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Holomorphic automorphisms of the loop space of $\mathbb{P}^n$

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The loop space $L\mathbb{P}^n$ of the complex projective space $\mathbb{P}^n$ consisting of all C^k or Sobolev $W^{k,p}$ maps $S^1 \to \mathbb{P}^n$ is an infinite dimensional complex manifold. We identify a class of holomorphic self-maps of $L\mathbb{P}^n$, including all automorphisms.

1. Introduction

Let M be a finite dimensional complex manifold. We fix a smoothness class C^k, $k = 1, 2, \ldots, \infty$, or Sobolev $W^{k,p}$, $k = 1, 2, \ldots$, $1 \le p < \infty$, and consider the loop space $LM = L_k M$, or $L_{k,p}M$, of all maps $S^1 \to M$ with the given regularity. It carries a natural complex Banach/Fréchet manifold structure (see [L]). In this paper, we identify a class of holomorphic self-maps of the loop space $L\mathbb{P}^n$ of the complex projective space $\mathbb{P}^n$. As a consequence, we compute the group $\mathrm{Aut}(L\mathbb{P}^n)$ of holomorphic automorphisms of $L\mathbb{P}^n$. This work was directly motivated by [MZ, Z1, LZ, Z2], in which certain subgroups of $\mathrm{Aut}(L\mathbb{P}^1)$ play a key role to study Dolbeault cohomology groups with values in line bundles over $L\mathbb{P}^1$.

There are two simple ways to construct holomorphic self-maps of a given loop space LM. First, such a map can be obtained from a family of holomorphic self-maps of M smoothly parameterized by $t \in S^1$. For example, let $G \simeq PGL(n+1, \mathbb{C})$ be the group of holomorphic automorphisms of $\mathbb{P}^n$. Its loop space LG with pointwise group operation is again a complex Lie group and acts on $L\mathbb{P}^n$ holomorphically; thus any element of LG can be considered as a holomorphic automorphism of $L\mathbb{P}^n$. Second, let $\mathscr{T}(S^1) = \mathscr{T}_k(S^1)$ resp. $\mathscr{T}_{k,p}(S^1)$ be the space of maps $\phi : S^1 \to S^1$ with

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the following properties: for any x in $L\mathbb{C} = L_k\mathbb{C}$ resp. $L_{k,p}\mathbb{C}$, the pull back $\phi^* x = x \circ \phi$ is still in $L\mathbb{C} = L_k\mathbb{C}$ resp. $L_{k,p}\mathbb{C}$, and the complex linear operator $L\mathbb{C} \ni x \mapsto x \circ \phi \in L\mathbb{C}$ is continuous. If we set the loop x above to be the inclusion $S^1 \to \mathbb{C}$, we see that any ϕ in $\mathscr{T}_k(S^1)$ resp. $\mathscr{T}_{k,p}(S^1)$ is a C^k resp. $W^{k,p}$ map. It is easy to verify that $\mathscr{T}_k(S^1) = C^k(S^1, S^1)$. Now any $\phi \in \mathscr{T}(S^1)$ induces a holomorphic map

$$f_{\phi, LM} : LM \ni x \mapsto x \circ \phi \in LM$$

(see Proposition 2.1). We shall write f_ϕ for $f_{\phi, L\mathbb{P}^n}$.

Recall that $H^2(L\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$ (see [CJ, Part II, Proposition 15.33] and [P, Theorem 13.14]). Any holomorphic self-map f of $L\mathbb{P}^n$ induces a homomorphism $[\omega] \mapsto m_f[\omega]$, where $[\omega] \in H^2(L\mathbb{P}^n, \mathbb{Z})$, and m_f is a non-negative integer. Our main result is

Theorem 1.1. *Let f be a holomorphic self-map of $L\mathbb{P}^n$. Then $m_f = 1$ if and only if there exist $\gamma \in LG$ and $\phi \in \mathscr{T}(S^1)$ such that*

$$f = \gamma \circ f_\phi. \tag{1}$$

Note that all constant loops are fixed points of f_ϕ, and γ does not share this property unless it is the identity. Hence the decomposition of f as in (1) is unique.

