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On curvature tensors of Hermitian manifolds

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In this article, we examine the behavior of the Riemannian and Hermitian curvature tensors of a Hermitian metric, when one of the curvature tensors obeys all the symmetry conditions of the curvature tensor of a Kähler metric. We will call such metrics G-Kähler-like or Kähler-like, for lack of better terminologies. Such metrics are always balanced when the manifold is compact, so in a way they are more special than balanced metrics, which drew a lot of attention in the study of non-Kähler Calabi-Yau manifolds. In particular we derive various formulas on the difference between the Riemannian and Hermitian curvature tensors in terms of the torsion of the Hermitian connection. We believe that these formulas could lead to further applications in the study of Hermitian geometry with curvature assumptions.

1. Introduction

In recent years, there has been much progress in the geometric analysis of Hermitian manifolds, with the intent of pushing analysis on Kähler manifolds to general Hermitian ones, and also with the study of non-Kähler Calabi-Yau manifolds from string theory. See for instance the work of Fu-Yau [5], Fu-Li-Yau [6], Fu [4], Fu-Wang-Wu [7], [8], Liu-Yang [16], [17], [18], Streets-Tian [**?**], Tosatti-Weinkove [23], [21], Guan-Sun [13], and the references therein.

Given a complex manifold M^n, a Hermitian metric g is just a Riemannian metric such that the almost complex structure J preserves the metric. There are two canonical connections associated with the metric, namely, the Hermitian (aka Chern) connection ∇^h and the Riemannian (aka Levi-Civita) connection ∇. The first is the unique connection that is compatible with both the metric and the complex structure, while the second is the only torsion-free connection that is compatible with the metric. Let us denote by R^h and R the curvature tensor of these two connections.

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Here R is just the Riemannian curvature tensor, and we extend it linearly over $\mathbb{C}$. Both R^h and R are anti-symmetric with respect to their first two or last two positions, and they are both real operators. R is also symmetric when its first two and last two positions are interchanged, and satisfies the Bianchi identity which means that when one positions is held fixed while the other three are cyclicly permuted, the sum is always zero.

Under the decomposition $TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, all the components of R^h vanish except $R^h_{X\overline{Y}Z\overline{W}}$ (plus the obvious variation by the skew-symmetries with respect to the first two or last two positions), where X, Y, Z and W are type $(1,0)$ tangent vectors. But in general it is not symmetric with respect to its first and third (or its second and fourth) positions.

When g is Kähler, which means $\nabla^h = \nabla$, we have $R^h = R$. In this case, the only non-trivial components are $R_{X\overline{Y}Z\overline{W}}$, and $R_{X\overline{Y}Z\overline{W}} = R_{Z\overline{Y}X\overline{W}}$. For a general Hermitian metric, however, components $R_{XYZ\overline{W}}$ and $R_{XY\overline{Z}\overline{W}}$ might be non-zero, and R may not be symmetric with respect to its first and third (or second and fourth) positions. The only known condition is that $R_{XYZW} = R_{\overline{XYZW}} = 0$, discovered by Gray in [12] (see also formula (19) in §3).

Of course when the metric g is Kähler, one has $R^h = R$. So a naive question is, when a Hermitian metric g satisfies $R^h = R$, must it be Kähler? We could not seem to find an answer to this question in the literature, to our surprise, so we took it to our own hands and the first result in this article is simply to give a positive answer to this question. That is, we have the following:

Theorem 1. *Given a Hermitian manifold (M^n, g), if its Riemannian curvature tensor R and its Hermitian curvature tensor R^h are equal, then g is Kähler.*

Next, we would like to know what will happen when both R and R^h satisfy all the symmetry conditions of the curvature tensor of a Kähler metric. To make things precise, let us first introduce the following notion:

Definition (Kähler-like and G-Kähler-like). *A Hermitian metric g will be called* Kähler-like, *if $R^h_{X\overline{Y}Z\overline{W}} = R^h_{Z\overline{Y}X\overline{W}}$ holds for any type $(1,0)$ tangent vectors X, Y, Z, and W. Similarly, if $R_{XY\overline{Z}\overline{W}} = R_{XYZ\overline{W}} = 0$ for any type $(1,0)$ tangent vectors X, Y, Z, and W, we will say that g is* Gray-Kähler-like, *or* G-Kähler-like *for short.*

Note that the above definition simply means that R^h (or R) satisfies all the symmetry conditions obeyed by the curvature tensors of Kähler manifolds. The G-Kähler-like condition was first introduced by Gray in [12] (condition (1) on p. 605).

When g is Kähler-like, by taking complex conjugations, we see that R^h is also symmetric with respect to its second and fourth positions, thus obeying all the symmetries of the curvature tensor of a Kähler metric.

Similarly, when g is G-Kähler-like, we have $R_{XY**} = 0$ by the aforementioned Gray's Theorem, so the only non-trivial components of R are in the form $R_{X\overline{Y}Z\overline{W}}$. Also, the vanishing of $R_{XZ\overline{Y}\overline{W}}$ plus the first Bianchi identity imply that $R_{Z\overline{Y}X\overline{W}} = R_{X\overline{Y}Z\overline{W}}$. So R obeys all the symmetries of the curvature tensor of a Kähler metric.

Of course being Kähler-like or G-Kähler-like does not mean that the metric g will have to be Kähler. There are plenty of non-Kähler Hermitian metrics g which are Kähler-like or G-Kähler-like. For instance, when $n \geq 2$, there are Hermitian manifolds that are non-Kähler but with $R^h = 0$ everywhere. Such manifolds are certainly Kähler-like. In [1], Boothby showed that a compact Hermitian manifold with $R^h = 0$ everywhere is the quotient of a complex Lie group by a discrete subgroup. For $n \geq 3$, there are such manifolds that are non-Kähler.

As we shall see in later sections, one can explicitly write down Hermitian metrics in dimension $n \geq 2$ that are Kähler-like or G-Kähler-like but non-Kähler. The first compact, non-Kähler example of G-Kähler-like manifolds was observed by Gray in [12]. It is the *Calabi threefolds*, a family of compact complex manifolds of dimension 3 with $c_1 = 0$ that are diffeomorphic to the product of a compact Riemann surface with a real 4-torus, discovered by Calabi in 1958 [2]. For the reader's convenience, we give a sketch of Calabi's construction in §3.

Theorem 1 implies that any Hermitian manifold (M^n, g) satisfying $R = R^h$ is Kähler. Note that under the assumption of $R = R^h$, the manifold is obviously both Kähler-like and G-Kähler-like in view of the above definition. In light of this, we raise the following natural question:

Conjecture 1. *If a Hermitian manifold (M^n, g) is both Kähler-like and G-Kähler-like, then g must be Kähler.*

At this point, we could not seem to establish a proof to this conjecture in its full generality. However, we are able to prove a partial result, which could be serving as a piece of supporting evidence. To be precise, we have the following:

Theorem 2. *Let (M^n, g) be a Hermitian manifold that is both Kähler-like and G-Kähler-like. If either M^n is compact or $n \leq 3$, then g is Kähler.*

Note that Kähler-like or G-Kähler-like metrics provide important classes of special Hermitian metrics. When the manifold is compact, either condition would imply that the metric is balanced, namely, $d(\omega^{n-1}) = 0$, where ω is the Kähler form. So each type is more special than being balanced for compact manifolds. More specifically, we have the following:

Theorem 3. *Let (M^n, g) be a compact Hermitian manifold. If it is either Kähler-like or G-Kähler-like, then it must be balanced.* [1]

In particular, on compact complex surfaces, Kähler-like or G-Kähler-like metrics are Kähler. In fact it was already observed in [24] that any compact G-Kähler-like Hermitian surface is Kähler. Note that the completeness assumption of the Hermitian metric plays an important role in view of the example of noncompact G-Kähler-like and non-Kähler surface we constructed in §3. In dimension 3 or higher, there are examples of compact non-Kähler manifolds that are Kähler-like or G-Kähler-like, e.g., the Iwasawa threefold is Hermitian flat (namely with vanishing R^h) thus Kähler-like; while the Calabi threefolds are G-Kähler-like. Note that the Calabi threefolds also have vanishing first Chern class. It would be a very interesting question to classify all compact three dimensional non-Kähler Hermitian manifolds that are Kähler-like or G-Kähler-like, especially for Calabi-Yau threefolds (namely those with trivial canonical bundle and finite fundamental group).

