Communications in
Analysis and Geometry
Volume 29, Number 1, 183–206, 2021

Shadows of graphical mean curvature flow

Wolfgang Maurer

We consider mean curvature flow of an initial surface that is the graph of a function over some domain of definition in $\mathbb{R}^n$. If the graph is not complete then we impose a constant Dirichlet boundary condition at the boundary of the surface. We establish longtime-existence of the flow and investigate the projection of the flowing surface onto $\mathbb{R}^n$, the shadow of the flow. This moving shadow can be seen as a weak solution for mean curvature flow of hypersurfaces in $\mathbb{R}^n$ with a Dirichlet boundary condition.

Furthermore, we provide a lemma of independent interest to locally mollify the boundary of an intersection of two smooth open sets in a way that respects curvature conditions.

1. Introduction

A family $(M_t)_{t\in(0,T)}$ of hypersurfaces of $\mathbb{R}^{n+1}$ is said to move by mean curvature flow if there is a map $X\colon M\times(0,T)\to\mathbb{R}^{n+1}$ such that $X(\cdot,t)$ is an immersion for all $t\in(0,T)$ with $X(M,t)=M_t$ and X solves

$$\frac{\mathrm{d}}{\mathrm{d}t}X(p,t)=-H(p,t)\,\nu(p,t),$$

where M is a n-dimensional manifold, $H(\cdot,t)$ is the mean curvature of M_t and $\nu(\cdot,t)$ its normal, such that $-H\nu$ is the mean curvature vector. If $M_t=\operatorname{graph} u(\cdot,t)$ for a family of functions $u(\cdot,t)\colon\Omega\subset\mathbb{R}^n\to\mathbb{R}$, then M_t moves by mean curvature flow if and only if u solves the graphical mean curvature flow equation, which is the parabolic partial differential equation

$$\text{(GMCF)}\qquad \frac{\mathrm{d}}{\mathrm{d}t}u=\sqrt{1+|Du|^2}\,\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

Graphical mean curvature flow was studied, e.g., in [5] by Ecker and Huisken. They proved long-time existence for the mean curvature flow of entire graphs, i.e. graphs of functions defined on all of $\mathbb{R}^n$, and showed that the solution stays graphical for all time. More recently, Sáez Trumper and

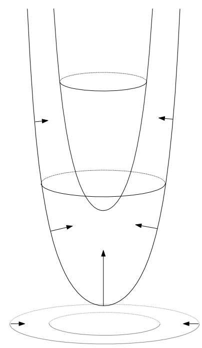

Figure 1: A rotationally symmetric "mean curvature flow without singularities" where we consider a graph over a ball. The graph moves by mean curvature flow and the shadow, i.e. the projection onto $\mathbb{R}^n$, evolves by mean curvature flow too, but in a weak sense. In this setting the graph will disappear to infinity in finite time, while the shadow develops a point singularity. Generally, any singularity occurring on the shadow-level will happen at infinity on the graph-level.

Schnürer proved in [11] a long-time existence result for complete graphs. Starting from an open set Ω_0 and a proper function $u_0\colon \Omega \to \mathbb{R}_+$, they showed the existence of a solution u to graphical mean curvature flow with initial data u_0, where $u(\cdot, t)$ is defined on an open set Ω_t for $t \geq 0$. This solution will not develop singularities on a finite level but it can disappear to infinity forming a singularity at infinity-level. It was observed that the sets $\partial\Omega_t$ can be interpreted as a weak solution to mean curvature flow, starting from $\partial\Omega_0$, and that it coincides almost everywhere with the level-set flow as long as the latter does not fatten (Figure 1).

In [3] and [4] existence results analogous to that of Sáez and Schnürer are proven for some fully nonlinear flows of non-compact convex surfaces. In [10] the situation is considered for very general normal velocities of homogeneity one.

In this article we consider graphical mean curvature flow with a Dirichlet boundary condition. The case of boundary conditions in this setting has not been dealt with before.

To avoid the need of carefully keeping track of the domain of definition we work with $[-\infty, \infty]$-valued functions u. Then the Dirichlet boundary value problem we consider can be phrased as follows: Given an open, smooth, and mean convex domain $\Omega \subset \mathbb{R}^n$ and an initial datum $u_0\colon \overline{\Omega} \to [-\infty, \infty]$, find a

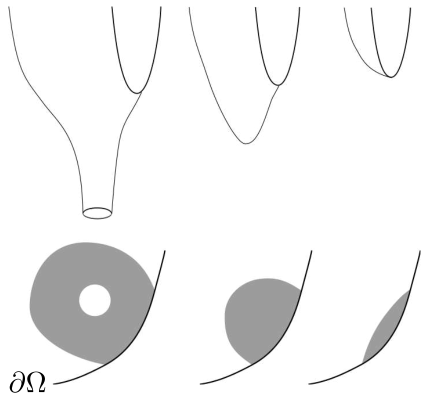

Figure 2: A "mean curvature flow without singularities" with Dirichlet boundary condition. The part of the graph going to $-\infty$ collapses in finite time.

continuous function $u\colon \bar{\Omega} \times [0, \infty) \to [-\infty, \infty]$ with the following properties:

$$\begin{cases} u \text{ solves GMCF} & \text{on } \{|u| < \infty\}, \\ u(x,t) = u_0(x) & \text{for } (x,t) \in \mathcal{P}(\Omega \times [0,\infty)). \end{cases} \tag{1.1}$$

Here $\mathcal{P}(\Omega \times [0, \infty))$ denotes the parabolic boundary of $\Omega \times [0, \infty)$, which is $\Omega \times \{0\} \cup \partial\Omega \times [0, \infty)$.

More geometrically speaking, the finite part of the graph of u, i.e. $\operatorname{graph} u \cap \bar{\Omega} \times [0, \infty) \times \mathbb{R}$, can be interpreted as an evolving (non-compact) hypersurface of $\mathbb{R}^{n+1}$ with boundary situated on $\partial\Omega \times \mathbb{R}$. It evolves by mean curvature flow and the boundary keeps fixed (cf. Figure 2).

A more precise formulation of the initial boundary value problem and the conditions needed for its solution in this article is provided in Theorem 2.1. It may be of interest that even in the case without boundary conditions ($\Omega = \mathbb{R}^n$) Theorem 2.1 is a slight generalization of the corresponding theorem in [11]. This is due to the fact that in that article they use the height function as a cut-off function. Instead one may use cut-off functions as in [5], thereby dropping the need for the assumptions of properness and boundedness from below in [11]. But as the author learned that was suggested to O. Schnürer by J. Metzger before.

As in the case without boundary in [11], the projections (shadows) of the graphical hypersurfaces are weak solutions to mean curvature flow. In the case of boundary conditions we prove that the shadows form weak solutions of mean curvature flow with Dirichlet boundary conditions: One can think of the boundary of the shadow as a (generalized) hypersurface of $\mathbb{R}^n$ that moves by mean curvature flow and whose boundary is held fixed on $\partial\Omega$ (cf. Figure 2). A precise formulation is given in Theorem 3.6.

At this stage a necessary condition on Ω already becomes apparent. Namely, we need to impose that $\partial\Omega$ has non-negative mean curvature $H[\partial\Omega] \geq 0$. Otherwise solutions of mean curvature flow starting from a hypersurface inside Ω with boundary held fixed on $\partial\Omega$ need not stay inside Ω. On the other hand it is well-known that even in the classical case of bounded Ω and bounded functions the initial boundary value problem (1.1) admits a solution for arbitrary boundary values if and only if $H[\partial\Omega] \geq 0$ ([9, Theorem 3], [7, Theorem 2.1]).

In section 2 the initial boundary value problem (1.1) is solved (Theorem 2.1). The technique used is the same as in [11]: It involves an approximation of the problem by bounded auxiliary problems and uses an Arzelà-Ascoli-argument and a priori estimates to pass to a limit. But in contrast to [11] we must prove local estimates at the boundary which we focus on in section 2. Also the approximation process becomes more difficult. The tool used here is provided by section 4.

