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The Weitzenböck formula for the Fueter-Dirac operator

Andrés J. Moreno and Henrique N. Sá Earp

We find a Weitzenböck formula for the Fueter-Dirac operator which controls infinitesimal deformations of an associative submanifold in a 7–manifold with a G_2–structure. We establish a vanishing theorem to conclude rigidity under some positivity assumptions on curvature, which are particularly mild in the nearly parallel case. As applications, we find a different proof of rigidity for one of Lotay's associatives in the round 7-sphere from those given by Kawai [14, 15]. We also provide simpler proofs of previous results by Gayet for the Bryant-Salamon metric [11]. Finally, we obtain an original example of a rigid associative in a compact manifold with locally conformal calibrated G_2-structure obtained by Fernández-Fino-Raffero [9].

1. Introduction

The theories of Riemannian holonomy and calibrated geometry are related by the fact each Riemannian manifold with reduced holonomy is equipped with a calibration. In particular, a reduction to the exceptional holonomy group G_2 can only occur in real dimension 7, in which case the relevant calibrations are a 3–form φ and its Hodge dual 4–form $\psi := *\varphi$, and their calibrated submanifolds are called *associative* and *coassociative*, respectively (cf. Definition 2.8). In this article, we propose a computational tool to study the deformation theory of associative submanifolds, in favourable cases of interest.

Let (M^7, φ) be a smooth manifold with G_2–structure. In [19], McLean proved that a class in the moduli space of associative deformations corresponds to a harmonic spinor of a twisted Dirac operator, under the torsion-free hypothesis $\nabla\varphi = 0$. Then, Akbulut and Salur [2, 3] generalized McLean's theorem for a general G_2-structure, identifying the tangent space at an associative submanifold Y^3 in (M^7, φ) with the kernel of

$$\not{D}_{\mathrm{A}} : \Omega^0(Y, NY) \to \Omega^0(Y, NY), \tag{1.1}$$

where $A = A_0 + a$, for A_0 the induced connection on NY and some $a \in \Omega^1(Y, \mathrm{ad}(NY))$. We obtain a Weitzenböck formula for the operator (1.1), that is, a relation between the second-order elliptic square ${\not{D}_{\mathrm{A}}}^2$ and the trace Laplacian $\nabla^*\nabla$ of the induced Levi-Civita connection on NY. Under suitable positivity assumptions on curvature, this implies *rigidity*, i.e., that Y has "essentially" no infinitesimal associative deformations, in the following sense. Denote by $G := \mathrm{Stab}(\varphi) \subset \mathrm{Aut}(M)$ the group of global automorphisms preserving φ. The infinitesimal associative deformations of Y consist of:

(i) *trivial* deformations given by the action of G on Y (see [15] and [21]);

(ii) *non-trivial* deformations, which depend intrinsically on the geometry of the associative submanifold.

For instance, in [15], an associative submanifold is considered rigid if all infinitesimal associative deformations are trivial; in the particular case of the homogeneous space $M = S^7$, the symmetry group of φ is $G = \mathrm{Spin}(7)$. On the other hand, Gayet [11] and McLean [19] consider a generic G_2–structure, i.e., without symmetries. So, G is 0–dimensional and Y is rigid if the space of nontrivial infinitesimal deformation vanishes.

The exposition is organised as follows. Section 1 is a proactive background review. We apply results from 4-dimensional spin geometry to obtain the explicit identification

$$NY \otimes_{\mathbb{R}} \mathbb{C} \cong S^+ \otimes_{\mathbb{C}} S^-,$$

between the normal bundle of Y and a spinor bundle $S = S^+ \oplus S^- \to Y$, in order to describe the Fueter-Dirac operator in detail. We then deduce some useful identities in G_2–geometry, following Karigiannis [13].

In Section 3, we calculate the general Weitzenböck formula for the operator (1.1):

$$\not{D}_A{}^2(\sigma) = \nabla^*\nabla\sigma + \frac{1}{4}k \cdot \sigma + \overline{\rho}(F^-)\sigma + P_1(\sigma) + P_2(\sigma) + P_3(\sigma), \tag{1.2}$$

where P_1, P_2 and P_3 are first order differential operators on NY, involving the torsion of the G_2–structure, and $\nabla^*\nabla$ is the connection Laplacian

$$\nabla^*\nabla n = -\sum \nabla_i^\perp \nabla_i^\perp n - \nabla^\perp_{\nabla_i e_i} n$$

in a global frame $\{e_i\}$ on the associative submanifold Y. The scalar curvature of Y is denoted by k, and the bundle map

$$\overline{\rho} : \Omega^2(Y, \mathrm{End}(S^-)) \to \mathrm{End}(S^+ \otimes S^-)$$

is defined by

$$\overline{\rho}(F^-) := \overline{\rho}\big(\sum (e_i \wedge e_j) \otimes F^-_{ij}\big) = \sum \Gamma_0(e_i)\Gamma_0(e_j) \otimes F^-_{ij}, \tag{1.3}$$

where $F^- \in \Omega^2(Y, \mathrm{End}(S^-))$ is the curvature of a connection on S^- and $\Gamma_0 : TY \to \mathrm{End}(S^+)$ is the spin structure on Y.
In Section 4, we specialise to the *nearly parallel* case, in which $d\varphi$ and ψ are collinear and the formula (1.2) simplifies significantly. For a generic nearly parallel G_2–structure, we obtain a vanishing theorem (Theorem 4.4) to conclude rigidity under suitable intrinsic geometric conditions on Y. As immediate applications, we propose alternative proofs of rigidity for the known cases of an associative SU(2)-orbit 3-sphere for Lotay's cocalibrated G_2-structure on S^7 studied by Kawai [14, 15, 17] and the associatives $S^3 \times \{0\}$ of the Bryant-Salamon metric studied by Gayet [11].

Finally, we obtain a hitherto unstudied rigid associative submanifold (Corollary 4.17) in a compact manifold S with locally conformal calibrated G_2-structure obtained from the 3-dimensional complex Heisenberg group by Fernández-Fino-Raffero [9]. In view of the systematic nature of their construction, our method lends itself to the production of many more such examples.

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2. Spin geometry, G_2–structures and the Fueter-Dirac operator

2.1. Spin geometry of four dimensional space

We begin by recalling some background and fixing notation, so the reader familiar with e.g. [22, Chapter 2] and [8, Chapter 3] may just skim through upon a first read.

On an inner product space $(V^n, \langle\cdot,\cdot\rangle)$, the Clifford algebra $\mathrm{Cl}(V)$ is a 2^n-dimensional associative algebra with unit 1, generated by the elements of some orthonormal basis $e_1, ..., e_n$ of V with relations

$$e_i^2 = -1, \quad e_i e_j = -e_j e_i \quad \text{for} \quad i \neq j.$$

A basis for $\mathrm{Cl}(V)$ is given by

$$e_0 = 1, \quad e_I = e_{i_1} \cdots e_{i_k}$$

where $I = \{i_1, ..., i_k\} \subset \{1, ..., n\}$ for $i_1 < \cdots < i_k$, and $\mathrm{Cl}(V)$ admits a natural involution

$$\alpha : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$$

defined by $\alpha(x) = \widetilde{x} := \sum_I \epsilon_I x_I e_I$, where $\epsilon_I := (-1)^{k(k+1)/2}$ and $x_I \in \mathbb{R}$ are the components of x in the basis $\{e_I\}$. Denote by $\deg(e_I) := |I|$ the degree of an element $e_I \in \mathrm{Cl}(V)$, by $\mathrm{Cl}_k(V)$ the subset of elements of degree k, and by $\mathrm{Cl}^0(V)$ and $\mathrm{Cl}^1(V)$ the subspaces of elements of even and odd degree, respectively.

Example 2.1. On $V = \mathbb{R}^4$ with the Euclidean inner product, we have $\mathrm{Cl}(V) = M_2(\mathbb{H})$, the 2×2 matrices with entries in the quaternions $\mathbb{H} =$

$\langle i, j, k\rangle$. The elements of $\mathrm{Cl}(V)$ are 1, e_i, $\{e_ie_j\}_{i<j}$, $\{e_ie_je_k\}_{i<j<k}$ and $e_1e_2e_3e_4$, with $i, j, k = 1, 2, 3, 4$, with generators

$$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} \quad \text{and} \quad e_4 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

and the involution $\alpha(A) = A^*$ is the transpose conjugation.

Denote the set of units of $\mathrm{Cl}(V)$ by $\mathrm{Cl}^\times(V)$. Considering the twisted adjoint representation $\widetilde{\mathrm{Ad}} : \mathrm{Cl}^\times(V) \to \mathrm{Gl}(\mathrm{Cl}(V))$ given by

$$\widetilde{\mathrm{Ad}}(x)y = ((x)^0 - (x)^1)y\widetilde{x},$$

where $(x)^0 \in \mathrm{Cl}^0(V)$ and $(x)^1 \in \mathrm{Cl}^1(V)$ are the even and odd parts of x, respectively. We define the *Spin group* of V:

$$\mathrm{Spin}(V) := \{x \in \mathrm{Cl}^0(V) |\, \widetilde{\mathrm{Ad}}(x)V = V,\, x\widetilde{x} = 1\}.$$

For $\dim V \geq 3$, $\mathrm{Spin}(V)$ is a compact, connected and simply connected Lie group, fitting in a short exact sequence [22, Lemma 4.25]

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(V) \to \mathrm{SO}(V) \to 1.$$

In particular, the following results hold in dimensions 3 and 4:

Lemma 2.2. *[22, Lemma 4.4] For every $x \in \mathrm{Sp}(1)$, there is a unique orthogonal matrix $\xi_0(x) \in \mathrm{SO}(3)$, such that $\xi_0(x)y = xy\widetilde{x}$, for all $y \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$, and the map $\xi_0 : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ is a surjective homomorphism with kernel $\{\pm 1\}$, hence*

$$\mathrm{SO}(3) \cong \mathrm{Sp}(1)/\mathbb{Z}_2 \quad \textit{and} \quad \mathrm{Spin}(3) \cong \mathrm{Sp}(1).$$

Lemma 2.3. *[22, Lemma 4.6] For every $x, y \in \mathrm{Sp}(1)$, there is a unique orthogonal matrix $\eta_0(x, y) \in \mathrm{SO}(4)$, such that $\eta_0(x, y)z = xz\widetilde{y}$, for all $z \in \mathbb{R}^4 \cong \mathbb{H}$, and the map $\eta_0 : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ is a surjective homomorphism with kernel $\{\pm(1, 1)\}$, hence*

$$\mathrm{SO}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)/\mathbb{Z}_2 \quad \textit{and} \quad \mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$$

The last lemma provides two natural surjective homomorphisms $\rho^{\pm}:\mathrm{SO}(4)\to\mathrm{SO}(3)$ and, therefore, two exact sequences

$$1\to\mathrm{Sp}(1)\xrightarrow{\iota^{\pm}}\mathrm{SO}(4)\xrightarrow{\rho^{\pm}}\mathrm{SO}(3)\to 1$$

where $\iota^{+}(v)=\eta_0([v,1])$ and $\iota^{-}(v)=\eta_0([1,v])$, interpreting η_0 as the induced homomorphism on the quotient $\mathrm{Sp}(1)\times_{\mathbb{Z}_2}\mathrm{Sp}(1)$. Those sequences are related to the SO(4)-action on the spaces of self-dual and anti-self-dual 2-forms of a 4-dimensional inner-product space.

An element $q\in\mathbb{H}$ in the canonical basis $q=t+xi+yj+zk=(t+xi)+(y+zi)j$ can be identified with the 2×2 complex matrix

$$A=\begin{pmatrix} t+xi & -y+zi \\ y+zi & t-xi \end{pmatrix},$$

with

$$\det A=t^2+x^2+y^2+z^2=|q|^2.$$

Since $A^*A=(\det A)I_2$, every $q\in\mathrm{Sp}(1)\cong S^3$ is identified with a unitary matrix with determinant 1, that is, $\mathrm{SU}(2)\cong\mathrm{Sp}(1)$.

Definition 2.4. Let V be a real inner product space of dimension $2n\equiv 2,4\mod 8$ or $2n+1\equiv 3\mod 8$. A *spin structure* on V is a quadruple (S,I,J,Γ), where S is a 2^{n+1}-dimensional real inner product space, I and J are two anti-commuting orthogonal complex structure

$$I^{-1}=I^*=-I,\quad J^{-1}=J^*=-J,\quad IJ=-JI,$$

and $\Gamma:V\to\mathrm{End}(S)$ is a real linear map such that for any $v\in V$ holds the following properties:

$$\Gamma(v)^*+\Gamma(v)=0,\quad \Gamma(v)^*\Gamma(v)=|v|^2\mathbb{1},\quad \Gamma(v)I=I\Gamma(v),\quad \Gamma(v)J=J\Gamma(v).$$

Example 2.5. For a vector space V of real dimension 4, using the identification $V\cong\mathbb{H}$ and defining $S=\mathbb{H}\oplus\mathbb{H}$, we have the maps $\Gamma:\mathbb{H}\to\mathrm{End}(\mathbb{H}\oplus\mathbb{H})$, $I,J:\mathbb{H}\oplus\mathbb{H}\to\mathbb{H}\oplus\mathbb{H}$ defined for $v,x,y\in\mathbb{H}$ by

$$\Gamma(v)(x,y)=(vy,-\bar{v}x),\quad I(x,y)=(xi,yi),\quad J(x,y)=(xj,yj).$$

It is interesting to note that

$$\Gamma(v)=\begin{pmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{pmatrix},$$

where $\gamma : \mathbb{H} \to \mathrm{End}(\mathbb{H})$ also satisfies

$$\gamma(v)^* + \gamma(v) = 0, \quad \gamma(v)^*\gamma(v) = |v|^2\mathbb{1}, \quad \forall v \in \mathbb{H}.$$

Given a spin structure on a 4-dimensional space V, consider $S = S^+ \oplus S^-$, where S^+ and S^- are copies of $\mathbb{C}^2$ with standard Hermitian metric $\langle \cdot, \cdot \rangle$. The associated symplectic form compatible with the almost complex structure $I : S^\pm \to S^\pm$ is defined by $\omega(x, y) := \langle x, Iy \rangle$. Now, consider the (real) 4-dimensional space $\mathrm{Hom}_I(S^+, S^-) = \mathrm{Re}(\mathrm{Hom}(S^+, S^-))$ of linear maps over the quaternions, where $\mathrm{Hom}(S^+, S^-)$ are complex linear maps. Unitary elements of $\mathrm{Hom}_I(S^+, S^-)$ preserve the Hermitian and symplectic structures, and $\gamma : V \to \mathrm{Hom}_I(S^+, S^-)$ defined above acts on the standard basis by

$$\gamma(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \gamma(e_2) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \gamma(e_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ \gamma(e_4) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Up to isomorphism, the above generate $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$, since the symmetry group $\mathrm{SU}(2)^+ \times \mathrm{SU}(2)^-$ of (S^+, S^-) is connected. Thus γ fixes the orientation of V and, using the sympletic form to identify S^+ with its dual, we have

$$V \otimes_{\mathbb{R}} \mathbb{C} \cong S^+ \otimes_{\mathbb{C}} S^-. \tag{2.1}$$

Moreover, given $v \in V$, consider the Hermitian adjoint $\gamma(v)^* : S^- \to S^+$ of the map $\gamma(v) : S^+ \to S^-$. Then, for orthonormal vectors $v, v' \in V$, the map $\gamma(v)^*\gamma(v')$ defines an endomorphism of S^+ which satisfies

$$\gamma(v)^*\gamma(v) = 1 \quad \text{and} \quad \gamma(v)^*\gamma(v') + \gamma^*(v')\gamma(v) = 0.$$

In particular, we have a natural action ρ of $\Lambda^2(V)$ on S^+ defined by

$$\rho(v \wedge v')s := -\gamma(v)^*\gamma(v')s \quad \text{for} \quad s \in S^+.$$

Now, with respect to the Euclidean metric, the 2–forms split as $\Lambda^2(V) = \Lambda^2_+(V) \oplus \Lambda^2_-(V)$, where $\Lambda^2_+(V)$ and $\Lambda^2_-(V)$ denote the self-dual and anti-self-dual forms, respectively:

$$\Lambda^2_\pm(V) := \{\beta \in \Lambda^2(V) \mid *\beta = \pm\beta\}.$$

We observe that $\Lambda^2_-(V)$ acts trivially on S^+, by direct inspection on basis elements:

$$\Lambda^2_-(V) = \mathrm{Span}\{e_1 \wedge e_2 - e_3 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3, e_1 \wedge e_3 - e_4 \wedge e_2\}$$

$$\rho(e_1 \wedge e_2 - e_3 \wedge e_4) = -\gamma(e_1)^*\gamma(e_2) + \gamma(e_3)^*\gamma(e_4)$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = 0,$$

$$\rho(e_1 \wedge e_4 - e_2 \wedge e_3) = -\gamma(e_1)^*\gamma(e_4) + \gamma(e_2)^*\gamma(e_3)$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0,$$

$$\rho(e_1 \wedge e_3 - e_4 \wedge e_2) = -\gamma(e_1)^*\gamma(e_3) + \gamma(e_4)^*\gamma(e_2)$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = 0.$$

Thus we get the isomorphisms $\Lambda^2_+(V) \to \mathfrak{su}(S^+)$ and $\Lambda^2_-(V) \to \mathfrak{su}(S^-)$.

2.2. G_2–manifolds and associative submanifolds

We first present some algebraic and geometric proprieties of manifolds with G_2–structures which can be found e.g. in [7, 12].

The octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^8$ are an 8-dimensional, non-associative division algebra. On the imaginary part $\mathrm{Im}(\mathbb{O}) = \mathbb{R}^7$, the cross product

$$\begin{array}{cccccc} \times & : & \mathbb{R}^7 \times \mathbb{R}^7 & \to & \mathbb{R}^7 \\ & & (u, v) & \mapsto & \mathrm{Im}(uv) \end{array}$$

corresponds to a 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$, defined by $\varphi_0(u, v, w) = \langle u \times v, w\rangle$ with the Euclidean inner product. In coordinates $(x_1, ..., x_7) \in \mathbb{R}^7$, we fix the convention

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \tag{2.2}$$

and accordingly its dual 4–form

$$\psi_0 := *\varphi_0 = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$

The Lie group G_2 can be defined as the stabiliser of φ_0 in $\mathrm{Gl}(7, \mathbb{R})$.

Definition 2.6. Let M be a smooth oriented 7–manifold. A G_2–structure is a 3–form $\varphi \in \Omega^3(M)$ such that, around every $p \in M$, there exists a local section f of the oriented frame bundle $P_{SO}(M)$ such that

$$\varphi_p = (f_p)^*\varphi_0.$$

The G_2–structure φ determines a Riemannian metric and a volume form by the relation (cf. [13])

$$(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = 6 g_\varphi(u, v) \, \mathrm{vol}_\varphi \, . \tag{2.3}$$

Consequently, φ induces a Hodge star operator $*_\varphi$ and the Levi-Civita connection ∇^φ, though for simplicity we omit henceforth the subscripts in $g := g_\varphi$, $* := *_\varphi$ and $\nabla := \nabla^\varphi$. Moreover, the model cross-product on $\mathbb{R}^7$ induces the bilinear map on vector fields

$$\begin{array}{rcl} P : \Omega^0(TM) \times \Omega^0(TM) & \to & \Omega^0(TM) \\ (u, v) & \mapsto & P(u, v) = u \times v. \end{array} \tag{2.4}$$

The G_2–structure φ is called *torsion-free* if $\nabla \varphi = 0$, in which case we say that (M, φ) is a G_2*–manifold*. This condition is equivalent to $\nabla P = 0$.

Remark 2.7. Regarding orientation conventions, some authors adopt the model 3-form to be

$$\varphi_0 = e^{567} + e^{125} + e^{136} + e^{246} + e^{147} - e^{345} - e^{237},$$

(cf. [19, Chapters 4 and 5]), which relates to (2.2) by the orientation-reversing automorphism of $\mathbb{R}^7$

$$\begin{pmatrix} & & & & I_3 \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & -1 & \end{pmatrix}.$$

In this case, relation (2.3) becomes

$$(u \lrcorner \varphi_0) \wedge (v \lrcorner \varphi_0) \wedge \varphi_0 = -6 g_0(u, v) \, \mathrm{vol}_{g_0} \, .$$

Unless otherwise stated, we adopt throughout the convention (2.2).

Definition 2.8. Let (M, φ) be a 7–manifold with G_2–structure. A 3–dimensional submanifold $Y \subset M$ is called *associative* if $\varphi|_Y \equiv \mathrm{vol}(Y)$.

The etymology of Definition 2.8 stems from the *associator* $\chi \in \Omega^3(M, TM)$, defined by

$$\psi(u, v, w, z) = *\varphi(u, v, w, z) = \langle \chi(u, v, w), z \rangle. \tag{2.5}$$

In a local orthonormal frame $\{e_i\}_{i=1\dots7}$ of TM, one has $\chi = -\sum_{i=1}^{7}(e_i \lrcorner \psi) \otimes e_i$. Expressing χ in terms of the cross product (c.f. [12]),

$$\chi(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle u, w \rangle v, \tag{2.6}$$

and using the relation (c.f.[12])

$$\varphi(u, v, w) + \frac{1}{4}|\chi(u, v, w)|^2 = |u \wedge v \wedge w|^2,$$

we see that the associative condition is equivalent to $\chi|_Y \equiv 0$.

Remark 2.9. In the sign convention of Remark 2.7, the associator is written as

$$\chi(u, v, w) = u \times (v \times w) + \langle u, v \rangle w - \langle u, w \rangle v$$

Lemma 2.10. *If Y is an associative submanifold, then there is a natural identification $TY \cong \Lambda^2_+(NY)$.*

Proof. Fix local orthonormal frames e_1, e_2, e_3 and $\eta_4, \eta_5, \eta_6, \eta_7$ of TY and NY, respectively, about a point $p \in Y$:

$$\varphi_p = e^{123} + e^1(\eta^{45} + \eta^{67}) + e^2(\eta^{46} + \eta^{75}) - e^3(\eta^{47} + \eta^{56}) \tag{2.7}$$

and

$$\begin{aligned} e_1 \lrcorner \varphi &= e^{23} + \eta^{45} + \eta^{67}, \\ e_2 \lrcorner \varphi &= e^{31} + \eta^{46} + \eta^{75}, \\ e_3 \lrcorner \varphi &= e^{12} - \eta^{47} - \eta^{56}. \end{aligned}$$

Denote $\omega_1 = (e_1 \lrcorner \varphi)|_{N_pY}$, $\omega_2 = (e_2 \lrcorner \varphi)|_{N_pY}$, $\omega_3 = -(e_3 \lrcorner \varphi)|_{N_pY}$ and define on each fibre the isomorphism $e_j \in T_pY \mapsto \omega_j \in \Lambda^2_+(N_pY)$, which obviously varies smoothly with p. $\square$

2.3. The twisted Dirac operator

The oriented orthonormal frame of TY has the form $\{e_1, e_2, e_3 = e_1 \times e_2\}$. So, with respect to the splitting $TM|_Y = TY \oplus NY$, the cross product induces maps

$$\Omega^0(TY) \times \Omega^0(TY) \to \Omega^0(TY), \quad \Omega^0(TY) \times \Omega^0(NY) \to \Omega^0(NY),$$
$$\Omega^0(NY) \times \Omega^0(NY) \to \Omega^0(TY).$$

In particular, the map $\gamma : \Omega^0(TY) \times \Omega^0(NY) \to \Omega^0(NY)$ endows NY with a Clifford bundle structure.
Since the Levi-Civita connection of (M, φ) induces metric connections on the bundles TY and NY, the composition

$$\Omega^0(NY) \xrightarrow{\nabla_{A_0}} \Omega^0(TY) \otimes \Omega^0(NY) \xrightarrow{\gamma} \Omega^0(NY) \tag{2.8}$$

defines a natural *Fueter-Dirac operator* $\not{D}_{A_0}(\sigma) := \gamma(\nabla_{A_0}(\sigma))$, where $A_0 \in \Omega^1(Y, \mathfrak{so}(4))$ denotes the connection induced on NY by the Levi-Civita connection ∇^φ of the G_2-metric of (M, φ). To simplify the notation, the twisted Dirac operator induced by the normal connection A_0 will be denoted just by $\not{D}$.
The normal bundle NY of an associative submanifold is trivial [7, Lemma 5.1, arXiv version: 1207.4470v3]. In particular, the second Stiefel-Whitney class $w_2(NY)$ vanishes, so there exists a spin structure on NY [16, Theorem 1.7]. This is equivalent to the existence of a map $\Gamma : NY \to \mathrm{End}(S)$ such that

$$\Gamma(\sigma) + \Gamma(\sigma)^* = 0 \quad \Gamma(\sigma)^*\Gamma(\sigma) = \langle\sigma, \sigma\rangle\mathbb{1} \quad \sigma \in \Omega^0(Y, NY),$$

where S is a vector bundle of (real) rank 8 and it splits into Γ–eigenbundles S^+ and S^- of rank 4. We saw in the last Section that the Spin structure induces an isomorphism

$$\rho_\pm : \Lambda^2_\pm(NY) \to \mathfrak{su}(S^\pm),$$

so, by Lemma 2.10, the spin structure $\Gamma_0 : TY \to \mathrm{End}(S^+)$ on TY coincides with the spin structure on NY via the projection $\mathrm{Spin}(4) = \mathrm{Spin}(3) \times \mathrm{Spin}(3)$. Defining the Clifford multiplication

$$\tau := \Gamma_0 \otimes \mathbb{1}_{S^-} : TY \to \mathrm{End}(S^+ \otimes S^-)$$

and using the spin connection ∇ on $S^+ \otimes S^-$,

$$\nabla(\sigma \otimes \varepsilon) = \nabla^+\sigma \otimes \varepsilon + \sigma \otimes \nabla^-\varepsilon,$$

we form the Dirac operator $D : \Omega^0(Y, S^+ \otimes S^-) \to \Omega^0(Y, S^+ \otimes S^-)$ by

$$D(\sigma \otimes \varepsilon) := \sum_{i=1}^{3} \tau(e_i)\nabla_i(\sigma \otimes \varepsilon).$$

Proposition 2.11. *Under the isomorphism* (2.1)*, we have* $NY \otimes_{\mathbb{R}} \mathbb{C} \cong S^+ \otimes_{\mathbb{C}} S^-$*, the Spin connection* ∇ *and the Clifford multiplication* τ *agree with the induced connection* $\nabla^\perp$ *on* NY *and* γ*, respectively.*

Proof. In fact, each section $\sigma \otimes \varepsilon$ of $S^+ \otimes_{\mathbb{C}} S^-$ induces a section $\nu = \sigma^* \otimes \varepsilon$ on $\mathrm{Hom}(S^+, S^-) \cong (S^+)^* \otimes S^-$ such that $\nu(\sigma) = \sigma^*(\sigma) \otimes \varepsilon = \varepsilon$, then

$$\nabla\nu = \nabla(\sigma^* \otimes \varepsilon) = (\nabla^+)^*\sigma^* \otimes \varepsilon + \sigma^* \otimes \nabla^-\varepsilon,$$

where $\nabla\nu$ is a section on $T^*Y \otimes \mathrm{Hom}(S^+, S^-)$, so, for each σ section on S^+

$$\begin{aligned}(\nabla\nu)(\sigma) &= (\nabla^+)^*\sigma^*(\sigma) \otimes \varepsilon + \sigma^*(\sigma) \otimes \nabla^-\varepsilon \\ &= [d\sigma^*(\sigma) - \sigma^*(\nabla^+\sigma)] \otimes \varepsilon + \sigma^*(\sigma) \otimes \nabla^-\varepsilon \\ &= -\nu(\nabla^+\sigma) + \nabla^-(\nu(\sigma)).\end{aligned}$$

On the other hand, the Spin connection ∇ is compatible with the induced connection $\nabla^\perp$, in the following sense. Given sections n of NY and σ of S^+, under the isomorphism $\Gamma : NY \to \mathrm{Hom}_J(S^+, S^-)$ induced by (2.1), one has $\nabla^-(\Gamma(n)\sigma) = \Gamma(\nabla^\perp n)\sigma + \Gamma(n)\nabla^+\sigma$, or, equivalently,

$$\Gamma(\nabla^\perp n)\sigma = -\Gamma(n)\nabla^+\sigma + \nabla^-(\Gamma(n)\sigma).$$

Therefore, $\nabla^\perp$ agrees with the Spin connection ∇ via the isomorphism Γ. Finally, with respect to the Clifford multiplications we have

$$\begin{array}{ccccc} TY & \xrightarrow{\Gamma_0} & \mathrm{End}(S^+) & \xhookrightarrow{\otimes \mathbb{1}_{\mathrm{End}(S^-)}} & \mathrm{End}(S^+ \otimes_{\mathbb{C}} S^-) \\ \downarrow{\scriptstyle\gamma} & & & \nearrow{\scriptstyle\cong} & \\ \mathrm{End}(NY \otimes_{\mathbb{R}} \mathbb{C}) & & & & \end{array}$$

and by Schur's lemma γ and τ are the same. $\square$

In conclusion, (2.8) defines a twisted Dirac operator.

2.4. Torsion tensor and local description of φ

We will briefly review the intrinsic torsion forms of a G_2–structure and define the full torsion tensor T_{ij}, using local coordinates, following [13]. Our goal is to derive Lemma 2.16, a set of 'Leibniz rules' for the covariant derivative and curvature operators with respect to the vector cross-product, which will be instrumental in Section 3.

As before, let (M,φ) be a smooth 7–manifold with G_2–structure. In a local coordinate system $(x_1,...,x_7)$, a differential k–form α on M will be written as

$$\alpha = \frac{1}{k!}\alpha_{i_1\cdots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}$$

where the sum is taken over all ordered subsets $\{i_1\cdots i_k\}\subset\{1,...,7\}$ and $\alpha_{i_1\cdots i_k}$ is skew-symmetric in all indices, i.e. $\alpha_{i_1\cdots i_k}=\alpha(e_{i_1},...,e_{i_k})$. A Riemannian metric g on M induces on $\Omega^k := \Omega^k(M)$ the metric $g(dx^i,dx^j):=g^{ij}$, where (g^{ij}) denotes the inverse of the matrix (g_{ij}).

A G_2–structure φ splits $\Omega^\bullet$ into orthogonal irreducible G_2 representations, with respect to its G_2–metric g. In particular,

$$\Omega^2=\Omega^2_7\oplus\Omega^2_{14} \quad \text{and} \quad \Omega^3=\Omega^3_1\oplus\Omega^3_7\oplus\Omega^3_{27},$$

where $\Omega^k_l\subset\Omega^k$ denotes (fibrewise) an irreducible G_2–submodule of dimension l, with an explicit description:

$$\begin{aligned}
\Omega^2_7 &= \{X\lrcorner\varphi; X\in\Omega^0(TM)\}\\
\Omega^2_{14} &= \{\beta\in\Omega^2; \beta\wedge\psi=0\}\\
\Omega^3_1 &= \{f\varphi; f\in C^\infty(M)\}\\
\Omega^3_7 &= \{X\lrcorner\psi; X\in\Omega^0(TM)\}\\
\Omega^3_{27} &= \{h_{ij}g^{jl}dx^i\wedge\left(\frac{\partial}{\partial x_l}\right)\lrcorner\varphi; h_{ij}=h_{ji}, \mathrm{tr}_g(h_{ij})=g^{ij}h_{ij}=0\}
\end{aligned} \tag{2.9}$$

The analogous decompositions of Ω^4 and Ω^5 are obtained from these by the Hodge isomorphism $*_\varphi:\Omega^k\to\Omega^{7-k}$. Decomposing $d\varphi\in\Omega^4$ and $d\psi\in\Omega^5$, we introduce the four *torsion forms* (cf. [5])

$$\tau_0\in\Omega^0_1,\quad \tau_1\in\Omega^1_7,\quad \tau_2\in\Omega^2_{14},\quad \tau_3\in\Omega^3_{27},$$

defined by

$$d\varphi=\tau_0\psi+3\tau_1\wedge\varphi+*\tau_3 \quad \text{and} \quad d\psi=4\tau_1\wedge\psi-*\tau_2.$$

Naturally, these forms arise from the theorem of Fernandez and Gray [10], asserting that a G_2–structure is torsion-free ($\nabla\varphi = 0$) if, and only if, φ is closed and co-closed. So, if either condition fails, the torsions $\nabla\varphi \in \Omega^1 \otimes \Omega^3$ and $\nabla\psi \in \Omega^1 \otimes \Omega^4$ can be expressed in terms of the four torsion forms.

Lemma 2.12 ([13], Lemma 2.24). *For any vector field, the 3-form $\nabla_X\varphi$ lies in the subspace Ω^3_7 of Ω^3.*

Proof. It suffices to consider a coordinate vector $X = e_l$ and check that $g(\nabla_l\varphi, \eta) = 0$ for an arbitrary $\eta \in \Omega^3_1 \oplus \Omega^3_{27}$. □

In a local frame $\{e_1, \dots, e_7\}$, denoting $\nabla_l\varphi := \frac{1}{6}\nabla_l\varphi_{abc}dx^a \wedge dx^b \wedge dx^c$ and identifying $\Omega^3_7 \cong \Omega^1$, we see from (2.9) that $\nabla_l\varphi$ is spanned by interior products $e_n \lrcorner \psi$, which defines a 2–tensor T_{lm} by

$$\nabla_l\varphi_{abc} =: T_{lm}g^{mn}\psi_{nabc}$$

called the *full torsion tensor.*

Proposition 2.13 ([13], Theorem 2.27). *The full torsion tensor T_{lm} is*

$$T_{lm} = \frac{\tau_0}{4}g_{lm} - (\tau_3)_{lm} - (\tau_1)_{lm} - \frac{1}{2}(\tau_2)_{lm}, \tag{2.10}$$

where τ_0 is a function, $g_{lm} = g(e_l, e_m)$, $\tau_1 = (\tau_1)_l dx^l$ is Ω^1_7-form which can be written as a Ω^2_7-form $\tau_1 = \frac{1}{2}(\tau_1)_{ab}dx^a \wedge dx^b$ with $(\tau_1)_{ab} = (\tau_1)_l g^{lk}\varphi_{kab}$, $\tau_2 = \frac{1}{2}(\tau_2)_{ab}dx^a \wedge dx^b$ and $\tau_3 = \frac{1}{2}(\tau_3)_{im}g^{ml}\varphi_{ljk}dx^i \wedge dx^j \wedge dx^k$ is a Ω^3_{27}-form.

