

# Ancient solutions of the mean curvature flow

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In this short article, we prove the existence of ancient solutions of the mean curvature flow that for  $t \rightarrow 0$  collapse to a round point, but for  $t \rightarrow -\infty$  become more and more oval: near the center they have asymptotic shrinkers modeled on round cylinders  $S^j \times \mathbb{R}^{n-j}$  and near the tips they have asymptotic translators modeled on  $\text{Bow}^{j+1} \times \mathbb{R}^{n-j-1}$ . We also obtain a characterization of the round shrinking sphere among ancient  $\alpha$ -Andrews flows, and logarithmic asymptotics.

## 1. Introduction

In this article, we study ancient solutions of the mean curvature flow. Recall that a one-parameter family of embedded hypersurfaces  $M_t \subset \mathbb{R}^{n+1}$  moves by mean curvature flow if the normal velocity at each point is given by the mean curvature vector. A solution is called ancient if it is defined on a time interval  $(-\infty, T)$ ,  $T \leq \infty$ . Ancient solutions typically arise in the study of singularities and of high curvature regions (see e.g. [7, 9, 11, 12, 17, 23, 24]). They also arise in conformal field theory, where they describe the ultraviolet regime of the boundary renormalization group equation (see e.g. [5, 8, 19]).

Daskalopoulos, Hamilton and Sesum have obtained a complete classification of ancient solutions in the case of closed embedded curves [6]. In higher dimensions, based on formal matched asymptotics, Angenent recently conjectured the existence of ancient ovals [3], i.e. ancient solutions that for  $t \rightarrow 0$  collapse to a round point, but for  $t \rightarrow -\infty$  become more and more oval in the sense that they look like round cylinders  $S^j \times \mathbb{R}^{n-j}$  near the central region and like translating solitons  $\text{Bow}^{j+1} \times \mathbb{R}^{n-j-1}$  near the tips.

In fact, variants of Angenent's conjecture have been proved already by White [24] and Wang [22]. Namely, by considering convex regions of increasing eccentricity and using a limiting argument, White proved the existence

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R.H. has been partially supported by NSF grant DMS-1406394. O.H. has been partially supported by NSF grant DMS-1105656.

of ancient flows of compact, convex sets that are not selfsimilar [24, p. 134].<sup>1</sup> Using a related construction, phrased in the language of PDE estimates for the level set equation, Wang proved the existence of ancient convex solutions of the mean curvature flow that are not  $k$ -rotationally symmetric [22, Thm. 4.1]. Moreover, he proved that one can always find a blowdown limit that is either a sphere, a cylinder, or a multiplicity two plane [22, Cor. 6.3].

The main purpose of this article is to carry out the above construction in more detail — including in particular the study of the geometry at the tips — and thus to prove the existence of ancient ovals of the form conjectured by Angenent. We phrase our variant of the construction in the framework of the estimates from Haslhofer-Kleiner [11], see also Section 2. This allows us to give a proof that is short and easy to read.

We write  $H$  for the mean curvature,  $A$  for the second fundamental form,  $\kappa_1 \leq \dots \leq \kappa_n$  for the principal curvatures, and  $\text{Bowl}$  for the unique rotationally invariant convex translating soliton [1], normalized such that the mean curvature at the tip equals one.

**Theorem 1.1 (Existence of ancient ovals).** *For every  $1 \leq j \leq n - 1$ , there exists an ancient solution  $\{M_t \subset \mathbb{R}^{n+1}\}_{t \in (-\infty, 0)}$  of the mean curvature flow with compact and strictly convex time slices that for  $t \rightarrow 0$  converges to a round point and for  $t \rightarrow -\infty$  has the following asymptotics:*

- *asymptotic shrinker: for  $\lambda \rightarrow \infty$  the parabolically rescaled flows  $\lambda^{-1}M_{\lambda^2 t}$  converge to the round shrinking cylinder  $S^j(\sqrt{2j|t|}) \times \mathbb{R}^{n-j}$ .*
- *asymptotic translators: given any direction  $v \in 0_{\mathbb{R}^{j+1}} \times \mathbb{S}^{n-j-1}$ , the blow-downs at the tip in direction  $v$  (see Claim 3.7) converge to  $\text{Bowl}(v)_t^{j+1} \times \mathbb{R}^{n-j-1}$ , where  $\text{Bowl}(v)_t^{j+1} \subset \mathbb{R}^{j+1} \times \langle v \rangle$  translates in direction  $-v$  and  $\mathbb{R}^{n-j-1}$  is the orthogonal complement of  $\langle v \rangle$  in  $\mathbb{R}^{n-j}$ .*

