The classification of conformal dynamical systems

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1 Introduction

Consider the group generated by reflections in a finite collection of disjoint circles in the plane. Under reflection through their partners, the circles nest down to a Cantor set on which every orbit is dense. Poincaré christened

such conformal dynamical systems *Kleinian groups* and noted they extend to isometries of hyperbolic 3-space.
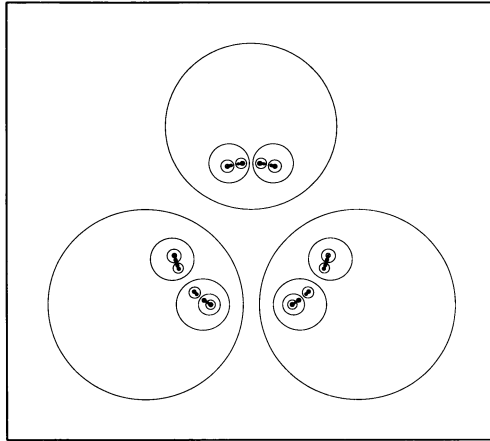

Figure 1. A reflection group.

The dynamics of a single conformal endomorphism of the sphere can exhibit similar structure, and these *iterated rational maps* were studied by Fatou and Julia in the 1920s.

Interest in iterated rational maps was rekindled in the 1980s, when Douady and Hubbard revealed the complexity of the family of quadratic polynomials. Contemporaneously, Thurston's 3-dimensional insights revolutionized the theory of Kleinian groups.

Sullivan then discovered a dictionary between these two classical subjects. Introducing quasiconformal methods to the setting of rational maps, he translated Ahlfors' finiteness theorem into a solution of the long-outstanding problem of wandering domains.

In this paper we illustrate the dictionary by discussing the classification problem for conformal dynamical systems.

We begin with the Teichmüller space of a dynamical system, which classifies deformations and leads to the finiteness theorems of Ahlfors and Sullivan. A more detailed picture is available for expanding dynamical systems, and we examine the conjecture that such systems are dense. Then we present the program for a topological classification of Kleinian groups via 3-manifolds, along with hints of a parallel theory for rational maps. The Hausdorff dimension, ergodic theory and local connectivity of limit sets and Julia sets are also discussed. Finally we describe the role of renormalization in recent progress on the classification of quadratic polynomials.

Notes and references are collected at the end.

2 Finiteness theorems and Teichmüller space

Definitions. A *Kleinian group* Γ is a discrete subgroup of the conformal automorphism group $\text{Aut}(\widehat{\mathbb{C}})$ of the Riemann sphere. We make the essential assumption that Γ is finitely generated. For convenience, we also assume Γ is torsion-free, orientation-preserving, and every abelian subgroup has infinite index. Elements of Γ can be represented concretely as Möbius transformations $\gamma(z) = (az + b)/(cz + d)$.

According to the dynamics of Γ, the sphere is partitioned into the *limit set* Λ and the *domain of discontinuity* Ω. The limit set is the locus of chaotic behavior. It is a compact, perfect set which can be defined as:

- the smallest closed Γ-invariant set with $|\Lambda| > 2$;

- the set of accumulation points of any orbit $\Gamma x \subset \widehat{\mathbb{C}}$;

- the closure of the set of repelling fixed points of $\gamma \in \Gamma$; or

- the set of points near which Γ does not form a normal family.[1]

On the other hand, Γ acts freely and conformally on $\Omega = \widehat{\mathbb{C}} - \Lambda$, preserving the *Poincaré metric* of constant curvature -1 provided by the uniformization theorem. Thus $X = \Omega/\Gamma$ is a disjoint union of hyperbolic Riemann surfaces.

A classical result in Kleinian groups is:

Theorem 2.1 (Ahlfors' finiteness theorem) *The quotient surface X is isomorphic to a finite union of compact Riemann surfaces with a finite number of points removed.*

In particular, every component of Ω has a nontrivial stabilizer; there is no *wandering component*.

Now let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a *rational map*, whose iterates $f^n = f \circ f \circ \ldots \circ f$ are to be studied. We will assume the topological degree d of f is at least two, so $f(z) = P(z)/Q(z)$ for two relatively prime polynomials with $d = \max(\deg P, \deg Q) \geq 2$.

We again obtain a partition of the sphere into the *Julia set J* and the *Fatou set Ω* according to the dynamics of f. The Julia set is a closed, perfect set, defined as

- the smallest closed set with $f^{-1}(J) = J$ and $|J| > 2$;

- the closure of the set of repelling periodic points of f; or

- the set of points near which $\langle f^n \rangle$ does not form a normal family.

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[1] A family of maps $\gamma_\alpha : U \to \widehat{\mathbb{C}}$ is *normal* if every sequence has a subsequence converging uniformly on compact subsets of U.

As f is proper, it maps components of Ω to components. Perceiving the analogy with Ahlfors' theorem, in the early 1980s Sullivan resolved a problem left open by Fatou and Julia in the 1920s by establishing:

Theorem 2.2 (No wandering domains) *Every component of the Fatou set eventually cycles, and there are only finitely many periodic components.*

One then has a fairly complete topological picture of the dynamics of f, since a component Ω_0 of the Fatou set of period p is of exactly one of the following types:

1. *An attractive basin:* there is an $x \in \Omega_0$, fixed by f^p, with

$$0 < |(f^p)'(x)| < 1,$$

   attracting all points of Ω_0 under iteration of f^p. (This is the generic case).

2. *A superattractive basin:* as above, but x is a critical point of f^p, so $(f^p)'(x) = 0$.

3. *A parabolic basin:* there is a point x in $\partial\Omega_0$ with $(f^p)'(x) = 1$, attracting all points of Ω_0.

4. *A Siegel disk:* Ω_0 is conformally isomorphic to the unit disk, and f^p acts by an irrational rotation.

5. *A Herman ring:* Ω_0 is isomorphic to an annulus, and f^p acts by an irrational rotation.

Teichmüller theory. Underlying these finiteness theorems is a description of the Teichmüller space of a conformal dynamical system.

It is a remarkable fact that a *measurable* field of tangent ellipses of bounded eccentricity determines a complex structure on the sphere. This ellipse field is recorded by a $(-1, 1)$-form $\mu(z)d\bar{z}/dz$ with $\|\mu\|_\infty < 1$. Since the sphere admits a unique complex structure, there is a homeomorphism $\phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that μ is the pullback of the standard structure. The conditions on μ imply that ϕ is a *quasiconformal map*: it has derivatives in L^2, and $\bar{\partial}\phi = \mu\partial\phi$.

The *Teichmüller space* Teich(Γ) classifies Kleinian groups Γ' equipped with quasiconformal conjugacies ϕ to Γ. Points in this space are determined by Γ-invariant complex structures μ on $\widehat{\mathbb{C}}$. The discussion is simplified by Sullivan's:

Theorem 2.3 (No invariant line fields) *The limit set of a finitely generated Kleinian group carries no measurable Γ-invariant field of tangent lines.*

Thus any Γ-invariant μ vanishes on the limit set (otherwise the major axes of the corresponding ellipse field would give invariant lines). On the other hand, $\mu|\Omega$ descends to a complex structure on the quotient Riemann surface X, yielding:

Corollary 2.4 *There is an isomorphism between the Teichmüller space of* Γ *and that of its quotient Riemann surface* $X = \Omega/\Gamma$:

$$\mathrm{Teich}(\Gamma) \cong \mathrm{Teich}(X).$$

Given a group G, let

$$\mathcal{V}(G) = \mathrm{Hom}(G, \mathrm{Aut}(\widehat{\mathbb{C}}))/\mathrm{conjugation}$$

denote the *representation variety* of G, and let $AH(G) \subset \mathcal{V}(G)$ be the subspace of discrete, faithful representations.

A conjugacy ϕ from Γ to Γ' determines an isomorphism of groups, so we have a map

$$\delta : \mathrm{Teich}(\Gamma) \to AH(\Gamma) \subset \mathcal{V}(\Gamma).$$

Since Γ is finitely generated, $\mathcal{V}(\Gamma)$ is finite-dimensional; the fibers of δ are discrete, and therefore the Teichmüller space of X is also finite-dimensional. In other words, each component of X has finitely many moduli, so it is isomorphic to a compact Riemann surface with a finite number of points removed.

This completes the proof of Ahlfors' finiteness theorem, except for the fact that infinitely many components of X might be triply-punctured spheres, which have no moduli; this possibility was ruled out by Greenberg.

The rigidity of the triply-punctured sphere is equivalent to the vanishing of any holomorphic quadratic differential with at worst simple poles at the punctures. By considering differentials of higher order, one obtains:

Theorem 2.5 (Bers) *If* Γ *is generated by* N *elements, then the hyperbolic area of* Ω/Γ *is bounded by* $4\pi(N-1)$.

The line fields problem for rational maps. The absence of invariant line fields is equivalent to the assertion that a finitely generated Kleinian group is quasiconformally *rigid* on its limit set. In particular, if $\Lambda = \widehat{\mathbb{C}}$ then Γ has a trivial Teichmüller space — it admits no deformations.

A literal translation of this result is false for rational maps, due to the Lattès examples.

Let $T = \mathbb{C}/L$ be a complex torus, and let $F : T \to T$ be the map $F(x) = nx$ for $n > 1$. The quotient of T by the equivalence relation $x \sim -x$ is the Riemann sphere; the quotient map $\wp : X \to \widehat{\mathbb{C}}$ can be given by the Weierstrass $\wp$-function, which presents T as a twofold cover of the sphere

branched over four points. Since $F(-x) = -F(x)$, the dynamical system F descends to a rational map f such that the diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\ F\ } & T \\
{\scriptstyle \wp}\Big\downarrow & & {\scriptstyle \wp}\Big\downarrow \\
\widehat{\mathbb{C}} & \xrightarrow{\ f\ } & \widehat{\mathbb{C}}
\end{array}
$$

commutes.

Repelling periodic points for F are dense on the torus, so the Julia set of f is the whole sphere. The map $x \mapsto nx$ preserves the family of horizontal lines in the plane, so F has an invariant line field on T (tangent to the foliation by parallel closed geodesics.) This line field descends to the Riemann sphere (with singularities at the four branch points) to give an invariant line field for f.

The existence of this line field reflects the fact that, although $J(f) = \widehat{\mathbb{C}}$, f is not *rigid*: it can be deformed by changing the shape of the torus T.

A fundamental problem is to show these are the only examples:

Conjecture 2.6 (No invariant line fields for rational maps) *The Julia set of a rational map f carries no invariant line field, except if f is a Lattès example.*

We will see later (Corollary 3.7) that this conjecture implies the density of expanding dynamical systems.

The Teichmüller space of f. The space $\operatorname{Teich}(f)$ classifies rational maps g equipped with quasiconformal conjugacies ϕ to f. It maps with discrete fibers to the moduli space

$$
\mathcal{V}_d = \operatorname{Rat}_d / \text{conjugation}
$$

of all rational maps of degree d, so it is finite-dimensional.

