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A homology vanishing theorem for graphs with positive curvature

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We prove a homology vanishing theorem for graphs with positive Bakry-Émery curvature, analogous to a classic result of Bochner on manifolds [3]. Specifically, we prove that if a graph has positive curvature at every vertex, then its first homology group is trivial, where the notion of homology that we use for graphs is the path homology developed by Grigor'yan, Lin, Muranov, and Yau [11]. We moreover prove that the fundamental group is finite for graphs with positive Bakry-Émery curvature, analogous to a classic result of Myers on manifolds [22]. The proofs draw on several separate areas of graph theory, including graph coverings, gain graphs, and cycle spaces, in addition to the Bakry-Émery curvature, path homology, and graph homotopy. The main results follow as a consequence of several different relationships developed among these different areas. Specifically, we show that a graph with positive curvature cannot have a non-trivial infinite cover preserving 3-cycles and 4-cycles, and give a combinatorial interpretation of the first path homology in terms of the cycle space of a graph. Furthermore, we relate gain graphs to graph homotopy and the fundamental group developed by Grigor'yan, Lin, Muranov, and Yau [12], and obtain an alternative proof of their result that the abelianization of the fundamental group of a graph is isomorphic to the first path homology over the integers.

1. Introduction

A significant theme in much of graph theory in recent years has been the application of tools and ideas from continuous geometry to discrete settings, most specifically to graphs. There has been growing interest both in the approximation of continuous spaces by discrete ones, and in the understanding of graphs via their geometric properties. For instance, a classical example resulting from this way of thinking is the well-known Cheeger inequality

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(see for instance [6]), which proves an isoperimetric inequality for graphs that was originally formulated for Riemannian manifolds.

One of the principal developments in this area concerns curvature for graphs. Numerous notions of curvature on graphs have been put forward [10, 23]. An important and very general notion of curvature for graphs has been defined via various formulas due to Bakry and Émery, which is called the Bakry-Émery curvature of a graph (see [1, 20, 24]).

In addition, there are various notions of homology and cohomology for graphs. Recent work has introduced one such theory called the path homology [13]. Path homology has been shown to be a non-trivial homology theory which is invariant under a notion of homotopy for graphs [12]. Using this homotopy theory, the fundamental group for a graph is defined in [12], and it is shown that the first path homology is isomorphic to the abelianization of this fundamental group. Furthermore, it satisfies nice functorial properties, namely the Künneth formulas hold for graph products [14]. For these reasons, it seems that the path homology is a more appropriate notion of homology for graphs than others that have been proposed. See [14] for a discussion of various homology theories for graphs and the advantages of the path homology.

In this paper, we prove an important connection between graph curvature and homology. Namely, we prove a homology vanishing theorem for graphs with positive Bakry-Émery curvature. Homology vanishing theorems are ubiquitous in continuous geometry, and give important structural information about manifolds. Our vanishing theorem is analogous to a fundamental result of Bochner on manifolds which states that a manifold with positive curvature at every point has trivial first homology [3].

Theorem 1.1. *If a finite graph G has positive Bakry-Émery curvature at every vertex, then its first path homology group is trivial.*

It turns out that Bakry-Émery curvature on graphs is also compatible with the notion of homotopy and fundamental group from [12]. Our homotopy theorem is analogous to a fundamental result of Myers on manifolds which states that the fundamental group of a manifold with positive curvature is finite [22].

Theorem 1.2. *If a finite graph G has positive Bakry-Émery curvature at every vertex, then its fundamental group $\pi_1(G)$ is finite.*

Showing that these classical theorems from geometry hold for graphs reinforces the idea that the path homology is a good notion of homology for graph theory.

While Theorems 1.1 and 1.2 can be considered the main results of this paper, our proofs are executed by developing relationships between several different areas of graph theory, which are interesting on their own. Particularly, we give a concrete, combinatorial interpretation of the first path homology of a graph in terms of the cycle space of the graph, which is the space of linear combinations of incidence vectors of cycles of a graph. Specifically, we prove that the first path homology is isomorphic to the cycle space modulo the space generated by simple cycles of length 3 and 4. It is an open question to find a similar nice interpretation of the path homology groups beyond the first.

Further, we develop relationships between the cycle space, and the notion of a gain graph which is a graph with edges labeled with elements of a group. The cycle space of a graph always has a basis of size equal to the cyclomatic number of the graph, but there are various different classes of cycle bases depending on certain properties. See [19] for a discussion of different kinds of cycle bases. One of our contributions is to give a new kind of cycle basis based on a gain graph, which we call a Γ-circuit generator, where Γ is the group associated with a gain graph.

In addition, we investigate the fundamental group as defined in [12], and give an interpretation of this group relating to gain graphs. Then, via results of [8], we are able to describe this fundamental group as the fundamental group of the topological space obtained by attaching a 2-cell to each cycle of length 3 or 4 in G. This allows us to connect graph coverings with the fundamental group. Our results also give an alternative proof of the result in [12] that the first path homology over $\mathbb{Z}$ is isomorphic to the abelianization of this fundamental group.

1.1. Organization and main results

The remainder of this paper will be organized as follows. In Section 2, we will give the technical preliminaries, including definitions and known results concerning gain graphs, cycle bases, covers of graphs, graph homology, and graph curvature. In addition to relevant known results, we present some new lemmas that will be useful later.

In Section 3.1, we prove a relationship between the path first homology group of a graph and its cycle space. Namely, we show that the first homology group $H_1(G, \mathbb{F})$, for a field $\mathbb{F}$, is isomorphic to the $\mathbb{F}$-cycle space modulo the space generated by all 3- and 4-cycles. This gives an interpretation of what the first path homology group is "measuring" in terms of a well-studied combinatorial concept–the cycle space.

Section 3.2 investigates the relationships between cycle spaces and gain graphs. In particular we prove that a collection of circuits generates $\mathbb{F}$-cycle space if and only if it is a Γ-circuit generator, where Γ is the additive group of the field $\mathbb{F}$.

Section 3.3 explores the relationship between gain graphs and covers of graphs. In particular, we use a construction due to Gross and Tucker [16] to produce coverings of graphs corresponding to gain functions, and show that those coverings preserve precisely the cycles that are balanced under the gain function.

In Section 3.4, we connect the Bakry-Émery curvature to coverings of graph. Using a known diameter bound involving the curvature [9, 21], we prove that a graph with positive curvature has no infinite covering that preserves 3- and 4-cycles.

In Section 3.5, we combine all these results to prove Theorem 1.1, the main result of the paper.

Section 3.6 investigates the fundamental group of a graph $\pi_1(G)$ under the notion of homotopy from [12]. This notion of homotopy treats 3- and 4-cycles as contractible subgraphs. We show that this fundamental group is isomorphic to a quotient of a canonical gain group balanced on the set of 3- and 4-cycles. Indeed, we define a generalization of the fundamental group, treating any arbitrary collection of cycles as contractible, and show that this is similarly isomorphic to a quotient of a canonical gain graph with the same set of cycles. As in classical topology, we connect the fundamental group with the universal covering allowing us to prove Theorem 1.2. We also show how our results give an alternate proof Theorem 4.23 of [12], which says that $H_1(G, \mathbb{Z})$ is the abelianization of $\pi_1(G)$.