Moreover, our solutions are  $\alpha$ -Andrews noncollapsed for some  $\alpha = \alpha(n) > 0$  (see Def. 1.3), are  $O_{j+1} \times O_{n-j}$ -symmetric, and also have the following additional properties:

- a) *uniformly  $(n - j + 1)$ -convex:  $\liminf_{t \rightarrow -\infty} \inf_{M_t} \frac{\kappa_1 + \dots + \kappa_{n-j+1}}{H} > 0$ ,*
- b) *unbounded rescaled diameter:  $\lim_{t \rightarrow -\infty} |t|^{-1/2} \text{diam}(M_t) = \infty$ ,*

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<sup>1</sup>Although not explicitly stated there, it of course follows from Huisken’s monotonicity formula [14] and the arguments in [23, 24], that the tangent flows at the singularity are shrinking spheres and the tangent flows at infinity are shrinking cylinders or a multiplicity two plane (since otherwise the whole flow would be self-similar).

c) *curvature decay of type II*:  $\limsup_{t \rightarrow -\infty} |t|^{1/2} \sup_{M_t} |A| = \infty$ .

**Remark 1.2.** For  $j = n - 1$  ( $n \geq 2$ ) our solutions are analogous to Perelman's example for three-dimensional Ricci flow [20, Ex. 1.4].

In physical terms, our solutions can be thought of as phase transitions between the sphere in the infrared and the cylinders in the ultraviolet.

Our proof is based on the recent estimates of Haslhofer-Kleiner [11], see also White [23, 24] and Huisken-Sinestrari [15, 16]. These estimates have been developed in the context of mean convex (i.e.  $H \geq 0$ ) mean curvature flows satisfying the conclusion of Andrews' beautiful noncollapsing result [2], see also White [23] and Sheng-Wang [21]. Let us now recall the definition:

**Definition 1.3 (Andrews condition [11, Def. 1.1]).** Let  $\alpha > 0$ . A mean convex mean curvature flow  $\{M_t\}$  is called  $\alpha$ -Andrews if for every  $p \in M_t$  there are interior and exterior balls tangent at  $p$  of radius at least  $\frac{\alpha}{H(p)}$ .

**Remark 1.4.** There is a more general notion for weak solutions, but by [11, Thm. 1.14] ancient  $\alpha$ -Andrews flows are automatically smooth until they become extinct. Also recall that, by the maximum principle, mean convexity and the Andrews condition are both preserved under mean curvature flow.

For comparison, we also prove the following theorem which characterizes the round shrinking sphere in the class of ancient  $\alpha$ -Andrews flows.

**Theorem 1.5 (Characterization of the sphere).** Let  $\{M_t \subset \mathbb{R}^{n+1}\}_{t \in (-\infty, 0)}$  be an ancient  $\alpha$ -Andrews flow and assume at least one of the following conditions is satisfied:

- a) *it is uniformly convex*:  $\liminf_{t \rightarrow -\infty} \inf_{M_t} \frac{\kappa_1}{H} > 0$ ,
- b) *it has bounded rescaled diameter*:  $\limsup_{t \rightarrow -\infty} |t|^{-1/2} \text{diam}(M_t) < \infty$ ,
- c) *the time slices are compact and the curvature decay is type I*:  
 $\limsup_{t \rightarrow -\infty} |t|^{1/2} \sup_{M_t} |A| < \infty$ .

Then  $\{M_t\}$  is a family of round shrinking spheres.

**Remark 1.6.** A related result for closed, ancient, convex solutions of the mean curvature flow has been announced recently by Huisken-Sinestrari [18]. Our proof based on the Andrews condition seems to be much shorter, though.

