Quantum cohomology and its associativity

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1 Introduction

The notion of quantum cohomology, which was first proposed in Witten's study of two dimensional nonlinear sigma models [W1], also see [Va], plays a fundamental role in understanding the phenomenon of mirror symmetry for Calabi-Yau manifolds. This phenomenon was first observed by physicists, who were motivated by their knowledge of topological field theory.

A topological sigma model theory starts with correlation functions. These are supposed to be computable as intersection numbers of cycles in the moduli space of holomorphic maps from Riemann surfaces to arbitrary manifolds. The key axiom of a topological field theory is the composition law for correlation functions. Basically, the composition law governs how correlation functions change under surgery of Riemann surfaces, such as gluing several Riemann surfaces together. One of its striking consequences, pointed out by Witten, is the associativity equation for quantum multiplications.

In last few years, there has been rapid progress on the mathematical aspects of topological field theory. A mathematical theory of quantum cohomology has been established. Solutions of the associativity (WDVV) equation have been constructed by using solutions of certain Cauchy-Riemann equations. Many applications of the associativity equation have been found in the study of integrable systems, algebraic geometry, etc.. In these notes, we will expose some of these recent developments, with an emphasis on the geometric aspects of quantum cohomology.

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The first step in the mathematical approach to topological field theory, is to define appropriate symplectic invariants which are the mathematical counterparts of correlation functions in Witten's topological sigma model (cf. [W1], [W2]). In the next section, we will review the construction of these invariants. Our method of defining these invariants is to show that for certain almost complex structure on a symplectic manifold V, an intersection theory on moduli spaces of generalized holomorphic maps can be built up.

To define the correlation functions, Witten constructed an appropriate topological sigma model Lagrangian and studied corresponding Feynman path integrals. At the moment, the latter do not exist in the rigorous mathematical sense. The goal of his physical discussions is to construct quantum field theory, at least formally. Witten also gave a formal reduction of the computation of the Feynman path integral to intersection theory on moduli spaces which are compact and smooth.

However, as Witten pointed out in [W2], his arguments for the reduction break down when the behavior of the moduli spaces is bad. In fact, moduli spaces of holomorphic maps are often noncompact and there are indeed contributions from the infinity, to the symplectic invariants defined in [RT1], [RT2] of the moduli spaces of holomorphic maps. Moreover, it is unclear how the symplectic structure of V is used in physical arguments. Since a sequence of holomorphic maps may not have any limit map on a general almost complex manifold, the role of symplectic structure seems to be essential. From the mathematical standpoint, it is still a very interesting problem how to define any useful invariants for general almost complex manifolds.

An algebraic geometer may be tempted to define the invariants directly by counting genuine genus g holomorphic curves in an algebraic manifold. As we will see, invariants defined in this way do not satisfy the composition law predicted by the physicists. Therefore, they are not the same as the correlation functions used by Witten. It was shown for the first time in [RT1], that the right invariants can be defined in terms of solutions of inhomogeneous Cauchy-Riemann equations. These invariants do indeed satisfy the composition law. The construction in [RT1] also provides ample examples of topological sigma models.

There are also purely mathematical motivations for a theory of symplectic invariants and quantum cohomology. Symplectic invariants (i.e., Gromov-Witten invariants) can be used to distinguish symplectic topology on differential manifolds. The problem of whether two symplectic manifolds of different shape are symplectically equivalent was first studied by Gromov in his seminal paper [Gr], and subsequently by Mcduff, Ruan, et. al. ([MS], [R1]). More recently, thanks to the new Seiberg-Witten invariants and the work of C. Taubes, many important results have been obtained on the structure of 4-dimensional symplectic manifolds. In particular, Taubes has proved that CP^2 has a unique symplectic structure.

Floer cohomology was introduced by A. Floer, in the course of his investigation on fixed points of symplectic diffeomorphisms or periodic orbits of

Hamiltonian systems. By the work of Floer, Hofer and Salamon, Ono, on any weakly monotone symplectic manifold, the number of fixed points for a nondegenerate symplectic diffeomorphism is not less than the total Betti number (cf. [F], [HS], [On]). This solves the Arnold conjecture for nondegenerate symplectic diffeomorphisms on weakly monotone manifolds.

For degenerate symplectic diffeomorphisms, the Arnold conjecture is largely open. It is expected that multiplicative structures on the Floer cohomology can be used to study the degenerate case (cf. [Ho], [OV]). As we will see, it turns out that the ring structure on the Floer cohomology is the same as the quantum ring (cf. section 9).

On algebraic manifolds, the GW-invariants generalize classical Schubert invariants. The composition law makes it possible to enumerate many Schubert invariants ([KM], [RT1]). Moreover, solutions of the associativity equation yield new Hodge bundles, and also produce a family of quantum ring structures. New geometric information concerning variation of Hodge structure can be obtained, via quantum cohomology and Mirror symmetry. For instance, the work of Candelas, et. al. ([COGP]) indicates that the GW-invariants on a Calabi-Yau can be computed via the variation of Hodge structures on its mirror manifold.

The organization of these notes is as follows:

In Section 3, we compute GW-invariants in several important cases. We give an application of the symplectic invariants to a problem on symplectic deformation equivalence, which was first studied by Ruan. We will also discuss briefly a very recent and remarkable result of C. Taubes.

In Section 4, we prove that our symplectic invariants satisfy the fundamental axiom – composition law.

The quantum cohomology ring structure is constructed on symplectic manifolds in Section 5. A few important examples are discussed and a quantized version of the Hard Lefschetz Theorem is formulated.

In Section 6, we discuss the WDVV equation and its relation to isomonodromic deformation, or equivalently Riemann-Hilbert problem. Formal series solutions of the WDVV equation are constructed on symplectic manifolds by using rational holomorphic maps.

Section 7 contains some applications of the associativity to enumerative geometry. In particular, we are able to show that the formal solutions of the WDVV equation constructed in Section 6 are indeed convergent in some region for a large class of Fano manifolds.

In section 8, we formulate a conjecture of Witten in terms of GW-invariants.

In section 9, we discuss the equivalence of the quantum cohomology ring and the exterior multiplication on the symplectic Floer cohomology.

The proof of the composition law is based on the technique of degeneration of domains for holomorphic maps. It is also natural to ask what would happen if we degenerate the target. It turns out that in this case, one can also obtain nice degeneration formulas, once one understands the combinatorial structure of maps in symplectic varieties with normal crossings. These are discussed in Section 10. There, we also give an application of such a degeneration formula to the convergence of formal soultions of the WDVV on Calabi-Yau manifolds.

In Section 11, we discuss the Mirror Symmetry Conjecture.

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Contents

2 The GW-invariants

In this section, we define the symplectic invariants of [RT1] and [RT2]. We will call them GW-invariants. Special cases of these invariants were first studied by Ruan in [R1], [R2] (also see [Gr], [MS]). More precisely, he defined two very important cases of these invariants when either the domain is S^2, or the target is a 4-dimensional symplectic manifold. The physical counterparts of the invariants are Witten's sigma model correlation functions (cf. [W1], [W2]). We will also discuss a few basic properties of the latter invariants and compute them in some cases.

Let (V, ω) be a compact symplectic manifold, $A \in H_2(V, Z)$ and J be an almost complex structure on V satisfying: $\omega(v, Jv) > 0$ for any nonzero tangent vector v, i.e., J is ω-tamed according to Gromov. As usual, $\mathcal{M}_{g,k}$ denotes the moduli space of genus g Riemann surfaces with k-marked points $(2g + k \geq 3)$ and $\overline{\mathcal{M}}_{g,k}$ denotes its Deligne-Mumford compactification. The universal family of curves $\pi : \overline{\mathcal{U}}_{g,k} \to \overline{\mathcal{M}}_{g,k}$ is a projective algebraic variety. We put $\mathcal{U}_{g,k} = \pi^{-1}(\mathcal{M}_{g,k})$. We fix an embedding $\overline{\mathcal{U}}_{g,k} \to \mathcal{P}^N$. Roughly speaking, an inhomongeneous term ν is an anti-J-linear section of $Hom(\pi_1^* T\mathcal{P}^N, \pi_2^* TV)$ over $\mathcal{P}^N \times V$, where π_i is the projection from $\mathcal{P}^N \times V$ to its i-th factor.

A marked Riemann surface in $\mathcal{M}_{g,k}$ will be denoted by $(\Sigma, x_1, \cdots, x_k)$, or simply $(\Sigma, \{x_i\})$ when there is no ambiquity. A (J,ν)-map from Σ is a continuous map $f : \Sigma \to V$, which is smooth on the regular part $\mathrm{Reg}(\Sigma)$ of Σ, satisfying the following inhomogeneous Cauchy-Riemann equation:

$$J \cdot df(x) + df \cdot j_\Sigma(x) = \nu(\phi(x), f(x)), \ x \in \mathrm{Reg}(\Sigma) \tag{2.1}$$

where j_Σ is the complex structure on Σ. In particular, if $\nu = 0$, we say that f is J-holomorphic . We put

$$\begin{aligned} \mathcal{M}_A(g,k,J,\nu) = \{(\Sigma, \{x_i\}, f) \mid (\Sigma, \{x_i\}) \in \mathrm{Reg}(\mathcal{M}_{g,k}), \ f : \Sigma \mapsto V \\ \text{is a } (J, v)\text{-map}, \ f_*[\Sigma] = A\} \end{aligned} \tag{2.2}$$

where $\mathrm{Reg}(\mathcal{M}_{g,k})$ denotes the regular part of $\mathcal{M}_{g,k}$.

Proposition 2.1. (cf. [RT2]) *For a generic pair (J,ν), the moduli space $\mathcal{M}_A(g,k,J,\nu)$ is an oriented smooth manifold of dimension* $2c_1(V)(A)+2(3-n)(g-1)+2k$.

The proof of this is standard and follows from an application of the Sard-Smale Transversality Theorem. Such a technique was used before by many people, notably, S. Donaldson, A. Floer, D. Mcduff, C. Taubes, K. Uhlenbeck, et. al..

By the Gromov-Uhlenbeck compactness theorem ([PW], [Ye] and [RT1, Proposition 3.1]), for any sequence $\{(\Sigma_i, x_{i1}, \cdots, x_{ik}, f_i)\}$, by taking a subsequence if necessary, we may assume that $(\Sigma_i, x_{i1}, \cdots, x_{ik})$ converges to some $(\Sigma_\infty, \cdots)$ in $\overline{\mathcal{M}}_{g,k}$, and f_i converges to a map f_∞ into V satisfying: (1) the domain Σ'_∞ of f_∞ is connected, and a union of Σ_∞ with some bubble trees of S^2's; (2) $f_\infty|_{\Sigma_\infty}$ is a (J,ν)-map and f_∞ restricts to J-holomorphic maps on bubble trees. Let $\{x_{\infty 1}, \cdots, x_{\infty k}\}$ be the limit in Σ'_∞ of the sequence $\{x_{i1}, \cdots, x_{ik}\}$. The limiting points are distinct. Some of them may lie in bubble trees instead of Σ_∞. Let G be the subgroup consisting of the automorphisms of the bubble trees which fix bubble components, all marked points $x_{\infty 1}, \cdots, x_{\infty k}$ and all singular points in Σ'_∞. It is easy to see that G is trivial if there are no bubbles. We say that two limits $(\Sigma'_\infty, x_{\infty 1}, \cdots, x_{\infty k}, f_\infty)$ and $(\tilde{\Sigma}'_\infty, \tilde{x}_{\infty 1}, \cdots, \tilde{x}_{\infty k}, \tilde{f}_\infty)$ are equivalent, if (1) $(\Sigma'_\infty, x_{\infty 1}, \cdots, x_{\infty k})$ is equal to $(\tilde{\Sigma}'_\infty, \tilde{x}_{\infty 1}, \cdots, \tilde{x}_{\infty k})$; (2) there is a $\sigma \in G$ such that $\tilde{f}_\infty \cdot \sigma = f_\infty$. Let $\overline{\mathcal{M}}_A(g,k,J,\nu)$ be the set of the limits of sequences in $\mathcal{M}_A(g,k,J,\nu)$ modulo the above equivalence. Then $\overline{\mathcal{M}}_A(g,k,J,\nu)$ is compact.

Consider the natural evaluation map

$$\begin{aligned} e : \mathcal{M}_A(g,k,J,\nu) \to \mathcal{M}_{g,k} \times V^k \\ e((\Sigma, x_1, \cdots, x_k, f)) = (\Sigma, x_1, \cdots, x_k) \times (f(x_1), \cdots, f(x_k)) \end{aligned} \tag{2.3}$$

This can be easily extended to a map, still denoted by e, from the compactification $\overline{\mathcal{M}}_A(g,k,J,\nu)$ into $\overline{\mathcal{M}}_{g,k} \times V^k$. To define our invariants, we need to prove that $e\left(\overline{\mathcal{M}}_A(g,k,J,\nu)\backslash\mathcal{M}_A(g,k,J,\nu)\right)$ is of smaller dimension in V^k.

Proposition 2.2. (cf. [RT2]) *Assume that (V,ω) is semi-positive. Then for a generic pair (J,ν), the image $e\left(\overline{\mathcal{M}}_A(g,k,J,\nu)\backslash\mathcal{M}_A(g,k,J,\nu)\right)$ is a set in $\overline{\mathcal{M}}_{g,k} \times V^k$ of dimension no more than $2c_1(V)(A) + 2(3-n)(g-1) + 2k - 2$.*

We say that (V,ω) is semi-positive if for a ω-tamed almost complex strcture J on V, there is no J-holomorphic map $f : S^2 \mapsto V$ satisfying: $3 - n \leq \int_{S^2} f^* c_1(V) < 0$, where $c_1(V)$ denotes the first Chern class of (V,ω). Clearly, this condition of semi-positivity holds for any symplectic manifolds of complex dimension ≤ 3. Furthermore, any algebraic manifolds with numerically positive anticanonical bundle, for instance, Calabi-Yau manifolds, are semi-positive.

As a matter of fact, the semi-positivity in Proposition 2.2 can be removed. This can be accomplished by generalizing a theorem of E.Bishop to symplectic manifolds. The Bishop theorem says that an analytic set in a projective space can be compactified by adding a set of codimension at least two, if the induced metric has finite volume.

Proof of Proposition 2.2: The basic idea of its proof is to stratify

$$\overline{\mathcal{M}}_A(g,k,J,\nu)\backslash\mathcal{M}_A(g,k,J,\nu)$$

according to the topology of the domain of limiting maps and the configuration of marked points. Let $\mathcal{M}_A(D,J,\nu)$ be one stratum. Then we first drop the multiplicity of bubbles and obtain a moduli space $\mathcal{M}_{A'}(D,J,\nu)$. By semi-positivity, we have $c_1(V)(A') \leq c_1(V)(A)$. For a generic (J,ν), by using the transversality arguments, one can show that $\dim(\mathcal{M}_{A'}(D,J,\nu))$ is no more than $2c_1(V)(A) + 2(3-n)(g-1) + 2k$, in fact, strictly smaller if the domain is in $\overline{\mathcal{M}}_{g,k}\backslash\mathcal{M}_{g,k}$. However, if the domain is not in $\overline{\mathcal{M}}_{g,k}\backslash\mathcal{M}_{g,k}$, then the parametrization group G is of dimension at least two, so the image $e(\mathcal{M}_{A'}(D,J,\nu))$ has codimension at least two. The proposition follows.

Now we can define the GW-invariant

$$\Psi^V_{(A,g,k)} : H_*(\overline{\mathcal{M}}_{g,k}, \mathbf{Q}) \times H_*(V, \mathbf{Z})^k \mapsto \mathbf{Q}$$

as follows: let $\alpha_1, \cdots, \alpha_k$ be any homology classes in $H_*(V,\mathbf{Z})$ and $[K]$ be a rational homology class in $H_*(\overline{\mathcal{M}}_{g,k}, \mathbf{Q})$. If

$$\operatorname{codim}([K]) + \sum_i \operatorname{codim}(\alpha_i) = 2c_1(V)(A) + 2(3-n)(g-1) + k, \qquad (2.4)$$

we choose a pseudo-submanifold representative, denoted by K, of $[K]$ such that its top strata lie in $\operatorname{Reg}(\mathcal{M}_{g,k})$. By (2.4) and Proposition 2.2, for generic pseudo-submanifolds $S_i : P_i \mapsto V$ representing α_i, $K \times \prod_i S_i(P_i)$

intersects $e(\overline{M}_A(g,k,J,\nu))$ transversally at finitely many smooth points of $\mathcal{M}_A(g,k,J,\nu)$. By Proposition 2.1, we can assign a sign to each of these intersection points. We define $\Psi^V_{(A,g,k)}([K];\alpha_1,\cdots\alpha_k)$ to be the algebraic sum of these points. If (2.4) does not hold, we simply define

$$\Psi^V_{(A,g,k)}([K];\alpha_1,\cdots,\alpha_k)=0.$$

For simplicity, we sometimes abbreviate this as $\Psi^V_{(A,g,k)}([K];\{\alpha_i\})$. Also we often write $\Psi^V_{(A,g,k)}(\{\alpha_i\})$ for $\Psi^V_{(A,g,k)}([K];\{\alpha_i\})$ in the case that K is the whole space $\overline{\mathcal{M}}_{g,k}$.

