Automorphic forms and Lie algebras

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Introduction

This paper is mainly an advertisement for one particular Lie algebra called the fake monster Lie algebra. The justification for looking at just one rather obscure object in what is supposed to be a general survey is that the fake monster Lie algebra has already led directly to the definition of vertex algebras, the definition of generalized Kac-Moody algebras, the proof of the moonshine conjectures, a new family of automorphic forms, and it is likely that there is still more to come from it.

The reader may wonder why we do not start by looking at a simpler example than the fake monster Lie algebra. The reason is that the fake

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monster is the simplest known example in the theory of non-affine Lie algebras; the other examples are even worse.

Here is a quick summary of the rest of this paper. The first sign of the existence of the fake monster came from Conway's discovery that the Dynkin diagram of the lattice $II_{25,1}$ is essentially the Leech lattice. We explain this in section 1. For any Dynkin diagram we can construct a Kac-Moody algebra, and a first approximation to the fake monster Lie algebra is the Kac-Moody algebra with Dynkin diagram the Leech lattice. It turns out that to get a really good Lie algebra we have to add a little bit more; more precisely, we have to add some "imaginary simple roots", to get a generalized Kac-Moody algebra. Next we can look at the "denominator function" of this fake monster Lie algebra. For affine Lie algebras the denominator function is a Jacobi form which can be written as an infinite product [K, chapter 13]. For the fake monster Lie algebra the denominator function turns out to be an automorphic form for an orthogonal group, which can be written as an infinite product. There is an infinite family of such automorphic forms, all of which have explicitly known zeros. (In fact it seems possible that automorphic forms constructed in a similar way account for all automorphic forms whose zeros have a "simple" description.) Finally we briefly mention some connections with other areas of mathematics, such as reflection groups and moduli spaces of algebraic surfaces.

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1 The Leech lattice and $II_{25,1}$

In this section we explain Conway's calculation of the reflection group of the even Lorentzian lattice $II_{25,1}$, so we first recall the definition of this lattice. A lattice is called even if the norms (v, v) of all vectors v are even and is called odd otherwise. The lattice is called unimodular if every element of the dual of L is given by the inner product with some element of L. The positive or negative definite lattices seem impossible to classify in dimensions above about 30 as there are too many of them, but the indefinite unimodular lattices have a very simple description: there is exactly one odd unimodular lattice $I_{r,s}$ of any given dimension $r+s$ and signature $r-s$ for positive integers r and s, and there is exactly one even unimodular lattice $II_{r,s}$ if the signature $r-s$ is divisible by 8 and no even unimodular ones otherwise. (Note that the II in $II_{r,s}$ is two I's and not a capital Π!) In particular there is a unique even unimodular 26 dimensional Lorentzian lattice $II_{25,1}$.

The odd lattice $I_{r,s}$ can be constructed easily as the set of vectors

$$(n_1, \dots, n_{r+s}) \in \mathbb{R}^{r,s} \text{ with all } n_i \in \mathbb{Z}.$$

(The norm of this vector in $\mathbb{R}^{r,s}$ is $n_1^2 + \cdots n_r^2 - n_{r+1}^2 - \cdots n_{r+s}^2$.) The even lattice can be constructed in the same way, except that the conditions on the coordinates are that their sum is even, and they are either all integers or all integers $+1/2$. Notice that the vector $(1/2, \dots, 1/2)$ has even norm $(r-s)/4$ because the signature $r-s$ is divisible by 8. For $r = 8, s = 0$ this is of course just the usual construction of the E_8 lattice. There is a second way to construct $II_{25,1}$: if Λ is the Leech lattice (the unique 24 dimensional even unimodular positive definite lattice with no roots) then $\Lambda \oplus II_{1,1}$ is an even 26 dimensional Lorentzian lattice and is therefore isomorphic to $II_{25,1}$. For the lattice $II_{1,1}$ it is convenient to use a different coordinate system: we represent its vectors as pairs $(m, n) \in \mathbb{Z}^2$ with the norm of (m, n) defined to be $-2mn$, so that $(1, 0)$ and $(0, 1)$ are norm 0 vectors. We use this to write vectors of $II_{25,1}$ in the form (λ, m, n), where $\lambda \in \Lambda$ and $m, n \in \mathbb{Z}$, where this vector has norm $\lambda^2 - 2mn$.

Conversely Conway showed [CS] that we can run this backwards and give a very short construction of the Leech lattice. If we let

$$\rho = (0, 1, 2, \dots, 22, 23, 24, 70) \in II_{25,1}$$

then $\rho^2 = 0$ and $\rho^\perp/\rho$ is isomorphic to the Leech lattice.

In the rest of this section we describe Vinberg's algorithm for finding simple roots of reflection groups and Conway's application of it. If L is any Lorentzian lattice the norm 0 vectors form two cones and the negative norm vectors are the "insides" of these cones. We select one cone and call it the positive cone C. The set of norm -1 vectors in C forms a copy of hyperbolic space H. (The metric on H is just the pseudo Riemannian metric on $L \otimes \mathbb{R}$ restricted to H, where it becomes Riemannian.) The group $Aut(L \otimes \mathbb{R})^+$ of all rotations of $L \otimes \mathbb{R}$ mapping C to itself acts on the hyperbolic space H, and is in fact the group of all isometries of H. In particular the group $Aut(L)^+$ of all automorphisms of L fixing the positive cone can be thought of as a discrete group of isometries of H. We let W be the subgroup of $Aut(L)^+$ generated by reflections. These reflections can be described as follows: if r is a positive norm vector of L such that $(r, r)|2(r, s)$ for all $s \in L$ then the reflection in $r^\perp$ given by $s \mapsto s - 2r(r, s)/(r, r)$ acts on $L \otimes \mathbb{R}$ and by restriction on H, where it is reflection in the hyper-space $r^\perp \cap H$.

The reflection hyperspaces divide hyperbolic space into cells called the Weyl chambers of W.

Just as in the case of finite Weyl groups the Weyl group W acts transitively on the Weyl chambers. We select one Weyl chamber D and call it the fundamental Weyl chamber. Then the full group $Aut(L)^+$ is the semidirect product of W and the subgroup of $Aut(L)^+$ fixing the fundamental Weyl chamber D. In particular if we can describe D this more or less determines the group $Aut(L)^+$.

Vinberg invented the following algorithm for finding the shape of D. First choose any point c in the fundamental Weyl chamber D. Then find the faces of D in order of their distance from c. Vinberg showed that a reflection hyperplane is a face of D if and only if it makes an angle of at most $\pi/2$ with each face of D nearer c (where the angle between faces means the angle as seen from inside D). This means we can find all the walls of D recursively in order of their distances from c.

It is convenient to rephrase this algorithm in terms of the lattice L rather then the hyperbolic space H. Instead of a point c we choose a vector ρ inside the closed positive cone C. For simplicity we assume that all the roots of W have norm 2 (which is the only case we will use later). Then we replace the distance from a hyperplane to the point c by the height $-(\rho, r)$ of r where r is any root having inner product at most 0 with ρ. Then as before we can find all the simple roots (i.e., positive roots r orthogonal to a face of D) in order of their heights, by observing that a positive root r is simple if and only if it has inner product at most 0 with all simple roots of smaller height.

We can now describe Conway's calculation [C] of the simple roots of the reflection group W of $II_{25,1}$ using Vinberg's algorithm. We choose the vector ρ to be $(0,0,1) \in \Lambda \oplus II_{1,1}$. There are no roots of height 0 because the Leech lattice Λ has no roots. The vectors $(\lambda, 1, \lambda^2/2-1)$ for $\lambda \in \Lambda$ are the norm 2 simple roots of height 1 and it is easy to check that they form a set isometric to the Leech lattice Λ.

Conway's amazing discovery was that these are the only simple roots of W. To see this suppose that (v, m, n) with $v^2 - 2mn = 2$ is any other simple root of height $m > 1$. Then (v, m, n) has inner product at most 0 with all the simple roots $(\lambda, 1, \lambda^2/2-1)$ for all $\lambda \in \Lambda$ and an easy calculation shows that this implies that $|v/m - \lambda| > \sqrt{2}$ for all λ. But Conway, Parker, and Sloane [CS, chapter 23] showed that the Leech lattice has covering radius exactly $\sqrt{2}$, in other words $\Lambda \otimes \mathbb{R}$ is just covered by closed balls of radius $\sqrt{2}$ around each lattice point of Λ. In particular there is some vector λ of Λ with $|v/m - \lambda| \leq \sqrt{2}$, and this contradiction proves that (v, m, n) cannot be a simple root.

To summarize, this gives a complete description of the group $Aut(II_{25,1})^+$ as a semidirect product $W.G$ where W is the reflection group with Dynkin diagram given by the Leech lattice as above, and where G is the group of automorphisms of this Dynkin diagram. It is not hard to see that G is just the group of all isometries of the "affine" Leech lattice (which is the Leech lattice with the origin "forgotten"), and so is a semidirect product $\Lambda.Aut(\Lambda)$ where $Aut(\Lambda)$ is the double cover $(\mathbb{Z}/2\mathbb{Z}).Co_1$ of Conway's largest sporadic simple group Co_1. So $Aut(II_{25,1})$ can be written as

$$W.\Lambda.(\mathbb{Z}/2\mathbb{Z}).Co_1.$$

This description of Co_1 as the group sitting at the top of $Aut(II_{25,1})$ seems to be the simplest and most natural description of any of the sporadic simple groups.