Theorem 1.5 shows that the additional properties a)–c) in Theorem 1.1 are in a sense sharp. Namely, if any of them is strengthened slightly, then this

forces the solution to be a family of round shrinking spheres. Nevertheless, one may wonder if there is a more quantitative growth rate for the diameter as predicted by the formal asymptotics of Angenent [3]. Indeed, by combining our existence result from Theorem 1.1 and the unique asymptotics result from Angenent-Daskalopoulos-Sesum [4], we obtain:

**Corollary 1.7 (Logarithmic asymptotics).** *The classical Angenent ovals, i.e. the solutions produced by Theorem 1.1 for  $j = n - 1$ , satisfy the diameter estimate*

$$(1.8) \quad 0 < \limsup_{t \rightarrow -\infty} \frac{\text{diam}(M_t)}{\sqrt{|t| \log |t|}} < \infty.$$

*Organization of the article:* In Section 2 we collect some preliminaries. In Section 3 we prove Theorem 1.1, and in Section 4 we prove Theorem 1.5.

**Acknowledgements.** We thank Bruce Kleiner, Hojoo Lee, Brian White, and the referees for useful comments.

## 2. Preliminaries

Three key ingredient in our proofs are the convexity estimate, the global convergence theorem, and the structure theorem for ancient solutions. The theory of mean convex mean curvature flow has been developed in the fundamental work of White [23, 24] and Huisken-Sinestrari [15, 16]. For our purpose it is most convenient to use the versions of these estimates obtained by the new approach of Haslhofer-Kleiner [11]. As usual, we use the notation  $P(p, t, r) = B(p, r) \times (t - r^2, t]$  for the parabolic ball centered at  $(p, t) \in \mathbb{R}^{n+1} \times \mathbb{R}$ , of size  $r > 0$ .

**Theorem 2.1 (Convexity estimate [11, Thm. 1.10], see also [24, Thm. 8] and [16, Thm. 1.1]).** *For all  $\varepsilon > 0$ ,  $\alpha > 0$ , there exists  $\eta = \eta(\varepsilon, \alpha) < \infty$  with the following property. If  $M_t$  is an  $\alpha$ -Andrews flow in a parabolic ball  $P(p, t, \eta r)$  centered at a point  $p \in M_t$  with  $H(p, t) \leq r^{-1}$ , then*

$$(2.2) \quad \kappa_1(p, t) \geq -\varepsilon r^{-1}.$$

**Theorem 2.3 (Global convergence [11, Thm. 1.12]).** *Let  $M_t^k$  be a sequence of  $\alpha$ -Andrews flows with  $\sup_k H(0, 0) < \infty$  that is defined in parabolic balls  $P(0, 0, r_k)$  centered at  $0 \in M_0^k$  with  $r_k \rightarrow \infty$ . Then there exists a smoothly convergent subsequence,  $M_t^{k_\varepsilon} \rightarrow M_t^\infty$  in  $C_{loc}^\infty$  on  $\mathbb{R}^{n+1} \times (-\infty, 0]$ .*

**Theorem 2.4 (Structure theorem for ancient solutions [11, Part of Thm. 1.14], see also [24, Thm. 1]).** *Ancient  $\alpha$ -Andrews flows in  $\mathbb{R}^{n+1}$  are convex and smooth until they become extinct. The only selfsimilarly shrinking ones are the sphere, the cylinders and the plane.*

Let us also recall the rigidity case of Hamilton’s Harnack inequality. Since we don’t know a-priori that the limits obtained using Theorem 2.3 have bounded curvature, this requires some minor adjustments.

**Theorem 2.5 (Rigidity of Hamilton’s Harnack inequality [9, Theorem B]).** *Let  $\{M_t \subset \mathbb{R}^{n+1}\}_{t \in (-\infty, \infty)}$  be a convex eternal mean curvature flow that satisfies Hamilton’s Harnack inequality. Assume that the mean curvature attains a critical value at a point in space-time. Then  $M_t$  is a translating soliton.*

**Remark 2.6.** Hamilton assumes that the mean curvature attains a maximum. However, his discussion of the equality case of the maximum principle goes through verbatim for critical values. Also, note that the strict maximum principle is local, and thus does not require curvature bounds.

### 3. Existence of ancient ovals

*Proof of Theorem 1.1.* Fix  $1 \leq j \leq n - 1$ . Although not strictly necessary, it is convenient to construct the solutions in an  $O_{j+1} \times O_{n-j}$  symmetric way.