In Section 3, we interpret the shadow as a weak solution to mean curvature flow of hypersurfaces in Ω with Dirichlet boundary condition on $\partial\Omega$ using a notion of weak solution based on the avoidance principle. Roughly, the definition of weak solution implies that any classical solution starting inside the shadow stays inside and any classical solution starting outside stays outside.

Finally, in section 4 we prove a lemma (Theorem 4.1) of independent interest, which allows to smooth out intersections of two sets. The versatile use of that result throughout the article shall demonstrate its usefulness. An intersection of two smooth open sets is (in general) not smooth at the intersection of the boundaries. Often times it is of interest to locally mollify intersections of smooth open sets at the intersections of their boundaries while preserving curvature conditions given by a curvature cone.

Definition 1.1. A *curvature cone* $\Gamma \subset \mathbb{R}^{n-1}$ is a convex cone which contains the open positive cone $\{x \in \mathbb{R}^{n-1} : x_i > 0 \text{ for all } 1 \leq i \leq n-1\}$ and is symmetric. Here symmetric means invariant when interchanging coordinates $x_i \leftrightarrow x_j$.

Many interesting conditions depending on the principal curvatures $\kappa_1, \ldots, \kappa_{n-1}$ of a hypersurface of $\mathbb{R}^n$ can be put in the form $(\kappa_1, \ldots, \kappa_{n-1}) \in \Gamma$ for an appropriate curvature cone Γ. Examples include convexity, mean convexity, k-convexity, and the condition $|A| \le CH$, where we denote by $|A|$ the norm of the second fundamental form. The case $\Gamma = \mathbb{R}^{n-1}$ is possible, too, and corresponds to "no conditions on the curvatures".

The problem of mollifying inside curvature cones was an open problem from the problem section of the conference *Geometric evolution equations* which took place in Konstanz in 2011. The proof of 4.1 uses distance functions to the boundaries and a mollified version of the minimum of these. It can be read independently of the other sections and is applied most notably in Section 2 in the approximation process. The author expects that this result is of great use and will be widely applicable.

This article emerged from the author's master thesis and is meant to gather the main results. The author wishes to thank Oliver Schnürer for supervising the thesis and for his great support. The author also likes to thank Ben Lambert for helpful advice.

2. Solution of the initial boundary value problem

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be open, smooth, and mean convex, that is, its boundary $\partial\Omega \in C^\infty$ has non-negative mean curvature $H[\partial\Omega] \ge 0$. Let $u_0 \colon \overline{\Omega} \to [-\infty, \infty]$ be continuous and assume u_0 is locally Lipschitz in the set $\{x \in \overline{\Omega} \colon |u_0(x)| < \infty\}$ and $u_0|_{\partial\Omega}$ is of class C^2 in $\{x \in \partial\Omega \colon |u_0(x)| < \infty\}$. Then there is a continuous function $u \colon \overline{\Omega} \times [0, \infty) \to [-\infty, \infty]$ which is smooth on $\big(\Omega \times (0, \infty)\big) \cap \{|u| < \infty\}$ and solves*

$$\begin{cases} \dot{u} = \sqrt{1 + |Du|^2}\, \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } \big(\Omega \times (0,\infty)\big) \cap \{|u| < \infty\}, \\ u(x,t) = u_0(x) & \text{for } (x,t) \in \mathcal{P}(\Omega \times (0,\infty)), \end{cases}$$

where $\mathcal{P}(\Omega \times (0, \infty)) := (\Omega \times \{0\}) \cup (\partial\Omega \times [0, \infty))$.

Proof. We approximate by auxiliary problems and use a priori estimates to pass to a limit which is the desired solution of the initial boundary value problem.

We cut off the initial function u_0 in height by considering

$$\bar{u}_{0,R} := \widetilde{\max}\left(\widetilde{\min}(u_0, R), -R\right)$$

for $R > 0$, where $\widetilde{\max}$ and $\widetilde{\min}$ are mollified versions of max and min respectively (defined analogously to (4.1) setting $\delta = 1/2$ there). Next, we cut off the domain of definition Ω by intersecting with a ball $B_{2R} \equiv B_{2R}(0)$

and using Theorem 4.1. This gives smooth open sets $\Omega \cap B_R \subset \Omega_R \subset \Omega \cap B_{2R}$ whose boundaries have non-negative mean curvature $H[\partial\Omega_R] \geq 0$. Finally, we restrict the functions $\bar{u}_{0,R}$ to Ω_R and take a mollification to find smooth functions $u_{0,R}$ defined on Ω_R satisfying $\|u_{0,R} - \bar{u}_{0,R}\|_{L^\infty(\Omega_R)} < R^{-1}$ and $\|u_{0,R} - \bar{u}_{0,R}\|_{C^2(\partial\Omega \cap B_R)} < R^{-1}$.

Now define u_R as the solution of the auxiliary problem

$$\begin{cases} \dot{u}_R = \sqrt{1+|Du_R|^2}\,\operatorname{div}\left(\frac{Du_R}{\sqrt{1+|Du_R|^2}}\right) & \text{in } \Omega_R \times (0,\infty), \\ u_R(x,t) = u_{0,R}(x) & \text{for } (x,t) \in \mathcal{P}(\Omega_R \times (0,\infty)). \end{cases}$$

By [7, Theorem 2.1], u_R is well-defined and satisfies

$$\begin{aligned} & u_R \in C^0(\overline{\Omega_R} \times [0,\infty)) \cap C^\infty(\Omega_R \times (0,\infty)) \\ \text{and} \quad & Du_R \in C^0(\overline{\Omega_R} \times [0,\infty)). \end{aligned}$$

(By de Giorgi-Nash-Moser-estimates the spatial derivative Du_R is even Hölder-continuous for some exponent.)

To be able to utilize the Arzelà-Ascoli-Theorem and to pass to a limit and obtain a solution of the initial problem, we are going to need local a priori estimates. These estimates will be local in space, time, and height: Due to the unboundedness in height we can only expect estimates at points (x,t) depending on $|u_R(x,t)|$.

Since by Lemma 2.2 and Lemma 2.3 we have local gradient bounds at the boundary, local gradient estimates easily follow from the results of Section 2 in [5]. Using spheres as barriers we obtain Hölder-estimates in time with exponent $1/2$ (cf. Section 6 of [11]). This is sufficient to apply an Arzelà-Ascoli argument.

To use Arzelà-Ascoli for unbounded functions, simply compose with a homeomorphism $\Phi\colon [-\infty,\infty] \to [-1,1]$ which is smooth on $(-\infty,\infty)$. Then the gradient and Hölder-in-time estimates give locally uniform estimates for $\Phi \circ u_R$ where $\Phi \circ u_R \in (-1+\varepsilon, 1-\varepsilon)$, which is sufficient: Locally in space-time there is for any $\varepsilon > 0$ a $\delta > 0$ such that for almost all $R \in \mathbb{N}$ we have

$$|(x,t) - (y,s)| < \delta \Rightarrow |\Phi \circ u_R(x,t) - \Phi \circ u_R(y,s)| < \varepsilon.$$

By Arzelà-Ascoli a subsequence of $(\Phi \circ u_R)_{R\in\mathbb{N}}$ converges locally uniformly to a continuous function on $\bar{\Omega} \times [0,\infty)$ as $R \to \infty$. This corresponds to pointwise convergence of a subsequence of (u_R) to a continuous function $u\colon \bar{\Omega} \times [0,\infty) \to [-\infty,\infty]$ and locally uniform convergence on $\bar{\Omega} \times [0,\infty) \cap \{|u| < \infty\}$.

Then, using the interior estimates for higher derivatives in [5], one has locally smooth convergence on $\Omega \times (0,\infty) \cap \{|u| < \infty\}$ and we see that u solves the same differential equation there as the u_R, which completes the proof. $\square$

Now we are going to establish the local a priori estimates at the boundary.