In [13, Lemma A.14], Karigiannis compiles several useful identities among the tensors g, φ and ψ:

$$\psi_{rstu}\psi_{abcd}g^{ra}g^{sb}g^{tc}g^{ud} = 168 \tag{2.11}$$

$$\psi_{rstu}\psi_{abcd}g^{sb}g^{tc}g^{ud} = 24g_{ra} \tag{2.12}$$

Differentiating (2.11) and (2.12), one obtains

$$\nabla_l\psi_{rstu}\psi_{abcd}g^{ra}g^{sb}g^{tc}g^{ud} = 0, \tag{2.13}$$

$$\nabla_l\psi_{rstu}\psi_{abcd}g^{sb}g^{tc}g^{ud} = -\psi_{rstu}\nabla_l\psi_{abcd}g^{sb}g^{tc}g^{ud}. \tag{2.14}$$

Lemma 2.14. *For any vector field X, the 4-form $\nabla_X \psi$ lies in the subspace Ω^4_7 of Ω^4.*

Proof. The proof is, basically, the same as in Lemma 2.12. Considering $X = e_l$ and applying (2.13), we have

$$g(\nabla_l \psi, \psi) = \frac{1}{24} \nabla_l \psi_{rstu} \psi_{abcd} g^{ra} g^{sb} g^{tc} g^{ud} = 0,$$

so $\nabla_l \psi \perp \Omega^4_1$. To see that $\nabla_l \psi \perp \Omega^4_{27}$, consider some $\eta \in \Omega^4_{27}$ in local form,

$$\eta = \frac{1}{3!} h_{ij} g^{jl} \psi_{labc} dx^i \wedge dx^a \wedge dx^b \wedge dx^c,$$

and take the inner product with $\nabla_l \psi$:

$$g(\nabla_l \psi, \eta) = \frac{1}{3!} \nabla_l \psi_{rstu} h^l_i \psi_{labc} g^{ri} g^{sa} g^{tb} g^{uc} = \frac{1}{3!} h^{lr} \nabla_l \psi_{rstu} \psi_{labc} g^{sa} g^{tb} g^{uc} = 0,$$

using that, $h^{lr} = g^{ri} h_{ij} g^{jl}$ is a symmetric $(0,2)$-tensor (since h_{ij} is a symmetric $(2,0)$-tensor), while $\nabla_l \psi_{rstu} \psi_{labc} g^{sa} g^{tb} g^{uc}$ is skew-symmetric in r and l, by (2.14). □

Using Lemma 2.14 above and the identity $*(X \lrcorner \psi) = \varphi \wedge X^\flat$ $(X \in \Omega^0(M))$, where $X^\flat$ is the 1–form defined by $X^\flat(Y) = g(X, Y)$, one has:

Corollary 2.15. *With the above notation,*

$$\nabla_l \psi_{rstu} = -T_{lr} \varphi_{stu} + T_{ls} \varphi_{rtu} - T_{lt} \varphi_{rsu} + T_{lu} \varphi_{rst}.$$

For a torsion-free G_2–structure, the cross-product (2.4) is parallel, so it satisfies the Leibniz rule

$$\nabla(u \times v) = \nabla u \times v + u \times \nabla v, \quad \forall u, v \in \Omega^0(TM).$$

In general, the action of ∇ on the cross product can be expressed in terms of the total torsion tensor:

Lemma 2.16. *For the vector fields $u, v, w, z \in \Omega^0(TM)$, we have*

(i) $\nabla_z(u \times v) = \nabla_z u \times v + u \times \nabla_z v + \sum_{m=1}^{7} T(z, e_m)\chi(e_m, u, v)$.

(ii) $R(w, z)(u \times v) = R(w, z)u \times v + u \times R(w, z)v + \mathcal{T}(w, z, u, v)$, *where* $\mathcal{T}$ *is given, in an orthonormal local frame* $\{e_1, ..., e_7\}$ *of* TM, *by*

$$\begin{aligned} \mathcal{T}(w, z, u, v) := \sum_{m=1}^{7} & T(z, e_m)(\nabla_w \psi)(e_m, u, v, \cdot)^\sharp \\ & - T(w, e_m)(\nabla_z \psi)(e_m, u, v, \cdot)^\sharp \\ & + \Big((\nabla_w T)(z, e_m) - (\nabla_z T)(w, e_m)\Big)\chi(e_m, u, v). \end{aligned} \tag{2.15}$$

Proof.

(i) The proof goes along the lines of [11, Lemma A.1], using the fact that the torsion $\nabla\varphi$ takes values in Ω^3_7 (c.f. Lemma 2.12). Consider normal coordinates $x_1, ..., x_7$ about a given $p \in M$ and an orthonormal frame $e_1, ..., e_7$. At the point p, we have:

$$\begin{aligned} \nabla_z(u \times v) =& \sum_{i=1}^{7} \nabla_z(\langle u \times v, e_i\rangle e_i) = \sum_{i=1}^{7} \nabla_z(\varphi(u, v, e_i)e_i) \\ =& \sum_{i=1}^{7} z(\varphi(u, v, e_i))e_i + \varphi(u, v, e_i)\nabla_z e_i \\ =& \sum_{i=1}^{7} \Big(\varphi(\nabla_z u, v, e_i) + \varphi(u, \nabla_z v, e_i) + \varphi(u, v, \nabla_z e_i) \\ & + (\nabla_z\varphi)(u, v, e_i)\Big) e_i \\ =& \sum_{i=1}^{7} \Bigg(\varphi(\nabla_z u, v, e_i) + \varphi(u, \nabla_z v, e_i) \\ & + \sum_{m=1}^{7} T(z, e_m)\psi(e_m, u, v, e_i)\Bigg) e_i \\ =& \nabla_z u \times v + u \times \nabla_z v + \sum_{m=1}^{7} T(z, e_m)\chi(e_m, u, v). \end{aligned}$$

Notice that we used $(\nabla_j e_i)_p = 0$ in the third and fourth equalities.

(ii) Using the first part, we have

$$\begin{aligned}\nabla_w\nabla_z(u\times v) =&\nabla_w\nabla_z u\times v+\nabla_z u\times\nabla_w v+\nabla_w u\times\nabla_z v+u\times\nabla_w\nabla_z v\\ &+\sum_{i,m=1}^{7}\Big(T(w,e_m)\Big(\psi(e_m,\nabla_z u,v,e_i)+\psi(e_m,u,\nabla_z v,e_i)\Big)\\ &+\Big((\nabla_w T)(z,e_m)+T(\nabla_w z,e_m)\Big)\psi(e_m,u,v,e_i)\\ &+T(z,e_m)\Big(\psi(e_m,\nabla_w u,v,e_i)+\psi(e_m,u,\nabla_w v,e_i)\\ &+(\nabla_w\psi)(e_m,u,v,e_i)\Big)\Big)e_i.\end{aligned}$$

Using symmetries of the curvature $R(w,z)=\nabla_w\nabla_z-\nabla_z\nabla_w-\nabla_{[w,z]}$ and the fact that ∇ is torsion-free, one has $[w,z]=\nabla_w z-\nabla_z w$, and we compute

$$\begin{aligned}R(w,z)(u\times v) =&R(w,z)u\times v+u\times R(w,z)v\\ &+\sum_{i,m=1}^{7}\Big(T(z,e_m)(\nabla_w\psi)(e_m,u,v,e_i)\\ &+\Big((\nabla_w T)(z,e_m)-(\nabla_z T)(w,e_m)\Big)\psi(e_m,u,v,e_i)\\ &-T(w,e_m)(\nabla_z\psi)(e_m,u,v,e_i)\Big)e_i,\end{aligned}$$

which concludes the proof.

□

3. The Fueter-Dirac Weitzenböck formula

We now address the general framework proposed by Akbulut and Salur [2, 3], in which the role of torsion in the associative deformation theory is captured by a *twisted* Fueter-Dirac operator. Given an associative submanifold Y^3 in (M,φ), the G_2–structure induces connections on the bundles NY and TY. Moreover, Proposition 2.11 gives an identification $NY\cong \mathrm{Re}(S^+\otimes_{\mathbb{C}} S^-)$, with the respective reductions $\Lambda^2_\pm(NY)\cong\mathfrak{su}(S^\pm)=\mathrm{ad}(S^\pm)$. We will refer to elements in the kernel $\ker \not{D}$ of the Dirac operator (2.8) as harmonic spinors twisted by S^-, or simply, *twisted harmonic spinors*.

Denote by $\mathcal{A}(S^\pm)$ the space of connections on each spinor bundle $S^\pm$, and let $A_0\in\Omega^1(Y,\mathfrak{so}(4))$ be the induced connection on NY, so that the isomorphism $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ gives a decomposition $A_0 = A_0^+ \oplus A_0^-$, with $A_0^\pm \in \mathcal{A}(S^\pm)$. Fixing these reference connections, each $\mathcal{A}(S^\pm)$ is an affine space modelled on $\Omega^1(Y, \mathrm{ad}(S^\pm))$, so a connection $A^\pm \in \mathcal{A}(S^\pm)$ is of the form

$$A^\pm = A_0^\pm + a^\pm \quad \text{for} \quad a^\pm \in \Omega^1(Y, \mathrm{ad}(S^\pm)).$$

Thus a connection on NY has the form

$$A = A_0 + a = (A_0^+ + a^+) \oplus (A_0^- + a^-) \quad \text{for} \quad a \in \Omega^1(Y, \mathrm{ad}(NY)).$$

Now, using the Clifford multiplication (indeed the cross-product), we define the *twisted Dirac operator*

$$\not{D}_{\mathrm{A}} := \sum_{j=1}^{3} e_i \times \nabla_{e_i} \quad : \quad \Omega^0(NY) \to \Omega^0(NY)$$

where $\nabla := \nabla_A$ is given by a connection on NY and the normal sections in $\ker(\not{D}_{\mathrm{A}})$ are called *harmonic spinors twisted by* (S^-, A). The following Definition is adopted from [2]:

Definition 3.1. Let Y be an associative submanifold of (M, φ). The *Fueter-Dirac operator* associated with Y is

$$\not{D}_{\mathrm{A}}\, \sigma := \sum_{i=1}^{3} e_i \times \nabla_{e_i}^{\perp} \sigma - e_i \times a(e_i)(\sigma), \tag{3.1}$$

where $a \in \Omega^1(Y, \mathrm{ad}(NY))$ defined by $a(e_i)(\sigma) = (\nabla_\sigma(e_i))^\perp$ is the normal component of $\nabla_\sigma(e_i)$, and ∇ is the Levi-Civita connection on M.

We know from [2, Theorem 6] that the linearisation of the deformation problem for an associative submanifold Y of (M, φ) at Y is identified with $\ker \not{D}_{\mathrm{A}}$, so this space is called the *infinitesimal deformation space* of Y. Our motivation is precisely the expectation that a Weitzenböck formula for (3.1), in favourable cases at least, can give information about the deformation space $\ker \not{D}_{\mathrm{A}}$.

3.1. The general squared Fueter-Dirac operator

Lemma 3.2. *Let $\{e_1, e_2, e_3\}$ and $\{\eta_4, ..., \eta_7\}$ be orthonormal frames of the vector bundles TY and NY, respectively. Then*

$$\not{D}_{\mathrm{A}}\,\sigma = \sum_{i=1}^{3} e_i \times \nabla^{\perp}_{e_i}\sigma - \sum_{k=4}^{7} (\nabla_\sigma \psi)(\eta_k, e_1, e_2, e_3)\eta_k \tag{3.2}$$

Proof. Since A_0 is the connection induced on NY by the Levi-Civita connection on M given by the G_2 metric g_φ, we have $\nabla_{A_0} = \nabla^{\perp}$. Now, for each $\sigma \in \Omega^0(NY)$,

$$\begin{aligned}\sum_{i=1}^{3} e_i \times a(e_i)(\sigma) =& e_1 \times (\nabla_\sigma e_1)^{\perp} + e_2 \times (\nabla_\sigma e_2)^{\perp} + e_3 \times (\nabla_\sigma e_3)^{\perp} \\ =& (e_2 \times e_3) \times (\nabla_\sigma e_1)^{\perp} + (e_3 \times e_1) \times (\nabla_\sigma e_2)^{\perp} \\ &+ (e_1 \times e_2) \times (\nabla_\sigma e_3)^{\perp} \\ =& \chi((\nabla_\sigma e_1)^{\perp}, e_2, e_3) + \chi((\nabla_\sigma e_2)^{\perp}, e_3, e_1) \\ &+ \chi((\nabla_\sigma e_3)^{\perp}, e_1, e_2) \\ =& (\diamondsuit).\end{aligned}$$

Since Y is associative exactly when $\chi|_{TY} = 0$, this implies

$$\chi((\nabla_\sigma e_i)^{\perp}, e_j, e_k) = \chi(\nabla_\sigma e_i, e_j, e_k).$$

Furthermore, the section $\chi(\nabla_\sigma(e_i), e_j, e_k)$ lies on the normal component, so

$$\begin{aligned}(\diamondsuit) =& \sum_{k=4}^{7} (\langle \chi(\nabla_\sigma(e_1), e_2, e_3), \eta_k\rangle + \langle \chi(e_1, \nabla_\sigma(e_2), e_3), \eta_k\rangle \\ &+ \langle \chi(e_1, e_2, \nabla_\sigma(e_3)), \eta_k\rangle)\eta_k \\ =& \sum_{k=4}^{7} (-(\nabla_\sigma \psi)(e_1, e_2, e_3, \eta_k) + \sigma(\psi(e_1, e_2, e_3, \eta_k)) \\ &- \psi(e_1, e_2, e_3, \nabla_\sigma(\eta_k)))\eta_k \\ =& \sum_{k=4}^{7} ((\nabla_\sigma \psi)(\eta_k, e_1, e_2, e_3)\eta_k.\end{aligned}$$

To obtain the second equality we used the covariant derivative of ψ:

$$(\nabla_\sigma\psi)(e_1,e_2,e_3,\eta_k) = \sigma(\psi(e_1,e_2,e_3,\eta_k)) - \psi(\nabla_\sigma e_1,e_2,e_3,\eta_k) - \cdots - \psi(e_1,e_2,e_3,\nabla_\sigma\eta_k)$$

and equation (2.5), and for the last one we used the skew-symmetry of $\nabla_\sigma\psi$ and the associativity condition $\chi(e_1,e_2,e_3)=0$. □

Corollary 3.3. *If φ is torsion free (i.e. $\nabla\varphi = 0$), then $a = A - A_0 = 0$.*

The purpose of this Section is to study in detail the expression for the squared Fueter-Dirac operator obtained from (3.2). Fix $p \in Y$ and choose local orthonormal frames $\{e_1,e_2,e_3\}$ and $\{\eta_4,\eta_5,\eta_6,\eta_7\}$ of TY and NY, respectively, such that

$$(\nabla_{e_i}e_j)_p = (\nabla_{e_i}\eta_k)_p = (\nabla_{\eta_l}\eta_k)_p = 0 \tag{3.3}$$

for all $i,j = 1,2,3$ and $k,l = 4,5,6,7$. Observe that, for any sections $\sigma,\eta \in \Omega^0(TM|_Y)$, one has

$$\nabla_\sigma(\eta) \in \Omega^0(TM|_Y) = \Omega^0(TY) \oplus \Omega^0(NY), \tag{3.4}$$

so both tangent and normal components of (3.3) vanish at p. Then the following holds at p:

$$\begin{aligned}
\not{D}_{\mathrm{A}}{}^2\sigma = {} & \sum_{i,j=1}^{3} e_i \times \nabla_i^\perp(e_j \times \nabla_j^\perp\sigma) - \sum_{j=1}^{3}\sum_{k=4}^{7}(\nabla_{e_j\times\nabla_j^\perp\sigma}\psi)(\eta_k,e_1,e_2,e_3)\eta_k \\
& - \sum_{i=1}^{3}\sum_{l=4}^{7} e_i \times \nabla_i^\perp\{(\nabla_\sigma\psi)(\eta_l,e_1,e_2,e_3)\eta_l\} \\
& + \sum_{k,l=4}^{7}(\nabla_{(\nabla_\sigma\psi)(\eta_l,e_1,e_2,e_3)\eta_l}\psi)(\eta_k,e_1,e_2,e_3)\eta_k,
\end{aligned} \tag{3.5}$$

and we organise that expression in five components:

$$
\begin{aligned}
{\not{D}_{\mathrm{A}}}^2\,\sigma = & \underbrace{\sum_{i,j=1}^{3} e_i \times (e_j \times \nabla_i^{\perp}\,\nabla_j^{\perp}\,\sigma)}_{\text{(I)}} \\
& + \underbrace{\sum_{i,j,l=1}^{3}\sum_{m=4}^{7} T_{il}\psi(e_l, e_j, \nabla_j^{\perp}\,\sigma, \eta_m) e_i \times \eta_m}_{\text{(II)}} \\
& - \underbrace{\sum_{j=1}^{3}\sum_{k,n=4}^{7} \varphi(e_j, \nabla_j^{\perp}\,\sigma, \eta_n)(\nabla_{\eta_n}\psi)(\eta_k, e_1, e_2, e_3)\eta_k}_{\text{(III)}} \\
& - \underbrace{\sum_{i=1}^{3}\sum_{l=4}^{7} e_i(\nabla_{\sigma}\psi(\eta_l, e_1, e_2, e_3))e_i \times \eta_l}_{\text{(IV)}} \\
& + \underbrace{\sum_{k,l=4}^{7} (\nabla_{\sigma}\psi)(\eta_l, e_1, e_2, e_3)(\nabla_{\eta_l}\psi)(\eta_k, e_1, e_2, e_3)\eta_k\,.}_{\text{(V)}}
\end{aligned}
$$

To obtain (I) and (II) we used Lemma 2.16 (i) and the property $(\nabla_i e_j)_p = 0$, whereas (IV) follows from the Leibniz rule for $\nabla^{\perp}$ and $(\nabla_i \eta_k)_p = 0$.

