Communications in
Analysis and Geometry
Volume 25, Number 3, 507–544, 2017

# The spectrum of geodesic balls on spherically symmetric manifolds

Denis Borisov and Pedro Freitas

We study the Dirichlet spectrum of the Laplace operator on geodesic balls centred at a pole of spherically symmetric manifolds. We first derive a Hadamard–type formula for the dependence of the first eigenvalue $\lambda_1$ on the radius $r$ of the ball, which allows us to obtain lower and upper bounds for $\lambda_1$ in specific cases. For the sphere and hyperbolic space, these bounds are asymptotically sharp as $r$ approaches zero and we see that while in two dimensions $\lambda_1$ is bounded from above by the first two terms in the asymptotics for small $r$, for dimensions four and higher the reverse inequality holds.

In the general case we derive the asymptotic expansion of $\lambda_1$ for small radius and determine the first three terms explicitly. For compact manifolds we carry out similar calculations as the radius of the geodesic ball approaches the diameter of the manifold. In the latter case we show that in even dimensions there will always exist logarithmic terms in these expansions.



*2000 Mathematics Subject Classification:* Primary 35P15; Secondary 58C40.
*Key words and phrases:* spectrum, geodesic balls, Hadamard formula, asymptotics.

## 1. Introduction

Within the last forty years several papers appeared in the literature devoted to the study of the first eigenvalue of geodesic disks on spherically symmetric manifolds, with particular emphasis on the case of spaces of constant curvature — see, for example, [Ba, BB, BCG, Ca, FH, Ga, Gr, K, M, MT, P1, P2, S, W]. In this particular instance, namely, hyperbolic space $\mathbb{H}^n$ and spheres $\mathbb{S}^n$, the solutions are known explicitly in terms of Legendre functions of the first kind, the eigenvalues then being given in terms of zeros of such functions. What is thus of interest is to obtain bounds and approximations which may be written explicitly in terms of more elementary functions, as in general computer packages may determine these zeros with the needed accuracy — it should, however, be noted that in limit cases such as when the radius becomes very large in hyperbolic space these computations may still pose some difficulties from a numerical point of view.

For general spherically symmetric manifolds, however, it will not be possible to write the eigenvalues of geodesic balls explicitly in terms of *known* functions, and it then becomes important to have accurate estimates for these quantities. Apart from their intrinsic interest, bounds of this type may also be used to estimate eigenvalues of balls on manifolds which are not necessarily spherically symmetric, by making use of the recent spectral comparison results established in [FMS].

A major difference when moving away from the Euclidean framework is that scaling the domain no longer translates into a mere scaling in spectral terms. More precisely, while in the former case we have $\lambda(\alpha\Omega) = \alpha^{-2}\lambda(\Omega)$ for a scaling of a domain $\Omega$ by a positive number $\alpha$ and any eigenvalue $\lambda$, for the latter the variation of eigenvalues will provide extra information on the metric.

With the above in mind, it makes thus sense as a first step in this direction to derive a Hadamard–type formula for the variation of the first eigenvalue of a geodesic ball with respect to the radius. This will in turn allow us to derive some new lower and upper bounds which, apart from improving existing results within certain ranges of the radius, also have the advantage of being explicit in the sense mentioned above. As an example of this, Theorem 3.3 gives lower and upper bounds for disks in hyperbolic space and spheres which agree with the first two terms in the asymptotics of the first eigenvalue as the radius approaches zero.

In fact, and as a consequence of these bounds, we see that for (non-Euclidean) constant curvature spaces, while in two dimensions the first eigenvalue is bounded from above by the first two terms in the asymptotics, this relation is reversed for dimensions greater than or equal to 4 — for $\mathbb{H}^3$ and $\mathbb{S}^3$ the expressions obtained are exact.

In the case of a general $n-$manifold, the first two terms in these asymptotic expansions of the $k^{\text{th}}$ eigenvalue $\lambda_k$ are known to be given by

$$\lambda_k(r) = \frac{1}{r^2}\gamma_k - \frac{1}{6}\mathcal{S}(p) + \mathrm{o}(1),$$

where $\gamma_k$ denotes the $k^{\text{th}}$ eigenvalue of the unit disk in $n-$dimensional Euclidean space and $\mathcal{S}(p)$ denotes the scalar curvature at the point $p$ — see [C, p. 318]. In the particular case of a spherically symmetric manifold we shall derive the full asymptotic expansion for the first eigenvalue of a disk of small radius $r$ centred at a pole and, for compact manifolds, also the expansion as this radius approaches the diameter of the manifold. We then determine the expression for the first three terms explicitly, from which we obtain, for instance, that under certain natural smoothness assumptions the third term in the expansion above is of order $r^2$ and the corresponding expansion for $\lambda_1$ is given by

$$\lambda_1(r) = \frac{j^2_{\frac{n}{2}-1,1}}{r^2} - \frac{1}{6}\mathcal{S}(p) + \left[\alpha_1\mathcal{S}^2(p) + \alpha_2\mathcal{S}''(p)\right] r^2 + \mathrm{o}(r^4),$$

where $\alpha_1$ and $\alpha_2$ are constants which depend only on the dimension — see Theorem 4.1 below for the details, including explicit expressions for these coefficients. The expressions for the asymptotics as the radius of the ball approaches the diameter are more involved (in particular, they depend in a nontrivial way on the dimension) and are presented in Theorem 5.1. Although this situation corresponds to the well–known singular perturbation problem of a manifold with a small hole whose volume approaches zero — see [F, C] and [MNP], for instance, and the references therein — we shall see that in all even dimensions logarithmic terms do appear in the expansions for $\lambda_1$. This comes as a surprise as, to the best of our knowledge, so far only in two dimensions was it known that a logarithmic term would be present (in fact the leading term), as a direct consequence of the singularity of the corresponding Green's function — see also Table II in [F] which contains an overview of the results for this type of problems.

The plan of the paper is as follows. In the next section we fix the notation and state some basic facts which will be used in the sequel. Section 3 contains the statement and derivation of the Hadamard-type formula, followed by its application to obtaining upper and lower bounds in Section 3.1. We then proceed to determine the asymptotic expansions mentioned above for small and maximal radius in Sections 4 and 5, respectively.

## 2. Notation and preliminaries

Given a sufficiently smooth function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \tag{2.1}$$

and $f(r) > 0$ for all $r$ in $(0, R)$ for some positive $R$ (possibly infinite), let $M$ be the spherically symmetric $n-$manifold with the metric $dr^2 + f^2(r)d\theta^2$ and $p$ be the point at the *North pole*. The above restriction on the second derivative of $f$ stems from the fact that the scalar curvature $\mathcal{S}$ at a point $p$ is given by [Pe, p. 69]

$$\mathcal{S}(p) = -2(n-1)\frac{f''(t)}{f(t)} + (n-1)(n-2)\frac{1 - \left[f'(t)\right]^2}{f^2(t)}$$

and it thus follows that for this to be finite at $t = 0$ one must have $f''(0) = 0$. It turns out that for the metric to be regular at $t$ equal to zero stronger restrictions have to be imposed on $f$. In particular, if it is to be smooth, then all even derivatives of $f$ must vanish at zero [Pe, pp. 12–13]. Throughout the paper we shall make this assumption, although it is clear that if $f^{(2k)}(0)$ is not zero for some $k$ larger than one then the terms appearing in the asymptotics in Section 4 should be modified accordingly.

We consider the geodesic ball $B(r)$ centred at $p$ and with radius $r$. The $n-$volume of this ball is then given by

$$V(r) = \omega_{n-1} \int_0^r f^{n-1}(t)dt,$$

while the $(n-1)-$volume of its boundary is given by

$$S(r) = \omega_{n-1} f^{n-1}(r),$$

where $\omega_{n-1}$ denotes the $(n-1)-$volume of the unit sphere $\mathbb{S}^{n-1}$.

In this setting, the first eigenvalue of the Laplace-Beltrami operator on $B(r)$ with Dirichlet boundary conditions is simple and the corresponding eigenfunction does not change sign and is also spherically symmetric. Denoting this eigenvalue by $\lambda = \lambda(r)$ and by $\psi$ a corresponding eigenfunction, we have that the pair $(\lambda, \psi)$ satisfies

$$(2.2)\qquad \begin{cases} -\left[f^{n-1}(t)\psi'(t)\right]' = \lambda f^{n-1}(t)\psi(t), & t \in (0,r) \\ \psi'(0) = \psi(r) = 0. \end{cases}$$

We can also interpret $\lambda$ and $\psi$ as the first eigenvalue and the associated eigenfunction of the operator $\mathcal{H}_r$, where $\mathcal{H}_r$ is introduced as the self-adjoint operator in $X_r^0$ associated with the closed lower-semibounded quadratic form $h_r[u] := \|u'\|^2_{X_r^0}$ on $X_r$. Here

$$X_r := \{u \in W^1_{2,loc}(0,r) : \ u(r) = 0, \ \|u\|_{X_r} < +\infty\},$$
$$\|u\|^2_{X_r} := \|u'\|^2_{X_r^0} + \|u\|^2_{X_r^0}, \quad \|u\|^2_{X_r^0} := \int_0^r f^{n-1}(t)|u(t)|^2\, dt,$$
$$X_r^0 := \{u \in L_{2,loc}(0,r) : \|u\|_{X_r^0} < +\infty\}.$$

In this way, $\lambda$ may also be obtained via its variational formulation which now becomes

$$(2.3)\qquad \lambda(r) = \inf_{u \in X_r} \frac{\displaystyle\int_0^r f^{n-1}(t)\left[u'(t)\right]^2 dt}{\displaystyle\int_0^r f^{n-1}(t) u^2(t) dt}.$$

## 3. A Hadamard-type formula for the first eigenvalue

While in the Euclidean case the first eigenvalue of balls centred at any point and with different radii are related to each other by a simple rescaling, this will not be the case in more general ambient spaces. As a consequence, the way in which the eigenvalue varies as a function of the radius of the ball will contain information about the ambient space. The purpose of this section is to establish a formula for such a variation in the more general situation of a spherically symmetric manifold with the centre of the ball placed at the North pole.