Finally, in Section 4, we discuss some open questions and possible weakening of the hypotheses of Theorem 1.1. We discuss a different notion of homology, and observe that Theorem 1.1 does not hold for this other definition. This strengthens the notion that the path homology has many advantages over other graph homology theories.

2. Preliminaries

We will denote a graph, either directed or undirected, as $G = (V, E)$. Two vertices are called *adjacent* if they are connected by an edge. The *neighborhood* of a vertex v is the set of vertices adjacent to v. The *degree* of a vertex v is the number of vertices adjacent to v. For a subset $S \subset V$, the subgraph of G *induced* by S is the subgraph of G obtained by deleting all vertices not in S (so the induced subgraph has vertex set S and all edges between those vertices that were present in S). A *circuit* or *simple cycle* C is a simple closed walk $(x_1, \ldots, x_n)$ of distinct vertices x_i adjacent to x_{i+1}, and x_n adjacent to x_1. A circuit with k vertices will be referred to as a k-cycle. We write $\mathcal{S}(G)$ for the set of all circuits in G.

2.1. Path homology of graphs

In this section, we will give definitions for a homology theory on graphs that has been developed in recent years, called *path homology.* See [11, 13, 14]. This homology theory is most naturally described for directed graph, and the homology of an undirected graph is obtained by orienting each edge of an undirected graph in both possible directions.

For a directed graph $G = (V, E)$ (without self-loops) we start by defining an *elementary* m*-path* on V to be a sequence $i_0, \ldots, i_m$ of $m+1$ vertices of V. For a field $\mathbb{F}$ we define the $\mathbb{F}$-linear space Λ_m to consist of all formal linear combinations of elementary m-paths with coefficients from $\mathbb{F}$. We identify an elementary m-path as an element of Λ_m denoted by $e_{i_0 \cdots i_m}$, and $\{e_{i_0 \cdots i_m} : i_0, \ldots, i_m \in V\}$ is a basis for Λ_m. Elements of Λ_m are called m*-paths*, and a typical m-path p can be written as

$$p = \sum_{i_0, \ldots, i_m \in V} a_{i_0 \cdots i_m} e_{i_0 \cdots i_m}, \quad a_{i_0 \cdots i_m} \in \mathbb{F}.$$

Note that Λ_0 is the set of all formal linear combinations of vertices in V.

We define the *boundary operator* $\partial : \Lambda_m \to \Lambda_{m-1}$ to be the $\mathbb{F}$-linear map that acts of elementary m-paths by

$$\partial e_{i_0 \cdots i_m} = \sum_{k=0}^{m} (-1)^k e_{i_0 \cdots \hat{i}_k \cdots i_m},$$

where $\hat{i}_k$ denotes the omission of index i_k.

For convenience, we define $\Lambda_{-1} = 0$ and $\partial : \Lambda_0 \to \Lambda_{-1}$ to be the zero map.

It can be checked that $\partial^2 = 0$, so that the Λ_m give a chain complex (see [14]). When it is important to make the distinction, we will use ∂_m to denote the boundary map on Λ_m, $\partial_m : \Lambda_m \to \Lambda_{m-1}$.

An elementary m-path $i_0 \cdots i_m$ is called *regular* if $i_k \neq i_{k+1}$ for all k, and is called *irregular* otherwise. Let I_m be the subspace of Λ_m spanned by all irregular m-paths, and define

$$\mathcal{R}_m = \Lambda_m / I_m.$$

The space $\mathcal{R}_m$ is isomorphic to the span of all regular m-paths, and the boundary map ∂ is naturally defined on $\mathcal{R}_m$, treating any irregular path resulting from applying ∂ as 0.

In the graph $G = (V, E)$, call an elementary m-path $i_0 \cdots i_m$ *allowed* if $i_k i_{k+1} \in E$ for all k. Define $\mathcal{A}_m$ to be the subspace of $\mathcal{R}_m$ given by

$$\mathcal{A}_m = \text{span}\{e_{i_0 \cdots i_m} : i_0 \cdots i_m \text{ is allowed}\}.$$

The boundary map ∂ on $\mathcal{A}_m$ is simply the restriction of the boundary map on $\mathcal{R}_m$, however, it can be the case that the boundary of an allowed m-path is not an allowed $(m-1)$-path. So we make one further restriction, and call an elementary m-path p *∂-invariant* if ∂p is allowed. We define

$$\Omega_m = \{p \in \mathcal{A}_m : \partial p \in \mathcal{A}_{m-1}\}.$$

Then it can be seen that $\partial \Omega_m \subseteq \Omega_{m-1}$. The Ω_m with the boundary map ∂ give us our chain complex of ∂-invariant allowed paths from which we will define our homology:

$$\cdots \Omega_m \xrightarrow{\partial} \Omega_{m-1} \xrightarrow{\partial} \cdots \to \Omega_1 \to \Omega_0 \to 0.$$

Observe that Ω_0 is the space of all formal linear combinations of vertices of G, and Ω_1 is space of all formal linear combinations of edges of G. We can now define the homology groups of this chain complex.

Definition 2.1 (path homology). The *path homology groups* of the graph G over the field $\mathbb{F}$ are

$$H_n(G, \mathbb{F}) = Ker\ \partial|_{\Omega_n} / Im\ \partial|_{\Omega_{n+1}}.$$

A standard fact is that $dim\ H_0(G, \mathbb{F})$ counts the number of connected components of G ([13]).

As a standard example of some of the interesting behavior of this homology, consider the directed 4-cycle pictured below.

$$\begin{tikzcd} y \arrow[r] & z \\ x \arrow[u] \arrow[r] & w \arrow[u] \end{tikzcd}$$

A 4-cycle with this orientation will be called a *square*. Note that the 2-paths e_{xyz} and e_{xwz} are allowed but not ∂-invariant, since the edge xz is missing from the graph. However, if we consider the linear combination $e_{xyz} - e_{xwz}$, then note

$$\begin{aligned}\partial(e_{xyz} - e_{xwz}) &= e_{yz} - e_{xz} + e_{xy} - (e_{wz} - e_{xz} + e_{xw}) \\ &= e_{yz} + e_{xy} - e_{wz} - e_{xw} \in \Omega_1.\end{aligned}$$

Thus $e_{xyz} - e_{xwz} \in \Omega_2$, and it turns out that $\Omega_2 = span\{e_{xyz} - e_{xwz}\}$. It can be seen that $Ker\ \partial_1$ is spanned by $e_{xy} + e_{yz} - e_{wz} - e_{xw}$, which is precisely $Im\ \partial_2$, so $H_1(G, \mathbb{F}) = 0$. A *triangle* is the oriented 3-cycle pictured below.

$$\begin{tikzcd} & y \arrow[dr] & \\ x \arrow[ur] \arrow[rr] & & z \end{tikzcd}$$

A directed graph with two vertices x, y and edges (x, y) and (y, x) we will refer to as a *2-cycle*. Similar computations show that H_1 is trivial for a triangle and 2-cycle as well. If the graph G is an oriented cycle other than the square, triangle, or 2-cycle, (in particular, any cycle of length more than 4), then $dim\ H_1(G, \mathbb{F}) = 1$. See [14] for details.