It is straightforward to verify that

$$f_\phi \circ \gamma = f_{\phi, LG}(\gamma) \circ f_\phi, \quad \gamma \in LG,\ \phi \in \mathscr{T}(S^1). \tag{2}$$

Let $\mathscr{D}(S^1) = \mathscr{D}_k(S^1)$ resp. $\mathscr{D}_{k,p}(S^1)$ be the space of bijections $\phi : S^1 \to S^1$ such that both ϕ and ϕ^{-1} are in $\mathscr{T}(S^1) = \mathscr{T}_k(S^1)$ resp. $\mathscr{T}_{k,p}(S^1)$. Then $\mathscr{D}_k(S^1)$ is the space of C^k diffeomorphisms of S^1. If $k = 2, 3, \ldots,$ or $k = p = 1$, then $\mathscr{D}_{k,p}(S^1)$ is the space of bijections $\phi : S^1 \to S^1$ such that $\phi, \phi^{-1} \in W^{k,p}(S^1, S^1)$; and $\mathscr{D}_{1,p}(S^1)$, where $p > 1$, is the space of bi-Lipschitz maps (see [V, Theorem 4] and [HS, Corollary 20.5]). Note that $f_\phi \in \mathrm{Aut}(L\mathbb{P}^n)$ if and only if $\phi \in \mathscr{D}(S^1)$, and $\mathscr{D}(S^1)$ can be considered as a subgroup of $\mathrm{Aut}(L\mathbb{P}^n)$.

Combination of Theorem 1.1 and (2) gives

Corollary 1.2. *The group $\mathrm{Aut}(L\mathbb{P}^n)$ is the semidirect product $LG \rtimes \mathscr{D}(S^1)$.*

The group of holomorphic automorphisms of a compact complex manifold is a complex Lie group. We would like to know whether $\mathrm{Aut}(L\mathbb{P}^n)$

can be endowed with a natural complex Lie group structure. Recall that LG is a complex Lie group. If $\mathscr{D}(S^1)$ can be endowed with a manifold structure such that $\mathrm{Aut}(L\mathbb{P}^n) = LG \rtimes \mathscr{D}(S^1)$ becomes a complex Lie group, then $\mathscr{D}(S^1) \simeq \mathrm{Aut}(L\mathbb{P}^n)/LG$ is a complex Lie group. The space $\mathscr{D}_k(S^1)$ can be considered as the open subset of L_kS^1 consisting of embedded loops. With this manifold structure, $\mathscr{D}_\infty(S^1)$ is a Lie group (and $\mathscr{D}_k(S^1)$, where $k < \infty$, is not a Lie group). It follows from [PS, Proposition 3.3.2] that we cannot endow $\mathscr{D}(S^1)$ with a complex Lie group structure such that the inclusion $\mathscr{D}_\infty(S^1) \to \mathscr{D}(S^1)$ is a Lie group homomorphism.

Let $\mathscr{C} = \mathscr{C}(L\mathbb{P}^n)$ be the set $\{\mu(\mathbb{P}^1)\}$ of curves in $L\mathbb{P}^n$, where μ ranges over all holomorphic embeddings $\mathbb{P}^1 \to L\mathbb{P}^n$ such that $\mu^* : H^2(L\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^1, \mathbb{Z})$ is an isomorphism, and let $\mathscr{H} = \mathscr{H}(L\mathbb{P}^n)$ be the space of holomorphic self-maps f of $L\mathbb{P}^n$ with the following property: for any curve $C \in \mathscr{C}$, there exists a curve $C' \in \mathscr{C}$ such that $f(C) \subset C'$. It turns out that any holomorphic self-map f of $L\mathbb{P}^n$ with $m_f = 1$ is in $\mathscr{H}$ (see Section 3). To prove Theorem 1.1, we only need to study maps in $\mathscr{H}$.