**Lemma 3.1.** *Let $\psi$ be a solution of equation (2.2) associated to the first eigenvalue $\lambda(r)$. Then, for any non–negative $r_0$,*

$$\lambda(r) = \frac{1}{r^2}\lim_{s\to r_0}\left[s^2\lambda(s)\right] + \frac{n-1}{2r^2}\int_{r_0}^{r}\left[\frac{t\displaystyle\int_0^t H(s)f^{n-1}(s)\psi^2(s)ds}{\displaystyle\int_0^t f^{n-1}(s)\psi^2(s)ds}\right]dt,$$

*where*

$$\begin{aligned}H(t) &= th'(t) + 2h(t)\\ h(t) &= g'(t) + \frac{n-1}{2}g^2(t)\\ g(t) &= \frac{f'(t)}{f(t)}.\end{aligned}$$

*In the particular case where $r_0$ is zero, the above expression simplifies to*

$$\lambda(r) = \frac{j^2_{\frac{n}{2}-1,1}}{r^2} + \frac{n-1}{2r^2}\int_0^r\left[\frac{t\displaystyle\int_0^t H(s)f^{n-1}(s)\psi^2(s)ds}{\displaystyle\int_0^t f^{n-1}(s)\psi^2(s)ds}\right]dt,$$

*where $j_{\frac{n}{2}-1,1}$ is the first zero of the Bessel function $J_{\frac{n}{2}-1}$.*

**Remark 3.2.** Note that the eigenfunction $\psi$ also depends on $r$ and that $H$ depends (linearly) on $n$.

*Proof.* From the variational formulation (2.3) we have

$$\lambda(\alpha r) = \inf_{u\in X_{\alpha r}}\frac{\displaystyle\int_0^{\alpha r} f^{n-1}(t)\left[u'(t)\right]^2dt}{\displaystyle\int_0^{\alpha r} f^{n-1}(t)u^2(t)dt} = \frac{1}{\alpha^2}\inf_{u\in X_r}\frac{\displaystyle\int_0^{r} f^{n-1}(\alpha t)\left[u'(t)\right]^2dt}{\displaystyle\int_0^{r} f^{n-1}(\alpha t)u^2(t)dt}$$

and

$$\lambda(\alpha r) \leqslant \frac{1}{\alpha^2}\frac{\displaystyle\int_0^{r} f^{n-1}(\alpha t)\left[\psi'(t)\right]^2dt}{\displaystyle\int_0^{r} f^{n-1}(\alpha t)\psi^2(t)dt} \tag{3.1}$$

where $\psi$ is a first eigenfunction corresponding to $\lambda$, that is, $\psi$ and $\lambda = \lambda(r)$ satisfy equation (2.2).

Multiply now equation (2.2) by $f^{n-1}(\alpha t)\psi(t)/f^{n-1}(t)$ and integrate between 0 and $r$ by parts twice to obtain

$$\begin{aligned}
&\int_0^r f^{n-1}(\alpha t)\psi^2(t)dt\ \lambda(r) = -\int_0^r \frac{f^{n-1}(\alpha t)\psi(t)}{f^{n-1}(t)}\left[f^{n-1}(t)\psi'(t)\right]' dt\\
&= (n-1)\int_0^r \left[\alpha f^{n-2}(\alpha t)f'(\alpha t) - f^{n-1}(\alpha t)\frac{f'(t)}{f(t)}\right]\psi(t)\psi'(t)dt\\
&\quad + \int_0^r f^{n-1}(\alpha t)\left[\psi'(t)\right]^2 dt\\
&= -\frac{n-1}{2}\int_0^r \left[\alpha^2(n-2)\left[\frac{f'(\alpha t)}{f(\alpha t)}\right]^2 + \alpha^2\frac{f''(\alpha t)}{f(\alpha t)}\right.\\
&\qquad\qquad \left. -\alpha(n-1)\frac{f'(t)f'(\alpha t)}{f(t)f(\alpha t)} - \frac{f''(t)}{f(t)} + \left[\frac{f'(t)}{f(t)}\right]^2\right]\\
&\quad \times f^{n-1}(\alpha t)\psi^2(t)dt + \int_0^r f^{n-1}(\alpha t)\left[\psi'(t)\right]^2 dt
\end{aligned}$$

Plugging this back into (3.1) yields

$$\begin{aligned}
\lambda(r) \geqslant \alpha^2\lambda(\alpha r) &- \frac{n-1}{2}\frac{\displaystyle\int_0^r \left[\alpha^2(n-2)\left[\frac{f'(\alpha t)}{f(\alpha t)}\right]^2 + \alpha^2\frac{f''(\alpha t)}{f(\alpha t)}\right.}{\displaystyle\int_0^r f^{n-1}(\alpha t)\psi^2(t)dt}\\
&- \frac{\displaystyle\left. \alpha(n-1)\frac{f'(t)f'(\alpha t)}{f(t)f(\alpha t)} - \frac{f''(t)}{f(t)} + \left[\frac{f'(t)}{f(t)}\right]^2\right] f^{n-1}(\alpha t)\psi^2(t)dt}{\displaystyle\int_0^r f^{n-1}(\alpha t)\psi^2(t)dt}.
\end{aligned}$$

Consider first the case of $\alpha$ smaller than one. Then

$$\begin{aligned}
\frac{\lambda(r)-\alpha^2\lambda(\alpha r)}{1-\alpha} \geqslant \frac{n-1}{2(\alpha-1)}&\frac{\displaystyle\int_0^r \left[\alpha^2(n-2)\left[\frac{f'(\alpha t)}{f(\alpha t)}\right]^2 + \alpha^2\frac{f''(\alpha t)}{f(\alpha t)}\right.}{\displaystyle\int_0^r f^{n-1}(\alpha t)\psi^2(t)dt}\\
&- \frac{\displaystyle\left. \alpha(n-1)\frac{f'(t)f'(\alpha t)}{f(t)f(\alpha t)} - \frac{f''(t)}{f(t)} + \left[\frac{f'(t)}{f(t)}\right]^2\right] f^{n-1}(\alpha t)\psi^2(t)dt}{\displaystyle\int_0^r f^{n-1}(\alpha t)\psi^2(t)dt}
\end{aligned}$$

and upon taking limits on both sides as $\alpha$ goes to $1^-$ we obtain, after some lengthy computations,

$$r\lambda'(r) + 2\lambda(r) \geqslant \frac{n-1}{2} \frac{\int_0^r \big[tg''(t) + (n-1)tg(t)g'(t) + 2g'(t)}{\int_0^r f^{n-1}(t)\psi^2(t)dt} + \frac{(n-1)g^2(t)\big] f^{n-1}(t)\psi^2(t)dt}{\int_0^r f^{n-1}(t)\psi^2(t)dt},$$

where $g(t) = f'(t)/f(t)$.

Now note that if we had taken $\alpha$ to be larger than one instead, then the inequalities above would be reversed. Thus, after taking limits, now with $\alpha$ approaching one from above, we would get the above inequality reversed. Hence this is an indentity, meaning we have

$$r\lambda'(r) + 2\lambda(r) = \frac{n-1}{2} \frac{\int_0^r \big[th'(t) + 2h(t)\big] f^{n-1}(t)\psi^2(t)dt}{\int_0^r f^{n-1}(t)\psi^2(t)dt},$$

with $h(t) = g'(t) + (n-1)g^2(t)/2$. Multiplying the above equation by $r$ and integrating between $r_0$ and $r$, yields the desired result. $\square$

### 3.1. Applications of Lemma 3.1 to hyperbolic space and spheres

As in the classical Hadamard formula for the variation of the first eigenvalue with respect to domain variations, the expressions in Lemma 3.1 depend on the eigenfuntion of the unperturbed domain. Although this may make it slightly difficult to use in general, there are situations for which, depending on the behaviour of the function $H$, it might be possible to derive simplified expressions for these formulae. The case of spaces of constant curvature is one such example as we shall now see.

**Theorem 3.3.** *The first eigenvalue of a ball of radius* $r$ *in* $\mathbb{H}^n$ $(0 < r)$ *or* $\mathbb{S}^n$ $(0 < r < \pi)$ *satisfies*

$$\frac{j_{0,1}^2}{r^2} + \frac{1}{4}\left[\frac{1}{r^2} - \frac{1}{\mathrm{s}^2(r)} \pm 1\right] \leqslant \lambda(r) \leqslant \frac{j_{0,1}^2}{r^2} \pm \frac{1}{3} \qquad (n = 2)$$

$$\lambda(r) = \frac{\pi^2}{r^2} \pm 1 \qquad (n = 3)$$

$$\frac{j_{\frac{n-2}{2},1}^2}{r^2} \pm \frac{n(n-1)}{6} \leqslant \lambda(r) \leqslant \frac{j_{\frac{n-2}{2},1}^2}{r^2} \pm \frac{(n-1)^2}{4} + \frac{(n-1)(n-3)}{4}\left[\frac{1}{\mathrm{s}^2(r)} - \frac{1}{r^2}\right] \qquad (4 \leqslant n),$$

*where* s *denotes* sinh *or* sin*, respectively, and in all indicated* $\pm$ *the plus and minus signs are to be considered for* $\mathbb{H}^n$ *and* $\mathbb{S}^n$*, respectively.*

**Remark 3.4.** In the case of the upper bounds in $\mathbb{H}^n$ $(n > 3)$, they approach $(n-1)^2/4$ as $r$ goes to infinity and are thus also asymptotically accurate in this limit. The bounds for $\mathbb{S}^n$ are not asymptotically accurate as $r$ approaches $\pi$ (except for $n = 3$), and indeed they may become quite poor in this limit.

**Remark 3.5.** Note also that the above bounds are similar to those in Theorem 5.2 in [Ga]. However, the latter do not display the correct asymptotic behaviour as $r$ goes to zero.

*Proof.* In the case of hyperbolic space with constant curvature $-1$, $f(t) = \sinh(t)$, and the function $H(t) = th'(t) + 2h(t)$ becomes

$$H(t) = (n-1)\coth^2(t) - \frac{1}{\sinh^2(t)}\left[2 + (n-3)t\coth(t)\right].$$

Its derivative may be written as

$$H'(t) = \frac{n-3}{2\sinh^4(t)}\left[2t(2+\cosh(2t)) - 3\sinh(2t)\right],$$

from which it follows that, for positive $t$, $H$ is decreasing for $n$ equal to two, identically equal to 2 for $n$ equal to three, and increasing for $n$ larger than three. Using these monotonicity properties in Lemma 3.1, together with

$$\lim_{t\to 0} H(t) = \frac{2n}{3} \quad \text{and} \quad \lim_{t\to\infty} H(t) = n-1,$$

yields the bounds for $\mathbb{H}^n$ given in Theorem 3.3.

Spheres with constant curvature 1 correspond to $f(t) = \sin(t)$, and now

$$H(t) = (n-1)\cot^2(t) - \frac{1}{\sin^2(t)}\left[2 + (n-3)t\cot(t)\right],$$

while

$$H'(t) = \frac{n-3}{2\sin^4(t)}\left[2t(2+\cos(2t)) - 3\sin(2t)\right].$$

We again obtain that, for $0 < t < \pi$, $H$ is decreasing when $n$ is two, constant $(-2)$ for $n$ equal to three and increasing for $n$ larger than 3. This, together with Lemma 3.1, yields the bounds for $\mathbb{S}^n$ given in Theorem 3.3. □

## 4. Asymptotic expansion for small radius

In this section we consider the case of a small geodesic disk centred at the North Pole in the case where its radius approaches zero. We assume $f$ to be sufficiently smooth for small $r$, and our aim is to obtain the asymptotic expansion for $\lambda(r)$ and the associated eigenfunction $\psi$.

Our main result here is the following.

