chi_{\{w<G\}} \quad \text{in } \mathbb{R}^n \tag{5.5}$$

among functions $w \leq G$ with $w = G$ at infinity. Note that we can now obtain the function, "V_R," that Caffarelli uses on page 9 of [4]. The relationship is simply:

$$V_R = w_R - G. \tag{5.6}$$

6. The mean value theorem

Finally, we can turn to the mean value theorem.

Lemma 6.1 (Ordering of sets**).** *For any $R < S$, we have*

$$\{w_R < G\} \subset \{w_S < G\}. \tag{6.1}$$

Proof. Let B_M be a ball that contains both $\{w_R < G\}$ and $\{w_S < G\}$. Then by the discussion in Section 2, we know w_R minimizes

$$\int_{B_M} a^{ij}D_iwD_jw - 2wR^{-n}$$

and w_S minimizes

$$\int_{B_M} a^{ij}D_iwD_jw - 2wS^{-n}.$$

Let $\Omega_1 \subset\subset B_M$ be the set $\{w_S > w_R\}$. Then it follows that

$$\int_{\Omega_1} a^{ij} D_i w_S D_j w_S - 2 w_S S^{-n} \leq \int_{\Omega_1} a^{ij} D_i w_R D_j w_R - 2 w_R S^{-n}, \tag{6.2}$$

which implies

$$\begin{aligned}\int_{\Omega_1} a^{ij} D_i w_S D_j w_S &\leq \int_{\Omega_1} a^{ij} D_i w_R D_j w_R + 2S^{-n} \int_{\Omega_1} (w_S - w_R) \\ &< \int_{\Omega_1} a^{ij} D_i w_R D_j w_R + 2R^{-n} \int_{\Omega_1} (w_S - w_R).\end{aligned}$$

Therefore, since $w_S \equiv w_R$ on $\partial\Omega_1$, and

$$\int_{\Omega_1} a^{ij} D_i w_S D_j w_S - 2 w_S R^{-n} < \int_{\Omega_1} a^{ij} D_i w_R D_j w_R - 2 w_R R^{-n}, \tag{6.3}$$

we contradict the fact that w_R is the minimizer of $\int a^{ij} D_i w D_j w - 2wR^{-n}$.
□

Lemma 6.2. *There exists a constant* $c = c(n, \lambda, \Lambda)$ *such that*

$$B_{cR} \subset \{G > w_R\}.$$

Proof. By Lemma (4.4), we already know that there exists a constant

$$C = C(n, \lambda, \Lambda),$$

such that $w_1(0) \leq C$. Then it is not hard to show that

$$\|w_1\|_{L^\infty(B_{1/2})} \leq \tilde{C}. \tag{6.4}$$

By [14] for any elliptic operator L with given λ and Λ, we have

$$\frac{c_1}{|x|^{n-2}} \leq G(x) \leq \frac{c_2}{|x|^{n-2}}. \tag{6.5}$$

By combining the last two equations it follows that there exists a constant $c = c(n, \lambda, \Lambda)$ such that

$$B_c \subset \{G > w_1\}.$$

It remains to show that this inclusion scales correctly.

Let $v_R := G - w_R$ (so $v_R = -V_R$). Then v_R satisfies

$$Lv_R = \delta - R^{-n}\chi_{\{v_R>0\}} \quad \text{in } \mathbb{R}^n. \tag{6.6}$$

Now observe that by scaling our operator L appropriately, we get an operator $\tilde{L}$ with the same ellipticity constants as L, such that

$$\tilde{L}\left(R^{n-2}v_R(Rx)\right) = \delta - \chi_{\{v_R(Rx)>0\}}. \tag{6.7}$$

So we have

$$B_c \subset \left\{x \,\middle|\, v_R(Rx) > 0\right\},$$

which implies

$$B_{cR} \subset \left\{v_R(x) > 0\right\}. \tag{6.8}$$

□

Suppose v is a supersolution to

$$Lv = 0,$$

i.e., $Lv \le 0$. Then for any $\phi \ge 0$, we have

$$\int_\Omega vL\phi \le 0. \tag{6.9}$$

If $R < S$, then we know that $w_R \ge w_S$, and so the function $\phi = w_R - w_S$ is a permissible test function. We also know

$$L\phi = R^{-n}\chi_{\{G>w_R\}} - S^{-n}\chi_{\{G>w_S\}}. \tag{6.10}$$

By observing that $v \equiv 1$ is both a supersolution and a subsolution and by plugging in our ϕ, we arrive at

$$R^{-n}|\{G > w_R\}| = S^{-n}|\{G > w_S\}|, \tag{6.11}$$

and this implies

$$L\phi = C\left[\frac{1}{|\{G > w_R\}|}\chi_{\{G>w_R\}} - \frac{1}{|\{G > w_S\}|}\chi_{\{G>w_S\}}\right]. \tag{6.12}$$