To describe it, we form an analogue of the quotient surface for a Kleinian group. Let $\widehat{J} \supset J$ denote the closure of the grand orbits of all critical points and periodic points of f.[2] Let $\widehat{\Omega} = \widehat{\mathbb{C}} - \widehat{J}$; then $f : \widehat{\Omega} \to \widehat{\Omega}$ is a covering map, hence a *local isometry* for the Poincaré metric. Decompose $\widehat{\Omega}$ as $\Omega^{\mathrm{dis}} \sqcup \Omega^{\mathrm{fol}}$ depending on whether or not the grand orbits are locally discrete.

The space $X = \Omega^{\mathrm{dis}}/f$ is a union of Riemann surfaces; each attracting or parabolic basin contributes a torus or cylinder with punctures to X.

The space Ω^{fol} consists of points tending to Siegel disks, Herman rings or superattracting cycles. Each component of Ω^{fol} is a dynamically foliated punctured disk or annulus.

Since we do not know if the Julia set of f carries an invariant line field or not, we need to include a factor $M_1(J, f)$ consisting of all f-invariant μ on J with $\|\mu\| < 1$. Then we have:

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[2] Two points x, y belong to the same *grand orbit* if $f^n(x) = f^m(y)$ for some $n, m > 0$.

Theorem 2.7 *The Teichmüller space of a rational map is canonically the product of the finite-dimensional Teichmüller space of the quotient surface Ω^{dis}/f and a pair of finite polydisks:*

$$\mathrm{Teich}(f) \cong \mathrm{Teich}(\Omega^{\mathrm{dis}}/f) \times \mathrm{Teich}(\Omega^{\mathrm{fol}}, f) \times M_1(J, f).$$

Proof of no wandering domains. We can now complete the proof of Sullivan's finiteness theorem. First, a wandering domain would give rise to an open disk (possibly with finitely many punctures) as a component of Ω^{dis}/f; since the Teichmüller space of a disk is infinite-dimensional, this is a contradiction. Each attracting basin or Herman ring contributes at least one complex modulus to f, so the total number of periodic regions of these two types is bounded. The remaining types of regions are bounded by counting critical points and indifferent cycles.

Using quasiconformal surgery, one has the sharper estimate:

Theorem 2.8 (Shishikura) *Let N be the total number of cycles of periodic components in the Fatou set, H the number of cycles of Herman rings, and I the number of nonparabolic indifferent cycles in the Julia set. Then*

$$N + H + I \leq 2\deg(f) - 2.$$

Examples. Let X be a hyperbolic Riemann surface of finite area. Then X can be presented as the quotient of the unit disk Δ by the action of a *Fuchsian group* $\Gamma \subset \mathrm{Aut}(\Delta)$. By Schwarz reflection Γ acts on the sphere as a Kleinian group, with limit set $\Lambda = S^1$. Letting $\overline{X}$ denote the complex conjugate of X, we have $\Omega/\Gamma = X \sqcup \overline{X}$ and thus $\mathrm{Teich}(\Gamma) = \mathrm{Teich}(X) \times \mathrm{Teich}(\overline{X})$.

As one moves away from the Fuchsian locus while remaining in $\mathrm{Teich}(\Gamma)$, the limit set becomes a fractal curve of dimension strictly greater than one.[3] This curve can be geometrically complicated but it remains a *quasicircle*: that is, it is a Jordan curve such that for any $x, y \in \Lambda$, the smaller subarc $\Lambda(x, y)$ joining x to y satisfies

$$\mathrm{diam}\, \Lambda(x, y) \leq K|x - y|.$$

Finally when one reaches the boundary of $\mathrm{Teich}(\Gamma)$ in $AH(\Gamma)$, some points in Λ can become identified and the topology of the limit set abruptly changes.

The case where X is a once-punctured torus is particularly easy to compute with, because $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$ is a free group; see Figure 2. The simplest type of point on the boundary of Teichmüller space is depicted in the final frame: a simple curve on the torus has been pinched to a *cusp*, splitting Ω into countably many components.

A proper holomorphic map or *Blaschke product* $f : \Delta \to \Delta$ is like a Fuchsian group; by Schwarz reflection again, it extends to a rational map on the sphere. If f has a fixed point in the disk then this point is attracting or superattracting and the Julia set $J = S^1$.

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[3]Poincaré writes "Ces deux domaines sont séparés par une ligne L, si l'on peut appeler cela une ligne."
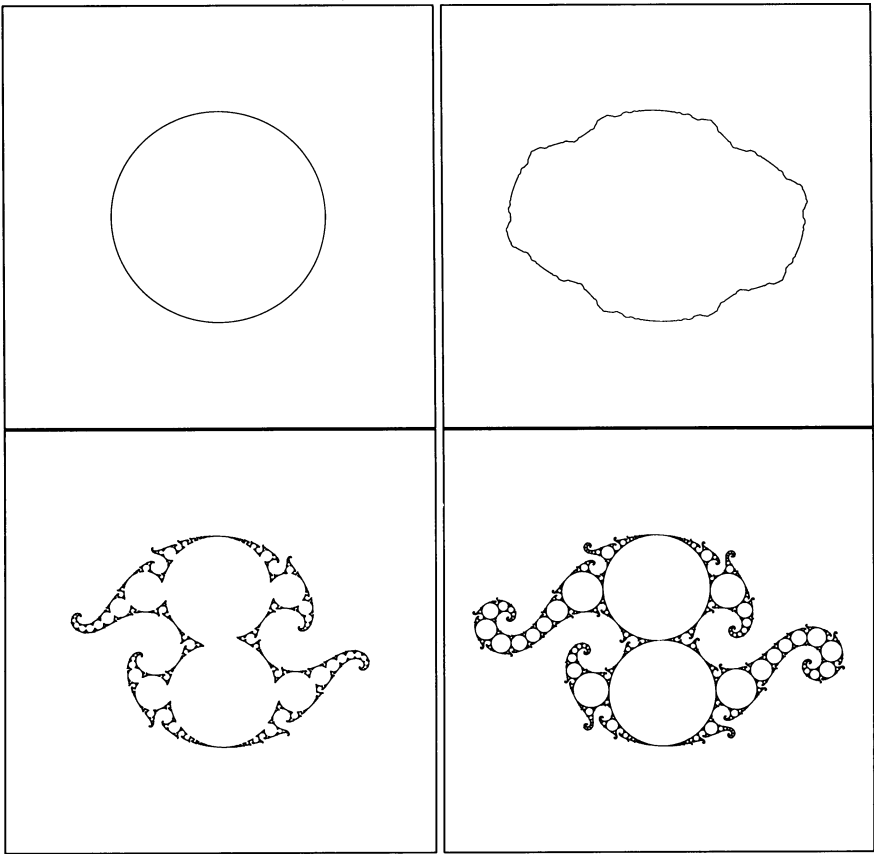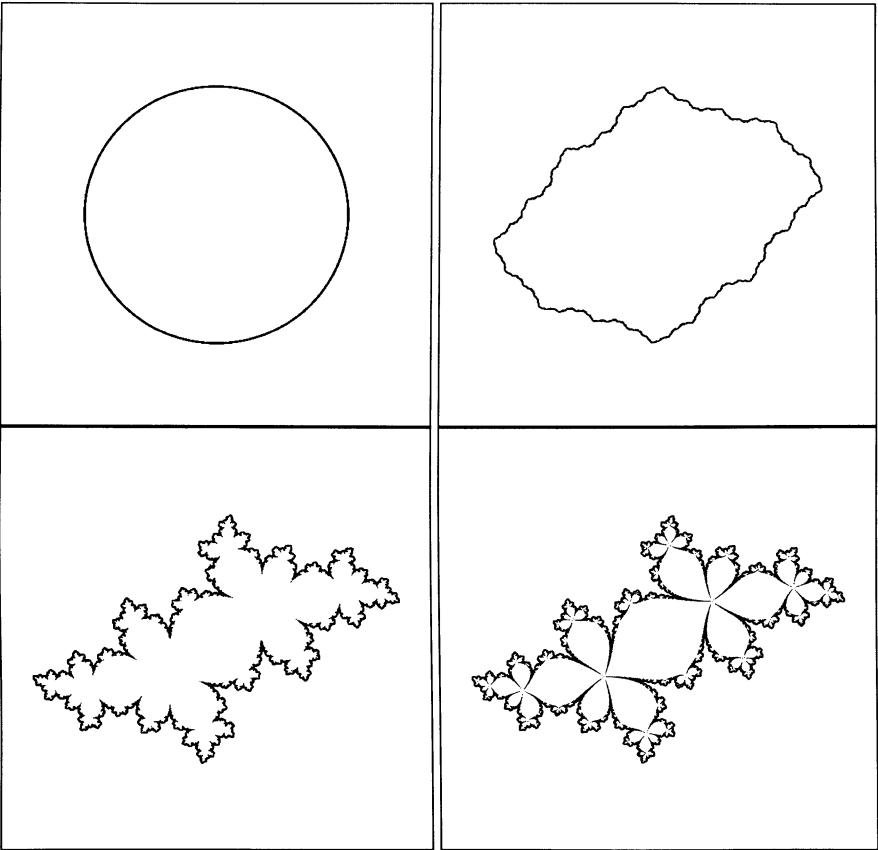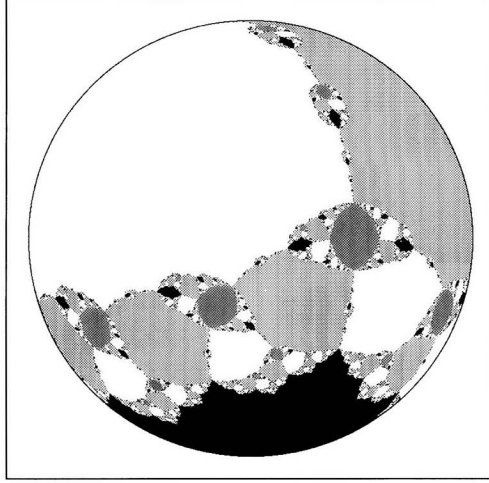
Figure 2. Limit sets.

Figure 3. Julia sets.

Figure 4. Newton's method for $4z^3 - 2z + 1$

The simplest Blaschke product is $f(z) = z^2$. For $|\lambda| < 1$ the maps $f_\lambda(z) = \lambda z + z^2$ are all quasiconformally conformally conjugate on their Julia sets, which are again fractal quasicircles. For $0 < |\lambda| < 1$, these maps belong to a single Teichmüller space; $z = 0$ is an attracting fixed point yielding a once-punctured torus as the quotient Riemann surface. Figure 3 depicts Julia sets for λ approaching $\exp(2\pi i 3/5)$; in the first three frames, $|\lambda| < 1$, while in in the final frame the attracting fixed point has become parabolic, splitting the Fatou set into countably many components, five of which meet at the origin. The pictures are centered at the critical point $z = -\lambda/2$.

Figure 4 depicts the dynamics of a more complicated rational map, namely Newton's method for a cubic polynomial. This map has superattracting cycles at the three roots of the polynomial, and another superattracting cycle of period two. (Initial guesses in the basin of this last cycle do not converge to a root.) Regions in the figure are shaded to indicate which of the four attractors they tend to.