In a larger context, recall that balanced metrics play an important role in the Strominger system ([20], [15], [5]). Mathematically, it is also intriguing to understand the moduli space of Calabi-Yau threefolds ([19] and [22]). From Theorem 3 we know that Kähler-like or G-Kähler-like metrics on closed Hermitian manifolds are more special than balanced ones. It might be interesting to know if Kähler-like or G-Kähler-like metrics on compact non-Kählerian Calabi-Yau threefolds can play a role in the study of Strominger system or the understanding of the moduli space of Calabi-Yau threefolds.

Next, let us consider the behavior of the Kähler-like or G-Kähler-like condition under conformal changes. Since balanced metrics are clearly unique (up to constant multiples) within each conformal class, by Theorem 3 we know that in the compact case there can be at most one such metric within

[1] An anonymous referee kindly brought to our attention that Corollary 4.5 of [17] also implies that any compact G-Kähler like manifold is balanced.

each conformal class. In the non-compact case, one can write down the equations and conclude that:

Theorem 4. *On a compact complex manifold M^n, each conformal class of Hermitian metrics contains at most one metric (up to constant multiples) that is Kähler-like or G-Kähler-like. For (M^n, g) non-compact and $\tilde{g} = e^{2u}g$ with $u \in C^\infty(M, \mathbb{R})$,*

1) if g is Kähler-like, then $\tilde{g}$ is Kähler-like if and only if $\partial\overline{\partial}u = 0$;

2) if g is G-Kähler-like, then $\tilde{g}$ is G-Kähler-like if and only if the function $\lambda = e^{-u}$ satisfies: $H_\lambda(X, Y) = 0$ and $\lambda H_\lambda(X, \overline{Y}) = \langle X, \overline{Y}\rangle |\nabla\lambda|^2$ for any type $(1, 0)$ tangent vectors X and Y. Here H_λ is the Hessian of λ.

In particular, $\lambda\Delta\lambda = n|\nabla\lambda|^2$ and $\Delta e^{(n-1)u} = 0$.

In [16], [17], and [18], Liu and Yang gave a detailed study of Hermitian manifolds and a thorough analysis on the relationship of various Ricci tensors arising from R^h and R. Here we will introduce the right notion of Riemannian bisectional curvature for R and compare it with the Hermitian bisectional curvature. The relationship between the two holomorphic sectional curvatures is particularly simple, and obey a monotonicity rule. See Theorem 7 in §5 for more details.

Let us remark that an interesting aspect of our work is to derive various formulas which characterize the difference between the Riemannian and Hermitian curvature tensors in terms of the torsion of Hermitian connection. We believe that these formulas could find further applications in the study of Hermitian geometry with curvature assumptions.

Next, we propose a natural question which should have been explored before, but again we could not seem to find it in the literature. It is well known that there are examples of non-Kähler Hermitian manifolds with everywhere vanishing Hermitian curvature tensor (i.e., with $R^h = 0$ everywhere). In the compact case such manifolds are all quotients of complex Lie groups, as proved in [1]. Naturally one would wonder if one can classify non-Kähler Hermitian manifolds with everywhere vanishing Riemannian curvature tensor (i.e., with $R = 0$ everywhere). We propose the following

Question 5. *Is there a characterization of Hermitian manifolds with vanishing Riemannian curvature tensor? In the compact case, it amounts to classify all compatible complex structures on the flat torus $T^{2n}_{\mathbb{R}}$.*

We will investigate Question 5 in a forthcoming work.

The paper is organized as follows: In Section 2, we start from the Cartan's structure equations and collect some preliminary results. In Section

3, we discuss the Riemannian curvature tensor R and the Hermitian curvature tensor R^h of a given Hermitian manifold, and pay special attention to the cases when one or both of these curvature tensors obey the symmetry conditions of the curvature tensor of a Kähler manifold. In Section 4, we give proofs to Theorems 1 and 2 stated in this section, and in Section 5, we examine the uniqueness problem for such metrics within a conformal class of Hermitian metrics. We also discuss the notion of bisectional curvature for the Riemannian curvature tensor R, and express the difference between the bisectional curvatures in terms of a quadratic formula of the torsion tensor. In particular, the holomorphic sectional curvatures of R and R^h obeys a simple monotonicity rule, with equality everywhere when and only when the metric is Kähler.

2. The structure equations of Hermitian manifolds

Let (M^n, g) be a Hermitian manifold, with $n \geq 2$. We will denote by ∇ and ∇^h the Riemannian and Hermitian connection of the metric g, and by R, R^h their curvatures, called the Riemannian or Hermitian curvature tensor, respectively.

Let $A = \nabla - \nabla^h$ and denote by T^h the torsion tensor of ∇^h:

$$T^h(X,Y) = \nabla^h_X Y - \nabla^h_Y X - [X,Y]$$

for any two tangent vectors X, Y on M. Since ∇ is torsion-free, the two tensors A and T^h are related by

$$A_X Y - A_Y X = -T^h(X,Y).$$

So T^h is the anti-symmetric part of A. On the other hand, the compatibility of the connections with the metric implies that

$$\langle A_X Y, Z\rangle + \langle A_Y X, Z\rangle = \langle X, T^h(Y,Z)\rangle + \langle Y, T^h(X,Z)\rangle$$

for any vector fields X, Y, and Z on M. So T^h completely determines A. Here $\langle , \rangle$ is the (real) inner product given by the Hermitian metric g.

While the difference of R and R^h is given by A and its first covariant derivative, it seems to us that the torsion tensor T^h would be easier to use in our context. Also, when the tangent frame is chosen to be orthogonal (unitary), the dependence of A on T^h takes the most convenient form. So we will use unitary coframes and focus on T^h from now on.

Let us complexify the tangent bundle and denote by $T^{1,0}M$ the bundle of complex tangent vector fields of type $(1,0)$, namely, complex vector fields in the form of $v - \sqrt{-1}Jv$, where v is any real vector field on M.

Suppose $\{e_1, \ldots, e_n\}$ is a frame of $T^{1,0}M$ in a neighborhood $M' \subseteq M$. Write $e = {}^t(e_1, \ldots, e_n)$ as a column vector. Denote by $\varphi = {}^t(\varphi_1, \ldots, \varphi_n)$ the column vector of $(1,0)$-forms in M' which is the coframe dual to e. For the Hermitian connection ∇^h of g, let us denote by θ, Θ the matrices of connection and curvature, respectively, and by τ the column vector of the torsion 2-forms, all under the local frame e. Then the structure equations are

$$d\varphi = - {}^t\theta \wedge \varphi + \tau, \tag{1}$$

$$d\theta = \theta \wedge \theta + \Theta. \tag{2}$$

Taking exterior differentiation of the above equations, we get the two Bianchi identities:

$$d\tau = - {}^t\theta \wedge \tau + {}^t\Theta \wedge \varphi, \tag{3}$$

$$d\Theta = \theta \wedge \Theta - \Theta \wedge \theta. \tag{4}$$

Note that under a frame change $\tilde{e} = Pe$, the corresponding forms are changed by

$$\tilde{\varphi} = {}^tP^{-1}\varphi, \quad \tilde{\theta} = P\theta P^{-1} + dPP^{-1}, \quad \tilde{\Theta} = P\Theta P^{-1}, \quad \tilde{\tau} = {}^tP^{-1}\tau.$$

In particular, the types of the 2-forms in Θ and τ are independent of the choice of the frame e.

We will denote by $\langle , \rangle$ the (real) inner product given by the Hermitian metric g, and by an abuse of notation, we will again denote by g the matrix $(\langle e_i, \overline{e}_j \rangle)$ of the metric under the frame e. The compatibility of ∇^h with the metric implies

$$\theta g + g\theta^* = dg, \qquad \Theta g + g\Theta^* = 0,$$

where $\theta^* = {}^t\overline{\theta}$. So when e is a unitary frame, $g = I$, and both θ and Θ are skew-Hermitian. While when e is holomorphic, θ is of type $(1,0)$, thus τ must be of type $(2,0)$, and Θ cannot have $(0,2)$ components, and its skew-Hermitian property for unitary frames implies that it cannot have $(2,0)$ components, either. So Θ must be of type $(1,1)$.

In particular, when e is holomorphic, we have

$$\theta = \partial g g^{-1}, \qquad \Theta = \overline{\partial}(\partial g g^{-1}).$$

We will write $\omega = \sqrt{-1}\ {}^t\varphi \wedge g\overline{\varphi}$ and introduce the following

$$\sigma = {}^t\tau \wedge g\overline{\tau}. \tag{5}$$

Both ω and σ are independent of the choice of the local frame, thus they are globally defined on M. ω is the Kähler (aka fundamental or Hermitian or metric) form of the Hermitian metric. It is everywhere positive definite. We will call σ the *torsion* $(2,2)$*-form*. It is a global, nonnegative $(2,2)$ form on M, and g is Kähler if and only if $\sigma = 0$ everywhere.