2.2. Graph homotopy and fundamental group

There is a separate notion of homotopy for graphs [12] under which the path homology is invariant. Via this homotopy, one can define a fundamental group of a graph. Namely, for a graph G, we specify a base vertex v_*, and define a based loop as a map $\phi : I_n \to G$ where I_n is a path on vertices $0, \ldots, n$, and ϕ satisfies $\phi(0) = \phi(n) = v_*$. Here, the map ϕ is a *graph map*, meaning that for $x \sim y$, either $\phi(x) \sim \phi(y)$ or $\phi(x) = \phi(y)$. Two loops are considered equivalent if there is a C-homotopy between them, where homotopy is defined in a way anologous to homotopy of algebraic topology. The exact definition of this is not needed here, but details can be found in [12]. We will make use of the following result from [12] to determine when two loops are equivalent. For our purposes, we can take this as the definition of C-homotopy. To state this, we need the following terminology: given a loop $\phi : I_n \to G$, the *word* of ϕ, denoted θ_ϕ is the sequence $v_0, \ldots, v_n$ with $v_i = \phi(i)$ for $i = 0, \ldots, n$.

Theorem 2.2 (Theorem 4.13 of [12]). *Two based loops $\phi : I_n \to G$ and $\psi : I_m \to G$ are C-homotopic if and only if the word θ_ψ can be obtained from θ_ϕ by a finite sequence of the following transformations and their inverses:*

1) $\cdots abc \cdots \mapsto \cdots ac \cdots$ *where (a, b, c) are vertices forming a triangle in G (and the $\cdots$ denotes the unchanged part of the word);*
2) $\cdots abc \cdots \mapsto \cdots adc \cdots$ *where (a, b, c, d) forms a square in G;*
3) $\cdots abcd \cdots \mapsto \cdots ad \cdots$ *where (a, b, c, d) is a square in G;*
4) $\cdots aba \cdots \mapsto \cdots a \cdots$ *if $a \sim b$;*
5) $\cdots aa \cdots \mapsto \cdots a \cdots$.

One interpretation of this is that triangles, squares, and single edges are *contractible* subgraphs of a graph.

The set of all equivalence classes of loops in G forms a group called the *fundamental group* of G, denoted $\pi_1(G)$. The group operation is concatenation of loops, the identity element is the trivial loop that maps all vertices to the base vertex, and the inverse of a loop is the loop traversed in reverse order. See [12] for details of why this is well-defined and forms a group.

2.3. Curvature bounds in graphs

For an undirected graph $G = (V, E)$, the *graph Laplacian* is the operator Δ on the space of functions $f : V \to \mathbb{R}$ given by

$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)).$$

The *Bakry-Émery operators* are defined via

$$\Gamma(f, g) := \frac{1}{2} \left(\Delta(fg) - f\Delta g - g\Delta f \right)$$
$$\Gamma_2(f, g) := \frac{1}{2} \left(\Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(g, \Delta f) \right).$$

We write $\Gamma(f) := \Gamma(f, f)$ and $\Gamma_2(f) = \Gamma_2(f, f)$.

Definition 2.3 (Bakry-Émery Curvature). A graph G is said to satisfy the *curvature dimension inequality* $CD(K, n)$ for some $K \in \mathbb{R}$ and $n \in (0, \infty]$ if for all f,

$$\Gamma_2(f) \geq \frac{1}{n} (\Delta f)^2 + K \cdot \Gamma(f).$$

Remark. A critical observation concerning the Bakry-Émery curvature is that it is a very local property. Indeed, the curvature of a vertex x is completely determined by the 2-ball centered at x, that is, the graph induced by the vertices at distance at most 2 from x. In fact, the full 2-ball is not even needed, but the "open" 2-ball, meaning the 2-ball deleting any edges between vertices at distance 2 from x. See [7, Section 2.1] for a more detailed discussion of this local aspect of the curvature. Note in particular, in this open 2-ball at a vertex, there will be no induced simple circuits of length 5 or more–any such long circuit will have a *chord*, or an edge between two vertices of the circuit, so that there is a smaller 3-cycle or 4-cycle within it. So in a sense, the curvature at x does not "see" induced cycles containing x of length longer than 4.

We state a diameter bound in terms of curvature, similar to the Bonnet-Myers theorem from geometry, proven for graphs in [21]. A similar result is found in [9]. We define the *diameter* of a graph G, denoted $diam(G)$ to be the maximum distance between two vertices in G, where distance is given by the minimum number of edges in a path connecting two vertices.

Theorem 2.4 (Bonnet-Myers Theorem, Corollary 2.2 of [21]). *Let G be a graph satisfying $CD(K,\infty)$ for some $K>0$, and let D_{max} be the maximum degree in G. Then*

$$diam(G) \leq \frac{2D_{max}}{K}.$$

2.4. Cycle space

In this section we give the necessary preliminaries associated with the cycle space of a graph. We take definitions primarily from [19].

In a directed graph $G=(V,E)$, an *oriented circuit* C is a subset of E that can be written $C=C^+\cup C^-$ consisting of *forward pointing* edges in C^+ and *backward pointing* edges in C^- such that reversing all the directed edges in C^- yields a simple directed cycle. The *incidence vector* ϕ_C of an oriented circuit C is a vector in $\{0,\pm 1\}^E$ such that an entry corresponding to an edge in C^+ is 1, corresponding to an edge in C^- is -1 an an edge not in C is 0.

Given a field $\mathbb{F}$, the *cycle space* of G over $\mathbb{F}$, or $\mathbb{F}$-*cycle space*, denoted $\mathcal{C}(G,\mathbb{F})$, is the subspace of $\mathbb{F}^E$ spanned by all incidence vectors of oriented circuits of G.

A *circulation* on G is a function $\phi : E \to \mathbb{F}$ satisfying

$$\sum_{\substack{x \\ (x,v)\in E}} \phi(x,v) = \sum_{\substack{x \\ (v,x)\in E}} \phi(v,x)$$

for all $v \in V$. Observe that the cycle space $\mathcal{C}(G,\mathbb{F})$ is simply the space of all circulations on G (see [19]).

Definition 2.5 ($\mathbb{F}$-cycle basis). A subset $\mathcal{B} \subset \mathcal{C}(G,\mathbb{F})$ is called an $\mathbb{F}$-*cycle basis* of G if $\mathcal{B}$ is a vector space basis of $\mathcal{C}(G,\mathbb{F})$. In the literature, a $\mathbb{Q}$-cycle basis is also called a *directed* cycle basis, where $\mathbb{Q}$ is the field of rational numbers, and an $\mathbb{F}_2$-cycle basis is also called an *undirected* cycle basis, where $\mathbb{F}_2$ is the field with 2 elements.