Lemma 3.4. *Denoting by $\nabla^*\nabla$ the Laplacian of the connection $\nabla^{\perp}$, by k the scalar curvature of Y, by F^- the curvature associated to the spin bundle S^-, and by $\overline{\rho}$ the natural extension $\rho \otimes \mathbb{1}_{\mathrm{End}(S^-)}$ of $\rho : \Omega^2(Y) \to \mathrm{End}(S^+)$, one has*

$$
\text{(I)} = \nabla^*\nabla\sigma + \frac{k}{4}\sigma + \overline{\rho}(F^-)\sigma.
$$

Proof. In terms of an orthonormal frame $\{e_1, e_2, e_3\}$ of TY,

$$
\begin{aligned}
(\mathrm{I}) &= \sum_{i=1}^{3} e_i \times (e_i \times \nabla_i^\perp \nabla_i^\perp \sigma) + \sum_{\substack{i,j=1 \\ i\neq j}}^{3} e_i \times (e_j \times \nabla_i^\perp \nabla_j^\perp \sigma) \\
&= -\sum_i \nabla_i^\perp \nabla_i^\perp \sigma - \sum_{i\neq j} (e_i \times e_j) \times \nabla_i^\perp \nabla_j^\perp \sigma \\
&= -\sum_i \nabla_i^\perp \nabla_i^\perp \sigma - \nabla^\perp_{\nabla_i^\top e_i} \sigma \\
&\quad - \sum_{i<j} (e_i \times e_j) \times (\nabla_i^\perp \nabla_j^\perp - \nabla_j^\perp \nabla_i^\perp - \nabla^\perp_{[e_i,e_j]})\sigma \\
&= \nabla^*\nabla\sigma - \sum_{i<j} (e_i \times e_j) \times R^\perp(e_i, e_j)\sigma.
\end{aligned}
$$

Here $R^\perp \in \Omega^0(\Lambda^2 T^*Y \otimes \mathrm{End}(NY))$ is the normal curvature of Y:

$$
R^\perp(e_i, e_j)\sigma = (\nabla_i^\perp \nabla_j^\perp - \nabla_j^\perp \nabla_i^\perp - \nabla^\perp_{[e_i,e_j]})\sigma. \tag{3.6}
$$

To obtain the second equality, we used (2.6) in each term of the form

$$
\begin{aligned}
e_i \times (e_i \times \nabla_i^\perp \nabla_i^\perp \sigma) &= -\chi(e_i, e_i, \nabla_i^\perp \nabla_i^\perp \sigma) \\
&\quad - \langle e_i, e_i\rangle \nabla_i^\perp \nabla_i^\perp \sigma + \langle e_i, \nabla_i^\perp \nabla_i^\perp \sigma\rangle e_i = -\nabla_i^\perp \nabla_i^\perp \sigma.
\end{aligned}
$$

Moreover, for $i \neq j$,

$$
\begin{aligned}
e_i \times (e_j \times \nabla_i^\perp \nabla_j^\perp \sigma) &= -\chi(e_i, e_j, \nabla_i^\perp \nabla_j^\perp \sigma) - \langle e_i, e_j\rangle \nabla_i^\perp \nabla_j^\perp \sigma \\
&\quad + \langle e_i, \nabla_i^\perp \nabla_j^\perp \sigma\rangle e_j \\
&= -\chi(\nabla_i^\perp \nabla_j^\perp \sigma, e_i, e_j) = \nabla_i^\perp \nabla_j^\perp \sigma \times (e_i \times e_j) \\
&= -(e_i \times e_j) \times \nabla_i^\perp \nabla_j^\perp \sigma.
\end{aligned}
$$

Since $\nabla := \nabla^{S^+\otimes S^-} = \nabla^+ \otimes \mathbb{1}_{S^-} + \mathbb{1}_{S^+} \otimes \nabla^-$ agrees with the induced connection $\nabla^\perp$, one has $R^\perp = F^\nabla = F^+ \otimes \mathbb{1}_{S^-} + \mathbb{1}_{S^+} \otimes F^-$, where $F^\pm$ is the curvature of the connection $\nabla^\pm$. Now, using Proposition 2.11, we identify the normal section σ with the section $\kappa \otimes \varepsilon \in \Omega^0(S^+ \otimes S^-)$, and recall that the Clifford product of the normal bundle $\gamma(e_i)\sigma = e_i \times \sigma$ coincides with Clifford multiplication

$$
\tau := \Gamma_0 \otimes \mathbb{1}_{S^-} : TY \to \mathrm{End}(S^+ \otimes S^-),
$$

where Γ_0 is the spin structure on TY. Defining

$$\mathcal{R}(\kappa \otimes \varepsilon) := \frac{1}{2} \sum_{ij} \tau(e_i)\tau(e_j) F_{ij}^{\nabla}(\kappa \otimes \varepsilon),$$

and using (2.6), we have

$$\begin{aligned} -\sum_{i<j} (e_i \times e_j) \times R^{\perp}(e_i, e_j)\sigma &= \sum_{i<j} e_i \times (e_j \times R^{\perp}(e_i, e_j))\sigma \\ &= \frac{1}{2} \sum_{ij} \gamma(e_i)\gamma(e_j)(R^{\perp}(e_i, e_j)\sigma) \\ &= \frac{1}{2} \sum_{ij} \tau(e_i)\tau(e_j) F_{ij}^{\nabla}(\kappa \otimes \varepsilon) = \mathcal{R}(\kappa \otimes \varepsilon) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{R}(\kappa \otimes \varepsilon) &= \frac{1}{2} \sum_{ij} (\Gamma_0 \otimes \mathbb{1}_{S^-})(e_i)(\Gamma_0 \otimes \mathbb{1}_{S^-})(e_j) F_{ij}^{\nabla}(\kappa \otimes \varepsilon) \\ &= \frac{1}{2} \sum_{ij} (\Gamma_0 \otimes \mathbb{1}_{S^-})(e_i)(\Gamma_0 \otimes \mathbb{1}_{S^-})(e_j)(F_{ij}^{+}\kappa \otimes \varepsilon + \kappa \otimes F_{ij}^{-}\varepsilon) \\ &= \underbrace{\frac{1}{2} \sum_{ij} (\Gamma_0(e_i)\Gamma_0(e_j) F_{ij}^{+}\kappa) \otimes \varepsilon}_{(\mathrm{I}')} + \underbrace{\frac{1}{2} \sum_{ij} (\Gamma_0(e_i)\Gamma_0(e_j)\kappa) \otimes F_{ij}^{-}\varepsilon}_{(\mathrm{I}'')}. \end{aligned}$$

Each endomorphism $F_{ij}^{+} : \Omega^0(S^+) \to \Omega^0(S^+)$ is given by the formula (c.f. [16, Theorem 4.15.])

$$F_{ij}^{+}\kappa = \frac{1}{2} \sum_{k<l} \langle R_{ij}(e_k), e_l \rangle \Gamma_0(e_k)\Gamma_0(e_l)\kappa,$$

where $R_{ij} = R(e_i, e_j)$ is the Riemann tensor of the induced connection on Y, with components $R_{ijk}^{l} = \langle R_{ij}(e_k), e_l \rangle$. Then, for the first term,

$$
\begin{aligned}
(\mathrm{I}') &= \frac{1}{8}\sum_{ijkl} R^l_{ijk}\Gamma_0(e_i)\Gamma_0(e_j)\Gamma_0(e_k)\Gamma_0(e_l)\kappa \\
&= \frac{1}{8}\sum_{l}\Bigg(\sum_{ij,(i=k)} R^l_{iji}\Gamma_0(e_i)\Gamma_0(e_j)\Gamma_0(e_i) + \sum_{ij,(j=k)} R^l_{ijj}\Gamma_0(e_i)\Gamma_0(e_j)\Gamma_0(e_j) \\
&\quad + \frac{1}{3}\sum_{i\neq j\neq k\neq i}(R^l_{ijk}+R^l_{jki}+R^l_{kij})\Gamma_0(e_i)\Gamma_0(e_j)\Gamma_0(e_k)\Bigg)\Gamma_0(e_l)\kappa \\
&= \frac{1}{8}\sum_{l}\Bigg(\sum_{ij} R^l_{iji}\Gamma_0(e_j) - \sum_{ij} R^l_{ijj}\Gamma_0(e_i)\Bigg)\Gamma_0(e_l)\kappa \\
&= \frac{1}{4}\sum_{ijl} R^l_{iji}\Gamma_0(e_j)\Gamma_0(e_l)\kappa = \frac{1}{8}\sum_{ijl} R^l_{iji}(\Gamma_0(e_j)\Gamma_0(e_l)+\Gamma_0(e_l)\Gamma_0(e_j))\kappa \\
&= -\frac{1}{4}\sum_{ijl} R^l_{iji}\delta^l_j\kappa = -\frac{1}{4}\sum_{ij} R^j_{iji}\kappa = \frac{1}{4}k\kappa,
\end{aligned}
$$

where k denotes the scalar curvature of Y. For the second term, recall that $\Omega^2(Y,\mathrm{End}(S^-)) \cong \Omega^2(Y)\otimes\Omega^0(\mathrm{End}(S^-))$, so

$$
F^- = \sum_{i<j}(e_i\wedge e_j)\otimes F^-_{ij}.
$$

Moreover, observe that Γ_0 (also γ) induces a map $\rho : \Omega^2(Y)\to\mathrm{End}(S^+)$ defined by

$$
\rho(\sum_{i<j}\eta_{ij}e_i\wedge e_j) := \sum_{i<j}\eta_{ij}\Gamma_0(e_i)\Gamma_0(e_j)
$$

and consider the extension

$$
\overline{\rho} := \rho\otimes\mathbb{1}_{\mathrm{End}(S^-)} : \Omega^2(Y,\mathrm{End}(S^-))\to\mathrm{End}(S^+\otimes S^-)
$$

given by

$$
\overline{\rho}(\sum_{i<j}(e_i\wedge e_j)\otimes F^-_{ij}) := \sum_{i<j}(\Gamma_0(e_i)\Gamma_0(e_j)\otimes F^-_{ij}).
$$

Then,

$$
\begin{aligned}
(\mathrm{I}'') &= \frac{1}{2}\sum_{ij}(\Gamma_0(e_i)\Gamma_0(e_j)\otimes F^-_{ij})(\kappa\otimes\varepsilon) \\
&= \frac{1}{2}\overline{\rho}(\sum_{ij}(e_i\wedge e_j)\otimes F^-_{ij})(\kappa\otimes\varepsilon) \\
&= \overline{\rho}(F^-)(\kappa\otimes\varepsilon).
\end{aligned}
$$

□

3.2. First-order corrections

The correction terms (II),...,(V) can be conveniently organised into three 1^{st} order differential operators P_1, P_2, P_3 on sections of NY.

Lemma 3.5.

$$(\mathrm{II}) = P_1(\sigma) := \sum_{i,j=1}^{3} T_{ii} e_j \times \nabla_j^{\perp} \sigma - T_{ji} e_j \times \nabla_i^{\perp} \sigma - 2 \sum_{(i,j,k)\in S_3^0} C_{ij} \nabla_k^{\perp} \sigma,$$

where S_3^0 are the even permutations in S_3, T_{ji} is the full torsion tensor and C_{ij} the anti-symmetric part of T_{ij}.

Proof. By Lemma 2.12, we have

$$(\mathrm{II}) = \sum_{i,j,n=1}^{3} \sum_{k=4}^{7} T(e_i, e_n)\psi(e_n, e_j, \nabla_j^{\perp} \sigma, \eta_k) e_i \times \eta_k = (*).$$

Since $\chi(e_n, e_j, \nabla_j^{\perp} \sigma) \in \Omega^0(NY)$, then using (2.6) we have

$$\begin{aligned}
(*) &= \sum_{i,j,n=1}^{3} T(e_i, e_n) e_i \times \chi(\nabla_j^{\perp} \sigma, e_n, e_j) \\
&= \sum_{i,j,n=1}^{3} -T(e_i, e_n) e_i \times (\nabla_j^{\perp} \sigma \times (e_n \times e_j)) \\
&= \sum_{i,j,n=1}^{3} T(e_i, e_n)(\chi(e_i, \nabla_j^{\perp} \sigma, e_n \times e_j) - \langle e_i, e_n \times e_j \rangle \nabla_j^{\perp} \sigma) \\
&= \sum_{i,j,n=1}^{3} T(e_i, e_n)(\nabla_j^{\perp} \sigma \times (e_i \times (e_n \times e_j)) - \varphi(e_i, e_n, e_j) \nabla_j^{\perp} \sigma)
\end{aligned}$$

Using relations $e_1 \times e_2 = e_3$ and $e_i \times (e_n \times e_j) = -\chi(e_i, e_n, e_j) - \langle e_i, e_n \rangle e_j + \langle e_i, e_j \rangle e_n$. The first term of the sum is equal to

$$\sum_{i,j=1}^{3} T_{ii} e_j \times \nabla_j^{\perp} \sigma - T_{ji} e_j \times \nabla_i^{\perp} \sigma.$$

Moreover, since $\varphi(e_1, e_2, e_3) = 1$, the second term becomes

$$-2 \sum_{(i,j,k)\in S_3^0}^{3} C_{ij} \nabla_k^{\perp} \sigma. \tag{3.7}$$

where $2C_{ij} = T_{ij} - T_{ji}$. □

Lemma 3.6. *With the above notation*

$$\sum_{k=4}^{7} (\nabla_n \psi)(\eta_k, e_1, e_2, e_3)\eta_k = -\sum_{k=4}^{7} T_{nk}\eta_k. \tag{3.8}$$

Proof. Since Y is associative, Corollary 2.15 gives $\nabla_n \psi_{k123} = -T_{nk}$. □

Denote the following two operators on NY, involving the full torsion tensor

$$P_2(\sigma) = \sum_{i=1}^{3} \sum_{l=4}^{7} ((\nabla_i T)(\sigma, \eta_l) + T(\nabla_i^{\perp} \sigma, \eta_l))e_i \times \eta_l,$$

$$P_3(\sigma) = \sum_{k,l=4}^{7} \left(T(\sigma, \eta_l) + \sum_{i=1}^{3} \varphi(e_i, \nabla_i^{\perp} \sigma, \eta_l) \right) T_{lk}\eta_k.$$

With this notation, we arrive at one of our main theorems:

Theorem 3.7. *The Weitzenböck formula for* (3.1) *is*

$$\not{D}_A{}^2(\sigma) = \nabla^*\nabla\sigma + \frac{1}{4}k \cdot \sigma + \overline{\rho}(F^-)\sigma + P_1(\sigma) + P_2(\sigma) + P_3(\sigma) \tag{3.9}$$

Proof. We examine the five components of $\not{D}_A{}^2$ as on page 172. Components (I) and (II) have been studied in Lemmata 3.4 and 3.5. Now, applying Lemma 3.6, we have

$$\text{(III)} = \sum_{i=1}^{3} \sum_{k,l=4}^{7} \varphi(e_i, \nabla_i^{\perp} \sigma, \eta_l) T_{lk}\eta_k.$$

As to (IV), for each $i = 1, 2, 3$ and $l = 4, 5, 6, 7$, we use Lemma 3.6 to find

$$e_i((\nabla_\sigma \psi)(\eta_l, e_1, e_2, e_3)) = -e_i(T(\sigma, \eta_l)) = -(\nabla_i T)(\sigma, \eta_l) - T(\nabla_i^{\perp} \sigma, \eta_l).$$

Then, indeed,

$$(\mathrm{IV}) = \sum_{i=1}^{3}\sum_{l=4}^{7}((\nabla_i T)(\sigma, \eta_l) + T(\nabla_i^\perp \sigma, \eta_l))e_i \times \eta_l = P_2(\sigma).$$

Finally, a simple calculation gives $(\mathrm{V}) = \sum_{k,l=4}^{7} T(\sigma, \eta_l) T_{lk} \eta_k$, and

$$(\mathrm{V}) + (\mathrm{III}) = \sum_{k,l=4}^{7} \Big(T(\sigma, \eta_l) + \sum_{i=1}^{3} \varphi(e_i, \nabla_i^\perp \sigma, \eta_l) \Big) T_{lk} \eta_k = P_3(\sigma)$$

□

Corollary 3.8. *Let* (M^7, φ) *be a* G_2*-manifold. Then,*

$$\not{D}_{\mathrm{A}}{}^2 = \not{D}^2 = \nabla^*\nabla + \frac{1}{4}k + \overline{\rho}(F^-)$$

In [11], Gayet obtains a Weitzenböck-type formula when the G_2-structure is torsion-free:

$$\not{D}^2 = \nabla^*\nabla + \mathcal{R} - \mathcal{A}. \tag{3.10}$$

The term $\mathcal{R}(\sigma) = \pi^\perp \sum_{i=1}^{3} R(e_i, \sigma) e_i$ can be seen as a partial Ricci operator, where R is the curvature tensor of g on M and $\pi^\perp$ is the orthogonal projection to NY, and

$$\mathcal{A} : \Omega^0(NY) \to \Omega^0(\mathrm{Sym}(TY)),$$

defined by $\mathcal{A}(\sigma) = S^t \circ S(\sigma)$, is a symmetric positive 0^{th}–order operator determined by the shape operator $S(\sigma)(X) = -(\nabla_X \sigma)^\top$. With these data, Gayet formulates a vanishing theorem for a compact associative submanifold Y of a G_2–manifold and proves that Y is rigid when the spectrum of the operator $\mathcal{R} - \mathcal{A}$ is positive. The advantage of formula (3.10) lies in the relation between the intrinsic and extrinsic geometries of the associative submanifold, because $\mathcal{R} - \mathcal{A}$ is obtained from a curvature term

$$-\sum_{i<j}^{3} (e_i \times e_j) \times R^\perp(e_i, e_j)\sigma. \tag{3.11}$$

While one cannot entirely apply his proof to the general case (because the full torsion tensor is nonzero), we are able to adapt some of its steps.