Theorem 2.3. ([RT2]) *Assume that (V,ω) is a compact semi-positive symplectic manifold. Then $\Psi^V_{(A,g,k)}$ is a symplectic invariant. In particular, it is independent of choices of (J,ν).*

This can be proved by constructing cobordisms. The basic techniques in the proof are versions of Proposition 2.1, 2.2 for cobordisms. For instance, let (J',ν') be another pair with properties stated in Proposition 2.2, 2.3. Then by using identical arguments in the proof of Proposition 2.2, 2.3, for a generic path $\{(J_t,\nu_t)\}$ from (J,ν) to (J',ν'), one can show the following:
(1) the set $\bigcup_t e(M_A(g,k,J_t,\nu_t))$ is a smooth submanifold in V^k with boundary

$$e(M_A(g,k,J,\nu))\cup(-e(M_A(g,k,J',\nu')));$$

(2) $\bigcup_t e(\overline{M}_A(g,k,J_t,\nu_t))$ is a compactification of $\bigcup_t e(M_A(g,k,J_t,\nu_t))$;
(3) $\bigcup_t e((\overline{M}_A(g,k,J_t,\nu_t)\backslash M_A(g,k,J_t,\nu_t))$ is of codimension at least two.

Hence, $K\times\prod_i S_i(P_i)$ has the same intersection number with both $e(\overline{M}_A(g,k,J,\nu))$ and $e(\overline{M}_A(g,k,J',\nu'))$ in V^k. It follows that

$$\Psi^V_{(A,g,k)}([K];\{\alpha_i\})$$

is independent of choices of (J,ν).

More recently, J. Li and the author can construct the GW-invariants for general symplectic manifolds.

It is possible to define the invariant $\Psi^V_{(A,g,k)}$ by using differential forms. The difficulty is to prove that the compactification $\overline{\mathcal{M}}_A(g,k,J,\nu)$ carries a fundamental class. This requires understanding how the strata in $\overline{\mathcal{M}}_A(g,k,J,\nu)$ are attached to each other. Analytically, it amounts to a delicate deformation theory near singular curves. It is proved in [RT3] that $\mathcal{M}_A(0,k,J,\nu)$ carries a fundamental class. Presumably, the proof can be extended to arbitary genus. Assuming that this is done, we should be able to prove

$$\begin{aligned}&\Psi^V_{(A,g,k)}([K];\alpha_1,\cdots,\alpha_k)\\&=\int_{\overline{\mathcal{M}}_A(g,k,J,\nu)}e^*(\pi_1^*PD([K])\wedge\pi_2^*\alpha_1^*\wedge\cdots\wedge\pi_{k+1}^*\alpha_k^*),\end{aligned}\tag{2.5}$$

where $PD([K])$ (resp. α_i^*) denotes the Poincare dual of $[K]$ (resp. α_i) in $\overline{\mathcal{M}}_{g,k}$ (resp. V) as usual, and π_j is the projection from $\overline{\mathcal{M}}_{g,k} \times V^k$ onto its j^{th}-factor.

Here are some basic properties of Ψ:

(1) $\Psi^V_{(A,g,k)} = 0$, *if either* $\omega(A) < 0$ *or* $c_1(V)(A) + (n-3)(1-g) < 0$, *in particular,* $\Psi_{(0,g,k)} = 0$ *for any* $g \geq 2$ *and* $n \geq 4$.

(2) $\Psi^V_{(A,g,k)}$ *is multilinear and supersymmetric on* $H_*(V, Z)^k$ *with respect to the* Z_2*-grading by even and odd degrees.*

(3) $\Psi^V_{(A,g,k)}$ *takes integer classes in* $H_*(\overline{\mathcal{M}}_{g,k}, Z) \times H_*(V, Z)^k$ *to integers, where the integer classes are those represented by cycles whose top strata lie in* $\operatorname{Reg}(\mathcal{M}_{g,k})$.

There is a natural map $\pi_k : \overline{\mathcal{M}}_{g,k} \mapsto \overline{\mathcal{M}}_{g,k-1}$: for $C = (\Sigma, x_1, \cdots, x_k)$ in $\overline{\mathcal{M}}_{g,k}$, if x_k is not in any rational components of Σ which contain only three distinguished points, then we define

$$\pi_k(C) = (\Sigma, x_1, \cdots, x_{k-1})$$

Note that a distinguished point of Σ is either a singular point or a marked point. If x_k is in one of the above rational components, we contract this component and obtain a stable curve C' in $\overline{\mathcal{M}}_{g,k-1}$, and define $\pi_k(C) = C'$. Clearly, π_k is continuous.

Then it follows easily from (2.5)

(4) *For any* $\alpha_1, \cdots, \alpha_{k-1}$ *in* $H_*(V, Z)$, *we have*

$$\begin{aligned}&\Psi^V_{(A,g,k)}([K]; \alpha_1, \cdots, \alpha_{k-1}, [V]) \\ =&\Psi^V_{(A,g,k-1)}(\pi_{k*}([K]); \alpha_1, \cdots, \alpha_{k-1})\end{aligned} \tag{2.6}$$

(5) *Let* α_n *be in* $H_{2n-2}(V, Z)$ *and* $[K]$ *in* $H_*(\overline{\mathcal{M}}_{g,k-1}, Q)$, *then*

$$\begin{aligned}&\Psi^V_{(A,g,k)}\left(\left[\pi_k^{-1}(K)\right]; \alpha_1, \cdots, \alpha_{k-1}, \alpha_k\right) \\ =&\hat{\alpha_k}(A)\Psi^V_{(A,g,k-1)}([K]; \alpha_1, \cdots, \alpha_{k-1})\end{aligned} \tag{2.7}$$

where $\hat{\alpha}_k$ *is the Poincare dual of* α_k.

It follows from (4) that $\Psi^V_{(A,0,3)}(\alpha_1, \alpha_2, [V])$ is equal to zero if $A \neq 0$ and the cap product $\alpha_1 \cap \alpha_2$ if $A = 0$.

These properties are formulated in [KM] as axioms. They (except (3)) are essentially stated in [W2] for the correlation functions in Witten's topological sigma model theory. Our proof of these properties is inspired by Witten's arguments in [W2].

Remark 2.4: The mixed invariant $\Phi_{(A,\omega,g)}$ in [RT1] can be identified with certain Ψ by choosing appropriate cycles $[K]$. More precisely, for any $k, l \geq 0$

with $2g + k \geq 3$, put $[K_{k,l}]$ to be the closure of the cycle $K^0_{k,l}$ in $\overline{\mathcal{M}}_{g,k+l}$, where $K^0_{k,l}$ is the set of all $(\Sigma, x_1, \cdots, x_{k+l})$ in $\mathcal{M}_{g,k+l}$ with $(\Sigma, x_1, \cdots, x_k)$ being a fixed point in $\mathcal{M}_{g,k}$. Then

$$\Psi^V_{(A,g,k+l)}([K_{k,l}]; \alpha_1, \cdots, \alpha_{k+l}) = \Phi_{(A,\omega,g)}(\alpha_1, \cdots, \alpha_k | \alpha_{k+1}, \cdots, \alpha_{k+l})$$

It follows from (4): $\Phi_{(A,\omega,g)}(\alpha_1, \cdots | \cdots, \alpha_{k+l})$ is 0 if $\dim(\alpha_{k+l}) > 2n - 2$ (cf. [RT1]).

To compute the GW-invariant, it is often more convenient to use holomorphic maps with respect to a particular complex structure, instead of (J, ν)-maps for a generic (J, ν) as in the definition. For instance, in case V is algebraic, a natural question is if Ψ can be defined in terms of holomorphic maps, i.e., Ψ is a classical enumerative invariant. The following theorem provides a partial answer to this question.

Theorem 2.5. *If V is a complex homogeneous manifold, then the genus zero GW-invariant $\Psi^V_{(A,0,k)}$ is enumerative, i.e., it can be defined to be the number of rational curves through given subvarieties in general position.*

This is proved in [RT1] for Grassmannian manifolds (also see [Ber]), and in [LT1] for general homogeneous manifolds. Also see [CF] for defining the invariants algebraically for flag manifolds. Since the tangent bundle is numerically positive for homogeneous manifolds, the moduli space of rational curves has expected dimension and its compactification is reasonably good. So one can use holomorphic maps to define $\Psi_{(A,0,k)}$. It was also stated in [KM] that the GW-invariants can be defined algebraically if the tangent bundle is positive in the suitable sense. In general, the genus zero GW-invariant $\Psi^V_{(A,0,k)}$ should be equal to the number of rational curves with homology class A, whenever the corresponding moduli space of rational curves has expected dimension. The latter should be true for any Fano manifold V and sufficiently large $c_1(V)(A)$.

It follows from Theorem 2.5 that if $V = CP^n$, $\Psi_{([\ell],0,3)}(H^{l_1}, H^{l_2}, H^{l_3}) = 1$ whenever $l_1 + l_2 + l_3 = 2n + 1$, where ℓ is a line in CP^n and H denotes the hyperplane class. Moreover, if $V = CP^2$, $\Psi^V_{(d[\ell],0,3d-1)}(H^2, \cdots, H^2)$ is the number of rational curves through $3d - 1$ points in general position.

Very little is known for higher genus invariants even in case of homogeneous manifolds. However, in [BDW], Bertram, Daskalopoulos and Wentworth defined invariants of higher genus on Grassmannian manifolds in terms of quotient sheaves. Later, Bertram proved that those invariants satisfy trace formulas [Ber]. It turns out that these trace formulas are identical to the Intrilligate-Vafa formulas for the GW-invariants, which were proved by B. Siebert and myself in [ST1]. Therefore, their invariants are the same as the GW-invariants $\Psi^V_{(A,g,k)}([pt]; \cdots)$, where pt denotes a point in $\mathcal{M}_{g,k}$.

Given a general algebraic manifold V, the moduli space $\mathcal{M}_A(g, k, J, 0)$ may not have expected dimension and its compactification may behave very badly. Therefore, one needs to find a subvariety (virtual moduli cycle) in $\overline{\mathcal{M}}_A(g,k,J,0)$, which plays the role of the moduli space for a generic (J,ν). A naive approach can be described as follows: let $\overline{\mathcal{U}}_{g,k}$ be the universal family of stable curves, and $\overline{\mathcal{U}}_A(g,k,J,0)$ be the fiber product of $\overline{\mathcal{M}}_A(g,k,J,0)$ and $\overline{\mathcal{U}}_{g,k}$ over $\overline{\mathcal{M}}_{g,k}$. Then there is a natural evaluation map e from $\overline{\mathcal{U}}_A(g,k,J,0)$ into V. Put $E = e^*TV$ and take its first direct image $\mathcal{R}^1\pi_{1*}E$ over $\overline{\mathcal{M}}_A(g,k,J,0)$, whose fiber over $(\Sigma,\cdots;f)$ is $H^1(\Sigma, f^*TV)$. The following is expected.

Assume that $\overline{\mathcal{M}}_A(g,k,J,0)$ is a d-dimensional Kähler orbifold, and $\mathcal{R}^1\pi_{1*}E$ is an orbifold bundle of rank r outside a subvariety of codimension greater than $r = d - c_1(V)(A) - (3-n)(g-1) - k$. Then

$$\Psi^V_{(A,g,k)}([K];\alpha_1,\cdots,\alpha_k) = \int_{\overline{\mathcal{M}}_A(g,k,J,\nu)} c_r(\mathcal{R}^1\pi_{1*}E)\wedge e^*(\pi_1^*PD([K])\wedge\pi_2^*\alpha_1^*\wedge\cdots\wedge\pi_{k+1}^*\alpha_k^*) \quad (2.8)$$

where e, π_j are as in (2.5), *and $c_r(\mathcal{R}^1\pi_{1*}E)$ is the orbifold Chern class.*

This means that $c_r(\mathcal{R}^1\pi_{1*}E)$ is Poincare dual to a subvariety in $\overline{\mathcal{M}}_A(g,k,J,0)$, which plays the role of the moduli space for a generic almost complex structure. In general, there are serious difficulties with this approach because $\overline{\mathcal{M}}_A(g,k,J,0)$ may have very bad singularities and the push-forward $\mathcal{R}^1\pi_{1*}E$ may not be a bundle. For instance, the rank of $\mathcal{R}^1\pi_{1*}E$ may be greater than r, the difference between $\dim\mathcal{M}_A(g,k,J,0)$ and the expected dimension. However, it is expected that an analog of intersection theory using normal cones (cf. [Fu]) can be developed to define the GW-invariant purely algebraically. Recently, Jun Li and I have constructed GW-invariants for any algebaric manifolds ([LT2]).

It follows from (2.8) that $\Psi^V_{(0,1,1)}([pt];[V])$ is equal to the Euler number of V. In a similar way, one can also compute $\Psi^V_{(0,1,1)}([\overline{\mathcal{M}}_{1,1}];\cdot)$ and $\Psi^V_{(0,g,0)}$ for $g\geq 2$ and $n\leq 3$.

3 More Examples and Applications

In this section, we compute the GW-invariants for some manifolds and give an application to a deformation equivalence problem.

Let $p: W\mapsto Z$ be a family of symplectic manifolds (V_z,ω_z), where Z is connected and $V_z = p^{-1}(z)$. Then the fundamental group $\pi_1(Z)$ induces a monodromy group Γ which acts on $H_*(V,\mathbf{Z})$, where $V = V_z$ for some z. The following can be easily proved by using the definition.

Theorem 3.1. *For any $\sigma\in\Gamma$, we have*

$$\Psi^V_{(\sigma(A),g,k)}([K];\sigma(\alpha_1),\cdots,\sigma(\alpha_k)) = \Psi^V_{(A,g,k)}([K];\alpha_1,\cdots,\alpha_k) \quad (3.1)$$

There is a useful application of this to Lefschetz pencils of algebraic varieties. Let $W\subset CP^N$ be an algebraic submanifold and V be a generic hyperplane section in W. The second Lefschetz theorem (cf. [AF]) says that the kernel $\mathrm{Ker}\,(H_n(V,\boldsymbol{Z}) \mapsto H_n(W,\boldsymbol{Z}))$, where $n = \dim V$, is generated by vanishing cycles e_i. One may arrange these vanishing cycles such that

(1) $e_i \cdot e_j = (-1)^{\frac{n(n-1)}{2}}(1+(-1)^n)\delta_{ij}$. Note that each e_i is represented by a real n-sphere, so $1+(-1)^n$ is the Euler number of this sphere.

(2) For each i, there is a σ_i in the monodromy group of this pencil satisfying:

$$\sigma_i(e_j) = e_j + (-1)^{\frac{(n+1)(n+2)}{2}}(e_j \cdot e_i)e_i,$$

in particular, if n is even, $\sigma(e_j) = e_j$ for $j \neq i$ and $\sigma_i(e_i) = -e_i$.

One can deduce from these and Theorem 3.1

Corollary 3.2. *Let V, W, e_i, etc. be as above. We further assume that A is invariant under the monodromy group. Then $\Psi^V_{(A,g,k)}$ vanishes on $e_i^{2m-1} \times H_*(V,\boldsymbol{Z})_i$ for each i and $m > 0$, where $H_*(V,\boldsymbol{Z})_i$ is the σ_i-invariant part of $H_*(V,\boldsymbol{Z})$.*

In particular, if V is a generic hypersurface in CP^{n+1}, then by the Lefschetz Theorem, $H_*(V,\boldsymbol{Z})$ is isomorphic to

$$\{[H]^i | 0 \le i \le n\} \oplus H_n(V,\boldsymbol{Z})_0$$

where H is a hyperplane in CP^{n+1} and $H_n(V,\boldsymbol{Z})_0$ is the primitive part, which is generated by vanishing cycles. Therefore, for any $n \ge 3$ and $\alpha_{l+1}, \cdots, \alpha_k$ in $H_n(V,\boldsymbol{Z})_0$, we have

$$\Psi^V_{(A,g,k)}(\,\cdot\,; H^{i_1}, \cdots, H^{i_l}, \alpha_{l+1}, \cdots, \alpha_k) = 0 \tag{3.2}$$

whenever $k-l$ is odd.

For a general algebraic manifold V, we denote by $H_{*,\mathrm{inv}}(V,\boldsymbol{Z})$ the subring of $H_*(V,\boldsymbol{Z})$ generated by those classes in $H_{2n-2}(V,\boldsymbol{Z})$, which can be represented by algebraic cycles, in other words, $H_{*,\mathrm{inv}}(V,\boldsymbol{Z})$ is generated by $H_{n-1,n-1}(V,\boldsymbol{Z})$ through cap products. Let $H_*(V,\boldsymbol{Q})_0$ be its orthogonal complement. The following is expected

For any $\alpha_1, \cdots, \alpha_k$ in $H_{*,\mathrm{inv}}(V,\boldsymbol{Z})$ and γ in $H_*(V,\boldsymbol{Q})_0$, we have

$$\Psi^V_{(A,g,k+1)}([K]; \alpha_1, \cdots, \alpha_k, \gamma) = 0 \tag{3.3}$$

whenever K is an analytic cycle.

We have already seen that this conjecture is true in many cases, such as complete intersections in any projective spaces.