According to [19], we define the determinant of a set of cycles. Let r be the cyclomatic number of G and let $\mathcal{B} \subset \mathcal{S}(G)$ be of size r. Corresponding to [19, Definition 22], consider the matrix $M(\mathcal{B},\mathbb{F})$ over the field $\mathbb{F}$ with the incidence vectors of $\mathcal{B}$ as columns. Let $M(\mathcal{B},\mathbb{F},T)$be the $r \times r$ submatrix that arises when deleting the arcs of the spanning tree T of G. Remark that $M(\mathcal{B},\mathbb{F})$ consists only of the entries 0 and ± 1. Now write

$$\det \mathcal{B} := |\det M(\mathcal{B},\mathbb{Q},T)|.$$

It is shown in [19] that $\det \mathcal{B}$ does not depend on the choice of the spanning tree T. The following theorem is a simple generalization of the characterization of directed and undirected cycle basis via determinants (see [19]).

Theorem 2.6. *A set $\mathcal{B} \subset \mathcal{C}(G,\mathbb{F})$ is an $\mathbb{F}$-cycle basis if and only if $\det \mathcal{B} \not\equiv 0 \mod \chi(\mathbb{F})$ where $\chi(\mathbb{F})$ denotes the characteristic of the field $\mathbb{F}$.*

Proof. It is easy to see that $\mathcal{B}$ is a $\mathbb{F}$-vector space basis of $\mathcal{C}(G,\mathbb{F})$ if and only if $M(\mathcal{B},\mathbb{F},T)$ is invertible as a matrix over $\mathbb{F}$ for a given spanning tree T of G. This holds true if and only if $\det M(\mathcal{B},\mathbb{F},T) \neq 0$ which is equivalent to

$$\det M(\mathcal{B},\mathbb{F},T) \not\equiv 0 \mod \chi(\mathbb{F}).$$

This directly implies the theorem. □

2.5. Gain graphs and circuit generators

Let $G = (V,E)$ be a undirected graph. We will denote by $\vec{E}$ the set that contains two directed arcs, one in each direction, for each edge in E. Let Γ

be a group. A *gain graph* is a triple (G, ϕ, Γ) where $\phi : \vec{E} \to \Gamma$ is a map satisfying $\phi(xy) = \phi(yx)^{-1}$ for all edges $(xy) \in \vec{E}$. The map ϕ is called the *gain function* of the gain graph. Denote by $\Phi(G, \Gamma)$ the set of all gain functions from the graph G to the group Γ.

Gain graphs have also been referred to as *voltage graphs*, and are special cases of *biased graphs* (see [25]). When Γ is a group of invertible linear transformations, they are also called *connection graphs* [5], and the map ϕ can be considered as a connection corresponding to a vector bundle on the graph [17].

For the most part, we will take terminology about gain graphs and biased graphs from [25]. A *theta graph* is the union of three internally disjoint simple paths that have the same two distinct endpoint vertices. A *biased graph* is a pair $(G, \mathcal{B})$ where $\mathcal{B} \subset \mathcal{S}(G)$ is a set of distinguished circuits, called *balanced circuits*, that form a *linear subclass* of circuits, that is, $\mathcal{B}$ has the property that if any two circuits of a theta graph are in $\mathcal{B}$, then so is the third. We say $\mathcal{B} \subset \mathcal{S}(G)$ is a *cyclomatic circuit set* if the cardinality of $\mathcal{B}$ equals the cyclomatic number of G, $|E| - |V| + c$ where c is the number of components of the graph. Typically, we will be thinking of connected graphs with $c = 1$.

Gain graphs are biased graphs in a natural way. Define the *order* of a circuit $C = (x_1, \ldots, x_n)$ under a gain function ϕ via

$$o_\phi(C) := \inf\left\{r > 0 : [\phi(x_1x_2)\ldots\phi(x_{n-1}x_n)\phi(x_nx_1)]^r = e_\Gamma\right\}.$$

We say the gain function ϕ is *balanced* on a circuit $C = (x_1, \ldots, x_n)$ if $o_\phi(C) = 1$. Denote by $\mathcal{B}(\phi)$ the set of balanced circuits. Then $(G, \mathcal{B}(\phi))$ defines a biased graph.

Definition 2.7 (Γ-circuit generator). We say a set $\mathcal{B}$ of circuits is a Γ-*circuit generator* if for all $\phi \in \Phi(G, \Gamma)$, ϕ balanced on $\mathcal{B}$ implies that ϕ is balanced on the entire graph G.

Definition 2.8 (Canonical gain graph). Let $G = (V, E)$ be a graph and let $\mathcal{B}$ be a linear subclass of circuits. We define the group $\Gamma(G, \mathcal{B})$ via the presentation

$$\Gamma(G, \mathcal{B}) = \langle \vec{E} | \mathcal{B} \rangle,$$

i.e., $\Gamma(G, \mathcal{B})$ is generated by the oriented edges and each circuit $C = (x_1, \ldots, x_n) \in \mathcal{B}$ gives a relation $(x_1x_2)\ldots(x_{n-1}x_n)(x_nx_1) = e_\Gamma$ where we identify $(xy) = (yx)^{-1} \in \vec{E}$. There is a natural gain function $\phi_{\mathcal{B}}$ given by the natural mapping of $\vec{E}$ into $\Gamma(G, \mathcal{B})$, and the corresponding gain graph

$(G, \phi_{\mathcal{B}}, \Gamma(G, \mathcal{B}))$ is called the *canonical gain graph* associated with the biased graph $(G, \mathcal{B})$.

2.6. Circuit preserving coverings

Let $G = (V, E)$ be a connected graph. Define the neighborhood of a vertex v to be the set of all vertices adjacent to v. If $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ is a graph, and $\Psi : \widetilde{G} \to G$ is a surjective graph homomorphism such that Ψ is locally bijective (i.e., Ψ is bijective when restricted to the neighborhood of a single vertex), then the pair $(\widetilde{G}, \Psi)$ is called a *covering* of G.

Since Ψ is locally bijective, than it can be seen that $|\Psi^{-1}(x)|$ is constant for all vertices $x \in V$. If this constant value is m, we say that $(\widetilde{G}, \Psi)$ is a covering with m *sheets*, or is an m*-sheeted* covering. Here, m can be infinite.

We call a covering $(\widetilde{G}, \Psi)$ *trivial* if Ψ restricted to any connected component of $\widetilde{G}$ is a graph isomorphism. We say it is *non-trivial* otherwise, i.e., if there is at least one connected component of $\widetilde{G}$ on which Ψ is not one-to-one.

Let $\mathcal{B} \subset \mathcal{S}(G)$ be a set of circuits. We say a covering $(\widetilde{G}, \Psi)$ is a $\mathcal{B}$ *preserving covering* of G if for all circuits $C = (x_1, \ldots, x_n) \in \mathcal{B}$ and all $\widetilde{x}_1 \in \widetilde{V}$ with $\Psi(\widetilde{x}_1) = x_1$, there exist a circuit $\widetilde{C} = (\widetilde{x}_1, \ldots, \widetilde{x}_n) \in \widetilde{V}$ s.t. $\Psi(\widetilde{x}_k) = x_k$ for all k. Note in particular that every circuit in the pre-image of C has length equal to the length of C.

