Communications in
Analysis and Geometry
Volume 24, Number 4, 821–841, 2016

Austere submanifolds in $\mathbb{C}P^n$

Marianty Ionel and Thomas A. Ivey

For an arbitrary submanifold $M \subset \mathbb{C}P^n$ we determine conditions under which it is *austere*, i.e., the normal bundle of M is special Lagrangian with respect to Stenzel's Ricci-flat Kähler metric on $T\mathbb{C}P^n$. We also classify austere surfaces in $\mathbb{C}P^n$.

1. Introduction

Special Lagrangian submanifolds were introduced in 1982 by Harvey and Lawson in their seminal paper [2]. They studied them in the more general context of calibrated submanifolds, which are a special class of minimal submanifolds. Calibrated submanifolds, in particular special Lagrangian submanifolds, play an important role in mirror symmetry and they have lately been the object of extensive study (see Joyce [6] and the references contained therein).

Let $\mathcal{X}$ be a Calabi-Yau manifold of complex dimension n with Kähler form Ω and holomorphic volume form Θ. Recall that an oriented submanifold L of real dimension n is *special Lagrangian* if it is calibrated by $\operatorname{Re}\Theta$. Harvey and Lawson showed that L is special Lagrangian if and only if $\Omega|_L \equiv 0$ and $\operatorname{Im}\Theta|_L \equiv 0$. (The same is true if we replace Θ by $e^{\mathrm{i}\phi}\Theta$, in which case L is said to be special Lagrangian *with phase* $e^{\mathrm{i}\phi}$.) In the same paper [2], Harvey and Lawson exhibited a construction of special Lagrangian submanifolds as total spaces of certain vector bundles. Specifically, they showed that the conormal bundle N^*M of an immersed submanifold $M^k \subset \mathbb{R}^n$ is special Lagrangian in $\mathbb{C}^n \cong T^*\mathbb{R}^n$ (with phase depending on k and n) if and only if M^k is *austere* in $\mathbb{R}^n$, i.e., the second fundamental form of M in any normal direction has its eigenvalues symmetrically arranged around 0. (This is equivalent to saying that all the odd-degree symmetric polynomials in the eigenvalues of the second fundamental form vanish identically.) Note that the austere condition implies that M is minimal, but in general is stronger than minimality.

In the early 1990s, Stenzel [10, 11] showed that the cotangent bundle of any compact rank one symmetric space can be endowed with a Ricci-flat Kähler metric, which is now called the *Stenzel metric*. Particular cases of

Stenzel metrics were initially discovered by Eguchi-Hanson on the cotangent bundle T^*S^2 and by Candela-de la Ossa on T^*S^3. In [7] Karigiannis and Min-Oo generalized Harvey and Lawson's construction to the cotangent bundle of S^n carrying the Stenzel metric. Specifically, they showed that the conormal bundle over an immersed submanifold $M \subset S^n$ is special Lagrangian with respect to some phase if and only if all the odd-degree symmetric polynomial in the eigenvalues of the second fundamental form, in any normal direction, vanish identically. In other words, this is the same austere condition as Harvey and Lawson found in [2] for $\mathbb{R}^n$. This is perhaps surprising, since the complex structure on T^*S^n is not the standard one (as it is in the case of $\mathbb{C}^n \cong T^*\mathbb{R}^n$) but instead is obtained by identifying it with a complex affine hyperquadric in $\mathbb{C}^{n+1}$.

In this paper, we further generalize the Harvey-Lawson construction to the case of $T^*\mathbb{C}P^n$, the cotangent bundle of complex projective space. We define $M \subset \mathbb{C}P^n$ to be *austere* if its conormal bundle N^*M is special Lagrangian in $T^*\mathbb{C}P^n$, with respect to the Stenzel metric. We will calculate conditions on the second fundamental form of M that are necessary and sufficient for M to be austere. (In fact, we will work on the normal bundle NM, using the standard metric on $\mathbb{C}P^n$ to identify $T^*\mathbb{C}P^n$ with $T\mathbb{C}P^n$.)

We now give a brief description of the contents of the paper.

- In section 2, we define a mapping Φ that identifies $T\mathbb{C}P^n$ with a Stein manifold which is the complement of a complex quadric in $\mathbb{C}P^n \times \mathbb{C}P^n$. (We do this in order to utilize the convenient expression of the Stenzel Kähler form given by Lee [8] in terms of coordinates on $\mathbb{C}P^n \times \mathbb{C}P^n$.) We calculate the differential of the restriction of Φ to NM using moving frames. We prove that if $M \subset \mathbb{C}P^n$ is an arbitrary immersed submanifold, then NM is a Lagrangian submanifold of $T\mathbb{C}P^n$ with respect to the Stenzel Kähler form (see Prop. 1).[1]
- In section 3, we determine the conditions under which an immersed submanifold $M \subset \mathbb{C}P^n$ is austere (see Theorem 4). A corollary of this result is that if $M \subset \mathbb{C}P^n$ is an arbitrary complex submanifold then M is austere (see Corollary 2).

[1] Note that while it is a basic result that the conormal bundle of any submanifold of a space $\mathfrak{X}$ is a Lagrangian submanifold of $T^*\mathfrak{X}$ with respect to its standard symplectic form, our Prop. 1 is non-trivial since the Stenzel Kähler form is not the standard symplectic structure.

- Finally, in section 4, we classify the austere surfaces in $\mathbb{C}P^n$, showing that they must be either holomorphic curves or totally geodesic (see Theorem 7).

While it might seem that there is a dearth of non-holomorphic examples of austere submanifolds in $\mathbb{C}P^n$, in subsequent papers we will investigate the solution space of the austere condition for real hypersurfaces in this geometry, and exhibit new examples of austere hypersurfaces in $\mathbb{C}P^2$ and higher dimensions (see, e.g., [5]).

Before proceeding with the calculations leading up to Proposition 1, we need to make a few remarks:

1) As indicated above, our calculations will be made using moving frames. Because the members of the moving frame are easier to differentiate when they take value in a fixed vector space, we will do most calculations on $\widehat{M} \subset S^{2n+1} \subset \mathbb{C}^{n+1}$, which is the inverse image of M under the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$. (This is the restriction of the projectivization map from $\mathbb{C}^{n+1}/\{0\}$ to $\mathbb{C}P^n$, which we also denote by π.) Similarly, we will do calculations on $NM \subset T\mathbb{C}P^n$ by regarding $T\mathbb{C}P^n$ as a quotient space and working on the inverse image. In more detail, let

$$B = \{(\zeta, \xi) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \,|\, \zeta \neq 0, \xi \cdot \overline{\zeta} = 0\},$$

   wherein the dot product is $\mathbb{C}$-bilinear. Recall that for a one-dimensional subspace $L \subset \mathbb{C}^{n+1}$, $T_L\mathbb{C}P^n$ is canonically defined as the set of $\mathbb{C}$-linear maps f from L to the quotient vector space $\mathbb{C}^{n+1}/L$. If f is such a map and ζ is a nonzero point on L, there is a unique ξ which projects to $f(\zeta)$ in the quotient vector space and satisfies $\xi \cdot \overline{\zeta} = 0$. Thus, $T\mathbb{C}P^n \cong B/\mathbb{C}^*$, where the $\mathbb{C}^*$ action on B is $(\zeta, \xi) \mapsto (\lambda\zeta, \lambda\xi)$ for $\lambda \in \mathbb{C}^*$.

2) Although the cotangent bundle of any complex manifold $\mathfrak{X}$ has a standard complex structure (obtained by identifying it with the bundle of $(1,0)$-forms), the complex structure underlying the Stenzel metric on $T^*\mathbb{C}P^n$ is not the standard one. For example, under the mapping Φ the image of the zero section is a totally real submanifold.

3) For an arbitrary submanifold $M \subset \mathbb{C}P^n$ and $p \in M$, let $\mathcal{H}_p$ and $\mathcal{N}_p$ be maximal J-invariant subspaces of T_pM and N_pM respectively (where J denotes the ambient complex structure). We'll assume that $\mathcal{H} =$

$\bigcup_{p\in M}\mathcal{H}_p$ and $\mathcal{N}=\bigcup_{p\in M}\mathcal{N}_p$ are smooth sub-bundles of TM and NM, and let $\mathcal{D}$ and $\mathcal{E}$ be their respective orthogonal complements. Thus, we have an orthogonal splitting

$$T\mathbb{C}P^n|_M = TM \oplus NM = (\mathcal{H}\oplus\mathcal{D})\oplus(\mathcal{E}\oplus\mathcal{N}) \tag{1}$$

such that $\mathcal{D}\oplus\mathcal{E}$ is J-invariant and $\operatorname{rk}\mathcal{D}=\operatorname{rk}\mathcal{E}\le n$. As we will see in the statement of Theorem 4, the austere condition along a fiber of NM depends on how the corresponding normal vector splits into components in $\mathcal{E}$ and $\mathcal{N}$.