Let us outline a heuristic proof as follows: For simplicity, we assume that $K = \overline{\mathcal{M}}_{o,k+1}$ and $H_2(V,\boldsymbol{Z}) = \boldsymbol{Z}$. Without loss of generality, we further assume that each α_i is represented by a subvariety, still denoted by α_i, in general position. Let Z be the moduli space of rational curves with homology class A, which intersect subvarieties α_i $(1 \leq i \leq k)$. Through the evaluation map, Z can be regarded as an effective holomorphic cycle in V, and $\Psi^V_{(A,o,k+1)}(\alpha_1, \cdots, \alpha_k, \gamma)$ can be defined as the intersection number of Z with γ. By the Lefschetz Theorem (cf. [GH]), γ is of the form $\tilde{\gamma} \cap H^l$, where $\tilde{\gamma}$ is a primitive cycle in V of real dimension $n + l'$ and H^l is an intersection of l ample divisors $(l \leq l')$. Note that $\dim_{\mathbf{R}} Z = n + 2l - l'$ is even, so $\dim_{\mathbf{R}}(\tilde{\gamma}) \leq 2n - 2$. Since $\tilde{\gamma}$ is primitive, for any smooth intersection S of $l' - l + 1$ ample divisors, γ can be represented by a cycle lying in $V \backslash S$.

On the other hand, since α_i are generic, Z should be contained in a smooth intersection of ample divisors in V. Using the fact that $\dim_{\mathbf{R}} \tilde{\gamma} = n + l' \leq 2n - 2$, one can embed Z as a subvariety in an intersection S of $l' - l + 1$ ample divisors. Therefore, $\gamma \cap Z = \emptyset$. This implies (3.3).

The only technical problem with these arguments is to construct Z rigorously. It can be overcome in view of my joint work with Jun Li in [LT2], where we construct the quantum cohomology on algebraic manifolds by purely algebraic method. One can also construct this cycle Z by using the results in [RT2].

Next we give an application to deformation equivalence of symplectic manifolds. Two symplectic manifolds (V, ω) and (V', ω') are deformation equivalent if and only if there is a diffeomorphism $\phi : V \mapsto V'$ and a family of symplectic forms ω_t such that $\omega_0 = \omega$ and $\omega_1 = \phi^* \omega'$. The classical Darboux theorem implies that all symplectic manifolds of the same dimension are locally isomorphic. Therefore, deformation equivalence is a global problem.

An important step was taken by J. Moser. He proved that (V, ω) and (V, ω') are equivalent if the cohomology class $[\omega]$ can be connected to $[\omega']$ in $H^2(V, \mathbf{R})$ by symplectic forms.

Usually, it is rather difficult to study such a deformation problem. For instance, in the complex surface theory, a long-standing problem is whether or not the moduli space of complex structures is always connnected on a given differential 4-manifold. Since any deformation of a Kähler surface is still Kähler, this can be related to the symplectic deformation problem. Nevertheless, Theorem 2.4 provides a tool for studying such a deformation problem, namely, two compact symplectic manifolds (V, ω) and (V', ω') are deformation equivalent only if the invariants Ψ^V and $\Psi^{V'}$ are the same. Ruan gave first two non-deformation equivalent, 6-dimensional symplectic manifolds, which are diffeomorphic to each other (cf. [R2], [MS]). Let Y be the Barlow surface, which is homeomorphic to the blow-up X of CP^2 at eight points. They have the same homology $H_*(X, \mathbf{Z}) = H_*(Y, \mathbf{Z})$, but they are not diffeomorphic. A result of C.T.C.Wall implies that $X' = X \times CP^1$ and $Y' = Y \times CP^1$ are diffeomorphic to each other. Ruan showed that $\Psi^{X'}_{(A,0,3)} \neq 0$ and $\Psi^{Y'}_{(A,0,3)} = 0$, where A represents the class of one of the blown-up points in X. Therefore, X' and Y' are not deformation equivalent. Very recently, M. Gross and Ruan can even construct two non-deformation equivalent Calabi-Yau 3-folds.

We recall a result of C.T.C. Wall, saying that two 4-manifolds X, Y are homeomorphic if and only if the stabilized manifolds $X' = X \times CP^1$, $Y' = Y \times CP^1$ are diffeomorphic. An analogy of this in symplectic topology would be ([R2]): given two homeomorphic symplectic 4-manifolds X and Y with $c_1(X) = c_1(Y)$, X is diffeomorphic to Y if and only if X' can be symplectically deformed to Y'. Note that X', Y' carry natural symplectic structures and $c_1(X) = c_1(Y)$ is necessary for X', Y' being deformation equivalent. Ruan's symplectic manifolds give the first indication that this might be true. Heuristically, the moduli space of horizontal rational curves in X' (resp. Y') is just X (resp. Y). If X' and Y' are deformation equivalent, then there is a cobordism from X to Y through moduli spaces of rational curves. It is not unreasonable to expect that such a cobordism is nice and imposes a strong relation between X and Y.

Using the GW-invariant Ψ, we will be able to prove this for simply-connected elliptic surfaces $E^n_{p,q}$ (p, q coprime). Here, E^1 is the blow-up of CP^2 at nine generic points, E^n denotes the fiber connected sum of n copies of E^1, and $E^n_{p,q}$ is obtained from E^n by logarithmic transformations along two smooth fibers with multiplicity p and q. Let A_p, A_q be the two multiple fibers in $E^n_{p,q}$. It is computed in [RT2]

$$\Psi^{E^n_{p,q}\times CP^1}_{(A_p,1,1)}(\overline{\mathcal{M}}_{1,1};\alpha) = 2pA\cdot\alpha,\ \Psi^{E^n_{p,q}\times CP^1}_{(A_q,1,1)}(\overline{\mathcal{M}}_{1,1};\alpha) = 2qA\cdot\alpha \qquad (3.4)$$

where $A = pA_p = qA_q$ is the fiber class and α is in $H_2(E^n_{p,q}, \boldsymbol{Z})$. This implies that the 3-folds $E^n_{p,q} \times CP^1$ are not deformation equivalent to each other. On the other hand, using the Donaldson invariant, Friedman and Morgan proved (cf. [FM]) that the surfaces $E^n_{p,q}$ are not diffeomorphic to each other, but they are homeomorphic for each fixed n.

Finally, we would like to mention a truely remarkable theorem, announced recently by C. Taubes ([Ta]). Taubes' theorem relates the GW invariant to the Seiberg-Witten invariant. Given a $Spin_C$ structure (denoted by $\mathcal{L}$) on V, we denote by $S^+_{\mathcal{L}}$ its positive spinor bundle. The Clifford multiplication induces a natural map $\tau : \mathrm{End}S^+_{\mathcal{L}} \mapsto \Lambda_+ \otimes \boldsymbol{C}$, which maps a self-adjoint endomorphism into an imaginary valued form. Then the Seiberg-Witten equations are

$$D_A\psi = 0 \text{ and } F^+_A = \frac{1}{4}\tau(\psi\otimes\psi^*) \qquad (3.5)$$

where A is a connection on $L = \det(S^+_{\mathcal{L}})$, F^+_A is the self-dual part of its curvature, and ψ is a section of $S^+_{\mathcal{L}}$. The Seiberg-Witten invariant $SW(\mathcal{L})$ is defined in terms of the solutions of (3.5). For instance, if the solutions of (3.5) are isolated, then $SW(\mathcal{L})$ just counts the number of solutions with appropriate sign (cf. [Ta], [W4]).

Theorem 3.3. ([Ta]) *Let V be a compact, oriented, symplectic 4-manifold with $b_2^+ \geq 3$, and $\mathcal{L}$ be a $Spin_C$-structure with $d = a\cdot a + c_1(V)(a) \geq 0$. Note that d is always even. Put $a \neq 0$ to be the Poincare dual to $\frac{1}{2}(c_1(L) + c_1(V))$*

and $g = 1 + d/2$. Then the SW-invariant $SW(\mathcal{L})$ can be identified with the GW-invariant $\Psi^V_{(a,g,d/2)}(\overline{\mathcal{M}}_{g,d/2}; [p], \cdots, [p])$, where p is a point in V.

This implies in particular that the GW-invariants in (3.6) are differential invariants. The Seiberg-Witten invariant is ± 1 for $L = K_V$, which is proved in [W4] for Kähler surfaces and in [Ta] for symplectic 4-manifolds, $\Psi^V_{(K_V, K_V^2+1, 0)} \neq 0$. This implies existence of a holomorphic curve of genus $K_V^2 + 1$. Moreover, it is observed in [W4] that all Kähler surfaces of general type satisfy: $SW(\mathcal{L}) = 0$ unless $L = K_V^{-1}$ or K_V, so we have $\Psi^V_{(A, 1+d/2, d/2)} = 0$ for any A dual to $mc_1(K_V)$ with $m > 1$ and d, g as in Theorem 3.4. Similar conclusions can be drawn for symplectic manifolds, too (cf. [Ta]).

4 The Composition Law

The composition law relates certain GW-invariants to counting degenerate holomorphic maps. Its classical cousin in enumerative algebraic geometry is the degeneration formula, which was only derived individually in very special cases. It was never developed into a systematic procedure as the composition law provides. One reason might be that classical counting of holomorphic curves, particularly of higher genus, can not have any simple composition laws! The other reason might be that it is very difficult to have a good deformation theory in algebraic geometry. Since our Ψ^V is a symplectic invariant, we only require a deformation theory in symplectic geometry. We will be able to establish this.

Recall that in the definition of $\Psi^V_{(A,g,k)}([K]; \cdots)$, we required that the top strata of K intersect with $\mathcal{M}_{g,k}$. Roughly speaking, the composition law means that this constrain on K can be dropped, i.e., the top strata of k can lie in $\overline{\mathcal{M}}_{g,k} \backslash \mathcal{M}_{g,k}$. Let us describe it in more details.

Assume $g = g_1 + g_2$ and $k = k_1 + k_2$ with $2g_i + k_i \geq 2$. Fix a decomposition $S = S_1 \cup S_2$ of $\{1, \cdots, k\}$ with $|S_i| = k_i$. Then there is a canonical embedding $\theta_S : \overline{\mathcal{M}}_{g_1, k_1+1} \times \overline{\mathcal{M}}_{g_2, k_2+1} \mapsto \overline{\mathcal{M}}_{g,k}$, which assigns to marked curves $(\Sigma_i; x^i_1, \cdots, x^i_{k_1+1})$ $(i = 1, 2)$, their union $\Sigma_1 \cup \Sigma_2$ with $x^1_{k_1+1}$ identified to x^2_1 and remaining points renumbered by $\{1, \cdots, k\}$ according to S.

There is another natural map $\mu : \overline{\mathcal{M}}_{g-1, k+2} \mapsto \overline{\mathcal{M}}_{g,k}$ by gluing together the last two marked points.

Choose a homogeneous basis $\{\beta_b\}_{1 \leq b \leq L}$ of $H_*(V, \mathbf{Z})$ modulo torsion. Let (η_{ab}) be its intersection matrix. Note that $\eta_{ab} = \beta_a \cdot \beta_b = 0$ if the dimensions of β_a and β_b are not complementary to each other. Put (η^{ab}) to be the inverse of (η_{ab}). Now we can state the composition law, which consists of two formulas.

Theorem 4.1. ([RT2]) *Let $[K_i] \in H_*(\overline{\mathcal{M}}_{g_i, k_i+1}, \mathbf{Q})$ $(i = 1, 2)$ and $[K_0] \in$*

$H_*(\overline{\mathcal{M}}_{g-1,k+2}, \boldsymbol{Q})$. *For any* $\alpha_1, \cdots, \alpha_k$ *in* $H_*(V, \boldsymbol{Z})$. *Then we have*

$$\begin{aligned}&\Psi^V_{(A,g,k)}\left(\theta_{S*}[K_1 \times K_2]; \{\alpha_i\}\right)\\ &= \sum_{A=A_1+A_2} \sum_{a,b} \Psi^V_{(A_1,g_1,k_1+1)}\left([K_1]; \{\alpha_i\}_{i\le k}, \beta_a\right) \eta^{ab} \\ &\qquad \times \Psi^V_{(A_2,g_2,k_2+1)}\left([K_2]; \beta_b, \{\alpha_j\}_{j>k}\right)\end{aligned} \tag{4.1}$$

$$\begin{aligned}&\Psi^V_{(A,g,k)}(\mu_*[K_0]; \alpha_1, \cdots, \alpha_k)\\ &= \sum_{a,b} \Psi^V_{(A,g-1,k+2)}([K_0]; \alpha_1, \cdots, \alpha_k, \beta_a, \beta_b)\eta^{ab}\end{aligned} \tag{4.2}$$

Let us sketch the proof of (4.1). The other case is similar. The proof is based on the Gromov-Uhlenbeck compactness theorem and a gluing theorem ([RT1]). We regard $K_1 \times K_2$ as a cycle in $\overline{\mathcal{M}}_{g,k}$. Let E_i be a sequence of cycles converging to $E_\infty = K_1 \times K_2$ such that the top strata of E_i lie in $\mathrm{Reg}(\mathcal{M}_{g,k})$. Put

$$\begin{aligned}\mathcal{N}_{A,i} = \{(\Sigma, x_1, \cdots, x_k, f) \in \mathcal{M}_A(g, k, J, \nu)\\ \mid (\Sigma, x_1, \cdots, x_k) \in E_i, f(x_i) \in C_i\}\end{aligned} \tag{4.3}$$

where C_i denotes a generic cycle representing α_i. Using Proposition 2.1, 2.2, one can show that for a generic pair (J, ν), each $\mathcal{N}_{A,i}$ consists of finitely many smooth points in $\mathcal{M}_A(g, k, J, \nu)$, consequently,

$$\Psi^V_{(A,g,k)}(\theta_*[K_1 \times K_2]; \alpha_1, \cdots, \alpha_k)$$

counts the number of elements in $\mathcal{N}_{A,i}$ with sign for each $i < \infty$. Note that $\theta_*[K_1 \times K_2] = [E_i]$.

Next we show that $\mathcal{N}_{A,i}$ converges to $\mathcal{N}_{A,\infty}$. Proposition 2.2 implies that $\lim \mathcal{N}_{A,i} \subset \mathcal{N}_{A,\infty}$. The following proposition is just Corollary 6.1 in [RT1] in slightly different form.

Proposition 4.2. *Let* $(\Sigma_\infty, x_{1\infty}, \cdots, x_{k\infty}, f_\infty)$ *be in* $\mathcal{N}_{A,\infty}$. *Assume that* f_∞ *is regular in the sense: the linearization* L_{f_∞} *of* (2.1) *at* f_∞ *has trivial cokernel* (cf. [RT1], p. 51-52 for details, and this assumption is true for a generic (J, ν))*. Then for i sufficiently large, there is a unique* $(\Sigma_i, x_{1i}, \cdots, x_{ki}, f_i)$ *in* $\mathcal{N}_{A,i}$ *which is close to* $(\Sigma_\infty, x_{1\infty}, \cdots, x_{k\infty}, f_\infty)$. *Moreover, orientations of the corresponding moduli spaces are the same at* $(\Sigma_i, x_{1i}, \cdots, x_{ki}, f_i)$ *and* $(\Sigma_\infty, x_{1\infty}, \cdots, x_{k\infty}, f_\infty)$.

This shows that $\lim \mathcal{N}_{A,i} = \mathcal{N}_{A,\infty}$, so $\Psi^V_{(A,g,k)}(\theta_{S*}[K_1 \times K_2]; \{\alpha_i\})$ is equal to the number of elements in $\mathcal{N}_{A,\infty}$, counted with sign. Each f_∞ in $\mathcal{N}_{A,\infty}$ consists of two (J, ν)-maps $f_{\infty 1}, f_{\infty 2}$ from two components of Σ_∞. Moreover, the image of $(f_{\infty 1}, f_{\infty 2})$ in $V \times V$ intersects with the diagonal class Δ. So the left side of (4.1) is the number of pairs of (J,ν)-maps $(f_{\infty 1}, f_{\infty 2})$ which intersect Δ, counted with sign. However, by constructing a cobordism as in the proof of Theorem 2.4, one can show that this number depends only the homology class $[\Delta]$ of Δ. On the other hand, by the Künneth formula,

$$[\Delta] = \sum_{a,b} \eta^{ab} \beta_a \otimes \beta_b$$

So (4.1) follows.

Corollary 4.3. *Let $\Phi_{(A,\omega,g)}$ be the mixed invariant in* [RT1] (see Remark 2.4). *Then for any $g = g_1 + g_2$ with $2g_1 + m \geq 3$ and $2g_2 + k - m \geq 3$, we have*

$$\begin{aligned} &\Phi_{(A,\omega,g)}(\{\alpha_i\}_{1\leq i\leq k}|\{\alpha_{k+j}\}_{1\leq j\leq l}) \\ &= \sum_{A=A_1+A_2} \sum_{a,b} \sum_{S=S_1\cup S_2} \epsilon(S) \Phi_{(A_1,\omega,g_1)}(\{\alpha_i\}_{1\leq i\leq m}|\{\alpha_j\}_{j\in S_1}) \\ &\qquad \times \Phi_{(A_2,\omega,g)}(\{\alpha_i\}_{m+1\leq i\leq k}|\{\alpha_j\}_{j\in S_2}) \end{aligned} \tag{4.4}$$

where $S = \{k+1, \cdots, k+l\}$, $S_1 \cap S_2 = \emptyset$, $\epsilon(S)$ denotes the sign of the permutation induced by S on those α_i with odd degree. In particular, all mixed invariants can be computed in terms of $\Psi^V_{(A,0,k)}(\overline{\mathcal{M}}_{0,k}; \cdots)$, i.e., primitive GW-invariants of genus zero.