3. Main results

3.1. Homology

In this section, we give a combinatorial interpretation of the first homology group in terms of the cycle space. We will state this for a general directed graph $G = (V, E)$.

Theorem 3.1. *Let $\mathbb{F}$ be a field with characteristic not equal to 2, and let $\mathcal{C}(G, \mathbb{F})$ denote the $\mathbb{F}$-cycle space of the directed graph G. Let TS denote the subspace of $\mathcal{C}(G, \mathbb{F})$ that is generated by all 2-cycles, triangles, and squares of G. Then*

$$H_1(G, \mathbb{F}) \cong \mathcal{C}(G, \mathbb{F})/TS.$$

The rest of this section will be devoted to proving this theorem.

Recall that by definition,

$$H_1(G, \mathbb{F}) = Ker\ \partial_1 / Im\ \partial_2$$

where ∂_1 denotes the boundary operator on 1-paths and ∂_2 the boundary operator on 2-paths.

Observe that the space Ω_1 of 1-paths can be naturally identified with the space of functions from the edge set to the field $\mathbb{F}$; that is

$$\Omega_1 \cong \{\phi : E \to \mathbb{F}\}.$$

Lemma 3.2. *$Ker\ \partial_1 \cong \mathcal{C}(G, \mathbb{F})$.*

Proof. The action of ∂_1 on Ω_1 can be given as

$$\partial_1 \phi = \sum_{(u,v)\in E} \phi(uv)(e_v - e_u) = \sum_v \left(\sum_{(x,v)\in E} \phi(xv) - \sum_{(v,x)\in E} \phi(vx) \right) e_v.$$

Therefore $\phi \in Ker\ \partial_1$ if and only if

$$\sum_{(x,v)\in E} \phi(xv) = \sum_{(v,x)\in E} \phi(vx) \text{ for all } v.$$

Thus the kernel of ∂_1 is exactly the space of all circulations on G, which is the cycle space $\mathcal{C}(G, \mathbb{F})$ (see the remark after the definition of cycle space in Section 2.4). This gives the lemma. $\square$

Lemma 3.3. *$Im\ \partial_2 \cong TS$.*

Proof. First, let ϕ be the incidence vector of a 2-cycle in G. That is we have edges xy and yx in G. Then ϕ is identified with the element $e_{yx} - e_{xy}$ of Ω_1. Note that e_{xyx} is allowed, and

$$\partial_2 e_{xyx} = e_{yx} - e_{xy}.$$

Thus $\phi \in Im\ \partial_2$.

Suppose we have a triangle in G consisting of edges xy, yz, and xz with incidence vector ϕ. Then

$$\partial_2(e_{xyz} - e_{zyx}) = e_{yz} - e_{xz} + e_{xy}$$

which is the triangle ϕ. So any triangle is contained in $Im\ \partial_2$.

Similarly let ϕ be a square of G with edges xy, yz, xw, wz. Then $e_{xyz} - e_{xwz}$ is allowed, and

$$\begin{aligned}\partial_2(e_{xyz} - e_{xwz}) &= e_{yz} - e_{xz} + e_{xy} - e_{wz} + e_{xz} - e_{xw} \\ &= e_{yz} + e_{xy} - e_{wz} - e_{xw}\end{aligned}$$

Therefore any square is in $Im\ \partial_2$ as well. It follows that the space $TS \subset Im\ \partial_2$.

Conversely, we must show $Im\ \partial_2 \subset TS$. Let $\phi \in Im\ \partial_2$ so we can write

$$\begin{aligned}\phi &= \partial_2\left(\sum_{xyz} a_{xyz}e_{xyz}\right) \\ &= \sum_{xyz} a_{xyz}(e_{yz} - e_{xz} + e_{xy}),\end{aligned}$$

where the sum is taken over ∂-invariant allowed paths xyz of G. We will split this sum into three cases. First, we may have that $z = x$, in which case xy and yx are both edges of G since the sum only includes allowed paths of G. In this case, e_{xx} vanishes (see Section 2.1). When $x \neq z$, then we must have xy and yz are edges of G, again, since Ω_2 consists only of ∂-invariant allowed elements. We then further split this sum depending on whether xz is an edge or not. That is, the sum above is equal to

$$\begin{aligned}&\sum_{xyx} a_{xyx}(e_{yx} - e_{xy}) + \sum_{\substack{xyz \\ z\neq x \\ xz\in E(G)}} a_{xyz}(e_{yz} - e_{xz} + e_{xy}) \\ &\quad + \sum_{\substack{xyz \\ z\neq x \\ xz\notin E(G)}} a_{xyz}(e_{yz} - e_{xz} + e_{xy}).\end{aligned}$$

If is clear that the first term is a linear combination of 2-cycles and the second is a linear combination of triangles.

For the last term, since it is allowed, any e_{xz} term must cancel. Thus, for any xyz for which a_{xyz} is non-zero in the last sum, there must be some other allowed 2-path in which e_{xz} shows up as a term. Namely, there exists $w \neq y$ such that xwz is allowed in G, and the coefficient $a_{xwz} = -a_{xyz}$. This then is a linear combination of squares of G. Thus $Im\ \partial_2 \subset TS$, and we have shown the lemma. □

Lemma 3.2 and Lemma 3.3 together immediately give the proof of Theorem 3.1.

Remark. If G is an undirected graph, then Theorem 3.1 still holds. To compute the homology, we simply treat each edge as two directed edges, one in each direction. Then all the 2-cycles cancel when we take the quotient, so the homology is the cycle space of the undirected graph modulo the 3-cycles and 4-cycles.

3.2. Cycle space and gain graphs

Theorem 3.4. *Let $\mathbb{F}$ be a field with additive group Γ. Let $\mathcal{B} \subset \mathcal{C}(G, \mathbb{F})$ be a collection of circuits of a graph G. T.f.a.e.:*

1) *$\mathcal{B}$ spans the $\mathbb{F}$-cycle space of G.*
2) *$\mathcal{B}$ is a Γ-circuit generator.*

Proof. $1 \Rightarrow 2$: We aim to show that every gain function ϕ is balanced on all cycles when assuming that ϕ is balanced on $\mathcal{B}$. Now, ϕ is balanced on C if and only $\phi(C) = 0$. Since ϕ is linear and $\phi(C) = 0$ for all $C \in \mathcal{B}$ due to assumption, we infer that $\phi(C) = 0$ for all $C \in \operatorname{span}(\mathcal{B}) = \mathcal{C}(G, \mathbb{F})$ since we assume that $\mathcal{B}$ spans the cycle space of G.