We gratefully acknowledge the comments and helpful discussions with the following mathematicians during our work on this paper: Henri Anciaux, Ronan Conlon, Spiro Karigiannis, JaeHyouk Lee, Conan Leung, Paul-Andi Nagy, and Pat Ryan.

2. Lagrangian submanifolds via normal bundles

Throughout what follows, $M\subset\mathbb{C}P^n$ will denote a submanifold of real dimension k. As above, let $\widehat{M}=\pi^{-1}(M)$ where $\pi: S^{2n+1}\to\mathbb{C}P^n$ is the Hopf fibration. (We will regard points of S^{2n+1}, as well as tangent vectors to the sphere, as vectors in $\mathbb{C}^{n+1}$.) Let $\mathcal{F}$ denote the bundle of oriented orthonormal frames $(e_0,\ldots,e_{2n})$ along $\widehat{M}$ that are *adapted* in the sense that $e_0,\ldots,e_k$ are tangent to $\widehat{M}$ and $e_0=\mathsf{i}\mathsf{z}$ is tangent to the fiber of π at $\mathsf{z}\in\widehat{M}$. (It follows that $e_1,\ldots,e_{2n}$ are orthogonal to the fiber of π, and so are called *horizontal* vectors.) The fibers of $\pi:\widehat{M}\to M$ are, of course, orbits of the S^1 action $\mathsf{z}\mapsto e^{\mathsf{i}\theta}\mathsf{z}$. This action extends to $\mathcal{F}$, by simultaneously multiplying all frame vectors by $e^{\mathsf{i}\theta}$, and preserves horizontality.

Given any $\mathsf{z}\in\widehat{M}$ and any normal vector $\mathsf{n}\in N_{\mathsf{z}}\widehat{M}$ (necessarily horizontal), there exists an adapted frame at z such that $\mathsf{n}=te_{2n}$ for some $t\in\mathbb{R}$. Thus, the mapping

$$\varrho : ((\mathsf{z}, e_0, e_1,\ldots,e_{2n}), t)\mapsto te_{2n}\in T_{\mathsf{z}}S^{2n+1}$$

is a submersion from $\mathcal{F}\times\mathbb{R}$ to $N\widehat{M}$. Because any normal vector $\nu\in N_{\pi(\mathsf{z})}M$ has a horizontal lift in $N_{\mathsf{z}}\widehat{M}$, there is also a submersion $\Pi: N\widehat{M}\to NM$ defined by $\mathsf{n}\mapsto\pi_*\mathsf{n}$. We define yet another submersion

$$\rho : ((\mathsf{z}, e_0, e_1,\ldots,e_{2n}), t)\mapsto (\mathsf{z}, te_{2n})\in B$$

that lifts the natural inclusion $\imath : NM \to T\mathbb{C}P^n$. In other words, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F} \times \mathbb{R} & \xrightarrow{\rho} & B \\ \varrho \downarrow & & \\ N\widehat{M} & & \downarrow \\ \Pi \downarrow & & \\ NM & \xrightarrow{\imath} & T\mathbb{C}P^n \end{array}$$

where the vertical map at right is quotient by the $\mathbb{C}^*$ action.

Let $\mathcal{M}$ denote the open subset in $\mathbb{C}P^n \times \mathbb{C}P^n$ defined in homogeneous coordinates by $\displaystyle\sum_{i=0}^{n} z_i w_i \neq 0$. (This space is denoted by M^{2n}_{II} in [8].) To generate an embedding of NM as a submanifold in the Stein manifold $\mathcal{M}$, we will compose ρ with a map $\widehat{\Phi} : B \to \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ defined by

$$\widehat{\Phi}(\zeta, \xi) = \left((\cosh \mu)\zeta + \mathrm{i} \left(\frac{\sinh \mu}{\mu} \right) \xi ; (\cosh \mu)\bar{\zeta} + \mathrm{i} \left(\frac{\sinh \mu}{\mu} \right) \bar{\xi} \right), \qquad \mu = \frac{|\xi|}{|\zeta|}. \tag{2}$$

(This is adapted from the work of Szöke [12].) Here and below, we write elements of $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ as an ordered pair of row vectors in $\mathbb{C}^{n+1}$ separated by a semicolon.

It is easy to check that, relative to the $\mathbb{C}^*$ action on B and projectivization on each factor in $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$, $\widehat{\Phi}$ covers a well-defined embedding $\Phi : T\mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$ that identifies $T\mathbb{C}P^n$ bijectively with $\mathcal{M}$. We will compute the differential of the composition of Φ with the inclusion $\imath$ by computing the differential of the composition of maps along the top edge of the following diagram

$$\begin{array}{ccccccc} \mathcal{F} \times \mathbb{R} & \xrightarrow{\rho} & B & \xrightarrow{\widehat{\Phi}} & \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} & & \\ \varrho \downarrow & & & & & \searrow \widehat{\mathfrak{A}} & \\ N\widehat{M} & & \downarrow & & \downarrow & & \\ \Pi \downarrow & & & & & & \\ NM & \xrightarrow{\imath} & T\mathbb{C}P^n & \xrightarrow{\Phi} & \mathbb{C}P^n \times \mathbb{C}P^n & \xrightarrow{\mathfrak{A}} & \mathbb{C}^{2n} \end{array} \tag{3}$$

in which $\mathfrak{A}$ denotes an affine coordinate chart (to be specified below) and $\widehat{\mathfrak{A}}$ is the corresponding map in terms of homogeneous coordinates. As we will see at the end of the next section, the differential along the top edge annihilates vectors that are tangent to the fibers of the map $\Pi \circ \varrho$.

2.1. Geometry of the frame bundle

On $\mathcal{F}$ we define a set of real-valued 1-forms by expanding the derivatives of the basepoint function and frame vectors in terms of the basis (over $\mathbb{R}$) for $\mathbb{C}^{n+1}$ provided by the frame itself. Taking the index ranges $0 \le a, b \le k$ and $k+1 \le \kappa, \lambda \le 2n$, and using the summation convention, we write

$$dz = e_a \omega^a, \quad de_a = -z\omega^a + e_b \psi^b_a + e_\lambda \psi^\lambda_a, \quad de_\kappa = e_b \psi^b_\kappa + e_\lambda \psi^\lambda_\kappa. \tag{4}$$

(In these equations each term is a vector-valued 1-form, i.e., a section of $\mathbb{C}^{n+1} \otimes T^*\mathcal{F}$. While we omit the tensor product symbol in these equations and write the vector factor on the left, it will occasionally be convenient to write the vector on the right, as in (7), (8), (9) below.) The last equation in (4) has no z terms on the right-hand side because

$$0 = \langle dz, e_\kappa \rangle = -\langle z, de_\kappa \rangle,$$

where $\langle , \rangle$ denotes the real-valued Euclidean inner product on $\mathbb{C}^{n+1}$.

The full oriented orthonormal frame bundle of S^{2n+1} may be identified with the special orthogonal group $SO(2n+2)$, by taking the basepoint and the frame vectors as successive rows of an orthogonal matrix. However, since by default we will regard the frame vectors as taking value in $\mathbb{C}^{n+1}$, to make the identification precise we introduce the convention that for a vector $v \in \mathbb{C}^N$,

$$\widehat{v} := (\operatorname{Re} v; \operatorname{Im} v) \in \mathbb{R}^{2N}.$$

Then, in terms of this notation,

$$\begin{pmatrix} \widehat{z} \\ \widehat{e_0} \\ \widehat{e_1} \\ \vdots \\ \widehat{e_{2n}} \end{pmatrix} \in SO(2n+2). \tag{5}$$

In this way, $\mathcal{F}$ is identified with a submanifold in $SO(2n+2)$, and in fact the 1-forms defined by (4) are the components of the Maurer-Cartan form of $SO(2n+2)$, pulled back to $\mathcal{F}$.