**Theorem 4.1.** *The first eigenvalue of the geodesic disk centred at the North Pole $p$ and with radius $r$ satisfies*

$$\lambda(r) = \frac{j^2_{\frac{n}{2}-1,1}}{r^2} - \frac{1}{6}\mathcal{S}(p) + \left[\alpha_1\mathcal{S}^2(p) + \alpha_2\mathcal{S}''(p)\right]r^2 + \mathrm{O}(r^4), \quad r \to 0^+,$$

*where the coefficients $\alpha_i$, $i = 1,2$ depend only on the dimension $n$ and are given by*

$$\alpha_1 = \frac{c_0^2}{270n^2(n-1)}\left[5\pi(n-1)(I_2 + j_{\frac{n-2}{2},1}I_3) - 3(n+2)I_1\right]$$

*and*

$$\alpha_2 = -\frac{c_0^2}{10}I_1,$$

*with*

$$c_0 = \frac{\sqrt{2}}{\left|J'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1})\right|},$$

$$I_1 = \int_0^1 \xi^3 J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\, d\xi,$$

$$I_2 = \int_0^1 \xi^3 J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi) Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\, d\xi,$$

$$I_3 = \int_0^1 \xi^4 J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi) Y'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\, d\xi.$$

**Remark 4.2.** The integrals $I_k$ $(k = 2, 3)$ above are, in general, not computable in terms of known constants. However, in dimension three it is possible to carry out all the computations explicitly to obtain that $\alpha_1$ vanishes in that case and thus

$$\lambda(r) = \frac{\pi^2}{r^2} - \frac{1}{6}\mathcal{S}(p) + \left(\frac{1}{2\pi^2} - \frac{1}{3}\right)\frac{\mathcal{S}''(p)}{10} r^2 + \mathrm{O}(r^4).$$

**Remark 4.3.** Although the above theorem provides only three terms in the asymptotics for the eigenvalue, our technique allows us to construct the complete asymptotic expansion, cf. equations (4.4), (4.5), (4.6), (4.7), (4.11), (4.12) below. In the theorem we calculate explicitly and in a convenient form only the first three terms, just to illustrate the result.

In order to handle the situation where $r \to 0^+$, it is natural to introduce a rescaled variable $\xi := tr^{-1}$ and to rewrite the eigenvalue problem (2.1) as

$$-\frac{d^2\psi}{d\xi^2} - (n-1)r\frac{f'(r\xi)}{f(r\xi)}\frac{d\psi}{d\xi} = r^2\lambda\psi \quad \text{in} \quad (0,1), \tag{4.1}$$

$$\psi'(0,r) = \psi(1,r) = 0. \tag{4.2}$$

In view of (2.1) we see that the coefficient at the first derivative behaves as

$$r\frac{f'(r\xi)}{f(r\xi)} = \frac{1}{\xi} + r^2 f_0(r\xi),$$

where the function $f_0(t)$ is bounded uniformly in $[0, r_0]$. We substitute the last identity into (4.1). It leads us to the equation

$$-\frac{d^2}{d\xi^2} - \frac{n-1}{\xi}\frac{d\psi}{d\xi} - (n-1)r^2 f_0(r\xi)\frac{du}{d\xi} = r^2\lambda\psi \quad \text{in} \quad (0,1), \tag{4.3}$$

with boundary conditions (4.2).

We introduce Hilbert spaces $Y_0$ and $Y_1$,

$$\begin{aligned} &Y_0 := \{u \in L_{2,loc}(0,1) : \|u\|_{Y_0} < \infty\},\\ &Y_1 := \{u \in W^1_{2,loc}(0,1) : \|u\|_{Y_1} < \infty\},\\ &(u,v)_{Y_0} := \int\limits_0^1 \xi^{n-1} u\overline{v}\, d\xi, \quad (u,v)^2_{Y_1} := (u',v')_{Y_0} + (u,v)_{Y_0}. \end{aligned}$$

We let

$$\mathcal{H}_* := -\frac{d^2}{d\xi^2} - \frac{n-1}{\xi}\frac{d}{d\xi} \quad \text{on} \quad (0,1)$$

with the boundary conditions (4.2). More precisely, the operator $\mathcal{H}_*$ is understood as the associated one with the quadratic form $h_0[u] := \|u'\|^2_{Y_0}$ on $Y_1$. By $\mathcal{D}(\mathcal{H}_*)$ we denote the domain of the operator $\mathcal{H}_*$.

**Lemma 4.4.** *For each $u \in \mathcal{D}(\mathcal{H}_*)$ the estimate*

$$\|f_0(r\,\cdot)u'\|_{Y_0} \leqslant C\big(\|\mathcal{H}_* u\|_{Y_0} + \|u\|_{Y_0}\big)$$

*holds true, where $C$ is a constant independent of $u$.*

*Proof.* The statement of the lemma follows easily from the two estimates

$$\begin{aligned} &h_0[u] = (\mathcal{H}_* u, u)_{Y_0} \leqslant \frac{1}{2}\big(\|\mathcal{H}_* u\|^2_{Y_0} + \|u\|^2_{\mathcal{H}_*}\big)\\ &\text{and} \quad \|f_0(r\,\cdot)u'\|_{Y_0} \leqslant C h_0[u], \end{aligned}$$

valid for each $u \in \mathcal{D}(\mathcal{H}_*)$. □

The last lemma implies that the operator $(n-1)f_0(r\xi)\frac{d}{d\xi}$ is $\mathcal{H}_*$-bounded with the bounds independent of $r$. Hence, we can consider problem (4.3), (4.2) as a small perturbation of the eigenvalue problem for $\mathcal{H}_*$. Therefore, the eigenvalue $\lambda(r)$ and the associated eigenfunction can be represented as the convergent series

$$\lambda(r) = r^{-2}\sum_{j=0}^{\infty} r^{2j}\lambda_j, \quad \psi(\xi,r) = \sum_{j=0}^{\infty} r^{2j}\psi_j(\xi,r), \tag{4.4}$$

where the latter converges in $Y_1$. We substitute these series into (4.3) and (4.2) to obtain the following equations for $\psi_j$,

$$\begin{aligned} &\mathcal{H}_*\psi_0 = \lambda_0\psi_0 \quad \text{and} \\ &(\mathcal{H}_* - \lambda_0)\psi_j = \lambda_j\psi_0 + \sum_{k=1}^{j-1}\lambda_k\psi_{j-k} + (n-1)f_0(r\,\cdot)\psi'_{j-1}. \end{aligned} \tag{4.5}$$

The ground state of $\mathcal{H}_*$ is expressed in terms of the Bessel function of the first kind,

$$\psi_0(\xi) = c_0\xi^{-\frac{n}{2}+1}J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi), \qquad \lambda_0 = j^2_{\frac{n}{2}-1,1}, \tag{4.6}$$

where $c_0$ is the normalization constant given in Theorem 4.1 ensuring

$$\|\psi_0\|_{Y_0} = 1. \tag{4.7}$$

Consider problem (4.5) for $\psi_1$,

$$(\mathcal{H}_* - \lambda_0)\psi_1 = \lambda_1\psi_0 + (n-1)f_0(r\,\cdot)\psi_0'. \tag{4.8}$$

The solvability condition to this problem is the orthogonality of the right hand side to $\psi_0$ in $Y_0$. With (4.7) taken into consideration this determines $\lambda_1$, which is thus given by

$$\lambda_1 = -(n-1)(f_0(r\,\cdot)\psi_0', \psi_0)_{Y_0}. \tag{4.9}$$

The solution to (4.8) exists and is defined up to an additive term $Cu_0$, which may be determined by the orthogonality condition

$$(\psi_1, \psi_0)_{Y_0} = 0. \tag{4.10}$$

In the same way we solve the succesion of the problems for $\psi_j$, $j \geqslant 2$. The corresponding equations (4.5) for $\psi_j$ are solvable, if their right–hand sides are orthogonal to $\psi_0$ in $Y_0$. These solvability conditions then yield the formulae for $\lambda_j$,

$$\lambda_j = -(n-1)(f_0(r\,\cdot)\psi'_{j-1}, \psi_0)_{Y_0}. \tag{4.11}$$

Here we have used that all the functions $\psi_j$ are defined modulo an additive term of the form $C\psi_0$, which are determined as above from the orthogonality conditions

$$(\psi_j, \psi_0)_{Y_0} = 0. \tag{4.12}$$

In this way all the coefficients of series (4.5) are uniquely determined. One can also check by induction that these coefficients are bounded uniformly in $r$ (the coefficients $\psi_j$ in $Y_1$-norm).

The coefficients of series (4.4) depend on $r$ through the coefficient $f_0(r\xi)$ in (4.3). On the other hand, if the function $f$ is smooth enough, we can simplify the asymptotic expansion. More precisely, we can replace $f_0(r\xi)$ by its Taylor formula as $r \to 0^+$ and substitute it into (4.5), (4.8), (4.9), (4.10), (4.11) and (4.12), leading to another asymptotic expansion in terms of powers of $r$. We employ this fact to construct leading terms of the asymptotics in a more explicit form than (4.4). Namely, we assume $f \in C^5[0, r_0]$ and $f^{(2)}(0) = 0$ with, as mentioned in Section 2, the latter identity reflecting the smoothness of the manifold at the pole. Hence

$$f_0(r\xi) = \frac{f'''(0)}{3}\xi + r^2\frac{3f^{(5)}(0) - 5(f'''(0))^2}{90}\xi^3 + \mathrm{O}(r^3), \quad r \to 0^+, \tag{4.13}$$

uniformly in $\xi \in [0, 1]$. We substitute this identity into (4.9),

$$\begin{aligned} \lambda_1(r) = &-\frac{(n-1)f'''(0)}{3}(\xi\psi'_0, \psi_0)_{Y_0} \\ &- r^2\frac{(n-1)\big(3f^{(5)}(0) - 5(f'''(0))^2\big)}{90}(\xi^3\psi'_0, \psi_0)_{Y_0} + \mathrm{O}(r^3). \end{aligned} \tag{4.14}$$

Integrating by parts and using (4.7), we get

$$(\xi\psi_0', \psi_0)_{Y_0} = \int_0^1 \xi^n \psi_0'(\xi)\psi_0(\xi)\, d\xi = -\frac{n}{2}\int_0^1 \xi^{n-1}\psi_0^2(\xi)\, d\xi = -\frac{n}{2},$$

$$\begin{aligned}(\xi^3\psi_0', \psi_0)_{Y_0} &= \int_0^1 \xi^{n+2}\psi_0'(\xi)\psi_0(\xi)\, d\xi = -\frac{n+2}{2}\int_0^1 \xi^{n+1}\psi_0^2(\xi)\, d\xi \\ &= -\frac{(n+2)c_0^2}{2}\int_0^1 s^3 J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}s)\, ds.\end{aligned} \tag{4.15}$$