This paper is organized as follows. In Section 2, we recall some relevant facts about loop spaces and prove two propositions which will be needed later on. Section 3 contains a complete proof of Theorem 1.1. In the final Section 4, we try to classify maps in $\mathscr{H}$.

We refer to [D, H] for the fundamentals of infinite dimensional holomorphy.

2. Preliminaries

Let $\Psi : M \to M'$ be a holomorphic map between finite dimensional complex manifolds. Then $L\Psi : LM \ni x \mapsto \Psi \circ x \in LM'$ is holomorphic, and L is functorial. For any $t \in S^1$, the evaluation map $E_t : LM \ni x \mapsto x(t) \in M$ is holomorphic (see [L]). The constant loops form a submanifold of LM, which can be identified with M. Note that all elements of LM can be represented by absolutely continuous maps.

Let $G \simeq PGL(n+1, \mathbb{C})$ be as in Section 1. Applying the functor L to the holomorphic action $G \times \mathbb{P}^n \to \mathbb{P}^n$, we obtain a holomorphic action $LG \times L\mathbb{P}^n \to L\mathbb{P}^n$.

Proposition 2.1. *Suppose M is an n-dimensional complex manifold, where $0 < n < \infty$, and $\phi : S^1 \to S^1$ is a map. Then $\phi \in \mathscr{T}(S^1)$ if and only if for any $x \in LM$, $x \circ \phi$ is still in LM, and the map $f_{\phi, LM} : LM \ni x \mapsto x \circ \phi \in LM$ is holomorphic.*

Proof. First we show that if $\phi \in \mathscr{T}(S^1)$ and $x \in LM$, then $x \circ \phi \in LM$. Let (U, Φ) be a coordinate chart of M, where $U \subset M$ is open, and Φ is a biholomorphic map from U to an open subset of $\mathbb{C}^n$. Then $LU \subset LM$ is open, and $L\Phi$ is a biholomorphic map from LU to an open subset of $L\mathbb{C}^n$. If $x \in LU$, then

$$x \circ \phi = (L\Phi)^{-1} \circ f_{\phi, L\mathbb{C}^n} \circ L\Phi(x) \in LU \subset LM.$$

For a general loop $x \in LM$ and any $t_0 \in S^1$, there exist a neighborhood $V' \subset S^1$ of $\phi(t_0)$, a coordinate chart (U, Φ) of M and $\tilde{x} \in LU$ such that $x(t) = \tilde{x}(t)$, $t \in V'$. Note that ϕ is continuous. Choose a neighborhood $V \subset S^1$ of t_0 such that $\phi(V) \subset V'$, then $x \circ \phi(t) = \tilde{x} \circ \phi(t)$, $t \in V$. So $x \circ \phi$ is C^k resp. $W^{k,p}$, i.e. $x \circ \phi \in LM$, and the map $f_{\phi, LM}$ is well-defined.

Next we investigate the relationship between maps $f_{\phi, LM}$ and $f_{\phi, L\mathbb{C}^n}$. Recall the complex structure of LM as constructed in [LS, Subsection 1.1], where an open neighborhood of $y \in LM$ is mapped by the local chart φ_y to an open subset of $C^k(y^*TM)$ resp. $W^{k,p}(y^*TM)$, the space of C^k resp. $W^{k,p}$ sections of the pull back of the tangent bundle of M by y. It is straightforward to verify that $\varphi_{y \circ \phi} \circ f_{\phi, LM} \circ \varphi_y^{-1}$ is precisely (the restriction of) the pull back

$$\phi^* : C^k(y^*TM) \to C^k(\phi^* y^*TM) \text{ resp. } W^{k,p}(y^*TM) \to W^{k,p}(\phi^* y^*TM).$$