Now, Equation (6.9) implies

$$(6.13)\ 0 \geq \int_\Omega vL\phi = C\left[\frac{1}{|\{G > w_R\}|}\int_{\{G>w_R\}} v - \frac{1}{|\{G > w_S\}|}\int_{\{G>w_S\}} v\right].$$

Therefore, we have established the following theorem:

Theorem 6.3 Mean value theorem for divergence form elliptic PDE. *Let L be any divergence form elliptic operator with ellipticity λ, Λ. For any $x_0 \in \Omega$, there exists an increasing family $D_R(x_0)$ which satisfies the following:*

(1) *$B_{cR}(x_0) \subset D_R(x_0) \subset B_{CR}(x_0)$, with c, C depending only on n, λ and Λ.*

(2) *For any v satisfying $Lv \geq 0$ and $R < S$, we have*

$$v(x_0) \leq \frac{1}{|D_R(x_0)|}\int_{|D_R(x_0)|} v \leq \frac{1}{|D_S(x_0)|}\int_{D_S(x_0)} v. \tag{6.14}$$

As on pages 9 and 10 of [4], (and as Littman *et al.* already observed using their own mean value theorem,) we have the following corollary:

Corollary 6.4 (Semicontinuous representative**).** *Any supersolution v, has a unique pointwise defined representative as*

$$v(x_0) := \lim_{R\downarrow 0}\frac{1}{|D_R(x_0)|}\int_{|D_R(x_0)|} v(x)dx. \tag{6.15}$$

This representative is lower semicontinuous:

$$v(x_0) \leq \lim_{x\to x_0} v(x) \tag{6.16}$$

for any x_0 in the domain.

We can also show the following analog of G.C. Evans' Theorem.

Corollary 6.5 (Analog of Evans' Theorem**).** *Let v be a supersolution to $Lv = 0$, and suppose that v restricted to the support of Lv is continuous. Then the representative of v given by Equation (6.16) is continuous.*

Proof. This proof is almost identical to the proof given on pages 10 and 11 of [4] for $L = \Delta$. □

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References

[1] I. Blank, *Sharp results for the regularity and stability of the free boundary in the obstacle problem*, Indiana Univ. Math. J. **50**(3)(2001), 1077–1112.

[2] I. Blank and Z. Hao, *Reifenberg flatness of free boundaries in obstacle problems with VMO ingredients*, `arXiv:1309.3077v2 [math.AP]`.

[3] I. Blank and K. Teka, *The Caffarelli alternative in measure for the nondivergence form elliptic obstacle problem with principal coefficients in VMO*, Comm. Partial Differ. Equ., **39**(2) (2014), 321–353.

[4] L.A. Caffarelli, *The obstacle problem*, The Fermi Lectures, Accademia Nazionale Dei Lincei Scuola Normale Superiore, 1998.

[5] L.A. Caffarelli and J.-M. Roquejoffre, *Uniform Hölder estimates in a class of elliptic systems and applications to singular limits in models for diffusion flames*, Arch. Ration. Mech. Anal. **183** (2007), 457–487.

[6] M. Focardi, M.S. Gelli, and E. Spadaro, *Monotonicity formulas for obstacle problems with Lipschitz coefficients*, `arXiv:1306.2127v1 [math.AP]`.

[7] A. Friedman, *Variational principles and free-boundary problems*, R.E. Krieger Pub. Co., Malabar, Florida, 1988.

[8] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd edn., Springer-Verlag, Berlin, 1983.

[9] E. Giusti, *Direct methods in the calculus of variations*, World Scientific, Singapore, 2003.

[10] Q. Han and F. Lin, *Elliptic partial differential equations*, AMS, New York, 2000.

[11] D. Kinderlehrer, *Personal communication.*

[12] D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, San Diego, 1980.

[13] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968.

[14] W. Littman, G. Stampacchia, and H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola. Norm. Sup. Pisa. Cl. Sci., **17**(1–2) (1963), 43–77.

[15] P. Mattila, *Geometry of sets and measures in euclidean spaces*, Cambridge University Press, Cambridge, UK, 1995.

[16] J.-F. Rodriguez, *Obstacle problems in mathematical physics*, Elsevier, Netherlands, 1987.

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