Next, let us consider the Riemannian (aka Levi-Civita) connection ∇ of g. We will use e and $\overline{e}$ as the frame on the complexified tangent bundle $TM \otimes \mathbb{C} = T^{1,0}M \oplus \overline{T^{1,0}M}$, so φ and $\overline{\varphi}$ form the coframe. Write

$$\nabla e = \theta_1 e + \overline{\theta_2}\overline{e}, \qquad \nabla \overline{e} = \theta_2 e + \overline{\theta_1}\overline{e}.$$

Then the matrices of connection and curvature for ∇ becomes:

$$\hat{\theta} = \begin{bmatrix} \theta_1 & \overline{\theta_2} \\ \theta_2 & \overline{\theta_1} \end{bmatrix}, \qquad \hat{\Theta} = \begin{bmatrix} \Theta_1 & \overline{\Theta}_2 \\ \Theta_2 & \overline{\Theta}_1 \end{bmatrix},$$

where

$$\Theta_1 = d\theta_1 - \theta_1 \wedge \theta_1 - \overline{\theta_2} \wedge \theta_2, \tag{6}$$

$$\Theta_2 = d\theta_2 - \theta_2 \wedge \theta_1 - \overline{\theta_1} \wedge \theta_2, \tag{7}$$

$$d\varphi = -\ {}^t\theta_1 \wedge \varphi - {}^t\theta_2 \wedge \overline{\varphi}. \tag{8}$$

and under the frame change $\tilde{e} = Pe$, $\overline{\tilde{e}} = \overline{P}\overline{e}$, the above matrices of forms are changed by

$$\tilde{\theta}_1 = P\theta_1 P^{-1} + dPP^{-1}, \quad \tilde{\theta}_2 = \overline{P}\theta_2 P^{-1}, \quad \tilde{\Theta}_1 = P\Theta_1 P^{-1}, \quad \tilde{\Theta}_2 = \overline{P}\Theta_2 P^{-1}.$$

We will write $\gamma = \theta_1 - \theta$. Then $\tilde{\gamma} = P\gamma P^{-1}$ so γ represents a tensor. The compatibility of ∇ with the metric implies

$$\theta_1 g + g\theta_1^* = dg, \qquad \theta_2 g + \overline{g}\ {}^t\theta_2 = 0,$$
$$\Theta_1 g + g\Theta_1^* = 0, \qquad \Theta_2 g + \overline{g}\ {}^t\Theta_2 = 0,$$

where $g = (\langle e_i, \overline{e}_j \rangle)$ and $\alpha^* = {}^t\overline{\alpha}$ as before. So when e is unitary, both θ_2 and Θ_2 are skew-symmetric, while θ_1, γ, and Θ_1 are skew-Hermitian.

Let us denote by $\gamma = \gamma' + \gamma''$ the decomposition into $(1,0)$ and $(0,1)$ parts. Note that the following two 2-forms are independent of the choice of the frame e, thus are globally defined on M:

$$\sigma_1 = -\sqrt{-1}\ \mathrm{tr}(\gamma' \wedge \gamma''), \qquad \sigma_2 = \sqrt{-1}\ \mathrm{tr}(\overline{\theta_2} \wedge \theta_2). \tag{9}$$

We will see that both are nonnegative $(1,1)$ forms, and vanish identically when and only when the metric is Kähler.

Lemma 1. *Each entry of* θ_2 *is a* $(1,0)$ *form, and the* $(0,2)$ *component of* Θ_2 *is zero.*

Proof. Let e be a local unitary frame. Write $\tau_i = \sum_{j,k=1}^{n} T^i_{jk}\varphi_j \wedge \varphi_k$, where $T^i_{jk} = -T^i_{kj}$. By (1) and (8), we get

$$^t\gamma \wedge \varphi + \tau + {}^t\theta_2 \wedge \overline{\varphi} = 0.$$

Let $\theta_2 = \theta_2' + \theta_2''$ be the decomposition into type $(1,0)$ and $(0,1)$, respectively. The above equation gives

$$^t\theta_2'' \wedge \overline{\varphi} = 0, \quad {}^t\gamma'' \wedge \varphi + {}^t\theta_2' \wedge \overline{\varphi} = 0, \quad {}^t\gamma' \wedge \varphi + \tau = 0. \tag{10}$$

Since e is unitary, both θ_2' and θ_2'' are skew-symmetric, and $\gamma'' = -\gamma'^*$. The first equation in (10) implies that $\theta_2'' = 0$. Now by (7), the $(0,2)$ part of Θ_2 vanishes.

□

Lemma 2. *Write* $\tau_i = \sum_{j,k=1}^{n} T^i_{jk}\varphi_j \wedge \varphi_k$ *with* $T^i_{jk} = -T^i_{kj}$ *under the frame* e *and its dual coframe* φ. *If* e *is unitary, then*

$$(\theta_2)_{ij} = \sum_{k=1}^{n} \overline{T^k_{ij}}\varphi_k, \qquad \gamma_{ij} = \sum_{k=1}^{n} (T^j_{ik}\varphi_k - \overline{T^i_{jk}}\overline{\varphi}_k). \tag{11}$$

Proof. Under a unitary frame, $\gamma'' = -\overline{{}^t\gamma'}$. So by the last two equations in (10) we get the coefficients of θ_2 and γ' under the frame. □

Lemma 3. σ_1 *and* σ_2 *are globally defined, nonnegative* $(1,1)$ *forms on* M. *The metric* g *is Kähler if and only if any one of the following vanishes identically:* τ, θ_2, γ', σ, σ_1, σ_2. *Also,* $d\sigma_2 = \sqrt{-1}\,tr(\overline{\Theta}_2\theta_2 - \overline{\theta_2}\Theta_2)$.

Proof. Under a frame change $\tilde{e} = Pe$, the matrices $\overline{\theta}_2 \wedge \theta_2$ and $-\gamma' \wedge \gamma''$ are changed into $P\overline{\theta_2} \wedge \theta_2 P^{-1}$ and $-P\gamma' \wedge \gamma'' P^{-1}$, respectively, so their traces,

σ_2 and σ_1, are globally defined $(1,1)$ forms on M. By (11), locally under any unitary frame e, they can be expressed as

$$\sigma_2 = \sqrt{-1} \sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} T_{ij}^{l} \overline{T_{ij}^{k}} \right) \varphi_k \wedge \overline{\varphi}_l, \tag{12}$$

$$\sigma_1 = \sqrt{-1} \sum_{k,l=1}^{n} \left(\sum_{i,j=1}^{n} T_{ik}^{j} \overline{T_{il}^{j}} \right) \varphi_k \wedge \overline{\varphi}_l. \tag{13}$$

Therefore both are everywhere nonnegative, and the vanishing of either of them is equivalent to the vanishing of τ. The identity on $d\sigma_2$ is a direct consequence of (7). □

Next, let us recall the *torsion* 1*-form* η which is defined to be the trace of γ' ([9]). Under any frame e, it has the expression:

$$\eta = \operatorname{tr}(\gamma') = \sum_{i,j=1}^{n} T_{ij}^{i} \varphi_j. \tag{14}$$

A direct computation shows that

$$\partial \omega^{n-1} = -2\eta \wedge \omega^{n-1}. \tag{15}$$

Recall that the metric g is said to be *balanced* if ω^{n-1} is closed. The above identity shows that g is balanced if and only if $\eta = 0$. When $n = 2$, $\eta = 0$ means $\tau = 0$, so balanced complex surfaces are Kähler. But for $n \geq 3$, η contains less information than τ.

Let us conclude this section by pointing out the following fact, which is probably well-known to experts in the field, but we give the outline of proof here for readers' convenience.

Lemma 4. *Given any point* p *in a Hermitian manifold* (M^n, g)*, there exists a unitary frame* e *in a neighborhood of* p *such that* $\theta|_p = 0$.

Proof. First we establish the following claim: Given any $n \times n$ complex matrix X, there exists a C^∞ map f from a small disc D in $\mathbb{C}$ into the unitary group $U(n)$ such that $f(0) = I$ and $\frac{\partial f}{\partial z}|_0 = X$.