The bundle y^*TM over S^1 is always trivial, and the above map can be considered as $f_{\phi, L\mathbb{C}^n}$. Now we have $\varphi_{y \circ \phi} \circ f_{\phi, LM} \circ \varphi_y^{-1} = f_{\phi, L\mathbb{C}^n}$. Thus $\phi \in \mathscr{T}(S^1)$ if and only if the map $f_{\phi, LM}$ is well-defined and holomorphic. $\square$

Let $i : \mathbb{P}^n \to L\mathbb{P}^n$ be the inclusion of the submanifold of constant loops. Since $E_t \circ f_\phi \circ i$ is the identity, the induced maps $i^* : H^2(L\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathbb{Z})$, $f_\phi^* : H^2(L\mathbb{P}^n, \mathbb{Z}) \to H^2(L\mathbb{P}^n, \mathbb{Z})$ and $E_t^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(L\mathbb{P}^n, \mathbb{Z})$ are all isomorphisms.

The following simple proposition will be very useful:

Proposition 2.2. *Let $\mu : \mathbb{P}^1 \to L\mathbb{P}^n$ and $\tau : \mathbb{P}^n \to L\mathbb{P}^n$ be holomorphic maps such that $\mu^* : H^2(L\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^1, \mathbb{Z})$ and $\tau^* : H^2(L\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^n, \mathbb{Z})$ are isomorphisms. Then:*

(a) *For any $t \in S^1$, the map $E_t \circ \mu : \mathbb{P}^1 \to \mathbb{P}^n$ is an embedding whose image is a projective line. In particular, μ itself is an embedding.*

(b) *There exists $\gamma \in LG$ (considered as an automorphism of $L\mathbb{P}^n$) such that $\tau = \gamma|_{\mathbb{P}^n}$.*

Proof. (a) Note that $(E_t \circ \mu)^* = \mu^* \circ E_t^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^1, \mathbb{Z})$ is an isomorphism, which maps the first Chern class of the hyperplane section bundle over $\mathbb{P}^n$ to the first Chern class of the hyperplane section bundle over $\mathbb{P}^1$. Therefore the inverse image of any hyperplane of $\mathbb{P}^n$ under $E_t \circ \mu$ is either a hyperplane (i.e. a single point) or the entire $\mathbb{P}^1$. The proof will be by induction on n. If $n = 1$, then $E_t \circ \mu$ is conformal. If $n > 1$, take a hyperplane $H \subset \mathbb{P}^n$ containing two different points in $E_t \circ \mu(\mathbb{P}^1)$. Then we must have $E_t \circ \mu(\mathbb{P}^1) \subset H \simeq \mathbb{P}^{n-1}$.

(b) The restriction of τ to any projective line in $\mathbb{P}^n$ can be considered as the map μ . In view of part (a), for any $t \in S^1$, $E_t \circ \tau : \mathbb{P}^n \to \mathbb{P}^n$ is injective; hence $E_t \circ \tau \in G$. Define $\gamma : S^1 \ni t \mapsto E_t \circ \tau \in G$. Then

$$\gamma(t)(\zeta) = \tau(\zeta)(t), \quad \zeta \in \mathbb{P}^n, t \in S^1. \tag{3}$$

Note that γ is C^k resp. $W^{k,p}$ (i.e. $\gamma \in LG$) if $\gamma(t)(\zeta_j)$, $j = 1, 2, \ldots, n+2$, are C^k resp. $W^{k,p}$ maps of t, where $\{\zeta_1, \zeta_2, \ldots, \zeta_{n+2}\} \subset \mathbb{P}^n$ is any given set of $n+2$ points in general position. It follows from (3) that $\gamma \in LG$ and $\tau = \gamma|_{\mathbb{P}^n}$. □

3. Proof of Theorem 1.1

We begin with two results concerning holomorphic self-maps of loop spaces of the type $f_{\phi,\,LM}$.