Given $u,v\in\Omega^0(TY)$, and their respective local extensions $\overline{u},\overline{v}$ to TM, the TY-valued 2–form $B(u,v):=\nabla_{\overline{u}}\overline{v}-\nabla^{\top}_{u}v$ relates to the shape operator by $\langle B(u,v),\sigma\rangle=\langle S_\sigma(u),v\rangle$, for $\sigma\in\Omega^0(TM)$. These data define a natural 0^{th}–order operator

$$\mathcal{B}(\sigma):=\sum_{i,j=1}^{3}(e_i\times e_j)\times B(e_j,S_\sigma(e_i)). \tag{3.12}$$

Proposition 3.9. *The curvature term in (3.11) can be rewritten as*

$$\begin{aligned}-\sum_{i<j}^{3}(e_i\times e_j)\times R^{\perp}(e_i,e_j)\sigma &= \mathcal{R}(\sigma)+\mathcal{B}(\sigma)\\ &\quad -\pi^{\perp}\Big(\sum_{i\in\mathbb{Z}_3}e_i\times\mathcal{T}(e_{i+1},\sigma,e_i,e_{i+1})\Big),\end{aligned} \tag{3.13}$$

where $\mathcal{R}(\sigma)=\pi^{\perp}\sum_{i=1}^{3}R(e_i,\sigma)e_i$ is the partial Ricci operator defined in (3.10) *and $\mathcal{T}$ is defined in* (2.15) *by*

$$\begin{aligned}&\mathcal{T}(e_{i+1},\sigma,e_i,e_{i+1})\\ &\quad:=\sum_{m=1}^{7}T(\sigma,e_m)(\nabla_{i+1}\psi)(e_m,e_i,e_{i+1},\cdot)^{\sharp}\\ &\qquad -T_{i+1m}(\nabla_\sigma\psi)(e_m,e_i,e_{i+1},\cdot)^{\sharp}\\ &\qquad +\Big((\nabla_{i+1}T)(\sigma,e_m)-(\nabla_\sigma T)(e_{i+1},e_m)\Big)\chi(e_m,e_i,e_{i+1}).\end{aligned}$$

Proof. Expanding the summands in the frame $\{\eta_4,\dots,\eta_7\}$ and using anti-symmetry of the mixed product and the Ricci equation, we have

$$\begin{aligned}-\sum_{i<j}^{3}(e_i\times e_j)\times R^{\perp}(e_i,e_j)\sigma &= -\frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\langle (e_i\times e_j)\times R^{\perp}(e_i,e_j)\sigma,\eta_k\rangle\eta_k\\ &= \frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\langle R^{\perp}(e_i,e_j)\sigma,(e_i\times e_j)\times\eta_k\rangle\eta_k\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\langle R(e_i,e_j)\sigma,(e_i\times e_j)\times\eta_k\rangle\eta_k \\
&\quad + \langle [S_\sigma, S_{(e_i\times e_j)\times\eta_k}]e_i,e_j\rangle\eta_k \\
&= \underbrace{-\frac{1}{2}\pi^\perp\sum_{i,j=1}^{3}(e_i\times e_j)\times R(e_i,e_j)\sigma}_{(\star)} \\
&\quad \underbrace{+\frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\langle [S_\sigma, S_{(e_i\times e_j)\times\eta_k}]e_i,e_j\rangle\eta_k}_{(\star\star)}\,.
\end{aligned}
$$

Applying the Bianchi identity $R(e_i,e_j)\sigma = -R(\sigma,e_i)e_j - R(e_j,\sigma)e_i$ to the first term, expanding the sum and using Lemma 2.16, we have:

$$
\begin{aligned}
(\star) &= \pi^\perp\sum_{i,j=1}^{3}(e_i\times e_j)\times R(e_j,\sigma)e_i \\
&= \pi^\perp(e_3\times R(e_2,\sigma)e_1 - e_2\times R(e_3,\sigma)e_1 - e_3\times R(e_1,\sigma)e_2 \\
&\quad + e_1\times R(e_3,\sigma)e_2 + e_2\times R(e_1,\sigma)e_3 - e_1\times R(e_2,\sigma)e_3) \\
&= \pi^\perp(\underbrace{-e_1\times[R(e_2,\sigma)e_1\times e_2 + e_1\times R(e_2,\sigma)e_2}_{\text{(I)}} + \mathcal{T}(e_2,\sigma,e_1,e_2)] \\
&\quad \underbrace{-e_2\times[R(e_3,\sigma)e_2\times e_3 + e_2\times R(e_3,\sigma)e_3}_{\text{(II)}} + \mathcal{T}(e_3,\sigma,e_2,e_3)] \\
&\quad - \underbrace{e_3\times[R(e_1,\sigma)e_3\times e_1 + e_3\times R(e_1,\sigma)e_1}_{\text{(III)}} + \mathcal{T}(e_1,\sigma,e_3,e_1)] \\
&\quad + e_3\times R(e_2,\sigma)e_1 + e_1\times R(e_3,\sigma)e_2 + e_2\times R(e_1,\sigma)e_3).
\end{aligned}
$$

Using the identity

$$
u\times(v\times w) + v\times(u\times w) = \langle u,w\rangle v + \langle v,w\rangle u - 2\langle u,v\rangle w,
$$

we check that

$$
\begin{aligned}
\text{(I)} &= -e_3\times R(e_2,\sigma)e_1 - (e_2,\sigma,e_1,e_2)e_1 + 2(e_2,\sigma,e_1,e_1)e_2 + R(e_2,\sigma)e_2 \\
\text{(II)} &= -e_1\times R(e_3,\sigma)e_2 - (e_3,\sigma,e_2,e_3)e_2 + 2(e_3,\sigma,e_2,e_2)e_3 + R(e_3,\sigma)e_3 \\
\text{(III)} &= -e_2\times R(e_1,\sigma)e_3 - (e_1,\sigma,e_3,e_1)e_3 + 2(e_1,\sigma,e_3,e_3)e_1 + R(e_1,\sigma)e_1,
\end{aligned}
$$

where $(e_1, \sigma, e_3, e_1) := \langle R(e_1, \sigma)e_3, e_1\rangle$. Cancelling terms and taking the orthogonal projection on (I) + (II) + (III), we find

$$(\star) = \mathcal{R}(\sigma) - \pi^{\perp}\Big(\sum e_i \times \mathcal{T}(e_{i+1}, \sigma, e_i, e_{i+1})\Big).$$

Finally, by the symmetry of S_σ and $S_{(e_i\times e_j)\times\eta_k}$, the second term is

$$\begin{aligned}
(\star\star) &= \frac{1}{2}\sum_{i,j=1}^{3}\sum_{k=4}^{7}\Big(\langle S_{(e_i\times e_j)\times\eta_k}(e_i), S_\sigma(e_j)\rangle - \langle S_\sigma(e_i), S_{(e_i\times e_j)\times\eta_k}(e_j)\rangle\Big)\eta_k\\
&= \sum_{i,j=1}^{3}\sum_{k=4}^{7}\Big(\langle S_{(e_i\times e_j)\times\eta_k}(e_i), S_\sigma(e_j)\rangle\Big)\eta_k\\
&= \sum_{i,j=1}^{3}\sum_{k=4}^{7}\Big(\langle B(e_i, S_\sigma(e_j)), (e_i\times e_j)\times\eta_k\rangle\Big)\eta_k\\
&= \sum_{i,j=1}^{3}\sum_{k=4}^{7}\varphi(e_i\times e_j, \eta_k, B(e_i, S_\sigma(e_j)))\eta_k\\
&= -\sum_{i,j=1}^{3}(e_i\times e_j)\times B(e_i, S_\sigma(e_j))\\
&= \mathcal{B}(\sigma).
\end{aligned}$$

□

4. The nearly parallel case and applications

The torsion-free condition for a G_2-structure is highly overdetermined, so examples are difficult to construct and seldom known explicitly. In terms of the Fernández-Gray classification recalled in Section 2.4, the next natural 'least-torsion' case consists of the so-called nearly parallel structures, for which the torsion forms τ_1, τ_2, τ_3 vanish and the remaining torsion is just a constant:

Definition 4.1. Let (M, φ) a manifold with a G_2–structure, φ is called *nearly parallel* if

$$d\varphi = \tau_0\psi,$$

with $\tau_0 \neq 0$ constant.

Regarding the deformations of associative submanifolds, our approach unifies previously known results by means of a Bochner-type vanishing theorem. This technique requires a certain 'positivity' of curvature, which can in practice be found in cases of interest studied by several authors.

4.1. Proof of the vanishing theorem

Following Proposition 2.13, the full torsion tensor in the nearly parallel case is given by $T_{ij} = \frac{\tau_0}{4} g_{ij}$, which drastically simplifies the Weitzenböck formula (3.9):

Proposition 4.2. *The Weitzenböck formula for the Fueter-Dirac operator* (3.1) *in the nearly parallel case is*

$$\not{D}_{\mathrm{A}}{}^2(\sigma) = \nabla^*\nabla\sigma + \frac{1}{4}k\cdot\sigma + \rho(F^-)\sigma + \tau_0\,\not{D}(\sigma) + \frac{\tau_0^2}{16}\cdot\sigma. \tag{4.1}$$

Proof. Given the orthonormal frame $\{e_1, e_2, e_3, \eta_4, ..., \eta_7\}$, it suffices to prove that the last three terms in (3.9) satisfy

$$(P_1 + P_2 + P_3)(\sigma) = \tau_0\,\not{D}(\sigma) + \frac{\tau_0^2}{16}\cdot\sigma.$$

At a point $p \in Y$, for P_1, we have $C_{ij} = 0$, because τ_1 and τ_2 are zero, then

$$\begin{aligned}\sum_{i,j=1}^{3} T_{ii}e_j \times \nabla_j^\perp\sigma - T_{ji}e_j \times \nabla_i^\perp\sigma &= \frac{3}{4}\tau_0\sum_{j=1}^{3} e_j\times\nabla_j^\perp\sigma - \frac{1}{4}\tau_0\sum_{j=1}^{3} e_j\times\nabla_j^\perp\sigma\\ &= \frac{1}{2}\tau_0\,\not{D}(\sigma).\end{aligned}$$

The next two components are

$$\begin{aligned}P_2(\sigma) &= \sum_{i=1}^{3}\sum_{l=4}^{7}((\nabla_i T)(\sigma,\eta_l) + T(\nabla_i^\perp\sigma,\eta_l))e_i\times\eta_l\\ &= \frac{\tau_0}{4}\sum_{i=1}^{3}\sum_{l=4}^{7} g(\nabla_i^\perp\sigma,\eta_l)e_i\times\eta_l\\ &= \frac{\tau_0}{4}\sum_{i=1}^{3} e_i\times\nabla_i^\perp\sigma = \frac{\tau_0}{4}\,\not{D}(\sigma),\end{aligned}$$

$$\begin{aligned}
P_3(\sigma) &= \sum_{k,l=4}^{7}\Big(T(\sigma,\eta_l)+\sum_{i=1}^{3}\varphi(e_i,\nabla_i^{\perp}\sigma,\eta_l)\Big)T_{lk}\eta_k \\
&= \frac{\tau_0}{4}\sum_{k,l=4}^{7}\Big(\frac{\tau_0}{4}g(\sigma,\eta_l)+\sum_{i=1}^{3}\varphi(e_i,\nabla_i\sigma,\eta_l)\Big)g(\eta_l,\eta_k)\eta_k \\
&= \frac{\tau_0}{4}\sum_{l=4}^{7}\Big(\frac{\tau_0}{4}g(\sigma,\eta_l)+\sum_{i=1}^{3}\varphi(e_i,\nabla_i\sigma,\eta_l)\Big)\eta_l \\
&= \frac{\tau_0^2}{16}\cdot\sigma+\frac{\tau_0}{4}\,\not{D}(\sigma).
\end{aligned}$$

□

Corollary 4.3. *Equation* (4.1) *can be rewritten as*

$$\not{D}_{\mathrm{A}}{}^2(\sigma)=\nabla^*\nabla\sigma+\mathcal{R}(\sigma)+\mathcal{B}(\sigma)+\tau_0\,\not{D}(\sigma)+\frac{\tau_0^2}{4}\cdot\sigma. \tag{4.2}$$

Proof. For a nearly parallel G_2-structure, the full torsion tensor is $T_{ij}=\frac{\tau_0}{4}g_{ij}$, thus $\nabla T=0$ and so:

$$\begin{aligned}
\sum_{i\in\mathbb{Z}_3} e_i\times\mathcal{T}(e_{i+1},\sigma,e_i,e_{i+1}) =&\frac{\tau_0}{4}\sum_{i\in\mathbb{Z}_3}\sum_{m,l=1}^{7}\Big(g(\sigma,e_m)(\nabla_{i+1}\psi)(e_m,e_i,e_{i+1},e_l) \\
&-g(e_{i+1},e_m)(\nabla_\sigma\psi)(e_m,e_i,e_{i+1},e_l)\Big)e_i\times e_l \\
=&\frac{\tau_0}{4}\sum_{i\in\mathbb{Z}_3}\sum_{l=1}^{7}\Big((\nabla_{i+1}\psi)(\sigma,e_i,e_{i+1},e_l) \\
&-(\nabla_\sigma\psi)(e_{i+1},e_i,e_{i+1},e_l)\Big)e_i\times e_l \\
=&\frac{\tau_0}{4}\sum_{i\in\mathbb{Z}_3}\sum_{l=1}^{7}\Big((\nabla_{i+1}\psi)(\sigma,e_i,e_{i+1},e_l)\Big)e_i\times e_l \\
=&-\frac{\tau_0}{4}\sum_{i\in\mathbb{Z}_3}\sum_{l=1}^{7}T(e_{i+1},e_{i+1})\varphi(\sigma,e_i,e_l)e_i\times e_l \\
=&-\frac{\tau_0^2}{16}\sum_{i\in\mathbb{Z}_3}g(e_{i+1},e_{i+1})e_i\times(\sigma\times e_i)=-\frac{3}{16}\tau_0^2\sigma
\end{aligned}$$

Here we used the skew-symmetry of $\nabla_\sigma\psi$ for the third equality and Corollary 2.15 for the fourth one. Equation (4.2) now follows from Proposition 3.9. □

Theorem 4.4. *Let (M,φ) be a 7-manifold with a nearly parallel* G_2*–structure. If $Y\subset M$ is a closed associative submanifold such that the operator $\overline{\rho}(F^-)$ associated to the curvature of the bundle S^- (c.f. (1.3) and Lemma 3.4) is bounded below by $-(\frac{k}{4}-\frac{3}{16}\tau_0^2)$, then Y is rigid.*

Proof. Let σ be a section of NY,

$$\begin{aligned}\Delta|\sigma|^2 &= \sum_i e_i e_i\langle\sigma,\sigma\rangle = 2\sum_i e_i\langle\nabla_i^\perp\sigma,\sigma\rangle\\ &= 2\sum_i\langle\nabla_i^\perp\nabla_i^\perp\sigma,\sigma\rangle + \langle\nabla_i^\perp\sigma,\nabla_i^\perp\sigma\rangle\\ &= -2\langle\nabla^*\nabla\sigma,\sigma\rangle + 2|\nabla^\perp\sigma|^2\\ &= -2\langle \not{D}_{\mathrm{A}}{}^2(\sigma),\sigma\rangle + \frac{k}{2}\langle\sigma,\sigma\rangle + 2\langle\overline{\rho}(F^-)\sigma,\sigma\rangle\\ &\quad + 2\tau_0\langle\not{D}(\sigma),\sigma\rangle + \frac{\tau_0^2}{8}|\sigma|^2 + 2|\nabla^\perp\sigma|^2.\end{aligned}$$

Taking $\sigma\in\ker\not{D}_{\mathrm{A}}$, equation (3.2) gives

$$\begin{aligned}\langle\not{D}(\sigma),\sigma\rangle &= \sum_{k=4}^{7}(\nabla_\sigma\psi)(\eta_k,e_1,e_2,e_3)\langle\eta_k,\sigma\rangle\\ &= -\sum_{k=4}^{7}T(\sigma,\eta_k)\langle\eta_k,\sigma\rangle = -\frac{\tau_0}{4}\sum_{k=4}^{7}\langle\sigma,\eta_k\rangle^2.\end{aligned} \tag{4.3}$$

By Stokes' theorem, it follows that

$$\begin{aligned}0 &= \int_Y\left(\frac{k}{4}|\sigma|^2 + \langle\overline{\rho}(F^-)\sigma,\sigma\rangle - \frac{\tau_0^2}{4}\sum_{k=4}^{7}\langle\sigma,\eta_k\rangle^2 + \frac{\tau_0^2}{16}|\sigma|^2 + |\nabla^\perp\sigma|^2\right)d\,\mathrm{vol}_Y\\ &= \int_Y(\left(\frac{k}{4}-\frac{3}{16}\tau_0^2\right)|\sigma|^2 + \langle\overline{\rho}(F^-)\sigma,\sigma\rangle + |\nabla^\perp\sigma|^2)d\,\mathrm{vol}_Y\,.\end{aligned}$$

By assumption, $\langle\overline{\rho}(F^-)\sigma,\sigma\rangle \geq -\left(\frac{k}{4}-\frac{3}{16}\tau_0^2\right)\langle\sigma,\sigma\rangle$, so $\nabla^\perp\sigma = 0$ and this implies $\not{D}(\sigma)=0$. Notice from Lemma 3.6 that the Fueter-Dirac operator is

$$\not{D}_{\mathrm{A}} = \not{D} + \frac{\tau_0}{4} \quad \text{with} \quad \tau_0\neq 0.$$

Then, from $\not{D}_{\mathrm{A}}(\sigma)=0$ it follows that $\sigma=0$, i.e. $\ker\not{D}_{\mathrm{A}}=\{0\}$. □

4.2. The associative submanifolds of the 7-sphere

In [17], Lotay defines a G_2–structure φ on S^7, writing $\mathbb{R}^8 \setminus \{0\} \cong \mathbb{R}^+ \times S^7$, such that

$$\Phi_0|_{(r,p)} = r^3 dr \wedge \varphi|_p + r^4 * \varphi|_p,$$

where Φ_0 is the Spin(7)–structure of $\mathbb{R}^8$, r the radial coordinate on $\mathbb{R}^+$ and $*$ the Hodge star on S^7 induced by the round metric. Since Φ_0 is closed, it follows that $d\varphi = 4 * \varphi$ i.e. φ is a nearly parallel G_2–structure.