To prove the claim, let $P = X - X^*$, $Q = i(X + X^*)$. Both are skew-Hermitian, thus in the Lie algebra of $U(n)$. So there are 1-parameter subgroups ϕ and ψ in $U(n)$ such that $\phi'(0) = P$ and $\psi'(0) = Q$. Now let $f(z) =$

$f(x+iy) = \phi(x)\psi(y)$. We have $f(0) = I$, and

$$\frac{\partial f}{\partial z}|_0 = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)|_0$$
$$= \frac{1}{2}(\phi'(0) - i\psi'(0)) = \frac{1}{2}(P - iQ) = X.$$

Now by taking matrix products, we know there exists a smooth map A from a small neighborhood of p in M^n into $U(n)$, such that $\frac{\partial A}{\partial z_i}|_p = X_i$, $1 \leq i \leq n$, for any prescribed complex $n \times n$ matrices $X_1, \ldots, X_n$.

Take any unitary local frame e near p. Write $\theta|_p = \sum_{i=1}^n(-X_i dz_i + X_i^* d\bar{z}_i)$. Then $\tilde{e} = Ae$ will satisfy $\tilde{\theta}|_p = (A\theta A^{-1} + dAA^{-1})|_p = \theta|_p + dA|_p = 0$. □

3. The Riemannian and Chern curvature tensors

Now we turn our attention to the curvature tensors. Denote by R^h, R the curvature tensor of the Hermitian connection ∇^h or the Riemannian connection ∇, respectively. We have

$$\Theta_{ij} = \sum_{k,l=1}^n R^h_{k\bar{l}i\bar{j}}\ \varphi_k \wedge \overline{\varphi_l}, \tag{16}$$

$$(\Theta_2)_{ij} = \sum_{k,l=1}^n \left(\frac{1}{2}R_{kli\bar{j}}\ \varphi_k \wedge \varphi_l + R_{k\bar{l}i\bar{j}}\ \varphi_k \wedge \overline{\varphi_l}\right), \tag{17}$$

$$(\Theta_1)_{ij} = \sum_{k,l=1}^n \left(\frac{1}{2}R_{kli\bar{j}}\ \varphi_k \wedge \varphi_l + R_{k\bar{l}i\bar{j}}\ \varphi_k \wedge \overline{\varphi_l} + \frac{1}{2}R_{\bar{k}\bar{l}i\bar{j}}\ \overline{\varphi_k} \wedge \overline{\varphi_l}\right). \tag{18}$$

Note that we have

$$R_{\overline{ij}\,\overline{kl}} = R_{ijkl} = 0, \tag{19}$$

because $\Theta_2^{0,2} = 0$ by Lemma 1. This property for general Hermitian metric was discovered by Gray in [12] (Theorem 3.1 on page 603), where it was stated as an equation with 8 real terms. (This perhaps once again illustrates the usefulness of writing things in complex coordinates instead of regarding M as a real manifold with an integrable almost complex structure J.)

From (16), (17), (18), and the definition of Kähler-like and G-Kähler-like in Section 1, it is easy to see that the following hold:

Lemma 5. *Given a Hermitian manifold (M^n, g), g is Kähler-like if and only if ${}^t\Theta \wedge \varphi = 0$, and g is G-Kähler-like if and only if $\Theta_2 = 0$.*

Note that the G-Kähler-like condition is equivalent to

$$R_{xyuv} = R_{xyJuJv}$$

for any real tangent vectors x, y, u, v on M. So this is just the symmetry condition introduced by Gray in [12] (formula (1) on page 605).

By Lemma 5 and (3), g being Kähler-like would mean that under any frame e, we have $d\tau = -\,{}^t\theta \wedge \tau$. By the structure equation (1)-(3), we know that under any unitary frame e, we have

$$\partial\omega = \sqrt{-1}\ {}^t\tau \wedge \overline{\varphi}, \quad \sqrt{-1}\partial\overline{\partial}\omega = {}^t\tau \wedge \overline{\tau} + {}^t\varphi \wedge \Theta \wedge \overline{\varphi}. \tag{20}$$

In particular, when g is Kähler-like, we have $\sqrt{-1}\partial\overline{\partial}\,\omega = \sigma$. In this case, if M^n is compact and admits a positive $(n-2, n-2)$ form χ that is $\partial\overline{\partial}$-closed, then we can integrate $\sigma \wedge \chi$ and conclude that σ must be 0, that is,

Theorem 6. *Let (M^n, g) be a Hermitian manifold that is Kähler-like. If M^n is compact and admits a positive, $\partial\overline{\partial}$-closed $(n-2, n-2)$ form χ, then g is Kähler. In particular, if M^n is compact, Kähler-like, and $\partial\overline{\partial}\,\omega^{n-2} = 0$, then g is Kähler. When $n = 2$, compactness implies that any Kähler-like metric is Kähler.*

In particular, if a compact complex threefold M^3 admits a Kähler-like metric that is non-Kähler, then M^3 can not have any pluriclosed metric (aka SKT metric, or Strongly Kähler with Torsion).

Boothby showed in [1] that any compact Hermitian manifold (M^n, g) with $R^h = 0$ must be a quotient of a complex Lie group. Of course any such manifold will be Kähler-like. One such example is the famous Iwasawa manifold:

Example (Iwasawa Manifold). Consider the complex Lie group G formed by all complex 3×3 matrices X in the form

$$X = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Denote by Γ the discrete subgroup of G of matrices with x, y, z all in $\mathbb{Z}+\sqrt{-1}\mathbb{Z}$. Γ acts on G by left multiplication, leaving the holomorphic 1-forms dx, dy, and $dz - xdy$ invariant. So the three 1-forms descend down to the quotient $M^3 = \Gamma\backslash G$ and form a global frame of the cotangent bundle. Using these three 1-forms to be the unitary frame, we get a Hermitian metric on M^3 that is Hermitian flat, namely, $R^h = 0$.

Note that the above manifold is non-Kähler but balanced. In general, it would be a very interesting problem to classify all compact 3-dimensional complex manifolds that admit non-Kähler, Kähler-like metrics.

Notice that when (M^n, g) is Kähler-like, we have $d\tau = -\,{}^t\theta \wedge \tau$, thus for a given point p in M, if e is a tangent frame such that $\theta|_p = 0$, then $\overline{e_l} T^k_{ij} = 0$. This implies that $\overline{\partial}\gamma' = 0$ at p. By taking trace, we get that $\overline{\partial}\eta = 0$.

Lemma 6. *If a Hermitian manifold (M^n, g) is Kähler-like, then its torsion 1-form η is holomorphic. The converse of this is also true if $n = 2$.*

Next let us consider the G-Kähler-like metrics, namely, those with $\Theta_2 = 0$. Of course we are only interested in those that are non-Kähler. The first G-Kähler-like but non-Kähler metric on compact complex manifold was observed by Gray in [12], on Calabi threefolds discovered in [2].

Example (Calabi threefolds). In 1958 Calabi [2] discovered that $\mathfrak{X} = \mathfrak{X}' \times T^4$, with $\mathfrak{X}'$ a hyperelliptic Riemann surface with odd genus $g \geq 3$ and T^4 a real 4-torus, can be given a complex structure J such that the resulting threefold $(\mathfrak{X}, J)$ admits no Kähler metrics. Later Gray [12] showed that there exists a Hermitian metric which is G-Kähler-like on $(\mathfrak{X}, J)$.

In more details, Calabi [2] proved that any orientable hypersurface M in $\mathbb{R}^7$ has a natural almost complex structure J induced from the space of purely imaginary octonions which is isomorphic to $\mathbb{R}^7$, and (M, g, J), with g the induced Riemannian metric from $\mathbb{R}^7$, has an almost Hermitian structure. It was further proved in [2] (see also Gray [11]) that (M, g, J) is Hermitian if and only if M is a minimal variety in $\mathbb{R}^7$. Based on these results, Calabi began with a compact hyperelliptic Riemann surface $\mathfrak{X}'$, for example, the Riemann surface defined by $\omega^2 = \prod_{i=1}^{8}(z - z_i)$ where z_i are distinct complex numbers, and constructed three linearly independent Abelian differentials which can be used to define an immersion F_1 from $\widetilde{\mathfrak{X}}'$, the universal Abelian covering of $\mathfrak{X}'$ locally into a minimal surface in $\mathbb{R}^3$. Moreover, the covering transformations of $F_1(\widetilde{\mathfrak{X}}') \subset \mathbb{R}^3$ are given by translations in $\mathbb{R}^3$, By the results just mentioned, the immersion $F := F_1 \times Id : \widetilde{\mathfrak{X}}' \times \mathbb{R}^4 \to \mathbb{R}^3 \times \mathbb{R}^4$ produces a Hermtian structure. It can be proved that this complex structure J is invariant under translations in $\mathbb{R}^7$, Therefore, we can descend it to get a compact Hermitian manifold $(\mathfrak{X}, J)$ where $\mathfrak{X} = \mathfrak{X}' \times T^4$. Calabi proved that $(\mathfrak{X}, J)$ does not admit any Kähler metrics. Historically, Calabi threefolds was the first nontrivial example of compact complex manifolds with zero first Chern class which are diffeomorphic to Kähler manifolds but admit no Kähler metrics.