Given any Dynkin diagram there is an associated Kac-Moody algebra. As the Dynkin diagram of the reflection group of $II_{25,1}$ is so nice, this suggests that the associated Kac-Moody algebra M_{KM} should also be very nice. This turns out to be almost but not quite true: we first have to modify the Dynkin diagram by adding some imaginary simple roots in order to get a nice Lie algebra (which will be the fake monster Lie algebra M, whose maximal Kac-Moody subalgebra is M_{KM}.). Before discussing this we recall some facts about Kac-Moody algebras.

2 Kac-Moody algebras

Suppose that G is a finite dimensional simple complex Lie algebra with Cartan subalgebra H. Then G has a symmetric invariant bilinear form $(,)$ which induces a form on H, which we use to identify H with its dual H^*. The roots $\alpha \in H^*$ are the nonzero eigenvalues of the adjoint action of H on G and are the roots of a finite reflection group W acting on H called the Weyl group of G. The simple roots α of some fundamental Weyl chamber D of W can be identified with the points of the Dynkin diagram of G.

We define the numbers a_{ij} to be the inner products (α_i, α_j) of the simple roots of G and are the entries of the "symmetrized Cartan matrix" A of G. We can and will normalize the inner product so that all the diagonal entries are positive reals. These numbers have the following properties:

$$\begin{aligned} &a_{ii} > 0 \\ &a_{ij} = a_{ji} \\ &a_{ij} \leq 0 \text{ if } i \neq j \\ &2a_{ij}/a_{ii} \in \mathbb{Z}. \end{aligned}$$

The Cartan matrix is normalized so that it has integral entries and diagonal entries all equal to 2 but is not symmetric; for our purposes it is better to use matrices that are symmetric but do not in general have integral entries or diagonal coefficients equal to 2. It is easy to get from a symmetrized Cartan matrix to the Cartan matrix just by multiplying all the rows by suitable constants.

We can recover the Lie algebra G from the matrix A as the Lie algebra generated by an $sl_2 = \langle e_i, f_i, h_i \rangle$ for each simple root α_i, subject to the following relations (due to Serre and Harish-Chandra) depending on the numbers a_{ij}:

$$\begin{aligned} [e_i, f_i] &= h_i \\ [e_i, f_j] &= 0 \text{ for } i \neq j \\ [h_i, e_j] &= a_{ij} e_j \\ [h_i, f_j] &= -a_{ij} f_j \\ Ad(e_i)^{1-2a_{ij}/a_{ii}} e_j &= 0 \\ Ad(f_i)^{1-2a_{ij}/a_{ii}} f_j &= 0 \end{aligned}$$

Kac and Moody noticed that we can define a Lie algebra in the same way for any matrix A satisfying the conditions above; these are the symmetrizable Kac-Moody algebras. (They also defined Lie algebras for non symmetrizable Cartan matrices, which we will not use.) Kac-Moody algebras have many of the properties of finite dimensional simple Lie algebras: we can define roots, Weyl chambers, Weyl groups, Cartan subalgebras, Verma modules, and so on by copying the usual definitions for finite dimensional Lie algebras. There is a Weyl-Kac character formula for the characters of some simple quotients of Verma modules, which for finite dimensional Lie algebras is just the usual Weyl character formula for finite dimensional representations. The only case of this we will use is the Weyl-Kac denominator formula, which is the Weyl-Kac character formula for the trivial one dimensional module of character 1, which says that

$$\sum_{w \in W} \det(w) w(e^\rho) = e^\rho \prod_{\alpha > 0} (1 - e^\alpha)^{mult(\alpha)}$$

where ρ is a special vector called the Weyl vector, the product is over all positive roots α, and *mult* is the multiplicity of a root, in other words the dimension of the corresponding root space.

Kac showed that we recover the Macdonald identities if we apply this denominator formula to certain special Kac-Moody algebras called affine Kac-Moody algebras, which are roughly central extensions of the Lie algebras $G[t, 1/t]$ of Laurent polynomials with coefficients in a finite dimensional Lie algebra G or twisted versions of these. For example the denominator formula of the Lie algebra $sl_2[t, 1/t]$ is the Jacobi triple product identity

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} z^n = \prod_{n>0}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}/z).$$

We would like to find some generalizations of the Macdonald identities corresponding to some Kac-Moody algebras other than the affine ones. To do this we need to find some Kac-Moody algebras for which both the root multiplicities $mult(\alpha)$ and the simple roots are known explicitly in some easy form. Knowing the simple roots is equivalent to knowing the Weyl group and hence the sum in the Weyl-Kac denominator formula, and knowing the root multiplicities is equivalent to knowing the product in the denominator formula. Unfortunately there are no known examples of Kac-Moody algebras (other than sums of finite dimensional and affine ones) for which both the simple roots and root multiplicities are known explicitly. It is of course always possible to calculate the multiplicity of any given root of any Lie algebra if we know the simple roots by using the denominator formula. This can be used to give either a recursive formula for the root multiplicities (due to Peterson) or a large and complicated alternating sum for the multiplicities. Unfortunately neither of these seems to give a satisfactory simple and explicit formula for the root multiplicities of any non affine Kac-Moody algebra. There have been several numerical calculations of the multiplicities of some of the easier Kac-Moody algebras, and the impression one gets from looking at these tables is that the root multiplicities look rather complicated and random.

3 Vertex algebras

As motivation for the definition of vertex algebras we first recall a short construction for the finite dimensional simple complex Lie algebras from their root lattice. We will just do the cases of the Lie algebras A_n, D_n, and E_6, E_7, E_8 (the others can be obtained as fixed points of these Lie algebras under diagram automorphisms). Suppose that L is the root lattice of G, so that the roots of G are exactly the norm 2 vectors of L. We construct a central extension $\hat{L}$ of L by a group of order 2 generated by an element ζ with the property that $e^a e^b = \zeta^{(a,b)} e^b e^a$ if e^a, e^b are lifts to $\hat{L}$ of $a, b \in L$. This central extension is unique up to (nonunique) isomorphism. We define G to be the Z-module

$$L \oplus \sum_{\alpha^2=2} e^\alpha$$

where the sum is over a set of lifts of the norm 2 vectors of L, and where we identify ζe^a with $-e^a$. We define the Lie bracket on G by

$$\begin{aligned}&[a,b] = 0 \text{ for } a,b \in L\\&[a,e^b] = -[e^b,a] = (a,b)e^b \text{ if } a,b \in L, (b,b) = 2\\&[e^a,e^b] = a \text{ if } a = -b, e^a e^b \text{ if } (a,b) = -1,\end{aligned}$$

and 0 otherwise.

Then it is not hard to check that this bracket is antisymmetric and satisfies the Jacobi identity, so it defines an integral form of the complex Lie algebras G. (Notice that if we did not first take a central extension of L the product would not be antisymmetric. In fact there is no canonical way to construct G from L because the automorphism group of L, which is more or less the Weyl group of G, is not usually a subgroup of the automorphism group of G but is only a subquotient.)

This construction gives a completely explicit basis for the Lie algebra G. We would like to do something similar for all Kac-Moody algebras. Unfortunately the construction above breaks down as soon as the lattice L is indefinite and there are norm 2 vectors a, b with $(a,b) \leq -2$ but $a \neq b$. The vertex algebra of a lattice will provide a generalization of the construction above for all lattices.

For simplicity we will only construct the vertex algebra of a lattice L in which all inner products are even; in general if some inner products are odd it is necessary to first replace L by a central extension $\hat{L}$ as above. We will also only do the construction over the complex numbers, although with slightly more effort it can be done over the integers. We define the underlying space of the vertex algebra V of L to be the universal commutative ring with derivation D generated by the complex group ring $\mathbb{C}(L)$ of L. It is not hard to work out the structure of the ring V: it is just the tensor product for the group ring of L with the symmetric algebra of the sum of a countable infinite number of copies L_i of L. It is best to think of V as a commutative ring together with an action of the additive formal group $\mathbb{C}$ of the complex numbers, the action of the formal group being given by the derivation D (which spans the Lie algebra of this formal group).

A vertex operator is just a formal series $v(z) = \sum_{n\in\mathbb{Z}} v_{-n-1}z^n$ where the v_n's are $\mathbb{C}$-linear operators from V to V such that for any $w \in V$ the elements $v_n w$ vanish for n sufficiently large. We can think of $v(z)$ as being a sort of formal operator valued meromorphic function of z, with v_n given formally as the residue of $v(z)z^n dz$ at $z = 0$. We will now define some vertex operators on V. For each $a \in L$ we define the vertex operator a^+ by $a^+(z) = \sum_{n\geq 0} D^n(e^a)z^n/n!$ (where the operator $D^n(e^a)$ is the operator of multiplication by the element $D^n(e^a)$). We define another vertex operator a^- for $a \in L$ by saying that $a^-(z)$ is the derivation from V to $V[z, 1/z]$ taking e^b to $z^{(a,b)}e^b$. Then we can check that all the vertex operators of the form a^+ commute with each other, all the operators $a^-(z)$ commute with each other. Finally we define the vertex operators $a(z)$ by $a(z) = a^+(z)a^-(z)$ for $a \in L$, so that the operators $a(y)$ and $b(z)$ formally commute with each other for any $a, b \in L$. in the sense that

$$(y-z)^N\big(a(y)b(z) - b(z)a(y)\big) = 0$$

for N sufficiently large depending on a and b. Notice that the coefficient of $y^m z^n$ in $a(y)b(z)$ is not usually the same as the coefficient of $y^m z^n$ in $b(z)a(y)$; it is only when both sides are multiplied by a power of $y - z$ that the coefficients become equal.