Then the right hand side in (4.8) casts into the form

$$\begin{aligned}&\lambda_1(r)\psi_0(\xi) + (n-1)f_0(r\xi)\psi_0'(\xi) \\ &= \frac{(n-1)f'''(0)}{3}\Big(\frac{n}{2}\psi_0(\xi) + \xi\psi_0'(\xi)\Big) + \mathrm{O}(r^2), \quad r \to 0^+,\end{aligned}$$

uniformly in $\xi \in [0,1]$, and the function $\psi_1$ may then be represented as

$$\psi_1(\xi, r) = \frac{(n-1)f'''(0)}{3}\Psi_1(\xi,\rho) + \mathrm{O}(r^2), \quad r \to 0^+, \tag{4.16}$$

uniformly in $\xi \in [0,1]$, where $\Psi_1$ is the solution to the equation

$$(\mathcal{H}_* - j^2_{\frac{n}{2}-1,1})\psi_1 = \frac{n}{2}\psi_0 + \xi\psi_0'$$

satisfying the orthogonality condition $(\Psi_1, \psi_0)_{Y_0} = 0$. The function $\Psi_1$ can be found explicitly,

$$\begin{aligned}\Psi_1(\xi) &= \frac{\pi c_0}{2}\xi^{-\frac{n}{2}+1}\widetilde{\Psi}_1(\xi), \\ \widetilde{\Psi}_1(\xi) &:= \frac{\xi^2}{2}J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi) - J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\widehat{\Psi}_1(\xi) \\ &\quad - c_1c_0^2 J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi), \\ \widehat{\Psi}_1(\xi) &:= \int_0^\xi sJ_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}s)(sY_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}s))'\, ds, \\ c_1 &:= \int_0^1 \left(\frac{\xi^3}{2}J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi) - \xi J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\widehat{\Psi}_1(\xi)\right) d\xi,\end{aligned}$$

where $Y_{\frac{n}{2}-1}$ is the Bessel function of second kind. This identity together with (4.16), (4.11) and (4.13) yield

$$\lambda_2(r) = -\frac{(n-1)^2(f'''(0))^2c_0}{9}\int_0^1 \xi^{\frac{n}{2}+1}J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\Psi_1'(\xi)\,d\xi + \mathrm{O}(r^2). \tag{4.17}$$

Integrating by parts, we obtain

$$\begin{aligned}
&\int_0^1 \xi^{\frac{n}{2}+1}J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\Psi_1'(\xi)\,d\xi\\
&= -\left(\frac{n}{2}+1\right)\int_0^1\left(\frac{\xi^3}{2}J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi) - \xi J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\widehat{\Psi}_1(\xi)\right)d\xi\\
&\quad - j_{\frac{n}{2}-1,1}\int_0^1\left(\frac{\xi^4}{2}J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)J'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\right.\\
&\qquad\qquad \left. - \xi^2 J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)J'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\widehat{\Psi}_1(\xi)\right)d\xi\\
&\quad + c_1c_0^2\int_0^1 \xi J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\left(\left(\frac{n}{2}+1\right)J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi) + j_{\frac{n}{2}-1,1}\xi J'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\right)d\xi.
\end{aligned}$$

Since

$$\begin{aligned}
&- j_{\frac{n}{2}-1,1}\int_0^1 \frac{\xi^4}{2}J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)J'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi\\
&= \frac{1}{6}\int_0^1 J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\big(\xi^4 Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\big)'\,d\xi,\\
&j_{\frac{n}{2}-1,1}\int_0^1 \xi^2 J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)J'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\widehat{\Psi}_1(\xi)\,d\xi\\
&= -\frac{1}{2}\int_0^1 \xi^3 J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\big(\xi Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\big)' - \int_0^1 \xi J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\widehat{\Psi}_1(\xi)\,d\xi,
\end{aligned}$$

$$\int_0^1 \xi J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi = \frac{1}{c_0^2},$$

$$\int_0^1 j_{\frac{n}{2}-1,1}\xi^2 J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)J'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi = -\int_0^1 \xi J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi = -\frac{1}{c_0^2},$$

we finally get

$$\begin{aligned}\int_0^1 \xi^{\frac{n}{2}+1} J_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\Psi_1'(\xi)\,d\xi = &-\frac{\pi c_0}{6}\int_0^1 \xi^3 J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi\\ &-\frac{\pi c_0}{6}\int_0^1 \xi^4 J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi.\end{aligned}$$

Formulae (4.14), (4.15), and (4.17) yield the desired asymptotics for $\lambda(r)$,

$$\begin{aligned}\lambda(r) &= \frac{j^2_{\frac{n}{2}-1,1}}{r^2} + \frac{n(n-1)f'''(0)}{6} + r^2\widetilde{\lambda}_2 + \mathrm{O}(r^4),\\ \widetilde{\lambda}_2 &= \frac{(n-1)(n+2)\big(3f^{(5)}(0) - 5(f'''(0))^2\big)c_0^2}{180}\int_0^1 \xi^3 J^2_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi\\ &\quad + \frac{\pi(n-1)^2(f'''(0))^2c_0^2}{54}\Bigg(\int_0^1 \xi^3 J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi\\ &\quad + j_{\frac{n}{2}-1,1}\int_0^1 \xi^4 J^3_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)Y'_{\frac{n}{2}-1}(j_{\frac{n}{2}-1,1}\xi)\,d\xi\Bigg).\end{aligned}$$

From the fact that the scalar curvature at a point $p$ is given by

$$\mathcal{S}(p) = -2(n-1)\frac{f''(t)}{f(t)} + (n-1)(n-2)\frac{1-\big[f'(t)\big]^2}{f^2(t)}$$

we obtain that at the North Pole $(t=0)$ this becomes $\mathcal{S}(p) = -n(n-1)f^{(3)}(0)$, while $\mathcal{S}'(p) = 0$ and

$$\mathcal{S}''(p) = \frac{n(n-1)}{3}\left[\Big(f^{(3)}(0)\Big)^2 - f^{(5)}(0)\right].$$

Solving this for $f^{(3)}$ and $f^{(5)}$ and substituting above yields the expressions for $\alpha_1$ and $\alpha_2$ in Theorem 4.1.

## 5. Asymptotic expansion for maximal radius

In this section we consider the case of a compact manifold with the function $f$ satisfying

$$f(R) = 0, \quad f'(R) = A < 0, \quad f''(R) = 0. \tag{5.1}$$

In contrast to the previous section, here we treat the case of the disk being close to the entire manifold, namely, $r \to R^-$. Note that since the disk is centred at the North pole, we have that the maximal radius $R$ equals the diameter of the manifold.

Our aim is to construct the asymptotic expansion for $\lambda(r)$ as $r \to R^-$. And the main result reads as follows.

**Theorem 5.1.** *The first eigenvalue of the geodesic disk centered at the North Pole $p$ and with radius $r$ satisfies as $r \to R^-$*

$$\begin{aligned} \lambda(r) = \frac{V(r)(2-n)}{B^{(1)}_{n,1}}(r-R)^{n-2}\Bigg(1 - \frac{B^{(1)}_{n,2}(2-n)}{B^{(1)}_{n,1}(4-n)}(r-R)^2 \\ + \left(\left(\frac{B^{(1)}_{n,2}(2-n)}{B^{(1)}_{n,1}(4-n)}\right)^2 - \frac{B^{(1)}_{n,3}(2-n)}{B^{(1)}_{n,1}(6-n)}\right)(r-R)^4\Bigg) \\ + \mathrm{O}\big((R-r)^{-n+3}\big), \end{aligned} \tag{5.2}$$

*for $n \geqslant 7$,*

$$\begin{aligned} \lambda(r) = -\frac{4V(R)}{B^{(1)}_{6,1}}(r-R)^4\Bigg[1 - \frac{2B^{(1)}_{6,2}}{B^{(1)}_{6,1}}(r-R)^2 + \frac{4B^{(1)}_{6,3}}{B^{(1)}_{6,1}}(r-R)^4\ln(R-r) \\ + \left(\frac{2B^{(1)}_{6,2}}{B^{(1)}_{6,1}}\right)^2(r-R)^4\Bigg] + \mathrm{O}\big((R-r)^9\big) \end{aligned} \tag{5.3}$$

*for* $n = 6$,

$$\lambda(r) = -\frac{3V(R)}{B_{5,1}^{(1)}}(r-R)^3 + \frac{9V(R)B_{5,2}^{(1)}}{\big(B_{5,1}^{(1)}\big)^2}(r-R)^5 + \left(B_1^{(2)} - \frac{9V(R)B_0^{(2)}}{\big(B_{5,1}^{(1)}\big)^2}\right)(r-R)^6 + \mathrm{O}\big((R-r)^7\big) \tag{5.4}$$

*for* $n = 5$,

$$\lambda(r) = -\frac{2V(R)}{B_{4,1}^{(1)}}(r-R)^2 - \frac{4V(R)B_{4,2}^{(1)}}{\big(B_{4,1}^{(1)}\big)^2}(r-R)^4\ln(R-r) + \left(B_3^{(2)} - \frac{4V(R)B_2^{(2)}}{\big(B_{4,1}^{(1)}\big)^2}\right)(r-R)^4 + \mathrm{O}\big((R-r)^5\ln^2(R-r)\big) \tag{5.5}$$

*for* $n = 4$,

$$\lambda(r) = -\frac{V(R)}{B_{3,1}^{(1)}}(r-R) + \left(B_6^{(2)} - \frac{V(R)B_4^{(2)}}{\big(B_{3,1}^{(1)}\big)^2}\right)(r-R)^2 + (B_5^{(2)} + B_7^{(2)} + B_{10}^{(2)})(r-R)^3 + \mathrm{O}\big((R-r)^4\big) \tag{5.6}$$

*for* $n = 3$,

$$\begin{aligned}\lambda(r) =& \frac{V(R)}{B_{2,1}^{(1)}}\ln^{-1}(R-r) + \left(\frac{B_{12}^{(2)}V^2(R)}{\big(B_{2,1}^{(1)}\big)^3} - \frac{B_{11}^{(2)}V(R)}{\big(B_{2,1}^{(1)}\big)^2}\right)\ln^{-2}(R-r) \\ &+ \left(\frac{B_{13}^{(2)}V^2(R)}{\big(B_{2,1}^{(1)}\big)^3} + \frac{\big(B_{11}^{(2)}\big)^2V(R)}{\big(B_{2,1}^{(1)}\big)^3} - 2\frac{\big(B_{12}^{(2)}\big)^2V^4(R)}{\big(B_{2,1}^{(1)}\big)^6}\right. \\ &\left.-\frac{B_{11}^{(2)}B_{12}^{(2)}V^2(R)}{\big(B_{2,1}^{(1)}\big)^4} + B_{14}^{(2)}\right)\ln^{-3}(R-r) + \mathrm{O}\big(\ln^{-4}(R-r)\big)\end{aligned} \tag{5.7}$$