Consider the 7–sphere as the homogeneous space $\mathrm{Spin}(7)/G_2$, viewing Spin(7) as the G_2 frame bundle over S^7. From the structure equations of Spin(7) [17, Chapter 4], the *second fundamental form* $B \in \Omega^0(\mathrm{Sym}^2(TY)^* \otimes NY)$ (c.f. [17, Definition 4.5]) satisfies

$$\sum_{i=1}^{3} e_i \times B(e_i, e_j) = 0. \tag{4.4}$$

Using (4.4), the operator $\mathcal{B}(\sigma)$ from (3.12) is given by

$$\begin{aligned}\mathcal{B}(\sigma) &= -\sum_{i=1}^{3} e_i \times \left(\sum_{j=1}^{3} e_j \times B(e_j, S_\sigma(e_i))\right) - \sum_{i,j=1}^{3} \langle e_i, e_j\rangle B(e_j, S_\sigma(e_i)) \\ &= -\sum_{i=1}^{3} B(e_i, S_\sigma(e_i))\end{aligned}$$

Taking the inner product with the section σ itself, one obtains the non-positivity property

$$\begin{aligned}\langle \mathcal{B}(\sigma), \sigma\rangle &= -\sum_{i=1}^{3} \langle B(e_i, S_\sigma(e_i)), \sigma\rangle \\ &= -\sum_{i=1}^{3} \langle S_\sigma(e_i), S_\sigma(e_i)\rangle = -\sum_{i=1}^{3} \|S_\sigma(e_i)\|^2 .\end{aligned}$$

Consider the action of SU(2) on S^7 given by

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} \mapsto \begin{pmatrix} az_1 + bz_2 \\ -\bar{b}z_1 + \bar{a}z_2 \\ az_3 + bz_4 \\ -\bar{b}z_3 + \bar{a}z_4 \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2). \tag{4.5}$$

By Corollary 4.3, we have,

$$\not{D}_{\mathrm{A}}{}^2(\sigma) = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) + \mathcal{B}(\sigma) + 4\,\not{D}_{\mathrm{A}}(\sigma),$$

or, in terms of the operator $\not{D}$,

$$\not{D}^2 = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) + \mathcal{B}(\sigma) + 2\,\not{D}(\sigma) + 3\sigma, \tag{4.6}$$

which coincides with the formula given by Kawai [15]. Consider the orbit $S^3 \subset S^7$ of the point $(1,0,0,0)$ under action (4.5). The tangent space to S^3 at a point $(z_1, z_2, 0, 0)$ is spanned by the vectors

$$X_1 = (z_2, -z_1, 0, 0), \quad X_2 = (iz_2, iz_1, 0, 0), \quad X_3 = (iz_1, -iz_2, 0, 0).$$

As the induced metric on S^3, from the round metric on S^7, coincides with the round metric of constant curvature 1, the following results of [4] can be adapted to our case.

Lemma 4.5. *The normal bundle NS^3 can be trivialized by parallel sections $\sigma_1, \ldots, \sigma_4$ of the connection $\nabla^\perp$.*

Proof. It suffices to show that the curvature operator $R^\perp$ vanishes (c.f. (3.6)). Let u, v be tangent vector fields of S^3, and σ a section of NS^3, then the Ricci equation gives

$$\begin{aligned} R^\perp(u,v)\sigma &= \sum_{k=4}^{7} \langle R^\perp(u,v)\sigma, \eta_k\rangle \eta_k \\ &= \sum_{k=4}^{7} (\langle R(u,v)\sigma, \eta_k\rangle + \langle [S_\sigma, S_{\eta_k}]u, v\rangle)\eta_k \\ &= \sum_{k=4}^{7} (\langle u,\sigma\rangle\langle v,\eta_k\rangle - \langle v,\sigma\rangle\langle u,\eta_k\rangle)\eta_k = 0. \end{aligned}$$

At the third equality we used the well-known facts that the metric on S^7 has constant sectional curvature equal to 1 and that $S^3 \subset S^7$ is a totally geodesic immersed submanifold. □

The following Weitzenböck formula relates the operator $D = \not{D} - \mathrm{Id}$ with the Laplacian of the connection $\nabla^\perp$ on NS^3.

Lemma 4.6. *On the normal bundle NS^3, the following formula holds:*

$$D^2 = \nabla^* \nabla + \mathrm{Id}\,. \tag{4.7}$$

Proof. In a local orthonormal frame e_1, e_2, e_3 around $p \in S^3$, we compute

$$\begin{aligned} D^2(\sigma) &= \not{D}^2(\sigma) - 2\not{D}(\sigma) + \sigma = \nabla^*\nabla\sigma + \mathcal{R}(\sigma) + 4\sigma \\ &= \nabla^*\nabla\sigma + \Big(\sum_{i=1}^{3} \langle \sigma, e_i\rangle e_i - \langle e_i, e_i\rangle \sigma\Big)^{\perp} + 4\sigma \\ &= \nabla^*\nabla\sigma + \sigma. \end{aligned}$$

□

Consider a basis $1 = f_0, f_1, f_2, \ldots$ of $L^2(S^3, \mathbb{R})$, consisting of eigenfunctions of the Laplace operator:

$$\Delta f_i = \lambda_i f_i.$$

As a direct consequence from Lemma 4.5 and (4.7), we obtain a natural eigenbasis for the operator D^2 on sections of NS^3:

Lemma 4.7. $D^2(f_i\sigma_k) = (\lambda_i + 1)(f_i\sigma_k)$.

Since the metric on S^3 has constant curvature 1, the eigenvalues of the Laplace operator on S^3 are

$$\lambda_k = k(k+2) \quad k \geq 0,$$

with multiplicities $m_k = (k+1)^2$ [25, Proposition 22.2 and Corollary 22.1]. Together with Lemma 4.7, this gives:

Corollary 4.8. *D^2 has eigenvalues $(k+1)^2$ with multiplicities $4(k+1)^2$, $k \geq 0$.*

In general, for an operator T and a vector u such that $T^2u = \mu^2 u$, if

$$v^{\pm} := (T \pm \mu)u \neq 0$$

then $v^{\pm}$ is an eigenvector of T with eigenvalue $\pm\mu$. Let us apply this principle to $T = D$, with $\mu_k^2 = (k+1)^2$ and $u_k = f_k\sigma_j$, for $j = 1, \ldots, 4$.

Let us first look at the case $k = 0$, in which $f_0 = 1$ and $\lambda_0 = 0$, so $u_0 = \sigma_j$ and $\mu_0^2 = 1$, i.e.,

$$v^{\pm} = (D \pm \mu_0)\sigma_j = D\sigma_j \pm \sigma_j.$$

Now, $\not{D}\,\sigma_j = 0$ by Lemma 4.5, so $D\sigma_j = -\sigma_j$ and therefore $v^+ = 0$ and $v^- = -2\sigma_j$. Accordingly, v^- is an eigenvector of D with eigenvalue $-\mu_0 = -1$. Since $v^- = -2\sigma_j$, for $j = 1, \ldots, 4$, the multiplicity of $-\mu_0 = -1$ is at least 4, but the multiplicity of $(-\mu_0)^2 = \mu_0^2 = 1$ is already 4, by Corollary 4.8, therefore the multiplicity of $-\mu_0 = -1$ is exactly 4.

Now, for $k \geq 1$, we take $u_k = f_k\sigma_j$ and $\mu_k = k+1$, and use the trivial fact that $e_i \times \sigma_j$ and σ_j are linearly independent for all i, j:

$$\begin{aligned} v_k^{\pm} =& (D \pm \mu_k)u_k = \not{D}\, u_k - (1 \mp \mu_k)u_k \\ =& \sum_{i=1}^{3} e_i(f_k)e_i \times \sigma_j - \underbrace{(1 \mp \mu_k)}_{\neq 0}\underbrace{f_k}_{\neq 0}\sigma_j \neq 0. \end{aligned}$$

Thus $v_k^{\pm}$ is an eigenvector of D with eigenvalue $\pm\mu_k$, and it follows that $v^{\pm}$ is an eigenvector of $\not{D}$ with eigenvalue $1 \pm \mu_k$, such that $m(1+\mu_k) + m(1-\mu_k) = 4(k+1)^2$. It remains to determine the multiplicities of the eigenvalues $1 \pm (k+1)$. We introduce the following notation, for $k \geq 1$:

$$\mu_0^+ := 1 - \mu_0 = 0, \quad \mu_k^+ := 1 + \mu_k = k+2, \quad \text{and} \quad \mu_{-k}^+ := 1 - \mu_k = -k.$$

From Corollary 4.8, multiplicities of opposite index add up as $m(\mu_k^+) + m(\mu_{-k}^+) = 4(k+1)^2$. Alternatively, in the sign convention of Remark 2.7, we denote the eigenvalues of $\not{D}$ by

$$\mu_0^- = 0, \quad \mu_{-k}^- = -k-2, \quad \text{and} \quad \mu_k^- = k, \quad k \geq 1,$$

and again we know $m(\mu_k^-) + m(\mu_{-k}^-) = 4(k+1)^2$.

Lemma 4.9. *The multiplicities in both sign conventions satisfy the following relations:*

$$m(\mu_{-k}^+) = m(\mu_k^-) = 2(k+1)(k+2), \quad k \geq 0.$$

and

$$m(\mu_k^+) = m(\mu_{-k}^-) = 2k(k+1), \quad k \geq 1.$$

Proof. From the above, the operator $\not{D} - \frac{3}{2}$ has eigenvalues

$$\alpha_0^+ = -\frac{3}{2}, \quad \alpha_k^+ = k + \frac{3}{2} - 1 \quad \text{and} \quad \alpha_{-k}^+ = -k - \frac{3}{2}.$$

Let $\alpha_k^- := -\alpha_{-k}^+$. Since $\mu_k^- = -\mu_{-k}^+$, we have $m(\alpha_k^\pm) = m(\mu_k^\pm)$, for all $k \in \mathbb{Z}$, and so

$$m(\alpha_k^\pm) + m(\alpha_{-k}^\pm) = 4(k+1)^2.$$

Now the claim clearly holds for $k = 0$ and, by induction on $k \geq 1$, we have

$$\begin{aligned} m(\mu_{-(k+1)}^+) &= m(\alpha_{-(k+1)}^+) = 4(k+2)^2 - m(\alpha_{(k+1)}^+) \\ &= 4(k+2)^2 - m(\alpha_k^-) = 4(k^2+4k+4) - 2(k+1)(k+2) \\ &= 2(k+2)(k+3). \end{aligned}$$

To obtain the second equality we used the relation

$$\alpha_{(k+1)}^+ = (k+1) + \frac{3}{2} - 1 = \alpha_k^-,$$

and for the last one we used the induction hypothesis on α_k^-. □

The group $\mathrm{Aut}(S^7, \varphi) = \mathrm{Spin}(7)$ of automorphisms of S^7 which fix the G_2–structure induces trivial associative deformations, and the associative 3–sphere is invariant by the action of the embedded subgroup $K = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2 \subset \mathrm{Spin}(7)$, where $\mathbb{Z}_2$ is generated by $(-1,-1,-1)$ [19, Theorem IV 1.38]. Therefore the space of infinitesimal associative deformations of S^3 has dimension at least $\dim(\mathrm{Spin}(7)/K) = 12$.

Corollary 4.10. *The* 3*-sphere in* S^7 *is rigid as an associative submanifold.*

Proof. Since μ_{-1}^+ is the eigenvalue corresponding to the space of infinitesimal associative deformations, by Lemma 4.9, $\dim(\ker \not{D}_{\mathrm{A}}) = m(\mu_{-1}^+) = 12$. □

4.3. The example of Bryant and Salamon

In [6], Bryant and Salamon constructed an example of a 7–manifold with constant scalar curvature and holonomy exactly G_2:

Theorem 4.11. *Let* (M^3, ds^2) *be a Riemannian* 3*-manifold with constant sectional curvature* $K = 1$*. Let* $\mathbf{S}(M) \to M$ *denote the standard spinor bundle, let* $r : \mathbf{S}(M) \to \mathbb{R}$ *be the squared Euclidean norm, and let* $d\sigma^2$ *denote* *the quadratic form of rank* 4 *on the total space of* $\mathbf{S}(M) \cong M^3 \times \mathbb{R}^4$ *which restricts to the standard flat metric on each fibre and whose null space at each point is the horizontal space of the standard spin connection. Then the following metric on* $\mathbf{S}(M)$ *is complete and has holonomy* G_2*:*

$$g = 3(r+1)^{2/3} ds^2 + 4(r+1)^{-1/3} d\sigma^2. \tag{4.8}$$

Here $r = |a|^2 = a\bar{a}$ *denotes squared radial distance and* $a : P_{\mathrm{Spin}(3)}(M) \times \mathbb{H} \to \mathbb{H}$ *is the projection onto the second factor.*

The corresponding associative submanifold is of the form $S^3 \times \{0\}$ for $0 \in \mathbb{R}^4$. Now, a compact spin manifold of positive scalar curvature admits no harmonic spinors (see e.g. [16]) – in fact, the same conclusion holds if the scalar curvature is just nonnegative and somewhere positive – but McLean showed in [19] that the moduli space of associative deformations at Y is the space of harmonic twisted spinors on Y, that is, the kernel of its Dirac operator.

Observe that the normal bundle of $S^3 \times \{0\}$ is isomorphic to the spinor bundle of S^3. In general, let Y^3 be an oriented Riemannian manifold and $\pi : P_{\mathrm{SO}}(Y) \to Y$ the frame bundle of oriented isometries. Now let $\xi : P_{\mathrm{Spin}}(Y) \to P_{\mathrm{SO}}(Y)$ be the Spin double cover of the bundle $P_{\mathrm{SO}}(Y)$. The normal bundle of S^3 (as a submanifold of $S^3 \times \mathbb{R}^4$) can be written as an associated bundle $NS^3 = P_{SO}(S^3) \times_{\mathrm{SO}(4)} \mathbb{H}$, via the representation $\varrho : \mathrm{SO}(4) \to \mathrm{Gl}(\mathbb{H})$, $\varrho([p,q])(v) = pv\bar{q}$. Now, the spinor bundle of S^3 can be written as $\mathbf{S}(S^3) = P_{\mathrm{Spin}}(S^3) \times_{\mathrm{Spin}(3)} \mathbb{H}$ via the inclusion $\iota^- : p \in \mathrm{Spin}(3) \hookrightarrow (1,p) \in \mathrm{Spin}(4)$ and the representation $\varsigma : \mathrm{Spin}(3) \times \mathrm{Spin}(3) \to \mathrm{Gl}(\mathbb{H})$, $\varsigma(p,q)(v) = v\bar{q}$. So the identification $\mathbb{R}^4 \cong \mathbb{H}$ gives a bundle map

$$\Phi : \mathbf{S}(S^3) \to NS^3$$

by $\Phi(\widetilde{p}, v) = (\xi(\widetilde{p}), v)$. Observe that Φ is well-defined:

$$\begin{aligned} \Phi(pg^{-1}, \varsigma(1,g)(v)) &= (\xi(p \cdot g), v\bar{g}) \\ &= (\xi(p)\xi_0(g), \varrho([1,g])(v)) \\ &= (\xi(p), v) \cdot g = \Phi(p,v). \end{aligned}$$

It is easy to check that Φ is a bundle isomorphism.

In those terms, we obtain a trivial alternative for Gayet's proof of rigidity of S^3 [11], using Theorem 4.4 and the fact that $\tau_0 = 0$.

Proposition 4.12. *Let $S^3 \times \mathbb{R}^4$ be the* G_2*–manifold with the metric* (4.8). *Then $S^3 \cong S^3 \times \{0\}$ is rigid as an associative submanifold.*

Proof. Since the normal bundle of $S^3 \times \{0\}$ coincides with the Spin bundle $\mathbf{S}(S^3)$, the term $\overline{\rho}(F^-)$ vanishes, and we conclude immediately from Theorem 4.4. □

4.4. Locally conformal calibrated case and applications

As an application of the Fueter-Dirac Weitzenböck formula (3.9) and Proposition 3.9, we focus on *locally conformal calibrated* G_2–structures, whose associated metric is (at least locally) conformal to a metric induced by a calibrated G_2–structure. We provide a novel example of a rigid associative submanifold, inside a compact manifold S with a locally conformal calibrated G_2–structure, studied by Fernández, Fino and Raffero [9].

Definition 4.13. A G_2–structure is *locally conformal calibrated* if it has vanishing torsion components $\tau_0 \equiv 0$ and $\tau_3 \equiv 0$, so

$$d\varphi = 3\tau_1 \wedge \varphi,$$
$$d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi.$$

A SU(3)–*structure* on a 6-manifold N is a pair $(\omega, \Omega_+) \in \Omega^2(N) \times \Omega^3(N)$ such that $\Omega_+ = \frac{1}{2}(\Omega + \overline{\Omega})$, where $\Omega \in \Omega^0(\Lambda^3(T^*N \otimes \mathbb{C}))$ is a decomposable complex 3-form and

$$\omega \wedge \Omega_+ = 0 \quad \text{and} \quad \frac{\omega^3}{6} = \frac{i}{8}\Omega \wedge \overline{\Omega} = \frac{1}{4}\Omega_+ \wedge \Omega_- \quad \text{with} \quad \Omega_- := \frac{1}{2i}(\Omega - \overline{\Omega}).$$

The SU(3)–structure (ω, Ω_+) is said to be *coupled* if $d\omega = c\Omega_+$ with c a non-zero real number. So, the product manifold $N \times S^1$ has a natural locally conformal calibrated G_2–structure defined by

$$\varphi = \omega \wedge dt + \Omega_+,$$

with $\tau_0 \equiv 0, \tau_3 \equiv 0$ and $\tau_1 = -\frac{c}{3}dt$.