Lemma 2.2 (local gradient estimates at the boundary). *Let Ω be as in the statement of Theorem 2.1, $x_0 \in \partial\Omega$, $r, T > 0$, $B_r := B_r(x_0) \cap \Omega$ and $\Gamma_r := \partial\Omega \cap B_r(x_0)$. Let $u \in C^{2;1}(B_{2r} \times (0,T)) \cap C^0(\overline{B_{2r}} \times [0,T])$ be a solution of*

$$\begin{cases} \dot{u} = \sqrt{1+|Du|^2}\, \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) & \text{in } B_{2r} \times (0,T), \\ u(x,0) = u_0(x) & \text{for } x \in B_{2r}, \\ u(x,t) = \varphi(x) & \text{for } x \in \Gamma_{2r},\ t > 0. \end{cases}$$

For the initial- and boundary data assume $u_0 \in \mathrm{Lip}(\overline{B_{2r}})$ and $\varphi \in C^2(\overline{B_{2r}})$. Assume $Du \in C^0(\overline{B_{2r}} \times [0,T])$. Then on $\Gamma_r \times [0,T]$ we have

$$|Du| \le C\left(n, \|u\|_{L^\infty(B_{2r}\times(0,T))}, \|u_0\|_{\mathrm{Lip}(B_{2r})}, \|\varphi\|_{C^2(\overline{B_{2r}})}, r, \Gamma_{3r}\right).$$

Proof. A non-localized version of the barrier can be found, e.g., in [6, Chapter 1.4]. The construction uses the distance function and its properties in a tubular neighborhood: There is an ε-neighborhood of Γ_{2r} such that the distance function d to the boundary part Γ_{3r} is of class C^2 on that ε-neighborhood. Moreover, $\Delta d \le 0$ as a consequence of $H[\partial\Omega] \ge 0$.

Let $M := \|u\|_{L^\infty} + \|\varphi\|_{L^\infty}$. We choose $\eta \in C^\infty(\overline{B_{2r}})$ with $0 \le \eta \le 1$, $\eta \equiv 0$ on B_r, and $\eta \equiv 1$ on $B_{2r} \setminus B_{\frac{3}{2}r}$. We define

$$h := \varphi + M\eta.$$

In the following we will restrict ourselves to a tubular neighborhood:

$$B_{2r}^{\varepsilon} := \{x \in \Omega\colon |x - x_0| < 2r, d(x) = \operatorname{dist}(x, \Gamma_{3r}) < \varepsilon\},$$

where $0 < \varepsilon < \frac{r}{2}$ is sufficiently small such that the properties mentioned in the beginning of the proof are satisfied. We will choose ε later.

We prove that a function of the form

$$w := h + \psi(d) = \varphi + M\eta + \psi(d)$$

is an upper barrier for u. Here, $\psi\colon [0,\varepsilon] \to \mathbb{R}_{\geq 0}$ is function to be chosen and of which we demand $\psi(0) = 0$, $\psi(\varepsilon) \geq M$, $\psi' \geq L$, and $\psi'' < 0$. We set $L := \sup|Du_0| + \sup|D\varphi|$. It will become obvious that the same method, we are going to use, can be applied to prove that $\varphi - M\eta - \psi(d)$ is a lower barrier for u. On $\Gamma_r \times [0,T]$ the upper and lower barriers take the same values as u and we therefore conclude

$$|Du| \leq |Dw| \leq |D\varphi| + \psi'(0).$$

Now we are left to establish that w is an upper barrier for u. First we check the boundary values, i.e. we check $w \geq u$ on $\mathcal{P}(B^{\varepsilon}_{2r} \times (0,T))$:

(i) For $x \in B^{\varepsilon}_{\frac{3}{2}r}$ let $\pi(x)$ be the orthogonal projection of x to the boundary Γ. We note $\pi(x) \in \Gamma_{2r}$. Also, $u_0 = \varphi$ on Γ_{2r} and we obtain

$$\begin{aligned} w(x,0) &\geq \varphi(x) + \psi(d(x)) \\ &\geq \varphi(\pi(x)) - d(x)\sup|D\varphi| + \psi(d(x)) \\ &\geq u_0(\pi(x)) - d(x)\sup|D\varphi| + Ld(x) \\ &= u_0(\pi(x)) + d(x)\sup|Du_0| \\ &\geq u_0(x). \end{aligned}$$

(ii) For $|x - x_0| \geq \frac{3}{2}r$ and $0 \leq t \leq T$:

$$w(x,t) = \varphi(x) + M + \psi(d(x)) \geq M - \|\varphi\|_{L^\infty} \geq u(x,t).$$

(iii) For $(x,t) \in \Gamma_{2r} \times [0,T]$:

$$w(x,t) = \varphi(x) + M\eta(x) \geq \varphi(x) = u(x,t).$$

(iv) For $d(x) = \varepsilon$ and $0 \leq t \leq T$:

$$\begin{aligned} w(x,t) &= \varphi(x) + M\eta(x) + \psi(\varepsilon) \geq \psi(\varepsilon) - \|\varphi\|_{L^\infty} \\ &\geq M - \|\varphi\|_{L^\infty} \geq u(x,t). \end{aligned}$$

Having checked the boundary values we now turn to the proof of

$$\dot{w} - \sqrt{1+|Dw|^2}\ \operatorname{div}\left(\frac{Dw}{\sqrt{1+|Dw|^2}}\right) \ge 0.$$

In the computations to come note that $|Dd| = 1$ and also $d_{ij}d^i = 0$.

$$\begin{aligned}
&\dot{w} - \sqrt{1+|Dw|^2}\ \operatorname{div}\left(\frac{Dw}{\sqrt{1+|Dw|^2}}\right) = -\left(\delta^{ij} - \frac{w^i w^j}{1+|Dw|^2}\right) w_{ij} \\
&= -\Delta h - \psi''(d) \underbrace{-\psi'(d)\Delta d}_{\ge 0} + \frac{w^i w^j h_{ij} + w^i w^j d_{ij}\psi'(d) + \psi''(d)\left\langle Dd, Dw\right\rangle^2}{1+|Dw|^2} \\
&\ge -c_0\left(n, \left\|D^2 h\right\|\right) + \psi'(d)\frac{h^i h^j d_{ij}}{1+|Dw|^2} - \psi''(d)\left(1 - \frac{\left(\left\langle Dd, Dh\right\rangle + \psi'(d)\right)^2}{1+|Dw|^2}\right).
\end{aligned}$$

Using $\psi'' < 0$ and

$$\begin{aligned}
&1 - \frac{\left(\left\langle Dd, Dh\right\rangle + \psi'(d)\right)^2}{1+|Dw|^2} \\
&= \frac{1}{1+|Dw|^2}\Big(1 + |Dh|^2 + 2\psi'(d)\left\langle Dh, Dd\right\rangle + \psi'(d)^2 \\
&\qquad - \left\langle Dd, Dh\right\rangle^2 - 2\psi'(d)\left\langle Dd, Dh\right\rangle - \psi'(d)^2\Big) \\
&= \frac{1}{1+|Dw|^2}\left(1 + |Dh|^2 - \left\langle Dd, Dh\right\rangle^2\right) \\
&\ge \frac{1}{1+|Dw|^2}
\end{aligned}$$

we obtain

$$\begin{aligned}
\text{(2.1)} \qquad &\dot{w} - \sqrt{1+|Dw|^2}\ \operatorname{div}\left(\frac{Dw}{\sqrt{1+|Dw|^2}}\right) \\
&\ge -c_0 + \psi'(d)\frac{h^i h^j d_{ij}}{1+|Dw|^2} + \frac{-\psi''(d)}{1+|Dw|^2}.
\end{aligned}$$