3   Expanding dynamics

Crucial to the modern theory of Kleinian groups is the insight that the action of Γ extends isometrically to *hyperbolic space* $\mathbb{H}^3$. Thus Γ determines a *Kleinian manifold*

$$\overline{M} = M \cup \partial M = (\mathbb{H}^3 \cup \Omega)/\Gamma$$

with a complete metric of constant negative curvature on its interior and a conformal structure on its boundary.

Conversely, let M be a 3-manifold with a metric of constant curvature -1; pursuant to our standing assumptions about Γ, we will always assume M is oriented and $\pi_1(M)$ is finitely generated. Then M gives rise to a Kleinian group, by letting $\Gamma = \pi_1(M)$ acting on the compactification of $\widetilde{M} \cong \mathbb{H}^3$ by the Riemann sphere.

Perhaps the most prominent gap in the dictionary is summed up by:

Question 3.1 *Is there a 3-dimensional geometric object naturally associated to a rational map f?*

The best behaved dynamical systems have *expanding* properties. For Kleinian groups, expansion is reflected in the geometry of M.

To describe this relation, let $\text{hull}(\Lambda) \subset \mathbb{H}^3$ denote the *convex hull* of the limit set, i.e. the smallest convex set containing all geodesics with both endpoints in Λ. Let $K = \text{hull}(\Lambda)/\Gamma \subset M$ be the *convex core* of M; it is the smallest convex set containing all the closed geodesics in M. We say M is *geometrically finite* if a unit neighborhood of K has finite volume. A *cusp* of M is a component of the thin part with parabolic fundamental group.

The following are equivalent:

- The convex core of M is compact.

- The Kleinian manifold $\overline{M}$ is compact.

- M is geometrically finite without cusps.

- For all $x \in \Lambda$ there is a $\gamma \in \Gamma$ such that $\|\gamma'(x)\| > 1$ with respect to the spherical metric.

In this case we say Γ is *expanding*, or *convex cocompact*.

The examples in Figure 2 are geometrically finite, but not expanding, because they have cusps. Limit sets of expanding groups appear in Figures 5 and 6 below.

For rational maps, the nonexpanding behavior is closely tied to the critical points of f. Let

$$P = \overline{\bigcup_{n>0,\, f'(c)=0} f^n(c)}$$

denote the *postcritical set* of f, i.e. the closure of the forward orbits of the critical points. Then the following are equivalent:

- $P \cap J = \emptyset$.

- There are no critical points or parabolic cycles in the Julia set.

- All critical points converge to attracting or superattracting cycles under iteration.

- For all $x \in J$ there is an $n > 0$ such that $\|(f^n)'(x)\| > 1$.

In this case we say f is *expanding*.[4] The Newton's method example in Figure 4 and the first three frames in Figure 3 are expanding.

Expanding dynamical systems have many robust features.

1. *Openness.* The space of expanding maps is open in $\mathcal{V}_d$, and the space of expanding Kleinian groups is open in $\mathcal{V}(G)$.

2. *Structural stability.* Any dynamical system close enough to an expanding one is quasiconformally conjugate to it on its limit set or Julia set.

3. *Finiteness.* An expanding dynamical system can be described by a finite amount of data. For a Kleinian group the topology of a convex submanifold suffices. For a rational map one may find a pair of finitely connected neighborhoods $U \subset V$ of the Julia set such that $f : U \to V$ is a covering map.

4. *Quasi-self-similarity.* Because of expansion, any small ball meeting the Julia set or limit set can be blown up to definite size with bounded distortion. Thus all features of J or Λ are replicated everywhere and at all scales.

5. *Fractal geometry.* Apart from the case where $\Lambda = \widehat{\mathbb{C}}$, the Julia set and limit set have Hausdorff dimension δ with $0 < \delta < 2$, and finite, positive Hausdorff measure in this dimension. The Hausdorff measure can be constructed dynamically as a weak limit of probability measures

$$\lim_{s \searrow \delta} \frac{1}{C_s} \sum_{\gamma(y)=x} \|\gamma'(y)\|^{-s} \delta_y$$

for Kleinian groups, and

$$\lim_{s \searrow \delta} \frac{1}{C_s} \sum_{f^n(y)=x} \|(f^n)'(y)\|^{-s} \delta_y$$

for rational maps; here x is any point with an infinite orbit outside the limit set or Julia set. (The constants C_s normalize the total mass to 1.)

If a rational map f is expanding, then there is a finite set A (the union of the attracting and superattracting cycles) such that $f^n(x)$ converges to this finite attractor for all x in the Fatou set (an open, dense full measure subset of the sphere.) Thus we have a simple picture of the asymptotic behavior of a typical point.

Experimentally, expanding groups and rational maps are ubiquitous. We can now state two central conjectures.

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[4]We avoid using the more common terminology *hyperbolic*, to minimize confusion with hyperbolic manifolds.

Conjecture 3.2 *Let G be a torsion-free group which does not contain $\mathbb{Z} \oplus \mathbb{Z}$. Then convex compact groups are dense in $AH(G)$.*

Conjecture 3.3 *For any degree d, the expanding rational maps are dense in the space of all rational maps of degree d.*

The theory of *holomorphic motions* is a basic tool in the construction of quasiconformal conjugacies, which leads to the following pair of tantalizing results:

Theorem 3.4 (Mañé-Sad-Sullivan) *Structural stability is dense in* Rat_d.

Theorem 3.5 (Sullivan) *A structurally stable Kleinian group is expanding.*

To prove the density of convex cocompact groups, it would suffice to translate the first result to the setting of Kleinian groups. The problem is that for rational maps, one can work in the full moduli space of rational maps $\mathcal{V}_d$, while for Kleinian groups one must restrict attention to the subspace $AH(G)$ of discrete faithful representations in $\mathcal{V}(G)$. The topology of $AH(G)$ is not well-understood; indeed conjecture 3.2 is equivalent to the assertion that $AH(G)$ is the closure of its interior.

Generalizing the second result to the setting of rational maps would prove the density of expanding dynamics. Here one knows:

Theorem 3.6 *There is an open dense set $U \subset \mathcal{V}_d$ such that any f, g in the same component U_0 of U are quasiconformally conjugate on the whole sphere. Thus U_0 is uniformized by the Teichmüller space of f.*

By counting dimensions, one can show that a structurally stable rational map f with no invariant line fields on its Julia set must be expanding. This yields:

Corollary 3.7 *The no invariant line fields conjecture implies the density of expanding rational maps.*

4 Topology of hyperbolic 3-manifolds

We now turn to the *classification problem* for rational maps and Kleinian groups. Ultimately, one would like to

1. attach a *combinatorial invariant* to each dynamical system,

2. describe all invariants which arise, and

3. parameterize all systems with the same invariants.

The last goal is fulfilled by the Teichmüller theory discussed in §2, so it is enough to find a combinatorial invariant which determines the *quasiconformal conjugacy class* of a rational map or Kleinian group.

This section presents the conjectural classification of Kleinian groups. The measurable and topological dynamics on the sphere are also briefly discussed.

3-manifolds. The classification of a Kleinian group Γ up to quasiconformal conjugacy is the same as the classification of $M = \mathbb{H}^3/\Gamma$ up to quasi-isometry. A combinatorial invariant for M can be presented in terms of 3-dimensional topology.

To begin with, suppose Γ is expanding. Then the desired combinatorial invariant is simply the compact Kleinian manifold $N = \overline{M}$, regarded as a topological space. It is not hard to see that the topology of N determines the quasi-isometry type of M. Indeed, given an end of M diffeomorphic to $S \times [0, \infty)$, choose a hyperbolic metric $\rho(x)dx^2$ on S; then the model metric $e^t\rho(x)dx^2 + dt^2$ is quasi-isometrically correct on this end.

Now suppose M is geometrically finite. Some additional information must be included to specify the cusps of M. In this case the combinatorial invariant is a *pared manifold* (N, P), where N is compact and P is a submanifold of ∂N. The pair (N, P) is determined by the conditions (a) M is diffeomorphic to the interior of N and (b) $\overline{M}$ is diffeomorphic to $N - P$. The *parabolic locus* P contains every torus component of ∂N; the rest of P consists of a finite number of annuli on ∂N. We then have:

Theorem 4.1 (Marden) *The topology of the pared manifold (N, P) determines M up to quasi-isometry.*

Finally assume only that M is an orientable hyperbolic 3-manifold with $\pi_1(M)$ finitely generated. In this case the conjectural invariant is a triple (N, P, ϵ), where ϵ is the *ending lamination*. To even define this invariant, we need to first assume:

Conjecture 4.2 (Tameness) *A hyperbolic 3-manifold M with finitely generated fundamental group is topologically tame: that is, M is diffeomorphic to the interior of a compact manifold N.*

It is known that M is homotopy equivalent to a compact manifold, but examples like the Whitehead continuum show the ends of a 3-manifold with finitely generated fundamental group need not be products.

Assuming M is diffeomorphic to $\mathrm{int}(N)$, we again can locate a parabolic locus $P \subset N$ specifying the finite set of cusps. A cusp can be thought of as a simple curve on ∂N whose length in M has shrunk to zero. The ending lamination records a generalization of this phenomenon, in which an "irrational" simple curve has been pinched.

Given a surface S, endowed with a hyperbolic metric for convenience, a *lamination* $\lambda \subset S$ is a closed subset which is a union of disjoint simple

geodesics. To construct the ending lamination, fix a diffeomorphism between int N and M, so that a neighborhood of P maps into the cusps of M. Consider any sequence of simple closed curves γ_n on a component S of $\partial N - P$ whose geodesic representatives in M converge to S. Then a subsequence converges, in a natural sense, to a lamination λ on S.[5]

The ending lamination ϵ is the union of all λ which arise this way.

Conjecture 4.3 (The ending lamination conjecture) *The pared manifold (N, P) and the ending lamination ϵ determine M up to quasi-isometry.*

This conjecture is open even for $N = S \times [0, 1]$ where S is a compact surface. Recently (March 1995) Minsky has announced:

Theorem 4.4 *The ending lamination conjecture is true for*

$$(N, P) = (S \times I, \partial S \times I),$$

where S is a torus with a disk removed.

The space of laminations on a punctured torus is a circle (since a lamination is determined by a line in $H_1(S, \partial S, \mathbb{R})$). Bers defined a compactification of Teichmüller space using limits of quasifuchsian groups; by Minsky's result, the compactifying points for a punctured torus are determined by their laminations, and we have:

Corollary 4.5 *The Bers' boundary of the Teichmüller space of a punctured torus is a Jordan curve.*

Here are two more general results bearing on the tameness and ending lamination conjectures.

A map $f : S \to N$ of a surface into a 3-manifold is *incompressible* if $f_* : \pi_1(S, p) \to \pi_1(N)$ is injective for all $p \in S$. A compact 3-manifold has *incompressible boundary* if $\partial N \subset N$ is incompressible. An open 3-manifold M has *incompressible ends* if it contains an embedded compact submanifold with incompressible boundary, such that $N \hookrightarrow M$ is a homotopy equivalence.