Gray [11] and [12] further investigated the curvature properties of such Hermitian manifolds. His result implies that on $(\mathfrak{X}, g, J)$ with g the the induced Riemannian metric from $\widetilde{\mathfrak{X}}' \times \mathbb{R}^4 \subset \mathbb{R}^7$, one has $R_{xyuv} = R_{xyJuJv}$ for any real tangent vectors x, y, u, v on $\mathfrak{X}$. This means that Calabi threefolds $(\mathfrak{X}, g, J)$ are G-Kähler-like.

In the non-compact case, however, even in dimension 2 there are lots of such examples. For instance, we have the following:

Example (G-Kähler-like surface). Consider the metric g on $\mathbb{C} \times \mathbb{H}$ given by

$$\omega_g = i(-iz_2 + i\overline{z}_2)^2 dz_1 \wedge d\overline{z}_1 + i dz_2 \wedge d\overline{z}_2,$$

where $\mathbb{H}$ is the upper half plane. Write $(-iz_2 + i\overline{z}_2)^2 = \lambda = e^{2u} > 0$ on $\mathbb{H}$. Then under the natural frame of $\{z_1, z_2\}$, we have

$$\theta_1 = \begin{bmatrix} du & -\lambda\overline{\mu} \\ \mu & 0 \end{bmatrix}, \qquad \theta_2 = \begin{bmatrix} 0 & -\lambda\nu \\ \nu & 0 \end{bmatrix},$$
$$\Theta_2 = d\theta_2 - \overline{\theta}_1 \wedge \theta_2 - \theta_2 \wedge \theta_1 = \begin{bmatrix} 0 & -d(\lambda\nu) - \lambda\nu du \\ d\nu - \nu du & 0 \end{bmatrix},$$

where $\mu = u_2 dz_1$, $\nu = u_{\overline{2}} dz_1$. Since $u = \ln(-iz_2 + i\overline{z}_2)$, $u_{\overline{2}2} = -u_{\overline{2}}u_2$, $u_{\overline{2}\overline{2}} = -(u_{\overline{2}})^2$, and $d\lambda = 2\lambda du$, we get $\Theta_2 = 0$, so the metric is G-Kähler-like.

Remark. For a non-compact complex manifold M^n, if g_0 is a Kähler metric on M and f is a holomorphic function M that is nowhere zero, then $g = |f|^2 g_0$ is Kähler-like, and is non-Kähler if $n \geq 2$ and f is not a constant.

Next let us compute the curvatures R and R^h in terms of the torsion components T^i_{jk} and their derivatives. We have the following:

Lemma 7. *Let (M^n, g) be a Hermitian manifold. Under any local unitary frame e, we have*

$$(21)\qquad 2T^k_{ij,\,\bar l} = R^h_{\bar j l i \bar k} - R^h_{i \bar l j \bar k},$$

$$(22)\qquad R_{ijk\bar l} = T^l_{ij,k} + T^l_{ri}T^r_{jk} - T^l_{rj}T^r_{ik},$$

$$(23)\qquad R_{ij\overline{kl}} = T^l_{ij,\bar k} - T^k_{ij,\bar l} + 2T^r_{ij}\overline{T^r_{kl}} + T^k_{ri}\overline{T^j_{rl}} + T^l_{rj}\overline{T^i_{rk}} - T^l_{ri}\overline{T^j_{rk}} - T^k_{rj}\overline{T^i_{rl}},$$

$$(24)\qquad R_{k\bar l i\bar j} = R^h_{k\bar l i\bar j} - T^j_{ik,\bar l} - \overline{T^i_{jl,\bar k}} + T^r_{ik}\overline{T^r_{jl}} - T^j_{rk}\overline{T^i_{rl}} - T^l_{ri}\overline{T^k_{rj}},$$

where the index r is summed over 1 through n, and the indices after the comma denote the covariant derivatives with respect to the Hermitian connection ∇^h.

Proof. Using the structure equations, the Bianchi identities, and Lemma 2, we get the above identities by a straight forward computation. □

As an immediate consequence of Lemma 7, we have the following:

Lemma 8. *Let (M^n, g) be a Hermitian manifold. If g is G-Kähler-like, then*

$$\overline{\partial}\eta \wedge \omega^{n-1} = -\eta \wedge \overline{\eta} \wedge \omega^{n-1}.$$

Proof. Let us fix any point p in M^n and let e be a unitary frame in a neighborhood of p such that $\theta|_p = 0$. Since $\Theta_2 = 0$, by taking $k = i$, $l = j$ in (23) of Lemma 7 and sum them over, we get

$$\sum_{i=1}^n \eta_{i,\bar i} = \sum_{r=1}^n |\eta_r|^2,$$

so Lemma 8 is proved. □

Proof of Theorem 3. Suppose (M^n, g) is a compact Hermitian manifold with $n \geq 2$. By (15), we get

$$\begin{aligned}\overline{\partial}\partial\omega^{n-1} &= \overline{\partial}(-2\eta \wedge \omega^{n-1})\\ &= -2\overline{\partial}\eta \wedge \omega^{n-1} - 4\eta \wedge \overline{\eta} \wedge \omega^{n-1}.\end{aligned}$$

If g is G-Kähler-like, then the above calculation leads to

$$\overline{\partial}\partial\omega^{n-1} = -2\eta \wedge \overline{\eta} \wedge \omega^{n-1} = -2||\eta||^2\omega^n.$$

Now since M is compact, integrating over M would yield $\eta = 0$.

If instead g is Kähler-like, then $\overline{\partial}\eta = 0$ by Lemma 6, and the above argument again yields $\eta = 0$. So in either case (M^n, g) is balanced. $\square$

4. The proofs of Theorems 1 and 2

In this section, we will give proofs to Theorems 1 and 2 stated in the introduction. First let us assume that (M^n, g) is a Hermitian manifold that is both Kähler-like and G-Kähler-like. Fix any point $p \in M$, and let e be a unitary frame near p such that $\theta|_p = 0$. Since g is Kähler-like, we have $R^h_{i\bar{j}k\bar{l}} = R^h_{k\bar{j}i\bar{l}}$, thus at the point p it holds

$$T^k_{ij,\ \bar{l}} = 0.$$

Therefore, by formula (23) in Lemma 7, we know that

Lemma 9. *If a Hermitian manifold (M^n, g) is both Kähler-like and G-Kähler-like, then under any unitary frame the following identity*

$$2\sum_{r=1}^{n} T^r_{ij}\overline{T^r_{kl}} = \sum_{r=1}^{n}\{T^l_{ri}\overline{T^j_{rk}} + T^k_{rj}\overline{T^i_{rl}} - T^k_{ri}\overline{T^j_{rl}} - T^l_{rj}\overline{T^i_{rk}}\} \tag{25}$$

holds for any indices i, j, k, l. In particular, M must be balanced.

Proof. We are only left to prove the last statement, namely, the torsion 1-form η is zero. By taking $k = i$, $l = j$ in (25) and sum over i and j, we get

$$2||T||^2 = 2||T||^2 - 2||\eta||^2,$$

where $||T||^2 = \sum_{i,j,k=1}^n |T^k_{ij}|^2$ and $||\eta||^2 = \sum_{k=1}^n |\sum_{i=1}^n T^i_{ik}|^2$. Hence $\eta = 0$. $\square$

Proof of Theorem 1. Let (M^n, g) be a Hermitian manifold such that the Riemannian curvature tensor R and the Hermitian curvature tensor R^h are equal to each other. Then both R and R^h satisfy all symmetry conditions of the curvature tensor of a Kähler manifold, so g will be both Kähler-like and G-Kähler-like. Thus by Lemma 9, we know that the formula (25) holds. Fix any p in M and choose a unitary frame e such that $\theta|_p = 0$. Since $R = R^h$, the formula (24) in Lemma 7 gives

$$\sum_{r=1}^{n} T_{ik}^{r}\overline{T_{jl}^{r}} = \sum_{r=1}^{n} \{T_{rk}^{j}\overline{T_{rl}^{i}} + T_{ri}^{l}\overline{T_{rj}^{k}}\} \tag{26}$$

for any indices i, j, k, l. By letting $j=i$, $l=k$ in (26), we get

$$\sum_{r=1}^{n} |T_{ik}^{r}|^2 = \sum_{r=1}^{n} \{|T_{rk}^{i}|^2 + |T_{ri}^{k}|^2\}.$$