$2 \Rightarrow 1$: We indirectly prove the claim. Assume $\mathcal{B}$ does not span the cycles space ($\mathcal{B}$ does not contain an $\mathbb{F}$-cycle basis). Then, there exists a basis $\widetilde{\mathcal{B}}$ and $C_0 \in \widetilde{\mathcal{B}}$ s.t. $\mathcal{B} \subset \operatorname{span}(\widetilde{\mathcal{B}} \setminus \{C_0\})$. The matrix $M(\widetilde{\mathcal{B}}, \mathbb{F})$ is a $r \times |E|$ matrix with full rank r. Hence, the multiplication with the gain functions $M(\widetilde{\mathcal{B}}, \mathbb{F}) : \Phi(G, \mathbb{F}) \to \mathbb{F}^{\widetilde{\mathcal{B}}}$ is surjective. In particular, there exists $\phi \in \Phi(G, \mathbb{F})$ such that for $C \in \widetilde{\mathcal{B}}$,

$$\phi(C) = \left[M(\widetilde{\mathcal{B}}, \mathbb{F})\phi\right](C) = \begin{cases} 1 & : C = C_0 \\ 0 & : C \in \widetilde{\mathcal{B}} \setminus \{C_0\} \end{cases}.$$

This implies $\phi(C) = 0$ for all $C \in \mathcal{B}$ and $\phi(C_0) = 1$ which proves that $\mathcal{B}$ is not a Γ-circuit generator. This finishes the proof. $\square$

3.3. Gain graphs and covering

For gain graphs, there is a natural construction of a covering of the graph that is derived from the gain function. This construction is given in [16] and is a variant on a construction from [15].

Definition 3.5. Let $G = (V, E)$ be a graph Let ϕ be a gain function on G into group Γ. We define the *(ordinary) derived graph*, denoted $G^\phi = (V^\phi, E^\phi)$

as

$$\begin{aligned} V^{\phi} &= V \times \Gamma \\ E^{\phi} &= \{\{(u,g),(v,g\phi(uv))\} : uv \in E, g \in \Gamma\}. \end{aligned}$$

There is a natural projection $\Psi : G^{\phi} \to G$ given by $\Psi((u,g)) = u$. It is clear, as noted in [16], that Ψ is a covering map, so that (G^{ϕ}, Ψ) is a covering of G with $|\Gamma|$ sheets.

Lemma 3.6. *Let (G, ϕ, Γ) be a gain graph, with (G^{ϕ}, Ψ) the corresponding ordinary derived covering. Given a circuit C of length n of G, the pre-image $\Psi^{-1}(C)$ consists of a collection of vertex disjoint circuits $\{\widetilde{C}\}$ where each $\widetilde{C}$ is of length $o_{\phi}(C) \cdot n$. In the case $o_{\phi}(C)$ is infinite, $\Psi^{-1}(C)$ contains an infinite path.*

Proof. Let $C = (x_1, \ldots, x_n)$ be a circuit of G, and fix $g \in \Gamma$. Set $g_1 = g$, and define $g_{j+1} = g_j \phi(x_j x_{j+1})$ where the index j on the x's is taken $\pmod{n}$. Then since $x_j x_{j+1} \in E$ for all j, then $\{(x_j, g_j), (x_{j+1}, g_{j+1})\} \in E^{\phi}$ for all j by definition. Now we ask, when (if ever) does the sequence $\widetilde{C} = ((x_1, g_1), (x_2, g_2), \ldots)$ return to its starting point at (x_1, g_1). Clearly, for this to be the case, the index j satisfies $j \equiv 1 \pmod{n}$, and every time j becomes $1 \pmod{n}$, the associated element gets mapped by $\phi(C)$. So the sequence comes back to (x_1, g_1) when the x_j have come back to x_1 $o_{\phi}(C)$ times. The sequence cannot intersect itself at any earlier point by minimality of $o_{\phi}(C)$. Therefore clearly $\widetilde{C}$ is a circuit of length $o_{\phi}(C) \cdot n$. If there are multiple distinct circuits in $\Psi^{-1}(C)$, it is clear that they are vertex disjoint by the definition of G^{ϕ}. □

We remark that for the derived cover (G^{ϕ}, Ψ), every circuit in Ψ^{-1} has the same length, $o_{\phi}(C) \cdot n$.

Corollary 3.7. *Let (G, ϕ, Γ) be a gain graph with (G^{ϕ}, Ψ) the associated ordinary derived covering.If $\mathcal{B}$ is a collection of circuits of G, then ϕ is balanced on $\mathcal{B}$ if and only if the covering from G^{ϕ} to G preserves $\mathcal{B}$.*

Proof. For the derived covering G^{ϕ}, a cycle C of length n is balanced if and only of $\phi(C)$ is the identity, which holds if and only if $o_{\phi}(C) = 1$ so that $\Psi^{-1}(C)$ is a collection of vertex disjoint cycles of length n by Lemma 3.6. This is the definition of C being preserved under the cover. □

Remark. It is proven in [16] that any regular covering of a graph is a derived covering for some group and some gain function. In addition, [16] defines a generalization of the ordinary derived graph called the *permutation derived graph* and shows that any covering of a graph, regular or not, arises as a permutation derived covering. The results above can also be done for permutation derived coverings as well (although the lengths of circuits in the pre-image of the covering map may not be constant).

3.4. Curvature and covering

Theorem 3.8. *Suppose a finite graph G satisfies $CD(K,\infty)$ for some $K > 0$. Then, there exists no infinite covering of G preserving all 3- and 4-cycles.*

Proof. Suppose there exists an infinite covering $(\widetilde{G},\Psi)$ of G preserving all 3- and 4-cycles. Consider any $x \in \widetilde{G}$ and its image $\Psi(x)$. Since Ψ preserves 3- and 4-cycles, then the image of any 3- or 4-cycle containing x in $\widetilde{G}$ is a 3- or 4-cycle containing $\Psi(x)$ in G. Then since Ψ is locally bijective and preserves these cycles, we see that the open 2-ball around x (see the remark after Definition 2.3) is isomorphic to the open 2-ball around $\Psi(x)$. This implies that $\widetilde{G}$ and G satisfy the same curvature bound $CD(K,\infty)$ (see the remark after Definition 2.3). Now, Theorem 2.4 implies $diam(\widetilde{G}) \leq \frac{2D_{max}}{K}$ and thus finiteness of $\widetilde{G}$. This is a contradiction and therefore proves that there is no infinite covering of G preserving 3- and 4-cycles. □

3.5. Proof of main result

We now have all the tools needed to prove the main result, which we restate as follows.

Theorem 3.9. *If G is a finite graph satisfying $CD(K,\infty)$ for some $K > 0$ and if $\mathbb{F}$ is a field with characteristic 0, then*

$$H_1(G,\mathbb{F}) = 0.$$

Proof. Suppose by way of contradiction that $H_1(G,\mathbb{F})$ is non-trivial. Let $\mathcal{B}$ be the collection of triangles and squares in G. Then by Theorem 3.1, the cycle space of G is not generated by $\mathcal{B}$. Then by Theorem 3.4, $\mathcal{B}$ is not a Γ-circuit generator where Γ is the additive group of $\mathbb{F}$. Thus there is some gain function $\phi : \vec{E} \to \Gamma$ that is balanced on all triangles and squares, but is unbalanced on some other cycle, call it C. Then we can construct the ordinary derived covering G^{ϕ} with projection Ψ of Section 3.3. Since ϕ is not balanced on C and since $\mathbb{F}$ has characteristic 0, then $o(\phi(C)) = \infty$.