For every  $\ell \in \mathbb{N}$  we construct a hypersurface  $M^\ell$  as follows. We take the product  $S^j(1) \times B^{n-j}(\ell)$ , where  $B^{n-j}(\ell)$  is the ball of radius  $\ell$ , and cap it off in a rotationally symmetric, strictly convex,  $\ell$ -independent way, say at a scale of length one. To see how the capping can be done, let  $\phi \in C^2([0, 1])$  be a strictly decreasing, nonnegative, concave function satisfying  $\phi(\xi)|_{[0, \delta]} = \sqrt{1 - \xi^2}$  and  $\phi(\xi)|_{[1-\delta, 1]} = (1 - \xi^2)^{1/3}$  for some  $\delta > 0$ . Using polar co-ordinates  $(r, \theta)$  for  $\mathbb{R}^{j+1}$  and polar co-ordinates  $(s, \mu)$  for  $\mathbb{R}^{n-j}$ , we can choose the caps to be the closure of the set

$$(3.1) \quad \begin{aligned} \{((r, \theta), (\ell + \phi(r), \mu)) \in \mathbb{R}^{j+1} \times \mathbb{R}^{n-j} \\ \text{for } r \in (0, 1], \theta \in S^j, \mu \in S^{n-j-1}\}. \end{aligned}$$

We therefore have:

- The hypersurfaces  $M^\ell$  are uniformly  $n - j + 1$  convex, i.e.  $\kappa_1 + \dots + \kappa_{n-j+1} \geq \beta H$  for some  $\beta = \beta(n) > 0$  uniformly for all  $\ell$ .

- There exists an  $\alpha = \alpha(n) > 0$  such that  $M^\ell$  is  $\alpha$ -Andrews for all  $\ell$ .

Let  $M_t^\ell$  be the mean curvature flow starting at  $M^\ell$  at  $t = 0$ . By Huisken’s classical theorem [13] (see also Andrews [2, Rmk. 6]) the flow becomes extinct in a round point in finite time  $T_{\text{ext}} < \infty$ . As  $S^n(1)$  lies entirely inside the domain bounded by  $M^\ell$ , while the cylinder  $S^j(1) \times \mathbb{R}^{n-j}$  lies entirely outside of it, and as both the flow of the sphere and the cylinder become extinct in time comparable to one, we see that  $T_{\text{ext}}$  is comparable to one.

Now, let  $\{\hat{M}_t^\ell\}_{t \in [T_\ell, 0]}$  (where  $T_\ell < -1$  denotes the new initial time) be the sequence of solutions obtained by parabolically rescaling  $M_t^\ell$  and shifting the time parameter such that:

- The flow becomes extinct at  $t = 0$ .
- The ratio of the major radius  $a(t) = \max_{x \in \hat{M}_t^\ell} (\sum_{i=j+2}^{n+1} x_i^2)^{1/2}$  and the minor radius  $b(t) = \max_{x \in \hat{M}_t^\ell} (\sum_{i=1}^{j+1} x_i^2)^{1/2}$  equals 2 for the first time at  $t = -1$ .

**Claim 3.2.** *There exists  $C < \infty$  independent of  $\ell$  such that*

$$(3.3) \quad C^{-1} \leq \text{diam}(\hat{M}_{-1}^\ell) \leq C.$$

*Proof.* Since the flow becomes extinct in roughly one unit of time, the diameter bounds follow from comparison with spheres. Indeed, if the diameter was too small then  $\hat{M}_{-1}^\ell$  would be surrounded by a sphere of very small radius, which becomes extinct too soon. Similarly, using convexity and  $\frac{a(-1)}{b(-1)} = 2$ , we see that if the diameter was too large,  $\hat{M}_{-1}^\ell$  would surround a sphere of large radius, contradicting again extinction in roughly one unit of time.  $\square$

**Claim 3.4.**  $\lim_{\ell \rightarrow \infty} T_\ell = -\infty$

*Proof.* Consider a time  $t_0 < -1$ . Using convexity and  $a(t_0)/b(t_0) \geq 2$ , we can put a sphere of radius  $b(t_0)/4$  inside  $\hat{M}_{t_0}$  at distance  $a(t_0)/2$  from the origin. Thus, by avoidance, it takes  $a(t)$  a time period of at least  $b(t_0)^2/32n$  to decrease by a factor one-half. On the other hand,  $b(t)$  decreases with time and by Claim 3.2, we know that  $b(t_0) \geq \delta$  for some  $\delta > 0$ . Thus, it takes the quotient  $\frac{a(t)}{b(t)}$  a time period of at least  $\delta^2/32n$  to decrease by a factor one-half. Since  $a(T_\ell)/b(T_\ell) \rightarrow \infty$  and  $a(-1)/b(-1) = 2$ , the claim follows.  $\square$

The same argument as in the proof of Claim 3.2 implies that there exist some  $C < \infty$  such for every  $\ell \in \mathbb{N}$  and  $-1 < t < 0$ ,  $d(\hat{M}_t^\ell, 0) \leq C\sqrt{|t|}$ .