If we sum over i and k, it leads to $||T||^2 = 2||T||^2$, hence $T=0$ and g is Kähler. This completes the proof of Theorem 1. □

Proof of Theorem 2. Now let us consider a Hermitian manifold (M^n, g) which is both Kähler-like and G-Kähler-like. By Lemma 9, we know that (25) holds and g is balanced. Letting $k=i$ and $l=j$ in (25), we get

$$2\sum_{r=1}^{n} |T_{ij}^{r}|^2 = \sum_{r=1}^{n} \{|T_{rj}^{i}|^2 + |T_{ri}^{j}|^2 - 2Re(\overline{T_{ri}^{i}}T_{rj}^{j})\}. \tag{27}$$

Also, by formula (22) in Lemma (7), we get $T_{ij,k}^{l} = \sum_{r=1}^{n}\{-T_{ri}^{l}T_{jk}^{r} + T_{rj}^{l}T_{ik}^{r}\}$ for any indices i, j, k, and l. Letting $l=j=s$ and sum over s, and using the fact that the metric is a balanced one, we get

$$\sum_{r,s=1}^{n} T_{ri}^{s}T_{sk}^{r} = 0 \tag{28}$$

for any indices i and k.

Now we are ready to prove Theorem 2. When $n=2$, balanced metrics are Kähler, so g is already Kähler. Now assume that $n=3$. Let (ijk) be any cyclic permutation of (123). Write $a_i = T_{jk}^{i}$, $b_i = T_{ij}^{j} = -T_{ik}^{k}$. The last equality holds true because of the fact that $\eta = 0$.

Since $T_{ij}^{k} = -T_{ji}^{k}$, the identities (27) and (28) lead to

$$|a_i|^2 + |a_j|^2 - 2|a_k|^2 = |b_i|^2 + |b_j|^2 - 2|b_k|^2, \tag{29}$$

$$b_i b_j = b_k a_k, \tag{30}$$

$$a_i a_j = b_k^2 \tag{31}$$

whenever (ijk) is a cyclic permutation of (123). Note that (31) is obtained by letting $i=k$ in (28) first.

If one of the b_i is zero, say, $b_1 = 0$, then by (30), $b_2b_3 = 0$. Without loss of generality, let us assume that $b_2 = 0$. If $b_3 \neq 0$, then by (30) $b_1b_2 = b_3a_3$ so $a_3 = 0$, and by (31), $a_1a_2 = b_3^2 \neq 0$. But then (29) gives $|a_1|^2 + |a_2|^2 = -2|b_3|^2$, a contradiction. So we must have $b_3 = 0$ as well. In this case, (31) implies that at least two of the a_i's must be zero, while (29) implies that the third one is also zero. So all a_i and b_i are zero, that is, $T = 0$, thus g is Kähler.

Now assume that $b_1b_2b_3 \neq 0$. Then by (30), we have $a_k = \frac{b_ib_j}{b_k}$. By letting $l = i$ in formula (25) in Lemma 9, we get through a direct computation that

$$b_j\overline{b_k} + b_i\overline{a_j} + a_k\overline{b_i} = 0.$$

Plugging in $a_j = (b_ib_k)/b_j$ and $a_k = (b_ib_j)/b_k$, we get

$$b_j\overline{b_k}\left(1 + \frac{|b_i|^2}{|b_j|^2} + \frac{|b_i|^2}{|b_k|^2}\right) = 0,$$

contradicting to the assumption that b_i's are non-zero. This completes the proof of Theorem 2 for the case when $n = 3$.

Now let us assume that $n \geq 2$ is arbitrary but M^n is compact. Let e be a local unitary frame. For a tensor P_{ij}^k of type (1,2), we will denote by

$$P_{ij,l}^k \quad \text{and} \quad P_{ij,\bar{l}}^k$$

the covariant differentiation of P with respect to the Hermitian connection ∇^h. So if e is a frame such that $\theta|_p = 0$ at the fixed point p, then at the point p we have $P_{ij,l}^k = e_l(P_{ij}^k)$ and $P_{ij,\bar{l}}^k = \overline{e}_l(P_{ij}^k)$. In particular, by the fact that g is both Kähler-like and G-Kähler-like, we get formula (25) and its special case (27), as well as the following

$$T_{ij,\bar{l}}^k = 0, \tag{32}$$

$$T_{ij,l}^k = \sum_{r=1}^n \{-T_{ri}^kT_{jl}^r + T_{rj}^kT_{il}^r\} \tag{33}$$

for any indices i, j, k, and l. Now we use the assumption that M is compact. Note that if f is any smooth function on M^n such that $Lf := \sum_{l=1}^n f_{,l\bar{l}} \geq 0$ everywhere, then by Bochner's Lemma (see [1]), $Lf = 0$ everywhere, and f is a constant. Consider the function $f = \sum_{i,j,k=1}^n |T_{ij}^k|^2$ under any unitary frame e. Then we have

$$\begin{aligned} Lf &= (T^k_{ij,l}\overline{T^k_{ij}})_{,\bar{l}} = |T^k_{ij,l}|^2 + (T^k_{ij,l})_{,\bar{l}}\,\overline{T^k_{ij}} \\ &= |T^k_{ij,l}|^2 + (-T^k_{ri}T^r_{jl} + T^k_{rj}T^r_{il})_{,\bar{l}}\overline{T^k_{ij}} = |T^k_{ij,l}|^2, \end{aligned}$$

where the third equality above is because of (33), while the others are because of (32). So we have $T^k_{ij,l} = 0$ and

$$\sum_{r=1}^{n} T^k_{ri}T^r_{jl} = \sum_{r=1}^{n} T^k_{rj}T^r_{il} \tag{34}$$

for any indices i, j, k, and l.

Write $V = T^{1,0}_p M$ for a given point $p\in M$, and then for any $X = \sum_i X_i e_i \in V$, let us denote by A_X the $n\times n$ matrix $(\sum_i X_i T^k_{ij})^n_{j,k=1}$, which represents a linear transformation from V into itself. By multiplying X_iX_l onto (34) and adding up i and l, we get $(A_X)^2 = 0$. Also, for $Y = \sum_i Y_i e_i$ in V, if we respectively multiplying X_iY_l or X_lY_i onto (34) and adding up i and l, we get $A_XA_Y = -A_YA_X$.

Claim 1: There exists $W\in V$ such that $A_Y(W) = 0$ for any Y in V.

To see this, it suffices to prove the following slightly more general statement about anti-commutative system of step-2 nilpotent matrices:

Claim 2: For any given integer m, Let $\{A_1,\ldots,A_m\}$ be a set of $n\times n$ complex matrices satisfying the condition

$$A_iA_j = -A_jA_i, \qquad \forall\ 1\le i,j\le m. \tag{35}$$

Then $\bigcap_{i=1}^m N(A_i)\neq 0$, where $N(A_i)$ denotes the kernel of A_i.

We will use induction on n to prove Claim 2. We may assume that these A_i are linearly independent, as otherwise we could just reduce the number m. When $n = 2$, since $A_1^2 = 0$, there exists non-singular 2×2 matrix P such that $PA_1P^{-1} = E$, where

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For any $i\ge 2$, since PA_iP^{-1} is nilpotent and anti-commutative with E, it must be in the form a_iE for some constant a_i. So all these A_i have common kernel.

For general n, let us assume that A_1 has the largest rank among all linear combinations of these A_i. By a base change, we know that there exists a non-singular matrix P such that PA_1P^{-1} takes the form

$$PA_1P^{-1} = \begin{bmatrix} 0 & 0 & I_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where I_k is the identity matrix and $k = \text{rank}(A_1)$ where $2k \leq n$. For any $i \geq 2$, since the rank of $\lambda A_1 + A_i$ is at most k for any $\lambda \in \mathbb{C}$, we know that the lower left corner of PA_iP^{-1} must be zero:

$$PA_iP^{-1} = \begin{bmatrix} B_i & * & * \\ 0 & 0 & * \\ 0 & 0 & -B_i \end{bmatrix}.$$

(The lower right corner is $-B_i$ because $A_1A_i = -A_iA_1$.) Note that these $k \times k$ matrices $\{B_i\}$ also satisfy (35), so by induction on n, we know that these B_i, thus all the A_i, will have a common kernel. This proves Claim 2, hence Claim 1.