Theorem 4.6 (Thurston, Bonahon) *Any hyperbolic 3-manifold with incompressible ends is topologically tame.*

Theorem 4.7 (Minsky) *Any two hyperbolic manifolds M_1 and M_2 with incompressible ends, injectivity radii bounded below and the same combinatorial invariant (N, ϵ) are quasi-isometric.*

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[5]The precise definition of convergence is that there is a projective measured lamination $[\lambda_0]$ with support λ such that $[\gamma_n] \to [\lambda_0]$.

All of these results are based on a study of incompressible surfaces $f :$ $S \to M$. The map f can always be deformed to a *pleated surface*, which induces a hyperbolic metric on S. Since the hyperbolic geometry of S is controlled (for example, the total area of S is $2\pi|\chi(S)|$, by Gauss-Bonnet), one obtains control on the geometry of M near $f(S)$. By constructing families of pleated surfaces exiting the ends of M, one obtains global estimates for its 3-dimensional geometry and topology.

Constructing hyperbolic manifolds. To have a more complete picture, we need to describe which combinatorial invariants can arise. To begin with, the data already discussed satisfies a few topological constraints.

- Since the universal cover of M is $\mathbb{H}^3$, N is *irreducible* — every embedded S^2 bounds a ball.

- By the classification of cusps, N is *atoroidal* — every $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(N)$ comes from a torus boundary component.

- The parabolic locus P is incompressible, it contains every torus component of ∂N, and any incompressible cylinder $f : (S^1 \times I, S^1 \times \partial I) \to (N, P)$ is homotopic into P.

Let us define a general pared manifold to be a pair (N, P) with the properties above. A 3-manifold is *Haken* if it can be built up from 3-balls inductively, by gluing along incompressible submanifolds of the boundary.

We can now state a remarkable existence theorem:

Theorem 4.8 (Thurston) *Every Haken pared manifold (N, P) can be realized by a geometrically finite hyperbolic manifold.*

Any irreducible 3-manifold with nonempty boundary is Haken. Thus one has a complete characterization of expanding Kleinian groups whose limit set is not the whole sphere:

Corollary 4.9 *A 3-manifold N is homeomorphic to the Kleinian manifold $\overline{M}$ of a convex cocompact group with $\Lambda \neq \widehat{\mathbb{C}}$ if and only if:*

- *N is compact, irreducible, orientable,*
- *$\partial N \neq \emptyset$ and*
- *$\pi_1(N)$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$.*

Corollary 4.10 *For any Kleinian group Γ, $AH(\Gamma)$ contains a geometrically finite group.*

Proof. If $M = \mathbb{H}^3/\Gamma$ is closed, then it is geometrically finite; otherwise M has the homotopy type of an irreducible atoroidal manifold with boundary, so Thurston's theorem applies. ■

Unfortunately a characterization of *closed* hyperbolic manifolds is not yet known. Thurston's geometrization program for 3-manifolds includes as a special case:

Conjecture 4.11 *A closed irreducible orientable 3-manifold is hyperbolic if and only if its fundamental group is infinite and does not contain $\mathbb{Z} \oplus \mathbb{Z}$.*

By Mostow rigidity these remaining cases account for only countably many Kleinian groups up to conjugacy.

To conclude we mention what is known about the ending lamination ϵ. The triple (N, P, ϵ) has some topological constraints which we will not make precise; we simply call a triple *admissible* if it satisfies them. Building on work of Thurston, Ohshika established:

Theorem 4.12 *Let (N, P) be a pared manifold with nonempty, incompressible boundary. Then every admissible triple (N, P, ϵ) can be realized by a hyperbolic manifold M.*

When $P = \emptyset$ the proof is by approximating M by convex cocompact manifolds. This yields:

Corollary 4.13 *Let N be a compact, orientable irreducible 3-manifold with incompressible boundary, such that $\pi_1(N)$ does not contain $\mathbb{Z} \oplus \mathbb{Z}$. Then the ending lamination conjecture implies the density of convex cocompact groups in $AH(\pi_1(N))$.*

Ergodic theory of Kleinian groups. Since Γ preserves the Lebesgue measure class on the sphere, it determines a measurable dynamical system. Here are two conjectured properties of the dynamics on the limit set.

Conjecture 4.14 (Ahlfors' measure zero problem) *If the limit set $\Lambda \neq \widehat{\mathbb{C}}$, then it has measure zero.*

Conjecture 4.15 (Ergodicity of the geodesic flow) *If $\Lambda = \widehat{\mathbb{C}}$, then Γ acts ergodically on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$. Equivalently, the geodesic flow on the unit tangent bundle $T_1(M)$ is ergodic.*

These conjectures are stronger than the No Invariant Line Fields Theorem 2.3. Indeed, suppose Λ carries an invariant line field; then for $(x, y) \in \Lambda \times \Lambda$ we can use parallel transport along the hyperbolic geodesic from x to y to measure the angle $\theta(x, y) \pmod{\pi}$ between the lines at x and y. The function $\theta(x, y)$ is Γ-invariant and essentially nonconstant, so Γ does not act ergodically on $\Lambda \times \Lambda$.

A triumph of the three-dimensional point has been its contributions to these originally two-dimensional problems. Due to work of Thurston, Bonahon and Canary, one now knows:

Theorem 4.16 *If $M = \mathbb{H}^3/\Gamma$ is topologically tame, then the conjectures above are true: either Λ has measure zero, or Γ acts ergodically on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$.*

Because of Theorem 4.6, the hypothesis can be verified in many cases: for example, the theorem applies to any Kleinian group Γ which is not isomorphic to a free product $A * B$. But it is still unknown if Ahlfors' conjecture is true for a Kleinian group Γ isomorphic to $\mathbb{Z} * \mathbb{Z}$.

On the other hand, by work of Sullivan, Tukia and Bishop-Jones one now knows:

Theorem 4.17 *The limit set of a finitely generated Kleinian group Γ has Hausdorff dimension 2 if and only if Γ is geometrically infinite.*

Local connectivity of limit sets. It is interesting to try to reconstruct the *topological dynamics* of a Kleinian group acting on the sphere from its combinatorial invariant (N, P, ϵ). This is possible when the limit set is locally connected — that is, when it has a basis of connected open sets.

The following result is well-known to experts:

Theorem 4.18 *If Γ is geometrically finite, then every component of Λ is locally connected.*

Proof. The group Γ is built up from groups with connected limit sets by the combination theorems, as can be seen by decomposing $\overline{M}$ along compressing disks. Thus one can reduce to the case where Λ is connected. The diameters of the components of Ω tend to zero, by a theorem of Maskit, so it suffices to show the boundary of every component Ω_0 of Ω is locally connected. We can assume Γ stabilizes Ω_0. If Ω has just two components, the limit set is a quasicircle by a theorem of Maskit and we are done. Otherwise we have a geometrically finite surface group with accidental parabolics; the limit set is also locally connected in this case, by a result of Abikoff.

∎

Peano curves. By work of Cannon-Thurston and Minsky, one also knows local connectivity of Λ and a topological model for the action when $\Gamma \cong \pi_1(S)$ is a surface group with injectivity radius bounded below. Sphere-filling Peano curves arise as special cases: that is, when $\Lambda = \widehat{\mathbb{C}}$ there is a surjective continuous map $f : S^1 \to \widehat{\mathbb{C}}$, conjugating a Fuchsian action of $\pi_1(S)$ to the action of Γ.

5   Topology of rational maps

For rational maps, even a conjectural combinatorial classification has yet to be formulated. In this section we discuss two cases where a rich theory is developing: expanding maps and polynomials.

Expanding components. Let $\mathrm{Exp}_d \subset \mathcal{V}_d$ denote the open set of expanding rational maps of degree d. An *expanding component* U is a connected component of Exp_d.

Any two maps in the same expanding component are quasiconformally conjugate on their Julia sets. An expanding component is partitioned into strata, each parameterized by a Teichmüller space. There is a unique top-dimensional stratum, characterized as those $f \in U$ such that the critical points have disjoint, infinite forward orbits.

One would like an invariant to classify expanding components. Such an invariant can be formulated when the Julia set is connected.
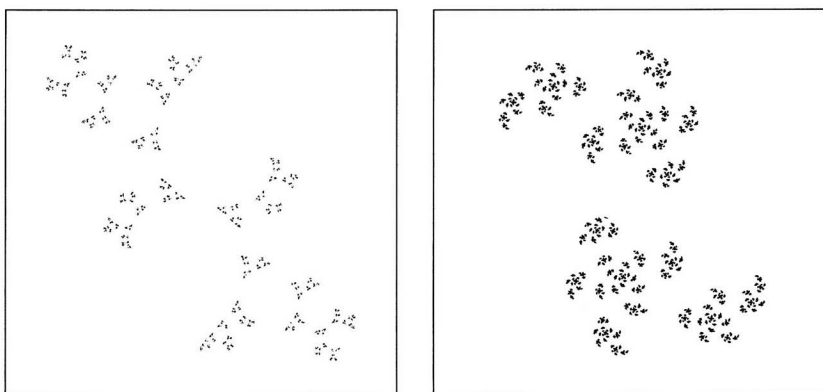

Figure 5. Cantor sets

Incompressible dynamics. Properties of a compact Kleinian manifold $\overline{M}$ are often reflected in the topology of its limit set. For example, $\overline{M}$ has incompressible boundary if and only if its limit set is *connected*. (Any essential loop in Ω represents an element of $\pi_1(\partial M)$ which is trivial in M.)

Figure 5 depicts the limit set for a *Schottky group* on the left, where $\overline{M}$ is a handlebody of genus two; this manifold has highly compressible boundary, and the limit set is a Cantor set. On the right is the Julia set for an quadratic polynomial whose finite critical point escapes to infinity; J is also a Cantor set, and it seems reasonable to regard this rational map as maximally compressible.

The classification of expanding components U is best understood in the case of *connected* Julia sets — that is, the incompressible case.

Indeed, if J is connected then there is a unique $f_0 \in U$, the *center* of U, such that all critical points eventually map to periodic critical points. The map f_0 also forms the unique zero-dimensional Teichmüller stratum in U. Then a finite combinatorial invariant for U is the simply *isotopy class* $[f_0]$ of the branched covering

$$f_0 : (S^2, P(f_0)) \to (S^2, P(f_0)).$$

(Here $P(f_0) = \bigcup_{n>0, f_0'(c)=0} f_0^n(c)$ is the postcritical set.)

Thurston's combinatorial characterization of rational maps

- shows $[f_0]$ determines the rational map f_0, and hence the expanding component U of which it is the center, uniquely; and

- describes all branched coverings which can arise.

To state this theorem, let $g : S^2 \to S^2$ be a topological branched covering of degree $d > 1$. The map g is *critically finite* if $|P(g)| < \infty$, where $P(g)$ is the union of the forward orbits of the branch points.