Letting  $x_t^\ell \in \hat{M}_t^\ell$  be a point with  $d(\hat{M}_t^\ell, 0) = d(x_t^\ell, 0)$ , we see that at every time of differentiability of the locally Lipschitz function  $t \mapsto d(\hat{M}_t^\ell, 0)$ ,  $H(x_t^\ell, t) = -\frac{d}{dt}d(\hat{M}_t^\ell, 0)$ , so  $\int_t^0 H(x_s^\ell, s)ds \leq C\sqrt{|t|}$ . Thus, for every  $\ell \in \mathbb{N}$  and  $k \in \mathbb{N}$  there exist  $t_k^\ell \in [-1/k, 0]$  and  $p_k^\ell \in \hat{M}_t^\ell$  such that  $d(p_k^\ell, 0) \leq C/\sqrt{k}$  and  $H(p_k^\ell, t_k^\ell) \leq C\sqrt{k}$ . Therefore, by the global convergence theorem (Theorem 2.3), and a diagonal argument, the sequence  $\hat{M}_t^\ell$  subconverges smoothly to an ancient  $\alpha$ -Andrews flow  $M_t$ , with convex compact time slices, that becomes extinct in a round point at the origin at time  $t = 0$ . In fact,  $M_t$  is strictly convex. To see this, note that  $H > 0$  everywhere (by the Andrews condition, as the flow is clearly not a static plane), and recall that mean curvature  $H$  and the second fundamental form  $A = A_j^i$  satisfy the evolution equations

$$(3.5) \quad \partial_t H = \Delta H + |A|^2 H, \quad \partial_t A = \Delta A + |A|^2 A.$$

Denoting by  $\kappa_1(x, t)$  the smallest principal curvature of  $M_t$ , if the flow was not strictly convex, the quantity  $\frac{\kappa_1}{H}$  would attain a global space-time minimum at some point  $(x, t)$ . The strict maximum principle for tensors (see e.g. [10] and [24, Appendix]) would then imply that the flow splits orthogonally as  $M_t = B_t \times \mathbb{R}$ , contradicting the fact that  $M_t$  has compact time slices. From the condition  $\frac{a(-1)}{b(-1)} = 2$  we see that  $M_t$  is certainly not a family of shrinking spheres. As we obtained  $M_t$  as a limit of rescalings of  $O_{j+1} \times O_{n-j}$  symmetric flows which are uniformly  $(n - j + 1)$ -convex,  $M_t$  has that symmetry and convexity property (i.e. property (a) in the statement of Theorem 1.1) as well. To finish the proof of Theorem 1.1, it remains to establish the other claimed properties for  $t \rightarrow -\infty$ .

Let us start with the asymptotic shrinker. For every  $\lambda > 0$ , the flow  $\bar{M}_t^\lambda = \lambda^{-1}M_{\lambda^2 t}$  becomes extinct at a round point  $(0, 0)$ . The argument after the proof of Claim 3.4 shows that for every  $\lambda > 0$  and  $k \in \mathbb{N}$  there exist  $t_k^\lambda \in [-1/k, 0]$  and  $p_k^\lambda \in \bar{M}_t^\lambda$  such that  $d(p_k^\lambda, 0) \leq C/\sqrt{k}$  and  $H(p_k^\lambda, t_k^\lambda) \leq C\sqrt{k}$ . Thus, by the global convergence theorem (Thm. 2.3), a diagonal argument, and Huisken’s monotonicity formula [14] (see also [11, App. B]) the flows  $\bar{M}_t^\lambda$  subconverge (as  $\lambda \rightarrow \infty$ ) to a nonempty selfsimilar shrinking flow  $N_t$ . By the structure theorem (Thm. 2.4),  $N_t$  must be a family of shrinking spheres, cylinders or a plane. As  $M_t$  becomes extinct in a round point, it follows from Huisken’s monotonicity formula that  $N_t$  is not the plane. The condition  $a(t)/b(t) \geq 2$  for all  $t \leq -1$  excludes the possibility of  $N_t$  being the sphere as well. Due to the  $O_{j+1} \times O_{n-j}$  symmetry, it therefore must be  $S^j(\sqrt{2j|t|}) \times \mathbb{R}^{n-j}$ . In particular, the limit is unique (the axis is also unique