By Lemma 3.6, $\Psi^{-1}(C)$ contains an infinite path, and by Corollary 3.7, Ψ preserves $\mathcal{B}$. But then Ψ is an infinite covering of G preserving all triangles and squares, so by Theorem 3.8, G cannot satisfy $CD(K,\infty)$ for some $K>0$ at every vertex. This implies the result. $\square$

3.6. Homotopy and fundamental group

In this section we examine the fundamental group of a graph as defined by Grigoryan et. al. [12], and connect this to the group for the canonical gain graph defined previously.

Recall from Theorem 2.2, the fundamental group of a graph can be described as the group of equivalence classes of loops, where loops are equivalent if their corresponding words differ by a finite sequence of application of some rules. These rules amount to triangles, squares, and trees being contractible. We will generalize this notion.

Definition 3.10. Let $\mathcal{B}$ be a collection of circuits of a graph G. Define $\pi_1(G,\mathcal{B})$ to be the group of equivalence classes of loops in G where two loops $\phi : I_n \to G$ and $\psi : I_m \to G$ are considered equivalent if the word θ_ψ can be obtained from θ_ϕ via a finite sequence of the following transformations and their inverses:

1) $\cdots av_1\cdots v_ib\cdots \mapsto \cdots aw_1\cdots w_jb\cdots$ where $(a,v_1,\ldots,v_i,b,w_j,\ldots,w_1)$ is a circuit from $\mathcal{B}$;
2) $\cdots aba\cdots \mapsto \cdots a\cdots$ if $a\sim b$;
3) $\cdots aa\cdots \mapsto \cdots a\cdots$

Observe then that the fundamental group $\pi_1(G)$ is precisely $\pi_1(G,\mathcal{B})$ where $\mathcal{B}$ is the collection of all triangles and squares in G.

Recall that, in Section 2, we defined for a family of circuits $\mathcal{B}$ the canonical gain group $\Gamma(G,\mathcal{B})$. Now, for a fixed spanning tree T of a graph G, we define the group $\Gamma(G,T,\mathcal{B})$ via the presentation

$$\Gamma(G,T,\mathcal{B}) = \langle \vec{E} \mid T, \mathcal{B}\rangle.$$

Theorem 3.11. *For any spanning tree T of G, we have*

$$\pi_1(G,\mathcal{B}) \cong \Gamma(G,T,\mathcal{B}).$$

In particular, the group $\Gamma(G,T,\mathcal{B})$ is independent of the spanning tree T up to isomorphism.

Proof. Let $\Gamma := \langle \vec{E} \mid T \rangle$ where we identify the edge xy with $(yx)^{-1}$. Define a map $\phi : \Gamma \to \pi_1(G, \mathcal{B})$ as follows. First, given $g \in \Gamma$, choose the shortest representative word $e_1 \cdots e_k$ without spanning tree edges, then associate to this word the loop given by starting at the base point v_*, and taking the unique path from v_* through T to the starting point of e_1, then go to the endpoint of e_1, and take the unique path in T from that vertex, to the starting vertex of e_2, continue in this manner until we reach the endpoint of e_k, and take the unique path in T from there to v_*. Then $\phi(e_1 \cdots e_k)$ is the equivalence class of this loop in $\pi_1(G, \mathcal{B})$.

Recall that $\Gamma(G, T, \mathcal{B}) = \langle \vec{E} \mid T, \mathcal{B} \rangle$,so by definition of a group presentation, is $\Gamma / \langle \mathcal{B} \rangle$ where $\langle \mathcal{B} \rangle$ is the normal closure of the set $\mathcal{B}$. So what we need to show is that ϕ is a well-defined surjective group homomorphism whose kernel is the normal closure of $\mathcal{B}$. Then we will be done by the first isomorphism theorem for groups.

Clearly, ϕ is well-defined. To see that it is a homomorphism, consider $\phi(e_1 \cdots e_j)\phi(e_{j+1} \cdots e_k)$. Since T is a spanning tree, the path in T from the endpoint of e_j to v_*, and from v_* to the start of e_{j+1} is equivalent to the path in T from the end of e_j to the start of e_{j+1}, possibly via application of the $\cdots aba \cdots \mapsto \cdots a \cdots$ rule of the definition of $\pi_1(G, \mathcal{B})$. Thus $\phi(e_1 \cdots e_j)\phi(e_{j+1} \cdots e_k) = \phi(e_1 \cdots e_k)$ as desired.

To show that ϕ is surjective, suppose the sequence $v_*, v_1, \ldots, v_k, v_*$ is the word of a loop in G. Then either $v_i = v_{i+1}$ or (v_i, v_{i+1}) is an edge of G. If $v_i = v_{i+1}$, we can get rid of one of these via the $\cdots aa \cdots \mapsto \cdots a \cdots$ rule. Due to the $\cdots aba \cdots \mapsto \cdots a \cdots$, we can assume without obstruction that the loop is non-backtracking, i.e., $v_{i+2} \neq v_i$ for all i. We consider the loop $v_* w_1 \cdots w_n v_* = \phi((v_*, v_1)(v_1, v_2) \cdots (v_{k-1}, v_k)(v_k, v_*))$. Since between every two vertices within a spanning tree there exists a unique non-backtracking path connecting both, we infer $w_i = v_i$ and $n = k$ meaning that ϕ maps g to the loop $v_* v_1 \cdots v_k v_*$. This proves surjectivity of ϕ.

Now, to see that $\mathcal{B} \subset Ker(\phi)$, suppose that if $e_1, \ldots, e_k = (u_1, u_2), \ldots, (u_k, u_1)$ are edges of a circuit from $\mathcal{B}$ with vertices $u_1, \ldots, u_k$. Then $\phi(e_1 \cdots e_k)$ is a loop whose word is has the form $v_* \cdots v_n u_1 u_2 \cdots u_k u_1 v_n \cdots v_*$, and by rules 1 and 2 of Definition 3.10, so this word belongs to $Ker(\phi)$.

Finally, we wish to show that $Ker(\phi)$ is a subset of the normal closure of $\mathcal{B}$. Let $w = e_1 \cdots e_k \in Ker(\phi)$. Then $\phi(w)$ is equivalent to the trivial loop, so this means that the word of the the loop $\phi(w)$ can be obtained form the trivial word v_* from a sequence transformations using the rules in Definition 3.10. Rules 2 and 3 can be ignored since these will not arise as images of words in Γ (they are equivalent to words in which these do not occur;

recall that xy and $(yx)^{-1}$ are identified). Then proceeding by induction on the applications of rule 1, this rule corresponds precisely to inserting edges of a circuit of $\mathcal{B}$ into the word w. So w is in the normal closure of $\mathcal{B}$. This implies the result. $\square$

The next corollary follows immediately. An equivalent formulation is found in [2, Theorem 5.11].