*for $n=2$. Here the constants are given by the formulae*

$$B^{(1)}_{n,1} := \frac{V^2(R)}{\omega_{n-1}A^{n-1}}, \quad B^{(1)}_{n,2} := -\frac{A_2 V^2(R)}{\omega_{n-1}A^{n-1}}, \quad B^{(1)}_{n,3} := \frac{(A_2^2 - A_4)V^2(R)}{\omega_{n-1}A^{n-1}},$$

$$A_2 := \frac{(n-1)f'''(0)}{3!A^{n-1}}, \quad A_4 := (n-1)\left(\frac{f^{(5)}(0)}{5!} + \frac{(n-2)\big(f'''(0)\big)^2}{12}\right),$$

$$B^{(2)}_0 := \int\limits_0^R \left(\frac{V^2(t)}{V'(t)} - B^{(1)}_{5,1}(t-R)^{-4} - B^{(1)}_{5,2}(t-R)^{-2}\right) dt - \frac{B^{(1)}_{5,1}}{3R^3} - \frac{B^{(1)}_{5,2}}{R},$$

$$B^{(2)}_1 := -\frac{9V^3(R)}{\omega_4\big(B^{(1)}_{5,1}\big)^3 A^4} \int\limits_0^R \frac{V(s)-V(R)}{V(s)} V(s)\,ds,$$

$$B^{(2)}_2 := \int\limits_0^R \left(\frac{V^2(t)}{V'(t)} - B^{(1)}_{4,1}(t-R)^{-3} - B^{(1)}_{4,2}(t-R)^{-1}\right) dt + \frac{B^{(1)}_{4,1}}{2R^2} - B^{(1)}_{4,2}\ln R,$$

$$B^{(2)}_3 := \int\limits_0^R \frac{V(s)-V(R)}{V'(s)} V(s)\,ds,$$

$$B^{(2)}_4 := \int\limits_0^R \left(\frac{V^2(t)}{V'(t)} - B^{(1)}_{3,1}(t-R)^{-2}\right) dt - B^{(1)}_{3,1}R^{-1},$$

$$B^{(2)}_5 := \frac{\omega_2 A^2 \big(B^{(1)}_{-2}\big)^3 - 3V(R)B^{(1)}_{3,2}B^{(1)}_{3,1} - 3V(R)\big(B^{(2)}_4\big)^2}{3\big(B^{(1)}_{3,1}\big)^3},$$

$$B^{(2)}_6 := -\frac{B^{(2)}_8 V^2(R)}{\big(B^{(1)}_{3,1}\big)^3}, \quad B^{(2)}_8 := \frac{V(R)}{\omega_2 A^2} \int\limits_0^R \frac{V(s)-V(R)}{V'(s)} V(s)\,ds,$$

$$B^{(2)}_7 := -\frac{2V^2(R)B^{(2)}_4 B^{(2)}_8}{\big(B^{(1)}_{3,1}\big)^4} - \left(\frac{B^{(2)}_9}{B^{(1)}_{3,1}} + \frac{B^{(2)}_4 B^{(2)}_8}{\big(B^{(1)}_{3,1}\big)^2}\right) \frac{V^2(R)}{\big(B^{(1)}_{3,1}\big)^2},$$

$$B^{(2)}_9 := \int\limits_0^R \frac{V(t)}{V'(t)} \left(\int\limits_0^t \frac{V(s)-V(t)}{V'(s)} V(s)\,ds - B^{(2)}_8 (t-R)^{-2}\right) dt + \frac{B^{(2)}_8}{R},$$

$$B^{(2)}_{10} := \frac{V^3(R)}{\omega_2\big(B^{(1)}_{3,1}\big)^3 A^2} \int\limits_0^R ds \frac{V(s)-V(R)}{V'(s)} \int\limits_0^s \frac{V(z)-V(s)}{V'(z)} V(z)\,dz,$$

$$B_{11}^{(2)} := -B_{2,1}^{(1)} \ln R + \int\limits_0^R \left( \frac{V^2(t)}{V'(t)} - B_{2,1}^{(1)}(t-R)^{-1} \right) dt,$$

$$B_{12}^{(2)} := -\frac{V(R)}{\omega_1 A} \int\limits_0^R \frac{V(s)-V(R)}{V'(s)} V(s)\, ds,$$

$$B_{13}^{(2)} := -B_{12}^{(2)} \ln R + \int\limits_0^R \left( \frac{V(t)}{V'(t)} \int\limits_0^t \frac{V(s)-V(t)}{V'(s)}\, ds - B_{12}^{(2)}(t-R)^{-1} \right) dt,$$

$$B_{14}^{(2)} := \frac{V^4(R)}{\omega_2 A^2 \big(B_{2,1}^{(1)}\big)^4} \int\limits_0^R ds \frac{V(s)-V(R)}{V'(s)} \int\limits_0^s \frac{V(z)-V(s)}{V'(z)} V(z)\, dz.$$

**Remark 5.2.** The first term in the asymptotics in the case of domains with a small hole was investigated by many authors — see [F] and the references therein, and also [C] and [MNP], for instance; the latter of these includes the full expansion in the case of a two-dimensional manifold with a small hole. With our approach we are able to obtain the complete asymptotic expansions in any dimension, cf. identities (5.8) below. As we see, for dimensions $2, 4$ and 6 logarithmic terms appear in these expansions. While for $n = 2$ this is quite natural as these terms are produced directly by the singularity of the Green function for such two-dimensional elliptic operators, for other dimensions their appearance is not so evident and, to our knowledge, was not known before. Moreover, provided $f$ is smooth enough, say, $f$ is infinitely differentiable, it can be shown that the complete asymptotic expansion for $\lambda(r)$ involves logarithmic terms in all even dimensions, see Remark 5.6.

First we construct the asymptotics formally and then we rigorously estimate the error terms. Since for $r = R$ the lowest eigenvalue of Laplace-Beltrami operator on the manifold is 0 and the associated eigenfunction is constant, we could assume that the leading term in the asymptotic expansion for $\psi(t,r)$ should be constant. On the other hand, the constant function does not satisfy the boundary condition on $t = r$ in (2.2). The usual way to achieve the desired boundary condition is to employ the boundary layer method [VL] or the matching of asymptotic expansion [I]. Here we do not go in this way since it is possible to include the inner expansion into the external one and to construct the full asymptotics as a series in terms of the variable $t$ without introducing the rescaled variable. In order to do it, we have to take the leading term in the expansion for $\psi$ in a special form.

Namely, we assume the following ansätzes,

$$\lambda(r) = \sum_{j=1}^{\infty} \mu_n^j(r)\lambda_j(r), \quad \psi(t,r) = \sum_{j=0}^{\infty} \mu_n^j(r)\psi_j(t,r), \tag{5.8}$$
$$\mu_n(r) := \begin{cases} \ln^{-1}(R-r), & n=2, \\ (2-n)(r-R)^{n-2}, & n \geqslant 3. \end{cases}$$

where $\lambda_j$ and $\psi_j$ are to be determined. We define the function $\psi_0$ as the solution to the boundary value problem

$$\begin{cases} -(f^{n-1}\psi_0')' = \mu\lambda_0 f^{n-1} \quad \text{in} \quad (0,r), \\ \psi_0'(0,r) = 0, \quad \psi_0(0,r) = 1, \end{cases}$$

given by the formula

$$\psi_0(t,r) = \lambda_0(r)\mu_n(r)\int_0^t \frac{V(s)}{V'(s)}\,ds, \quad \lambda_0(r) := \left(\mu_n(r)\int_0^r \frac{V(s)}{V'(s)}\,ds\right)^{-1}. \tag{5.9}$$

It follows from (5.1) that

$$f(t) = A(t-R) + \mathrm{o}(t-R), \qquad t \to R^-. \tag{5.10}$$

$$\mu_n(r)\int_0^r \frac{V(s)}{V'(s)}\,ds = \frac{V(R)}{\omega_{n-1}A^{n-1}} + \mathrm{o}(1), \quad r \to R^-,$$
$$\lambda_0(r) = \frac{\omega_{n-1}A^{n-1}}{V(R)}(1+\mathrm{o}(1)), \qquad r \to R^-, \tag{5.11}$$

It is also obvious that $\psi_0 \in C^2[0,R]$ and

$$\|\psi_0\|_{C[0,R]} \leqslant C,$$

where $C$ is a constant independent of $r$.

We plug in series (5.8) into the eigenvalue problem (2.2) and equate the coefficients of like powers of $\mu$, leading us to the boundary value problems for $\psi_j$,

$$
\begin{aligned}
&-(f^{n-1}\psi_1')' = \lambda_1 f^{n-1}\psi_0 - \lambda_0 f^{n-1} && \text{in} \quad (0,r), \\
&-(f^{n-1}\psi_j')' = \lambda_j f^{n-1}\psi_0 + f^{n-1}\sum_{k=1}^{j-1}\lambda_k\psi_{j-k} && \text{in} \quad (0,r), \quad j \geqslant 2, \\
&\psi_j'(0,r) = \psi_j(r,r) = 0, && j \geqslant 1.
\end{aligned} \tag{5.12}
$$

**Lemma 5.3.** *Let $g \in C[0,R]$ and*

$$\int_0^r f^{n-1}(t)g(t)\,dt = 0. \tag{5.13}$$

*Then the boundary value problem*

$$-(f^{n-1}u')' = f^{n-1}g \quad \text{in} \quad (0,r) \qquad u'(0) = u(r) = 0, \tag{5.14}$$

*has the unique solution given by the formula*

$$u(t) = \mathcal{L}[g](t), \quad \mathcal{L}[g](t) := \int_t^r f^{-n+1}(s)\left(\int_0^s f^{n-1}(z)g(z)\,dz\right)ds. \tag{5.15}$$

*It belongs to $C^2[0,r]$ and satisfies the uniform in $r$ estimate*

$$\|u\|_{C^1[0,R]} \leqslant C\|h\|_{C[0,R]}. \tag{5.16}$$

*Proof.* It is clear that the function $u$ defined by (5.15) solves (5.14) and belongs to $C^2[0,R]$. Let us prove estimate (5.16). Due to (5.13) we have

$$u'(t) = -f^{-n+1}(t)\int_0^t f^{n-1}(s)g(s)\,ds = f^{-n+1}(t)\int_t^r f^{n-1}(s)g(s)\,ds.$$

By the boundedness of $g$ and the positiveness of $f$ these formulae imply

$$
\begin{aligned}
&|u'(t)| \leqslant \frac{V(t)}{V'(t)}\|g\|_{C[0,R]}, \\
&|u'(t)| \leqslant \frac{V(r)-V(t)}{V'(t)}\|g\|_{C[0,R]} \leqslant \frac{V(R)-V(t)}{V'(t)}\|g\|_{C[0,R]}.
\end{aligned}
$$

Hence,

$$|u'(t)| \leqslant \frac{\min\{V(t), V(R)-V(t)\}}{V'(t)} \|g\|_{C[0,R]} \leqslant C\|g\|_{C[0,R]},$$

where the constant $C$ is independent of $r$ and $g$. Employing the last estimate and the identity

$$u(t) = -\int\limits_t^r u'(s)\,ds,$$

we arrive at (5.16). □

We employ the last lemma to solve the problems (5.12). In order for the series (5.8) to be asymptotic, the coefficients $\lambda_j$ and $\psi_j$ should be bounded uniformly in $r$. Hence, to have the function $\psi_1$ bounded, the right hand side of the equation for $\psi_1$ should satisfy (5.13). This implies the formula for $\lambda_1$,

$$\lambda_1(r) = \frac{\lambda_0(r)V(r)}{\int\limits_0^r V'(s)\psi_0(s,r)\,ds}.$$

We shall now compute the denominator in the last formula. To this end, integrate by parts taking into consideration the definition of $\psi_0$,

$$\int\limits_0^r V'(t)\psi_0(t,r)\,dt = \lambda_0\mu G(r), \quad G(t) := \int\limits_0^t \frac{V^2(s)}{V'(s)}\,ds. \tag{5.17}$$

Together with the definition of $\lambda_0$ in (5.9), this allows us to rewrite the formula for $\lambda_1$,

$$\lambda_1(r) = \frac{V(r)}{\mu_n(r)G(r)}. \tag{5.18}$$

Again by (5.10) we see that

$$\lambda_1(r) = \frac{\omega_{n-1}A^{n-1}}{V(R)} + \mathrm{o}(\mu).$$

Since condition (5.13) is satisfied, the solution to the equation for $\psi_1$ in (5.12) is given by the identity

$$\psi_1 = \mathcal{L}[\lambda_1 f^{n-1}\psi_0 - \lambda_0 f^{n-1}] \in C^2[0,R]$$

and this function is bounded uniformly in $r$,

$$\|\psi_1\|_{C^1[0,R]} \leqslant C.$$

In the same way we solve problems (5.12) for $j \geqslant 2$. We first write condition (5.13) that determines $\lambda_j$,

$$\lambda_j = -\frac{1}{\int\limits_0^r f^{n-1}\psi_0\,dt} \sum_{k=1}^{j-1} \lambda_k \int\limits_0^r f^{n-1}\psi_{j-k}\,dt = -\frac{1}{\lambda_0 \mu G} \sum_{k=1}^{j-1} \lambda_k \int\limits_0^r V'\psi_{j-k}\,dt, \tag{5.19}$$

where we have used (5.17). Provided the functions $\psi_k$ and $\lambda_k$, $k \geqslant j-1$, are bounded uniformly in $r$ (the former in the $C^1[0,R]$-norm), by (5.10) we obtain that $\lambda_j$ is also bounded uniformly in $r$. Then, by Lemma 5.3, the function $\psi_j$ reads as follows,

$$\psi_j = \mathcal{L}\left[\lambda_j\psi_0 + \sum_{k=1}^{j-1} \lambda_k\psi_{j-k}\right]. \tag{5.20}$$

It belongs to $C^2[0,R]$ and is bounded uniformly in $r$ in the $C^1[0,R]$-norm.

