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On the stable 4-genus of knots with indefinite Seifert form

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Under a simple assumption on the Seifert form, we characterise knots whose stable topological 4-genus coincides with the genus.

1. Introduction

The topological 4-genus $g_4(K)$ of a knot K is the minimal genus of a topological, locally flat surface embedded in the 4-ball with boundary K. A well-known theorem due to Freedman asserts that knots with trivial Alexander polynomial bound a locally flat disc in the 4-ball [2]. Unlike for the classical genus g, there is no known algorithm that determines the topological 4-genus of a knot. The signature bound by Kauffman and Taylor [6], $|\sigma(K)| \leq 2g_4(K)$, fails to be sharp for the simplest knots, such as the figure-eight knot. As we will see, the signature bound becomes much more effective when the topological 4-genus is replaced by its stable version $\widehat{g}_4$ defined by Livingston [7]:

$$\widehat{g}_4(K) = \lim_{n\to\infty} \frac{1}{n} g_4(K^n).$$

Here K^n denotes the n-times iterated connected sum of K. The existence of $\widehat{g}_4$ follows from general principles on subadditive functions (see Theorem 1 in [7]).

Theorem 1. *Let $\Sigma \subset \mathbb{R}^3$ be a minimal genus Seifert surface for a knot K. Assume that the Seifert form associated with Σ takes the value $+1$ or -1. Then the following are equivalent:*

(i) $\widehat{g}_4(K) = g(K)$,

(ii) $|\sigma(K)| = 2g(K)$.

Corollary 1. *Let $\Sigma \subset \mathbb{R}^3$ be a minimal genus Seifert surface for a knot K. If Σ contains two embedded annuli with framings $+1$ and -1, then $\widehat{g}_4(K) < g(K)$.*

The second condition of Theorem 1 clearly implies the first one, by the following chain of (in)equalities:

$$n2g(K) = n|\sigma(K)| = |\sigma(K^n)| \leq 2g_4(K^n) \leq 2g(K^n) = n2g(K).$$

We do not know whether the reverse implication holds without any additional assumption on the Seifert form.

Question. *Does there exist a knot K with $|\sigma(K)| < 2g(K)$ and $\widehat{g}_4(K) = g(K)$?*

As pointed out by the referee, this is closely related to a result by Gilmer (see [5]): for every $n \in \mathbb{N}$, there exists a knot K with $\sigma(K) = 0$ yet $g_4(K^n) = g(K^n) = n$. However, these examples do not answer the question since the condition $\widehat{g}_4(K) = g(K)$ is potentially stronger than each individual condition $g_4(K^n) = g(K^n)$.

We conclude the introduction with an application concerning positive braid knots, i.e. knots which are closures of a positive braids. As shown in [1], the only positive braid knots with $|\sigma(K)| = 2g(K)$ are torus knots of type $T(2,n)$ ($n \in \mathbb{N}$), $T(3,4)$ and $T(3,5)$. Moreover, positive braid knots have a canonical Seifert surface (in fact, a fibre surface), which always contains a Hopf band with framing $+1$.

Corollary 2. *Let K be a positive braid knot. Then $\widehat{g}_4(K) = g(K)$, if and only if K is a torus knot of type $T(2,n)$ ($n \in \mathbb{N}$), $T(3,4)$ or $T(3,5)$.*

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2. Constructing tori with slice boundary

Let $K \subset S^3$ be a knot with minimal genus Seifert surface Σ. The Seifert form $V : H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}) \to \mathbb{Z}$ is defined by linear extension of the formula

$$V([x],[y]) = \mathrm{lk}(x, y^+),$$

valid for simple closed curves $x, y \subset \Sigma$. Here lk denotes the linking number and y^+ is a push-off of the curve y in the positive direction with respect to a fixed orientation of Σ. The number $V([x],[x]) \in \mathbb{Z}$ is called self-linking or framing of the curve x. The signature $\sigma(K)$ of K is defined as the number of positive eigenvalues minus the number of negative eigenvalues of the symmetrised Seifert form $V+V^T$. The Alexander polynomial of K is defined as $\Delta_K(t)=\det(\sqrt{t}V-\frac{1}{\sqrt{t}}V^T)$. Throughout this section, we will assume that

(i) the symmetrised Seifert form on $H_1(\Sigma,\mathbb{Z})$ is indefinite, i.e.

$$|\sigma(K)|<2g(K),$$

(ii) there exists an element $\alpha\in H_1(\Sigma,\mathbb{Z})$ with self-linking $+1$ (the case of framing -1 can be reduced to this by taking the mirror image of Σ).

Let Σ^n be the Seifert surface for K^n obtained by n-times iterated boundary connected sum of Σ and let $V_n: H_1(\Sigma^n,\mathbb{Z})\times H_1(\Sigma^n,\mathbb{Z})\to\mathbb{Z}$ be the corresponding Seifert form. We define

$\mathcal{F}(\Sigma)=\{m\in\mathbb{Z}\,|$ there exist a number $n\in\mathbb{N}$ and an element $\gamma\in H_1(\Sigma^n,\mathbb{Z})$ with $V_n(\gamma)=m\}$.