Corollary 3.12. *Let $\mathcal{B}$ be the collection of triangles and squares of G, and T any spanning tree of G. Then*

$$\pi_1(G) \cong \Gamma(G, T, \mathcal{B})$$

where $\pi_1(G)$ denotes the fundamental group of G from [12].

In [8], DeVos, Funk, and Pivotto make use of the group we are calling $\Gamma(G, T, \mathcal{B})$ to determine when a biased graph comes from a gain graph. As a step in this, they prove that this group is isomorphic to the fundamental group of the topological space obtained by attaching a 2-cell to every circuit of $\mathcal{B}$ (see the proof of Theorem 2.1 of [8]). Thus we have the following.

Corollary 3.13. *Let K be the 2-cell complex obtained by attaching disc to each circuit of $\mathcal{B}$. Then $\pi_1(G, \mathcal{B})$ is isomorphic to the fundamental group of this topological space. In particular, the fundamental group $\pi_1(G)$ of [12] is isomorphic to the fundamental group of the space obtained by attaching a disc to each triangle and square of G.*

In particular, there is a canonical 1-1 correspondence between coverings of the 2-cell complex K, and coverings of G preserving $\mathcal{B}$. Therefore, we can now characterize the existence of an infinite connected covering of G preserving $\mathcal{B}$.

Corollary 3.14. *Let $G = (V, E)$ be a connected graph and let $\mathcal{B}$ be a set of circuits. T.f.a.e:*

1) *There exists no infinite connected covering of G preserving $\mathcal{B}$*
2) *The fundamental group $\pi_1(G, \mathcal{B})$ is finite.*

Proof. Due to the 1-1 correspondence between coverings of the 2-cell complex K and $\mathcal{B}$ preserving coverings of G, the first statement is equivalent to finiteness of the universal cover of K, which is equivalent to finiteness of the fundamental group of K. This implies the corollary since the fundamental group of K is isomorphic to $\pi_1(G, \mathcal{B})$. $\square$

Combining this corollary with Theorem 3.8, we immediately obtain the following relation between curvature and the fundamental group.

Corollary 3.15. *Suppose a finite graph satisfies $CD(K, \infty)$ for some $K > 0$. Then, $\pi_1(G)$ is finite, where $\pi_1(G)$ denotes the fundamental group of G from [12].*

We now characterize the abelianization of the fundamental group $\pi_1(G, \mathcal{B})$.

Proposition 3.16. *Let $\mathcal{B}$ be any collection of cycles of G. Then*

$$Ab\ \pi_1(G, \mathcal{B}) \cong \mathcal{C}(G, \mathbb{Z})/\langle \mathcal{B} \rangle$$

where Ab denotes the abelianization of the group, and $\langle \mathcal{B} \rangle$ denotes the set of all integer linear combinations of cycles in $\mathcal{B}$.

Proof. Let T be a spanning tree. Due to Theorem 3.11,

$$\pi_1(G, \mathcal{B}) \cong \langle \vec{E} \mid T, \mathcal{B} \rangle.$$

Let $\Gamma := \langle \vec{E} \mid T \rangle$. We observe that $\mathrm{Ab}\,\Gamma \cong \mathcal{C}(G, \mathbb{Z})$. Abelianization of $\pi_1(G, \mathcal{B})$ yields

$$\begin{aligned} \mathrm{Ab}\ \pi_1(G, \mathcal{B}) &\cong \langle \vec{E} \mid T, \mathcal{B}, \{sts^{-1}t^{-1}\} \rangle \\ &\cong \frac{\langle \vec{E} \mid T, \{sts^{-1}t^{-1}\} \rangle}{\langle \mathcal{B} \rangle} \cong \frac{\mathrm{Ab}\,\Gamma}{\langle \mathcal{B} \rangle} \cong \frac{\mathcal{C}(G, \mathbb{Z})}{\langle \mathcal{B} \rangle} \end{aligned}$$

which finishes the proof. $\square$

It is known from Theorem 4.23 of [12] that

$$\mathrm{Ab}\ \pi_1(G) \cong H_1(G, \mathbb{Z}).$$

This result now also follows from Theorem 3.1, Theorem 3.11, and Proposition 3.16 taken together, so we have come up with an alternative proof of this result.

4. Conclusion

We conclude with some discussion of some open questions, possible weakening of hypotheses in the main result, and other notions of homology.

First, we point out that, of course, the converses of Theorems 1.1 and 1.2 do not hold. Indeed, it is well-known that in trees other than paths or the star on 4 vertices, there will typically be vertices with negative curvature (see, for instance [7]). However, all trees have trivial first homology [11] and trivial fundamental group.

It is natural to ask if the hypotheses of Theorem 1.1 can be weakened, but still obtain trivial first homology. Any cycle of length greater than 4 has non-negative curvature everywhere, but has non-trivial first homology. This shows that we cannot replace that hypothesis of "positive" with "non-negative." It is of interest to ask if it is sufficient to assume non-negative curvature with the added condition that at least one vertex has positive curvature.

Another result due to Bochner [4] is that a Riemannian manifold of non-negative curvature has finite-dimensional first homology group. We conjecture that this holds for graphs as well.

4.1. A remark on clique homology

Another commonly used notion of homology in graph theory is the *clique homology* coming from the *clique complex*, or *flag complex* of the graph. In this theory, the chain complex is

$$\cdots C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$$

where C_n is the space of all formal $\mathbb{F}$-linear combinations of n-cliques of the graph G. (Hence it is still the case that C_1 is all formal linear combinations of edges, and C_0 all formal linear combinations of vertices.) The boundary map ∂ of a clique is the sum of all its "faces," viewing the graph as a cell complex with an n-cell filling each n-clique. Then the clique homology groups are defined in the same way,

$$H_n^{\text{clique}}(G, \mathbb{F}) = Ker\ \partial|_{C_n} / Im\ \partial|_{C_{n+1}}.$$

The action of the boundary operator here is essentially the same as the boundary operator for the path homology that we hav given. Thus, it is still the case that

$$Ker\ \partial_1 \cong \mathcal{C}(G, \mathbb{F})$$

but now the space C_2 is the linear combinations of 3-cliques (which are 3-cycles). So it becomes apparent that

$$Im\ \partial_2 \cong T$$

where T is the space spanned by all 3-cycles of the graph. Thus we have the following theorem.

Theorem 4.1. *Let $\mathbb{F}$ be a field with characteristic not equal to 2, and let $\mathcal{C}(G, \mathbb{F})$ denote the $\mathbb{F}$-cycle space of G. Let T denote the subspace of $\mathcal{C}(G, \mathbb{F})$ that is generated by all simple triangles of G. Then*

$$H_1^{clique}(G, \mathbb{F}) \cong \mathcal{C}(G, \mathbb{F})/T.$$

Hence for both the path and clique homology theories, the first homology group "counts" cycles of the graph, but there are certain types of cycles ignored depending on the theory; clique homology does not see triangles, and path homology sees neither triangles nor squares.

Observe in particular that the homology vanishing theorem for path homology, Theorem 1.1, does not hold for the clique homology (a simple 4-cycle being a counterexample). We take this as further evidence that the path homology is an appropriate homology theory for graph theory.

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