Proposition 3.1. *Let f be a holomorphic self-map of LM. Then $f = f_{\phi,\,LM}$ for some $\phi \in \mathscr{T}(S^1)$ if and only if for any $t \in S^1$, there exists $t' \in S^1$ such that $E_t \circ f = E_{t'}$.*

Proof. If $f = f_{\phi,\,LM}$, then $E_t \circ f = E_{\phi(t)}$. For the other direction, define the map $\phi : S^1 \ni t \mapsto t' \in S^1$. Then $f(x) = x \circ \phi$. It follows from Proposition 2.1 that $\phi \in \mathscr{T}(S^1)$. □

Lemma 3.2. *Let f be a self-map of $L\mathbb{C}^n$. Then $f = f_{\phi,\,L\mathbb{C}^n}$ for some $\phi \in \mathscr{T}(S^1)$ if and only if f is continuous complex linear and*

$$f(x)(S^1) \subset x(S^1) \tag{4}$$

for all $x \in L\mathbb{C}^n$.

Proof. One direction being trivial, we shall only verify the sufficiency part of the claim. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{C}^n \subset L\mathbb{C}^n$. For any

$x \in L\mathbb{C}^n$, we write $x = \sum_{j=1}^n x_j e_j$, where $x_j \in L\mathbb{C}$. In view of (4), we have $f(x_j e_j) = y_j e_j$, where $y_j \in L\mathbb{C}$ and $y_j(S^1) \subset x_j(S^1)$. Thus f induces continuous complex linear maps $f_j : L\mathbb{C} \ni x_j \mapsto y_j \in L\mathbb{C}$, $j = 1, 2, \ldots, n$. Now $f(x) = \sum_{j=1}^n f_j(x_j)e_j$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Then $f_j(L\mathbb{C}^*) \subset L\mathbb{C}^*$ and $f_j(1) = 1$. It follows from [Z1, Lemma 3.1] that for any $t \in S^1$, there exists $t_j \in S^1$ such that $E_t \circ f_j = E_{t_j}$. Therefore

$$E_t \circ f(x) = \sum_{j=1}^n x_j(t_j)e_j. \tag{5}$$

Let $x_0 \in L\mathbb{C}$ be an embedded loop. Setting $x = \sum_{j=1}^n x_0 e_j$ in (4) and (5), we obtain that $x_0(t_1) = x_0(t_2) = \cdots = x_0(t_n)$. Hence $t_1 = t_2 = \cdots = t_n$ and $E_t \circ f = E_{t_1}$. By Proposition 3.1, $f = f_{\phi, L\mathbb{C}^n}$ for some $\phi \in \mathscr{T}(S^1)$. □

Proposition 3.3. *Let x and y be two different points of $L\mathbb{P}^n$. Then there exists a curve $C \in \mathscr{C}$ through both x and y if and only if $x(t) \neq y(t)$ for all $t \in S^1$.*

Proof. The "only if" direction follows from Proposition 2.2(a). To show the "if" direction, consider the natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ and the holomorphic map $L\pi : L(\mathbb{C}^{n+1} \setminus \{0\}) \to L\mathbb{P}^n$. Let $\tilde{x}, \tilde{y} \in L(\mathbb{C}^{n+1} \setminus \{0\})$ be such that $\tilde{x}(t)$ and $\tilde{y}(t)$ are linearly independent for all $t \in S^1$. Then the map

$$\mu = \mu_{\tilde{x}, \tilde{y}} : \mathbb{P}^1 \ni [Z_0, Z_1] \mapsto L\pi\,(Z_0\tilde{x} + Z_1\tilde{y}) \in L\mathbb{P}^n, \tag{6}$$

where Z_0, Z_1 are homogeneous coordinates on $\mathbb{P}^1$, is well-defined and holomorphic. For any $t \in S^1$, the map $E_t \circ \mu$ is an embedding whose image is a projective line in $\mathbb{P}^n$; thus μ is an embedding and $\mu^* : H^2(L\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^1, \mathbb{Z})$ is an isomorphism.