Roughly speaking, such a primitive invariant counts the number of rational curves in V through certain cycles in general position.

Mcduff and Salamon in [MS], Liu Gang in [Li] gave an alternative proof of Corollary 4.3 in case $g = 0$ and $k = 4$.

For each $k \geq 4$, there is a fibration $\pi : \overline{\mathcal{M}}_{0,k} \mapsto \overline{\mathcal{M}}_{0,k-1}$ such that all fibers over $\mathcal{M}_{0,k-1}$ are CP^1. Then one can show that any $[K]$ in $H_*(\overline{\mathcal{M}}_{0,k}, Q)$ can be represented by a cycle in $\overline{\mathcal{M}}_{0,k-1}$, as long as $[K]$ is not of top dimension. Therefore, one can deduce from the composition law

Corollary 4.4. *All genus zero GW-invariants $\Psi^V_{(A,0,k)}$ can be reduced to primitive GW-invariants.*

Later we will see that all genus zero GW-invariants can be actually reduced to finitely many primitive GW-invariants for a large class of Fano manifolds.

5 Quantum Cohomology

Let (V,ω) be a semi-positive, compact symplectic manifold. The quantum cohomology on V is the cohomology $H^*(V, Z\{H_2(V)\})$ with a ring structure defined by GW-invariants. The Novikov ring $Z\{H_2(V)\}$, which first appeared in Novikov's study of Morse theory for multivalued functions (cf. [No]), can be described as follows (cf. [HS], [Pu], [RT1]): choose a basis $q_1, \cdots, q_s$ of $H_2(V, Z)$, we identify the monomial $q^d = q_1^{d_1} \cdots q_s^{d_s}$ with the sum $\sum_{i=1}^s d_i q_i$. This turns $H_2(V)$ into a multiplicative ring, i.e., $q^d \cdot q^{d'} = q^{d+d'}$. This multiplicative ring has a natural grading defined by $\deg(q^d) = 2c_1(V)(\sum d_i q_i)$. Then $Z\{H_2(V)\}$ is the graded homogeneous ring generated by all formal power series $\sum_{d=(d_1,\cdots,d_s)} n_d q^d$ satisfying: $n_d \in Z$, all q^d with $n_d \neq 0$ have the same degree and the number of n_d with $\omega(\sum d_i q_i) \leq c$ is finite for any $c > 0$. If V is a Fano manifold or a monotone symplectic manifold, then $Z\{H_2(V)\}$ is just a group ring.

Now we can define a ring structure on $H^*(V, Z\{H_2(V)\})$. For any α^*, β^* in $H^*(V, Z)$, we define the quantum multiplication $\alpha^* \bullet \beta^*$ by

$$\alpha^* \bullet \beta^*(\gamma) = \sum_{A \in H_2(V,Z)} \Psi^V_{(A,0,3)}(\alpha, \beta, \gamma) q^A \tag{5.1}$$

where $\gamma \in H_*(V, Z)$. Equivalently, if $\{\beta_a\}$ is a basis as in Theorem 4.1,

$$\alpha^* \bullet \beta^* = \sum_A \sum_{a,b} \Psi^V_{(A,0,3)}(\alpha, \beta, \beta_a)\, \eta^{ab}\, \beta_b^*\, q^A \tag{5.2}$$

Note that if $A = \sum a_i q_i$, we identify A with $(a_1, \cdots, a_s)$. In general, any α^*, β^* in $H^*(V, Z\{H_2(V)\})$ can be written:

$$\alpha^* = \sum_d \alpha_d^*\, q^d, \quad \beta^* = \sum \beta_{d'}^*\, q^{d'}$$

where α_d^*, $\beta_{d'}^*$ are in $H^*(V, Z)$. We define

$$\alpha^* \bullet \beta^* = \sum_{d,d'} \alpha_d^* \bullet \beta_{d'}^*\, q^{d+d'} \tag{5.3}$$

Recall that the degree of $\alpha_d^* q^d$ is $\deg(\alpha^*) + \deg(q^d)$. It follows that the multiplication preserves the degree. However, it is not clear at all if the multiplication is associative. Given α^*, β^*, γ^* in $H_*(V, Z)$ and any δ in $H_*(V, Z)$, we have

$$((\alpha^* \bullet \beta^*) \bullet \gamma^*)(\delta) = \sum_{A,B} \sum_{a,b} \Psi^V_{(A,0,3)}(\alpha, \beta, \beta_a)\eta^{ab}\Psi^V_{(A,0,3)}(\beta_b, \gamma, \delta)$$

$$(\alpha^* \bullet (\beta^* \bullet \gamma^*))(\delta) = \sum_{A,B} \sum_{a,b} \Psi^V_{(A,0,3)}(\alpha, \beta_a, \delta)\eta^{ab}\Psi^V_{(B,0,3)}(\beta, \gamma, \beta_b)$$

So the associativity means that for any fixed A in $H_2(V, Z)$,

$$\sum_{A_1+A_2=A} \sum_{a,b} \Psi^V_{(A_1,0,3)}(\alpha, \beta, \beta_a)\eta^{ab}\Psi^V_{(A_2,0,3)}(\beta_b, \gamma, \delta)$$
$$= \sum_{A_1+A_2=A} \sum_{a,b} \Psi^V_{(A_1,0,3)}(\alpha, \beta_a, \delta)\eta^{ab}\Psi^V_{(A_2,0,3)}(\beta, \gamma, \beta_b) \tag{5.4}$$

But by either Theorem 4.1 or Corollary 4.3, both sides of (5.4) are equal to $\Psi^V_{(A,0,4)}([p];\alpha,\beta,\gamma,\delta)$. Therefore, we have

Theorem 5.1. ([RT1, Theorem 8.4]) *The quantum multiplication $\bullet$ is associative, consequently, there is an associative, supercommutative, graded ring structure, i.e., quantum ring structure, on $H_*(V, Z\{H_2(V)\})$.*

In physics and sometimes mathematical literatures, one substitutes q by $(e^{-t\omega(q_1)}, \cdots, e^{-t\omega(q_s)})$, so the quantum product becomes

$$\alpha^* \cdot \beta^* = \sum_A \Psi^V_{(A,0,3)}(\alpha,\beta,\beta_a)\eta^{ab}\beta_b^* e^{-t\omega(A)} \tag{5.5}$$

In particular, this converges to the classical cup product as $t \to \infty$. If $c_1(V) > 0$, then

$$\alpha^* \cdot \beta^* = \alpha^* \cup \beta^* + \sum_{c_1(V)(A)>0} \phi_A$$

where ϕ_A is of degree $\deg(\alpha^* \cup \beta^*) - 2c_1(V)(A)$.

Examples:
(1) ([GK]) Let F_{n+1} be the Flag manifold of all sequences of subspaces

$$E_1 \subset E_2 \subset \cdots \subset E_n$$

of C^{n+1} with $\dim_C E_j = j$. Its cohomology ring is generated by the first Chern classes u_j of the universal line bundles $L_j \mapsto F_{n+1}$ with fiber E_{j+1}/E_j. Note that $\sum u_j = 0$. Givental and Kim observed that there is a natural isomorphism

$$H^*(F_{n+1}, Z\{H_2(F_{n+1})\}) = \frac{Z[u_0, \cdots, u_n, q_0, \cdots, q_n]}{\mathcal{I}}$$

where $\mathcal{I}$ denotes the ideal generated by coefficients c_j of the characteristic polynomial

$$\det(A_n + \lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n+1},$$

and A_n is defined by induction: $A_0 = u_0$, $A_1 = u_0u_1 + q_1$, $\cdots$, $A_k = u_kA_{k-1} + q_kA_{k-2}, \cdots$. Note that these c_j are the Poisson commuting integrals of the Toda lattice studied by J. Moser (cf. [GK], [MS]). The quantum cohomology has been computed (in a heuristic way) for partial Flag manifolds by Astashkevich and Sadov [AS]. Recently, these computations have been rigorously verified (cf. [CF], [Lu]).

Before we discuss further examples, we give a simple lemma on the general structure of quantum cohomology of Fano manifolds. This lemma is proved in [ST1], but we will state it slightly differently.

Lemma 5.4. *Let (V,ω) be a compact, semipositive symplectic manifold, such as Fano manifolds. If its ordinary cohomology ring has a presentation*

$Z[x_1, \cdots, x_L]/\mathcal{I}$, where $\mathcal{I}$ is an ideal $\{f_1, \cdots, f_m\}$ generated by homogeous polynomials $f_1, \cdots, f_m$, then the quantum ring is isomorphic to

$$\frac{Z[x_1, \cdots, x_L, q]}{\{F_1(x,q), \cdots, F_m(x,q)\}}$$

where $F_i(x,q)$ are homogeneous polynomial in x, q such that $F(x,0) = f(x)$.

(2) The quantum cohomology of the Grassmannian $G(r,n)$ has been computed in [ST1], [W3], [Pu]. Let S be the tautological bundle over $G(r,n)$ of complex k-planes in C^n. It is known that $H^*(G(r,n), Z)$ is given by

$$\frac{Z[x_1, \cdots, x_r]}{\{s_{n-r+1}, \cdots, s_n\}}$$

where s_j are Segre classes, defined inductively by $s_j = -x_1 s_{j-1} - \cdots - x_{j-1}s_1 - x_j$. In fact, x_i corresponds to the i-th Chern class $c_i(S)$ ($i = 1, \cdots, r$). Using Lemma 5.4, one can prove (see [ST1])

$$H^*(G(r,n), Z\{H_2(G(r,n))\}) = \frac{Z[x_1, \cdots, x_r, q]}{\{s_{n-r+1}, \cdots, s_{n-1}, s_n + (-1)^r q\}}.$$

(3) Batyrev has computed the quantum cohomology of toric Fano varieties (cf. [Ba]). More recently, Morrison and Plesser were able to generalize Batyrev's computations to singular toric varieties (cf. [MP]).

(4) Let V be a hypersurface in CP^{n+1} ($n \geq 3$) of degree $d \leq n+1$. It is a Fano manifold. Its rational cohomology is generated by the hyperplane class H and the primitive cohomology $H^n(V, Q)_0$, with the relations:

$$H^{n+1} = 0, \; H \cup \alpha = 0, \; \alpha \cup \beta = \frac{1}{d}\left(\int_V \alpha \wedge \beta\right) H^n$$

for $\alpha, \beta \in H^n(V, Q)_0$. Since all primitive classes are Poincare dual to vanishing cycles, using the discussion in section 3, one can show

$$H_{n+1} = \sum_{i=1}^{[\frac{n+1}{r}]} a_i H_{n+1-ir} q^i, \; H \bullet \alpha = \max\{0, d-n\} \tilde{\alpha}\, q,$$

$$\alpha \bullet \beta = \frac{1}{d}\int_V \alpha \wedge \beta \left(H_n - \sum_{i=1}^{[\frac{n}{r}]} b_i H_{n-ir} q^i\right)$$

where H_l denotes the quantum product of l classes H, $r = n+2-d$ is the index and $\alpha, \tilde{\alpha}, \beta \in H^n(V, Q)_0$.

If $r \geq 2$, then $H \bullet \alpha = 0$. By the associativity,

$$0 = (H \bullet \alpha) \bullet \beta = \frac{1}{d}\int_V \alpha \wedge \beta \left(H_{n+1} - \sum_{i=1}^{[\frac{n}{r}]} b_i H_{n+1-ir} q^i\right)$$

It follows that $a_{n+1} = 0$, $a_i = b_i$ for $1 \leq i \leq n$, whenever $H^*(V, \boldsymbol{Q})_0$ is non-trivial. The coefficient $a_1 = b_1$ encodes the contribution of lines in V. By a standard computation, one can deduce from the above a result of Beauville [Bel]: in case $n \geq 2d - 3$, the quantum cohomology of V over rational numbers is generated by H and $H^n(V, \boldsymbol{Q})_0$ with relations

$$H_{n+1} = d^d H_{d-1} q, \; H \bullet \alpha = 0, \; \alpha \bullet \beta = \frac{1}{d} \int_V \alpha \wedge \beta \left(H_n - d^d H_{d-2} q\right)$$

for $\alpha, \beta \in H^n(V, \boldsymbol{Q})_0$.

If $r = 1$, i.e., $d = n + 1$, then for any β in $H^n(V, \boldsymbol{Q})$,

$$\tilde{\alpha} \cup \beta([V]) = \Psi^V_{(1,0,3)}(H, \alpha, \beta).$$

It is shown in [Bel] that

$$\Psi^V_{(1,0,3)}(H, \alpha, \beta) = -(n+1)^n \int_V \alpha \wedge \beta.$$

Therefore, for any primitive class α, we have

$$\left(H + (n+1)^{(n+1)}\right) \bullet \alpha = 0.$$

In the preliminary version of this paper, I dropped the term $(n + 1)^{(n+1)}$ in the above equation. This error was pointed to me by A. Collino.

Using the above equations, one can deduce

$$\begin{aligned} &a_i = b_i + (n+1)^{(n+1)} b_{i-1} \text{ for } 2 \leq i \leq n+1, \\ &A_1 = b_1 - (n+1)^{(n+1)}. \end{aligned}$$

These discussions are valid for complete intersections in $\boldsymbol{C}P^{n+l}$.

In general, let V be any compact Kähler manifold and $\omega \in H^2(V, \boldsymbol{R})$ be a Kähler class. The Hard Lefschetz Theorem states that $H^*(V, \boldsymbol{R})$ has the following decomposition

$$H^*(V, \boldsymbol{R}) = \oplus_{k=1}^{[\frac{n}{2}]} \oplus_{i=0}^{n-k} \omega^i \cup P^k(V), \tag{5.6}$$

where $P^k(V)$ is the k^{th}-primitive part of $H^k(V, \boldsymbol{R})$ with respect to the Kähler class ω. In view of (3.3) and its generalization, we propose

Conjecture 5.5. *Let V be a compact Kähler manifold. For each k, we denote by $\{H^2(V, \boldsymbol{R}), P^k(V)\}$ the subspace of $H^*(V, \boldsymbol{R})$ spanned by classes of the form $\alpha^i \cup \beta$, where $\alpha \in H^2(V, \boldsymbol{R})$ and $\beta \in P^k(V)$. Then*

$$H^2(V, \boldsymbol{R}) \bullet \{H^2(V, \boldsymbol{R}), P^k(V)\} \subset \{H^2(V, \boldsymbol{R}), P^k(V)\}. \tag{5.7}$$

This can be thought as a quantized version of the Hard Lefschetz Theorem. If this is true, then the quantum multiplication on $H^*(V, \boldsymbol{Z}\{H_2(V)\})$ induces a subring structure on $H^*_{\text{inv}}(V, \boldsymbol{Z}\{H_2(V)\})$, which is just

$$H^*_{\text{inv}}(V, \boldsymbol{R}) \otimes \boldsymbol{Z}\{H_2(V)\}.$$

If V is a Fano manifold with $H^2(V, \mathbf{Z}) = \mathbf{Z}$, then the conjecture means that for any $k \geq 3$,

$$\oplus_{i=0}^{n-k} c_1(V)_i \bullet P^k(V) \;=\; \oplus_{i=0}^{n-k} c_1(V)^i \cup P^k(V). \tag{5.8}$$

In this case, one may further expect that $c_1(V) \bullet \alpha = c_1(V) \cup \alpha$ for any $\alpha \in P^k(V)$.

Finally, we would like to mention that Crauder and Miranda [CM] computed quantum cohomology on rational surfaces, such as cubic surfaces in $\mathbf{C}P^3$. Recently, Siebert and the author proved a recursion formula for the quantum cohomology of the moduli spaces of stable bundles over Riemann surfaces Σ_g (cf. [ST2]). In view of [Do], [DS], this computes certain product structure on the instanton Floer cohomology on 3-manifolds $S^1 \times \Sigma_g$.

6 The WDVV Equation

In fact, there is a family of quantum multiplications, containing $\bullet$ of the last section as a special case. These quantum multiplications make use of all the primitive GW-invariants, while $\bullet$ uses only the invariant $\Psi^V_{(A,0,3)}$.