We can now define a vertex algebra (roughly) as a vector space V such that for each element v of V we are given a vertex operator $v(z)$, and these vertex operators all formally commute with each other. We also require that V should have an identity element 1 such that $v(0)1 = v$ and $1(z)v = v$. The best way to think of a vertex algebra is as a sort of commutative ring with a formal action of the group $\mathbb{C}$, where the action of $z \in \mathbb{C}$ on v is $v(z)$, and $v(z)w$ is the ring product of $v(z)$ and w. If the vertex operator $v(z)$ is "holomorphic" for all v, which means that $v_n = 0$ for $n \geq 0$, then a vertex algebra is just a commutative ring with derivation and the ring product is defined by $vw = v_{-1}w$. In general we can think of V as behaving like a commutative ring whose multiplication is not defined everywhere because it has poles; if we perturb one of the elements v, w by an element z of the group $\mathbb{C}$ then we get a meromorphic function $v(z)w$ of z (behaving like the product of v acted on by z and of w) which will in general have a pole at $z = 0$.

Suppose we have a vector space V acted on by a set of commuting operators, such that V is generated as a vector space by the images of an element $1 \in V$ under the ring R generated by these operators. Then it is easy to see that the map from R to V taking r to $r(1)$ is an isomorphism of vector spaces and hence gives V the structure of a commutative ring with unit element 1. There is a similar theorem for vertex operators: if V is a vector space acted on by an operator D with a compatible set of commuting vertex operators acting on it, such that V is generated from an element $1 \in V$ with $D(1) = 0$ by the components of these vertex operators, then we can make V into a vertex algebra. If we apply this to the space V above constructed from a lattice and to the commuting set of operators $a(z)$ for $a \in L$ we see that V can be given a vertex algebra structure. If the inner product on L is identically zero then this vertex algebra structure on V is essentially the same as the commutative ring structure defined above; in general the vertex algebra structure defined by some other inner product on V can be thought of as a sort of "meromorphic deformation" of this ring structure.

If u, v, and w are elements of a vertex algebra then $u(x)v(y)w = v(y)u(x)w$ is a meromorphic function of x and y with poles only at $x = 0$, $y = 0$, and $x = y$. This function is also equal to $(u(x-y)v)(y)w$, which is easy to understand if one interprets $u(x)$ as the action of the group element x on the element u. (This is easier to see if we denote the action of a group element x on a ring element u by u^x, when it becomes $u^x v^y w = (u^{x/y}v)^y w$.) If $f(x)$ is any function of x with poles only at $x = 0$ or $x = y$ then Cauchy's formula shows that the residue at 0 plus the residue at y is the integral around a large circle of $f(x)dx/2\pi i$. If we apply this to $f(x) = u(x)v(y)w$ and then take residues at $y = 0$, we find that

$$\int_y \int_x (u(x-y)v)(y)w\,dx\,dy = \int_y \int_x u(x)v(y)w\,dx\,dy - \int_y \int_x v(y)u(x)w\,dx\,dy$$

where the paths of integration of x are a small circle around y, a large circle around both y and 0, and a small circle around 0 in the 3 integrals, or in other words

$$(u_0 v)_0 w = u_0(v_0 w) - v_0(u_0 w).$$

If we first multiply the integrands by $(x-y)^q x^m y^n$ we get more complicated identities involving the operators $u_i v$ and some binomial coefficients, which are (non trivially) equivalent to the identities first used to define vertex algebras in [B86].

4 The no-ghost theorem and I. Frenkel's upper bound

Recall that the Virasoro algebra is a central extension of the Lie algebra of polynomial complex vector fields on the circle, which is spanned by elements L_i, $i \in \mathbb{Z}$, with $[L_i, L_j] = (i-j)L_{i+j}$. The vertex algebra of a nonsingular lattice L has a natural action of the Virasoro algebra on it and we define the physical subspace M of $V(L)$ to be the quotient of P^1/DP^0 by the kernel of a certain bilinear form, where P^i is the space of (lowest weight) vectors with $L_0(v) = iv$, $L_i(v) = 0$, $i > 0$. (The motivation for this comes from string theory, where this space is roughly the space of physical states of a chiral string moving on the torus $L \otimes \mathbb{R}/L$.) The vertex algebra $V(L)$ also has a natural L-grading induced by the obvious L grading on the group ring of L. If L is 26 dimensional and Lorentzian then the no-ghost theorem states (among other things) that the piece of degree $\alpha \in L, \alpha \neq 0$ has dimension $p_{24}(1-\alpha^2/2)$, where $p_{24}(n)$ is the number of partitions of n into parts of 24 colors. (In spite of several statements to the contrary in the mathematical literature, the original proof of the no-ghost theorem by Goddard and Thorn is mathematically rigorous.)

I. Frenkel showed [F] that if L is the lattice $II_{25,1}$ then the Kac-Moody algebra M_{KM} (with Dynkin diagram given by the Dynkin diagram of $II_{25,1}$) can be embedded as a subspace of the space of physical states of $V(II_{25,1})$, and in particular this gives $p_{24}(1-\alpha^2/2)$ as an upper bound on the multiplicities of the root spaces of this Lie algebra. (This work of Frenkel's was the main motivation for the definition of vertex algebras.)

Frenkel's method gives the same upper bound for the multiplicities of the roots of any Kac-Moody algebra of rank 26 all of whose roots have norm 2. For Kac-Moody algebras of rank k not equal to 26 it gives the weaker bound $p_{k-1}(1-\alpha^2/2) - p_{k-1}(-\alpha^2/2)$ which is slightly larger than $p_{k-2}(1-\alpha^2/2)$. There are examples of Lie algebras of ranks not equal to 26, (such as E_{10}, due to Kac and Wakimoto), some of whose root multiplicities are strictly larger than $p_{k-2}(1-\alpha^2/2)$, so the upper bound given by the no ghost theorem in rank 26 cannot be generalized in the obvious way to all ranks.

We can next ask how good Frenkel's upper bound $p_{24}(1-\alpha^2/2)$ for the root multiplicities of M_{KM} is. If we calculate the multiplicities of the roots of M_{KM} (for example by using the Peterson recursion formula) we find the following results. All norm 2 vectors have multiplicity 1, equal to Frenkel's upper bound $p_{24}(1-2/2)$. For norm 0 vectors there are 24 orbits of primitive norm 0 vectors corresponding to the 24 Niemeier lattices (24 dimensional even unimodular positive definite lattices). The correspondence is given as follows: if w is a nonzero norm 0 vector in $II_{25,1}$ then the quotient $w^\perp/w$ of the orthogonal complement $w^\perp$ of w by the space generated by w is a Niemeier lattice, and conversely if N is a Niemeier lattice then $N \oplus II_{1,1}$ is a 26 dimensional even Lorentzian lattice and is therefore isomorphic to $II_{25,1}$. A norm 0 vector in the $II_{1,1}$ gives a norm 0 vector w in $II_{25,1}$ with $w^\perp/w = N$. If z is any nonzero norm 0 vector in $II_{25,1}$ then it is not hard to check using the theory of affine Lie algebras that the multiplicity of the root nz is equal to the rank of the corresponding Niemeier lattice, which is 0 for the Leech lattice and 24 for any other Niemeier lattice. So the multiplicity of the vector nz is equal to Frenkel's upper bound $p_{24}(1-0^2/2) = 24$ except when z is a norm 0 vector corresponding to the Leech lattice. For norm -2 vectors the calculations take much more effort. There are 121 orbits of norm -2 vectors. One of the orbits has multiplicity 0, one has multiplicity 276, and the other 119 orbits all have multiplicity $324 = p_{24}(1-(-2)/2)$ equal to Frenkel's upper bound. Similarly there are 665 orbits of norm -4 vectors, all but 3 of which turn out to have multiplicity equal to $3200 = p_{24}(1-(-4)/2)$. Furthermore when calculating these multiplicities using (say) the Peterson recursion formula, it is apparent that the "deficiencies" in the multiplicities of vectors of negative norm are "caused" by the fact that the multiplicities of the norm 0 vectors corresponding to the Leech lattice are 0 rather than 24. This suggests that if we somehow added a 24 dimensional root space for each multiple of the norm 0 vector ρ then we would get a Lie algebra whose root multiplicities were exactly equal to Frenkel's upper bound. We already have a space M containing M_{KM} given by the space of all physical states of a chiral string in 26 dimensions. The remarks and calculations above strongly suggest that M itself should be a Lie algebra with simple roots given by the simple roots of M_{KM} together with multiples of ρ.

It is easy to construct a Lie algebra product on M using the theory of vertex algebras (which is not surprising given that vertex algebras were invented partly to construct such a Lie bracket). At the end of section 3 we saw that the vertex algebra product $u_0v = u_0(v)$ satisfies the identity

$$(u_0v)_0w = u_0(v_0w) - v_0(u_0w)$$

which is one version of the Jacobi identity, but it is not antisymmetric. However the sum $u_0v + v_0u$ does at least lie in the image of the operator $D = L_{-1}$, and using this it is easy to check that defining $[u, v]$ to be u_0v defines a Lie bracket on the quotient V/DV for any vertex algebra V (where DV is the image of V under the derivation D). The space M is a subquotient of the Lie algebra V/DV where V is the vertex algebra of the lattice $II_{25,1}$, so this defines a Lie algebra structure on M. (In fact M is even a subalgebra of V/DV, but this is harder to prove and depends heavily on the structure of M described later.)

There is another way to construct the Lie algebra M using semi-infinite cohomology. It follows from [FGZ] that the vector space M can be identified with a certain semi-infinite cohomology group, which therefore has a natural Lie algebra structure. Lian and Zuckerman study this in [LZ] and reconstruct this Lie algebra from an algebraic structure called a Gerstenhaber algebra on the full semi-infinite cohomology. In the case of the fake monster Lie algebra M we do not really get anything new because the semi-infinite cohomology with its algebraic structure can be reconstructed from the Lie algebra M with its bilinear form, but in more general cases this is no longer true, and semi-infinite cohomology is probably the correct way to construct Lie algebras.