Consider a finite system Γ of disjoint simple closed curves γ on $S^2 - P(g)$, no two parallel, and none contractible or peripheral (i.e. enclosing a single point of $P(g)$). The system Γ is *g-invariant* if every component α of $g^{-1}(\gamma)$ is contractible, peripheral or isotopic to some $\delta \in \Gamma$. Define $A(\Gamma) : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ by the formula

$$A_{\delta\gamma} \;=\; \sum_\alpha \frac{1}{\deg(g : \alpha \to \gamma)}$$

where the sum is taken over components α of $g^{-1}(\gamma)$ which are isotopic to δ. Let $\lambda(\Gamma)$ denote the spectral radius of $A_{\delta\gamma}$. Then we have:

Theorem 5.1 (Thurston) *A critically finite branched cover $g : S^2 \to S^2$ is realized by an expanding rational map f if and only if:*

- *every branch point of g eventually maps to a periodic branch point, and*

- *$\lambda(\Gamma) < 1$ for every g-invariant curve system Γ.*

The rational map f is unique up to conformal conjugacy.

A slightly more general statement characterizes *all* critically finite (not necessarily expanding) rational maps.

The fact that $\lambda(\Gamma) < 1$ for a rational map is already an interesting observation. This can be seen by thickening the curves in Γ to annuli C_γ, and observing that the geometry of $P(f)$ gives an upper bound for the vector of conformal moduli $[m_\gamma] = [\mathrm{mod}(C_\gamma)]$. The vector $[A_{\delta\gamma} m_\gamma]$ is a strict lower bound for the moduli of annuli which can be manufactured from the preimages of C_γ under f, so the Perron-Frobenius eigenvalue of $A_{\delta\gamma}$ must be contracting.

Thurston's criterion is rather implicit. Even fixing the degree d of g and the size $p = |P(g)|$ of the postcritical set, there are usually infinitely many isotopy classes of branched coverings g, only finitely many of which represent rational maps. For a more satisfactory classification, one needs to solve the following combinatorial problem:

Problem 5.2 *Find an efficient way to organize and enumerate all critically finite rational maps of a given degree.*
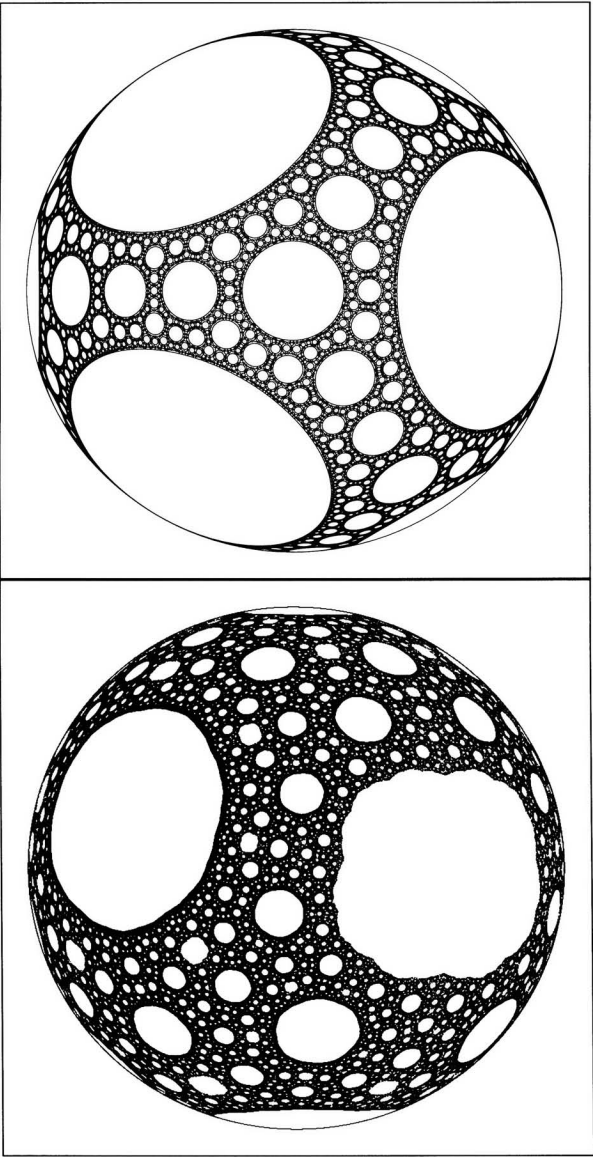
Figure 6. Sierpiński curves.

Acylindrical dynamics. A 3-manifold N with incompressible boundary is *acylindrical* if any map $f : (S^1 \times I, \partial(S^1 \times I)) \to (N, \partial N)$, injective on π_1, is homotopic into ∂N. A compact Kleinian manifold $\overline{M}$ is acylindrical if and only if its limit set Λ is a *Sierpiński curve*. This means Λ is obtained from the sphere by removing a countable dense set of open disks, bounded by disjoint Jordan curves whose diameters tend to zero.

Explicit examples of expanding rational maps with Sierpiński Julia sets have been given by Milnor-Tan and Pilgrim, and the notion of a "cylinder" for rational maps is under development.

Thurston showed that degeneration of Kleinian groups is related to cylinders, and that $AH(\pi_1(\overline{M}))$ is *compact* when $\overline{M}$ is acylindrical.

Question 5.3 *Let U be an expanding component with center f_0. If the Julia set of f_0 is a Sierpiński curve, is $\overline{U} \subset \mathcal{V}_d$ compact?*

At present, not one expanding component is known to have compact closure.

Figure 6 depicts the limit set of an acylindrical Kleinian group above, and the Julia set of an acylindrical expanding rational map below. By a result of Whyburn, these two fractal sets are homeomorphic.

A sequence of rational maps can degenerate by collapsing towards a map of lower degree. Morgan and Shalen have used $\mathbb{R}$-trees to study degenerations of hyperbolic manifolds, and Shishikura applied simplicial trees to some special degenerations of rational maps, so perhaps trees can help resolve the question above.

The general case. A compact 3-manifold with *compressible* boundary can be cut along a finite number of compressing disks to obtain incompressible pieces. Inverting this procedure, combination theorems of Klein and Maskit allow one to build up a Kleinian group with disconnected limit set from a finite number of subgroups with connected limit sets.

It should be possible to treat expanding rational maps with disconnected Julia sets by an analogous procedure.

Problem 5.4 *Develop decomposition and combination theorems for rational maps.*

One would like a systematic theory including existing constructions like *mating* and *tuning*, as well a theory for disconnected Julia sets — just as the Haken theory discusses cutting along general incompressible subsurfaces, not just disks. For example, some degree two rational maps can be expressed as matings in more than one way, just as a 3-manifold can contain several embedded incompressible surfaces.

Problem 5.5 *Give a combinatorial invariant for a general rational map f.*

One expects that every Kleinian group is a limit of geometrically finite groups of *fixed* topological type. In contrast to this, a rational map with every cycle repelling cannot belong to the closure of any expanding component $U \subset \mathcal{V}_d$. Thus it is challenging to predict even the *topology* of a sequence of expanding maps f_n converging to f.

A combinatorial invariant for rational maps of degree two near a Lattès example has been developed by Bernard.

Polynomials. Using laminations, one can formulate a combinatorial invariant for many polynomials.

Let $f(z) = z^d + a_2 z^{d-2} + \ldots + a_d$ be a monic centered polynomial of degree $d > 1$ with connected Julia set. The set of all such polynomials forms the *connectedness locus* $\mathcal{C}_d$. The coefficients $(a_2, \ldots, a_d)$ of f determine an embedding of $\mathcal{C}_d$ as a compact, connected subset of $\mathbb{C}^{d-1}$. Any polynomial with connected Julia set is conformally conjugate to one in $\mathcal{C}_d$.

The *filled Julia set* $K(f) \subset \mathbb{C}$ is the union of $J(f)$ and the bounded components of the Fatou set; it is compact set bounded by $J(f)$. The complement $\widehat{\mathbb{C}} - K(f)$ is the basin of attraction of the point at infinity. By the Riemann mapping theorem, there is a unique conformal map

$$\phi_f : (\mathbb{C} - \overline{\Delta}) \to (\mathbb{C} - K(f))$$

such that $\phi_f(z)/z \to 1$ as $z \to \infty$. The map ϕ_f conjugates z^d to f; that is, that is, $f(\phi_f(z)) = \phi_f(z^d)$.

This conjugacy facilitates the encoding of the combinatorics of f by a lamination. As can be seen in Figure 3, as f varies in $\mathcal{C}_d$ the topology of the Julia set $J(f)$ can change from a circle to a circle with many points identified. The lamination records these identifications.

Let $S^1 = \mathbb{R}/\mathbb{Z}$, and identify S^1 with the boundary of the unit disk Δ via the map $t \mapsto \exp(2\pi it)$. For $t \in S^1$, let

$$R_t = \{\phi_f(r \exp(2\pi it)) : 1 < r < \infty\}$$

be the *external ray* at angle t. Clearly $f(R_t) = R_{dt}$. The ray R_t *lands* at $p \in J(f)$ if

$$\lim_{r \searrow 1} \phi_f(r \exp(2\pi it)) = p.$$

It is known that R_t lands for every rational angle t, and if t is periodic under $t \mapsto dt$, then the landing point p is a parabolic or repelling periodic point for f.

A *lamination* $\lambda \subset S^1 \times S^1$ is an equivalence relation such that the convex hulls of distinct equivalence classes are disjoint. The *rational lamination* $\lambda_{\mathbb{Q}}(f)$ is defined by $t \sim t'$ if $t = t'$, or if t and t' are rational and the external rays R_t and $R_{t'}$ land at the same point in the Julia set $J(f)$. (See Figure 7 for an example where $\{1/7, 2/7, 4/17\}$ is a single equivalence class under $\lambda_{\mathbb{Q}}(f)$.)
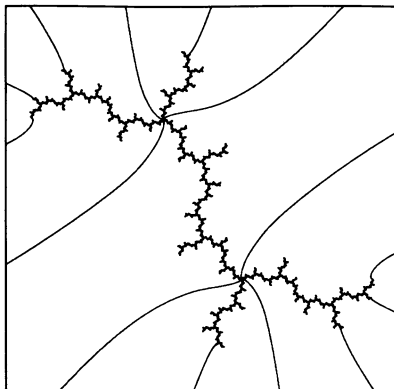
Figure 7. External rays for $f(z) = z^2 + i$.

A map $f \in \mathcal{C}_d$ with no indifferent cycles is *combinatorially rigid* if it is determined up to quasiconformal conjugacy on its Julia set by its rational lamination. More precisely, for any $g \in \mathcal{C}_d$ with no indifferent cycles such that $\lambda_{\mathbb{Q}}(g) = \lambda_{\mathbb{Q}}(f)$, the conformal map

$$\phi_g \circ \phi_f^{-1} : (\mathbb{C} - K(f)) \to (\mathbb{C} - K(g))$$

should extend to a quasiconformal map on the whole Riemann sphere.

We can now state:

Conjecture 5.6 (Combinatorial rigidity) *Every polynomial with connected Julia set and no indifferent cycle is combinatorially rigid.*

One also conjectures a relative version of the density of expanding rational maps.