Not all of these 1-forms are linearly independent on $\mathcal{F}$. We will need to clarify the dependencies among the 1-forms on $\mathcal{F}$ involved in the differential of the components z and e_{2n} of the map ρ, which are

$$\omega^0, \omega^\alpha, \psi^0_{2n}, \psi^\alpha_{2n}, \psi^\nu_{2n}.$$

(Here, we introduce new index ranges $1 \leq \alpha, \beta \leq k$ and $k+1 \leq \mu, \nu < 2n$.) Recalling that $e_0 = \mathrm{i}\mathsf{z}$, we have

$$\psi^0_{2n} = \langle e_0, de_{2n}\rangle = -\langle \mathsf{z}, d(\mathrm{i}e_{2n})\rangle = \langle d\mathsf{z}, \mathrm{i}e_{2n}\rangle.$$

Thus, if we define functions $r_\alpha = \langle \mathrm{i}e_{2n}, e_\alpha\rangle$ on $\mathcal{F}$, then $\psi^0_{2n} = r_\alpha \omega^\alpha$. (Note that if M is a complex submanifold then all $r_\alpha = 0$.) We also have

$$0 = d\langle e_{2n}, d\mathsf{z}\rangle = \psi^a_{2n} \wedge \omega^a = (\psi^\alpha_{2n} - r_\alpha \omega^0) \wedge \omega^\alpha,$$

which shows that

$$\psi^\alpha_{2n} = r_\alpha \omega^0 - h_{\alpha\beta}\omega^\beta$$

for some functions $h_{\alpha\beta} = h_{\beta\alpha}$. In fact, one can check that if we define $\underline{e}_\alpha = \pi_* e_\alpha$ and $\underline{e}_{2n} = \pi_* e_{2n}$, then $h_{\alpha\beta} = \underline{e}_{2n} \cdot \mathrm{II}(\underline{e}_\alpha, \underline{e}_\beta)$, where II is the second fundamental form of M at the point $\pi(\mathsf{z})$. Putting these results together, we have

$$d\mathsf{z} = \mathrm{i}\mathsf{z}\omega^0 + e_\alpha \omega^\alpha, \quad de_{2n} = \mathrm{i}\mathsf{z}(r_\alpha\omega^\alpha) + e_\alpha(r_\alpha\omega^0 - h_{\alpha\beta}\omega^\beta) + e_\mu \psi^\mu_{2n}. \tag{6}$$

2.2. Computing the differential

We now use the above formulas to compute

$$\begin{aligned} d(\widehat{\Phi} \circ \rho) = \big(&\cosh t\, d\mathsf{z} + \mathrm{i}\sinh t\, de_{2n} + (\sinh t\, \mathsf{z} + \mathrm{i}\cosh t\, e_{2n})dt; \\ &\cosh t\, d\bar{\mathsf{z}} + \mathrm{i}\sinh t\, d\overline{e_{2n}} + (\sinh t\, \bar{\mathsf{z}} + \mathrm{i}\cosh t\, \overline{e_{2n}})dt\big), \end{aligned}$$

giving

$$\begin{aligned} &d(\widehat{\Phi} \circ \rho) = \cosh t\, [\omega^0,\, \omega^\alpha,\, \psi^\mu_{2n},\, dt] \\ &\otimes \left[\begin{pmatrix} \mathrm{i} & \mathrm{i}\tau r_\beta & 0 & 0 \\ -\tau r_\alpha & \delta_{\alpha\beta} - \mathrm{i}\tau h_{\alpha\beta} & 0 & 0 \\ 0 & 0 & \mathrm{i}\tau\delta_{\mu\nu} & 0 \\ \tau & 0 & 0 & \mathrm{i} \end{pmatrix} \begin{bmatrix} \mathsf{z} \\ e_\beta \\ e_\nu \\ e_{2n} \end{bmatrix} ; \begin{pmatrix} -\mathrm{i} & \mathrm{i}\tau r_\beta & 0 & 0 \\ \tau r_\alpha & \delta_{\alpha\beta} - \mathrm{i}\tau h_{\alpha\beta} & 0 & 0 \\ 0 & 0 & \mathrm{i}\tau\delta_{\mu\nu} & 0 \\ \tau & 0 & 0 & \mathrm{i} \end{pmatrix} \begin{bmatrix} \bar{\mathsf{z}} \\ \overline{e_\beta} \\ \overline{e_\nu} \\ \overline{e_{2n}} \end{bmatrix}\right]. \end{aligned} \tag{7}$$

(The matrices are partitioned into block form, with column and row widths 1, k, $n-k-1$ and 1; we have also introduced the abbreviation $\tau = \tanh t$.)

We will use the isometry group $U(n+1)$ to simplify these matrices, by moving any horizontal vector in $N\widehat{M}$ into a standard position. For, given any point $\mathsf{z} \in \widehat{M}$ and a horizontal vector $\mathsf{h} \in T_{\mathsf{z}}S^{2n+1}$, we can arrange that

$$\mathsf{z} = \begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}, \qquad \mathsf{h} = \begin{bmatrix}0 & \dots & 0 & \mathrm{i}\end{bmatrix}.$$

Thus, from now on we will assume that $\mathsf{z} = E_0$ and $e_{2n} = \mathrm{i}E_n$, where $E_0, \dots,$ E_n denote the elementary basis row vectors of $\mathbb{C}^{n+1}$. Then

$$\widehat{\Phi} \circ \rho\,((\mathsf{z}, e_0, \dots, e_{2n}), t) = (\cosh t\,, 0, \dots, 0, -\sinh t\,; \cosh t\,, 0, \dots, 0, \sinh t\,).$$

Let $o \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ be the point on the right-hand side. For any t this is in the domain of the map

$$\widehat{\mathfrak{A}} : (z_0, \dots, z_n; w_0, \dots, w_n) \mapsto \left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}; \frac{w_1}{w_0}, \dots, \frac{w_n}{w_0}\right).$$

which covers the affine coordinate chart $\mathfrak{A}$. We compute the differential of $\widehat{\mathfrak{A}}$, and evaluate at o:

$$d\widehat{\mathfrak{A}}_o = \operatorname{sech} t(dz_1, \dots, dz_{n-1}, dz_n + \tau dz_0; dw_1, \dots, dw_{n-1}, dw_n - \tau dw_0).$$

To compute the differential of $\widehat{\mathfrak{A}} \circ \widehat{\Phi} \circ \rho$, we apply the $\mathbb{C}$-linear map $d\widehat{\mathfrak{A}}_o$ to the vector factors in the tensor product in (7). For example,

$$\begin{aligned} d\widehat{\mathfrak{A}}_o(e_\alpha; 0) &= (\widecheck{e_\alpha}; 0), & d\widehat{\mathfrak{A}}_o(e_{2n}; 0) &= (\mathrm{i}\widecheck{E_n}; 0), \\ d\widehat{\mathfrak{A}}_o(\mathsf{z}; 0) &= (\tau\widecheck{E_n}; 0), & d\widehat{\mathfrak{A}}_o(0; \bar{\mathsf{z}}) &= (0, -\tau\widecheck{E_n}), \end{aligned}$$

where the $\widecheck{\ }$ indicates the result of deleting the first entry from a row vector in $\mathbb{C}^{n+1}$, and we use the fact that, because the vectors e_α and e_ν are horizontal at $\mathsf{z} = E_0$, their first entries are zero. Computing in this way, we obtain

$$(8) \quad d(\widehat{\mathfrak{A}} \circ \widehat{\Phi} \circ \rho) = [\omega^0, \omega^\alpha, \psi^\mu_{2n}, dt] \otimes \begin{pmatrix} \mathrm{i}\tau(\widecheck{E_n} + r_\beta\widecheck{e_\beta}) & ; & \mathrm{i}\tau(\widecheck{E_n} + r_\beta\widecheck{\overline{e_\beta}}) \\ \widecheck{e_\alpha} - \mathrm{i}\tau h_{\alpha\beta}\widecheck{e_\beta} - \tau^2 r_\alpha \widecheck{E_n} & ; & \widecheck{\overline{e_\alpha}} - \mathrm{i}\tau h_{\alpha\beta}\widecheck{\overline{e_\beta}} - \tau^2 r_\alpha \widecheck{E_n} \\ \mathrm{i}\tau\widecheck{e_\mu} & ; & \mathrm{i}\tau\widecheck{\overline{e_\mu}} \\ (\tau^2 - 1)\widecheck{E_n} & ; & (1 - \tau^2)\widecheck{E_n} \end{pmatrix}.$$

While the 1-forms $\omega^0, \omega^\alpha, \psi^\mu_{2n}$ and dt are linearly independent and span the semibasic 1-forms for ϱ, it is evident from the first equation in (6) that ω^0

is not semibasic for $\Pi \circ \varrho$. In fact, if v is the vector field on $\mathcal{F} \times \mathbb{R}$ that generates the S^1 action under which $NM = N\widehat{M}/S^1$, then

$$\mathsf{v} \lrcorner \omega^0 = 1, \qquad \mathsf{v} \lrcorner \omega^\alpha = 0, \qquad \mathsf{v} \lrcorner \psi^\mu_{2n} = r_\mu := \langle \mathrm{i}e_{2n}, e_\mu \rangle.$$

Using these formulas, it is easy to check that v is in the kernel of the differential (8). In fact, the r_μ give the coefficients under which the top row of the matrix is a linear combination of the third set of rows below it, since $\breve{E_n} + r_\beta \breve{e}_\beta = -\mathrm{i}\breve{e_{2n}} + r_\beta \breve{e}_\beta = -r_\mu e_\mu$.