Let $\{\beta_a\}_{1 \leq a \leq L}$ be an integral basis of $H_*(V, \mathbf{Z})$ modulo torsion. Any $w \in H^*(V, \mathbf{C})$ can be written as $\sum t_a \beta_a^*$. Clearly, $w \in H^*(V, \mathbf{Z})$ if all t_a are integers. We define the quantum multiplication $\bullet_w$ by

$$\begin{aligned} & \alpha^* \bullet_w \beta^*(\gamma) \\ = & \sum_A \sum_{k \geq 0} \frac{\epsilon(\{a_i\})}{k!} \Psi^V_{(A,0,k+3)}(\alpha, \beta, \gamma, \beta_{a_1}, \cdots, \beta_{a_k}) t_{a_1} \cdots t_{a_k} \, q^A \end{aligned} \tag{6.1}$$

where $\alpha, \beta, \gamma \in H_*(V, \mathbf{Z})$, and $\epsilon(\{a_i\})$ is the sign of the induced permutation on odd dimensional β_a. Obviously, this multiplication reduces to $\bullet$ at $w = 0$. As we argued in the last section, the associativity of $\bullet_w$ is equivalent to

$$\begin{aligned} & \sum_{A=A_1+A_2} \sum_{a,b} \sum_{\{1,\cdots,k\}=S_1 \cup S_2} \epsilon(S_1, S_2) \Psi^V_{(A_1,0,|S_1|+3)}(\alpha, \beta, \beta_a, \{\beta_{a_i}\}_{a_i \in S_1}) \, \eta^{ab} \\ & \qquad \times \Psi^V_{(A_2,0,|S_2|+3)}(\beta_b, \gamma, \delta, \{\beta_{a_i}\}_{a_i \in S_2}) \\ = & \sum_{A=A_1+A_2} \sum_{a,b} \sum_{\{1,\cdots,k\}=S_1 \cup S_2} (-1)^{\deg(\beta^* \cup \gamma^*)} \epsilon(S_1, S_2) \\ & \qquad \times \Psi^V_{(A_1,0,|S_1|+3)}(\alpha, \gamma, \beta_a, \{\beta_{a_i}\}_{a_i \in S_1}) \, \eta^{ab} \\ & \qquad \times \Psi^V_{(A_2,0,|S_2|+3)}(\beta_b, \beta, \delta, \{\beta_{a_i}\}_{a_i \in S_2}) \end{aligned} \tag{6.2}$$

for all $\alpha, \beta, \gamma, \delta$, where $\epsilon(S_1, S_2)$ is the sign of the permutation induced by S_1, S_2 on odd dimensional β_a. However, (6.2) follows easily from Corollary 4.3 or Theorem 4.1. Therefore, we have

Theorem 6.1. *The quantum multiplications $\bullet_w$ are associative.*

The associativity (6.2) can be restated in terms of a solution of WDVV equations. To avoid additional technicalities in using superstructures, we confine ourselves to the case $w \in W = H^{\text{even}}(V, \boldsymbol{C})$. The general case can be treated similarly (cf. [KM]).

Following Witten, we define a generating function (prepotential in physical literature)

$$\Phi^V(w,q) = \sum_{A \in H_2(V,\boldsymbol{Z})} \sum_{k \geq 3} \frac{1}{k!} \Psi^V_{(A,0,k)}(\overline{\mathcal{M}}_{0,k}; w, \cdots, w)\, q^A \tag{6.3}$$

This function is a formal power series in w, q. In fact, the generating function in [W2] is equal to $\Phi^V(w, (1, \cdots, 1))$, i.e., Φ^V specified at $q = (1, \cdots, 1)$. Its convergence remains to be a difficult problem in general (cf. section 7). By inserting q, we gain the advantage of avoiding the convergence problem. We may arrange the basis $\{\beta_a\}$ such that β_a is an even class if and only if $a \leq N$. Then for any $a, b, c \leq N$,

$$(\beta_a \bullet_w \beta_b)(\beta_c) = \frac{\partial^3 \Phi^V}{\partial t_a \partial t_b \partial t_c}(w) \tag{6.4}$$

Then (6.2) implies

Theorem 6.2. ([RT1]) *Let V be a semi-positive symplectic manifold. Then for any specified number q, Φ^V_w satisfies the associativity equation:*

$$\sum_{a,b} \frac{\partial^3 \Phi^V}{\partial t_i \partial t_j \partial t_a} \eta^{ab} \frac{\partial^3 \Phi^V}{\partial t_k \partial t_l \partial t_b} = \sum_{a,b} \frac{\partial^3 \Phi^V}{\partial t_i \partial t_k \partial t_a} \eta^{ab} \frac{\partial^3 \Phi^V}{\partial t_j \partial t_l \partial t_b} \tag{6.5}$$

where $1 \leq i, j, k, l \leq N$.

According to Dubrovin (cf. [Du], [RT1], [MS]), a solution of (6.4) gives rise to a family of flat connections. If we define $\Gamma = \{\Gamma^c_{ab}\}$ by

$$\Gamma^c_{ab} = \sum_e \eta^{ce} \frac{\partial^3 \Phi^V}{\partial t_a \partial t_b \partial t_e}$$

then $\nabla^\epsilon = d + \epsilon\Gamma$ defines a family of connections on the tangent bundle TW over W, where d denotes the standard differentiation on W. Then (6.5) is the same as flatness of ∇^ϵ. In fact, Dubrovin observed

Theorem 6.3. *Each ∇^ϵ is a torsion-free, flat connection satisfying:*

$$d\langle \alpha, \beta \rangle = \langle \nabla^\epsilon \alpha, \beta \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product on W induced by (η_{ab}). Moreover, $\nabla^\epsilon \mathbf{1} = \sqrt{-1}$ for the identity $\mathbf{1}$ on W.

The generating function Φ^V has the following scaling property: let $\sigma(\lambda)$ ($\lambda \in C^*$) be the scaling map:

$$\sigma(\lambda)(t_1, \cdots, t_N) = (\lambda^{\deg(\beta_1^*)-2} t_1, \cdots, \lambda^{\deg(\beta_N^*)-2} t_N) \tag{6.6}$$

Then

$$\Phi^V(\sigma(\lambda)(w), \lambda^{-2c_1(V)} q) = \lambda^{2(\dim_C V - 3)} \Phi^V(w, q) \tag{6.7}$$

Note that $\lambda^{-2c_1(V)} q = (\lambda^{-2c_1(V)(q_1)} q_1, \cdots, \lambda^{-2c_1(V)(q_s)} q_s)$. Consequently, the Dubrovin connection satisfies:

$$\Gamma_{ab}^c(\sigma(\lambda)w, \lambda^{-2c_1(V)} q) = \lambda^{\deg(\beta_c^*)-\deg(\beta_a^*)-\deg(\beta_b^*)} \Gamma_{ab}^c \tag{6.8}$$

These scaling properties follow simply from the definition of Ψ^V. Infinitesimally, (6.8) can be reformulated ([Du]) as

$$\mathcal{L}_X \Phi^V = 2(\dim_C V - 3)\Phi^V \tag{6.9}$$

where $\mathcal{L}_X$ denotes the Lie derivative, and X is the vector field

$$\sum_{a \leq N} (\deg(\beta_a^*) - 2) t_a \partial_a - \sum_s 2c_1(V)(q_s) \partial_{q_s} \tag{6.10}$$

The Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation is just the associativity equation (6.4) together with the scaling property (6.9). It follows from Theorem 6.3 and (6.9) that quantum multiplications $\bullet_w$ induce the structure of a so-called Frobenius algebra on W (see [Du], [KM]).

In order to further investigate the prepotential, we arrange the basis $\{\beta_a\}$ such that $\deg(\beta_N^*) = 2n$ and $\deg(\beta_a^*) = 2n - 2$ whenever $N - s \leq a < N$. Then using Remark 2.4, one can deduce

$$\begin{aligned} & \frac{\partial^3 \Phi^V}{\partial t_a \partial t_b \partial t_c}(t_1, \cdots, t_N; q) \\ = & \frac{\partial^3 \Phi^V}{\partial t_a \partial t_b \partial t_c}(t_1, \cdots, t_{N-s-1}, 0, \cdots, 0; e^{t_{N-s}} q_1, \cdots, e^{t_{N-1}} q_s) \end{aligned} \tag{6.11}$$

It follows that

$$\Phi^V(t_1, \cdots, t_N; q) - \Phi^V(t_1, \cdots, t_{N-s-1}, 0, \cdots, 0; e^{t_{N-s}} q_1, \cdots, e^{t_{N-1}} q_s)$$

is a quadratic polynomial.

Examples: (1) $V = CP^n$, choose $\beta_a = H^{n+1-a}$ $(1 \leq a \leq n+1)$, where H is the hyperplane class. For $n = 2$, we have

$$\Phi^{CP^2}(t_1, t_2, t_3; 1) = \frac{1}{2}(t_1 t_3^2 + t_3 t_2^2) + \sum_{k \geq 1} \frac{n_k t_1^{3k-1}}{(3k-1)!} e^{k t_2} \tag{6.12}$$

where n_k is the number of rational curves of degree k in CP^2 through $3d-1$ points in general position.

In general, let $S_{n,k}$ be the set of $(n-1)$-tuples $\{k_a\}_{1\le a\le n-1}$ with $k_i \ge 0$ and $\sum_{i=1}^{n-1} ik_{n-i} = (n+1)k + n - 3$. Then we have

$$\Phi^{CP^n}(t_1, \cdots, t_{n+1}; 1) = \frac{1}{6}\sum_{a+b+c=2n+3} t_a t_b t_c + \sum_{k\ge1}\sum_{\{k_a\}\in S_{n,k}} \frac{\sigma_k^{CP^n}(k_1, \cdots, k_{n-1}) t_1^{k_1}\cdots t_{n-1}^{k_{n-1}}}{k_1!\cdots k_{n-1}!} e^{kt_n} \tag{6.13}$$

where $\sigma_k^V(k_1, \cdots, k_{n-1})$ denotes the number of degree k rational curves in V through k_1 points, ... k_{n-1} subspaces of dimension $n-2$ in general position.

(2) Let V be a hypersurface in CP^{n+1} of degree $d \le n+2$. Then the hyperplane section $H \cap V$ generates a subring $H_{\rm inv}(V, Z)$ of $H_*(V, Z)$, which is invariant under the monodromy group. Let $\Phi_{\rm inv}^V$ be the restriction of Φ^V to $H_{\rm inv}(V, C)$. Then by the discussions in section 3, $\Phi_{\rm inv}^V$ also satisfies the WDVV equation. Furthermore, if we put $S_{V,k}$ to be the set of $n-1$-tuples $\{k_a\}_{1\le a\le n-1}$ with $k_i \ge 0$ and $\sum_{i=1}^{n-1} ik_{n-i} = (n+2-d)k + n - 3 > 0$, then

$$\Phi_{\rm inv}^V(t_1, \cdots, t_{n+1}; 1) = \frac{d}{6}\sum_{a+b+c=2n+3} t_a t_b t_c + \sum_{k\ge1}\sum_{\{k_a\}\in S_{V,k}} \frac{\sigma_k^V(k_1, \cdots, k_{n-1}) t_1^{k_1}\cdots t_{n-1}^{k_{n-1}}}{k_1!\cdots k_{n-1}!} e^{kt_n} \tag{6.14}$$

If V is a Calabi-Yau 3-fold, i.e., $d = 5, n = 3$, it has been proved (cf. [AM], [Vo]) that

$$\frac{\partial^3\Phi^V(t_1, t_2, t_3, t_4)}{\partial t_3^3} = 5 + \sum_{k\ge1} \frac{k^3 n_k e^{kt_3}}{1 - e^{kt_3}} \tag{6.15}$$

$$\frac{\partial^3\Phi^V(t_1, t_2, t_3, t_4)}{\partial t_i \partial t_j \partial t_k} = \frac{5}{2}\frac{\partial^3}{\partial t_i \partial t_j \partial t_k}(t_1 t_4^2 + 2t_2 t_3 t_4), \ (i, j, k) \ne (3, 3, 3),$$

where n_k denotes the number of irreducible rational curves in V.

In the following, we will take $q = (1, \cdots, 1)$.

Consider the open subset M in $H^*(V, C)$ where Φ^V is well-defined and the quantum multiplication by $X(w)$ on $H^*(V, C)$ has no multiple spectrum, i.e., semi-simple according to B. Dubrovin. Therefore, we can find new local coordinates $u_1, \cdots, u_N$ of M, which are simply eigenvalues of the quantum multiplication $X(w)\bullet_w$ on $H^*(V, C)$. Dubrovin proved that in these new coordinates $u_1, \cdots, u_N$, the prepotential Φ^V can be extended to be a meromorphic function on $(CP^1)^N$. When the action $X(w)\bullet_w$ is nilpotent, like in the case of Calabi-Yau manifolds, it is unclear how to find such new coordinates that Φ^V can be extended.

To find his extension, Dubrovin identifies the WDVV equation with a certain Riemann-Hilbert problem ([Du]).

Denote by ∂_a the differentiation $\frac{\partial}{\partial t_a}$ in the direction of β_a. Then flatness of the Dubrovin connection ∇^ϵ is equivalent to the compatibility of the linear system

$$\partial_a \xi^b \; = \; - \epsilon \Gamma^b_{ac} \xi^c \tag{6.16}$$

This gives a "Lax pair" for the associativity equation (6.5), where ϵ plays the role of the spectral parameter. For any ϵ , this system has N linearly independent solutions, which give rise to a flat coordinate system for ∇^ϵ. In fact, the solutions $\{\xi^a\}$ of (6.16) correspond one-to-one to flat sections $\xi = \xi^a \partial_a$ of ∇^ϵ.

Next, the standard machinery of scaling reduction of integrable systems suggests to add a differential equation in the spectral parameter ϵ for the section $\xi = \xi(t, \epsilon)$ (cf. Flaschka and Newell [FN], Jimbo et. al. [JMS], Its and Novokshenov [IN]).

Lemma 6.4. ([Du, Proposition 3.1]) *The WDVV equation is equivalent to compatibility of the system* (6.16) *with the equation*

$$2\epsilon \frac{\partial \xi^a}{\partial \epsilon} = \epsilon X^p \Gamma^a_{bp} \xi^b + \partial_b X^a \xi^b \tag{6.17}$$

where X is defined in (6.10), *consequently,* $\partial_b X^a = (\deg(\beta_a) - 2)\delta_{ab}$.

Proof: Introduce differential operators D_b and D_ϵ as follows:

$$\begin{aligned} D_b \xi^a &= \partial_b \xi^a + \epsilon \Gamma^a_{bc} \xi^c \\ D_\epsilon \xi^a &= \frac{\partial \xi^a}{\partial \epsilon} - \frac{1}{2}\left(X^b \Gamma^a_{bc} \xi^c + \frac{1}{\epsilon} \partial_b X^a \xi^b \right) \end{aligned} \tag{6.18}$$

Then compatibility simply means that $[D_b, D_\epsilon] = 0$, namely, they commute with each other. A straightforward computation shows

$$\begin{aligned} &(D_b D_\epsilon - D_\epsilon D_b)\xi^a \\ = &-\frac{1}{2}\left(\partial_b X^p \Gamma^a_{pc}\xi^c + \partial_c X^p \Gamma^a_{pb}\xi^c - \partial_c X^a \Gamma^c_{pb}\xi^p + X^p \partial_b \Gamma^a_{pc}\xi^c\right) \\ &- \Gamma^a_{bc}\xi^c - \frac{\epsilon}{2}(\Gamma^a_{bc}\Gamma^c_{pe} - \Gamma^c_{be}\Gamma^a_{pc})\xi^e X^p \qquad (6.19) \\ = &-\frac{1}{2}\left((\deg(\beta_b) + \deg(\beta_c) - \deg(\beta_a))\Gamma^a_{bc} + X^p \partial_p \Gamma^a_{bc}\right)\xi^c \qquad \text{(associativity)} \\ = &0 \qquad \text{(homogeneity of } \Gamma^a_{bc}) \end{aligned}$$

The lemma is proved.

The compatibility of (6.17) can be reformulated (cf. [KM]) as vanishing of the curvature of ∇ on $H^*(V, C) \times C$, where ∇ is the connection such that D_b, D_ϵ are its induced differentiations on components ξ^a of any section $\xi = \xi^a \partial_a$.

The basic idea (also see [FN], [IN], [JMS]) is to parametrize the solutions of the WDVV equation by the monodromy data of the operator (6.11). The problem is to find the monodromy. This was done by Dubrovin under a certain semi-simplicity assumption on the quantum product $\bullet_t$.

Consider the operator

$$\Lambda = 2\frac{d}{d\epsilon} - U - \frac{1}{\epsilon}V, \tag{6.20}$$

where U and V are two complex $N \times N$ ϵ-independent matrices, U is a diagonal matrix with distinct eigenvalues, and V is skew-symmetric. The solutions of $\Lambda\varphi = 0$ are analytic multivalued functions in $C\backslash\{0\}$. The monodromy of these solutions is called monodromy of Λ.

Assume that for a given w in $H^*(V, C)$, the multiplication $X \bullet_w$, where X is as in (6.10), induces a semi-simple $N \times N$ matrix. Then the eigenvalues $u_1, \cdots, u_N$ of this matrix give rise to new coordinates near w in $H^*(V, C)$. One can show that the intersection form (flat metric) η is diagonalized under the new coordinates $u_1, \cdots, u_N$, i.e.,

$$\eta(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}) = \eta_{ii}\delta_{ij}. \tag{6.21}$$

Put $\tilde{\partial}_i = \frac{1}{\sqrt{\eta_{ii}}}\frac{\partial}{\partial u_i}$, then $\eta(\tilde{\partial}_i, \tilde{\partial}_j) = \delta_{ij}$. Moreover, we have

$$\tilde{\partial}_i X = (u_j - u_i)\gamma_{ij}\tilde{\partial}_j, \; \gamma_{ij} = \gamma_{ji} = \frac{1}{\sqrt{\eta_{jj}}}\frac{\partial\sqrt{\eta_{ii}}}{\partial u_j}. \tag{6.22}$$

These γ_{ij} are called the rotation coefficients.