For $\delta, \sigma > 0$ we choose $\varepsilon = \frac{1}{\sqrt{\sigma}}$ and

$$\begin{aligned} \psi(d) &= \delta \log(1+\sigma d) \quad \text{and conclude} \\ \psi'(d) &= \delta \frac{\sigma}{1+\sigma d} \geq \delta \frac{\sigma}{1+\sqrt{\sigma}}, \\ -\psi''(d) &= \delta \frac{\sigma^2}{(1+\sigma d)^2} = \frac{1}{\delta} \psi'(d)^2. \end{aligned}$$

The special choice of ψ and ε allows ψ'' to become big relative to $(\psi')^2$ (δ small) while still making $\psi(\varepsilon)$ and ψ' big by choosing $\sigma = \sigma(\delta)$ afterwards. First we choose $\delta := \frac{1}{2c_0+2}$, then $\sigma \geq \sigma_0(\delta, \|Dh\|_{L^\infty}) > 0$ such that

$$\begin{aligned} 1 + |Dw|^2 &= 1 + |Dh|^2 + 2\psi'(d) \langle Dh, Dd \rangle + \psi'(d)^2 \\ &\leq 2\psi'(d)^2 \end{aligned}$$

and consequently

$$\frac{-\psi''(d)}{1+|Dw|^2} \geq \frac{1}{\delta} \psi'(d)^2 \frac{1}{2\psi'(d)^2} = \frac{1}{2\delta} = c_0 + 1. \tag{2.2}$$

For the second summand on the right-hand side of (2.1) we may use

$$\psi'(d)^{-1} \left(1 + |Dw|^2\right) = \psi'(d)^{-1} \left(1 + |Dh|^2\right) + 2 \langle Dh, Dd \rangle + \psi'(d) \xrightarrow{\sigma \to \infty} \infty.$$

Choosing $\sigma = \sigma(\delta, \|Dh\|_{L^\infty}, \left\|D^2 d\right\|_{L^\infty})$ even greater, if needed, we arrive at

$$\psi'(d) \frac{h^i h^j d_{ij}}{1+|Dw|^2} > -\frac{1}{2}. \tag{2.3}$$

Putting together (2.1), (2.2), and (2.3) the desired differential inequality follows:

$$\dot{w} - \sqrt{1+|Dw|^2} \ \mathrm{div} \left(\frac{Dw}{\sqrt{1+|Dw|^2}} \right) > 0.$$

To satisfy the assumptions $\psi(\varepsilon) \geq M$ and $\psi' \geq L$, we must also take into account M and L in our choice of σ.

$$\begin{aligned} \sigma &\equiv \sigma(\delta, \|Dh\|_{L^\infty}, \left\|D^2 d\right\|_{L^\infty}, M, L) \\ &\equiv \sigma(n, \|Dh\|_{C^1}, \Gamma_{3r}, \|u\|_{L^\infty}, \|Du_0\|_{L^\infty}, \|D\varphi\|_{L^\infty}) \\ &\equiv \sigma\left(n, \|u\|_{L^\infty}, \|Du_0\|_{L^\infty}, \|\varphi\|_{C^2}, r, \Gamma_{3r}\right). \end{aligned}$$

As was mentioned above the barrier argument yields on $\Gamma_r \times [0,T]$

$$\begin{aligned} |Du| &\le |D\varphi| + \psi'(0) = |D\varphi| + \delta\sigma \\ &\le C\left(n, \|u\|_{L^\infty}, \|Du_0\|_{L^\infty}, \|\varphi\|_{C^2}, \|f\|_{L^\infty}, r, \Gamma_{3r}\right). \end{aligned}$$

□

To apply Lemma 2.2 we need local L^∞-estimates on u. For this purpose we have the following

Lemma 2.3 (L^∞-estimates near the boundary). *Let Ω be as before, $R, T > 0$ and x_0, B_R and Γ_R defined as in Lemma 2.2. Let $u \in C^{2;1}(B_R \times (0,T)) \cap C^0(\overline{B_R} \times [0,T])$ be a solution of* (GMCF) *on $B_R \times (0,T)$. Then there exists $r > 0$ dependent on B_R, Γ_R and T, such that*

$$\sup_{B_r \times (0,T)} |u| \le C,$$

where $C > 0$ depends on the same quantities as r and additionally depends on $\sup_{(B_R \times \{0\}) \cup (\Gamma_R \times (0,T))} |u|$.

Proof. Without loss of generality we may assume $x_0 = 0$ and that the inner normal to $\partial\Omega$ at x_0 is $e_n = (0, \ldots, 0, 1)$. We are going to write the boundary locally as a graph: There is an open ball B in $\mathbb{R}^{n-1}$ with center at the origin and $s > 0$, such that

$$B_R \cap (B \times (-s,s)) = \{(\hat{x}, x^n) \in B \times (-s,s) \colon h(\hat{x}) < x^n\}$$

and $\Gamma_R \cap (B \times (-s,s)) = \operatorname{graph} h$ for some function $h \in C^\infty(\overline{B})$. Because of $H[\partial\Omega] \ge 0$, h satisfies the differential inequality

$$\operatorname{div}\left(\frac{Dh}{\sqrt{1+|Dh|^2}}\right) \ge 0.$$

Take $0 \le \eta \in C_0^\infty(B)$, $\eta \not\equiv 0$, such that $v_0 := h + \eta$ fulfills $|v_0| < s$. Finally, define $v \in C^\infty(B \times (-1, T+1)) \cap C^0(\overline{B} \times [-1, T+1])$ to be the solution of

$$\begin{cases} \dot{v} = \sqrt{1+|Dv|^2}\,\operatorname{div}\left(\frac{Dv}{\sqrt{1+|Dv|^2}}\right) & \text{in } B \times (-1, T+1), \\ v(\hat{x}, t) = v_0(\hat{x}) & \text{for } (\hat{x}, t) \in \mathcal{P}(B \times (-1, T+1)). \end{cases}$$

This way we have $v \in C^\infty(\overline{B} \times [0,T])$. By the maximum principle we also have $|v| < s$, and by the strong maximum principle $h(\hat{x}) < v(\hat{x}, t)$ holds for all $x \in B$ and $t > -1$.

Define Q by

$$Q := \{(\hat{x}, x^n, t) \in B \times (-s, s) \times (0, T) \colon h(\hat{x}) < x^n < v(\hat{x}, t)\},$$

and choose $r > 0$ from our assertion such that $B_r \times (0, T) \subset Q$ and $B_r \times (0, T)$ has positive distance to $\operatorname{graph} v$.

Now we construct a barrier on Q with the aid of the function v. On Q define

$$w(x, t) := (v(\hat{x}, t) - x^n)^{-1} \equiv (v(x, t) - x^n)^{-1},$$

setting $v(x, t) \equiv v(\hat{x}, t)$ for $(x, t) \in Q$. One easily verifies that the level-sets of w move by mean curvature, so that w solves

$$\dot{w} - \left(\delta^{ij} - \frac{w^i w^j}{|Dw|^2}\right) w_{ij} = 0.$$

Using this fact we calculate

$$\begin{aligned}
\dot{w} - \sqrt{1 + |Dw|^2}\, \operatorname{div}\left(\frac{Dw}{\sqrt{1 + |Dw|^2}}\right) &= \dot{w} - \left(\delta^{ij} - \frac{w^i w^j}{1 + |Dw|^2}\right) w_{ij} \\
&= \left(-\frac{w^i w^j}{|Dw|^2} + \frac{w^i w^j}{1 + |Dw|^2}\right) w_{ij} = -\frac{w^i w^j}{|Dw|^2 \left(1 + |Dw|^2\right)} w_{ij} \\
&= \frac{1}{1 + |Dw|^2} \left(\frac{v_{ij} w^i w^j}{(v - x^n)^2 |Dw|^2} - 2 |Dw|^2 (v - x^n)\right) \\
&\geq -\frac{\left\|D^2 v\right\|_{L^\infty(Q)}}{|Dw|^2 (v - x^n)^2} - 2(v - x^n) \\
&= -\frac{(v - x^n)^2}{1 + |Dv|^2} \left\|D^2 v\right\|_{L^\infty(Q)} - 2 (v - x^n) \\
&\geq -4s^2 \left\|D^2 v\right\|_{L^\infty(Q)} - 4s \geq -c
\end{aligned}$$

where $c > 0$ is a constant that ultimately only depends on B_R, Γ_R and T through the above construction. Finally,

$$w^\pm := \pm \left(w + ct + \sup_{(B_R \times \{0\}) \cup (\Gamma_R \times (0,T))} |u|\right)$$

are upper and lower barriers for u on Q respectively. Observe

$$w(y, t) \to \infty \text{ for } y \to x \in \mathcal{P}(Q) \setminus \left[\left(B_R \times \{0\}\right) \cup \left(\Gamma_R \times (0, T)\right)\right].$$

Now, on $B_r \times [0, T]$, $w^{\pm}$ and therefore $|u|$ are bounded by a constant as in the assertion. $\square$

Remark 2.4. It is worth mentioning that the solution is not unique in general, as was pointed out to the author by O. Schnürer. The author found a short proof.