Conjecture 5.7 *Expanding polynomials are dense in the connectedness locus $\mathcal{C}_d$ and in the space of all polynomials of degree d.*

Measure of Julia sets. A direct translation the Ahlfors measure zero conjecture to the setting of rational maps appeared plausible for many years. Recently however the following result has been announced:

Theorem 5.8 (Nowicki-van Strien) *Let $d > 0$ be an even integer, and let $f(z) = z^d + c$ where $c \in \mathbb{R}$. Suppose the closest returns of $f^n(0)$ to the origin occur along the Fibonacci sequence $n = 1, 2, 3, 5, 8, \dots$. Then for d large enough, the Julia set of f has positive measure.*

A pair of Fibonacci maps can also be combined to give a polynomial which is not ergodic on its Julia set. On the other hand, Ahlfors' conjecture

was motivated by the problem of invariant complex structures on the limit set, and these Fibonacci maps carry no invariant line fields.

Question 5.9 *When $J(f)$ has positive measure, does it have finitely many ergodic components? Is their number bounded by the number of critical points?*

Though one expects the expanding maps to be open and dense in $\mathcal{V}_d$, they do not fill out the space of rational maps in the sense of measure. In fact a random rational map has a definite chance of being ergodic:

Theorem 5.10 (Rees) *The set of rational maps such that $J(f) = \widehat{\mathbb{C}}$ and f acts ergodically on the sphere has positive measure in the space of all rational maps of degree d.*

Local connectivity of Julia sets. If $f \in \mathcal{C}_d$ and its Julia set is locally connected, then $J(f)$ is homeomorphic to S^1/λ for some lamination λ. However, it is known that there are Julia sets which are *not* locally connected. A concrete example is provided by $f(z) = e^{2\pi i t}z + z^2$, where θ is an irrational number which is very well approximated by rationals — e.g. $t = \sum 1/a_n$ where $\{a_1, a_2, a_3, \dots\} = \{2, 2^2, 2^{2^2}, \dots\}$.

Expanding rational maps are better behaved. A rational map f is *geometrically finite* if every critical point either has a finite forward orbit, or converges to an attracting, superattracting or parabolic cycle. Any expanding map is geometrically finite, and we have:

Theorem 5.11 (Tan-Yin) *Every preperiodic component of the Julia set of a geometrically finite rational map is locally connected.*

6   Renormalization

Much progress towards the density of expanding dynamics and combinatorial rigidity has been recently achieved in the setting of quadratic polynomials. The methods also shed light on questions of local connectivity and measure. The very formulation of the results involves the idea of *renormalization*.

For quadratic polynomials $P_c(z) = z^2 + c$, the connectedness locus $\mathcal{C}_2$ is the same as the *Mandelbrot set* $M \subset \mathbb{C}$. The Mandelbrot set can also be described as the set of c such that $P_c^n(0)$ does not tend to infinity. By work of Douady, Hubbard and Yoccoz we have:

Theorem 6.1 *Let c belong to the Mandelbrot set M. Then either*

- P_c *has an indifferent cycle and M is locally connected at c; or*

- P_c *has no indifferent cycle, and M is locally connected at c if and only if P_c is combinatorially rigid.*

Corollary 6.2 (Douady-Hubbard) *Local connectivity of the Mandelbrot set implies the density of expanding maps in the space of quadratic polynomials.*

Proof. If M is locally connected then we have combinatorial rigidity in degree two.

Density of expanding dynamics can only fail if there is a component Q of $\text{int}(M)$ consisting of a single quasiconformal conjugacy class, such that $f_0 \in Q$ has all finite cycles repelling and an invariant line field on its Julia set. The points $B_n \subset \partial Q$ with an indifferent cycle of period n are nowhere dense, so there is is a $g \in \partial Q$ with no indifferent cycle. The rational lamination changes only at parabolic points, so $\lambda_{\mathbb{Q}}(g) = \lambda_{\mathbb{Q}}(f_0)$.

Combinatorial rigidity would then imply g and f_0 are quasiconformally conjugate on their Julia sets. But then $J(g)$ would carry an invariant line field, contradicting the fact that $g \in \partial M$.

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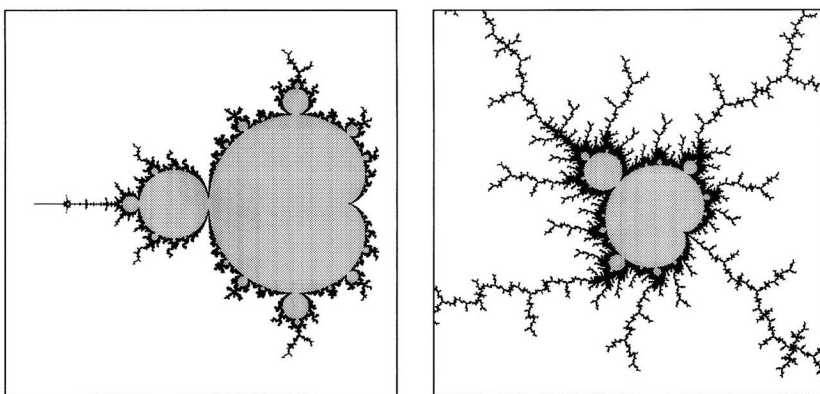Figure 8. Small copies of M in M.

Quadratic-like maps. A remarkable feature of the Mandelbrot set is that it contains countably many small copies of itself (Figure 8). To explain these and other reappearances of M, Douady and Hubbard introduced the theory of *polynomial-like maps*. A polynomial-like map $f : U \to V$ is a proper holomorphic map between disks in the plane, such that $\overline{U}$ a compact subset of V. The filled Julia set of f is defined by $K(f) = \bigcap_1^\infty f^{-n}(V)$, and the Julia set $J(f)$ is defined as its boundary. The filled Julia set consists of the points which never escape from the domain of f under iteration. A basic theorem asserts that $f|K(f)$ is quasiconformally conjugate to $p|K(p)$ for some polynomial p.

Consider now a quadratic polynomial $f(z) = z^2 + c$ with connected Julia set. A *quadratic-like* map is a polynomial-like map of degree two. We say an iterate f^n is *renormalizable* if there are disks U_n and V_n, containing the critical point $z = 0$, such that $f^n : U_n \to V_n$ is a quadratic-like map with connected Julia set. (Note that f^n itself has degree 2^n.) If there are infinitely many such n, then f is *infinitely renormalizable*.

A point $s \in M$ is *superstable* of period p if $z = 0$ has period p under $P_s(z)$. Douady and Hubbard construct for each such s a *tuning homeomorphism* $M \to M_s \subset M$, written $c \mapsto s * c$, such that if $c \neq 1/4$, then $P^p_{s*c}(z)$ is renormalizable and conjugate on its filled Julia set to $P_c(z)$. Thus renormalization explains the small copies of M inside of M. Any infinitely renormalizable map lies inside a countable nest of small Mandelbrot sets.

Example. The two left frames in Figure 9 depict the filled Julia sets of $f_i(z) = z^2 + s_i$, $i = 1, 2$, where s_1 and s_2 are superstable of period two and three respectively; the latter is known as Douady's rabbit. The upper right frame depicts the filled Julia set for $f(z) = z^2 + s_1 * s_2$; here the critical point has period six, and f^2 is renormalizable. The final frame shows the filled Julia set of a renormalization $f^2 : U_2 \to V_2$; it is homeomorphic to the rabbit.
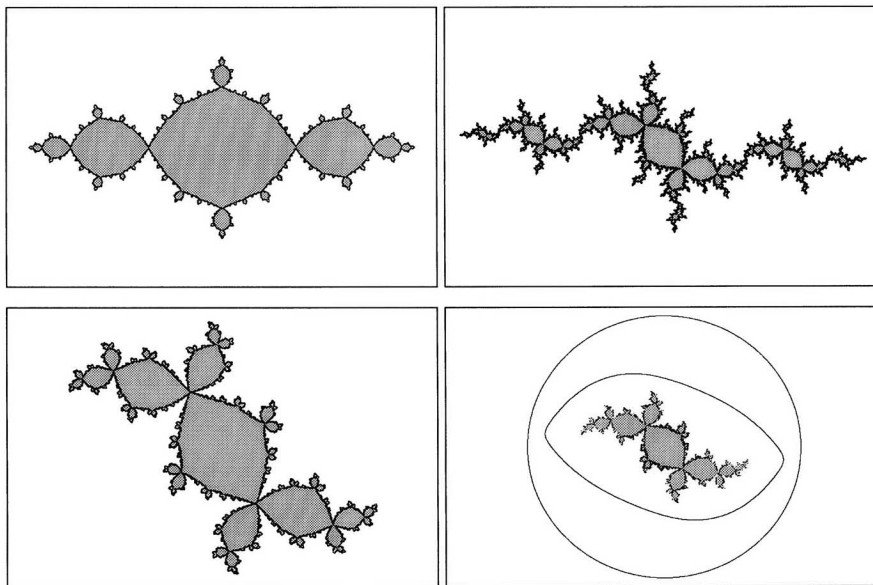

Figure 9. Renormalization.

We can now state two recent results towards the classification of quadratic polynomials.

Theorem 6.3 (Yoccoz) *Suppose $c \in M$ and $f(z) = z^2 + c$ has no indifferent cycle. Then either:*

- *f is combinatorially rigid, $J(f)$ is locally connected and M is locally connected at c, or*

- *f is infinitely renormalizable.*

Theorem 6.4 (Lyubich) *Combinatorial rigidity holds for uncountably many infinitely renormalizable quadratic polynomials $f(z) = z^2 + c$ with explicitly described combinatorial classes.*

Carrying Yoccoz's argument further, Lyubich and Shishikura proved independently:

Theorem 6.5 *If $f(z) = z^2 + c$ has no indifferent cycle and is only finite renormalizable, then $J(f)$ has measure zero.*

Renormalization operators. Renormalization is closely related to *universality* in dynamics.

To explain this phenomenon, for $c \in M \cap \mathbb{R} = [-2, 1/4]$ let $A_c \subset \mathbb{R}$ denote the set of accumulation points of $P_c^n(0)$. The set A_c often plays the role of an attractor for P_c over the reals; for example, it coincides with the attracting cycle if P_c has one.
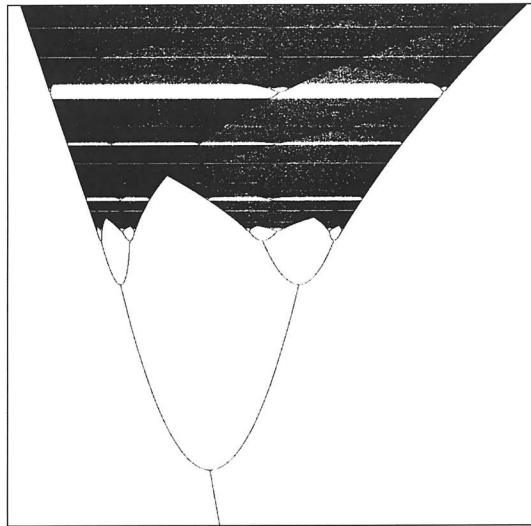

Figure 10. Bifurcation diagram.