There is an alternative proof of Claim 2, which is constructive in nature and might be interesting in its own right. [2] The proof goes as follows:

Let us suppose that dim $V = n$ and rank$A_X = k > 0$, then $n = 2k + l$, with $l \neq 0$. In this situation we can find a basis $\{v_1, \ldots, v_k, x_1, \ldots, x_l, y_1, \ldots, y_k\}$ of V such that $\{v_1, \ldots, v_k\}$ is a basis for $V_1 = \text{Im}A_X$, $x_1 = X$, and such that $A_X y_i = v_i$ and $\{v_1, \ldots, v_k, x_1, \ldots, x_l\}$ is a basis for kerA_X. Note that in our situation we have $(A_Y)^2 = 0$, $A_YA_Z = -A_ZA_Y$, and $A_YZ = -A_ZY$. Of course, $A_{\mu Y+\nu Z} = \mu A_Y + \nu A_Z$. We take $W_1 = A_{y_1}A_{y_2} \cdots A_{y_{k-1}}(v_k)$. Because $A_{y_i}v_j = A_{y_i}A_Xy_j = A_XA_{y_j}y_i = -A_{y_j}v_i$, $i \neq j$, note that

$$W_1 = (-1)^{k-i}A_{y_1} \cdots A_{y_{i-1}}A_{y_{i+1}} \cdots A_{y_k}(v_i).$$

Also note that $A_{v_i}v_j = A_{v_i}A_Xy_j = -A_{v_i}A_{y_j}X = A_{y_j}A_{v_i}X = -A_{y_j}A_Xv_i = 0$. Moreover, $A_{x_i}v_j = A_{x_i}A_Xy_j = -A_{y_j}A_Xv_i = 0$. Therefore $A_{v_i}W_1 = 0$ and $A_{x_i}W_1 = 0$. Another remark is that $A_{y_i}v_i = 0$. Therefore, $A_{y_i}W_1 = 0$. In summary, $A_ZW_1 = 0$, for all $Z \in V$. If $W_1 \neq 0$, we would have proved Claim 2.

If $W_1 = 0$, we choose a set of $k - 1$ indices. For instance, we consider the indices $1, \ldots, k-2, k-1$ and take the element $W_2 = A_{y_1}A_{y_2} \cdots A_{y_{k-2}}(v_{k-1})$. We already have $A_{y_k}W_2 = 0$, because $W_1 = 0$. Moreover, as before, $A_{v_i}W_2 =$

[2] The authors are indebted to an anonymous referee for suggesting the alternative proof.

0, $A_{x_j}W_2 = 0$ and $A_{y_i}W_2 = 0$, for $i = 1;, \dots, k$ and $j = 1, \dots, l$. If $W_2 \neq 0$ for some set of $k-1$ indices, we would have proved the claim. If $W_2 = 0$ for all set of $k-1$ indices, we would choose a set of $k-2$ indices, etc. For some set of $k-j$ indices, we have to obtain $W_{j+1} \neq 0$. In the worst situation, we would have done k steps and $W_{k+1} = v_1 \neq 0$. Because of the previous steps $A_Z v_1 = 0$, for all $Z \in V$ and the claim would be proved.

Now that Claim 1 is established, there exists non-zero tangent vector X in $T^{1,0}_p M$ such that $T^k_{Xj} = 0$ for any j, k. Let us choose unitary frame e so that X is parallel to e_n. Then we have $T^k_{nj} = 0$ for any j, k. Let $i = n$ in (27), we get

$$\sum_{r=1}^{n} |T^n_{rj}|^2 = 0$$

for any j. Therefore $T^n_{jk} = 0$ for any j, k. That is, the components of the torsion tensor $T^k_{ij} = 0$ whenever any of the indices is n. Repeating this argument, we conclude that $T^k_{ij} = 0$ whenever any of the indices is greater than 2. Then T must be 0 since g is balanced, and this completes the proof that g must be Kähler. □

5. The conformal change of metrics and bisectional curvatures

Let (M^n, g) be a Hermitian manifold, $u \in C^\infty(M)$ a real-valued smooth function, and $\tilde{g} = e^{2u} g$ a conformal change of the metric.

Let e be (the column vector of) a local unitary frame of g, with (the column vector of) the dual coframe φ. Then $\tilde{e} = e^{-u} e$ and $\tilde{\varphi} = e^u \varphi$ are local unitary frame and coframe with respect to the metric $\tilde{g}$.

Denote by $\tilde{\theta}$ and $\tilde{\Theta}$ the matrix of Hermitian connection and Hermitian curvature of the metric $\tilde{g}$ with respect to the unitary frame $\tilde{e}$, then it is easy to see that

$$\tilde{\theta} = \theta + (\partial u - \overline{\partial} u) I, \qquad \tilde{\Theta} = \Theta - 2\partial\overline{\partial} u I,$$

where θ and Θ are the matrix of Hermitian connection and Hermitian curvature of g under e. From that, we get

$$\tilde{\tau} = e^u(\tau + 2\partial u \wedge \varphi) \quad \text{and}$$

$$e^u \tilde{T}^i_{jk} = T^i_{jk} + u_j \delta_{ik} - u_k \delta_{ij} \tag{36}$$

where $u_j = e_j(u)$. Using Lemma 2, we get the following:

Lemma 10. *Let e, $\tilde{e} = e^{-u}e$ be the local unitary frames for g and $\tilde{g} = e^{2u}g$, respectively. Then the connection matrixes are related as*

$$\tilde{\theta}_1 = \theta_1 + v\ {}^t\varphi - \overline{\varphi}\ v^*, \tag{37}$$

$$\tilde{\theta}_2 = \theta_2 + \overline{v}\ {}^t\varphi - \varphi\ v^* \tag{38}$$

where $v = {}^t(u_1, \ldots, u_n)$.

Now we are ready to prove Theorem 4.

Proof of Theorem 4. When M^n is a compact complex manifold, by Theorem 3, Kähler-like or G-Kähler-like metrics are balanced, and balanced metrics are clearly unique (up to constant multiples) within each conformal class, so each conformal class of Hermitian metrics on M^n can contain at most one (up to constant multiples) Kähler-like or G-Kähler-like metric.

Now assume that (M^n, g) is a non-compact Hermitian manifold. Let u be a real-valued smooth function on M^n and $\tilde{g} = e^{2u}g$ be a conformal change of g. As in the above, let e be a local unitary frame of g with dual coframe φ, then $\tilde{e} = e^{-u}e$ and $\tilde{\varphi} = e^u\varphi$ are unitary frame and coframe for $\tilde{g}$. We have $\tilde{\Theta} = \Theta - 2\partial\overline{\partial}uI$. So when g is Kähler-like, which means that ${}^t\Theta \wedge \varphi = 0$, the metric $\tilde{g}$ will be Kähler-like if and only ${}^t\tilde{\Theta} \wedge \tilde{\varphi} = 0$, which is equivalent to $\partial\overline{\partial}u = 0$.

Next let us assume that $\tilde{\Theta}_2 - \Theta_2 = 0$. By Lemma 10 and a somewhat lengthy but straight forward computation, we get the following equations for $\lambda = e^{-u}$:

$$e_i(\lambda_j) - \theta_{jk}(e_i)\lambda_k + T^k_{ij}\lambda_k = 0,$$
$$\overline{e}_i(\lambda_j) - \theta_{jk}(\overline{e}_i)\lambda_k - \overline{T}^j_{ik}\lambda_k - T^i_{jk}\overline{\lambda}_k = 2\delta_{ij}|\lambda_k|^2/\lambda$$

for any i, j. Note that the index k is summed up in the above identities. Let H_λ be the Hessian of the function λ with respect to the Riemannian metric g, then the above equations are simply saying that

$$H_\lambda(X, Y) = 0, \qquad \lambda H_\lambda(X, \overline{Y}) = \langle X, \overline{Y}\rangle |\nabla\lambda|^2$$

for any type $(1, 0)$ tangent vectors X and Y. In particular, one has $\lambda\Delta\lambda = n|\nabla\lambda|^2$, and $\Delta e^{(n-1)u} = 0$, where $\Delta\lambda$ and $\nabla\lambda$ are the Laplacian and gradient of λ with respect to the Riemannian metric g. This completes the proof of Theorem 4. □

As a consequence, since there is no non-constant positive harmonic function on the Euclidean space, we know any Hermitian metric conformal to the complex Euclidean metric g_0 on $\mathbb{C}^n$ cannot be G-Kähler-like unless it is a constant multiple of g_0. The same is true for any G-Kähler-like manifold (M^n, g) that is complete and with nonnegative Ricci curvature, for exactly the same reason.