Example 4.14. [9, Example 3.3] Consider the 6–dimensional Lie algebra $\mathfrak{n}_{28}$, and let $\{e_1, ..., e_6\}$ be a SU(3)–basis. With respect to the dual basis

$\{e^1, ..., e^6\}$, the structure equations of $\mathfrak{n}_{28}$ are

$$(4.9) \qquad (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}),$$

and we denote its components by $de^i := 0$, for $i = 1, \ldots, 4$, $de^5 := e^{13} - e^{24}$ and $de^6 := e^{14} + e^{23}$. The pair

$$(4.10) \qquad \omega = e^{12} + e^{34} - e^{56} \quad \text{and} \quad \Omega_+ = e^{136} - e^{145} - e^{235} - e^{246}$$

defines a coupled $SU(3)$–structure on $\mathfrak{n}_{28}$ with $d\omega = -\Omega_+$. Denote by G the 3-dimensional complex Heisenberg group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{n}_{28}$ given by

$$G = \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix}; \quad z_1, z_2, z_3 \in \mathbb{C} \right\}.$$

The structure equations (4.9) can be rewritten as

$$dz_1 = e^1 + ie^2, \quad dz_2 = e^3 + ie^4 \quad dz_3 + z_1 dz_2 = e^5 + ie^6.$$

By [18, Theorem 7], G admits a uniform discrete subgroup $\Gamma \subset G$, i.e., a discrete subgroup such that $\Gamma\backslash G$ is compact, the elements of which have $z_1, z_2, z_3 \in \mathbb{Z}[i]$. The left-invariant forms ω and Ω_+ on G are well defined in the quotient $\Gamma\backslash G$. Consider the automorphism $\nu : G \to G$ defined by

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\nu} \begin{pmatrix} 1 & iz_1 & z_3 \\ 0 & 1 & -iz_2 \\ 0 & 0 & 1 \end{pmatrix},$$

and denote by $\mathrm{Diff}_\nu := \langle (p, t) \mapsto (\nu(p), t + 1) \rangle$ the infinite cyclic subgroup of diffeomorphisms of $(\Gamma\backslash G) \times \mathbb{R}$. The manifold

$$S = \Big((\Gamma\backslash G) \times \mathbb{R} \Big) / \mathrm{Diff}_\nu$$

is endowed with a locally conformal calibrated G_2–structure as follows: for the left-invariant coframe given in (4.9), we have

$$\nu^*(e_1) = -e_2, \; \nu^*(e_2) = e_1, \; \nu^*(e_3) = e_4,$$
$$\nu^*(e_4) = -e_3, \; \nu^*(e_5) = e_5, \; \nu^*(e_6) = e_6.$$

Hence $\nu^*\omega = \omega$ and $\nu^*\Omega_+ = \Omega_+$, for (ω, Ω_+) defined in (4.10). Denoting by $p_1 : (\Gamma\backslash G) \times \mathbb{R} \to \Gamma\backslash G$ the projection onto the first factor, the forms

$p_1^*\omega \in \Omega^2((\Gamma\backslash G)\times\mathbb{R})$ and $p_1^*\Omega_+ \in \Omega^3((\Gamma\backslash G)\times\mathbb{R})$ are invariant under $\sim_\nu$. Therefore, we have differential forms $\widetilde{\omega} \in \Omega^2(S)$ and $\widetilde{\Omega}_+ \in \Omega^3(S)$ satisfying the same relations as (ω, Ω_+) from (4.10). In this setup, the 3-form

$$\widetilde{\varphi} = \widetilde{\omega} \wedge e^7 + \widetilde{\Omega}_+ \tag{4.11}$$

defines a locally conformal calibrated G_2-structure on S. Here e^7 denotes the pullback of the canonical closed 1-form on $\mathbb{R}$ by the projection $p_2 : (\Gamma\backslash G)\times\mathbb{R} \to \mathbb{R}$. The torsion forms of $\widetilde{\varphi}$ are

$$\tau_1 = \frac{1}{3}e^7, \quad \tau_2 = \widetilde{\alpha} \quad \text{where} \quad \alpha = -\frac{4}{3}\left(e^{12} + e^{34} + 2e^{56}\right)$$

and, by Proposition 2.13, the full torsion tensor is

$$T = \widetilde{\beta}, \quad \text{with} \quad \beta = e^{12} + e^{34} + e^{56}.$$

The 7-manifold from Example 4.14 contains an associative submanifold, corresponding to a particular Lie subalgebra:

Example 4.15. Consider the Abelian subalgebra $\mathfrak{n}_{28}' = \mathrm{Span}(e_5, e_6) \subset \mathfrak{n}_{28}$ and its respective Lie group $G' = [G, G] = \exp(\mathfrak{n}_{28}') \subset G$, which is generated by the commutator $[g, h] = ghg^{-1}h^{-1}$. Since G' is obtained as the maximal integral submanifold of G given by the left-invariant distribution

$$\Delta(g) = (dL_g)_1\mathfrak{n}_{28} \quad \text{for} \quad g \in G,$$

i.e. $(L_h)_*(\Delta(g)) \subset \Delta(hg)$ (c.f. [24, Theorem 6.5]), we get an integral distribution $\overline{\Delta}$ on $\Gamma\backslash G$. Representing G' by

$$G' = \left\{ \begin{pmatrix} 1 & 0 & z_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad z_3 \in \mathbb{C} \right\},$$

we see that, for each $p = \Gamma g' \in \Gamma\backslash G'$, we have $T_p(\Gamma\backslash G') = \overline{\Delta}(\Gamma g')$, and so $\Gamma\backslash G'$ is a compact embedded submanifold of $\Gamma\backslash G$. Now $\nu|_{G'} = Id$ and the quotient map $(\Gamma\backslash G)\times\mathbb{R} \to S$ is a local diffeomorphism, so

$$Y = \Big((\Gamma\backslash G') \times \mathbb{R}\Big)/\mathrm{Diff}_\nu \cong (\Gamma\backslash G') \times S^1$$

is a compact embedded submanifold of S. Moreover,

$$T_{(p,t)}Y = T_p(\Gamma\backslash G') \oplus T_t\mathbb{R} \cong \mathfrak{n}_{28}' \oplus \mathbb{R},$$

and indeed $\widetilde{\varphi}|_{T_pY} \equiv \mathrm{vol}(e_5, e_6, e_7)$. Hence, Y is a closed associative submanifold of S.

Now, we assess formula (3.1) of Section 3.2 for Example 4.15. The first correction term is

$$\begin{aligned} P_1(\sigma) &= -T_{56} e_5 \times \nabla_6^{\perp}\sigma - T_{65} e_6 \times \nabla_5^{\perp}\sigma - 2T_{56}\nabla_7^{\perp}\sigma \\ &= -(e_7 \times e_6) \times \nabla_6^{\perp}\sigma - (e_7 \times e_5) \times \nabla_5^{\perp}\sigma - 2\nabla_7^{\perp}\sigma \\ &= e_7 \times \not{D}(\sigma) - \nabla_7^{\perp}\sigma. \end{aligned}$$

Here, to obtain the second equality we used the associative relation $e_5 \times e_6 = -e_7$ and for the last one we used the identity $(u \times v) \times w = -u \times (v \times w)$, for mutually orthonormal u, v, w. To calculate P_2, we need the covariant derivative of the total torsion tensor T

$$\nabla_i T_{kl} = e_i(T_{kl}) - \Gamma_{ik}^m T_{ml} - \Gamma_{il}^m T_{km} = -\Gamma_{ik}^m T_{ml} - \Gamma_{il}^m T_{km}. \tag{4.12}$$

Since S is locally isometric to $G \times \mathbb{R}$, the Christoffel symbols of the G_2-metric on S are defined by the structure constants of the Lie algebra $\mathfrak{n}_{28}$ (cf. [20]):

$$\Gamma_{ij}^k = \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) \quad \text{with} \quad \alpha_{ijk} = \langle [e_i, e_j], e_k \rangle.$$

Applying this to Example 4.14, we find

$$\begin{aligned} \Gamma_{13}^5 = \Gamma_{23}^6 = \Gamma_{36}^2 = \Gamma_{42}^5 = \Gamma_{63}^2 = \Gamma_{52}^4 &= -\frac{1}{2} \\ \Gamma_{14}^6 = \Gamma_{25}^4 = \Gamma_{35}^1 = \Gamma_{46}^1 = \Gamma_{64}^1 = \Gamma_{53}^1 &= -\frac{1}{2} \\ \Gamma_{16}^4 = \Gamma_{24}^5 = \Gamma_{31}^5 = \Gamma_{41}^6 = \Gamma_{61}^4 = \Gamma_{51}^3 &= +\frac{1}{2} \\ \Gamma_{15}^3 = \Gamma_{26}^3 = \Gamma_{32}^6 = \Gamma_{45}^2 = \Gamma_{62}^3 = \Gamma_{54}^2 &= +\frac{1}{2} \end{aligned}$$

$$\Gamma_{ij}^k = 0, \ \text{otherwise}.$$

Using the cross product defined by (4.11) and the above Christoffel symbols, we have:

$$\nabla_l e_{i+5} = \nabla_{i+5} e_l = \frac{(-1)^i}{2} e_{6-i} \times e_l \quad \text{for} \quad i = 0, 1 \quad \text{and} \quad l = 1, 2, 3, 4. \tag{4.13}$$

Notice that the full torsion tensor of the G_2–structure (4.11) can be written as

$$T(u,v) = -\langle e_7 \times u^\top, v^\top\rangle + \langle e_7 \times u^\perp, v^\perp\rangle \quad \text{for} \quad u,v \in \Omega^0(TS|_Y) \tag{4.14}$$

where $u^\top$ and $u^\perp$ are the tangent and normal components of u, respectively. Combining these facts with Lemma 2.16 (i), we have

(4.15)

$$\begin{aligned}\nabla_u(v\times w) &= \nabla_u v \times w + v \times \nabla_u w + \sum_{i=1}^{7} T(u,e_m)\chi(e_m,v,w)\\ &= \nabla_u v \times w + v \times \nabla_u w - \chi(e_7\times u^\top, v, w) + \chi(e_7 \times u^\perp, v, w).\end{aligned}$$

Now, for P_2 we obtain:

$$\begin{aligned}P_2(\sigma) &= \sum_{i=5}^{7}\sum_{k=1}^{4} e_i(T(\sigma,e_k))e_i \times e_k\\ &= \sum_{i=5}^{7}\sum_{k=1}^{4} e_i \times \Big(\nabla_i^\perp(T(\sigma,e_k)e_k) - T(\sigma,e_k)\nabla_i^\perp e_k\Big)\\ &= \sum_{i=5}^{7} e_i \times \nabla_i^\perp(e_7\times\sigma) - \sum_{i=0,1}\sum_{k=1}^{4}\langle e_7\times\sigma, e_k\rangle \frac{(-1)^i}{2} e_{i+5}\times(e_{6-i}\times e_k)\\ &= \sum_{i=5}^{7} e_i \times (e_7 \times \nabla_i^\perp \sigma) - e_i \times \chi(e_7\times e_i, e_7, \sigma)\\ &\quad - \sum_{i=0,1}\frac{(-1)^i}{2} e_{i+5}\times(e_{6-i}\times(e_7\times\sigma))\\ &= -2\nabla_7^\perp\sigma + \sum_{i=5}^{7} -e_7\times(e_i\times\nabla_i^\perp\sigma)\\ &\quad \underbrace{- \sum_{i=0,1} e_{i+5}\times\chi(e_7\times e_{i+5}, e_7,\sigma) + \frac{(-1)^i}{2}(e_{i+5}\times e_{6-i})\times(e_7\times\sigma)}_{(\star)}\\ &= -e_7 \times \not{D}(\sigma) - 2\nabla_7^\perp\sigma - 3\sigma\end{aligned}$$

For the third equality, we used (4.14) in the first term and (4.13) in the second one. The fourth equality follows from (4.15) and, finally, a short calculation gives:

$$
\begin{aligned}
(\star) &= \sum_{i=0,1} -e_{i+5} \times ((e_7 \times e_{i+5}) \times (e_7 \times \sigma)) + \frac{(-1)^i}{2}(e_{i+5} \times e_{6-i}) \times (e_7 \times \sigma) \\
&= \sum_{i=0,1} -((e_{i+5} \times e_7) \times e_{i+5}) \times (e_7 \times \sigma) + \frac{(-1)^i}{2}(e_{i+5} \times e_{6-i}) \times (e_7 \times \sigma) \\
&= -((e_5 \times e_7) \times e_5) \times (e_7 \times \sigma) + \frac{1}{2}(e_5 \times e_6) \times (e_7 \times \sigma) \\
&\quad - ((e_6 \times e_7) \times e_6) \times (e_7 \times \sigma) - \frac{1}{2}(e_6 \times e_5) \times (e_7 \times \sigma) \\
&= \sigma + \frac{1}{2}\sigma + \sigma + \frac{1}{2}\sigma = 3\sigma.
\end{aligned}
$$

Finally, for P_3, we have

$$
\begin{aligned}
P_3(\sigma) &= \sum_{k,l=1}^{4} \Big(T(\sigma, e_k) + \sum_{i=5}^{7} \widetilde{\varphi}(e_i, \nabla_i^{\perp}\sigma, e_k)\Big) T_{kl} e_l \\
&= \sum_{k=1}^{4} \Big(\langle e_7 \times \sigma, e_k\rangle + \sum_{i=5}^{7} \langle e_i \times \nabla_i^{\perp}\sigma, e_k\rangle\Big) e_7 \times e_k \\
&= e_7 \times (e_7 \times \sigma) + e_7 \times \not{D}(\sigma) = -\sigma + e_7 \times \not{D}(\sigma)
\end{aligned}
$$

Now, writing the curvature tensor as

$$
R(e_i, e_j)e_k = \sum_{l,m=1}^{7} \Big(\Gamma_{jk}^{l}\Gamma_{il}^{m} - \Gamma_{ik}^{l}\Gamma_{jl}^{m} - (\Gamma_{ij}^{l} - \Gamma_{ji}^{l})\Gamma_{lk}^{m}\Big) e_m
$$

and using the last expression, we have

$$
\begin{aligned}
R(e_5, \sigma)e_5 &= \sum_{l,m=1}^{7} \sum_{j=1}^{4} \sigma^j \Big(\Gamma_{j5}^{l}\Gamma_{5l}^{m} - \Gamma_{55}^{l}\Gamma_{jl}^{m} - (\Gamma_{5j}^{l} - \Gamma_{j5}^{l})\Gamma_{l5}^{m}\Big) e_m \\
&= \sum_{l,m=1}^{7} \sum_{j=1}^{4} \sigma^j \Big(\Gamma_{j5}^{l}\Gamma_{5l}^{m}\Big) e_m \\
&= \sigma^1\Gamma_{15}^{3}\Gamma_{53}^{1}e_1 + \sigma^2\Gamma_{25}^{4}\Gamma_{54}^{2}e_2 + \sigma^3\Gamma_{35}^{1}\Gamma_{51}^{3}e_3 + \sigma^4\Gamma_{45}^{2}\Gamma_{52}^{4}e_4 = -\frac{\sigma}{4},
\end{aligned}
$$

$$\begin{aligned}
R(e_6,\sigma)e_6 &= \sum_{l,m=1}^{7}\sum_{j=1}^{4}\sigma^j\Big(\Gamma^l_{j6}\Gamma^m_{6l}-\Gamma^l_{66}\Gamma^m_{jl}-(\Gamma^l_{6j}-\Gamma^l_{j6})\Gamma^m_{l6}\Big)e_m\\
&= \sum_{l,m=1}^{7}\sum_{j=1}^{4}\sigma^j\Big(\Gamma^l_{j6}\Gamma^m_{6l}\Big)e_m\\
&= \sigma^1\Gamma^4_{16}\Gamma^1_{64}e_1+\sigma^2\Gamma^3_{26}\Gamma^2_{63}e_2+\sigma^3\Gamma^2_{36}\Gamma^3_{62}e_3+\sigma^4\Gamma^1_{46}\Gamma^4_{61}e_4=-\frac{\sigma}{4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{R}(\sigma) &= \Big(R(e_5,\sigma)e_5+R(e_6,\sigma)e_6+R(e_7,\sigma)e_7\Big)^{\perp}=-\frac{1}{4}\sigma-\frac{1}{4}\sigma+0\\
&= -\frac{1}{2}\sigma.
\end{aligned}$$

Now, we assess the operator $\mathcal{T}$ defined in equation (2.15) for a pair $e_i, e_j \in \Omega^0(TY)$ and $\sigma \in \Omega^0(NY)$:

$$\begin{aligned}
\mathcal{T}(e_j,\sigma,e_i,e_j) = \sum_{m=1}^{7}&\underbrace{T(\sigma,e_m)\nabla_j\psi(e_m,e_i,e_j,\cdot)^\sharp}_{\text{(I)}}-\underbrace{T(e_j,e_m)\nabla_\sigma\psi(e_m,e_i,e_j,\cdot)^\sharp}_{\text{(II)}}\\
&+\underbrace{\Big(\nabla_jT(\sigma,e_m)-\nabla_\sigma T(e_j,e_m)\Big)\chi(e_m,e_i,e_j)}_{\text{(III)}}.
\end{aligned}$$

We will use throughout the proof both the expression of $\nabla\psi$ in terms of T and φ from Corollary 2.15 and the expression for T given in (4.14). For the first term,

$$\begin{aligned}
\text{(I)} &= \sum_{m=1}^{7}\langle e_7\times\sigma,e_m\rangle\nabla_j\psi(e_m,e_i,e_j,\cdot)^\sharp=\nabla_j\psi(e_7\times\sigma,e_i,e_j,\cdot)^\sharp\\
&= -T(e_j,e_7\times\sigma)\varphi(e_i,e_j,\cdot)^\sharp+T(e_j,e_i)\varphi(e_7\times\sigma,e_j,\cdot)^\sharp\\
&\quad -T(e_j,e_j)\varphi(e_7\times\sigma,e_i,\cdot)^\sharp+T(e_j,\cdot)^\sharp\varphi(e_7\times\sigma,e_i,e_j)\\
&= -\langle e_7\times e_j,e_i\rangle(e_7\times\sigma)\times e_j=\langle e_7\times e_j,e_i\rangle(e_7\times e_j)\times\sigma.
\end{aligned}$$

Here we used the vanishings $T(e_j, e_7 \times \sigma) = 0$, again by (4.14), $T(e_j, e_j) = 0$, by skew-symmetry, and $\varphi(e_7 \times \sigma, e_i, e_j) = \langle e_i \times e_j, e_7 \times \sigma\rangle = 0$, by orthogonality.