Consider two so called *grim reaper curves* lying next to each other, i.e. the graph of the function

$$u_0(x) := -\log|\sin x| \qquad \text{for } 0 \neq x \in (-\pi, \pi).$$

We can write down a translating solution to (GMCF) with initial data u_0:

$$v(x,t) := t - \log|\sin x| \qquad \text{for } 0 \neq x \in (-\pi, \pi),\ t \in \mathbb{R}.$$

But the solution u we have constructed in the proof of Theorem 2.1 differs from this translating solution v: The two grim reapers get connected at infinity. This is because we cut off the function u_0 at some height R in the approximation process. To see that the approximating solutions $\tilde{u}_R := u_R|_{(-\pi,\pi)}$ do not converge to v we may consider the integral of the difference:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\pi}^{\pi} v(\cdot,t) - \tilde{u}_R(\cdot,t) = \int_{\operatorname{graph} v(\cdot,t)} K[v] \quad - \int_{\operatorname{graph} \tilde{u}_R(\cdot,t)} K[\tilde{u}_R] \geq 2\pi - \pi = \pi,$$

where $K[v], K[\tilde{u}_R]$ are the respective curvatures of the graphs. By the maximum principle the convergence $\tilde{u}_R \to u$ is monotone and thus $\int_{-\pi}^{\pi} \tilde{u}_R(\cdot,t) \to \int_{-\pi}^{\pi} u(\cdot,t)$ as $R \uparrow \infty$. We conclude $\int_{-\pi}^{\pi}(v-u)(\cdot,t) \geq \pi t$ and therefore $u \neq v$.

3. The shadowflow

In this section we are going to investigate the projections/shadows of graphical mean curvature flow, that is the sets $\{|u| < \infty\}$ where u is as in Theorem 2.1. We show that this shadow is a weak solution of mean curvature flow, where we use the following notion of weak solution which is based on the avoidance principle.

For this section we do not need to assume constant Dirichlet boundary values.

Let Ω be as in the last section.

Definition 3.1 (Weak solutions). A family $(A_t)_{t\in[0,\infty)}$ of open subsets of $\overline{\Omega}$ is called

- a supersolution to mean curvature flow if the following holds: For any $0 \leq a < b < \infty$ and any family $(B_t)_{t\in[a,b]}$ of (bounded) open sets, such that $B_t \Subset \Omega$ with $\partial B_t \in C^\infty$ for all $t \in [a, b]$, and such that $(\partial B_t)_{t\in[a,b]}$ is a classical solution to mean curvature flow, we have

$$\overline{B_a} \subset A_a \ \Rightarrow \ B_b \subset A_b.$$

- a subsolution to mean curvature flow, if $(\overline{\Omega} \setminus \overline{A_t})_{t\in[0,\infty)}$ is a supersolution.
- a weak solution of mean curvature flow, if $(A_t)_{t\in[0,\infty)}$ is both a super- and subsolution.

We shall call $(\partial\Omega \cap A_t)_{t\in[0,\infty)}$ the boundary values of $(A_t)_{t\in[0,\infty)}$.

Remarks 3.2.

(i) In the above setting in the definition of supersolution we even have $\overline{B_t} \subset A_t$ for all $t \in [a, b]$, when $(A_t)_t$ is a supersolution. To see that the closure is contained one can use, for instance, the translation invariance of classical flows.

(ii) If (A_t) is a family of open subsets of $\overline{\Omega}$ such that (∂A_t) is a classical solution to mean curvature flow, then, by the avoidance principle, (A_t) is a weak solution.

(iii) Let $x \in \partial A_t \cap \Omega$ and suppose that ∂A_t is smooth in a ball $B_r(x) \subset \mathbb{R}^n$. If (A_t) is a weak solution and ∂A_t is smooth in a spacetime-neighborhood of (x, t), then (∂A_t) solves mean curvature flow at (x, t) classically: We may assume, that $\partial A_t \cap B_r(x)$ is the graph of a smooth function. Using the result of the next section we find two smooth open sets, one lying inside $A_t \cap B_r$, the other inside $B_r(x) \setminus A_t$, and such that their boundaries coincide in a neighborhood of x with ∂A_t. Taking r to be small enough, small translations of these two open sets away from ∂A_t serve as barriers. This way it can be seen, that the normal velocity of ∂A_t at (x, t) coincides with the mean curvature at that point.

(iv) Similar weak notions of mean curvature flow are the set-theoretic subsolutions of Ilmanen ([8]) or more generally the barriers of De Giorgi. Both of them were compared to the level-set flow (see [1] for a comparison of De Giorgi's barriers to level-set flow). (See [12] for a definition of set-theoretic subsolutions including boundary values.)

(v) Note that our definition of weak solutions is not very useful where $\partial A_t \subset \partial\Omega$ (taking the boundary ∂A_t relative to $\overline{\Omega}$). This is because there is no space left for a classical solution, that could possibly push ∂A_t inwards into Ω. To circumvent this, one could compare with classical solutions with boundary values that may not be written as the boundary of an open set and which can intersect ∂A_t. But this would cause trouble in the methods we are going to use next. Another way would be to compare ∂A_t not only with classical solutions in Ω but with classical solutions that are boundaries of open subsets in $\mathbb{R}^n$ and which do not intersect the boundary values. This viewpoint has the disadvantage of not being intrinsically in $\overline{\Omega}$. The methods presented in the following also work if one adopts this definition for weak solutions.

Proposition 3.3. *For any open set $A \subset \Omega$ there exists a weak solution of mean curvature flow $(A_t)_{t\in[0,\infty)}$ with $A_0 = A$. Furthermore there is a smallest such weak solution $(A_t)_{t\in[0,\infty)}$: For any weak solution $(A'_t)_{t\in[0,\infty)}$ with $A \subset A'_0$ we have $A_t \subset A'_t$ for all $t \in [0,\infty)$.*

Proof. A similar argument is given in [12].