On the other hand, by Theorem 4, we could draw the following conclusion:

Example (G-Kähler-like metrics conformal to the Euclidean metric). Let $M^n \subset \mathbb{C}^n$ be an open subset not equal to $\mathbb{C}^n$. Let g_0 be the restriction on M^n of the complex Euclidean metric. For any $p \in \mathbb{C}^n \setminus M$, one can check directly that the metric $\tilde{g} = \frac{1}{|z-p|^4} g_0$ on M^n is G-Kähler-like. Conversely, if $\tilde{g} = e^{2u} g_0$ is G-Kähler-like on M^n, then by Theorem 4, we know that the function $\lambda = e^{-u}$ satisfies

$$\frac{\partial^2 \lambda}{\partial z_i \partial z_j} = 0, \qquad \lambda \frac{\partial^2 \lambda}{\partial z_i \partial \overline{z}_j} = 2\delta_{ij} \sum_{k=1}^{n} |\lambda_i|^2.$$

From this it follows that there must be a constant $c > 0$ and a point $p \in \mathbb{C}^n \setminus M$ such that $\lambda = c|z-p|^2$, hence $e^{2u} = \frac{1}{c^2 |z-p|^4}$.

Our next goal is to introduce the right notion of bisectional curvature and holomorphic sectional curvature. The novelty here is only the definition of (Riemannian) bisectional curvature.

We have two natural candidates for defining the Riemannian bisectional curvature, namely, $R_{X\overline{X}Y\overline{Y}}$ and $R_{X\overline{Y}Y\overline{X}}$. In the Kähler case, or more generally the G-Kähler-like case, they are equal to each other, and in general, their difference is

$$R_{XY\overline{XY}} = R_{X\overline{X}Y\overline{Y}} - R_{X\overline{Y}Y\overline{X}}.$$

This gives us a one-parameter family of choices of Riemannian bisectional curvature B_a for any real number a:

Definition (Bisectional curvatures). *Given a Hermitian manifold (M^n, g), and given any two non-zero type $(1,0)$ tangent vectors X, Y at p in M, the (Hermitian) bisectional curvature $B^h(X,Y)$ and the Riemannian bisectional curvature $B_a(X,Y)$ in the directions of X and Y are defined as*

$$B^h(X,Y) = \frac{R^h_{X\overline{X}Y\overline{Y}}}{|X|^2 |Y|^2}, \qquad B_a(X,Y) = \frac{a R_{X\overline{X}Y\overline{Y}} + (1-a) R_{X\overline{Y}Y\overline{X}}}{|X|^2 |Y|^2}.$$

The (Hermitian) holomorphic sectional curvature and Riemannian holomorphic sectional curvature in the direction of X are defined by $H^h(X) = B^h(X,X)$ and $H(X) = B_a(X,X)$, respectively.

Note that $B_a(X,Y)$ and $B^h(X,Y)$ are both real valued, and $B_a(X,Y) = B_a(Y,X)$, but in general $B^h(X,Y) \neq B^h(Y,X)$. When the metric is Kähler-like, B^h is symmetric, and when the metric is G-Kähler-like, B_a is independent of a.

The Riemannian bisectional curvature B_a gives us a couple of Ricci type curvature tensor:

$$Ric_a(X) = \sum_{i=1}^{n} B_a(X,e_i) = aRic_1(X) + (1-a)Ric_0(X)$$

where e is a unitary frame. Clear they are independent of the choice of the unitary frame.

Lemma 11. *On a Hermitian manifold (M^n,g), if $X = \frac{1}{\sqrt{2}}(u - iJu)$ and $Y = \frac{1}{\sqrt{2}}(v - iJv)$, where u and v are real tangent vectors, then we have*

$$-R_{X\overline{X}Y\overline{Y}} + 2R_{X\overline{Y}Y\overline{X}} = -\frac{1}{2}\{R(u,v) + R(Ju,Jv) + R(Ju,v) + R(u,Jv)\}$$

where $R(u,v)$ stands for R_{uvuv}. Therefore

$$B_{-1}(X,Y) = \frac{1}{2}\sin^2\phi_{uv}\{K_{u\wedge v} + K_{Ju\wedge Jv}\} + \frac{1}{2}\sin^2\phi_{uJv}\{K_{Ju\wedge v} + K_{u\wedge Jv}\}$$

where $K_{u\wedge v} = -R(u,v)/|u\wedge v|^2$ is the sectional curvature of the plane spanned by u and v, and ϕ_{uv} denotes the angle between u and v. In particular, if (M^n,g) has positive (negative, nonnegative, or nonpositive) sectional curvature, then it will have positive (negative, nonnegative, or nonpositive) Riemannian bisectional curvature B_{-1}.

Proof. A straightforward computation leads to the above identities. □

In particular, we have

$$Ric_{-1}(X) = -Ric_0(X) + 2Ric_1(X) = \frac{1}{2}\{Ric(u) + Ric(Ju)\} \tag{39}$$

where $Ric(u)$ stands for the Ricci curvature in the direction of u. This means that in the non-Kähler case, the trace of the Riemannian bisectional curvature B_{-1} is only the J-invariant part of the Ricci curvature, which may not control the full Ricci curvature tensor, even though the scalar curvature is controlled by it:

$$\sum_{i,j=1}^{n} B_{-1}(e_i, e_j) = \frac{1}{2} Scal \tag{40}$$

where $\{e_i\}$ is any unitary frame and $Scal$ stands for the scalar curvature of the Riemannian metric g.

Next we want to examine the relationship between $B_a(X,Y)$ and $B^h(X,Y)$. As a direct consequence of the definitions and Lemma 7, we get through a direct computation that the following holds:

Theorem 7. *For any type $(1,0)$ tangent vectors X, Y at a point p in a Hermitian manifold (M^n, g), it holds*

$$\frac{1}{2}(R^h_{X\overline{X}Y\overline{Y}} + R^h_{Y\overline{Y}X\overline{X}}) - R_{X\overline{Y}Y\overline{X}} = \sum_{k=1}^{n} \{|T^k_{XY}|^2 + 2Re(T^Y_{kY}\overline{T^X_{kX}})\}, \tag{41}$$

$$R^h_{X\overline{Y}Y\overline{X}} - R_{X\overline{X}Y\overline{Y}} = \sum_{k=1}^{n} \{|T^Y_{kX}|^2 + |T^X_{kY}|^2 - |T^k_{XY}|^2\}, \tag{42}$$

where $\{e_i\}$ is a unitary frame and

$$T^X_{YZ} = \sum_{i,j,k=1}^{n} T^i_{jk}\overline{X}_i Y_j Z_k, \quad X = \sum_{i=1}^{n} X_i e_i, Y = \sum_{i=1}^{n} Y_i e_i, \quad \textit{and} \quad Z = \sum_{i=1}^{n} Z_i e_i.$$

In particular, the holomorphic sectional curvature satisfies the monotonicity condition

$$R^h_{X\overline{X}X\overline{X}} - R_{X\overline{X}X\overline{X}} = 2\sum_{k=1}^{n} |T^X_{kX}|^2 \geq 0. \tag{43}$$

Moreover, if the equality always holds, then $T = 0$ and g is Kähler.

Notice that if we write $x = \frac{1}{2}(X + \overline{X})$ and $y = \frac{1}{2}(Y + \overline{Y})$, then

$$\sum_{k=1}^{n} |T^X_{kY}|^2 = 2||(\nabla_x J)(y)||^2.$$

So the difference between the holomorphic sectional curvatures is measured by the norm square of the covariant differentiation of the almost complex structure. Note that $\nabla J = 0$ means that g is Kähler.

Formula 41 is particularly interesting. It says that the difference between the symmetrized Hermitian bisectional curvature and the Riemannian bisectional curvature B_0 is a quadratic expression of the torsion tensor, and it does not involve the derivatives of the torsion. Perhaps we should use B_0 to be the Riemannian bisectional curvature, even though it is not clear to us whether B_0 can be expressed as a positive linear combination of sectional curvature terms as in the Kähler case.

For Ricci curvature tensors, Liu and Yang wrote a nice paper recently [17] in which they systematically studied all 6 possible Ricci tensors, and wrote down their explicit relationship. So we will not get into Ricci or scalar curvature here.

To close this article, let us leave the readers with the following vague question, namely, can we further study Kähler-like and G-Kähler-like metrics on compact non-Kählerian complex manifold of dimension 3 that is Calabi-Yau, that is, with trivial canonical line bundle and finite fundamental group? Is there a role that Kähler-like or G-Kähler-like metrics can play in the Strominger system ([20], [5]) on such manifolds?

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