A classical experiment is to plot A_c as a function of c. Figure 10 renders the set $\{(x, c) \ : \ x \in A_c, -0.6 < c < -2.0\}$. Initially A_c is a single attracting

fixed point; as c decreases (moving vertically in the figure), the attractor undergoes a sequence of period doublings, accumulating to a Cantor set at the parameter $c_F \approx -1.401155189$. In physical terms c_F marks a *phase transition* from periodicity to chaos in this family. The *Feigenbaum polynomial* $f(z) = z^2 + c_F$ is infinitely renormalizable; moreover the renormalization of f^2 is topologically conjugate to f itself.

In other families of real dynamical systems one finds cascades of period doublings remarkably similar to the quadratic family: for example the Hausdorff dimension of A_c and the pace of period doubling are independent of the particular family examined, and thus *universal*.

In the late 1970s Feigenbaum and Coullet-Tresser offered an explanation of this universality in terms of a fixed point for renormalization. Sullivan has recently given a rigorous proof of many aspects of this explanation; here is a precise statement in the language of quadratic-like mappings that follows from his work.

A quadratic-like map is *real* if $f(\bar{z}) = \overline{f(z)}$. Let $\mathcal{G}$ denote the set of real quadratic-like maps f with connected Julia sets, up to affine conjugacy near $K(f)$. We say $[f_n] \to [f]$ in $\mathcal{G}$ if there are representatives converging uniformly on a neighborhood of $K(f)$. Let $\mathcal{G}^{(p)} \subset \mathcal{G}$ denote the set of f such that f^p is renormalizable. The *renormalization operator*

$$\mathcal{R}_p : \mathcal{G}^{(p)} \to \mathcal{G}$$

sends f to the germ of its quadratic-like restriction $f^p : U_p \to V_p$. We may now state:

Theorem 6.6 (Real fixed points of renormalization) *Suppose the critical point $z = 0$ has period p under $z \mapsto z^2 + s$, where $s \in \mathbb{R}$. Then:*

1. *The limit $s^\infty = \lim s^{*n}$ exists, and is the unique real fixed point of tuning by s;*

2. *if $f(z) = z^2 + s^\infty$, then $\mathcal{R}_p^n(f) \to F$, where F is a fixed-point of renormalization; and*

3. *$\mathcal{R}_p^n(g) \to F$ for any $g \in \mathcal{G}$ which is topologically conjugate to f on $K(g) \cap \mathbb{R}$.*

Corollary 6.7 *The real fixed points of $\mathcal{R}_p$ are classified by the real superstable points $s \in M$ of period p.*

Example. For the Feigenbaum polynomial $f(z) = z^2 + c_F$, we have $c_F = (-1)^\infty$, and after rescaling the renormalizations of f^{2^n} converge to a solution F of the functional equation $F \circ F(z) = \alpha F(\alpha^{-1}z)$.

3-manifolds which fiber over the circle. There is an intriguing analogy between the renormalization operators and the mapping class group of a surface. Consider a closed surface S of genus 2 or more, and let $\psi : S \to S$ be a pseudo-Anosov mapping class. Then ψ acts on $AH(\pi_1(S))$, and a

fixed point corresponds to a discrete faithful representation satisfying the functional equation

$$\rho \circ \psi_*(\gamma) = \alpha^{-1}\rho(\gamma)\alpha$$

for some $\alpha \in \mathrm{Aut}(\widehat{\mathbb{C}})$. Although $\Gamma = \rho(\pi_1(S))$ is geometrically infinite, the group generated by Γ and α is *expanding, cocompact*, and gives a hyperbolic structure on the closed 3-manifold T_ψ which fibers over the circle with monodromy ψ.

Similarly, a fixed point for the renormalization operator $\mathcal{R}_p$ gives a solution to the functional equation

$$F^p(z) = \alpha^{-1}F(\alpha(z)),$$

and while the critical point of F is recurrent, the dynamical system generated by F and $z \mapsto \alpha z$ has good expansion properties.

To complete the classification of quadratic polynomials, establish local connectivity of the Mandelbrot and prove the density of expanding dynamics, it remains only to prove combinatorial rigidity in the infinitely renormalizable case. Although Theorem 6.4 handles many cases, even the following is still open:

Question 6.8 *Is the Feigenbaum polynomial combinatorially rigid?*

Understanding the expansion implicit in renormalization may be useful in resolving these questions.

7   Notes and references.

§1 Introduction. Classical references for conformal dynamical systems as seen by Poincaré, Fatou and Julia include [Poin], [Julia], and [Fatou]. The dictionary between rational maps and Kleinian groups was introduced in [Sul4] and is discussed at length in [Sul2]. The scope of conformal dynamics is broader than our focus here; it includes real one-dimensional dynamics and codimension-one holomorphic foliations, as well iterated entire functions, correspondences, and higher-dimensional Kleinian groups.

§2 Finiteness theorems and Teichmüller space. Ahlfors' finiteness theorem and the Ahlfors measure zero problem appear in [Ah]; the number of triply-punctured spheres is bounded in [Gr]. Sullivan's no wandering domains theorem appears in [Sul4] (see also [Bers3]), and the Teichmüller space of a holomorphic dynamical system is developed in [McS]. Bers' area theorem and Shishikura's bounds are in [Bers1] and [Shi1].

§3 Expanding dynamics. A candidate 3-dimensional object for a rational maps is constructed in [LM]. A prototype for the conjecture on the density of convex cocompact groups is [Bers2, Conjecture II].

The relation of hyperbolicity to structural stability is discussed for Kleinian groups in [Sul5], and for rational maps in [MSS], [McS] and [Mc4]. Theorem 3.6 and its Corollary can be found in [McS].

Density of hyperbolicity has been announced for the family of entire functions $f_\lambda(z) = \lambda \tan z$ [KK].

§4 Topology of hyperbolic 3-manifolds. Marden's work on the topology of geometrically finite 3-manifolds appears in [Mar]. The basic reference for the ending lamination, pleated surfaces, topological tameness and their relation to the Ahlfors conjecture is [Th1]. Important further developments appear in [Bon], [Oh], [Min1], [Can] and [Min2]. The Whitehead continuum (whose complement gives a manifold with a wild end) is discussed in [Rol, §3I].

Thurston's geometrization theorem for Haken manifolds is discussed in [Mor1], [Th2] and [Th3]; a new approach to the the proof is surveyed in [Mc3].

The Hausdorff dimension of limit sets is studied in [Sul1], [Sul3], [Tu] and [BJ].

The local connectivity of geometrically finite limit sets (Theorem 4.18) is assembled from [AM], [Mask2, Theorem 6], [Mask1, Theorem 2] and [Ab, Theorem 3]; see also [Fl]. A more detailed proof is given in [AM]. Equivariant Peano curves are discussed in [Th2] and [CT]; [Min1] contains more general local connectivity results.

§5 Topology of rational maps. Thurston's characterization of critically finite rational maps appears in [Th4], [DH2]; it is compared to the geometrization of 3-manifolds in [Mc2]. The problem of enumerating critically finite *polynomials* is addressed in [BFH] and [Po]; [Rees2] studies the combinatorics of rational maps of degree two.

Sierpiński Julia sets and components of the space of expanding maps are discussed in [Mc1, §4] (which includes a version of Question 5.3), [Mil2], [Pil], [Mak] and [Pet]. Sierpiński's curve first appears in [Sie]; its uniqueness is proved in [Why]. For the role of trees in conformal dynamics, see [Mor2], [Shi2].

Tuning, mating and other surgery constructions can be found in [Dou1] and [Tan]. Combinatorics near a degree two Lattès example is developed in [Ber].

The landing of external rays is discussed in [Mil3, §18], [Hub] and [Mc4, §6]. A proof that C_d *cellular*, and therefore connected, appears in [Lav].

A polynomial Julia set of positive measure is constructed in [NS], and an example which is not ergodic on its Julia set appears in [St]. Positive measure sets of ergodic maps are constructed in [Rees1].

For local connectivity of geometrically finite Julia sets, see [TY].

§6 Renormalization. External angles for Julia sets and the Mandelbrot set are studied in [DH1], which includes a proof that local connectivity of M implies the density of expanding polynomials. A more recent treatment

appears in [Dou2]. See [Mil1] for the tuning operators and self-similarity of the Mandelbrot set.

Background on complex renormalization can be found in [Mc4]; for a historical collection including experimental work on physical systems, see [Cvi]. Work of Yoccoz on the local connectivity of $J(f)$ and M is presented in [Yoc], [Mil4] and [Hub]. These breakthroughs were foreshadowed by work of Branner and Hubbard on cubic polynomials with disconnected Julia sets [BH]. Theorem 6.5 on the measure of Julia sets appears in [Lyu2]. For Lyubich's results on infinitely renormalizable quadratic polynomials, see [Lyu1].

Sullivan's work on renormalization appears in [Sul6], and is expanded upon in [MeSt]. A parallel approach to renormalization and 3-manifolds which fiber over the circle is the subject of [Mc5]. Theorem 6.6 is implicit in Sullivan's work and stated explicitly in [Mc5, Thm 7.16].

The illustrations. Figure 2 and the left frame of Figure 5 were drawn using a computer program written by David Wright [MMW]. The remaining pictures were produced by the author. In Figure 6, the Kleinian group is generated by reflections through the sides of a regular stunted ideal tetrahedron with dihedral angles $2\pi/7$; the rational map is $f(z) \approx -0.695620(z - 1)^2(z + 2)/(3z - 2)$. The second frame of Figure 8 is a blowup around $c \approx -0.160 + 1.034i$, a superstable point of period 4. The parameters for Figure 9 are $s_1 = -1$, $s_2 \approx -0.22561 + 0.744862i$ and $s_1 * s_2 \approx -1.13000 + 0.24033i$.

Bibliography

[Ab]   W. Abikoff. On boundaries of Teichmüller spaces and on Kleinian groups: III. *Acta Math.* **134**(1975), 211–237.

[AM]   W. Abikoff and B. Maskit. Geometric decompositions of Kleinian groups. *Amer. J. Math* **99**(1977), 687–698.

[Ah]   L. Ahlfors. Finitely generated Kleinian groups. *Amer. J. of Math.* **86**(1964), 413–429.

[AM]   J. Anderson and B. Maskit. On the local connectivity of limit sets of Kleinian groups. *Preprint, 1995.*

[Ber]   J. Bernard. *Dynamique des perturbations d'un example de Lattès.* Thesis, Univ. Paris-Sud Orsay, 1994.

[Bers1] L. Bers. Inequalities for finitely generated Kleinian groups. *J. d'Analyse Math.* **18**(1967), 23–41.

[Bers2] L. Bers. On boundaries of Teichmüller spaces and on kleinian groups: I. *Annals of Math.* **91**(1970), 570–600.

[Bers3] L. Bers. On Sullivan's proof of the finiteness theorem and the eventual periodicity theorem. *Am. J. Math.* **109**(1987), 833–852.