Let $\mathcal{B}^{(0)}$ be the set of all families of the form $(B_t)_{t\in[0,b]}$ ($b > 0$ may differ from family to family) with the following properties: $B_t \Subset \Omega$ are open, $\overline{B_0} \subset A$, and $\partial B_t \in C^\infty$ fulfills mean curvature flow. Define for $0 \le t < \infty$

$$A_t^{(0)} := \bigcup\{B\colon B = B_t \text{ for some family } (B_t)_{t\in[0,b]} \in \mathcal{B}^{(0)} \text{ with } b \ge t\}.$$

That is $(A_t^{(0)})_{t\in[0,\infty)}$ is the smallest set containing all classical solutions which start at time 0 inside A (in the sense made precise above). Next we add all classical solutions which start inside some $A_t^{(0)}$ and then iterate this process. Thus, we inductively define $\mathcal{B}^{(k)}$ and $A^{(k)}$ by setting $\mathcal{B}^{(k)}$ to be the set of all families of the form $(B_t)_{t\in[a,b]}$ where $0 < a < b$ are arbitrary, $B_t \Subset \Omega$ are open, $\overline{B_a} \subset A_a^{(k-1)}$, and $\partial B_t \in C^\infty$ fulfills mean curvature flow. Then set for $0 \le t < \infty$

$$A_t^{(k)} := \bigcup\{B\colon B = B_t \text{ for some family } (B_t)_{t\in[a,b]} \in \mathcal{B}^{(k)} \text{ with } a \le t \le b\}.$$

Finally define the open sets $A_t := \bigcup_{k\in\mathbb{N}} A_t^{(k)}$. Note that $A_t^{(k)}$ is a non-decreasing sequence of open sets. As a union of subsolutions (A_t) is again a subsolution. To see that (A_t) is a supersolution let $(B_t)_{t\in[a,b]}$ be a classical solution and let $\overline{B_a} \subset A_a$. Then by compactness $\overline{B_a} \subset A_a^{(k)}$ for some $k \in \mathbb{N}$. Therefore $\overline{B_b} \subset A_b^{(k+1)} \subset A_b$ and hence (A_t) is a supersolution.

The second assertion is obvious from the construction. □

The following lemma concerning weak solutions will be useful. The result is trivial for classical solutions.

Lemma 3.4. *Suppose $(A_t)_{t\in[0,\infty)}$ is a family of open subsets of $\overline{\Omega}$, such that $(A_t\times\mathbb{R})_t$ is a weak solution of mean curvature flow in $\overline{\Omega}\times\mathbb{R}$.*

Then $(A_t)_t$ is a weak solution of mean curvature flow in $\overline{\Omega}$.

Proof. We only show, that (A_t) is a supersolution using the fact that $(A_t\times\mathbb{R})$ is a supersolution.

Let $(B_t)_{t\in[a,b]}$ be a family of open sets such that $\partial B_t\in C^\infty$ fulfills mean curvature flow, and $\overline{B_a}\subset A_a$. We need to show $B_b\subset A_b$. In fact we prove $B_b\times\mathbb{R}\subset A_b\times\mathbb{R}$. For this we approximate $B_a\times\mathbb{R}$ by bounded sets.

For $R>0$ take $\hat K^R$ to be a smoothed intersection of $B_a\times\mathbb{R}$ with $\{|x^{n+1}|<2R\}$ containing $B_a\times[-R,R]$ (use Theorem 4.1). We may take the same closing ends for different $R>1$, to give curvature bounds on $\partial\hat K^R$ independent of R. Then define $K^R:=\{x\in\hat K^R\colon\ \mathrm{dist}(x,\partial\hat K^R)>R^{-1}\}$. Then we still have uniform curvature bounds for $R>R_0$ sufficiently large. Thus, by Proposition 4.1 of [5] there are classical solutions $(M_t^R)_{t\in[a,a+\tau]}$ of mean curvature flow with $M_a^R=\partial K^R$ for some $\tau>0$ independent of R. These solutions (M_t^R) are written as graphs over the $\partial\hat K^R$, which contain $\partial B_a\times[-R,R]$.

By interior estimates of [5] and the uniqueness of the limit (see below) we find for $R\to\infty$ local convergence as graphs over $\partial B_a\times\mathbb{R}$. This gives a solution of mean curvature flow which is written as a graph over $\partial B_a\times\mathbb{R}$ and starts from there. Hence it coincides with $(\partial B_t\times\mathbb{R})_{t\in[a,a+\tau]}$ by uniqueness (see remark 3.5).

Thus, we have shown that the corresponding flows $(K_t^R)_{t\in[a,a+\tau]}$ of the open sets starting from K^R satisfy

$$\bigcup_{R>1}K_t^R=B_t\times\mathbb{R},\qquad\text{for } t\in[a,a+\tau].$$

Let $(W_t)_{t\in[a,b]}$ be the smallest weak solution in $\mathbb{R}^{n+1}$ with $W_a=B_a\times\mathbb{R}$. The argument above has shown $W_t=B_t\times\mathbb{R}$ for $t\in[a,a+\tau]$. The same argument shows, that the maximal time-interval on which (W_t) and $(B_t\times\mathbb{R})$ coincide is right-open. It is easy to see that this maximal interval is also right-closed: Suppose $W_t=B_t\times\mathbb{R}$ for $t<t_0$. Since (W_t) is the smallest solution $W_{t_0}\subset B_{t_0}\times\mathbb{R}$. To show the reverse inclusion let $x\in B_{t_0}\times\mathbb{R}$. Since $\bigcup_{t\in(a,b)}B_t\times\mathbb{R}\times\{t\}$ is open we find a ball-solution of mean curvature flow

centered at x that is contained in $(B_t \times \mathbb{R})_t$ and contains (x, t_0). This shows $x \in W_{t_0}$ since $W_t = B_t \times \mathbb{R}$ for $t < t_0$ and W_t is a supersolution and hence contains the ball-solution.

Summarizing we find $W_t = B_t \times \mathbb{R}$ for all t in an right-open, right-closed subset of $[a, b]$ which contains a, i.e. for all $t \in [a, b]$. Hence, the smallest weak solution starting from $B_a \times \mathbb{R}$ is $(B_t \times \mathbb{R})_t$. As $A_t \times \mathbb{R}$ is a supersolution with $B_a \times \mathbb{R} \subset A_a \times \mathbb{R}$ we conclude $B_b \times \mathbb{R} \subset A_b \times \mathbb{R}$. □

Remark 3.5. A simple argument shows the uniqueness of mean curvature flow in the case of cylinders of the form $N \times \mathbb{R}$ with smooth section N: Write a hypothetically non-cylindrical solution as a graph over the initial cylinder. By the strong maximum principle the difference to the ordinary cylindrical solution attains no interior maximum. Then translate the non-cylindrical solution along the cylinder and again use interior estimates to find convergence to a new solution, and do the translation in such a way that the difference of the limit to the cylindrical solution attains an interior maximum. This contradicts the strong maximum principle.

A shadowflow is a weak solution:

Theorem 3.6. *Let* $u\colon \overline{\Omega} \times [0, \infty) \to [-\infty, \infty]$ *be continuous, smooth in the set* $\{(x, t) \in \Omega \times (0, \infty)\colon |u(x, t)| < \infty\}$ *and suppose* u *satisfies* (GMCF) *in this set. Define* A_t *to be the projection of* $\operatorname{graph} u(\cdot, t) \cap \mathbb{R}^{n+1}$ *onto* $\overline{\Omega}$, *that is*

$$A_t = \{x \in \overline{\Omega}\colon |u(x, t)| < \infty\}.$$

Then $(A_t)_{t \in [0,\infty)}$ *is a weak solution of mean curvature flow with boundary values* $(\{x \in \partial\Omega\colon |u(x, t)| < \infty\})_{t \in [0,\infty)}$.

Proof. By Lemma 3.4 it suffices to show that $A_t \times \mathbb{R}$ is a weak solution.

First we prove that $(A_t \times \mathbb{R})_t$ is a subsolution. So let $(B_t)_{t \in [a,b]}$ be a family of open bounded subsets of $\mathbb{R}^{n+1}$ such that the boundaries $(\partial B_t)_{t \in [a,b]}$ form a smooth solution of mean curvature flow. Suppose $\overline{B_a} \subset (\Omega \setminus \overline{A_a}) \times \mathbb{R}$. Assume for contradiction that there is $X \in B_b$ such that $X \notin (\Omega \setminus \overline{A_b}) \times \mathbb{R}$. By openness of B_b we may assume $X \in A_b \times \mathbb{R}$. Taking a vertical translation of (B_t) we may further assume $X \in \operatorname{graph} u(\cdot, b)$. But because $\operatorname{graph} u(\cdot, t)$ is a smooth solution to mean curvature flow this contradicts the avoidance principle.