The map $L\pi$ is surjective (for C^k loops see [LS, Lemma 2.2], and the proof of [LS, Lemma 2.2] also works for $W^{k,p}$ loops). Take $\tilde{x} \in (L\pi)^{-1}(x)$ and $\tilde{y} \in (L\pi)^{-1}(y)$ in (6), then the image of μ is a curve $C \in \mathscr{C}$ through both x and y. □

From now on, we shall concentrate on holomorphic self-maps f of $L\mathbb{P}^n$, and curves in $\mathscr{C}$ will be needed throughout the rest of the paper.

Proposition 3.4. *If $m_f = 0$, then f is constant.*

Proof. If $\eta : \mathbb{P}^1 \to \mathbb{P}^n$ is a holomorphic map such that $\eta^* : H^2(\mathbb{P}^n, \mathbb{Z}) \to H^2(\mathbb{P}^1, \mathbb{Z})$ is zero, then η is constant. So for any curve $C \in \mathscr{C}$ and any $t \in S^1$, the map $E_t \circ f|_C : C \to \mathbb{P}^n$ is constant; hence $f|_C$ is constant. It follows from Proposition 3.3 that for any $x \in L\mathbb{P}^n$ and any constant loop $\zeta \in \mathbb{P}^n \setminus x(S^1)$, there exists a curve $C \in \mathscr{C}$ through both x and ζ; thus f is constant. □

Recall the mapping space $\mathscr{H}$ as defined in Section 1. If $f \in \mathscr{H}$, then the topological degree of $f|_C : C \to C'$ is m_f. In particular, $f(C) = C'$ if $m_f \geq 1$, and $f|_C$ is one-to-one if $m_f = 1$. By Proposition 2.2(a), any holomorphic self-map f of $L\mathbb{P}^n$ with $m_f = 1$ is in $\mathscr{H}$. Hence $LG \subset \mathscr{H}$, and $f_\phi \in \mathscr{H}$ for any $\phi \in \mathscr{T}(S^1)$.

Proposition 3.5. *Suppose $f \in \mathscr{H}$, $m_f \geq 1$, $x \in L\mathbb{P}^n$, and $\zeta \in \mathbb{P}^n \setminus x(S^1)$ is a constant loop. If either $m_f = 1$ or $f(x) \notin f(\mathbb{P}^n)$, then $f(x)(t) \neq f(\zeta)(t)$ for all $t \in S^1$.*

Proof. By Proposition 3.3, there exists a curve $C \in \mathscr{C}$ through both x and ζ. If either $m_f = 1$ or $f(x) \notin f(\mathbb{P}^n)$, then $f(x) \neq f(\zeta)$. Since both $f(x)$ and $f(\zeta)$ are in $f(C) \in \mathscr{C}$, it follows from Proposition 3.3 that $f(x)(t) \neq f(\zeta)(t)$ for all $t \in S^1$. □

Theorem 3.6. *Let f be a holomorphic self-map of $L\mathbb{P}^n$. Then $f = f_\phi$ for some $\phi \in \mathscr{T}(S^1)$ if and only if every constant loop is a fixed point of f.*

Proof. The necessity is obvious. Regarding sufficiency, note that $f|_{\mathbb{P}^n}$ is the identity; hence $m_f = 1$ and $f \in \mathscr{H}$. It follows from Proposition 3.5 that

$$f(x)(S^1) \subset x(S^1) \tag{7}$$

for all $x \in L\mathbb{P}^n$. Let

$$U_0 = \{[Z_0, Z_1, \ldots, Z_n] \in \mathbb{P}^n : Z_0 \neq 0\},$$

where $Z_0, Z_1, \ldots, Z_n$ are homogeneous coordinates on $\mathbb{P}^n$. Now consider U_0 as $\mathbb{C}^n$. It follows from (7) that $f(L\mathbb{C}^n) \subset L\mathbb{C}^n$. Next we show that $f|_{L\mathbb{C}^n}$ is complex linear.