Thus, in terms of the diagram (3), $\widehat{\mathfrak{A}} \circ \widehat{\Phi} \circ \rho$ covers a well-defined map $\mathfrak{A} \circ \Phi \circ \imath$ from NM to $\mathbb{C}^{2n}$. Matters being so, we will expand the differential just in terms of the 1-forms ω^α, dt and $\widetilde{\psi}^\mu_{2n} := \psi^\mu_{2n} - r_\mu \omega^0$, as

$$(9) \quad d(\widehat{\mathfrak{A}} \circ \widehat{\Phi} \circ \rho) = [\omega^\alpha,\ \widetilde{\psi}^\mu_{2n},\ dt] \otimes \begin{pmatrix} \breve{e_\alpha} - \mathrm{i}\tau h_{\alpha\beta}\breve{e}_\beta - \tau^2 r_\alpha \breve{E_n} & ; & \breve{\overline{e_\alpha}} - \mathrm{i}\tau h_{\alpha\beta}\breve{\overline{e_\beta}} - \tau^2 r_\alpha \breve{E_n} \\ \mathrm{i}\tau \breve{e}_\mu & ; & \mathrm{i}\tau \breve{\overline{e_\mu}} \\ (\tau^2 - 1)\breve{E_n} & ; & (1-\tau^2)\breve{E_n} \end{pmatrix}.$$

2.3. The Stenzel Kähler form

A convenient explicit description of the Stenzel metric in local coordinates on $\mathbb{C}P^n \times \mathbb{C}P^n$ is given by T-C. Lee [8]; we briefly reproduce it here for the sake of the calculations in §2.4. Lee defines two functions on $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$,

$$\mathcal{A} = \sum_{j,k=0}^{n} |z_j w_k|^2, \qquad \mathcal{B} = \mathsf{z} \cdot \mathsf{w} = \sum_{j=0}^{n} z_j w_j,$$

which are homogeneous of degree 4 and 2 respectively; then the exhaustion function $\mathcal{N} = \mathcal{A}/|\mathcal{B}|^2$ is well-defined and smooth on $\mathcal{M}$. The Kähler potential $f(\mathcal{N})$ for the Stenzel metric satisfies $f' = \mathcal{N}^{-1/2}$. Using this, we calculate the Kähler form in terms of affine coordinates $Z_a = z_a/z_0$ and $W_a = w_a/w_0$, where we now take $1 \le a, b \le n$.

To start with,

$$\overline{\partial}\mathcal{A} = (1 + |W|^2) Z_b \, d\overline{Z}_b + (1 + |Z|^2) W_b \, d\overline{W}_b$$

where $|Z|^2 = \sum_a Z_a \overline{Z}_a$ and $|W|^2 = \sum_a W_a \overline{W}_a$; then

$$\begin{aligned}(10)\qquad \partial\overline{\partial}\mathcal{A} &= (1+|W|^2)dZ_b \wedge d\overline{Z}_b + \overline{W}_a Z_b\, dW_a \wedge d\overline{Z}_b \\ &\quad + \overline{Z}_a W_b\, dZ_a \wedge d\overline{W}_b + (1+|Z|^2)dW_b \wedge d\overline{W}_b \\ &= (dZ\, dW) \wedge \begin{pmatrix} (1+|W|^2)I_n & \overline{Z}^T W \\ \overline{W}^T Z & (1+|Z|^2)I_n \end{pmatrix} \begin{pmatrix} d\overline{Z}^T \\ d\overline{W}^T \end{pmatrix},\end{aligned}$$

where we regard Z and W as row vectors of length n, I_n is the $n \times n$ identity matrix, and ^{T} indicates transpose. Following Lee, we identify the $(1,1)$-form $\partial\overline{\partial}\mathcal{A}$ with the hermitian matrix in (10) that gives its coefficients with respect to the affine coordinate differentials; similarly, we identify $\partial\mathcal{A}$ with the row vector $\big((1+|W|^2)\overline{Z}\,;\,(1+|Z|^2)\overline{W}\big)$ of length $2n$. With these conventions, we identify $\partial\overline{\partial} f$ with the hermitian matrix

$$\begin{aligned}\mathsf{G} = \frac{f'}{|\mathcal{B}|^2}\Bigg[&\partial\overline{\partial}\mathcal{A} - \frac{1}{\mathcal{A}}(\partial\mathcal{A})^T\overline{\partial}\mathcal{A} \\ &+ \left(\frac{f''}{|\mathcal{B}|^2 f'} + \frac{1}{\mathcal{A}}\right)(\partial\mathcal{A} - (\mathcal{A}/\mathcal{B})\partial\mathcal{B})^T \left(\overline{\partial}\mathcal{A} - (\mathcal{A}/\overline{\mathcal{B}})\overline{\partial}\overline{\mathcal{B}}\right)\Bigg].\end{aligned}$$

We will only need the value of the metric at the point $\breve{o} = \widehat{\mathfrak{A}}(o)$, where $W_n = \tau = \tanh t$, $Z_n = -\tau$ and all other coordinates are zero. At this point, we have

$$(11)\qquad \mathsf{G} = \frac{1}{1-\tau^4}\left[(1+\tau^2)I_{2n} + \begin{pmatrix} (q-\tau^2)\mathsf{M} & -q\mathsf{M} \\ -q\mathsf{M} & (q-\tau^2)\mathsf{M} \end{pmatrix}\right],$$

where

$$\mathsf{M} = \breve{E}_n^{\,T}\breve{E}_n, \qquad \text{and} \quad q = \frac{2\tau^2}{(1-\tau^2)^2}.$$

2.4. Checking Lagrangian-ness

In this section, we prove:

Proposition 1. *If $M \subset \mathbb{C}P^n$ is an arbitrary submanifold, then $\Phi(NM)$ is a Lagrangian submanifold of $\mathcal{M}$.*

Proof. We will identify a real tangent vector $\mathsf{v} \in T_{\breve{o}}\mathcal{M}$ with the row vector in $\mathbb{C}^{2n}$ given by $\mathsf{v} \lrcorner (dZ; dW)$. With this convention, the Stenzel metric g

satisfies

$$g(\mathsf{v},\mathsf{w}) = 2\,\mathrm{Re}(\bar{\mathsf{v}}\mathsf{G}\mathsf{w}^T), \tag{12}$$

and its Kähler form $\Omega = \mathrm{i}\partial\bar{\partial}f$ satisfies

$$\Omega(\mathsf{v},\mathsf{w}) = -2\,\mathrm{Im}(\bar{\mathsf{v}}\mathsf{G}\mathsf{w}^T), \quad \forall \mathsf{v},\mathsf{w} \in T_{\breve{o}}\mathcal{M}, \tag{13}$$

The process of verifying that Ω vanishes on the tangent space to $\Phi(NM)$ is made simpler by decomposing tangent vectors into vertical and horizontal pieces. (By vertical vectors, we mean those tangent to the images under Φ of the fibers of $T\mathbb{C}P^n$, and the horizontal vectors are those in the orthogonal complement with respect to g.) Computing

$$\left.\frac{d}{ds}\right|_{s=0} \widehat{\mathfrak{A}} \circ \widehat{\Phi}(\zeta, \xi + s\eta)$$

when $\zeta = E_0$, $\xi = \mathrm{i}tE_n$ and $\eta \in \mathbb{C}^{n+1}$ ranges over all vectors satisfying $\eta \cdot \bar{\zeta} = 0$, shows that the space of vertical tangent vectors at $\breve{o}$ consists of all vectors of the form $(\mathsf{z}; -\bar{\mathsf{z}})$ for $\mathsf{z} \in \mathbb{C}^n$. Then, noting the special form of G in (11), it is easy to check using (12) that the space of horizontal vectors consists of all vectors of the form $(\mathsf{z}; \bar{\mathsf{z}})$. It is also easy to check that $\Omega(\mathsf{v},\mathsf{w}) = 0$ whenever v and w are both vertical or both horizontal.