due to symmetry), and thus a full limit. The existence of an asymptotic cylinder clearly implies that the solution has unbounded rescaled diameter, i.e. proves property (b). Property (c) follows from property (b) by integration, as the velocity of the flow is (by definition) the mean curvature.

Finally, let us discuss the asymptotic translators. Fix a direction  $v \in 0_{\mathbb{R}^{j+1}} \times \mathbb{S}^{n-j-1}$ . Let  $p_t$  be the unique point at the tip of  $M_t$  in direction  $v$ , i.e. the unique point in  $M_t$  that can be written as  $\mu v$  for some  $\mu > 0$ . We now perform the modified type II blow-down as follows. Pick times  $t_k$  such that

$$(3.6) \quad |t_k|^{1/2}H(p_{t_k}, t_k) = \max_{t \in [-k, -1]} |t|^{1/2}H(p_t).$$

**Claim 3.7.** *Let  $\hat{M}_{\hat{t}}^k$  be the sequence of flows obtained by shifting  $(p_{t_k}, t_k)$  to the origin and normalizing  $\lambda_k = H(p_{t_k}, t_k)$  to one, explicitly  $\hat{M}_{\hat{t}}^k = \lambda_k \cdot (M_{\hat{t}/\lambda_k^2 + t_k} - p_{t_k})$ . Then  $\hat{M}_{\hat{t}}^k$  converges to  $\text{Bowl}(v)_{\hat{t}}^{j+1} \times \mathbb{R}^{n-j-1}$ .*

*Proof.* Let  $r(t) = d(p_t, 0)$ . By the proof of property (b) and (c) we actually have  $\lim_{t \rightarrow -\infty} |t|^{-1/2}r(t) = \infty$  and  $\limsup_{t \rightarrow -\infty} |t|^{1/2}H(p_t) = \infty$ , as points of bounded rescaled distance to zero must have a nonzero  $\mathbb{R}^{j+1}$  component (since the shrinker at  $-\infty$  is the cylinder). In particular,  $t_k \rightarrow -\infty$  as  $k \rightarrow \infty$ . By construction the flows  $\hat{M}_{\hat{t}}^k$  satisfy  $H(0, 0) = 1$  and are defined for  $-\infty < \hat{t} < \lambda_k^2|t_k| \rightarrow \infty$ . Moreover, by condition (3.6) we have:

$$(3.8) \quad \hat{H}^2(\hat{p}, \hat{t}) = \frac{H^2(p, \hat{t}/\lambda_k^2 + t_k)}{\lambda_k^2} \leq \left| \frac{t_k}{\hat{t}/\lambda_k^2 + t_k} \right| = \left| \frac{1}{1 - \hat{t}/(\lambda_k^2|t_k|)} \right| \rightarrow 1$$

for  $p$  at the tip and all  $\hat{t}$  with  $0 \leq \hat{t} < \lambda_k^2(|t_k| - 1)$ .

By the global convergence theorem (Thm. 2.3),  $\hat{M}_{\hat{t}}^k$  subconverges to an eternal convex  $\alpha$ -Andrews flow  $\{N_t\}_{t \in (-\infty, \infty)}$  with  $H > 0$  everywhere (as it is not flat). By symmetry,  $\hat{M}_{\hat{t}}^k \cap (\{0\}_{\mathbb{R}^{j+1}} \times \mathbb{R}^{n-j})$  is an  $n - j - 1$  sphere of radius  $\hat{r}_k$ , and as

$$(3.9) \quad \hat{r}_k = \lambda_k r(t_k) = \lambda_k |t_k|^{1/2} |t_k|^{-1/2} r(t_k) \rightarrow \infty,$$

this implies that if  $0 \leq \kappa_1(x, t) \leq \kappa_2(x, t) \leq \dots \leq \kappa_n(x, t)$  are the principal curvatures of  $N_t$  at  $x \in N_t$ , then  $\kappa_1(0, 0) = \dots = \kappa_{n-j-1}(0, 0) = 0$ . Recall