We would now like to work out the structure of M. As M is close to being a Kac-Moody algebra this suggests that we should try to "force" M to be a Kac-Moody algebra and find out what the obstruction to this is. We summarize the facts we know about M: it is graded by $II_{25,1}$ with root multiplicities given by $p_{24}(1-\alpha^2/2)$, it has an invariant bilinear form $(,)$ induced by the form on the vertex algebra V, it has an involution ω induced by the automorphism -1 of $II_{25,1}$ (or more precisely by a lift of this to a double cover of $II_{25,1}$), and the contravariant form $(a,b)_0 = (a,\omega(b))$ is positive definite on the root spaces of all nonzero roots. (The form $(,)_0$ is not positive definite on the zero weight space M_0; this weight space is 26 dimensional and the form $(,)_0$ on it is Lorentzian. So M is an infinite dimensional Lorentzian space under $(,)_0$, in other words, it has a negative norm vector whose orthogonal complement is positive definite.) We try to make M into a Kac-Moody algebra by finding the elements e_i and then defining the remaining generators f_i and h_i by $f_i = \omega(e_i), h_i = [e_i, f_i]$. Any Kac-Moody algebra G can be written as a direct sum of subspaces $E\oplus H\oplus F$ where H is the Cartan subalgebra and E and F are the sums of the root spaces of the positive and negative roots. (Of course G is not the Lie algebra sum of the subalgebras E, F, and H, but only the vector space sum.) If we choose a positive integer n_i for each simple root α_i then we can $\mathbb{Z}$-grade G by giving e_i degree n_i, f_i degree $-n_i$, and h_i degree 0. Then $E = \oplus_{n>0}E_n$ is the sum of the positive degree subspaces E_n of G. The elements e_i are then a minimal set of generators of the space E, in the sense that the elements e_i of any fixed degree n are a basis for the quotient of E_n by the elements in the subalgebra generated by the E_k's for $k < n$. This suggests that we can recursively construct the elements e_i.

We carry out this process for the Lie algebra M. We grade it by choosing any negative norm vector v which has nonzero inner product with all norm 2 vectors, and defining the degree of M_r to be (r, v), so that we get a subalgebra $E = \oplus_{n>0} E_n$ of the positive root spaces of M. We recursively construct the elements e_i in E_n as a basis for the space of elements of E_n that are orthogonal to the subalgebra generated by all the e_i's we have previously constructed in E_k for $k < n$. Then all the e_i's we construct in this way generate E, essentially because the form $(,)_0$ is positive definite on E_n so that any space E_n is the sum of any subspace and its orthogonal complement. We can also arrange that the e_k's we construct are orthonormal and are eigenvectors of $H = M_0$. Finally we define the generators f_i and h_i in terms of e_i by $f_i = \omega(e_i), h_i = [e_i f_i]$ as above, and define the elements a_{ij} of a matrix A by $a_{ij} = (h_i, h_j)$. Now we ask whether the elements a_{ij} and the generators e_i, f_i, and h_i satisfy the conditions for Kac-Moody algebras in section 2. It turns out that they do not quite: instead they satisfy the slightly weaker conditions

$$a_{ij} = a_{ji}$$

$$a_{ij} \leq 0 \text{ if } i \neq j$$

$$2a_{ij}/a_{ii} \in \mathbb{Z} \text{ if } a_{ii} > 0$$

$$[e_i, f_i] = h_i$$

$$[e_i, f_j] = 0 \text{ for } i \neq j$$

$$[h_i, e_j] = a_{ij} e_j$$

$$[h_i, f_j] = -a_{ij} f_j$$

$$Ad(e_i)^{1-2a_{ij}/a_{ii}} e_j = 0 \text{ if } a_{ii} > 0,\ [e_i, e_j] = 0 \text{ if } a_{ij} = 0.$$

$$Ad(f_i)^{1-2a_{ij}/a_{ii}} f_j = 0 \text{ if } a_{ii} > 0,\ [f_i, f_j] = 0 \text{ if } a_{ij} = 0.$$

If all the diagonal entries a_{ii} are positive, in other words if all the simple roots are real (positive norm), then these conditions are exactly those defining symmetrizable Kac-Moody algebras. We will call any Lie algebra generated by relations satisfying the conditions above a generalized Kac-Moody algebra. (In fact generalized Kac-Moody algebras are slightly more general than this, because we also allow a few extra operations, such as adding outer derivations, quotienting by subspaces of H in the center of G, and taking central extensions.)

As the name implies, the theory of generalized Kac-Moody algebras is similar to that of Kac-Moody algebras (and finite dimensional simple Lie algebras), with some minor changes. The main difference is that we may have imaginary (norm ≤ 0) simple roots. These contribute some extra terms to the sum in the Weyl-Kac character formula. In particular the denominator formula now looks like

$$\sum_{w\in W} \det(w) \sum_{S} (-1)^{|S|} w(e^{\rho+\sum S}) = e^{\rho} \prod_{\alpha>0} (1-e^{\alpha})^{mult(\alpha)}.$$

The inner sum is over sets S of pairwise orthogonal imaginary simple roots, of cardinality $|S|$ and sum σS. Notice that knowing the sum on the left is essentially equivalent to knowing all the simple roots, because the real simple roots determine the Weyl group W in the outer sum, and the imaginary simple roots determine the inner sum. There is an explicit example of such a denominator formula at the end of this section.

Roughly speaking the imaginary simple roots do not really add anything very interesting to G; the complexity of G is closely related to the complexity of the Weyl group W of G, which only depends on the real simple roots of W. The imaginary simple roots only contribute some extra flab, which usually looks a bit like free or free abelian Lie algebras [J]. The main advantage of generalized Kac-Moody algebras is that, as we have seen above for M, it is sometimes possible to prove that some naturally occurring Lie algebra is a generalized Kac-Moody algebra using only general properties of M (in particular the existence of an "almost positive definite" contravariant bilinear form $(,)_0$). Most "natural" examples of Kac-Moody algebras other than affine ones (such as the Kac-Moody algebra M_{KM} with Dynkin diagram the Leech lattice, or the Frenkel-Feingold algebra [FF]) seem to be the subalgebra of some more natural generalized Kac-Moody algebra (such as the fake monster Lie algebra M) generated by some of the real simple root spaces. See the end of this section for an example.

So far we have seen that M is a generalized Kac-Moody algebra with known root multiplicities, so the next obvious thing to do is to try to work out the simple roots of M. The fact that we will be able to do this for M may be misleading: there are very few known examples of generalized Kac-Moody algebras for which the simple roots and root multiplicities are both known. For example it is possible to construct a generalized Kac-Moody algebra similar to M for any Lorentzian lattice L, but if L is any Lorentzian lattice other than $II_{25,1}$ then the simple roots of the generalized Kac-Moody algebra of L seem to be too messy to describe explicitly. Most of the several hundred known examples of generalized Kac-Moody superalgebras whose simple roots and root multiplicities are both known explicitly can be obtained by "twisting" the denominator formula for the fake monster Lie algebra in some way.

There are some obvious simple roots of M: the simple roots $(\lambda, 1, \lambda^2/2-1)$ of M_{KM} are the real simple roots of M, and the positive multiples of the norm 0 Weyl vector $\rho = (0,0,1)$ of M are simple roots of multiplicity 24. The calculations of root multiplicities above turn out to be equivalent to saying that M has no simple roots of norm -2 or -4, and suggest that M has no simple roots other than the ones of norm 2 or 0 above. This can be proved roughly as follows. We can get information about the imaginary simple roots using the fact that they appear in the sum defining the denominator function. The denominator function is not easy to describe completely, so the the key step is to look at the denominator function restricted to the 2 dimensional space of vectors of the form $(0, \sigma, \tau) \in II_{25,1} \otimes \mathbb{C}$ with $\Im(\sigma) > 0$, $\Im(\tau) > 0$. The restriction to this space of the infinite product defining the denominator function is

$$p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1-p^m q^n)^{c'(mn)}$$

where $p = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$, and the numbers $c'(mn)$ are defined by

$$\sum_{n\in\mathbb{Z}} c'(n)q^n = \Theta_\Lambda(\tau)/\Delta(\tau) = j(\tau)-720 = q^{-1}+24+196884q+21493760q^2+\cdots$$

The function $1/\Delta(\tau)$ has the numbers $p_{24}(n+1)$ as the coefficient of q^n, and the function Θ_Λ is the theta function of the Leech lattice, which appears because the Leech lattice is the kernel of the projection of $II_{25,1}$ to the 2 dimensional lattice $II_{1,1}$. Their product is, up to a constant, the elliptic modular function $j(\tau)$. This infinite product can be evaluated explicitly by rewriting it as

$$p^{-1} \exp\Big(\sum_{m>0} p^m T_m(j(\tau)-720)\Big)$$

where the T_m's are Hecke operators. The point is that if T_m is applied to a modular function such as j then the result is still a modular function (and in fact a polynomial in j). In particular with respect to the variable τ the infinite product behaves like a modular form of weight 12. It is also antisymmetric in σ and τ, and these two conditions (plus some technical conditions about growth at infinity) are sufficient to characterize it up to multiplication by a constant. The result is that

$$p^{-1} \prod_{m>0, n\in\mathbb{Z}} (1-p^m q^n)^{c'(mn)} = \Delta(\sigma)\Delta(\tau)(j(\sigma)-j(\tau)).$$

This identity turns out to mean that in some sense the multiplicities of the root spaces of imaginary simple roots of $II_{25,1}$ with the same image in $II_{1,1}$ have an "average" value of 0 (in some rather weird sense of the word average). As the multiplicity of a simple root is the dimension of the simple root space and hence nonnegative, the only way in which the simple root multiplicities can have an average value of 0 is if they are all 0. In other words there are no simple roots of negative norm. For more precise details of this argument see [B90].