In conclusion to the formal constructing we prove that series (5.8) are formal asymptotic solutions to (2.2). For $N \geqslant 0$ we let

$$\lambda^{(N)}(r) := \sum_{j=1}^{N} \mu_n^j(r)\lambda_j(r), \quad \psi^{(N)}(t,r) := \sum_{j=0}^{N} \mu_n^j(r)\psi_j(t,r). \tag{5.21}$$

**Lemma 5.4.** *Given any $N \geqslant 0$, for the functions $\lambda^{(N)}$ and $\psi^{(N)}$ the convergences*

$$\lambda^{(N)} \to 0, \quad \|\psi^{(N)} - \psi_0\|_{C^2[0,r]} \to 0, \quad \varepsilon \to 0^+, \tag{5.22}$$

*and the equation*

$$(\mathcal{H}_r - \lambda^{(N)})\psi^{(N)} = h^{(N)} \tag{5.23}$$

*hold true. The function $h_N \in C[0,r]$ satisfies the estimate*

$$\|h_N\|_{C[0,r]} \leqslant C_N \mu_n^{N+1}, \tag{5.24}$$

*where $C_N$ is a constant independent of $\mu_n$ and $\varepsilon$.*

*Proof.* The convergences (5.22) follow directly from the uniform boundedness of $\lambda_j$ and $\|\psi_j\|_{C^1[0,R]}$ in $r$.

Employing boundary value problems (5.12) for $\psi_j$, by direct calculations we check that

$$h^{(0)} = \lambda_0 \mu_n \psi_0, \quad h^{(N)} = \sum_{\substack{1\leqslant k,j\leqslant N\\ k+j\geqslant N+1}} \mu_n^{k+j} \lambda_k \psi_j, \quad N \geqslant 1.$$

satisfy the boundary value problem (5.23). Estimate (5.24) follows directly from the last identity and the aforementioned boundedness of $\lambda_j$ and $\psi_j$. □

We proceed to the justification of the asymptotics. We first prove two auxiliary lemmas characterizing $\lambda(r)$.

**Lemma 5.5.** *The eigenvalue $\lambda(r)$ is the only one of the problem (2.2) which converges to zero as $\varepsilon \to 0^+$. It is simple and satisfies the estimate*

$$\lambda(r) \leqslant \frac{\mu_n(r)\lambda_1(r)}{1 + \mu_n \lambda_1(\mu_n) G^{-1}(r) \int\limits_0^r \frac{G^2(t)}{G'(t)}\, dt}, \tag{5.25}$$

$$0 \leqslant G^{-1}(r) \int\limits_0^r \frac{G^2(t)}{G'(t)}\, dt \leqslant \begin{cases} C\mu_n(r), & n = 2, 3, \\ C\mu_n(r) \ln \mu_n(r), & n = 4, \\ C\mu_n^{\frac{2}{n-2}}(r), & n \geqslant 5, \end{cases} \tag{5.26}$$

*where $C$ is a constant independent of $\mu_n$.*

*Proof.* We first prove the upper bound for $\lambda(r)$. Using $\psi_0$ as a test function in the Rayleigh quotient (2.3), we obtain

$$\lambda(r) \leqslant \frac{\int\limits_0^r f^{n-1} (\psi_0')^2\, dt}{\int\limits_0^r f^{n-1} \psi_0^2\, dt} = \frac{G(r)}{\int\limits_0^r V'(t) \left( \int\limits_t^r \frac{V(s)}{V'(s)}\, ds \right)^2 dt}. \tag{5.27}$$

The denominator may be simplified by integration by parts as follows

$$\int_0^r V'(t)\left(\int_t^r \frac{V(s)}{V'(s)}\,ds\right)^2 dt = 2\int_0^r G'(t)\int_t^r \frac{V(s)}{V'(s)}\,ds\,dt = 2\int_0^r \frac{G(t)V(t)}{V'(t)}\,ds\,dt$$
$$= \frac{G^2(r)}{V(r)} + \int_0^r \frac{V'(t)G^2(t)}{V^2(t)}\,dt.$$

Substituting this identity into (5.27), we arrive at the first estimate in (5.25). Let us prove the second one.

By (5.10) we have

$$G(r) = \frac{V^2(r)}{\mu_n}(1+\mathrm{o}(1)).$$

It follows from the definition of $G$ and (5.10) that

$$G^{-1}(r)\int_0^r \frac{G^2(t)}{G'(t)}\,dt = G^{-1}(r)\left(\int_0^{R/2} + \int_{R/2}^r\right)\frac{V'(t)G^2(t)}{V(t)}\,dt$$
$$= G^{-1}(r)\int_{R/2}^r \frac{V'(t)G^2(t)}{V(t)}\,dt + C\mu_n(r)$$
$$\leqslant C\mu_n(r)\left(\int_{R/2}^r f^{n-1}(t)G^2(t)\,dt + 1\right),$$

$$\mu_n(r)\int_{R/2}^r f^{n-1}(t)G^2(t)\,dt \leqslant C\mu_n(r)\int_{R/2}^r f^{n-1}(t)\left(\int_{R/2}^t \frac{ds}{f^{n-1}(s)}\right)^2 dt$$
$$\leqslant \begin{cases} C\mu_n(r), & n=2,3,\\ C\mu_n(r)\ln\mu_n(r), & n=4,\\ C\mu_n^{\frac{2}{n-2}}(r), & n\geqslant 5,\end{cases}$$

where $C$ denotes various inessential constants independent of $\mu_n$. The second estimate in (5.25) is proven.

Estimate (5.25) yields that $\lambda(r)\to 0^+$ as $r\to R^-$. It remains to prove that there are no other eigenvalues of (2.2) converging to zero.

Let $\lambda_*(r)$ be an eigenvalue of (2.2) converging to zero and $\psi_*(t,r)$ be an associated eigenfunction. We normalize $\psi_*$ by the condition

$$\max_{[0,r]} |\psi_*(\cdot,r)| = 1. \tag{5.28}$$

Then we represent $\psi_*$ as

$$\psi_*(t,r) = a_*(R) + \widetilde{\psi}_*(t,r), \quad a_*(r) := r^{-1}\int_0^r \psi_*\,dt, \quad \int_0^r f^{n-1}\widetilde{\psi}_*\,dt = 0.$$

We observe that $a_*$ and $\widetilde{\psi}_*$ are bounded uniformly in $t$ and $r$. Hence, we can apply Lemma 5.3 to $g = \lambda_*\widetilde{\psi}_*$ and represent $\psi_*$ as

$$\psi_* = \frac{\lambda_* a_*}{\lambda_0\mu_n}\psi_0 + \widehat{\psi}_*, \quad \|\widehat{\psi}_*\|_{Y_0} \leqslant C\lambda_*, \tag{5.29}$$

where $C$ is a constant independent of $r$. Now we employ normalization (5.28),

$$1 \geqslant |\psi_*(0,r)| = \left|\frac{\lambda_* a_*}{\lambda_0\mu_n}\psi_0(0,r) + \widehat{\psi}_*(0,r)\right| \geqslant \frac{\lambda_* a_*}{\lambda_0\mu_n} - C\lambda_*$$

that implies

$$\lambda_* a_* \leqslant C_1\lambda_0\mu_n,$$

where $C_1$ is a constant independent of $\varepsilon$. At the same time,

$$1 = |\psi_*(t_0,r)| \leqslant \frac{\lambda_* a_*}{\lambda_0\mu_n}\psi_0(0,r) + C\lambda_* = \frac{\lambda_* a_*}{\lambda_0\mu_n} + C\lambda_*$$

and therefore

$$\lambda_* a_* \geqslant C_2\lambda_0\mu_n,$$

where $C_2$ is a constant independent of $r$. Hence, the first term in the right hand side of (5.29) is of order O(1) while $\widetilde{\psi}_*$ is of order O($r$). If we assume now that there are two eigenvalues of (2.2) converging to zero, then the associated eigenfunctions satisfy (5.29). At the same time, this contradicts the fact that these eigenfunctions should be linear independent. □

By the proven lemma the closest to $\lambda^{(N)}$ eigenvalue of $\mathcal{H}_r$ is $\lambda(r)$. Hence,

$$\|(\mathcal{H}_r-\lambda^{(N)})^{-1}\|=\frac{1}{|\lambda^{(N)}(r)-\lambda(r)|},$$

and by Lemma 5.4 it follows

$$\begin{aligned}\|\psi^{(N)}\|_{X_r^0}&\leqslant\|(\mathcal{H}_r-\lambda^{(N)})^{-1}\|\,\|h^{(N)}\|_{X_r^0}\leqslant\frac{\|h^{(N)}\|_{X_r^0}}{|\lambda^{(N)}(r)-\lambda(r)|}\\&\leqslant\frac{C_N R}{|\lambda^{(N)}(r)-\lambda(r)|},\\\|\psi^{(N)}\|_{X_r^0}&=\|\psi_0\|_{X_r^0}+\mathrm{o}(1).\end{aligned}\tag{5.30}$$

We calculate the norm $\|\psi_0\|_{X_r^0}$ by integration by parts and employing (5.11),

$$\begin{aligned}\|\psi_0\|^2_{X_r^0}&=\int\limits_0^r f^{n-1}(t)\psi_0^2(t,r)\,dt=\frac{2\lambda_0^2\mu_n^2}{\omega_{n-1}}\int\limits_0^r\frac{V^2(t)}{V'(t)}\left(\int\limits_t^r\frac{V(s)}{V'(s)}\,ds\right)\,dt\\&=\frac{2\lambda_0^2\mu_n^2V^3(r)}{\omega_{n-1}}\int\limits_{R/2}^r\frac{1}{V'(t)}\left(\int\limits_{R/2}^t\frac{ds}{V'(s)}\right)\,dt\cdot(1+\mathrm{o}(1))\\&=\frac{2V(R)}{\omega_{n-1}}(1+\mathrm{o}(1)).\end{aligned}$$

We substitute the obtained identity into (5.30),

$$\frac{2V(R)}{\omega_{n-1}}(1+\mathrm{o}(1))\leqslant\frac{C_N R\mu_n^{N+1}}{|\lambda(r)-\lambda^{(N)}(r)|},\quad|\lambda(r)-\lambda^{(N)}(r)|=\mathrm{O}(\mu_n^{N+1}),\tag{5.31}$$

that justifies asymptotics (5.8) for $\lambda(r)$.