that  $N_t$  is convex and that  $A = A_j^t$  satisfies the evolution equation

$$(3.10) \quad \partial_t A = \Delta A + |A|^2 A.$$

As the quantities  $\frac{\kappa_1}{H}, \dots, \frac{\kappa_{n-j-1}}{H}$  obtain a global (space-time) minimum 0 at  $(0, 0)$ , the strict maximum principle for tensors (see e.g. [10] and [24, Appendix]) implies that the flow  $N_t$  splits orthogonally as  $N_t = B_t \times \mathbb{R}^{n-j-1}$ , where  $B_t$  is strictly convex.

Note that  $B_t$  is  $O_{j+1}$  symmetric. In particular,  $\nabla H = 0$  at the origin. Since  $B_t$  arises as a smooth limit of compact solutions it satisfies Hamilton’s Harnack inequality [9, Thm. A], in particular  $\partial_t H \geq 0$ . Together with equation (3.8) this implies  $\partial_t H = 0$  at the origin (for all times  $t > 0$ ). Thus, by the equality case of Hamilton’s Harnack inequality (Thm. 2.5),  $B_t$  must be a translating soliton. By rotational symmetry, it must be the bowl. Finally, due to uniqueness, the subsequential limit is actually a full limit.  $\square$

This finishes the proof of Theorem 1.1.  $\square$

**Remark 3.11.** Contrary to the standard type II blow-up procedure (see e.g. [16, Sec. 4]), we could not select points  $(p, t)$  with maximal  $|t|^{1/2}H(p, t)$  over all times and points in those time slots. Since we didn’t know a-priori the curvature is maximal at the tip, we needed the full strength of the global convergence theorem (Thm. 2.3) to pass to a smooth limit.

### 4. Charaterizations of shrinking spheres

*Proof of Theorem 1.5.* By the structure theorem for ancient solutions (Theorem 2.4) ancient  $\alpha$ -Andrews flows are convex and smooth until they become extinct. Arguing as in the proof of Theorem 1.1, we can find an asymptotic shrinker. Namely, there is a sequence of of positive numbers  $\{\lambda_k\}_{k=1}^\infty$  with  $\lambda_k \rightarrow \infty$  such that  $\lambda_k^{-1}M_{-\lambda_k^2 t} \rightarrow N_t$ , where  $N_t$  is either a round shrinking cylinder  $S^j \times \mathbb{R}^{n-j}$  or a round shrinking sphere  $S^n$ . However, any of the assumptions (a)–(c) excludes the cylinders. Indeed, as the condition  $\frac{\kappa_1}{H} \geq \delta > 0$  is preserved under scalings and smooth limits,  $N_t$  must satisfy it, so in particular, it can not split off a line. Assumption (b) implies that there exists some  $C < \infty$  such that for every  $k$  sufficiently large  $\text{diam}(\lambda_k^{-1}M_{-\lambda_k^2 t}) \leq C$ . But then  $\text{diam}(N_{-1}) \leq C$  so it can not be a cylinder in case (b) either. Assumption (c) in turn implies assumption (b), as assuming there exist some  $C < \infty$  and  $T > -\infty$  such that  $|A| \leq C|t|^{-1/2}$  for  $t < T$ , and letting

$D = \text{diam}(M_T)$ , we get by integration

$$(4.1) \quad \text{diam}(M_t) \leq D + 4C\sqrt{n|t|}$$

for every  $t < T$ . In all cases, we conclude that the asymptotic shrinker  $N_t$  is a round shrinking sphere. In particular, it follows that  $M_t$  is compact. Thus, by Huisken's classical theorem [13],  $M_t$  becomes extinct in a round point.

Now since  $N_t$  is a family of round shrinking spheres and  $M_t$  shrinks to a round point, it follows from the scale invariance of Huisken's entropy that the entropy (w.r.t. the extinction point in space time) is in fact constant along the flow  $M_t$ . By the equality case of Huisken's monotonicity formula [14] (see also [11, App. B]) the flow  $M_t$  must be a shrinker, which as observed above becomes extinct at a round point, i.e. it must be a family of round shrinking spheres.  $\square$

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RECEIVED JANUARY 8, 2014