Equation (9) shows that the tangent space to $\Phi(NM)$ at $\breve{o}$ is spanned by purely vertical vectors $(\mathrm{i}\breve{e_\mu}; \mathrm{i}\breve{\overline{e_\mu}})$ and $(\breve{E_n}; -\breve{E_n})$, and the 'mixed' vectors

$$\mathsf{u}_\alpha := (\breve{e_\alpha} - \tau^2 r_\alpha \breve{E_n}; \breve{\overline{e_\alpha}} - \tau^2 r_\alpha \breve{E_n}) - \tau h_{\alpha\beta}(\mathrm{i}\breve{e_\beta}; \mathrm{i}\breve{\overline{e_\beta}}).$$

(Note that the first term is horizontal and the second is vertical.) When evaluating Ω on the pairing of u_α with a purely vertical vector, we only need the horizontal part of u_α. For example, we compute using (11) and (13) that

$$\begin{aligned}
&\Omega(\mathsf{u}_\alpha, (\breve{E_n}; -\breve{E_n}))\\
&= \frac{-2}{1-\tau^4}\,\mathrm{Im}\Big[(1+\tau^2)\Big((\overline{\breve{\overline{e_\alpha}}} - \tau^2 r_\alpha \breve{E_n})\cdot \breve{E_n} + (\breve{e_\alpha} - \tau^2 r_\alpha \breve{E_n})\cdot(-\breve{E_n})\Big)\\
&\qquad + (q-\tau^2)\Big((\overline{\breve{\overline{e_\alpha}}} - \tau^2 r_\alpha \breve{E_n})\cdot \mathsf{M}\breve{E_n} + (\breve{e_\alpha} - \tau^2 r_\alpha \breve{E_n})\cdot \mathsf{M}(-\breve{E_n})\Big)\\
&\qquad - q\Big((\overline{\breve{\overline{e_\alpha}}} - \tau^2 r_\alpha \breve{E_n})\cdot \mathsf{M}(-\breve{E_n}) + (\breve{e_\alpha} - \tau^2 r_\alpha \breve{E_n})\cdot \mathsf{M}\breve{E_n}\Big)\Big] = 0.
\end{aligned}$$

In fact, the terms on each line inside the square brackets cancel out because $\mathsf{M}\breve{E_n} = \breve{E_n}$ and because

$$0 = \langle e_\alpha, e_{2n}\rangle = \mathrm{Re}(\breve{e_\alpha}\cdot \mathrm{i}\breve{E_n}),$$

so that $\check{e}_\alpha \cdot \check{E}_n$ is real (and equal to $-r_\alpha$). Pairing with the other vertical vectors, we get

$$
\begin{aligned}
&\Omega(\mathsf{u}_\alpha,\ (\mathrm{i}\check{e}_\mu; \mathrm{i}\check{\overline{e}}_\mu)) \\
&= \frac{-2}{1-\tau^4} \operatorname{Im} \Big[(1+\tau^2) \Big((\overline{\check{e}_\alpha} - \tau^2 r_\alpha \overline{\check{E}_n}) \cdot \mathrm{i}\check{e}_\mu + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathrm{i}\check{\overline{e}}_\mu \Big) \\
&\qquad + (q-\tau^2) \Big((\overline{\check{e}_\alpha} - \tau^2 r_\alpha \overline{\check{E}_n}) \cdot \mathrm{i}\mathsf{M}\check{e}_\mu + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathrm{i}\mathsf{M}\check{\overline{e}}_\mu \Big) \\
&\qquad - q \Big((\overline{\check{e}_\alpha} - \tau^2 r_\alpha \overline{\check{E}_n}) \cdot \mathrm{i}\check{\overline{e}}_\mu + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathrm{i}\check{e}_\mu \Big) \Big]
\end{aligned}
$$

Again, $\check{e}_\mu \cdot \check{E}_n = -r_\mu$ is real, and so $\mathsf{M}\check{e}_\mu = -r_\mu \check{E}_n$; we also note that

$$
0 = \langle e_\alpha, e_\mu \rangle = \operatorname{Re}(\overline{\check{e}_\alpha} \cdot \check{\overline{e}}_\mu) = \tfrac{1}{2}(\overline{\check{e}_\alpha} \cdot \check{e}_\mu + \check{e}_\alpha \cdot \check{\overline{e}}_\mu).
$$

Thus,

$$
\begin{aligned}
&\Omega(\mathsf{u}_\alpha,\ (\mathrm{i}\check{e}_\mu; \mathrm{i}\check{\overline{e}}_\mu)) \\
&= \frac{-4}{1-\tau^4} \Big[(1+\tau^2)\tau^2 r_\alpha r_\mu + (q-\tau^2)(1+\tau^2) r_\alpha r_\mu - q(1+\tau^2) r_\alpha r_\mu \Big] = 0.
\end{aligned}
$$

It remains to check that Ω is zero on a pair $\mathsf{u}_\alpha, \mathsf{u}_\beta$ of 'mixed' vectors. Let

$$
\mathsf{v}_\alpha = (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n; \overline{\check{e}_\alpha} - \tau^2 r_\alpha \overline{\check{E}_n}),
$$

the horizontal part of u_α, and let $\mathsf{w}_a = (\mathrm{i}\check{e}_\beta; \mathrm{i}\check{\overline{e}}_\beta)$. First, we compute that

$$
\begin{aligned}
&\Omega(\mathsf{v}_\alpha, \mathsf{w}_\beta) \\
&= \frac{-2}{1-\tau^4} \operatorname{Im} \Big[(1+\tau^2) \Big((\overline{\check{e}_\alpha} - \tau^2 r_\alpha \overline{\check{E}_n}) \cdot \mathrm{i}\check{e}_\beta + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathrm{i}\check{\overline{e}}_\beta \Big) \\
&\qquad + (q-\tau^2) \Big((\overline{\check{e}_\alpha} - \tau^2 r_\alpha \overline{\check{E}_n}) \cdot \mathrm{i}\mathsf{M}\check{e}_\beta + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathrm{i}\mathsf{M}\check{\overline{e}}_\beta \Big) \\
&\qquad - q \Big((\overline{\check{e}_\alpha} - \tau^2 r_\alpha \overline{\check{E}_n}) \cdot \mathrm{i}\mathsf{M}\check{\overline{e}}_\beta + (\check{e}_\alpha - \tau^2 r_\alpha \check{E}_n) \cdot \mathrm{i}\mathsf{M}\check{e}_\beta \Big) \Big] \\
&= \frac{-4}{1-\tau^4} \Big[(1+\tau^2)(\delta_{\alpha\beta} + \tau^2 r_\alpha r_\beta) + (q-\tau^2)(1+\tau^2) r_\alpha r_\beta - q(1+\tau^2) r_\alpha r_\beta \Big] \\
&= \frac{-4}{1-\tau^2} \delta_{\alpha\beta},
\end{aligned}
$$

where we have used the fact that $\mathsf{M}e_\beta = -r_\beta \breve{E}_n$ and $\delta_{\alpha\beta} = \langle e_\alpha, e_\beta\rangle = \mathrm{Re}(\overline{\breve{e}_\alpha} \cdot \breve{e}_\beta)$. Then

$$\begin{aligned}
\Omega(\mathsf{u}_\alpha, \mathsf{u}_\beta) &= \Omega(\mathsf{v}_\alpha - \tau h_{\alpha\gamma}\mathsf{w}_\gamma, \mathsf{v}_\beta - \tau h_{\beta\epsilon}\mathsf{w}_\epsilon)\\
&= -\tau h_{\alpha\gamma}\Omega(\mathsf{w}_\gamma, \mathsf{v}_\beta) - \tau h_{\beta\epsilon}\Omega(\mathsf{v}_\alpha, \mathsf{w}_\epsilon)\\
&= \frac{-4}{1-\tau^2}\,(\tau h_{\alpha\gamma}\delta_{\gamma\beta} - \tau h_{\beta\epsilon}\delta_{\alpha\epsilon}) = 0,
\end{aligned}$$

where we use index ranges $1 \leq \alpha, \beta, \gamma, \epsilon \leq k$. □

3. The austerity condition

Let S be the matrix on the right in (9). Then the pullback of the holomorphic volume form Θ under $\mathfrak{A} \circ \Phi \circ \imath$ equals $\det \mathsf{S}$ times a real volume form on NM. In this section we will evaluate this determinant, and find conditions under which it has constant phase. In what follows, we again represent an arbitrary normal vector $\nu \in NM$ by its horizontal lift in $T_{\mathsf{z}}\widehat{M}$, which is given by te_{2n} for some $t \in \mathbb{R}$ and some adapted frame at z.