Let us justify asymptotics (5.8) for $\psi$. By [K, Ch. V, Sec. 3.5, Eq. (3.21)] we have the representation

$$\psi^{(N)}=\frac{(h^{(N)},\psi)_{X_r^0}}{\lambda(r)-\lambda^{(N)}(r)}\psi+\mathcal{R}(r)h^{(N)},\tag{5.32}$$

where $\mathcal{R}(r)$ is an operator in $X_r^0$ bounded uniformly in $r$ and mapping $X_r^0$ into the orthogonal complement of $\psi$ in $X_r^0$, and $\psi$ is supposed to be normalized in $X_r^0$. Therefore,

$$(\psi^{(N)},\psi)_{X_r^0}=\frac{(h^{(N)},\psi)_{X_r^0}}{\lambda(r)-\lambda^{(N)}(r)},\quad \|\mathcal{R}(r)h^{(N)}\|_{X_r^0}=\mathrm{O}(\mu_n^{N+1}). \tag{5.33}$$

It follows from definition (5.21) of $\psi^{(N)}$ and the boundedness of $\|\psi_j\|_{C[0,R]}$ that for each $N\geqslant 0$

$$\frac{(h^{(N)},\psi)_{X_r^0}}{\lambda(r)-\lambda^{(N)}(r)}-\frac{(h^{(N+1)},\psi)_{X_r^0}}{\lambda(r)-\lambda^{(N+1)}(r)}=\mathrm{O}(\mu_n^{N+1}).$$

Hence, there exists a function $b(\mu_n)$ such that for each $N\geqslant 0$

$$\frac{(h^{(N)},\psi)_{X_r^0}}{\lambda(r)-\lambda^{(N)}(r)}=b(\mu_n)+\mathrm{O}(\mu_n^{N+1}).$$

We substitute this identity and the second relation from (5.33) into (5.32),

$$b(\mu_n)\psi=\psi^{(N)}+\mathrm{O}(\mu_n^{N+1})$$

in the norm of $X_r^0$. This identity is also valid in the norm of $X_r$, since by (5.31) the equations for $\psi$ and $\psi^{(N)}$

$$\begin{aligned}\mathcal{H}_r(\psi^{(N)}-\psi)&=\lambda^{(N)}(\psi^{(N)}-\psi)+(\lambda^{(N)}-\lambda)\psi+h^{(N)},\\ \|(\psi^{(N)}-\psi)'\|_{X_r^0}&=\lambda^{(N)}\|\psi^{(N)}-\psi\|^2_{X_r^0}+(\lambda^{(N)}-\lambda)(\psi,\psi^{(N)}-\psi)_{X_r^0}\\ &\quad+(h^{(N)},\psi^{(N)}-\psi)_{X_r^0}=\mathrm{O}(\mu_n^{2N+2}).\end{aligned}$$

The justification is complete.

As in the previous section, let us calculate the leading terms of asymptotics (5.8) in a more explicit form. We assume that $f\in C^6[0,R]$ and $f^{(4)}(0)=f''(0)=0$. Then

$$\begin{aligned}f(t)&=A(t-R)+\frac{f'''(0)}{3!}(t-R)^3\\ &\quad+\frac{f^{(5)}(0)}{5!}(t-R)^5+\mathrm{O}\big((t-R)^6\big),\quad t\to R^-,\\ f^{n-1}(t)&=A^{n-1}(t-R)^{n-1}\big(1+A_2(t-R)^2\\ &\quad+A_4(t-R)^4+\mathrm{O}((t-R)^5)\big),\quad t\to R^-,\end{aligned}$$

We employ the identity

$$V(r) = V(R) - w_{n-1} \int_r^R f^{n-1}(t)\, dt,$$

to obtain

$$V(t) = V(R) - \frac{\omega_{n-1} A^{n-1}}{n}(t-R)^n - \frac{\omega_{n-1} A^{n-1} A_2}{n+2}(t-R)^{n+2} + \mathrm{O}\big((t-R)^{n+4}\big), \tag{5.34}$$

and

$$\frac{V^2(t)}{V'(t)} = B^{(1)}_{n,1}(t-R)^{-n+1} + B^{(1)}_{n,2}(t-R)^{-n+3} + B^{(1)}_{n,3}(t-R)^{-n+5} + B^{(1)}_{n,4}(t-R) + B^{(1)}_{n,5}(t-R)^3 + B^{(1)}_{n,6}(t-R)^{(n+1)} + \mathrm{O}\big((R-t)^{-n+6} + (R-t)^5 + (R-t)^{n+3}\big), \quad t \to R^-, \tag{5.35}$$

where

$$B^{(1)}_{n,4} := -\frac{2V(R)}{A^{n-1}n}, \quad B^{(1)}_{n,5} := \frac{2A_2 V(R)}{A^{n-1}}\left(\frac{1}{n} - \frac{A^{n-1}}{n+2}\right), \quad B^{(1)}_{n,6} := \frac{\omega_{n-1}}{A^{n-1}n^2}.$$

We recall that other constants $B^{(1)}_{n,j}$ and $B^{(2)}_j$ were defined in the formulation of Theorem 5.1.

**Remark 5.6.** Provided $f$ is smooth enough and all its even derivatives vanish at $R$, we can write the next terms in expansion (5.35). They will be of order $\mathrm{O}((t-R)^{-n+2k+1})$, $k \geqslant 0$, and for even dimensions we obtain a term of order $\mathrm{O}((t-R)^{-1})$. After integration, this will produce the logarithmic term in the expansion for $G(r)$ as $r \to R^-$, giving rise to such terms appearing in the expansion for $\lambda(r)$.

Suppose $n \geqslant 6$. Then by (5.8)

$$\lambda(r) = \mu_n(r)\lambda_1(\rho) + \mathrm{O}\big((R-r)^{2n-4}\big), \quad r \to R^-. \tag{5.36}$$

Let us identify the asymptotic behavior of $\lambda_1(r)$ as $r \to R^-$. We first employ (5.17) and integrate (5.35) to do it for $G(r)$,

$$G(r) = B^{(1)}_{n,1}\mu_n^{-1}(r) + B^{(1)}_{n,2}\mu_{n-2}^{-1}(r) + B^{(1)}_{n,3}\mu_{n-4}^{-1}(r) + \mathrm{O}\big((R-r)^{-n+7}+1\big), \quad r \to R^-.$$

Thus, by (5.18), (5.34) we get

$$\mu_n(r)\lambda_1(r) = \frac{V(R)}{B^{(1)}_{n,1}}\mu_n(r)\left(1 - \frac{B^{(1)}_{n,2}}{B^{(1)}_{n,1}}\frac{\mu_n(r)}{\mu_{n-2}(r)} - \frac{B^{(1)}_{n,3}}{B^{(1)}_{n,1}}\frac{\mu_n(r)}{\mu_{n-4}(r)} \left(\frac{B^{(1)}_{n,2}}{B^{(1)}_{n,1}}\frac{\mu_n(r)}{\mu_{n-2}(r)}\right)^2\right) + \mathrm{O}\big((R-r)^5 + (R-r)^{n-2}\big), \quad r \to R^-.$$

Together with (5.36), this implies (5.2) and (5.3).

In order to calculate similar three-terms asymptotics for $\lambda(r)$ for low dimensions $n = 2, 3, 4, 5$ we cannot neglect higher terms of asymptotics (5.8) as in (5.36). The reason is that higher terms also contribute to the desired asymptotics. In what follows we consider separately each dimension.

Consider first the case $n = 5$. We take first two terms in (5.8),

$$\lambda(r) = \mu_5(r)\lambda_1(r) + \mu_5^2(r)\lambda_2(r) + \mathrm{O}\big((R-r)^9\big), \quad r \to R^-. \tag{5.37}$$

Integrating (5.35) and proceeding as above, as $r \to R^-$ we get

$$G(r) = -\frac{B^{(1)}_{5,1}}{3}(r-R)^{-3} - B^{(1)}_{5,2}(r-R)^{-1} + B^{(2)}_0 + \mathrm{O}\big((R-r)^2\big), \tag{5.38}$$

and hence

$$\mu_5(r)\lambda_1(r) = -\frac{3V(R)}{B^{(1)}_{5,1}}(r-R)^3 + \frac{9V(R)B^{(1)}_{5,2}}{\big(B^{(1)}_{5,1}\big)^2}(r-R)^5 - \frac{9V(R)B^{(2)}_0}{\big(B^{(1)}_{5,1}\big)^2}(r-R)^6 + \mathrm{O}\big((R-r)^7\big), \quad r \to R^-, \tag{5.39}$$

To find out similar formula for $\mu_5^2\lambda_2$ in (5.37), we first convert expression (5.19) for $\lambda_2$ to a more convenient form. Employing (5.19), (5.20), (5.9), and (5.15) and integrating by parts, we get

$$
\begin{aligned}
\mu_n^2(r)\lambda_2(r) &= -\frac{\mu_n(r)\lambda_1(r)}{\lambda_0(r)G(r)}\int\limits_0^r V'(t)\psi_1(t,r)\,dt \\
&= \frac{\mu_n(r)\lambda_1(r)}{\lambda_0(r)G(r)}\int\limits_0^r V(t)\psi_1'(t,r)\,dt \\
&= \frac{\mu_n(r)\lambda_1^2(r)}{\lambda_0(r)G(r)}\int\limits_0^r dt\frac{V(t)}{V'(t)}\int\limits_0^t V'(s)\psi_0(s,r)\,ds \\
&= \frac{\mu_n^2(r)\lambda_1^2(r)}{G(r)}\int\limits_0^r dt\frac{V(t)}{V'(t)}\int\limits_0^t \frac{V(s)-V(t)}{V'(s)}V(s)\,ds.
\end{aligned} \tag{5.40}
$$

We observe that this formula is valid for all $n \geqslant 2$. Together with (5.34), (5.35), (5.39), (5.38) it yields

$$
\mu_5^2\lambda_2(r) = B_1^{(2)}(r-R)^6 + \mathrm{O}\big((r-R)^7\big), \quad r \to R^-.
$$

By (5.37), (5.39) it implies (5.4).

The case $n = 4$ can be treated in the same way as the case $n = 5$ and below we provide only the main formulas. The analogue of (5.38) reads as

$$
G(r) = -\frac{B_{-3}^{(1)}}{2}(r-R)^{-2} + B_{4,2}^{(1)}\ln(R-r) + B_2^{(2)} + \mathrm{O}\big((R-r)^2\big), \quad r \to R^-,
$$

Formulae for $\mu_4\lambda_1$, $\mu_4^2\lambda_2$ follow from above one, (5.40),

$$
\begin{aligned}
\mu_4(r)\lambda_1(r) = &-\frac{2V(R)}{B_{4,1}^{(1)}}(r-R)^2 - \frac{4V(R)}{\big(B_{4,1}^{(1)}\big)^2}\big(B_{4,2}^{(1)}\ln(R-r) + B_2^{(2)}\big) \\
&+ \mathrm{O}\big((R-r)^6\ln^2(R-r)\big), \quad r \to R^-,
\end{aligned}
$$

and

$$
\mu_4^2(r)\lambda_2(r) = B_3^{(2)}(r-R)^4 + \mathrm{O}\big((R-r)^6|\ln(R-r)|\big), \quad r \to R^-.
$$

Hence, by (5.8), we obtain (5.5).