The denominator formula for M can now be written out explicitly as we know both the simple roots and the root multiplicities, and it is ([B90])

$$e^{\rho}\prod_{r>0}(1-e^{r})^{p_{24}(1-r^2/2)} = \sum_{w\in W n\in Z} \det(w)\tau(n)e^{w(n\rho)}$$

where $\tau(n)$ is the Ramanujan tau function defined by

$$\sum_{n\in\mathbb{Z}}\tau(n)q^n = q\prod_{n>0}(1-q^n)^{24} = q-24q^2+\cdots .$$

The terms with $n=1$ on the right hand side are exactly what one would get for the denominator formula of the Kac-Moody algebra M_{KM}; the terms with $n>1$ are the extra correction terms coming from the imaginary simple roots (= positive multiples of ρ) of M.

We have seen that the Kac-Moody algebra with Dynkin diagram given by the Leech lattice is best thought of as a subalgebra of a larger generalized Kac-Moody algebra whose root multiplicities are known explicitly. A similar phenomenon also happens for some other non affine Kac-Moody algebras; they are best thought of as large subalgebras of generalized Kac-Moody algebras. We will illustrate this by discussing the case of Frenkel and Feingold's algebra [FF] with Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$$

which can be thought of as the affine A_1 Dynkin diagram with an extra node attached. Frenkel and Feingold calculated many of the root multiplicities of roots λ, and observed that many of them were given by values $1,1,2,3,5,7,11,\ldots$ of the partition function $p(1-\lambda^2/2)$. A suitable generalized Kac-Moody algebra containing it was described by Niemann in his thesis [N]. The root lattice of this Lie algebra is $K\oplus II_{1,1}$ where K is generated by x and y with $(x,x)=4$, $(y,y)=6$, $(x,y)=-1$. This lattice has determinant 23 and no roots, and corresponds to a non principal ideal in the imaginary quadratic field of discriminant −23. We define $p_\sigma(n)$ by

$$\sum_n p_\sigma(n)q^n = \frac{1}{\eta(\tau)\eta(23\tau)} = q^{-1} + 1 + 2q + 3q^2 + 5q^3 + 7q^4 + 11q^5 + \cdots$$

Notice that the first 23 values of $p_\sigma(n)$ are the same as the values of the partition function $p(n+1)$. Niemann showed that there is a generalized Kac-Moody algebra with root lattice K, whose root multiplicities are given by

$$mult(\lambda) = p_\sigma(-\lambda^2/2)$$

if $\lambda \in K, \lambda \notin 23K'$ and by

$$mult(\lambda) = p_\sigma(-\lambda^2/2) + p_\sigma(-\lambda^2/46)$$

if $\lambda \in 23K'$. Moreover this Lie algebra has a norm 0 Weyl vector and its simple roots can be described explicitly in a way similar to those of the fake monster Lie algebra. In particular the norm 2 simple roots correspond to points of the lattice K, and there are also some norm 46 simple roots corresponding to two cosets of K, and some norm 0 simple roots of multiplicity 1 or 2 corresponding to positive multiplicities of the Weyl vector. If we take the norm 2 simple roots corresponding to 0 and the 2 basis vectors of K we find they have the same Dynkin diagram as the Feingold-Frenkel Lie algebra, so in particular the Feingold-Frenkel Lie algebra is a subalgebra and its root multiplicities are bounded by the multiplicities given above. It is also easy to check that in some sense the Feingold-Frenkel Lie algebra accounts for all the "small" roots, so the first few root multiplicities are given exactly by $p_\sigma(-\lambda^2/2)$, which are values of the partition function, and this explains the observation that the Feingold-Frenkel Lie algebra's root multiplicities are often given by values of the partition function. So this answers Frenkel and Feingold's question about whether it is possible to write down an explicit denominator formula for their Lie algebra, as long as we are willing to modify their question a bit and look at a slightly bigger Lie algebra.

Niemann also proved similar results for some other Kac-Moody algebras obtained by adding an extra node to an affine Dynkin diagram.

5 Relations with moonshine

The identities above are used in the proof of the moonshine conjectures, which state that the monster sporadic group has an infinite dimensional graded representation $V = \oplus V_n$ such that the traces of elements of the monster on this representation are given by certain Hauptmoduls. In particular the dimensions of the graded pieces V_n are given by the coefficients of q^{n-1} of the elliptic modular function $j(\tau) - 744$. As there are already several survey articles ([B94], [LZ], [J], [G], [Y]) about the proof of the moonshine conjectures we will only briefly discuss its relation with the fake monster Lie algebra.

A candidate for the representation V was constructed by Frenkel, Lepowsky and Meurman [FLM] and has the structure of a vertex algebra [B86]. This vertex algebra is a twisted version of the vertex algebra of the Leech lattice. If we look at the restriction of the denominator function of the fake monster Lie algebra to the 2 dimensional subspace (see the end of section 4) we see that it says an infinite product is equal to a simple expression, so we can ask if this restriction is itself the denominator function of some generalized Kac-Moody algebra. This is not quite right, but if we divide both sides by $\Delta(\sigma)\Delta(\tau)$ we find the identity

$$p^{-1} \prod_{m>0, n \in Z} (1 - p^m q^n)^{c(mn)} = j(p) - j(q).$$

This is the denominator formula of a $II_{1,1}$-graded generalized Kac-Moody algebra whose piece of degree $(m,n) \neq (0,0)$ is equal to the coefficient $c(mn)$ of q^{mn} in $j(\tau) - 744 = \sum_n c(n) q^n = q^{-1} + 196884q + \cdots$. These numbers are exactly the same as the dimensions of the graded pieces of V, and using this as a hint it is easy to guess how to construct a generalized Kac-Moody algebra from V called the monster Lie algebra: we first tensor the vertex algebra V with the vertex algebra of $II_{1,1}$ to get a vertex algebra similar to that of the lattice $II_{25,1}$, and then we apply the same construction to $V \otimes V(II_{1,1})$ that we used to construct the fake monster Lie algebra from $V(II_{25,1})$. In other words the monster Lie algebra is just the space of physical states in $V \otimes V(II_{25,1})$. As V is a twisted version of $V(\Lambda)$, we see that the monster Lie algebra is a twisted version of the fake monster Lie algebra; more precisely, both Lie algebras have automorphisms of order 2 such that the fixed point subalgebras are isomorphic. The monster Lie algebra can also be constructed via semi infinite cohomology as in [FGZ, LZ].

The monster Lie algebra by construction has an action of the monster on it with the property that the pieces of degrees (a,b) and (c,d) are isomorphic as representations of the monster whenever $ab = cd \neq 0$. This can be used to prove Conway and Norton's moonshine conjectures; see [B92], [J], [G], [B94].

6 The denominator function

The sum in the denominator formula for the fake monster Lie algebra defines a function Φ on a subset of $II_{25,1} \otimes \mathbb{C}$. It is not hard to check that this sum converges whenever the imaginary part of the argument lies in the interior of the cone C. The fact that Φ can also be written as an infinite product suggests that we can find the zeros of Φ by looking at the zeros of the factors in the infinite product. This turns out to be incorrect because the infinite product does not converge everywhere and the function Φ has zeros outside the region of convergence. (This is a quite common phenomenon; for example, it also happens for the Riemann zeta function!) We can find one zero of Φ which cannot be seen in any of its factors as follows.

First suppose that we consider Φ just for purely imaginary values of its argument v. Then we can work out the exact region of convergence of the infinite product, because we know the asymptotic behavior of the coefficients $p_{24}(1+n)$ by the Hardy-Ramanujan theorem, and this region turns out to be exactly the region with $v^2 > 2$. (Notice the region is $v^2 > 2$ rather than $v^2 < -2$ because v is purely imaginary.) Next we can see that Φ vanishes whenever v is purely imaginary and $v^2 = 2$. To see this recall the well known lemma from complex analysis which says that if $f(z) = \sum a_n z^n$ is a power series with radius of convergence r and all the a_n's are non negative, then f has a singularity at $z = r$. All the coefficients of the series for $-\log(\Phi)$ are non negative (as can be seen by taking the log of the infinite product and using the fact that $p_{24}((1+n)$ is nonnegative), so by using a higher dimensional version of the lemma mentioned above we see that $\log(\Phi(v))$ has a singularity whenever v is purely imaginary and $v^2 = 2$. However Φ itself is holomorphic at these points as this is inside the region of convergence of the infinite sum defining Φ, so the only way that $\log(\Phi(v))$ can be singular is if Φ vanishes at these points. Finally as Φ is holomorphic it must vanish at all points of the divisor $v^2 = 2$ inside the region where it is defined. Notice that this divisor is not a zero of any factor of the infinite product for Φ and (hence) lies entirely outside the region of convergence of this infinite product.