Now to see that $(A_t \times \mathbb{R})_t$ is a supersolution, suppose $\overline{B_a} \subset A_a \times \mathbb{R}$. By performing a translation we may assume $u(X_1, \ldots, X_n, a) < X_{n+1}$ for all $X \in \overline{B_a}$, i.e. $\overline{B_a}$ is above $\operatorname{graph} u(\cdot, a)$. But then by the avoidance principle

$\overline{B_b}$ is above graph $u(\cdot,b)$. We conclude $\overline{B_b} \subset \{u(\cdot,b) < \infty\} \times \mathbb{R}$. On the other hand one can perform a translation in the other direction such that $\overline{B_a}$ is below graph $u(\cdot,a)$. Then similar as above $\overline{B_b} \subset \{u(\cdot,b) > -\infty\} \times \mathbb{R}$. Taking both facts together we conclude $\overline{B_b} \subset \{|u(\cdot,b)| < \infty\} \times \mathbb{R} = A_b \times \mathbb{R}$. □

4. Smoothing intersections respecting curvature conditions

Theorem 4.1. *Let $A, B \subset \mathbb{R}^n$ be open, $\partial A, \partial B \in C^\infty$ and suppose $A \cap B \neq \emptyset$ is bounded. Then for any $\varepsilon > 0$ there is an open set Ω with $\partial\Omega \in C^\infty$ such that*

$$(A \cap B) \setminus (\partial A \cap \partial B)_\varepsilon \subset \Omega \subset A \cap B,$$

where $(\partial A \cap \partial B)_\varepsilon := \{x \in \mathbb{R}^n\colon\ \operatorname{dist}(x, \partial A \cap \partial B) < \varepsilon\}$.

Moreover, if $\Gamma \subset \mathbb{R}^{n-1}$ is an open or closed curvature cone (cf. Definition 1.1), and if the principal curvatures of ∂A and ∂B at every point are contained in Γ, then Ω can be chosen such that the principal curvatures of $\partial\Omega$ at any point are contained in Γ, too.

We observe the following

Lemma 4.2. *Let $\Gamma \subset \mathbb{R}^{n-1}$ be an open or closed curvature cone. Then the set of real symmetric $(n-1) \times (n-1)$-matrices with eigenvalues in Γ is convex.*

Proof. Let Γ be open. (The case of closed Γ is handled similar.) Let $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ be the signed distance function to $\partial\Gamma$ (which we assume to be nonempty) such that $\lambda \in \Gamma \iff f(\lambda) > 0$. By the convexity of Γ, f is concave. By the symmetry of Γ, f is symmetric. And because Γ contains the positive cone f is increasing in each component of its argument. Then $F(A) := f(\lambda(A))$ is a concave function on the set of symmetric matrices, where $\lambda(A)$ denotes the eigenvalues of A (see e.g. [2, end of §3]). Thus $\{A\colon F(A) > 0\}$ is a convex subset of the symmetric matrices, which completes the proof. □

Proof of Theorem 4.1. The idea is to use distance functions and take a mollified version of min (denoted $\widetilde{\min}$) and to define

$$\Omega := \{\widetilde{\min}(\operatorname{dist}_{\partial A}, \operatorname{dist}_{\partial B}) > 0\}$$

though we will not directly use the distance functions.

1. Altered distance functions and reference neighborhoods. First note that since $A \cap B$ is bounded it suffices to consider everything in a large ball. Let $d_A \in C^\infty(\mathbb{R}^n)$ be such that $d_A < 0$ in $\mathbb{R}^n \setminus \overline{A}$ and in our large ball we have $d_A > 0$ in A and d_A coincides with the signed distance function in a tubular neighborhood of ∂A. Let $g \in C^\infty(\mathbb{R})$ be such that $g(0) = 0$, $1 \le g' \le 2$ and $g''(s) \le -C$ for $|s| < \varepsilon(C)$, where we choose $C > 0$ later, and set

$$a := g \circ d_A.$$

We derive

$$\begin{aligned} Da &= g'(d_A)\, Dd_A, \\ -D^2 a &= -g''(d_A)\, Dd_A \otimes Dd_A + g'(d_A)\,(-D^2 d_A). \end{aligned}$$

In a tubular neighborhood (in our large ball) Da is an eigenvector of $-D^2 a$ with eigenvalue $-g''(d_A)$. The remaining eigenvalues are $g'(d_A)$ times the corresponding eigenvalues of $-D^2 d_A$ which are

$$\frac{\kappa_i \circ \pi}{1 - d_A \cdot \kappa_i \circ \pi} \qquad (i = 1, \ldots, n-1).$$

Here κ_i are the principal curvatures at the boundary ∂A and π denotes the closest point projection onto the boundary. Note that inside $\overline{A}$ these eigenvalues of $-D^2 d_A$ are greater or equal to the principal curvatures of the boundary, independent of their sign. Because Γ contains the positive cone, we find the eigenvalues of $-D^2 a$ which correspond to eigenvectors orthogonal to Da in Γ.

Now choose $C > 0$ from above such that the eigenvalue $-g'' \ge C$ of $-D^2 a$ which corresponds to the eigenvector Da is the largest eigenvalue of $-D^2 a$ in a neighborhood of ∂A (still restricted to a large ball). We refer to this neighborhood restricted to $\overline{A}$ as the reference neighborhood of ∂A (in our large ball). In the reference neighborhood the eigenvalues of $-D^2 a|_V$ are contained in Γ for any $(n-1)$-dimensional hyperplane V.

Analogously we define b with respect to ∂B and the reference neighborhood of ∂B.

2. Construction. Let $f \in C^\infty(\mathbb{R})$ be a function with the following properties

(i) $\min\{s, 0\} - 1 < f(s) < \min\{s, 0\}$ for $|s| < 1$,

(ii) $f(s) = \min\{s, 0\}$ for $|s| \ge 1$,

(iii) $0 \le f' \le 1$,

(iv) $f'' \le 0$.

Define

$$\begin{aligned}\Phi\colon (0,1)\times\mathbb{R}^n &\to \mathbb{R},\\ (\delta,x) &\mapsto \delta f\left(\frac{a(x)-b(x)}{\delta}\right)+b(x).\end{aligned}\tag{4.1}$$

This is a mollified version of $\min(a,b)$ with parameter δ. In fact

$$\min\{a(x),b(x)\}-\delta<\Phi(\delta,x)\le\min\{a(x),b(x)\}\tag{4.2}$$

for all $x\in\mathbb{R}^n$ and $\delta\in(0,1)$. We will choose Ω from the assertion of the form

$$\Omega_\delta := \{x\in\mathbb{R}^n\colon \Phi(\delta,x)>0\}$$

for an appropriate choice of δ.

3. Inclusions. Because of $A\cap B=\{\min(a,b)>0\}$ it is obvious from (4.2) that $\Omega_\delta\subset A\cap B$ holds for all $\delta\in(0,1)$. To check the other inclusion let $x\in A\cap B\setminus(\partial A\cap\partial B)_\varepsilon$. By continuity there is $0<\delta_1=\delta_1(\varepsilon)$ independent of x, such that

$$\max\{a(x),b(x)\}>\delta_1.\tag{4.3}$$

Now choose $\delta\le\delta_1/2$ and distinguish two cases: Suppose $|a(x)-b(x)|>\delta$. Then by property (ii) of f we find

$$\Phi(\delta,x)=\min\{a(x),b(x)\}>0.$$

If on the other hand $|a(x)-b(x)|\le\delta$ then by (4.2), (4.3), and $\delta\le\delta_1/2$

$$\Phi(\delta,x)>\min\{a(x),b(x)\}-\delta\ge\delta_1-2\delta\ge 0.$$

In summary we find the claimed inclusions, provided that δ is sufficiently small.