Lemma 1. *$\mathcal{F}(\Sigma)=\mathbb{Z}$.*

Proof. The set $\mathcal{F}(\Sigma)$ is closed under addition and contains $+1$. Therefore, we need only construct an element with negative self-linking in $H_1(\Sigma,\mathbb{Z})$. The symmetrised Seifert form $q=V+V^T$ being indefinite and non-degenerate (the latter is true for all Seifert surfaces with one boundary component), there exists a vector $\beta\in H_1(\Sigma,\mathbb{R})$ with $q(\beta)<0$. Since negative vectors for q form an open cone in $H_1(\Sigma,\mathbb{R})$, we may choose $\beta\in H_1(\Sigma,\mathbb{Z})$. □

Remark. For surfaces Σ with one boundary component, every primitive element of $H_1(\Sigma,\mathbb{Z})$ can be represented by a simple closed oriented curve in Σ (see e.g. [8]). In particular, elements with self-linking ±1 can be represented by simple closed curves. Combining this with Lemma 1, we conclude that there exists a number $n\in\mathbb{N}$ and two embedded annuli $A,B\subset\Sigma^n$ with framings $+1$ and -1, respectively.

Lemma 2. *There exists a number $N\in\mathbb{N}$ and an embedded torus $T\subset\Sigma^N$ with one boundary component whose Alexander polynomial is trivial.*

Proof. Let $S\subset\Sigma$ be an embedded torus with one boundary component, viewed as a union of two embedded annuli $C,D\subset S$ that meet in a square. Let $\begin{pmatrix} a & b \\ b+1 & d\end{pmatrix}$ be the matrix representing the Seifert form on $H_1(T,\mathbb{Z})$ with respect to a pair of oriented core curves of C and D. By adding a suitable number of copies of A or B to C and D in a power Σ^n, far away from the initial torus $S \subset \Sigma$, we may impose the individual framings of C and D to be b and $b+1$, without changing the mutual linking of C and D. Thus we obtain an embedded torus $T \subset \Sigma^N$ with Seifert form $V = \begin{pmatrix} b & b \\ b+1 & b+1 \end{pmatrix}$. The Alexander polynomial of the boundary link $L = \partial T$ can be computed as

$$\Delta_L(t) = \det\left(\sqrt{t}V - \frac{1}{\sqrt{t}}V^T\right) = 1.$$

□

In order to prove Theorem 1, we need to invoke Freedman's result ([2], see also [3] and [4]): knots with trivial Alexander polynomial are topologically slice.

Proof of Theorem 1. As mentioned in the introduction, the condition $|\sigma(K)| = 2g(K)$ implies $\widehat{g}_4(K) = g(K)$, without any assumption on the Seifert surface Σ. For the reverse implication, we assume $|\sigma(K)| < 2g(K)$ and prove $\widehat{g}_4(K) < g(K)$. By Lemma 2, there exists a number $N \in \mathbb{N}$ and an embedded torus $T \subset \Sigma^N$ with one boundary component $L = \partial T$ and $\Delta_L(t) = 1$. According to Freedman, there exists a topological, locally flat disc D embedded in the 4-ball with boundary L. We may assume that the interior of D is contained in the interior of the 4-ball. Now the union of D and $\Sigma^N \setminus T$ is a topological, locally flat surface embedded in the 4-ball with boundary K^N and genus $Ng(K) - 1$. Therefore,

$$\widehat{g}_4(K) \leq g(K) - \frac{1}{N} < g(K).$$

□

References

[1] S. Baader, *Positive braids of maximal signature*, Enseign. Math., **59** (2013), no. 3–4, 351–358.

[2] M. Freedman, *The disk theorem for four-dimensional manifolds*, Proceedings of the International Congress of Mathematicians in Warsaw (1983), 647–663.

[3] M. Freedman and F. Quinn, *Topology of 4-manifolds*, Princeton University Press, Princeton, NJ, 1990.

[4] S. Garoufalidis and P. Teichner, *On knots with trivial Alexander polynomial*, J. Differential Geom., **67** (2004), no. 1, 167–193.

[5] P. Gilmer, *On the slice genus of knots*, Invent. Math., **66** (1982), no. 2, 191–197.

[6] L. Kauffman and L. Taylor, *Signature of links*, Trans. Amer. Math. Soc., **216** (1976), 351–365.

[7] C. Livingston, *The stable 4-genus of knots*, Alg. Geom. Top., **10** (2010), 2191–2202.

[8] W. H. Meeks and J. Patrusky, *Representing homology classes by embedded circles on a compact surface*, Illinois J. Math., **22** (1978), no. 2, 262–269.

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