We proceed to the case $n = 3$. In contrast to all previous cases, here we have to deal with first three terms in (5.8), namely,

$$\lambda(r) = \mu_3(r)\lambda_1(r)+\mu_3^2(r)\lambda_2(r)+\mu_3^3(r)\lambda_3(r)+\mathrm{O}\big((R-r)^4\big), \quad r\to R^-. \tag{5.41}$$

First two terms can be treated as above,

$$G(r) = -B_{3,1}^{(1)}(r-R)^{-1} + B_4^{(2)} + B_{3,2}^{(1)}(r-R) + \mathrm{O}\big((R-r)^2\big), \quad r\to R^-,$$

and

$$\begin{aligned}\mu_3(r)\lambda_1(r) = {} & -\frac{V(R)}{B_{3,1}^{(1)}}(r-R) - \frac{V(R)B_4^{(2)}}{\big(B_{3,1}^{(1)}\big)^2}(r-R)^2 \\ & + B_5^{(2)}(r-R)^3 + \mathrm{O}\big((R-r)^3\big), \quad r\to R^-,\end{aligned} \tag{5.42}$$

$$\mu_3^2\lambda_2(r) = B_6^{(2)}(r-R)^2 + B_7^{(2)}(r-R)^3 + \mathrm{O}\big((R-r)^4\big), \quad r\to R^-, \tag{5.43}$$

To obtain similar formula for the third term in the right hand side of (5.41), we again first convert formula $\lambda_3$ by integration by parts, as it was done in (5.40). By (5.19) we have

$$\mu_n^3(r)\lambda_3(r) = -\frac{\mu_n^2(r)}{\lambda_0(r)G(r)}\left(\lambda_1(r)\int_0^r V'(t)\psi_2(t,r)\,dt \right. \\ \left. +\lambda_2(r)\int_0^r V'(t)\psi_1(t,r)\,dt\right).$$

Integrating by parts and employing (5.20), (5.15), we obtain

$$\begin{aligned}&\int_0^r V'(t)\psi_2'(t,r)\,dt = -\int_0^r V(t)\psi_2'(t,r)\,dt \\ &= \int_0^r dt\frac{V(t)}{V'(t)}\int_0^t V'(s)\big(\lambda_2(r)\psi_0(s,r)+\lambda_1(r)\psi_1(s,r)\big)\,ds \\ &= -\int_0^r dt\frac{V(t)}{V'(t)}\int_0^t \big(V(s)-V(t)\big)\big(\lambda_2(r)\psi_0'(s,r)+\lambda_1(r)\psi_1'(s,r)\big)\,ds.\end{aligned}$$

This identity and (5.40) yield

$$\mu_n^3(r)\lambda_3(r) = -\frac{\mu_n^3\lambda_1(r)^3}{G(r)}\int\limits_0^r dt\frac{V(t)}{V'(t)} \times \int\limits_0^t ds\frac{V(s)-V(t)}{V'(s)}\int\limits_0^s \frac{V(z)-V(s)}{V'(z)}V(z)\,dz. \tag{5.44}$$

Hence,

$$\mu_3^3(r)\lambda_3(r) = B_{10}^{(2)}(r-R)^3 + \mathrm{O}\big((R-r)^4\big), \quad r\to R^-,$$

and by (5.41), (5.42), (5.43) it yields (5.6).

It remains to consider the case $n=2$. As in (5.41), we take first three terms in (5.8),

$$\lambda(r) = \mu_2(r)\lambda_1(r) + \mu_2^2(r)\lambda_2(r) + \mu_2^3(r)\lambda_3(r) + \mathrm{O}\big(\ln^{-4}(R-r)\big), \quad r\to R^-. \tag{5.45}$$

It follows from (5.35), (5.17) that

$$G(r) = B_{2,1}^{(1)}\ln(R-r) + B_{11}^{(2)} + \mathrm{O}\big((R-r)^2\big), \quad r\to R^-, \tag{5.46}$$

Hence, by (5.18), (5.34),

$$\mu_2(r)\lambda_1(r) = \frac{V(R)}{B_{2,1}^{(1)}}\ln^{-1}(R-r) - \frac{B_{11}^{(2)}V(R)}{\big(B_{2,1}^{(1)}\big)^2}\ln^{-2}(R-r) + \frac{\big(B_{11}^{(2)}\big)^2V(R)}{\big(B_{2,1}^{(1)}\big)^3}\ln^{-3}(R-r) + \mathrm{O}\big((R-r)^2\ln^{-1}(R-r)\big), \quad r\to R^-. \tag{5.47}$$

Employing (5.40), in the same way we get

$$\mu_2^2(r)\lambda_2(r) = \frac{B_{12}^{(2)}V^2(R)}{\big(B_{2,1}^{(1)}\big)^3}\ln^{-2}(R-r) + \left(\frac{B_{13}^{(2)}V^2(R)}{\big(B_{2,1}^{(1)}\big)^3} - 2\frac{\big(B_{12}^{(2)}\big)^2V^4(R)}{\big(B_{2,1}^{(1)}\big)^6} - \frac{B_{11}^{(2)}B_{12}^{(2)}V^2(R)}{\big(B_{2,1}^{(1)}\big)^4}\right)\ln^{-3}(R-r) + \mathrm{O}\big(\ln^{-4}(R-r)\big), \quad r\to R^-, \tag{5.48}$$

And (5.44), (5.46), (5.47) yield

$$\mu_2^3(r)\lambda_3(r) = B_{14}^{(2)} \ln^{-3}(R-r) + \mathrm{O}\big(\ln^{-4}(R-r)\big), \quad r \to R^-,$$

The last identity and (5.45), (5.47), (5.48) imply (5.7).

## Acknowledgments

D.B. was partially supported by RFBR (grant no. 13-01-00081-a). P.F. was partially supported by the Fundação para a Ciência e Tecnologia, Portugal, through project PTDC/MAT-CAL/4334/2014. This work was initiated during the visit of D.B. to the Group of Mathematical Physics, University of Lisbon and D.B. thanks for the hospitality extended to him. Part of this work was finished while P.F. was visiting the Laboratoire Jacques-Louis Lions at the University Pierre et Marie Curie, Paris, and he would like to thank the people there for their hospitality.

## References

[Ba] F. E. Baginski, *Upper and lower bounds for eigenvalues of the Laplacian on a spherical cap*, Quart. Appl. Math. **48** (1990), 569–573 and Errata: *Upper and lower bounds for eigenvalues of the Laplacian on a spherical cap*, Quart. Appl. Math. **48** (1990), 569–573. Quart. Appl. Math. **49** (1991), no. 2, 399.

[BB] C. S. Barroso and G. P. Bessa, *A note on the first eigenvalue of spherically symmetric manifolds*, XIV School on Differential Geometry, Mat. Contemp. **30** (2006), 63–69.

[BCG] C. Betz, G. A. Cámera, and H. Gzyl, *Bounds for the first eigenvalue of a spherical cap*, Appl. Math. Optim. **10** (1983), no. 3, 193–202.

[Ca] G. A. Cámera, *On a Sturm-Liouville problem*, Appl. Math. Optim. **30** (1994), 159–169.

[C] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics **115**, Academic Press, Inc., Orlando, FL, 1984.

[F] M. Flucher, *Approximation of Dirichlet eigenvalues on domains with small holes*, J. Math. Anal. Appl. **193** (1995), 169–199.

[FMS] P. Freitas, J. Mao, and I. Salavessa, *Spherical symmetrization and the first eigenvalue of geodesic disks on manifolds*, Calc. Var. Partial Differential Equations **51** (2014), 701–724.

[FH] S. Friedland and W. K. Hayman, *Eigenvalue inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions*, Comment. Math. Helv. **51** (1976), 133–161.

[Ga] M. E. Gage, *Upper bounds for the first eigenvalue of the Laplace–Beltrami operator*, Indiana Univ. Math. J. **29** (1980), 897–912.

[Gr] A. Grigor'yan, *Isoperimetric inequalities and capacities on Riemannian manifolds*, The Maz'ya anniversary collection, Vol. 1 (Rostock, 1998), 139–153, Oper. Theory Adv. Appl. **109**, Birkhäuser, Basel, 1999.

[I] A. M. Il'in, *Matching of asymptotic expansions of solutions of boundary value problems*, Translations of Mathematical Monographs **102**, Providence, RI: American Mathematical Society, 1992.

[K] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, New York 1966.

[K] N. N. Katz, *An upper bound for small eigenvalues of the Laplacian*, Osaka J. Math. **40** (2003), 449–454.

[MT] T. Matsuzawa and S. Tanno, *Estimates of the first eigenvalue of a big cup domain of a 2-sphere*, Compositio Math. **47** (1982), 95–100.

[MNP] V. Mazya, S. Nazarov, and B. Plamenevskij, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains*, Vol. I, Translated from the German by Plamenevskij, Operator Theory: Advances and Applications, Vol. 112 (Birkhäuser, 2000).

[M] H. P. McKean, *An upper bound to the spectrum of* $\Delta$ *on a manifold of negative curvature*, J. Differential Geometry **4** (1970), 359–366.

[P1] M. A. Pinsky, *The spectrum of the Laplacian on a manifold of negative curvature I*, J. Differential Geometry **13** (1978), 87–91.

[P2] M. A. Pinsky, *An upper bound for the first eigenvalue of a spherical cap*, Appl. Math. Optim. **30** (1994), 171–174.

[Pe] P. Petersen, *Riemannian Geometry* 2nd. Edition, Springer, New York 2006.

[S] S. Sato, *Barta's inequalities and the first eigenvalue of a cap domain of a 2-sphere*, Math. Z. **181** (1982), 313–318.

[VL] M. I. Vishik and L. A. Lyusternik, *Regular degeneration and the boundary layer for linear differential equations with small parameter*, Amer. Math. Soc. Transl. Ser. 2 **35** (1962), 239–364.

[W] F. Y. Wang, *Estimates of the first Dirichlet eigenvalue by using diffusion processes*, Probab. Theory Related Fields **101** (1995), 363–369.

Institute of Mathematics, Ufa Scientific Center, RAS
Chernyshevsky str. 112, 450008, Ufa, Russia
and Bashkir State Pedagogical University
October St. 3a, 450000 Ufa, Russia
and University of Hradec Králové
Rokitanského 62, 500 03, Hradec Králové, Czech Republic
*E-mail address*: borisovdi@yandex.ru

Departamento de Matemática, Instituto Superior Técnico
Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
and Grupo de Física Matemática, Faculdade de Ciências, Universidade de Lisboa, Campo Grande, Edifício C6, 1749-016 Lisboa, Portugal
*E-mail address*: psfreitas@fc.ul.pt

Received January 22, 2015