First, we consider the special case when $\nu \in \mathcal{N}$. Then $r_\alpha = 0$ for all α, and we can factor S as

$$\begin{pmatrix} I_k - \mathrm{i}\tau H & 0 & 0\\ 0 & \mathrm{i}\tau I_{2n-k-1} & 0\\ 0 & 0 & \mathrm{i}(1-\tau^2)\end{pmatrix}\begin{pmatrix} \breve{e}_\alpha & ; & \overline{\breve{e}_\alpha}\\ \breve{e}_\mu & ; & \overline{\breve{e}_\mu}\\ \mathrm{i}\breve{E}_n & ; & -\mathrm{i}\overline{\breve{E}_n}\end{pmatrix}, \tag{14}$$

where I_k is the $k \times k$ identity matrix and H is the matrix with entries $h_{\alpha\beta}$. Letting V be the matrix on the right in (14), we note that

$$\tfrac{1}{2}V\begin{pmatrix} I_n & -\mathrm{i}I_n\\ I_n & \mathrm{i}I_n\end{pmatrix} = \begin{pmatrix} \overset{\frown}{\breve{e}_\alpha}\\ \overset{\frown}{\breve{e}_\mu}\\ \overset{\frown}{\mathrm{i}\breve{E}_n}\end{pmatrix}. \tag{15}$$

The matrix on the right-hand side of (15) lies in $O(2n)$, but it has determinant $(-1)^n$, since when we substitute our particular frame into (5), we obtain

$$1 = \det\begin{pmatrix} 1 & 0\ldots & 0 & 0\ldots\\ 0 & 0\ldots & 1 & 0\ldots\\ 0 & \mathrm{Re}\,\breve{e}_\alpha & 0 & \mathrm{Im}\,\breve{e}_\alpha\\ 0 & \mathrm{Re}\,\breve{e}_\mu & 0 & \mathrm{Im}\,\breve{e}_\mu\\ 0 & 0\ldots & 0 & \breve{E}_n\end{pmatrix} = (-1)^n\det\begin{pmatrix} \mathrm{Re}\,\breve{e}_\alpha & \mathrm{Im}\,\breve{e}_\alpha\\ \mathrm{Re}\,\breve{e}_\mu & \mathrm{Im}\,\breve{e}_\mu\\ 0\ldots & \breve{E}_n\end{pmatrix}.$$

Taking determinants on each side in (15) gives $(\mathrm{i}/2)^n \det V = (-1)^n$, so that $\det V = (2\mathrm{i})^n$. Thus,

$$\det \mathsf{S} = (-2)^n \mathrm{i}^{n-k} \tau^{2n-k-1}(1-\tau^2)\det(I_k - \mathrm{i}\tau H). \tag{16}$$

It is clear that the real part of $\mathrm{i}^{n-k}\det \mathsf{S}$ is nonzero for values of τ in an open interval containing zero. On the other hand, by diagonalizing H it is easy to see that

$$\operatorname{Im}\det(I_k - \mathrm{i}\tau H) = \sum_{j=1}^{\lfloor (k+1)/2 \rfloor} (-1)^j \tau^{2j-1} H^{(2j-1)}, \tag{17}$$

where $H^{(2j-1)}$ denotes the elementary symmetric polynomial of degree $2j-1$ in the eigenvalues of H. Thus, we conclude that

> For $\nu \in \mathcal{N}$, the imaginary part of $\mathrm{i}^{n-k}\Omega$ vanishes along the line in NM spanned by ν if and only if all odd-degree elementary symmetric polynomials in the eigenvalues of H vanish (where H represents $\nu \cdot \mathrm{II}$ with respect to an orthonormal basis).

Corollary 2. *If $M \subset \mathbb{C}P^n$ is a complex submanifold, then M is austere.*

Proof. Because the complex structure J is parallel along M, the second fundamental form satisfies $\mathrm{II}(X, \mathsf{J}Y) = \mathsf{J}\,\mathrm{II}(X,Y) = \mathrm{II}(\mathsf{J}X, Y)$. If matrix J represents the complex structure with respect to an orthonormal basis, then $HJ = J^T H = -JH$, and hence $JHJ = -J^2 H = H$. So,

$$\det(I_k - \mathrm{i}\tau H) = \det(I_k - \mathrm{i}\tau J^{-1}HJ) = \det(I_k + \mathrm{i}\tau JHJ) = \det(I_k + \mathrm{i}\tau H),$$

and so $\operatorname{Im}\det(I_k - \mathrm{i}\tau H) = 0$. □

Now consider the more general case, when ν has a non-zero component in $\mathcal{E}$, and thus $\mathsf{J}\nu$ has a nonzero orthogonal projection onto $\mathcal{D}$. We will further specialize the orthonormal frame so that

$$\mathrm{i}e_{2n} = \cos\theta\, e_1 + \sin\theta\, e_{2n-1},$$

where θ is the angle between $\mathsf{J}\nu$ and the tangent space to M. Since we also have $e_{2n} = \mathrm{i}E_n$, then $r_1 = \cos\theta$, $r_{2n-1} = \sin\theta$ and all other r_α, r_μ are zero.

To simplify calculating $\det \mathsf{S}$, we modify the matrix by adding $\tau^2 r_1/(\tau^2 - 1)$ times row $2n$ to row 1, giving

$$\mathsf{S}' = \begin{pmatrix} \breve{e_\alpha} - \mathrm{i}\tau \breve{H_\alpha} & ; & \overline{\breve{e_\alpha}} - \mathrm{i}\tau \overline{\breve{h_\alpha}} - 2\tau^2 r_\alpha \breve{E_n} \\ \mathrm{i}\tau \breve{e_\mu} & ; & \mathrm{i}\tau \overline{\breve{e_\mu}} \\ (\tau^2-1)\breve{E_n} & ; & (1-\tau^2)\breve{E_n} \end{pmatrix},$$

where we introduce the abbreviation $\breve{H_\alpha} = h_{\alpha\beta}\breve{e_\beta}$. Again, only coefficient $r_1 = \cos\theta$ is nonzero, and we expand the determinant in terms of it. Letting S_0 denote the matrix in (14), we have

$$\begin{aligned} \det \mathsf{S} &= \det \mathsf{S}' \\ &= \det \mathsf{S}_0 + (-1)^{2n-1}(-2\tau^2\cos\theta) \det \begin{pmatrix} \breve{e_\beta} - \mathrm{i}\tau \breve{H_\beta} & ; & \overline{\breve{e_\beta}} - \mathrm{i}\tau \overline{\breve{h_\beta}} \\ \mathrm{i}\tau \breve{e_\mu} & ; & \mathrm{i}\tau \overline{\breve{e_\mu}} \\ (\tau^2-1)\breve{E_n} & ; & 0 \end{pmatrix}, \end{aligned}$$

where we now use the index range $2 \le \beta \le k$, and $\breve{}$ indicates the result of deleting the first and last entries from a vector in $\mathbb{C}^{n+1}$. Thus, the matrix on the right is $2n-1 \times 2n-1$; moreover, in the bottom row only the nth entry is nonzero, so we may use a cofactor expansion to write

$$\begin{aligned} \det \mathsf{S} &= \det \mathsf{S}_0 + \Bigg(2\tau^2\cos\theta(-1)^{2n-2+n-1}(\tau^2-1) \\ &\qquad \cdot \det \begin{pmatrix} \widetilde{e_\beta} - \mathrm{i}\tau \widetilde{H_\beta} & ; & \overline{\widetilde{e_\beta}} - \mathrm{i}\tau \overline{\widetilde{h_\beta}} \\ \mathrm{i}\tau \widetilde{e_\mu} & ; & \mathrm{i}\tau \overline{\widetilde{e_\mu}} \end{pmatrix} \Bigg) \\ &= \det \mathsf{S}_0 + \Bigg(2\cos\theta(-1)^{n-3}\tau^2(\tau^2-1) \\ &\qquad \cdot \det \begin{pmatrix} I_{k-1} - \mathrm{i}\tau\widetilde{H} & 0 \\ 0 & \mathrm{i}\tau I_{2n-k-1} \end{pmatrix} \det \begin{pmatrix} \widetilde{e_\beta} & ; & \overline{\widetilde{e_\beta}} \\ \widetilde{e_\mu} & ; & \overline{\widetilde{e_\mu}} \end{pmatrix} \Bigg), \end{aligned}$$

where $\widetilde{H}$ is the $(k-1)\times(k-1)$ matrix obtained from H by deleting the first row and column.

Lemma 3. *Let* $\widetilde{V} = \begin{pmatrix} \widetilde{e_\beta} & ; & \overline{\widetilde{e_\beta}} \\ \widetilde{e_\mu} & ; & \overline{\widetilde{e_\mu}} \end{pmatrix}$. *Then* $\det \widetilde{V} = (-2\mathrm{i})^{n-1}\cos\theta$.