The fact that Φ vanishes for $v^2 = 2$ suggests that perhaps Φ satisfies some sort functional equation forcing it to vanish at these points, perhaps something like $\Phi(2v/(v,v)) = f(v)\Phi(v)$ where f is some function not equal to 1 when $(v,v) = 2$. We can guess the exact form of f by looking at Φ restricted to the 2 dimensional space of vectors $(0,\sigma,\tau)$, where we evaluated Φ explicitly as $\Delta(\sigma)\Delta(\tau)(j(\sigma) - j(\tau))$. Using the functional equations $\Delta(-1/\tau) = \tau^{12}\Delta(\tau)$, $j(-1/\tau) = j(\tau)$, and the fact that if $v = (0,\sigma,\tau)$

then $v^2 = -2\sigma\tau$ we see that on this 2 dimensional subspace Φ satisfies the functional equation

$$\Phi(2v/(v,v)) = -((v,v)/2)^{12}\Phi(v).$$

Once we have guessed the correct form of the functional equation above it is surprisingly easy to prove. As before we restrict to purely imaginary values of v because if we prove it for these v it will follow for all v by analytic continuation. We first observe that both $\Phi(v)$ and $(v,v)^{-12}\Phi(2v/(v,v))$ are solutions to the wave equation. For Φ this follows from the fact that it is a sum of terms of the form $e^{2\pi i(z,v)}$ with $z^2 = 0$, which are all solutions of the wave equation. If Φ is any solution of the wave equation in $\mathbb{R}^{25,1}$ then $(v,v)^{1-\dim(\mathbb{R}^{25,1})/2}\Phi(2v/(v,v))$ is automatically also a solution by the standard behavior of the wave operator under the conformal transformation $v \mapsto 2v/(v,v)$. (More generally we get an action of not just isometries but also the whole conformal group on the space of solutions of the wave equation.) Secondly, the fact that Φ vanishes for $v^2 = 2$ easily implies that both Φ and $-((v,v)/2)^{-12}\Phi(2v/(v,v))$ have the same partial derivatives of order at most 1 on this surface. (It is obvious they both vanish there and have vanishing first partial derivatives tangent to this surface, so we only need to check the first derivatives normal to the surface which is not hard.) Hence Φ and $-((v,v)/2)^{-12}\Phi(2v/(v,v))$ are both analytic solutions to a second order differential equation and both have the same derivatives of order at most 1 along some (non characteristic) Cauchy surface, so by the Cauchy Kovalevsky theorem they are equal.

7 The automorphic form Φ

We have seen that Φ satisfies the functional equation

$$\Phi(2v/(v,v)) = -((v,v)/2)^{12}\Phi(v),$$

and it is easy to check that it also satisfies the functional equations $\Phi(v+\lambda) = \Phi(v)$ for $\lambda \in II_{25,1}$ and $\Phi(w(v)) = \det(w)\Phi(v)$ for $w \in Aut(II_{25,1})^+$. We can work out the group generated by these three sorts of transformations and it turns out to be isomorphic to the group $Aut(II_{26,2})^+$, which is a discrete subgroup of the group $O_{26,2}(\mathbb{R})$ of conformal transformations of (a conformal completion of) $\mathbb{R}^{25,1}$. The action of $O_{26,2}(\mathbb{R})^+$ on the domain of definition of Φ is given as follows. We can identify this domain of vectors $v \in II_{25,1} \otimes \mathbb{C}$ with $\Im(v) \in C$ with a subset of norm 0 vectors in the projective space $P((II_{25,1} \oplus II_{1,1}) \otimes \mathbb{C})$ by mapping v to the norm 0 vector $(v, 1, v^2/2) \in II_{25,1} \oplus II_{1,1} = II_{26,2}$. The image is the set of vectors in projective space represented by norm 0 vectors whose real and imaginary parts form a positively oriented orthonormal base for a negative definite subspace of $\mathbb{R}^{26,2}$. As $O_{26,2}(\mathbb{R})^+$ obviously acts naturally on this space we get an action on the domain of definition of Φ.

We can think of the domain of Φ as a generalization of the upper half plane, and $O_{26,2}(\mathbb{R})^+$ as a generalization of the group $SL_2(\mathbb{Z})$ acting on the upper half plane. The function Φ should then be thought of as a generalization of a modular form on the upper half plane; these generalizations are called automorphic forms. We can summarize much of what we have discussed so far by saying that the denominator function of the fake monster Lie algebra is an automorphic form of weight 12 for the discrete subgroup $Aut(II_{26,2})^+$ of $O_{26,2}(\mathbb{R})^+$.

8 The zeros of Φ

We saw above that the naive guess for the zeros of Φ was wrong: there are zeros which cannot be seen in the factors of Φ. However we can now find all the zeros using the fact that Φ is an automorphic form. We have seen that Φ has a zero along the divisor $v^2 = 2$, and obviously also has zeros along all conjugates of this divisor under the group $Aut(II_{26,2})^+$ because of the transformations of Φ under this group. If we identify the domain of Φ with a subset of complex projective space as above these conjugates are easy to visualize: the divisor $v^2 = 2$ is just the set of points $(v, 1, v^2/2)$ orthogonal to the norm 2 vector $(0, 1, -1) \in II_{26,2}$, and $Aut(II_{26,2})^+$ acts transitively on these norm 2 vectors, so the zeros of Φ conjugate to $v^2 = 2$ just correspond to pairs $\{r, -r\}$ of norm 2 vectors in $II_{26,2}$. These are exactly the zeros of Φ that are "forced" by its functional equations. Now we want to see that Φ has no other zeros. Recall that the zero $v^2 = 2$ of Φ was somehow very closely related to the asymptotic behavior of the function $p_{24}(1 + n)$. The Hardy-Ramanujan-Rademacher circle method gives a much finer description of this asymptotic behavior as

$$p_{24}(1+n) = 2\pi n^{-13/2} \sum_{k>0} \frac{I_{13}(4\pi\sqrt{n}/k)}{k} \sum_{0 \le h,h' < k hh' \equiv -1 \bmod k} e^{2\pi i(nh+h')/k}$$

where $I_{13}(z) = -iJ_{13}(iz)$ is the modified Bessel function of order 13. (In particular $p_{24}(1 + n)$ is asymptotic to $2^{-1/2} n^{-27/4} e^{4\pi\sqrt{n}}$ for large n.) The zero $v^2 = 2$ corresponds to the dominant term in the asymptotic expansion, and it turns out that all other zeros are related to smaller terms in this expansion. The reason for this is that all singularities of a periodic function, such as $\log(\Phi)$, are very closely related to the asymptotic behavior of its Fourier coefficients (and in the same way the local behavior of any function depends on the growth at infinity of its Fourier transform). It turns out to be rather hard to describe the zeros of Φ directly like this, although it would in principle be possible. However it is not hard to see using these ideas that all zeros of Φ must be divisors corresponding to some positive norm vectors in $II_{26,2}$, perhaps of norms greater than 2. But every such divisor is conjugate under the group $Aut(II_{26,2})$ to some divisor intersecting the region of convergence of the infinite product for Φ. As it is easy to see that none of the factors of Φ have any zeros corresponding to vectors of $II_{26,2}$ of norms greater than 2, this shows that the only zeros of Φ are the ones we have already found.

9 Siegel theta functions

The proof that Φ is an automorphic form and has zeros as above is very indirect, using the no-ghost theorem, vertex algebras, generalized Kac-Moody algebras, and so on. As the statement only involves automorphic forms, we can ask if it is possible to prove that Φ is an automorphic form using only automorphic form theory. This can be done roughly as follows. (We will only sketch this method briefly, because we will see later in this section that there is a better way of proving this.) First by extending the argument in the previous section, we can prove that Φ (defined as the infinite product) can be analytically continued from the region of convergence of the infinite product to the full domain of Φ. Secondly we can write Φ in the form $\exp(\sum_{m>0} T_m(f))$ where f is a certain Jacobi form with poles at cusps and the T_m's are Hecke operators acting on this Jacobi form. (The idea of considering $\sum_{m>0} T_m(f)$ for holomorphic Jacobi forms f was used by Maass in his work on the Saito-Kurokawa conjecture described in [EZ] and then extended by Gritsenko [Gr] to Jacobi forms on higher dimensional lattices.) Notice that this is a sort of blown up version of the formula we found for the restriction of Φ to a 2 dimensional subspace in section 4. This second expression for Φ easily implies that it transforms nicely under a certain subgroup $SL_2(\mathbb{Z})$ of $Aut(II_{26,2})^+$. By looking at the Fourier coefficients of Φ it is easy to check that it also transforms nicely under $Aut(II_{25,1})^+$. As this group together with $SL_2(\mathbb{Z})$ generate $Aut(II_{26,2})^+$ we see that Φ transforms correctly under all elements of $Aut(II_{26,2})^+$ and hence shows that Φ is an automorphic form.

The advantage of the proof sketched above is that it does not really depend on all the special properties of the Leech lattice, and can be extended to more general lattices than $II_{25,1}$ and more general exponents that $p_{24}(1+n)$. If we carry out this extension for even unimodular lattices we find:

Theorem [B95]. *Suppose that L is the even unimodular lattice $II_{s+2,2}$. Suppose that $f(\tau) = \sum_n c(n)q^n$ is a meromorphic modular form of weight $-s/2$ for $SL_2(\mathbb{Z})$ with integer coefficients, with poles only at cusps, and with $24|c(0)$ if $s = 0$. There is a unique vector $\rho \in L$ such that*

$$\Phi(v) = e^{-2\pi i(\rho,v)} \prod_{r>0}(1 - e^{-2\pi i(r,v)})^{c(-(r,r)/2)}$$

is a meromorphic automorphic form of weight $c(0)/2$ for $Aut(II_{s+2,2})^+$.

It is also possible to describe the zeros and poles of Φ explicitly. The denominator function for the fake monster Lie algebra is the special case of the theorem above with $s = 24$, $f(\tau) = 1/\Delta(\tau)$.

The proof sketched above is not very enlightening, as it is essentially a brute force check that Φ transforms correctly under a set of generators of $Aut(II_{s+2,2})^+$. It is possible to extend this a bit to non unimodular lattices, but it becomes increasingly complicated as the level and determinant increase. Gritsenko and Nikulin [GN] have worked out a some higher level examples explicitly using the method above. In a recent preprint [HM], Harvey and Moore have sketched a conceptual proof of the fact that Φ is an automorphic form, and also given a better explanation of why its only zeros are as described above, using Siegel theta functions of indefinite lattices (which seem to have been independently rediscovered by physicists working in string theory). Their method generalizes much better to higher levels; see [B].