For any section $\xi = \xi^a\partial_a = \zeta^i\tilde{\partial}_i$, $D_\epsilon\xi^a = 0$ for all a if and only if for all i,

$$\Lambda_u\zeta^i = (2\frac{d}{d\epsilon} - U - \frac{1}{\epsilon}V)\zeta^i = 0, \tag{6.23}$$

where $U = \mathrm{diag}(u_1, \cdots, u_N)$ and $V = ((u_j - u_i)\gamma_{ij})_{1\le i,j\le N}$ (cf. [Du]). Since Λ_u is just the restriction of the flat connection ∇ to $\{u\} \times C$, the monodromy of Λ_u is independent of u. Thus solutions of the WDVV equation can be embedded into the space of isomonodromy deformations of operators Λ of the form (6.20). Each operator Λ has two singularities in the ϵ-sphere $C \cup \infty$: the regular singularity at $\epsilon = 0$ and the irregular one at $\epsilon = \infty$. The monodromy of Λ at $\epsilon = 0$ is determined by the matrix V. The monodromy at $\epsilon = \infty$ is specified by a $N \times N$ Stokes matrix (cf. [Du]). By analyzing the Stokes matrix at ∞ and using a result of T. Miwa, Dubrovin showed

Theorem 6.5 ([Du]). *Let Φ be any solution of the WDVV equation such that the induced multiplication by X is semi-simple. Then the corresponding matrix-valued function V, in coordinates $u_1, \cdots, u_N$, extends to a meromorphic function on the universal covering of $CP^{N-1}\backslash\{\mathrm{diagonals}\}$.*

One can also identify the WDVV with the equation of isomonodromy deformations of an operator with regular singularities (cf. [Du]). This can be done by using the Laplace transformation. We will only carry out formal arguments here. These arguments can be made rigorous.

We denote by $L(f)$ the Laplace transform of f, namely,

$$L(f)(\lambda) = \int e^{-\lambda\epsilon} f(\epsilon) d\epsilon \tag{6.24}$$

Put $\zeta = L(\xi) = L(\xi^a)\partial_a$. Then $D_\epsilon \xi = 0$ implies

$$(2\lambda\delta_c^a - X^p\Gamma_{pc}^a)\frac{\partial\zeta^c}{\partial\lambda} - \deg(\beta_a)\zeta^a = 0, \tag{6.25}$$

in coordinates $u_1, \cdots, u_N$, this is the same as

$$\frac{\partial\zeta}{\partial\lambda} + (2\lambda - U)^{-1}(V - 2I)\zeta = 0, \tag{6.26}$$

The points $\lambda = u_1, \cdots, u_N, \infty$ are regular singularities of the coefficients. The monodromy preserving deformations of (6.26) were described by Schlesinger at the begining of this century. In modern language, they form a moduli space of flat holomorphic bundles, over punctured spheres, with fixed holonomy at each puncture. Therefore, the solution Φ^V of the WDVV equation gives rise to a meromorphic flat connection on the moduli space, which can be extended to be quasi-projective.

Finally, we briefly discuss the semi-simplicity assumption. We say that V is semi-simple in the sense of Dubrovin, if for a generic w, the quntum multiplication $X\bullet_w$ on $H^*(V, C)$ has only simple eigenvalues.

If V is a Calabi-Yau manifold, then for dimensional reason, the GW-invariant

$$\Psi^V_{(A,0,k+3)}(\alpha, \beta, \gamma, \{\beta_{a_i}\})$$

in (6.1) vanishes unless $\deg\alpha^* + \deg\beta^* + \deg\gamma^* \leq \dim_C V$. This implies that $X\bullet_w$ corresponds to certain lower triangular matrix with respect to a basis $\{\beta_a\}$ of $H^*(V, C)$. So the multiplication $X\bullet_w$ always has only one eigenvalue. Therefore, no Calabi-Yau manifold is semi-simple.

However, we expect that the semi-simplicity condition holds for Fano manifolds. If so, the above discussion indicates that counting rational curves in a Fano manifold can be identified with a deformation problem of flat bundles over a punctured sphere. In a sense, this can be thought as the Mirror symmetry phenomenon for Fano manifolds.

Let V be an n-dimensional Fano manifold. We put $\omega = \frac{1}{r}c_1(V)$, where r is the index. By the Hard Lefschetz Theorem, we have the decomposition

$$H^*(V, C) = \oplus_{k=0}^{[\frac{n}{2}]} \oplus_{i=0}^{n-k} \omega^i \cup P^k(V, C),$$

where $P^k(V, \boldsymbol{C}) = \{\alpha \in H^*(V, \boldsymbol{C}) \mid \omega^{n-k+1} \cup \alpha = 0\}$ is the k^{th}-primitive part of $H^*(V, \boldsymbol{C})$. For each k, we choose a basis $\{\beta_{ik}\}_{1 \le i \le N_k}$ of $P^k(V)$. Clearly, we have

$$N_0 = 1, \ N = \sum_{k=0}^{[\frac{n}{2}]} (n-k) N_k.$$

As before, we denote by $\omega_l = \omega \bullet \cdots \bullet \omega$ the quantum product of ω by l times. Since

$$\omega_l \bullet \beta_{ik} = \omega^l \cup \beta_{ik} + \text{ terms of lower degree}$$

$\{\omega_l \bullet \beta_{ik}\}$ forms a basis of $H^*(V, \boldsymbol{C})$.

Let U_w be the $N \times N$-matrix representing the quantum multiplication $X(w) \bullet_w$ in the basis $\{\omega_l \bullet \beta_{ik}\}$, and let h_w be its characteristic polynomial, i.e., $h_w(\lambda) = \det(\lambda I - U_w)$. Obviously, V is semi-simple if and only if h_w is minimal for a generic w, namely, h_w has only simple roots.

If $V = \boldsymbol{C}P^n$, then $N = n+1$ and $h_0(\lambda) = \lambda^{n+1} - (-2(n+1))^{n+1}$. The roots of h_0 are $-2(n+1)e^{\frac{2\pi k \sqrt{-1}}{n+1}}$, where $k = 0, 1, \cdots, n$. So $\boldsymbol{C}P^n$ is semi-simple.

Next we take V to be the n-dimensional quadratic hypersurface in $\boldsymbol{C}P^{n+1}$ with $n \ge 3$. Then $N = n + 2 - \frac{1}{2}(1 - (-1)^n)$. If n is odd, $N = n+1$ and consequently, $H^*(V, \boldsymbol{Q})$ is generated by $H^2(V, \boldsymbol{Z})$. A simple computation shows that $h_0(\lambda) = \lambda^{n+1} - 4(-2n)^n \lambda$. Its roots are 0, $-2n(4e^{2\pi k \sqrt{-1}})^{\frac{1}{n}}$ $(k = 0, 1, \cdots, n-1)$, so V is semi-simple. If n is even, $H^*(V, \boldsymbol{C}) = H^*_{\text{inv}}(V, \boldsymbol{C}) \oplus H^*(V, \boldsymbol{C})_0$. Using that $c_1(V) \bullet \alpha = 0$ for $\alpha \in H^*(V, \boldsymbol{C})_0$, one can easily show that $h_0(\lambda) = \lambda^{n+2} - 4(-2n)^n \lambda^2$. So h_0 has only one multiple root 0 with multiplicity two. However, V is still semi-simple in the sense of Dubrovin. To see this, we denote by ω the positive generator of $H^2(V, \boldsymbol{Z})$. Note that $c_1(V) = n\omega$. Put $w = t\omega^n$. Then $X(w) = (2n-2)t\omega^n - 2c_1(V)$. An easy computation shows that h_w has simple roots for t sufficiently small. So V is semi-simple.

Conjecture 6.6. *Any Fano manifold is semi-simple in the sense of Dubrovin.*

Using the discussion in section 3, 5, one may be able to check this for Fano hypersurfaces in projective spaces.

There is a weaker version of the above conjecture. Let Φ^V_{inv} be the restriction of the prepotential to $H^*_{\text{inv}}(V, \boldsymbol{C})$. Assuming (3.3), we can easily deduce that Φ^V_{inv} also satisfies the WDVV equation with the scaling field X_{inv} on $H^*_{\text{inv}}(V, \boldsymbol{C})$, which is induced by X. Then one can ask if $X_{\text{inv}} \bullet_w$, which acts on $H^*_{\text{inv}}(V, \boldsymbol{C})$, is semi-simple for a generic w in $H^*_{\text{inv}}(V, \boldsymbol{C})$, where V is any Fano manifold. It does not seem to be very hard to check this for hypersurfaces, in view of Corollary 3.2 and Example (4) in section 5. For instance, if V is a cubic hypersurface in $\boldsymbol{C}P^4$, one can easily show that the quantum multiplication by X_{inv} has only simple eigenvalues for a generic w.

7 Applications to Enumerative Geometry

The associativity law has some direct and interesting applications in counting holomorphic curves in algebraic varieties. In this section, we will count the number of rational curves on Fano manifolds, such as hypersurfaces, assuming that $\Psi^V_{(A,0,k)}$ actually counts the number of rational curves in those algebraic varieties. It is believed that this assumption holds at least for rational curves of large degree on Fano manifolds. We will always denote by $H^*_{\rm inv}(V,\mathbf{Z})$ the ring generated by $H^2(V,\mathbf{Z})$ modulo torsion.

For simplicity, we will start with a Fano manifold V with $H_2(V,\mathbf{Z})=\mathbf{Z}$ such that the associativity law holds for the restriction of Φ^V to $H^*_{\rm inv}(V,\mathbf{C})$, for example, complete intersections in $\mathbf{C}P^{n+1}$ ($n\geq 3$) or Fano manifolds satisfying (3.3). We will abbreviate $\Psi^V_{(A,0,k)}(H^{j_1},\cdots,H^{j_k})$ by $\Psi^V_A(j_1,\cdots,j_k)$, where H is the Poincare dual of the positive generator in $H^2(V,\mathbf{Z})$. Roughly speaking, $\Psi^V_A(j_1,\cdots,j_k)$ counts the number of rational curves in V with homology class A through subvarieties H^{j_1},...,H^{j_k}.

Theorem 7.1. *Let V be as above. Then for $j_1\geq\cdots\geq j_k\geq 2$,*

$$\begin{aligned}
&\Psi^V_d(j_1,j_2,j_3,j_4,\cdots,j_k) - \Psi^V_d(j_1,j_2+1,j_3-1,j_4,\cdots,j_k)\\
&= d\left(\Psi^V_d(j_1+j_3-1,j_2,j_4,\cdots,j_k) - \Psi^V_d(j_1+j_2,j_3-1,j_4,\cdots,j_k)\right) \qquad (7.1)\\
&+\sum_{m=1}^{d-1}\sum_{i=0}^{n}\sum_{\{4,\cdots,k\}=S_1\cup S_2} m\left(\Psi^V_{d-m}(j_1,j_2,i,\{j_a\}_{a\in S_1})\cdot\Psi^V_m(j_3-1,n-i,\{j_a\}_{a\in S_2})\right.\\
&\qquad\left.-\Psi^V_{d-m}(j_1,j_3-1,i,\{j_a\}_{a\in S_1})\cdot\Psi^V_m(j_2,n-i,\{j_a\}_{a\in S_1})\right)
\end{aligned}$$

In particular, there is a recursion formula for computing σ^V_d.

Proof: The proof follows directly from the associativity law. Special cases of (7.1) have been derived before. In case $V=\mathbf{C}P^n$, (7.1) is derived in [KM] and [RT1], in case V is a hypersurface of odd dimension, (7.1) is derived in [RT1]. By Property (1) in section 2, we have $\sigma^V_0(j_1,\cdots,j_l)=\deg(V)$ or 0 according to whether or not $l=3$ and $j_1+j_2+j_3=n$. Moreover, $H^i\cap H^j=\deg(V)$ if $i+j=n$, $=0$ if $i+j\neq 0$. Then using (2.6) to eliminate H, one can easily deduce (7.1) from (6.2) with $\alpha=H^{j_1}$, $\beta=H^{j_2}$, $\gamma=H^{j_3-1}$, $\delta=H$ and $\{\beta_{a_i}\}=\{H^{j_i}\}_{4\leq i\leq k}$.

An easy corollary of this theorem is that all primitive GW-invariants $\Psi^V_A(j_1,\cdots,j_k)$ can be computed in terms of 3-point invariants $\Psi^V_d(H^i,H^j,H^k)$ with $i+j+k=2n-rd\geq 0$, where r is the index of V, i.e., $c_1(V)=rH^*$. In particular, if V is a hypersurface in $\mathbf{C}P^{n+1}$ of degree $\leq\frac{n+3}{2}$, then all primitive GW-invariants of genus zero can be explicitely computed (cf. section 5, example 4). In case $V = CP^2$, (7.1) becomes

$$n_d = \frac{1}{2} \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} \frac{(d_1d_2(3dd_1d_2 - 2d^2 + 6d_1d_2)(3d-4)!}{(3d_1-1)!(3d_2-1)!} n_{d_1} n_{d_2} \tag{7.2}$$

where n_d are given in last section. This formula was first observed by Kontsevich (cf. [KM]).

Theorem 7.1 can be used to derive an estimate on growth of degrees of moduli spaces of degree d rational curves, or equivalently, growth of numbers of rational curves through subvarieties as d tends to infinity. More precisely, for each n, there are positive constants $\lambda_1, \cdots, \lambda_{n-1}$ such that

$$\frac{\sigma_d^V(k_1, \cdots, k_{n-1})}{k_1! \cdots k_{n-1}!} \leq \lambda_1^{k_1} \cdots \lambda_{n-1}^{k_{n-1}} \tag{7.3}$$

It follows immediately from this that the formal series $\Phi^V_{\mathrm{inv}}(t_1, \cdots, t_{n+1}, 1)$ is convergent whenever $|t_1 e^{\frac{t_n}{n+2-d}}| < \lambda_1$, ..., $|t_{n-1} e^{\frac{(n-1)t_n}{n+2-d}}| < \lambda_{n-1}$. It seems that the generating function Φ^V is also convergent in some region of $H^*(V, C)$, since all other cohomology classes are dominated by powers of the Poincare dual of the hyperplane section. It is not hard to give λ_i explicitly. However, it seems unclear what is exactly the asymptotic growth of σ^V_d.

To generalize Theorem 7.1 to any Fano manifolds, we need further information on the GW-invariants Ψ^V. Assuming that $H^2(V, Z)$ generates the cohomology ring and using the associativity law, Kontsevich and Manin showed (see [KM]) that $\Psi^V_{(A,0,k)}(\{\alpha_i\})$ can be computed in terms of 3-point invariants for rational curves. However, the assumption is too strong for most Fano manifolds. A natural question is whether or not one can compute the restriction of Ψ^V to the part $H^*_{\mathrm{inv}}(V, Z)$, which is generated by $H^2(V, Z)$. The key question is if the restriction Ψ^V_{inv} satisfies the WDVV equation, which would follow from (3.3) for rational curves. Therefore, one can show

Theorem 7.2. *Let V be a Fano manifold such that* (3.3) *is true for rational curves. Then any invariant $\Psi^V_{(A,0,k)}(\{\alpha_i\})$, where $\{\alpha_i\} \subset H^*_{\mathrm{inv}}(V, Z)$, can be computed in terms of 3-point invariants for rational curves. In particular, the number of rational curves with sufficiently large degree can be computed.*

Finally, let us say a few words on higher genus cases. The composition law implies that all GW-invariants with fixed conformal structure can be computed by taking the "trace" on $\Psi^V_{(A,0,k)}$. However, it is still unclear how $\Psi^V_{(A,g,k)}([p]; \cdots)$ can be used to count the number of irreducible genus g algebraic curves in V with fixed conformal structure.

8 The Generalized Witten Conjecture

In this section, using the GW-invariant Ψ^V, we put a conjecture of Witten ([W2]) on a rigorous mathematical footing. We will also verify this conjecture mathematically for small genus. Most arguments in this section are due to Witten.