Let $x = (x_1, \ldots, x_n) \in L\mathbb{C}^n$ and $y = (y_1, \ldots, y_n) \in L(\mathbb{C}^n \setminus \{0\})$. If we choose $\tilde{x} = (1, x_1, \ldots, x_n)$ and $\tilde{y} = (0, y_1, \ldots, y_n)$ in (6), then the image of μ is a curve $C_{x,y} \in \mathscr{C}$, and

$$C_{x,y} \cap L\mathbb{C}^n = C_{x,y} \setminus \{L\pi(\tilde{y})\} = \{x + \lambda y \in L\mathbb{C}^n : \lambda \in \mathbb{C}\},$$

where $L\pi(\tilde{y}) \in L(\mathbb{P}^n \setminus \mathbb{C}^n)$. Note that $f(C_{x,y})$ is also a curve in $\mathscr{C}$, and f maps $C_{x,y}$ conformally onto $f(C_{x,y})$. By Proposition 2.2(a), for any $t \in S^1$,

$E_t \circ f(C_{x,y})$ is a projective line in $\mathbb{P}^n$. In view of (7), we have

$$E_t \circ f(C_{x,y}) \cap \mathbb{C}^n = \{E_t \circ f(x + \lambda y) : \lambda \in \mathbb{C}\}. \tag{8}$$

The above set must be an affine line of $\mathbb{C}^n$. As a function of λ, $E_t \circ f(x + \lambda y)$ maps $\mathbb{C}$ bijectively onto the affine line in (8); therefore it is a polynomial of degree one. So

$$E_t \circ f(x + \lambda y) = [f(x + y)(t) - f(x)(t)]\, \lambda + f(x)(t)$$

for all $t \in S^1$. Thus

$$f(x + \lambda y) = [f(x + y) - f(x)]\, \lambda + f(x),$$

i.e. $\mathbb{C} \ni \lambda \mapsto f(x + \lambda y) \in L\mathbb{C}^n$ is a polynomial of degree one, where $x \in L\mathbb{C}^n$ and $y \in L(\mathbb{C}^n \setminus \{0\})$. Since $L(\mathbb{C}^n \setminus \{0\})$ is dense in $L\mathbb{C}^n$, $f(x + \lambda y)$ is a polynomial of λ of degree less than or equal to one for all $x, y \in L\mathbb{C}^n$. Hence $f|_{L\mathbb{C}^n}$ is a polynomial of degree one (see [H, Section 2.2]). As $f(0) = 0$, $f|_{L\mathbb{C}^n}$ is complex linear.

By (7) and Lemma 3.2, we have $f|_{L\mathbb{C}^n} = f_{\phi,\, L\mathbb{C}^n}$ for some $\phi \in \mathscr{T}(S^1)$; thus $f = f_\phi$ on the connected manifold $L\mathbb{P}^n$. □

Proof of Theorem 1.1. The sufficiency is obvious. Regarding necessity, by Proposition 2.2(b), there exists $\gamma \in LG$ such that $f|_{\mathbb{P}^n} = \gamma|_{\mathbb{P}^n}$. Then every constant loop is a fixed point of $\gamma^{-1} \circ f$. It follows from Theorem 3.6 that $\gamma^{-1} \circ f = f_\phi$ for some $\phi \in \mathscr{T}(S^1)$, i.e. $f = \gamma \circ f_\phi$. □

4. The mapping space $\mathscr{H}$

In this section, we continue to study maps in $\mathscr{H}$. Recall that all holomorphic self-maps f of $L\mathbb{P}^n$ with $m_f \leq 1$ are in $\mathscr{H}$.