The Siegel theta function $\Theta_L(\tau_1, \tau_2, V)$ of a lattice $L = II_{r,s}$ (which we will assume is even and unimodular for simplicity) is defined as

$$\Theta_L(\tau_1, \tau_2, V) = \sum_{\mathfrak{l}\in L} \exp(2\pi i\tau_1\mathfrak{l}_1^2/2 + 2\pi i\tau_2\mathfrak{l}_2^2/2)$$

where $\Im(\tau_1) > 0$, $\Im(\tau_2) < 0$, V is a negative definite subspace of $L \otimes \mathbb{R}$ of maximal dimension n, and $\mathfrak{l}_1$ and $\mathfrak{l}_2$ are the projections of $\mathfrak{l}$ into $V^\perp$ and V. If L is positive definite this is just the usual theta function of L, as it does not depend on τ_2 or V, and for indefinite lattice the series still converges absolutely because of the condition that $\Im(\tau)_2 < 0$. It satisfies the functional equations

$$\Theta_L\left(a\tau_1 + \frac{b}{c}\tau_1 + d, a\tau_2 + \frac{b}{c}\tau_2 + d, V\right)$$
$$= ((c\tau_1 + d)/i)^{r/2}((c\tau_2 + d)i)^{s/2}\Theta_L(\tau_1, \tau_2, V),$$

and is obviously invariant under the natural action of $Aut(II_{r,s})$ on V.

Now suppose that f is any linear functional from functions of τ_1 and τ_2 to $\mathbb{C}$. Then $f(\Theta_L)(V)$ is a function of V which is automatically invariant under $Aut(II_{r,s})$.

Harvey and Moore make the following choices for L and f. They choose L to be a lattice $II_{s+2,2}$. The set of subspaces V can be identified with the hermitian symmetric space of L as follows: the norm 0 vector $x + iy \in L \otimes \mathbb{C}$ representing a point of the projective space $P(L \otimes \mathbb{C})$ corresponds to the negative definite space spanned by x and y. The linear functional they use is (more or less) given by

$$f(\Theta_L(\tau_1, \tau_2, V)) = \int_D \Theta_L(\tau, \bar{\tau}, v)g(\tau)d\tau d\bar{\tau}/\Im(\tau)$$

where D is a fundamental domain for the action of $SL_2(\mathbb{Z})$ on the upper half plane and f is a meromorphic modular form of weight $-s/2$ with poles only at cusps. The function $\Im(\tau)\Theta_L$ transforms like a modular form of weight $s/2$ and the differential $d\tau d\bar{\tau}/\Im(\tau)^2$ is invariant under $SL_2(\mathbb{Z})$ so the integrand is invariant under $SL_2(\mathbb{Z})$ and the integral does not depend on the choice of fundamental domain D. The integral needs to be interpreted rather carefully as it is wildly divergent because of terms of the form $e^{2\pi in\tau}$ with $n < 0$. Harvey and Moore get round this by first integrating with respect to $\Re(\tau)$ and only then integrating with respect to $\Im(\tau)$ (although this still leaves a problem with integrating $1/\Im(\tau)$ from 1 to infinity, which they deal with by first subtracting a suitable function from the integrand).

The integral above is similar to the integral used by Niwa [Ni] in his work on the Shimura correspondence. This can be used to explain the formal similarities observed in [B95] between the Shimura correspondence and the infinite products above; see section 11 below. In particular Harvey and Moore's formula can be thought of as a sort of version of the Howe correspondence for the dual reductive pair $O_{s+2,2}(\mathbb{R})$ and $SL_2(\mathbb{R})$ for functions with singularities.

Harvey and Moore formally evaluate this integral and find that it is essentially given by the logarithm of the absolute value of the function Φ

of the theorem above, plus a few elementary factors which account for the fact that the integral is invariant under $Aut(II_{r,s})^+$ while Φ is not quite invariant.

Jorgenson and Todorov found a quite different way of constructing similar automorphic functions on moduli spaces as analytic discriminants [JT]. The relation to the work above is that the moduli spaces of Enriques surfaces and polarized K3 surfaces are (roughly) quotients of the hermitian symmetric spaces of lattices of signature $(n, 2)$ for $n = 10$ or 18. It seems likely that some of the functions constructed by Jorgensen and Todorov can also be constructed using some variation of Harvey and Moore's method.

Automorphic forms which are infinite products have recently turned up in several papers by physicists on string theory, but I do not understand this well enough to report on it. See [HM] and [D] for example.

10 Some superalgebras of rank 10

Harvey and Moore's formula for Φ has the big advantage that it generalizes easily to arbitrary lattices L of any dimension, signature, and determinant; see [B] for details. We will give a few examples in this section.

One extra feature that appears in the higher level case is that an automorphic form can have several apparently quite different infinite product expansions, converging in different regions. A classical example of this is the different product expansions for the theta function of a one dimensional lattice. If we put

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2}$$

then it is a modular form for the group $\Gamma(2)$, and has the following infinite product expansions at cusps of $\Gamma(2)$.

$$\begin{aligned}
\theta(\tau) &= 1 - 2q^{1/2} + 2q^{4/2} - 2q^{9/2} + \cdots \\
&= (1 - q^{1/2})^2(1-q)(1-q^{3/2})^2(1-q^2)\cdots \\
(\tau/i)^{-1/2}\theta(-1/\tau) &= 2q^{1^2/8} + 2q^{3^2/8} + 2q^{5^2/8} + \cdots \\
&= 2q^{1/8}(1-q)^{-1}(1-q^2)(1-q^3)^{-1}(1-q^4)\cdots
\end{aligned}$$

So two apparently quite different infinite products are really the same function expanded around different cusps.

We will take L to be the lattice $II_{9,1} \oplus II_{1,1}(2)$, where $II_{1,1}(2)$ is the lattice $II_{1,1}$ with all norms multiplied by 2. We define a vector valued modular form F with components f_{ij} for $i, j \in \mathbb{Z}/2\mathbb{Z}$ by

$$
\begin{aligned}
f_{00}(\tau) &= 8\eta(2\tau)^8/\eta(\tau)^{16} = 8 + 128q + 1152q^2 + \cdots \\
f_{10}(\tau) &= f_{01}(\tau) = -8\eta(2\tau)^8/\eta(\tau)^{16} = -8 - 128q - 1152q^2 - \cdots \\
f_{11}(\tau) &= 8\eta(2\tau)^8/\eta(\tau)^{16} + \eta(\tau/2)^8/\eta(\tau)^{16} = q^{-1/2} + 36q^{1/2} + 402q^{3/2} + \cdots
\end{aligned}
$$

Then there is an automorphic form such that the log of its absolute value is (more or less) given by

$$\int_D \Theta_L(\tau, \bar{\tau}, v) F(\tau) d\tau d\bar{\tau}/\Im(\tau)$$

(where the Siegel theta function is now a certain vector valued function taking values in the group ring of L'/L).

By theorem 13.3 of [B] we can find an infinite product expansion of this automorphic form for every primitive norm 0 vector of L (or equivalently to every one dimensional cusp). Define $c(n)$ by

$$\sum_n c(n)q^n = f_{00}(\tau) + f_{11}(\tau) = q^{-1/2} + 8 + 36q^{1/2} + O(q).$$

Then the infinite product at one cusp (for K the even sublattice of $I_{9,1}$) is

$$
\begin{aligned}
& e^{2\pi i(\rho,v)} \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} (1 - e^{2\pi i(v,\lambda)})^{\pm c(-\lambda^2/2)} \\
&= \sum_{w \in G} \det(w) e^{2\pi i(w(\rho),v)} \prod_n (1 - e^{2\pi i n(w(\rho),v)})^{(-1)^n 8}
\end{aligned}
$$

where the sign in the exponent is 1 if $\lambda \in K$ or if λ has odd norm, and 1 if λ has even norm but is not in K. The group G is the reflection group generated by reflections of the norm 1 vectors of K. The infinite product at the other cusp (with $K = II_{9,1}$) is

$$\prod_{\lambda \in K (\lambda, W) > 0} (1 - e^{(v,\lambda)})^{c(-\lambda^2/2)} (1 + e^{(v,\lambda)})^{-c(-\lambda^2/2)} = 1 + \sum_\lambda a(\lambda) e^{2\pi i(v,\lambda)}$$

where $a(\lambda)$ is 1 if $\lambda = 0$, the coefficient of q^n of

$$\eta(\tau)^{16}/\eta(2\tau)^8$$

if λ is n times a primitive norm 0 vector in the closure of the positive cone C, and 0 otherwise.

In both cases we can identify the infinite product as a sum using the fact that it is an automorphic form of singular weight, so that all its Fourier coefficients corresponding to values of λ of nonzero norm vanish, and we get the formulas above. Both of these formulas are denominator formulas for generalized Kac-Moody superalgebras of rank 10 [R], and both superalgebras can be constructed as spaces of states of a superstring moving on a 10 dimensional torus.

The automorphic form can also be considered as a function on the period space of marked Enriques surfaces as in [B96], which can be used to show that the moduli space of Enriques surfaces is quasiaffine.

11 The Shimura correspondence

The Shimura correspondence is a map from modular forms of weight $k + 1/2$ to modular forms of weight $2k$. Shimura's original definition in [S] was rather roundabout and involved taking an eigenform of weight $k + 1/2$ under the Hecke operators, taking the corresponding Euler product, changing it in a mysterious way to a new Euler product, and then using Weil's theorem to reconstruct a modular form from this new Euler product. Niwa [Ni] and Kohnen [Ko] reformulated Shimura's result and found a more straightforward way of constructing Shimura's map. Combining their results in the level 1 case (and making an easy extension to the case $k = 0$) we get the following theorem as a special case.