Using this lemma (to be proved later), and the formula (16) for $\det \mathsf{S}_0$, we have

$$\begin{aligned}\det \mathsf{S} &= \det \mathsf{S}_0 + \Big(2(-1)^{n-3}\cos^2\theta\tau^2(\tau^2-1)(\mathrm{i}\tau)^{2n-k-1}(-2\mathrm{i})^{n-1}\\ &\qquad\qquad \cdot \det(I_{k-1} - \mathrm{i}\tau\widetilde{H})\Big)\\ &= (-2)^n\tau^{2n-k-1}\Big[\mathrm{i}^{n-k}(1-\tau^2)\det(I_k - \mathrm{i}\tau H)\\ &\qquad\qquad + \mathrm{i}^{2n-k-1}(-\mathrm{i})^{n-1}\tau^2(1-\tau^2)\cos^2\theta\det(I_{k-1}-\mathrm{i}\tau\widetilde{H})\Big]\\ &= 2^n\mathrm{i}^{n-k}\tau^{2n-k-1}(1-\tau^2)\Big[\det(I_k - \mathrm{i}\tau H) + \tau^2\cos^2\theta\det(I_{k-1}-\mathrm{i}\tau\widetilde{H})\Big].\end{aligned}$$

As in (17),

$$\begin{aligned}&\operatorname{Im}\Big[\det(I_k - \mathrm{i}\tau H) + \tau^2\cos^2\theta\det(I_{k-1}-\mathrm{i}\tau\widetilde{H})\Big]\\ &= \sum_{j=0}^{\lfloor (k-1)/2\rfloor}(-1)^{j+1}\tau^{2j+1}H^{(2j+1)} + \cos^2\theta\sum_{j=1}^{\lfloor k/2\rfloor}(-1)^j\tau^{2j+1}\widetilde{H}^{(2j-1)}\\ &= -\tau H^{(1)} + \tau^3(H^{(3)} - \cos^2\theta\widetilde{H}^{(1)}) - \tau^5(H^{(5)} - \cos^2\theta\widetilde{H}^{(3)}) + \cdots\end{aligned}$$

Thus, for $\nu \notin \mathcal{N}$ the imaginary part of $\mathrm{i}^{n-k}\Theta$ vanishes along the line in NM spanned by ν if and only if $H^{(1)} = 0$, $H^{(3)} = \cos^2\theta\widetilde{H}^{(1)}$, and so on, up to $H^{(k)} = \cos^2\theta\widetilde{H}^{(k-2)}$ if k is odd, or $0 = \cos^2\theta\widetilde{H}^{(k-1)}$ if k is even. On the other hand, if $\nu \in \mathcal{N}$ then $\cos\theta = 0$ and these conditions simplify to the requirement that all the odd degree symmetric polynomials in the eigenvalues of H vanish—i.e., the conclusion we reached for the special case above. We can therefore summarize our calculations as follows:

Theorem 4. *A submanifold $M \subset \mathbb{C}P^n$ of real dimension k is austere if and only if, for every unit normal vector $\nu \in N_pM$ and every point $p \in M$,*

$$H^{(2j+1)} = \cos^2\theta\,\widetilde{H}^{(2j-1)}, \qquad 0 \le j \le \lfloor k/2\rfloor, \tag{18}$$

where θ is the angle between $\mathsf{J}\nu$ and T_pM, H is the quadratic form $\nu \cdot \mathrm{II}$, $\widetilde{H}$ denotes the restriction of $\nu\cdot\mathrm{II}$ to the subspace of T_pM orthogonal to $\mathsf{J}\nu$, and the odd-degree symmetric polynomials in (18) *are understood to be zero if the degree is negative or larger than the size of the representative matrix.*

By setting $j = 0$ in (18), we obtain:

Corollary 5. *If $M \subset \mathbb{C}P^n$ is austere, then M is minimal.*

The following examples may help us understand the austerity condition:

Example 1. Assume $M \subset \mathbb{C}P^n$ is a hypersurface. Then $\mathcal{N}$ has rank zero, $\mathcal{D}$ and $\mathcal{E}$ have rank one, and $\mathsf{J} : \mathcal{D} \to \mathcal{E}$. Hence $\theta = 0$, and we may write the austere conditions as

$$A^{(2j+1)} = \widetilde{A}^{(2j-1)}, \qquad 0 \le j \le n-1,$$

where A denotes the scalar-valued second fundamental form of M and $\widetilde{A}$ is its restriction to the holomorphic distribution $\mathcal{H}$.

Example 2. Assume $M \subset \mathbb{C}P^n$ is a curve. Then M is austere if and only if it is a geodesic.

Example 3. Assume $M \subset \mathbb{C}P^n$ is a surface which is not a holomorphic curve. Hence, $\mathcal{H}$ has rank zero. Then M is austere if and only if M is minimal and the "highest-degree condition", obtained by setting $j = 1$ in (18), holds; this condition is that $\mathrm{II}^\nu(\mathsf{v},\mathsf{v}) = 0$, where $\mathsf{v} \in T_pM$ is the vector orthogonal to $\mathsf{J}\nu$, and ν runs over the unit circle bundle in $\mathcal{E}_p$ for all $p \in M$.

Austere surfaces are discussed in more detail in the next section.

Proof of Lemma 3. As observed after equation (15),

$$(-1)^n = \det\begin{pmatrix} \widehat{\breve{e}_\alpha} \\ \widehat{\breve{e}_\mu} \\ \widehat{\mathrm{i}\breve{E}_n} \end{pmatrix} = \det\begin{pmatrix} \mathrm{Re}\,\widetilde{e_1} & -\cos\theta & \mathrm{Im}\,\widetilde{e_1} & 0 \\ \mathrm{Re}\,\widetilde{e_\beta} & 0 & \mathrm{Im}\,\widetilde{e_\beta} & 0 \\ \mathrm{Re}\,\widetilde{e_\lambda} & 0 & \mathrm{Im}\,\widetilde{e_\lambda} & 0 \\ \mathrm{Re}\,\widetilde{e_{2n-1}} & -\sin\theta & \mathrm{Im}\,\widetilde{e_{2n-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where we take index ranges $1 < \beta \le k$ and $k < \lambda < 2n-1$. Because the first and last entries of e_β and e_λ are zero, vectors $\widetilde{e_\beta}$ and $\widetilde{e_\lambda}$ are mutually orthogonal unit vectors in $\mathbb{C}^{n-1}$. Since $\widetilde{e_1}$ and $\widetilde{e_{2n-1}}$ are orthogonal to all of them, these two vectors must be linearly dependent over $\mathbb{R}$. Since $\widetilde{e_{2n-1}}$ is a unit vector with $\widetilde{e_{2n-1}} \cdot \breve{E}_n = -\sin\theta$, then $|\widetilde{e_{2n-1}}|^2 = \cos^2\theta \ne 0$. If we set $\widetilde{e_1} = a\widetilde{e_{2n-1}}$ for a scalar a, then solving $0 = \langle \breve{e}_1, \widetilde{e_{2n-1}} \rangle$ gives $a = -\tan\theta$. Thus, adding $\tan\theta$ times the second-last row to the first row in the matrix gives

$$(-1)^n = \det\begin{pmatrix} 0 & -\sec\theta & 0 & 0 \\ \mathrm{Re}\,\widetilde{e_\beta} & 0 & \mathrm{Im}\,\widetilde{e_\beta} & 0 \\ \mathrm{Re}\,\widetilde{e_\lambda} & 0 & \mathrm{Im}\,\widetilde{e_\lambda} & 0 \\ \mathrm{Re}\,\widetilde{e_{2n-1}} & -\sin\theta & \mathrm{Im}\,\widetilde{e_{2n-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (-1)^n \sec\theta \det\begin{pmatrix} \widehat{\breve{e}_\beta} \\ \widehat{\breve{e}_\lambda} \\ \widehat{e_{2n-1}} \end{pmatrix}.$$

On the other hand, as in (15) we have

$$\tfrac{1}{2}\widetilde{V}\begin{pmatrix} I_{n-1} & -\mathrm{i}I_{n-1} \\ I_{n-1} & \mathrm{i}I_{n-1}\end{pmatrix} = \begin{pmatrix} \widecheck{\widetilde{e_\beta}} \\ \widecheck{\widetilde{e_\lambda}} \\ \widehat{e_{2n-1}} \end{pmatrix}.$$

Taking determinants on each side and solving gives the desired formula for $\det \widetilde{V}$. □

4. Classification of austere surfaces

In this section we classify surfaces in $\mathbb{C}P^n$ that satisfy the austere condition of Theorem 4.

Proposition 6. *Let $M \subset \mathbb{C}P^n$ be an austere surface such that $\mathcal{H} = 0$ at every point. Then M is totally geodesic.*

Proof. By assumption, the splitting (1) implies $TM = \mathcal{D}$, $NM = \mathcal{E} \oplus \mathcal{N}$, where $\mathcal{D}, \mathcal{E}$ are rank 2 and $\mathcal{D} \oplus \mathcal{E}$ is J-invariant. Let $p \in M$ be an arbitrary point and let $\nu \in N_pM$ be an arbitrary unit normal vector. At p we will construct a orthonormal basis for $T_p\mathbb{C}P^n$ of the form $(\mathsf{e}_1, \mathsf{J}\mathsf{e}_1, \mathsf{e}_2, \mathsf{J}\mathsf{e}_2, \ldots, \mathsf{e}_n, \mathsf{J}\mathsf{e}_n)$, which we will refer to as a *unitary frame*.