4. Smoothness. We compute

$$D\Phi=\left(f\left(\frac{a-b}{\delta}\right)-f'\left(\frac{a-b}{\delta}\right)\frac{a-b}{\delta},\ f'\left(\frac{a-b}{\delta}\right)(Da-Db)+Db\right).$$

Assuming $\Phi=0$ implies

$$f\left(\frac{a-b}{\delta}\right)=-\frac{b}{\delta},$$

and therefore

$$\partial_\delta \Phi = 0 \iff b + \underbrace{f'\left(\frac{a-b}{\delta}\right)}_{\in[0,1]}(a-b) = 0.$$

This occurs only if one of the following is true (note that $\Phi = 0$ already implies $a, b \geq 0$, by (4.2))

(I) $a = 0$ and $b = 0$

(II) $a = 0$ and $f'(\frac{a-b}{\delta}) = 1$

(III) $b = 0$ and $f'(\frac{a-b}{\delta}) = 0$.

(I) implies $\Phi \neq 0$, a contradiction. In case (II) we find $\partial_x \Phi = Da \neq 0$, because $Da \neq 0$ on ∂A, which is where $a = 0$ holds. Case (III) is treated analogously to case (II).

Summarizing we obtain $D\Phi \neq 0$ where $\Phi = 0$. The implicit function theorem shows, that $\Phi^{-1}(0)$ is a smooth n-dimensional submanifold of $(0,1) \times \mathbb{R}^n$ with normal $\frac{D\Phi}{|D\Phi|}$.

By applying Sard's theorem to the mapping $\Phi^{-1}(0) \to (0,1)$, $(\delta, x) \mapsto \delta$ one can show, that for almost all $\delta \in (0,1)$, $\partial_x \Phi(\delta, \cdot) \neq 0$ where $\Phi(\delta, \cdot) = 0$, so that by the implicit function theorem again Ω_δ is smooth for almost all $\delta \in (0,1)$ and $\partial \Omega_\delta = \{x \in \mathbb{R}^n \colon \Phi(\delta, x) = 0\}$.

We choose $\Omega := \Omega_{\delta_0}$ for such an $\delta_0 \in (0,1)$ sufficiently small, that the asserted inclusions hold.

5. Curvature condition. Let $x_0 \in \partial\Omega$ and assume without loss of generality $x_0 = 0$ and further assume that the tangent space of $\partial\Omega$ at $x_0 = 0$ is orthogonal to e_n. We identify the tangent space and $\mathbb{R}^{n-1} \equiv \{(\hat{x}, x^n) \in \mathbb{R}^n \colon x^n = 0\}$. Locally $\partial\Omega$ is a graph over the tangent space at x_0: Let $w \in C^\infty(\mathbb{R}^{n-1})$ and $r > 0$ with

$$B_r(0) \cap \Omega = B_r(0) \cap \{(\hat{x}, x^n) \in \mathbb{R}^n \colon w(\hat{x}) < x^n\}.$$

We write $\Phi_{\delta_0} \equiv \Phi(\delta_0, \cdot)$ and $\hat{D}$ for differentiation with respect to the first $n-1$ components.

The following holds in a neighborhood of $0 \in \mathbb{R}^{n-1}$:

$$\Phi_{\delta_0}(\hat{x}, w(\hat{x})) = 0, \tag{4.4}$$

$$\hat{D}w(0) = 0, \tag{4.5}$$

$$\partial_n \Phi_{\delta_0}(0) = |D\Phi_{\delta_0}(0)|\,. \tag{4.6}$$

Differentiating (4.4) twice and using (4.5) and (4.6) we obtain

$$\hat{D}^2 w(0) = -\frac{\hat{D}^2 \Phi_{\delta_0}(0)}{|D\Phi_{\delta_0}(0)|}$$

Note that the eigenvalues of $\hat{D}^2 w(0)$ are the principle curvatures of $\partial\Omega$ at x_0. Hence, it remains to prove that the eigenvalues of $-\hat{D}^2\Phi_{\delta_0}(0)$ are contained in Γ.

We compute

$$(4.7) \qquad \begin{aligned} -\hat{D}^2\Phi_{\delta_0} = & -\frac{1}{\delta_0} f''\left(\frac{a-b}{\delta_0}\right)(\hat{D}a - \hat{D}b)\otimes(\hat{D}a - \hat{D}b) \\ & - \left(f'\left(\frac{a-b}{\delta_0}\right)\left(\hat{D}^2 a - \hat{D}^2 b\right) + \hat{D}^2 b\right). \end{aligned}$$

The matrix $(\hat{D}a - \hat{D}b)\otimes(\hat{D}a - \hat{D}b)$ is positive semi-definite and $-f'' \geq 0$ (property (iv)). By the min-max-characterization of eigenvalues and since Γ contains the positive cone it therefore suffices to consider the second term in (4.7). We distinguish three cases, below. But first we observe the following: We may assume that $\varepsilon > 0$ is so small that $(\partial A \cap \partial B)_\varepsilon \cap (A \cap B)$ is contained in the intersection of the reference neighborhoods of ∂A and ∂B. Then $x_0 = 0 \in \partial\Omega$ is in ∂A or ∂B or in the reference neighborhoods of both ∂A and ∂B.

(I) $a(0) - b(0) \geq \delta_0$: Then

$$f'\left(\frac{a(0) - b(0)}{\delta_0}\right) = 0.$$

The term in question becomes $-\hat{D}^2 b$. Moreover, in this case $0 = \Phi_{\delta_0}(0) = b(0)$ holds. That is why we are in the reference neighborhood of ∂B, where the eigenvalues of $-\hat{D}^2 b$ are contained in Γ.

(II) $a(0) - b(0) \leq -\delta_0$: This case is treated similar.

(III) $|a(0) - b(0)| < \delta_0$: Here, by property (i) of f, $0 \notin \partial A, \partial B$. Therefore 0 must be in the reference neighborhoods of both ∂A and ∂B. As a convex combination of two matrices, whose eigenvalues are contained in Γ, the eigenvalues of

$$-\left(f'\left(\frac{a-b}{\delta_0}\right)\left(\hat{D}^2 a - \hat{D}^2 b\right) + \hat{D}^2 b\right),$$

the matrix in question, are contained in Γ by Lemma 4.2.

In any case, the principal curvatures of $\partial\Omega$ at the point $x_0 = 0$ are contained in Γ. □

References

[1] G. Bellettini and M. Novaga, *Comparison results between minimal barriers and viscosity solutions for geometric evolutions*, Annali della Scuola Normale Superiore di Pisa – Classe di Scienze **26** (1998), no. 1, 97–131.

[2] L. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Function of the eigenvalues of the Hessian*, Acta Math. **155** (1985), 261–301.

[3] K. Choi and P. Daskalopoulos, *The Q_k flow on complete non-compact graphs*, `arXiv:1603.03453v1`, (10 Mar 2016).

[4] K. Choi, P. Daskalopoulos, L. Kim, and K. Lee, *The evolution of complete non-compact graphs by powers of Gauss curvature*, `arXiv:1603.04286v1`, (14 Mar 2016).

[5] K. Ecker and G. Huisken, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), no. 3, 547–569.

[6] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific Publishing Co. Ptc. Ltd., (2003), `ISBN 981-238-043-4`.

[7] G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, Journal of Differential Equations **77** (1989), 369–378.

[8] T. Ilmanen, *The level-set flow on a manifold*, Proc. Symp. Pure Math. **54** Pt. 1 (1993), 193–204.

[9] H. Jenkins and J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions*, Journal für die reine und angewandte Mathematik **229** (1968), 170–187.

[10] L. Xiao, *General curvature flow without singularities*, `arXiv:1604.05743v1`, (19 Apr 2016).

[11] M. Sáez Trumper and O. Schnürer, *Mean curvature flow without singularities*, J. Differential Geom. 97 (2014), no. 3, 545-570.

[12] B. White, *The topology of hypersurfaces moving by mean curvature*, Communication in Analysis and Geometry **3** (1995), no. 2, 317–333.

Fachbereich Mathematik und Statistik, Universität Konstanz
78457 Konstanz, Germany
E-mail address: wolfgang.maurer@uni-konstanz.de

Received April 19, 2016
Accepted June 19, 2018