Theorem. *Suppose that $f(\tau) = \sum_n c(n)q^n$ is a modular form of weight $k + 1/2 > 0$ for $\Gamma_0(4)$ (with k even) such that the Fourier coefficients $c(n)$ vanish unless $n \equiv 0, 1 \bmod 4$. Then the function $\Phi(\tau)$ defined by*

$$\Phi(\tau) = -c(0)B_k/2k + \sum_{n\neq 0} q^n \sum_{0<d|n} d^{k-1}c(n^2/d^2)$$

is a modular form of weight $2k$ if $k > 0$. If $k = 0$ and all the coefficients $c(n)$ are integers then for some rational h the functions $q^h \exp(\Phi(\tau)$ is a modular form of weight $c(0)$ for some character of $SL_2(\mathbb{Z})$ (at least if we first remove the infinite constant term from the expression for Φ).

Example. If $k = 0$ then we take $f(\tau) = \theta(\tau) = \sum_n q^{n^2}$. Then

$$\Phi(\tau) = \log\left(\prod_{n>0}(1-q^n)^2\right)$$

so that $q^h \exp(\Phi(\tau))$ is $\eta(\tau)^2$ and $h = 1/12$.

Example. Put $f_{13/2} = \theta F(\theta^4 - 16F)(\theta^4 - 2F)$ of weight $6 + 1/2$ where $F(\tau) = \sum_{n>0} \sigma_1(2n+1)q^{2n+1}$. Then $\Phi(\tau)$ must be a form of weight 12, so must be (a multiple of) $\Delta(\tau)$. We find

$$\begin{aligned} f_{13/2}(\tau) &= q - 56q^4 + 120q^5 - 240q^8 + 9q^9 + 1440q^{12} - 1320q^{13} \\ &\quad - 704q^{16} + O(q^{17}) \\ \Delta(\tau) &= \sum_n \tau(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + O(q^5) \end{aligned}$$

and can check explicitly in the example above that

$$\tau(n) = \sum_{d|n} d^5 c(n^2/d^2).$$

Niwa showed that the function Φ could be obtained by considering the integral of $\bar{\Theta}(\tau)f(\tau)$, where Θ is the Siegel theta function of a 3 dimensional lattice. This is very similar to Harvey and Moore's formula in section 9, and in fact both formulas turn out to be special cases of a sort of Howe correspondence for modular forms, possibly with poles at cusps [B]. In particular the Shimura correspondence stated above still works perfectly well if the form f is allowed to have poles at cusps, except that the form Φ will now have singularities at imaginary quadratic irrationals coming from the poles of f. The singularities will be poles of order k if $k > 0$ and logarithmic singularities (corresponding to the poles and zeros of $q^h \exp(\Phi)$) if $k = 0$.

Example. Put

$$\begin{aligned} f(\tau) &= f_{13/2}(\tau)E_8(4\tau)/\Delta(4\tau) + 6720\sum_n H(2,n)q^n \\ &= q^{-3} + 64q - 32384q^4 + 131535q^5 - 4257024q^8 + 11535936q^9 + O(q^{12}) \\ &= \sum_n c(n)q^n \end{aligned}$$

(where $H(2,n) = L(-1,\chi_n)$ is Cohen's function [Co]) so that F has weight $5/2$. We see that $\Psi(\tau)$ should is a modular form of weight $2(5/2 - 1/2) = 4$, and we can work out the singularities and find that they are poles of order 2 at the conjugates of a cube root of 1. We find that $\Psi(\tau)$ is

$$\begin{aligned} 64\Delta(\tau)/E_4(\tau)^2 &= 64(q - 504q^2 + 180252q^3 - 56364992q^4 + O(q^5)) \\ &= \sum_n q^n \sum_{d|n} dc(n^2/d^2) \end{aligned}$$

which has poles of order 2 at cube roots of 1 and their conjugates because E_4 has zeros of order 1 at these points.

Example. Now we look at an example where $k = 0$ and f has poles at the cusp. We take

$$\begin{aligned} f(\tau) &= -2f_{13/2}(\tau)E_6(4\tau)/\Delta(4\tau) - 108\theta(\tau) \\ &= -2q^{-3} + 4 + 504q - 53496q^4 + 171990q^5 \\ &\quad - 3414528q^8 + 8192504q^9 + O(q^{12}) \\ &= \sum_n c(n)q^n. \end{aligned}$$

This time $q^h \exp(\Psi)$ should again have weight equal to the constant term 4, and also turns out to have poles of order 2 at conjugates of a cube root of 1. These poles come from the term $-2q^{-3}$ as follows: the coefficient -2 is the order of the zero, and the exponent 3 is the discriminant of the quadratic equations whose roots are conjugates of a cube root of 1. We find that

$$\begin{aligned} q^h \exp(\Psi) &= \Delta(\tau)/E_4(\tau)^2 \\ &= q\prod_{n>0}(1-q^n)^{c(n^2)} \end{aligned}$$

is the same as the function in the previous example up to a factor of 64. So both the coefficients of this function and the exponents in its infinite product expansion can be given in terms of modular forms of half integral weight with poles at the cusps.

As we already know an infinite product for Δ we can use the example above to find an explicit infinite product for the Eisenstein series $E_4(\tau)$ and hence also for the elliptic modular function $j(\tau) = E_4(\tau)^3/\Delta(\tau)$. More generally we can find an infinite product expansion for any level 1 modular function with integral coefficients all of whose zeros and poles are at imaginary quadratic irrationals or cusps.

12 Finiteness theorems

One can ask whether the examples discussed above are isolated and exceptional objects or whether they are part of an infinite family. It is possible to generalize most of the constructions above to produce a few hundred similar examples of things similar to (say) the fake monster Lie algebra. However there are several theorems suggesting that the total number of examples like this may be finite.

The nice behavior of the monster Lie algebra depends on the fact that the reflection group of $II_{25,1}$ is very nice, and in particular on the fact that it has a (Weyl) vector ρ which has bounded inner product with all simple roots. If such a vector exists in some Lorentzian lattice L with negative norm then there are only a finite number of simple roots of L and the reflection group of L has finite index in the full automorphism group. V. Nikulin has shown [N] that there are essentially only a finite number of such lattices, up to multiplication by constants, and Esselmann has shown that the only one of dimension greater than 20 is the 22 dimensional lattice of determinant 4 consisting of the even vectors of $I_{21,1}$. Nikulin recently extended his theorem to cover the case when the vector ρ has zero norm. The largest lattice in this case is presumably the lattice $II_{25,1}$ though this has not been proved. So Nikulin's work suggests that there may be only a finite number of interesting Lie algebras similar to the monster Lie algebra. However there are some examples of generalized Kac-Moody algebras with no real roots which still have known simple roots and root multiplicities and Weyl vectors equal to 0, so it is still conceivable (but unlikely) that there are an infinite number of these. Nikulin and Gritsenko have given some examples of generalized Kac-Moody algebras related to some hyperbolic reflection groups in [GN] and have suggested that maybe most crystallographic reflection groups in Lorentzian lattices L which have finite index in the full automorphism group are associated with some nice generalized Kac-Moody algebra or superalgebra whose denominator formula is an automorphic form. In particular they show that the Siegel modular form Δ_5 (one of the standard generators of the ring of Siegel modular forms of genus 2) can be written as such an infinite product.

Many of the Lie algebras similar to the fake monster Lie algebra (and all the known ones of rank at least 3) are closely related to certain products of (positive and negative) powers of η functions with multiplicative coefficients. For example the fake monster Lie algebra itself is related to the function $\Delta(\tau) = \eta(\tau)^{24}$, whose coefficients $\tau(n)$ were proved to be multiplicative by Mordell and Hecke. Y. Martin [M] has recently found all such products of eta functions and in particular has proved that there are only a finite number of them. This again hints that there are only finitely many analogues of the fake monster Lie algebra.

Some of the generalized Kac-Moody algebras of rank 2, such as the monster Lie algebra itself, are not related to multiplicative products of eta functions. However most of the known ones that are not are instead closely related to certain Hauptmoduls of genus 0 congruence subgroups of $SL_2(\mathbb{R})$; for example, the monster Lie algebra is related to the Hauptmodul $j(\tau)$ of the subgroup $SL_2(\mathbb{Z})$. It is easy to find an infinite number of non congruence genus 0 subgroups, but J. G. Thompson showed that there are only a finite number of conjugacy classes of congruence subgroups of any given genus.

The theorem in section 9 produces an infinite supply of examples of automorphic forms with infinite product expansions. Unfortunately most of these cannot possibly be the denominator formulas of generalized Kac-Moody algebras. The point is that most of the infinite products involve vectors of norm at least 4 in the lattice, so such vectors would have to be roots of any generalized Kac-Moody algebra. However if v is a positive root of a generalized Kac-Moody algebra, then $(v,v)|2(v,w)$ for any other root w. This means that if the root lattice is unimodular then positive norm vectors cannot have norms greater than 2. In fact the only infinite products given by the theorem in section 9 which can be the denominator formulas of generalized Kac-Moody algebras are the ones associated to the fake monster Lie algebra and the monster Lie algebra that we have already seen above. There are of course many automorphic forms of higher level which have nice infinite product expansions, and it seems conceivable that there may be infinite families of these with no positive norm vectors appearing in the infinite product. These could then be interpreted as the denominator formulas for generalized Kac-Moody superalgebras with known differences of the root multiplicities and simple root multiplicities.

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