[BFH]   Bielefeld, Y. Fisher, and J. Hubbard. The classification of critically preperiodic polynomials as dynamical systems. *J. AMS* **5**(1992), 721–762.

[BJ]   C. J. Bishop and P. W. Jones. Hausdorff dimension and Kleinian groups. *Preprint.*

[Bon]   F. Bonahon. Bouts des variétés hyperboliques de dimension 3. *Annals of Math.* **124**(1986), 71–158.

[BH]   B. Branner and J. H. Hubbard. The iteration of cubic polynomials, Part II: Patterns and parapatterns. *Acta Mathematica* **169**(1992), 229–325.

[Can]   R. D. Canary. Ends of hyperbolic 3-manifolds. *J. of the AMS* **6**(1993), 1–35.

[CT]   J. W. Cannon and W. P. Thurston. Group invariant Peano curves. *Preprint.*

[Cvi]   P. Cvitanović. *Universality in Chaos.* Adam Hilger Ltd, 1984.

[Dou1] A. Douady. Chirurgie sur les applications holomorphes. In *Proceedings of the International Congress of Mathematicians*, pages 724–738. American Math. Soc., 1986.

[Dou2] A. Douady. Descriptions of compact sets in C. In L. R. Goldberg and A. V. Phillips, editors, *Topological Methods in Modern Mathematics*, pages 429–465. Publish or Perish, Inc., 1993.

[DH1]   A. Douady and J. Hubbard. *Étude dynamique des polynômes complexes.* Pub. Math. d'Orsay, 1984.

[DH2]   A. Douady and J. Hubbard. A proof of Thurston's topological characterization of rational maps. *Acta Math.* **171**(1993), 263–297.

[Fatou] P. Fatou. Sur les équations fonctionnelles. *Bull. Sci. Math. France* **47-48**(1919-20), 161–271, 33–94, 208–314.

[Fl]   W. J. Floyd. Group completions and limit sets of Kleinian groups. *Invent. math.* **57**(1980), 205–218.

[Gr]   L. Greenberg. On a theorem of Ahlfors and conjugate subgroups of Kleinian groups. *Amer. J. Math* **89**(1967), 56–68.

[Hub]   J. H. Hubbard. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In L. R. Goldberg and A. V. Phillips, editors, *Topological Methods in Modern Mathematics*, pages 467–511. Publish or Perish, Inc., 1993.

[Julia] G. Julia. Mémoire sur l'itération des applications fonctionnelles. *J. Math. Pures et Appl.* **8**(1918), 47–245.

[KK]   L. Keen and J. Kotus. Dynamics of the family $\lambda \tan z$. *Preprint 1994.*

[Lav]   P. Lavaurs. *Systèmes dynamiques holomorphes: explosion de points périodiques paraboliques.* Thesis, Univ. Paris-Sud Orsay, 1989.

[Lyu1] M. Lyubich. Geometry of quadratic polynomials: Moduli, rigidity, and local connectivity. *Stony Brook IMS Preprint 1993/9.*

[Lyu2] M. Lyubich. On the Lebesgue measure of the Julia set of a quadratic polynomial. *Stony Brook IMS Preprint 1991/10.*

[LM]   M. Lyubich and Y. Minsky. Laminations in holomorphic dynamics. *Stony Brook IMS Preprint 1994/20.*

[Mak]   P. Makienko. Unbounded components in parameter space of rational maps. *ICTP Trieste Preprint IC/93-84.*

[MSS]   R. Mañé, P. Sad, and D. Sullivan. On the dynamics of rational maps. *Ann. Sci. Éc. Norm. Sup.* **16**(1983), 193–217.

[Mar]   A. Marden. The geometry of finitely generated kleinian groups. *Annals of Math.* **99**(1974), 383–462.

[Mask1]   B. Maskit. On boundaries of Teichmüller spaces and on kleinian groups: II. *Annals of Math.* **91**(1970), 607–639.

[Mask2]   B. Maskit. Intersection of component subgroups of Kleinian groups. In *Discontinuous Groups and Riemann Surfaces*, pages 349–367. Princeton Annals of Math Studies 79, 1974.

[Mc1]   C. McMullen. Automorphisms of rational maps. In *Holomorphic Functions and Moduli I*, pages 31–60. Springer-Verlag, 1988.

[Mc2]   C. McMullen. Rational maps and Kleinian groups. In *Proceedings of the International Congress of Mathematicians Kyoto 1990*, pages 889–900. Springer-Verlag, 1991.

[Mc3]   C. McMullen. Riemann surfaces and the geometrization of 3-manifolds. *Bull. AMS* **27**(1992), 207–216.

[Mc4]   C. McMullen. *Complex Dynamics and Renormalization.* Annals of Math Studies 135, Princeton University Press, 1994.

[Mc5]   C. McMullen. *Renormalization and 3-Manifolds which Fiber over the Circle.* To appear, Annals of Math. Studies, Princeton University Press.

[McS]   C. McMullen and D. Sullivan. Quasiconformal homeomorphisms and dynamics III: The Teichmüller space of a holomorphic dynamical system. *Preprint.*

[MeSt]   W. de Melo and S. van Strien. *One-Dimensional Dynamics.* Springer-Verlag, 1993.

[Mil1]   J. Milnor. Self-similarity and hairiness in the Mandelbrot set. In M. C. Tangora, editor, *Computers in Geometry and Topology*, pages 211–259. Lect. Notes Pure Appl. Math., Dekker, 1989.

[Mil2]   J. Milnor. Geometry and dynamics of quadratic rational maps. *Experimental Math.* **2**(1993), 37–83.

[Mil3] J. Milnor. Dynamics in one complex variable: Introductory lectures. *Stony Brook IMS Preprint 1990/5.*

[Mil4] J. Milnor. Local connectivity of Julia sets: Expository lectures. *Stony Brook IMS Preprint 1992/11.*

[Min1] Y. Minsky. On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds. *J. of the AMS* **7**(1994), 539–588.

[Min2] Y. Minsky. The classification of punctured torus groups. *Preprint.*

[Mor1] J. Morgan. On Thurston's uniformization theorem for three dimensional manifolds. In *The Smith Conjecture*, pages 37–125. Academic Press, 1984.

[Mor2] J. Morgan. Trees and hyperbolic geometry. In *Proceedings of the International Congress of Mathematicians Berkeley 1986*, pages 590–597. American Math. Soc., 1987.

[MMW] D. Mumford, C. McMullen, and D. Wright. Limit sets of free two-generator kleinian groups. *Preprint.*

[NS] T. Nowicki and S. van Strien. Polynomial maps with a Julia set of positive Lebesgue measure: Fibonacci maps. *Preprint, 1993.*

[Oh] K. Ohshika. Ending laminations and boundaries for deformation spaces of Kleinian groups. *J. London Math. Soc.* **42**(1990), 111–121.

[Pet] C. L. Petersen. No elliptic limits for quadratic maps. *Preprint, MAT-Report No. 1994-45.*

[Pil] K. Pilgrim. *Cylinders for Iterated Rational Maps.* Thesis, University of California, Berkeley, 1994.

[Poin] H. Poincaré. Mémoire sur les groupes kleinéens. *Acta Math.* **3**(1883), 49–92.

[Po] A. Poirier. On post critically finite polynomials, Part two: Hubbard trees. *Preprint Stony Brook IMS 1993/5.*

[Rees1] M. Rees. Positive measure sets of ergodic rational maps. *Ann. scient. Éc. Norm. Sup.* **19**(1986), 383–407.

[Rees2] M. Rees. A partial description of parameter space of rational maps of degree two: Part I. *Acta Math.* **168**(1992), 11–87.

[Rol] D. Rolfsen. *Knots and Links.* Publish or Perish, 1976.

[Shi1] M. Shishikura. On the quasiconformal surgery of rational functions. *Ann. Sci. Éc. Norm. Sup.* **20**(1987), 1–30.

[Shi2]   M. Shishikura. Trees associated with the configuration of Herman
          rings. *Ergod. Th. & Dynam. Sys* **9**(1989), 543–560.

[Sie]   W. Sierpiński. Sur une courbe cantorienne qui contient une image
          biunivoque et continue de toute courbe donnée. In *Oeuvres Choisies,*
          t. II, pages 107–119. PWN – Éditions sci. de Pologne, 1975.

[St]   S. van Strien. Transitive rational maps which are not ergodic. *To*
          appear, Ergod. Th. & Dynam. Sys.

[Sul1]   D. Sullivan. Growth of positive harmonic functions and Kleinian
          group limit sets of zero planar measure and Hausdorff dimension
          two. In *Geometry at Utrecht*, pages 127–144. Springer-Verlag Lec-
          ture Notes in Math. 894, 1981.

[Sul2]   D. Sullivan. Conformal dynamical systems. In *Geometric Dynamics*,
          pages 725–752. Springer-Verlag Lecture Notes No. 1007, 1983.

[Sul3]   D. Sullivan. Entropy, Hausdorff measures new and old and limit
          sets of geometrically finite Kleinian groups. *Acta Math.* **153**(1984),
          259–277.

[Sul4]   D. Sullivan. Quasiconformal homeomorphisms and dynamics I: So-
          lution of the Fatou-Julia problem on wandering domains. *Annals of*
          Math. **122**(1985), 401–418.

[Sul5]   D. Sullivan. Quasiconformal homeomorphisms and dynamics II:
          Structural stability implies hyperbolicity for Kleinian groups. *Acta*
          Math. **155**(1985), 243–260.

[Sul6]   D. Sullivan. Bounds, quadratic differentials and renormalization
          conjectures. In F. Browder, editor, *Mathematics into the Twenty-*
          first Century: 1988 Centennial Symposium, August 8-12, pages 417–
          466. Amer. Math. Soc., 1992.

[Tan]   Tan L. Matings of quadratic polynomials. *Ergod. Th. & Dynam.*
          Sys **12**(1992), 589–620.

[TY]   Tan L. and Yin Y. Local connectivity of the Julia set for geometri-
          cally finite rational maps. *ENS Lyon Preprint UMPA-94-121.*

[Th1]   W. P. Thurston. *Geometry and Topology of Three-Manifolds.*
          Princeton lecture notes, 1979.

[Th2]   W. P. Thurston. Three-dimensional manifolds, Kleinian groups and
          hyperbolic geometry. *Bull. AMS* **6**(1982), 357–381.

[Th3]   W. P. Thurston. Hyperbolic structures on 3-manifolds I: Deforma-
          tions of acylindrical manifolds. *Annals of Math.* **124**(1986), 203–246.

[Th4]   W. P. Thurston. On the combinatorics and dynamics of iterated rational maps. *Preprint.*

[Tu]   P. Tukia. The Hausdorff dimension of the limit set of a geometrically finite Kleinian group. *Acta Math.* **152**(1984), 127–140.

[Why]   G. T. Whyburn. Topological characterization of the Sierpiński curve. *Fund. Math.* **45**(1958), 320–324.

[Yoc]   J.-C. Yoccoz. Sur la connexité locale des ensembles de Julia et du lieu de connexité des polynômes. *In preparation.*