As before, we denote by $\overline{\mathcal{U}}_{g,k}$ the universal family of curves over $\overline{\mathcal{M}}_{g,k}$. Then each marked point x_i gives rise to a section, still denoted by x_i, of the fibration $\overline{\mathcal{U}}_{g,k} \mapsto \overline{\mathcal{M}}_{g,k}$. If $\mathcal{K}_{\mathcal{U}|\mathcal{M}}$ denotes the cotangent bundle to fibers of this fibration, we put $\mathcal{L}_{(i)} = x_i^*(\mathcal{K}_{\mathcal{U}|\mathcal{M}})$. Following Witten, we put

$$\langle \tau_{d_1,\alpha_1}\tau_{d_2,\alpha_2}\cdots\tau_{d_k,\alpha_k}\rangle_g(q) \;=\; \sum_{A\in H_2(V,\mathbf{Z})} \Psi^V_{(A,g,k)}([K_{d_1,\cdots,d_k}];\{\alpha_i\})\,q^A \quad (8.1)$$

where $\alpha_i \in H_*(V,\mathbf{Q})$ and $[K_{d_1,\cdots,d_k}]$ is the Poincare dual of $c_1(\mathcal{L}_{(1)})^{d_1}\cup\cdots\cup c_1(\mathcal{L}_{(k)})^{d_k}$. Symbolically, $\tau_{d,\alpha}$'s denote "quantum field theory operators". As before, choose a basis $\{\beta_a\}_{1\le a\le N}$ of $H_{*,\mathrm{even}}(V,\mathbf{Z})$ modulo torsion. We introduce formal variables $t_{r,a}$, where $r = 0,1,2,\cdots$ and $1\le a\le N$. Witten's generating function (cf. [W2]) is now simply defined to be

$$F^V(t_{r,a};q) = \left\langle e^{\sum_{r,a} t_{r,a}\tau_{r,\beta_a}}\right\rangle = \sum_{n_{r,a}}\prod_{r,a}\frac{(t_{r,a})^{n_{r,a}}}{n_{r,a}!}\left\langle\prod_{r,a}\tau_{r,\beta_a}^{n_{r,a}}\right\rangle \quad (8.2)$$

where the $n_{r,a}$ are arbitrary collections of nonnegative integers, almost all zero, labeled by r,a. The summation in (8.2) is over all values of the genus g and all homotopy classes A of (J,ν)-maps. Sometimes, we write F_g^V to be the part of F^V involving only GW-invariants of genus g. It is clear that F^V is a generalization of Φ^V in the last section. Indeed this generalized function contains more information on the underlying manifold, for instance, using Taubes' theorem (section 3), one can show that for an algebraic surface V of general type,

$$F^V(t_{r,a};q) \;=\; F^V(t_{r,a};0) + q^{K_V}e^{\tau_{0,0}} + \cdots, \quad (8.3)$$

while Φ^V depends only on the intersection form of V.

One of Witten's goals is to find out the equations which F^V satisfies. The case that V is a point corresponds to the 2-dimensional topological gravity, where F^V is governed by the KdV hierarchy, conjectured by Witten ([W2]) and verified by Kontsevich ([Ko]). In general, it is not clear what equations F^V would solve, though there are partial results for $V = \mathbf{C}P^1$ (see [Ho]). However, in [W2], Witten made a conjecture on F^V, which we will describe now.

Assume that $\beta_1 = [V]$. Following Witten's arguments in [W2], one can deduce from (2.6) that F^V satisfies the generalized string equation:

$$\frac{\partial F^V}{\partial t_{0,1}} = \frac{1}{2}\eta_{ab}t_{0,a}t_{0,b} + \sum_{i=0}^{\infty}\sum_a t_{i+1,a}\frac{\partial F^V}{\partial t_{i,a}} \quad (8.4)$$

Following Witten, one can introduce

$$U = \frac{\partial^2 F^V}{\partial t_{0,1}\partial t_{0,\sigma}},\; U' = \frac{\partial^3 F^V}{\partial t_{0,1}^2\partial t_{0,\sigma}},\;\cdots,\; U_\sigma^{(l)} = \frac{\partial^{l+2}F^V}{\partial t_{0,1}^{l+1}\partial t_{0,\sigma}},\ \text{for } l\ge 0 \quad (8.5)$$

We will regard $U^{(l)}$ to be of degree l. By a differential function of degree k we mean a function $G(U, U', U'', \cdots)$ of degree k in that sense. In particular, any function of the form $G(U)$ is of degree zero, and $(U')^2$ has degree two.

Witten's Conjecture: *For every $g \geq 0$, there are differential functions $G_{m,a,n,b}(U_\sigma, U'_\sigma, \cdots)$ of degree $2g$ such that*

$$\frac{\partial^2 F_g}{\partial t_{m,a} \partial t_{n,b}} = G_{m,a,n,b}(U_\sigma, U'_\sigma, \cdots) \tag{8.6}$$

up to and including terms of genus g.

To explain the significance of this conjecture, we quote from Witten ([W2]): if V is a point, it is a consequence of the KdV hierarchy. Indeed, the KdV hierarchy has a stronger property — $G_{m,a,n,b}$ is a differential function of degree at most $2(m+n)$ even for g tending to infinity.

It was pointed out by Witten that the composition law implies

$$\frac{\partial^3 F_0}{\partial t_{d_1,a_1} \partial t_{d_2,a_2} \partial t_{d_3,a_3}} = \sum_{a,b} \frac{\partial^2 F_0}{\partial t_{d_1-1,a_1} \partial t_{0,a}} \eta^{ab} \frac{\partial^3 F_0}{\partial t_{0,b} \partial t_{d_2,a_2} \partial t_{d_3,a_3}} \tag{8.7}$$

and consequently, the conjecture for $g = 0$.

In [RT2], we showed a weaker version of the conjecture for $g \leq 2$. The special case $g = 1$ and $V = CP^1$ was checked in [W2].

9 Quantum Cohomology and Symplectic Floer Cohomology

The symplectic Floer cohomology was defined for a hamiltonian symplectomorphism whose fixed points are nondegenerate. A crucial step to a solution of the Arnold conjecture is to show that the Floer cohomology is the same as the ordinary cohomology. Furthermore, Floer also introduced a multiplication on the Floer cohomology, *extrinsic multiplication*, by the ordinary cohomology. This multiplication is very useful for the solution of the Arnold conjecture in the case of degenerate fixed points [F], [OV]. On the other hand, there is a natural way to define an intrinsic multiplication on the Floer cohomology by using perturbed J-holomorphic maps from a pair of pants to the symplectic manifold. In fact, such an intrinsic multiplication has been formally defined by Betz and Rade [BR]. It has been conjectured that all these multiplications (intrinsic and extrinsic) are the same as the quantum multiplication. For the extrinsic multiplication, a physical argument was given by Sadov [Sa]. In [Pu], Piunikhin outlined a different approach towards the proof of the equivalence of the extrinsic multiplication and the quantum multiplication by using intersection theory. Recently, this conjecture was solved independently by Piunikhin, Salamon, Schwarz [PSS] and Ruan, Tian [RT3].

The two proofs are completely different. In [RT3], the authors established the symplectic Floer cohomology for Bott-type hamiltonian symplectomorphisms and then the equivalence follows as a corollary, while the method in [PSS] is more direct.

Given any function H on $V \times S^1$, we can associate a vector field X_H as follows:

$$\omega(X_H(z,t), v) = v(H)(z,t), \text{ for any } v \in T_z V \tag{9.1}$$

We call H a periodic hamiltonian and X_H the hamiltonian vector field associated to H. Let $\phi_t(H)$ be the integral flow of the hamiltonian vector field X_H. Then $\phi_1(H)$ is a hamiltonian symplectomorphism.

Definition 9.1. *A periodic hamiltonian H is of Bott-type if and only if the fixed-point set $F(\phi_1(H))$ of $\phi_1(H)$ is nondegenerate in the sense of Bott, i.e., $F(\phi_1(H))$ consists of smooth submanifolds in V and $\phi_1(H)$ is nondegenerate in the normal directions of these submanifolds.*

From now on, we fix a Bott-type periodic hamiltonian function H. Let $\mathcal{L}(V)$ be the space of contractible maps (sometimes called contractible loops) from S^1 into V and $\tilde{\mathcal{L}}(V)$ be the universal covering of $\mathcal{L}(V)$, namely, $\tilde{\mathcal{L}}(V)$ is as follows:

$$\tilde{\mathcal{L}}(V) = \{(u,x) | u \in \mathcal{L}(V), x : D^2 \to V \text{ such that } u = x|_{\partial D^2}\}/\sim, \tag{9.2}$$

where the equivalence relation $\sim$ is the homotopic equivalence of x. The covering group of $\tilde{\mathcal{L}}$ over $\mathcal{L}$ is $\pi_2(V)$. We can define a symplectic action functional on $\tilde{\mathcal{L}}(V)$,

$$a_H((u,x)) = -\int_{D^2} x^*\omega + \int_0^{2\pi} H(t,u(t))dt \tag{9.3}$$

It follows from the closedness of ω that a_H descends to the quotient space by $\sim$. The Euler-Lagrange equation of a_H is

$$\dot{u} + X_H(t,u(t)) = 0 \tag{9.4}$$

Let $\boldsymbol{R}(H)$ be the critical point set of a_H, i.e., the set of smooth contractible loops satisfying the Euler-Lagrange equation.

We can assign a Conley-Zehnder index to each component of $\boldsymbol{R}(H)$. So

$$\boldsymbol{R}(H) = \cup_i \boldsymbol{R}_i(H),$$

where $\boldsymbol{R}_i(H)$ consists of critical points in $\boldsymbol{R}(H)$ with the Conley-Zehnder index i.

Consider the space of trajectories

$$\tilde{\mathcal{N}} = \bigcup \tilde{\mathcal{N}}(i,j),$$

where

$$\tilde{\mathcal{N}}(i,j) = \{u \in \tilde{\mathcal{N}};\ \lim_{s\to-\infty} u_s \in \boldsymbol{R}_i(H),\ \lim_{s\to\infty} u_s \in \boldsymbol{R}_j(H)\}.$$

Clearly, $\boldsymbol{R}^1$ acts on $\tilde{\mathcal{N}}$ by translations in time. Let $\mathcal{N} = \tilde{\mathcal{N}}/\boldsymbol{R}^1$ and $\mathcal{N}(i,j) = \tilde{\mathcal{N}}(i,j)/\boldsymbol{R}^1$. For a generic J, $\mathcal{N}(i,j)$ is a smooth, oriented manifold of dimension $i-j-1+\dim \boldsymbol{R}_i(H)$. Furthermore, it admits a compactification $\overline{\mathcal{N}}(i,j)$, which is an abstract geometric chain (cf. [RT3, Theorem 2.4]). An abstract geometric chain of dimension n is a finite simplicial complex P such that its singular set P_s is a subcomplex and $\dim(P_s) \leq n-2$, $\dim(P_s \cap \partial P) \leq n-3$ (cf. [Fut]). A geometric chain is a pair (P, f), where P is an abstract geometric chain and f is a map from P.

One defines the boundary maps

$$\partial_-^i : \overline{\mathcal{N}}(i,j) \to \boldsymbol{R}_i(H),\ \partial_+^j : \overline{\mathcal{N}}(i,j) \to \boldsymbol{R}_j(H).$$

For a generic J, the boundary maps $\partial_\pm^i$ are smooth and transverse to each other whenever they have the same target space.

The group $\Gamma_0 = \{A \in \pi_2(V); c_1(V)(A) = 0\}$ acts on $\tilde{\mathcal{L}}(V)$ and generates (possibly) infinitely many components in $\boldsymbol{R}_i$. We denote by $\boldsymbol{R}_i^s(H)$ the components of $\boldsymbol{R}_i(H)$, so $\boldsymbol{R}_i(H) = \bigcup \boldsymbol{R}_i^s(H)$. For each $R_i^s(H)$, let $C_m(R_i^s(H), \boldsymbol{Z})$ be the space of geometric chains in $R_i^s(H)$ transverse to all the boundary maps into $R_i^s(H)$. Then $H_*(C_*(R_i^s(H), \boldsymbol{Z}), \partial) = H_*(R_i^s(H), \boldsymbol{Z})$. Now we define a boundary map:

$$\partial_k : C_m(R_i(H), \boldsymbol{Z}) = \oplus C_m(R_i^s(H), \boldsymbol{Z}) \mapsto C_{m+k-1}(R_{i-k}, \boldsymbol{Z})$$
$$\partial_k((P, f)) = (P \times_{R_i(H)} \overline{\mathcal{N}}(i, i-k), \partial_+^{i-k} \cdot \pi_2)$$

where π_2 is the natural projection from the fiber product $P \times_{R_i(H)} \overline{\mathcal{N}}(i, i-k)$ onto the second factor. We define the cochain complex $C^*(R_i(H), \boldsymbol{Z})$ of cochains $f = \sum_s f_s$, where $f_s \in \mathrm{Hom}(C_j(R_i^s(H), \boldsymbol{Z}), \boldsymbol{Z})$ satisfying:

$$\#\{s | f_s \neq 0, a_H(R_i^s) \geq c\} < \infty$$

for any c. Define the coboundary map δ_k by $\delta_k(\alpha) = \alpha \partial_k$. Furthermore, we define $\partial_0 = (-1)^{n+j}\partial$ and δ_0 similarly. Put $\delta^H = \oplus \delta_k$, then $\delta^H \delta^H = 0$. We define $HF^*(V, H)$ to be the cohomology of $(C^*(R_*(H), \boldsymbol{Z}), \delta^H)$. This is just the Floer cohomology in case H is nondegenerate. It is proved in [RT3] that $HF^*(V, H)$ is independent of particular choices of Bott-type Hamiltonian functions. Therefore, we may often write $HF^*(V)$ for $HF^*(V, H)$. Choosing $H = 0$, one can show that $HF^*(V) = H^*(V, \boldsymbol{Z}\{H_2(V)\})$.

One can introduce a multiplicative structure on $HF^*(V)$ in two ways: extrinsic and intrinsic. Here we just discuss the extrinsic multiplication. We refer the readers to [RT3] for the intrinsic one. The extrinsic product is an action of $H^*(V, \boldsymbol{Z})$ on $HF^*(V)$. Let A be a generic cocycle representing $\alpha^* \in H^k(V, \boldsymbol{Z})$ and $\gamma \in C^{m-i}(R_i(H), \boldsymbol{Z})$, then we define $A(\gamma) \in C^{m+k}(V, H)$

as follows: let $A(\gamma) = \sum A^j(\gamma)$, where $A^j(\gamma) \in C^{m+k-j}(R_j, \boldsymbol{Z})$. Consider the evaluation map

$$e : \tilde{\mathcal{N}}(j,i) \to V$$

by $e(u) = u(0,0)$. One can compactify $\tilde{\mathcal{N}}(j,i)$ by adding boundary components

$$\mathcal{N}(j,k_1) \times \cdots \times \mathcal{N}(k_{p-1},k_p) \times \tilde{\mathcal{N}}(k_p,k_{p+1}) \times \mathcal{N}(k_{p+1},k_{p+2}) \times \cdots \times \mathcal{N}(k_s,i).$$

We denote this compactification by $\overline{\mathcal{N}}^*(j,i)$. Then $\overline{\mathcal{N}}^*(j,i)$ is an abstract geometric chain (cf. [RT3, Lemma 3.1]) and e extends over $\overline{\mathcal{N}}^*(j,i)$.

For any geometric chain $(P,f) \in C_{m+k-j}(R_j^s(H), \boldsymbol{Z})$, the fiber product

$$P \times_{R_j^s} \overline{\mathcal{N}}^*(j,i)$$

is an abstract geometric chain of dimension $m+k$. Together with the map

$$\partial^+ \times e : P \times_{R_j^s} \overline{\mathcal{N}}^*(j,i) \to R_i(H) \times V,$$

it gives a geometric chain of $R_i(H) \times V$. We define

$$A(\gamma)((P,f)) = \gamma \times A((P \times_{R_j^s} \overline{\mathcal{N}}^*(j,i), \partial^+ \times e))$$

One can show that $\delta A \cdot - A \cdot \delta = 0$, i.e., $A\cdot$ is a cochain map. Therefore, it induces an action of α^* on $H^*(V, \boldsymbol{Z})$, the extrinsic product, on the Bott-type Floer cohomology $HF^*(V,H)$. Moreover, this action is independent of H. Choosing a nondegenerate H, one can easily show that this extrinsic multiplication is the same as Floer's. On the other hand, when $H = 0$, the extrinsic multiplication is just the quantum multiplication, if we view α^* as an element of the quantum cohomology. Therefore, we have

Theorem 9.2. *The extrinsic multiplication is equivalent to the quantum multiplication.*

This allows us to compute the multiplicative structure on the Floer cohomology.

10 Degenerating Families

A symplectic degeneration is a fibration $\pi : \mathcal{V} \mapsto \Delta \subset \boldsymbol{C}$ satisfying: $\mathcal{V}$ is a symplectic manifold and $V_t = \pi^{-1}(t)$ is a symplectic submanifold if $t \neq 0$, a union of symplectic submanifolds with simple normal crossings if $t = 0$. It is not hard to find an ω-tamed almost complex structure J on $\mathcal{V}$, such that π is J-holomorphic. Degenerations of algebraic manifolds are examples of such a degeneration. In this section, we will loosely discuss how GW-invariants on a smooth fiber are related to those on the singular fiber in a symplectic degeneration $\pi : \mathcal{V} \mapsto \Delta$. For simplicity, we assume that the central fiber V_0

has no triple points, i.e., no three components intersect. In fact, any degeneration can be decomposed into finitely many such simple degenerations, so the general case can be reduced to special cases by induction. For simplicity, we will only consider the case of rational curves here.