Fix a unit vector $\mathsf{e}_1 \in \mathcal{D}_p$, and let θ be the angle between $\mathsf{J}\mathsf{e}_1$ and $\mathcal{D}_p$. This is the *Kähler angle*, which is independent of the choice of e_1 and nonzero by assumption. Thus, $\mathsf{J}\mathsf{e}_1$ has a nonzero orthogonal projection onto $\mathcal{E}_p$; let w be the unit vector in the direction opposite to this projection, and let $\mathsf{e}_2 \in \mathcal{E}_p$ be a choice of unit vector orthogonal to w. Then $\mathsf{e}_1, \mathsf{J}\mathsf{e}_1, \mathsf{e}_2$ are linearly independent, and thus $(\mathsf{e}_1, \mathsf{J}\mathsf{e}_1, \mathsf{e}_2, \mathsf{J}\mathsf{e}_2)$ is an orthonormal basis for $\mathcal{D}_p \oplus \mathcal{E}_p$. Since $\langle \mathsf{w}, \mathsf{J}\mathsf{e}_1\rangle = -\sin\theta$, we must have $\mathsf{w} = -\sin\theta\, \mathsf{J}\mathsf{e}_1 \pm \cos\theta \mathsf{J}\mathsf{e}_2$. We adjust the sense of e_2 so that

$$\mathsf{w} = -\sin\theta\, \mathsf{J}\mathsf{e}_1 + \cos\theta \mathsf{J}\mathsf{e}_2. \tag{19}$$

If we define

$$\mathsf{v} := \cos\theta \mathsf{J}\mathsf{e}_1 + \sin\theta \mathsf{J}\mathsf{e}_2$$

then v is orthogonal to w and e_2, and thus $(\mathsf{e}_1, \mathsf{v})$ is an orthonormal basis for $\mathcal{D}_p = T_pM$.

We now choose the remaining vectors of the unitary frame so that e_3 is the unit vector in the direction of the orthogonal projection of ν onto $\mathcal{N}_p$, and $\mathcal{N}_p$ is spanned by $\mathsf{e}_3, \mathsf{J}\mathsf{e}_3, \ldots, \mathsf{e}_n, \mathsf{J}\mathsf{e}_n$. Let ψ be the angle between ν and

$\mathcal{N}_p$, and let II^ν be the quadratic form on T_pM given by $\nu \cdot \mathrm{II}$. If $\psi = 0$ (i.e., $\nu \in \mathcal{N}_p$) then the austere condition reduces to $\operatorname{tr} \mathrm{II}^\nu = 0$. We are interested in what additional conditions arise when $\psi \neq 0$, so we assume this from now on.

Let u be the unit vector in the direction of the orthogonal projection of ν onto $\mathcal{E}_p$. Then

$$\nu = \cos\psi\, \mathsf{e}_3 + \sin\psi\, \mathsf{u}. \tag{20}$$

Let φ be the angle such that $\mathsf{u} = \cos\varphi\, \mathsf{e}_2 + \sin\varphi \mathsf{w}$. When we substitute this, and then (19), into (20) we obtain

$$\nu = \cos\psi\, \mathsf{e}_3 + \sin\psi(\cos\varphi\, \mathsf{e}_2 + \sin\varphi\, \mathsf{w}) \tag{21}$$

$$= \cos\psi\, \mathsf{e}_3 + \sin\psi(\cos\varphi\, \mathsf{e}_2 - \sin\varphi \sin\theta\, \mathsf{J}\mathsf{e}_1 + \sin\varphi\cos\theta\, \mathsf{J}\mathsf{e}_2). \tag{22}$$

It is important to note that, while fixing e_3, we can vary the normal vector ν by varying the angles ψ and φ independently.

As in Example 3, the austere condition requires that M be minimal and that $\mathrm{II}^\nu(\xi,\xi) = 0$, where $\xi \in T_pM$ is a unit vector orthogonal to $\mathsf{J}\nu$. By computing $\mathsf{J}\nu$ using (22), one can verify that

$$\xi = \cos\varphi\, \mathsf{e}_1 - \sin\varphi\, \mathsf{v}. \tag{23}$$

To compute II^ν, we express the second fundamental form in our basis normal directions as

$$\mathrm{II}^{\mathsf{w}} = \begin{pmatrix} a_1 & b_1 \\ b_1 & -a_1 \end{pmatrix}, \quad \mathrm{II}^{\mathsf{e}_2} = \begin{pmatrix} a_2 & b_2 \\ b_2 & -a_2 \end{pmatrix}, \quad \mathrm{II}^{\mathsf{e}_3} = \begin{pmatrix} a_3 & b_3 \\ b_3 & -a_3 \end{pmatrix},$$

where each quadratic form is represented by a matrix with respect to the orthonormal basis $(\mathsf{e}_1, \mathsf{v})$.

Now using (21) and (23), the condition $\mathrm{II}^\nu(\xi,\xi) = 0$ can be expanded as

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \end{bmatrix} (\sin\psi\sin\varphi\, \mathrm{II}^{\mathsf{w}} + \sin\psi\cos\varphi\, \mathrm{II}^{\mathsf{e}_2} + \cos\psi\, \mathrm{II}^{\mathsf{e}_3}) \begin{bmatrix} \cos\varphi \\ -\sin\varphi \end{bmatrix}.$$

Since this equation must be satisfied for all angles φ and ψ (provided $\sin\psi \neq 0$), we can take particular values. Setting $\varphi = 0$ yields

$$a_2 \sin\psi + a_3 \cos\psi = 0,$$

which can only hold for all ψ if $a_2 = a_3 = 0$. Setting $\varphi = \pi/2$ yields

$$a_1 \sin\psi + a_3 \cos\psi = 0,$$

so that a_1 must also vanish. Taking $a_1 = a_2 = a_3 = 0$ into account, the condition becomes

$$\sin\varphi\cos\varphi(\sin\psi(b_1\sin\varphi + b_2\cos\varphi) + b_3\cos\psi) = 0 \qquad \forall\varphi,\psi,$$

which implies that $b_1 = b_2 = b_3 = 0$.

Since ν is an arbitrary normal direction, e_3 can range over all of $\mathcal{N}_p$, and p is arbitrary, we conclude that M is totally geodesic. □

Theorem 7. *If $M \subset \mathbb{C}P^n$ is a connected austere surface, then M is either a holomorphic curve or an open subset of a real projective plane $\mathbb{R}P^2 \subset \mathbb{C}P^2$, where $\mathbb{C}P^2$ is embedded as a complex linear subspace in $\mathbb{C}P^n$.*

Proof. If M is austere, then M is minimal and hence real-analytic. Thus, the points where T_pM is J-invariant form an open and closed subset of M. By Proposition 6, M is either a holomorphic curve or totally geodesic. If M is totally geodesic, then by a well-known result of Wolf [13], M is either an open set of a complex line in $\mathbb{C}P^n$, or the real part of a complex projective plane. □

References

[1] R. L. Bryant, *Some remarks on the geometry of austere manifolds*, Bol. Soc. Bras. Mat. **21** (1991), 133–157.

[2] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[3] M. Ionel and T. Ivey, *Austere submanifolds of dimension four: examples and maximal types*, Illinois Math J. **54** (2010), 713–746.

[4] M. Ionel and T. Ivey, *Ruled austere submanifolds of dimension four*, Differential Geometry and its Applications **30** (2012), 588–603.

[5] T. Ivey, *Stark hypersurfaces in complex projective space*, Contemporary Mathematics **674** (2016), 75–88.

[6] D. Joyce, *Riemannian Holonomy Groups and Calibrated Geometry*, Oxford University Press, 2007.

[7] S. Karigiannis and M. Min-Oo, *Calibrated subbundles in noncompact manifolds of special holonomy*, Ann. Global Anal. Geom. **28** (2005), 371–394.

[8] T.-C. Lee, *Complete Ricci flat Kähler metric on M_{I}^n, M_{II}^{2n}, M_{III}^{4n}*, Pacific J. Math. **185** (1998), 315–326.

[9] G. Patrizio and P-M. Wong, *Stein manifolds with compact symmetric center*, Math. Ann. **289** (1991), 355–382.

[10] M. Stenzel, *Kähler Structures on the cotangent bundles of real analytic riemannian manifolds*, PhD thesis, Massachusetts Institute of Technology, 1990.

[11] M. Stenzel, *Ricci-flat metrics on the complexification of a compact rank one symmetric space*, Manuscripta Math. **80** (1993), 151–163.

[12] R. Szöke, *Complex structures on cotangent bundles of Riemannian manifolds*, Math. Ann. **291** (1991), 409–428.

[13] J. A. Wolf, *Geodesic spheres in Grassmann manifolds* and *Elliptic spaces in Grassmann manifolds*, Illinois Math. J. **7** (1963), 425–462.

Institute of Mathematics
Federal University of Rio de Janeiro, Brazil
E-mail address: marianty@im.ufrj.br

Department of Mathematics, College of Charleston
66 George St., Charleston SC 29424, USA
E-mail address: iveyt@cofc.edu

Received March 18, 2014