For the second term,

$$\begin{aligned}
\text{(II)} =& \sum_{m=1}^{7} \langle e_7 \times e_j, e_m \rangle \nabla_\sigma \psi(e_m, e_i, e_j, \cdot)^\sharp = \nabla_\sigma \psi(e_7 \times e_j, e_i, e_j, \cdot)^\sharp \\
=& - T(\sigma, e_7 \times e_j)\varphi(e_i, e_j, \cdot)^\sharp + T(\sigma, e_i)\varphi(e_7 \times e_j, e_j, \cdot)^\sharp \\
& - T(\sigma, e_j)\varphi(e_7 \times e_j, e_i, \cdot)^\sharp + T(\sigma, \cdot)^\sharp \varphi(e_7 \times e_j, e_i, e_j) \\
=& - \langle e_7 \times \sigma, \cdot \rangle^\sharp \langle (e_7 \times e_j) \times e_i, e_j \rangle = -\langle (e_7 \times e_j) \times e_i, e_j \rangle e_7 \times \sigma.
\end{aligned}$$

Again the vanishings $T(\sigma, e_7 \times e_j) = T(\sigma, e_i) = T(\sigma, e_j) = 0$ follow from (4.14).

For the third term, we use the derivatives (4.12) of the torsion tensor:

$$\begin{aligned}
\text{(III)} &= - \sum_{m=1}^{7} \Big(T(\sigma, \nabla_j e_m) - T(e_j, \nabla_\sigma e_m) \Big) \chi(e_m, e_i, e_j) \\
&= - \sum_{m=1}^{7} \Big(\langle e_7 \times \sigma, \nabla_j e_m \rangle + \langle e_7 \times e_j, \nabla_\sigma e_m \rangle \Big) \chi(e_m, e_i, e_j).
\end{aligned}$$

We now apply (I), (II) and (III) for $i = 5$ and $j = 6$:

$$\begin{aligned}
\mathcal{T}(e_6, \sigma, e_5, e_6) =& \langle e_7 \times e_6, e_5 \rangle (e_7 \times e_6) \times \sigma + \langle (e_7 \times e_6) \times e_5, e_6 \rangle e_7 \times \sigma \\
& - \sum_{m=1}^{7} \Big(\langle e_7 \times \sigma, \nabla_6 e_m \rangle + \langle e_7 \times e_6, \nabla_\sigma e_m \rangle \Big) \chi(e_m, e_5, e_6) \\
=& \, e_5 \times \sigma - \sum_{m=1}^{7} \left(-\frac{1}{2} \langle e_7 \times \sigma, e_5 \times e_m \rangle + \langle e_5, \nabla_\sigma e_m \rangle \right) \chi(e_m, e_5, e_6) \\
=& \, e_5 \times \sigma - \sum_{m=1}^{7} \Big(\frac{1}{2} \langle e_5 \times (e_7 \times \sigma), e_m \rangle + \sigma \langle e_5, e_m \rangle \\
& - \langle \nabla_\sigma e_5, e_m \rangle \Big) \chi(e_m, e_5, e_6) \\
=& \, e_5 \times \sigma - \sum_{m=1}^{7} \Big(-\frac{1}{2} \langle e_6 \times \sigma, e_m \rangle - \frac{1}{2} \langle e_6 \times \sigma, e_m \rangle \Big) \chi(e_m, e_5, e_6) \\
=& \, e_5 \times \sigma + \chi(e_6 \times \sigma, e_5, e_6) = e_5 \times \sigma - (e_6 \times \sigma) \times (e_5 \times e_6) \\
=& \, e_5 \times \sigma + (e_6 \times \sigma) \times e_7 \\
=& \, 2 e_5 \times \sigma.
\end{aligned}$$

Here we used repeatedly that $e_5 \times e_6 = -e_7$ and $e_i \times (e_j \times \sigma) = -e_j \times (e_i \times \sigma)$ for $i \neq j$. At the second and fourth lines we applied again (4.13), and at the third line we used the compatibility of the Riemannian connection.

For $j=7$ and $i=6$, we have trivially $\mathcal{T}(e_7,\sigma,e_6,e_7)=0$. Finally, for $j=5$ and $i=7$, we have

$$\begin{aligned}
\mathcal{T}(e_5,\sigma,e_7,e_5) &= \langle e_7\times e_5, e_7\rangle (e_7\times e_5)\times\sigma + \langle (e_7\times e_5)\times e_7, e_5\rangle e_7\times\sigma \\
&\quad - \sum_{m=1}^{7}\left(\langle e_7\times\sigma, \nabla_5 e_m\rangle + \langle e_7\times e_5, \nabla_\sigma e_m\rangle\right)\chi(e_m,e_7,e_5) \\
&= \langle e_6, e_7\rangle e_6\times\sigma - \langle e_6\times e_7, e_5\rangle e_7\times\sigma \\
&\quad - \sum_{m=1}^{7}\Big(\frac{1}{2}\langle e_7\times\sigma, e_6\times e_m\rangle - \langle e_6, \nabla_\sigma e_m\rangle\Big)\chi(e_m,e_7,e_5) \\
&= e_7\times\sigma - \sum_{m=1}^{7}\Big(-\frac{1}{2}\langle e_6\times(e_7\times\sigma), e_m\rangle - \sigma\langle e_6, e_m\rangle \\
&\quad + \langle \nabla_\sigma e_6, e_m\rangle\Big)\chi(e_m,e_7,e_5) \\
&= e_7\times\sigma + \frac{1}{2}\sum_{m=1}^{7}\left(\langle e_5\times\sigma, e_m\rangle + \langle e_5\times\sigma, e_m\rangle\right)\chi(e_m,e_7,e_5) \\
&= e_7\times\sigma + \chi(e_5\times\sigma, e_7, e_5) = e_7\times\sigma - (e_5\times\sigma)\times(e_7\times e_5) \\
&= e_7\times\sigma + (e_5\times\sigma)\times e_6 = 2e_7\times\sigma.
\end{aligned}$$

Therefore,

$$\Big(\sum_{i\in\mathbb{Z}_3} e_{i+5}\times\mathcal{T}(e_{i+6},\sigma,e_{i+5},e_{i+6})\Big)^{\perp} \quad = \quad -4\sigma.$$

Following the notation of [7, §5.3], we define an operator

$$\not{D}^c(\sigma) := e_5\times\nabla_5^\perp\sigma + e_6\times\nabla_6^\perp\sigma,$$

and recall that the cross-product by e_7 defines an almost complex structure on $T(\Gamma\backslash G)$ denoted by $J(\sigma):=e_7\times\sigma$. Then (3.2) becomes

$$\not{D}_{\mathrm{A}}(\sigma) = \not{D}^c(\sigma) + J(\dot\sigma) + J(\sigma),$$

where $\dot\sigma := \nabla_7^\perp\sigma$. To simplify notation, let $\|\cdot\|$ and $\langle\langle\cdot,\cdot\rangle\rangle$ denote the L^2-norm and inner product of sections, respectively (the integral of the corresponding pointwise quantity over the associative submanifold). The next Lemma gathers some relations between the operators $\not{D}$, J and ∇; although some of them will not be used in this article, we state them anyway as a curiosity.

Lemma 4.16. *With the above notation, we have the following properties:*

(i) $\not{D}^c \circ J(\sigma) = -J \circ \not{D}^c(\sigma) + 2\sigma$.

(ii) $\langle\langle \not{D}^c(\sigma), \eta\rangle\rangle = \langle\langle \sigma, \not{D}^c(\eta)\rangle\rangle + 2\langle\langle \sigma, J(\eta)\rangle\rangle$.

(iii) $\langle\langle \not{D}^c(\sigma), J(\dot{\sigma})\rangle\rangle = 0$.

(iv) $\langle\langle \dot{\sigma}, \sigma\rangle\rangle = 0$ *and* $\langle\langle \not{D}^c(\sigma), J(\sigma)\rangle\rangle \leq 0$.

Proof. (i) Using Lemma 2.16 (i), we have,

$$\begin{aligned}\not{D}^c \circ J(\sigma) &= -J \circ \not{D}^c(\sigma) - T_{65} e_6 \times (e_5 \times (e_7 \times \sigma)) \\ &\quad - T_{56} e_5 \times (e_6 \times (e_7 \times \sigma)) \\ &= -J \circ \not{D}^c(\sigma) + 2T_{56}(e_5 \times e_6) \times (e_7 \times \sigma) \\ &= -J \circ \not{D}^c(\sigma) + 2 \cdot \sigma.\end{aligned}$$

(ii) Using the Leibniz rule (4.15) and the following trivial calculation,

$$\begin{aligned}\chi(e_7 \times e_i, e_i, \eta) =& \chi(\eta, e_7 \times e_i, e_i) = -\eta \times ((e_7 \times e_i) \times e_i) \\ =& -\eta \times (e_i \times (e_i \times e_7)) = -e_7 \times \eta,\end{aligned}$$

we have:

$$\begin{aligned}\langle \not{D}^c(\sigma), \eta\rangle_p &= -\sum_{i=5}^{6} \langle \nabla_i^\perp \sigma, e_i \times \eta\rangle_p \\ &= -\sum_{i=5}^{6} \{e_i \langle \sigma, e_i \times \eta\rangle - \langle \sigma, \nabla_i^\perp (e_i \times \eta)\rangle\}_p \\ &= \operatorname{div}(\sigma \times \eta)_p - \sum_{i=5}^{6} \langle \sigma, e_i \times \nabla_i^\perp \eta - \chi(e_7 \times e_i, e_i, \eta)\rangle\}_p \\ &= \operatorname{div}(\sigma \times \eta)_p + \langle \sigma, \not{D}^c(\eta)\rangle_p + 2\langle \sigma, e_7 \times \eta\rangle_p.\end{aligned}$$

(iii) Using (i) and (ii), one has $\langle\langle \not{D}^c(\sigma), J(\dot{\sigma})\rangle\rangle = \langle\langle J(\sigma), \not{D}^c(\dot{\sigma})\rangle\rangle$, and, by the vanishing of the normal curvature tensor $R^\perp(e_i, e_7)\sigma = 0$ for $i = 5, 6$, we have $\nabla_i^\perp \nabla_7^\perp \sigma = \nabla_7^\perp \nabla_i^\perp \sigma$. From Lemma 2.16 (i) and the compatibility of $\nabla^\perp$ with the induced metric in NY, we have:

$$\langle \not{D}^c(\sigma), J(\dot\sigma)\rangle_p = \sum_{i=5}^{7}\langle J(\sigma), e_i \times \nabla^{\perp}_7\nabla^{\perp}_i\sigma\rangle_p = \sum_{i=5}^{7}\langle J(\sigma), \nabla^{\perp}_7(e_i \times \nabla^{\perp}_i\sigma)\rangle_p$$
$$= -\langle \nabla^{\perp}_7(J(\sigma)), \not{D}^c(\sigma)\rangle_p + e_7\langle J(\sigma), \not{D}^c(\sigma)\rangle_p$$
$$= -\langle J(\dot\sigma), \not{D}^c(\sigma)\rangle_p + \operatorname{div}(\langle J(\sigma), \not{D}^c(\sigma)\rangle e_7)_p.$$

(iv) Again by compatibility of $\nabla^{\perp}$ with the metric on NY, we have $2\langle\dot\sigma, \sigma\rangle = 2\langle\nabla^{\perp}_7\,\sigma, \sigma\rangle = e_7|\sigma|^2$. Now Stokes' Theorem gives

$$\langle\langle\dot\sigma, \sigma\rangle\rangle = \frac{1}{2}\int_Y e_7|\sigma|^2 d\,\mathrm{vol}_Y = \frac{1}{2}\int_Y \operatorname{div}(|\sigma|^2 e_7) d\,\mathrm{vol}_Y = 0. \tag{4.16}$$

Computing the L^2-norm for $\not{D}_{\mathrm{A}}(\sigma)$, we have

$$\left\|\not{D}_{\mathrm{A}}(\sigma)\right\|^2 = \left\|\not{D}^c(\sigma)\right\|^2 + \|\dot\sigma\|^2 + \|\sigma\|^2$$
$$+ 2\langle\langle\not{D}^c(\sigma), J(\dot\sigma)\rangle\rangle + 2\langle\langle\not{D}^c(\sigma), J(\sigma)\rangle\rangle + 2\langle\langle\dot\sigma, \sigma\rangle\rangle,$$

and from Lemma 4.16(iii) and equation (4.16) it follows that

$$\left\|\not{D}_{\mathrm{A}}(\sigma)\right\|^2 = \left\|\not{D}^c(\sigma)\right\|^2 + \|\dot\sigma\|^2 + \|\sigma\|^2 + 2\langle\langle\not{D}^c(\sigma), J(\sigma)\rangle\rangle.$$

Therefore, by the triangle inequality, $\langle\langle\not{D}^c(\sigma), J(\sigma)\rangle\rangle \leq 0$.

□

Corollary 4.17. *The submanifold Y of Example 4.15 is rigid.*

Proof. Notice that the operator $\mathcal{B}$ vanishes on Y, as can be seen from

$$\begin{aligned}
\mathcal{B}(\sigma) &= \sum_{i,j=5}^{7}(e_i \times e_j) \times B(e_j, S_\sigma(e_i)) \\
&= \sum_{k=1}^{4}\sum_{i,j=5}^{7}\langle S_{e_k}(e_j), S_\sigma(e_i)\rangle(e_i \times e_j) \times e_k \\
&= -\sum_{k=1}^{4}\sum_{i,j,l=5}^{7}\Gamma^l_{jk}\langle e_l, S_\sigma(e_i)\rangle(e_i \times e_j) \times e_k = 0,
\end{aligned}$$

since, $\Gamma^l_{jk} = 0$ for $j, l = 5, 6, 7$ and $k = 1, ..., 4$. Applying equation (3.9), Lemma 4.3 and the previous calculation, we obtain the Weitzenböck formula

$$\not{D}_{\mathrm{A}}{}^2(\sigma) = \nabla^*\nabla\sigma + e_7 \times \not{D}(\sigma) - 3\nabla^{\perp}_7\sigma - \frac{1}{2}\sigma.$$

Taking the inner product with σ and integrating over Y,

$$\begin{aligned}\int_Y \langle {\not{D}_{\mathrm{A}}}^2(\sigma),\sigma\rangle d\,\mathrm{vol}_Y &= \int_Y \langle \nabla^*\nabla\sigma,\sigma\rangle d\,\mathrm{vol}_Y + \int_Y \langle e_7\times \not{D}(\sigma),\sigma\rangle d\,\mathrm{vol}_Y \\ &\quad - \int_Y 3\langle \nabla_7^{\perp}\sigma,\sigma\rangle d\,\mathrm{vol}_Y - \int_Y \frac{1}{2}\langle\sigma,\sigma\rangle d\,\mathrm{vol}_Y \\ &\geq \int_Y \langle e_7\times \not{D}(\sigma),\sigma\rangle d\,\mathrm{vol}_Y - 3\int_Y \langle\dot{\sigma},\sigma\rangle d\,\mathrm{vol}_Y \\ &\quad - \int_Y \frac{1}{2}\langle\sigma,\sigma\rangle d\,\mathrm{vol}_Y\,.\end{aligned}$$

From Lemma 4.16 (iv), we conclude that

$$\int_Y \langle {\not{D}_{\mathrm{A}}}^2(\sigma),\sigma\rangle d\,\mathrm{vol}_Y \geq \int_Y \langle e_7\times \not{D}(\sigma),\sigma\rangle d\,\mathrm{vol}_Y - \frac{1}{2}\int_Y \langle\sigma,\sigma\rangle d\,\mathrm{vol}_Y\,. \tag{4.17}$$

So, for $\sigma\in\ker \not{D}_{\mathrm{A}}$, we have $\not{D}(\sigma) = -e_7\times\sigma$ and, replacing that in (4.17), we get the inequality

$$0 \geq -\int_Y \langle e_7\times(e_7\times\sigma),\sigma\rangle d\,\mathrm{vol}_Y - \frac{1}{2}\int_Y \langle\sigma,\sigma\rangle d\,\mathrm{vol}_Y = \frac{1}{2}\int_Y \langle\sigma,\sigma\rangle d\,\mathrm{vol}_Y\,.$$

Then $\sigma = 0$ and therefore Y is rigid. □

Afterword

In many cases a Weitzenböck formula is a useful tool to rule out parallel spinors, but in full generality equation (3.9) has the drawback of first order terms with unpredictable spectrum. In the nearly parallel case (4.1), however, the Weitzenböck formula is very similar to the formula for a parameterized connection with skew-torsion symmetric tensor of Agricola and Friedrich [1]. In this context and under favourable assumptions, it is possible to control the spectrum of $\not{D}$. Using the Weitzenböck formula of [11], we have

$${\not{D}_{\mathrm{A}}}^2 = \nabla^*\nabla + \mathcal{R} + \mathcal{B} + \tau_0\,\not{D} + \frac{\tau_0^2}{4}$$

so, when a normal section lies in $\ker \not{D}_{\mathrm{A}}$, it corresponds to the eigenvalue $\frac{\tau_0}{4}$ of $\not{D}$. Therefore, nontrivial deformations for an associative submanifold are in direct correspondence with elements of that eigenspace.

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Imecc, University of Campinas
Campinas, 13083-859, Brazil
E-mail address: amoreno@unicamp.br
E-mail address: henrique.saearp@ime.unicamp.br

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