Theorem 4.1. *Let $f \in \mathscr{H} = \mathscr{H}(L\mathbb{P}^n)$.*

(a) *If $n \geq 2$, then $m_f \leq 1$.*

(b) *If $n = 1$ and $m_f \geq 2$, then $f(L\mathbb{P}^1) = f(\mathbb{P}^1)$ (i.e. $f(C) = f(\mathbb{P}^1)$ for all $C \in \mathscr{C}$).*

Proof. (a) If f is constant, then $m_f = 0$. If f is non-constant, by Proposition 3.4 we have $m_f \geq 1$. Fix $t \in S^1$ and define $\alpha = E_t \circ f|_{\mathbb{P}^n} : \mathbb{P}^n \to \mathbb{P}^n$. Then α induces a homomorphism $[\omega] \mapsto m_f[\omega]$, where $[\omega] \in H^2(\mathbb{P}^n, \mathbb{Z})$. So the topological degree of α is $m_f^n \neq 0$. Let $\zeta^* \in \mathbb{P}^n$ be a regular value of α. Take $\zeta \in \alpha^{-1}(\zeta^*)$ and choose a hyperplane $H \subset \mathbb{P}^n$ such that $\zeta \notin H$. For any $w \in H$, the projective line $P_{\zeta,w}$ through both ζ and w is in $\mathscr{C}$; thus $f(P_{\zeta,w}) \in \mathscr{C}$. In view of Proposition 2.2(a), $\alpha(P_{\zeta,w})$ is a projective line of $\mathbb{P}^n$. The topological degree of the map $P_{\zeta,w} \overset{\alpha}{\to} \alpha(P_{\zeta,w})$ is m_f. If $m_f \geq 2$, then $P_{\zeta,w} \setminus \{\zeta\}$ must contain at least one point in $\alpha^{-1}(\zeta^*)$. For different $w \in H$, the sets $P_{\zeta,w} \setminus \{\zeta\}$ are disjoint; hence $\alpha^{-1}(\zeta^*)$ is not a finite set, which is a contradiction.

(b) Note that $f(\mathbb{P}^1) \in \mathscr{C}$. By Proposition 2.2(b) (in which we choose τ to be a suitable holomorphic embedding $\mathbb{P}^1 \to L\mathbb{P}^1$ with image $f(\mathbb{P}^1)$), there exists $\gamma \in LG$ such that $\gamma(\mathbb{P}^1) = f(\mathbb{P}^1)$; thus $\gamma^{-1} \circ f(\mathbb{P}^1) = \mathbb{P}^1$. Without loss of generality, we may assume that $f(\mathbb{P}^1) = \mathbb{P}^1$.

Let $\rho = f|_{\mathbb{P}^1}$. Then $\rho : \mathbb{P}^1 \to \mathbb{P}^1$ is an m_f-sheeted branched covering. Choose a non-empty open subset $W \subset \mathbb{P}^1$ such that $\rho|_W$ is one-to-one. Then $LW \subset L\mathbb{P}^1$ is a non-empty open subset. We claim that

$$f(x) \in \mathbb{P}^1 \text{ for } x \in LW. \tag{9}$$

Otherwise there would exist $x_0 \in LW$ such that $y_0 = f(x_0) \notin \mathbb{P}^1$. The set $y_0(S^1)$ is not finite; thus we can find a regular value ζ^* of ρ in $y_0(S^1)$. Take $\zeta_1, \zeta_2 \in \rho^{-1}(\zeta^*)$, $\zeta_1 \neq \zeta_2$. It follows from Proposition 3.5 that $\zeta_1, \zeta_2 \in x_0(S^1) \subset W$; then $\rho|_W$ is not one-to-one, which is a contradiction.

By (9), maps $E_t \circ f|_{LW}$ are independent of $t \in S^1$; hence maps $E_t \circ f$ are independent of t on the connected manifold $L\mathbb{P}^1$, i.e. $f(L\mathbb{P}^1) \subset \mathbb{P}^1$. □

The maps as in Theorem 4.1(b) do exist: Let $\rho : \mathbb{P}^1 \to \mathbb{P}^1 \subset L\mathbb{P}^1$ be a holomorphic map with topological degree $m \geq 2$, $t \in S^1$ and $\gamma \in LG$. Then $f = \gamma \circ \rho \circ E_t \in \mathscr{H}(L\mathbb{P}^1)$ and $m_f = m$.

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