First we introduce a notion of marked rational V_0-trees, or simply V_0-trees. Let $V_0 = \bigcup_{i=1}^m V_{0i}$, where V_{0i} are irreducible. A V_0-tree T consists of an ordinary tree with branches T_0, T_j $(1 \le j \le l)$ with a marking $(i(j), a_j)$ attached to each T_j, satisfying: (1) $1 \le i(j) \le m$, $a_j \in H_2(V_{0i(j)}, \mathbf{Z})$; (2) If $T_j \cap T_{j'} \neq \emptyset$, then $i(j) \neq i(j')$; (3) For any j, the number of j' with fixed $i' = i(j')$ and $T_{j'} \cap T_j \neq \emptyset$ is the same as the intersection number $a_j \cdot V_{0i'}$ in $\mathcal{V}$. As usual, such a tree is rational if it does not contain any cycles, or equivalently, l is equal to the number of intersection points in T. Two V_0-trees T and T' are isomorphic if there is a tree isomorphism $\phi : \{T_j\} \mapsto \{T_j'\}$ such that $\phi(T_0) = T_0'$ and $(i(j), a_j) = (i'(j'), a'_{j'})$ whenever $\phi(T_j) = T'_{j'}$. Denote by G_T the group of automorphisms of T.

Rational V_0-trees arise naturally as configurations of limiting holomorphic maps. Fix $A \in H_2(\mathcal{V}, \mathbf{Z})$ and a generic inhomogeneous term ν on $\mathcal{V}$ compatible with the fibration. Denote by $\mathcal{M}_{A,t}$ the moduli space of (J,ν)-maps f from S^2 into V_t with homology class A when viewed as a map to $\mathcal{V}$. Let $f_t \in \mathcal{M}_{A,t}$, by the Gromov-Uhlenbeck compactness theorem and taking a subsequence if necessary, we may assume that f_t converges to f_0 in V_0. The map f_0 consists of a (J,ν)-map f_{00} and J-holomorphic maps f_{0j} from S^2, i.e., bubbles, where $1 \le j \le l$. Suppose that the sequence $\{f_t\}$ and (J,ν) are generic. Then images of f_{0j} intersect each other transversally. Now we associate f_0 with a rational tree $T(f_0)$ as follows: branches of $T(f_0)$ correspond to components of f_0 with $T(f_0)_0$ associated to f_{00}, $T(f_0)_j$ intersects with $T(f_0)_{j'}$ if and only if $Im(f_{0j})$ intersects $Im(f_{0j'})$, for each branch $T(f_0)_j$, the marking $(i(j), a_j)$ is given by the V_i in which f_{0j} lies and the homology class of f_{0j} in V_i. Obviously, $T(f_0)$ is rational. It can be proved (cf. [Ti1]) that $T(f_0)$ is actually a V_0-tree. This follows from that f_0 is the limit of smooth maps from S^2 and the total space $\mathcal{V}$ is smooth.

Let $\mathcal{T}_A$ be the set of equivalence classes of rational V_0-trees T with $A = \sum_{j=0}^{|T|-1} a_j$, where $|T|$ is the number of branches in T. For any $T \in \mathcal{T}_A$, put

$$\mathcal{M}_{T,0} = \{f_0 = \cup_j f_{0j} \mid f_{00} : \text{ a } (J,\nu) - \text{map}\}.$$

Note that $-\sum_{j=0}^{|T|-1} a_j \cdot V_{0i(j)}$ is twice the number $s(T)$ of intersection points in T. The rationality implies that $s(T) = |T| - 1$. Then one can show that the "virtual" dimension of $\mathcal{M}_{T,0}$ is given by

$$\begin{aligned} &\dim \mathcal{M}_{T,0} \\ =&\sum_{j=0}^{|T|} \left(2c_1(V_{i(j)})(a_j) + 2n - (n-1)a_j \cdot (V_0 - V_{0i(j)})\right) - 6(|T|-1) \\ =&2c_1(\mathcal{V})(A) + 2n + 2(n-3)(|T| - s(T) - 1) = \dim \mathcal{M}_{A,t} \end{aligned} \qquad (10.1)$$

In fact, using the gluing theorem in [RT1], one can show

Lemma 10.1. *Assume that $\mathcal{V}$ is semi-positive. We denote by* $\lim \mathcal{M}_{A,t}$ *the set of limits of maps in* $\mathcal{M}_{A,t}$. *Then for a generic* (J,ν), $\lim \mathcal{M}_{A,t}$ *coincides with* $\bigcup_{T\in\mathcal{T}_A} \mathcal{M}_{T,0}$ *outside finitely many subsets of codimension at least two.*

Following the arguments in defining the GW-invariant, one can define an invariant $\Psi^{V_0}_{(T,k)}$ on $H_*(\mathcal{V},Z)^{\otimes k}$ for each $T \in \mathcal{T}_A$. Roughly speaking, if $\alpha_1,\cdots,\alpha_k$ are in $H_*(\mathcal{V},Z)$, $\Psi^{V_0}_{(T,k)}(\{\alpha_i\})$ counts with sign the number of $(f;x_1,\cdots,x_k)$ satisfying: (1) $f \in \mathcal{M}_{T,0}$; (2) x_i are smooth points in the domain of f; (3) $f(x_i) \in \alpha_i$. It follows from Lemma 10.1,

Theorem 10.2. *Let $\mathcal{V}$ be a fibration as above. Then we have the following degeneration formula:*

$$\Psi^{V_t}_{(A,0,k)} = \sum_{T\in\mathcal{T}_A} \Psi^{V_0}_{(T,k)} \tag{10.2}$$

where $t \neq 0$.

This theorem has many applications. For instance, one can apply this to certain Calabi-Yau 3-folds V, such as quintics in CP^4 and prove that the GW-potential Φ^V is convergent for $Re(t_i)$ sufficiently large, or equivalently, the number of rational curves in V of degree d grows exponentially with d. The idea is simple: let V be a quintic, then it has a semistable degeneration into a quartic plus a hyperplane. Formula (10.2) tells us that the number of rational curves in V can be bounded in terms of the number of rational curves in a quartic and a hyperplane. The last two manifolds are all Fano, so we can apply results in section 7.

11 Mirror Symmetry and Calabi-Yau Manifolds

A two-dimensional conformal topological field theory consists of a linear superspace with a nondegenerate inner product and a family of correlations satisfying certain axioms. According to physicists, each Calabi-Yau manifold V gives rise to two conformal topological field theories, called A- and B-models (cf. [W5]). The correlation functions of the A-model are determinated by counting holomorphic maps from a Riemann surface into V, while the correlation functions of the B-model can be computed by calculating periods of classical differential forms. Mirror Symmetry gives a correspondence between those two models. One remarkable consequence of Mirror Symmetry is the prediction of Candelas, de la Ossa, Green and Parkes on the number of rational curves in a generic quintic hypersurface V in CP^4 ([COGP]). They predicted that the generating function Φ^V in section 6 can be found by studying the 4-th order linear differential operators satisfied by some hypergeometric functions, namely,

$$\sum_{k=0}^{\infty} \frac{(5k)!\, q^k}{(k!)^5}$$

consequently, they computed the number n_d of rational curves in V of degree d for all d. Only the first three numbers have been computed rigorously by mathematicians ([Kz], [ES]). They do coincide with the prediction of Candelas, etc.. However, Kontsevich has made some progress on the number of higher degree [Ko2]. Since 1991, more explicit Calabi-Yau manifolds have been studied and the number of rational curves on them have been computed based on the Mirror Symmetry Hypothesis (cf. [CKM], [HKTY]). Also certain Hodge theoretic explanation of B-models has been provided based on work of Griffiths, etc. (cf. [Mo]). However, Mirror Symmetry remains largely a mystery. Mathematical aspects of the A-model are essentially formal besides some special computations in concrete examples. The most interesting part on rational curves and geometry of Calabi-Yau manifolds still needs basic mathematical understanding. In this section, we will give a brief discussion on the Mirror Symmetry Conjecture in mathematical terms.

Let V be a Calabi-Yau manifold, i.e., a compact Kähler manifold with vanishing real first Chern class. It is known (cf. [Be2]) that a finite covering of V admits a decomposition $T^k_C \times V_1 \times V_2$, where V_1 is a hyperKähler manifold and $h^{2,0}(V, C) = 0$.

Now we assume that $\dim_C V = 3$, then there is no factor V_1 in the above decomposition, so without loss of generality, we assume that V is simply connected, which implies that $h^{1,0}(V, C) = 0$ and $h^{3,0}(V, C) = 1$. Let $\mathcal{M}_V$ be the moduli space of complex structures on V. By the Bogomolov-Tian-Todorov theorem, $T_V \mathcal{M}_V$ is naturally isomorphic to $H^1(V, T_V)$, the infinitesimal deformation space. To avoid technicalities, we assume that there is a universal family $\pi : \mathcal{U}_V \mapsto \mathcal{M}_V$. Then the direct image $\mathcal{F}^0 = \mathcal{R}^3 \pi_* C_{\mathcal{U}_V}$ is a holomorphic bundle over $\mathcal{M}_V$. There is a canonical flat connection ∇^B, namely, the Gauss-Manin connection, on $\mathcal{F}^0$ such that all integral cohomology classes are parallel sections. The bundle $\mathcal{F}^0$ has a natural filtration

$$\mathcal{F}^3 \subset \mathcal{F}^2 \subset \mathcal{F}^1 \subset \mathcal{F}^0$$

where $\mathcal{F}^k$ is the subbundle with fibers $\mathcal{F}^k|_V = \oplus_{p \geq k} H^{p,3-p}(V, C)$. The connection ∇^B is compatible with this filtration in the sense (Griffiths transversality): ∇^B can be written as $\nabla^{LC} + \theta + \overline{\theta}$, where ∇^{LC} is a hermitian connection and θ is holomorphic such that for all $k > 0$,

$$\theta : \mathcal{F}^k \mapsto \mathcal{F}^{k-1} \otimes \Omega^1_{\mathcal{U}_V|\mathcal{M}_V}, \ \theta^2 = 0$$

Note that $T_{\mathcal{U}_V|\mathcal{M}_V}$ is naturally isomorphic to $(\mathcal{F}^2/\mathcal{F}^3) \otimes (\mathcal{F}^3)^*$. It follows that θ is locally determinated by a holomorphic section, still denoted by θ, in the symmetric product $S^3 \Omega^1_{\mathcal{U}_V|\mathcal{M}_V}$.

On the other hand, the prepotential Φ^V in section 6 induces a flat connection ∇^A, at least formally, on $\mathcal{H}^0 = \oplus_{0\le k\le 3} H^k(V,\Omega^k)$ over $H^1(V,\Omega^1)$. If we write $\nabla^A = \nabla^{LC} + \Gamma$, then

$$\begin{aligned}\eta_{ae}\Gamma^e_{bc}(t) &= \frac{\partial^3\phi^V}{\partial t_a\partial t_b\partial t_c}(t)\\ &= \psi^V_{(0,0,3)}(\beta_a,\beta_b,\beta_c) + \sum_{A\neq 0}\frac{\beta^*_a(A)\beta^*_b(A)\beta^*_c(A)\, n_A e^{t^*(A)}}{1-e^{t^*(A)}}\end{aligned}$$

where n_A is the number of irreducible rational curves in V with homology class A.

As pointed out in the last section, if V is any complete intersection in CP^N, such as a quintic hypersurface, the connection ∇^A is actually well-defined in the region $\{\omega \in H^1(V,\Omega^1) | Re(\omega) << 0\}$. It is flat and is compatible with the obvious filtration $\mathcal{H}^p = \oplus_{0\le k\le 3-p} H^k(V,\Omega^k)$. The following conjecture is folklore.

Conjecture 11.1: *∇^A can be always extended to be a flat meromorphic connection on a quasi-projective manifold, moreover, it is often the same as ∇^B of another Calabi-Yau manifold V' via variation of Hodge structures.*

This is one important part of the Mirror Symmetry Conjecture. The other part is to show that the natural coordinates in the definition of ∇^A are the same as those flat coordinates, which were predicted by physicists (cf. [COGP]) and mathematically reformulated in [Mo]. In particular, the Mirror Symmetry Conjecture implies that the prepotential F^A on V can be identified with certain F^B on V' after an appropriate change of variables (cf. [COGP], [Mo]), where the Calabi-Yau manifold V' is given in Conjecture 11.1 and called the mirror of V. The prepotential F^B is much easier to calculate. For instance, in [COGP], Candelas et. al. computed the number of rational curves in a quintic hypersurface by calculating F^B on the mirror manifold (assuming that it exists).

It is obvious that the part of the conjecture on the extension of ∇^A is more tractable. Let us provide an evidence for this conjecture as follows.

Let V be a Calabi-Yau 3-fold and C_i $(1 \le i \le k)$ be smooth rational curves in V with $H^1(C_i, N_{C_i|V}) = 0$, where $N_{C_i|V}$ denotes the normal bundle of C_i in V. Contracting these rational curves, one obtains a 3-fold V_0 with k ordinary double points. A smoothing of V_0 is a smooth 4-fold $\pi : \mathcal{V} \mapsto \Delta \subset C$ such that its generic fiber $\pi^{-1}(t)$ is smooth and $V_0 = \pi^{-1}(0)$. Put V_t to be the generic fiber. This process from V to V_t is actually an analytic surgery. It provides a method of finding new complex manifolds, which is due to Clemens (see [Fr]). It is shown by Friedman, Kawamata, Tian ([Fr], [Ka], [Ti2]) that V_0 has a smoothing if and only if there are nonzero λ_i satisfying: $\sum_i \lambda_i[C_i] = 0$ in $H_2(V,C)$. Such V and V_t are said to be adjacent to each other. In general, V_t may not be projective. We say that two Calabi-Yau 3-folds V and V' are united if there are finitely many Calabi-Yau 3-folds V_α

($\alpha = 1, \cdots, L$) such that $V_1 = V$, $V_L = V'$ and V_α, $V_{\alpha+1}$ are adjacent to each other for all α. In fact, R. Friedman, M. Reid speculated independently that any two Calabi-Yau 3-folds are united.

Now suppose that two Calabi-Yau 3-folds V, V' are united. We will see that the GW-invariants Ψ^V and $\Psi^{V'}$ are related in a simple way. It is clear that we only need to show this in case V and V' are adjacent. Without loss of generality, assume that $V' = V_t$ as above. There is a natural map $p: H_2(V, \mathbf{Z}) \mapsto H_2(V', \mathbf{Z})$. One can show that the kernel of p is $\sum_{i=1}^k \mathbf{Z}[C_i]$, i.e.,

$$H_2(V', \mathbf{Z}) = H_2(V, \mathbf{Z}) / \sum_{i=1}^{k} \mathbf{Z}[C_i]$$

For simplicity, we may assume that $H_2(V', \mathbf{Z})$ is torsion-free.

By perturbing the complex structure of V outside C_i, one may arrange all holomorphic rational curves other than C_i to lie entirely in $V \setminus \bigcup_i C_i$. These curves will survive during the blow-down and smoothing, so they are in 1-1 correspondence with rational curves in V'. Then for any nonzero A' in $H_2(V, \mathbf{Z})$, we have

$$n_{A'} = \sum_{p(A)=A'} n_A,$$

where $n_{A'}$ (resp. n_A) is the number of irreducible rational curves in V' (resp. V) with homology class A' (resp. A).

On the other hand, if $p(A) = 0$, then $n_A = \#\{C_i | [C_i] = A\}$, in particular, $n_{[mC_i]} = 0$ if there is no C_j with $[C_j] = m[C_i]$. Here we used the fact that V' is projective.

For any t with Poincare dual t^* in $H^2(V, \mathbf{C})$, we have

$$\begin{aligned} \frac{\partial^3 \Phi^V}{\partial t_a \partial t_b \partial t_c(t)} &= \Psi^V_{(0,0,3)}(\beta_a, \beta_b, \beta_c) + \sum_{p(A) \neq 0} \frac{\beta_a^*(A)\beta_b^*(A)\beta_c^*(A)\, n_A e^{t^*(A)}}{1 - e^{t^*(A)}} \\ &\quad + \sum_{A=[C_i]} \frac{\beta_a^*(A)\beta_b^*(A)\beta_c^*(A)\, n_A e^{t^*(A)}}{1 - e^{t^*(A)}} \\ &= \Phi_1^V + \Phi_2^V \end{aligned}$$

We may identify any t' in $H_{n-2}(V', \mathbf{C})$ with a point in $H_{n-2}(V, \mathbf{C})$. Then $t'^*([C_i]) = 0$. It follows from the above that as t tends to t', $\Phi_1^V(t)$ converges to $\Phi^{V'}(t')$. On the other hand, $\Phi_2^V(t)$ has a simple pole at t'.

This shows how the connection ∇^A changes, when one Calabi-Yau manifold V moves to another Calabi-Yau V'. Symmetrically, when V' moves to V through degeneration and then small resolution, the connection ∇^B changes in a similar way as the above. This makes us wonder if the above motion from one Calabi-Yau to another is preserved under Mirror Symmetry, more precisely, if V moves to V' through blow-down and smoothing, then the mirror of V goes to the mirror of V' through degeneration and small resolution.

It seems to be even more interesting to see what would happen when one of united Calabi-Yau 3-folds, say V', is not projective, for example, $H_2(V', Z) = 0$. In this case, V' has to be diffeomorphic to the connected sum of finitely many copies of $S^3 \times S^3$.

Nevertheless, for the first part of Conjecture 11.1, the central problem is to find the monodromy group of ∇^A. The above method of studying adjacent Calabi-Yau manifolds may provide an insight into this problem.

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