Wavelets, paraproducts, and Navier-Stokes equations

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Foreword

These notes are obviously not the written version of the two talks I gave on April 20, 1996, in the Harvard-MIT seminar.

These two talks were directed to a broad audience and I wanted to present a large panorama of applications of wavelets to science and technology.

In contrast here I wished to focus on one specific application of wavelet analysis. I decided to address a hot issue : the relevance of wavelets for Navier-Stokes equations.

These notes will lead the reader to much unpublished material on Navier-Stokes equations, and the debate will not be settled.

I would like to thank the organizers of the Harvard-MIT seminar for their warm hospitality and David Jerison for his patience and

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Last but not least, Isabelle Paisant did a splendid job in typing these notes.

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1 Introduction

The relevance of wavelet analysis in Navier-Stokes equations is a serious issue. Big claims were not followed by any serious progress. The first goal of these notes is to clarify these matters.

The second goal is to pave the way to some striking results obtained by P.G. Lemarié-Rieusset and his group. On the way the fundamental achievements by T. Kato will be revisited.

All these results are concerning the same problem, namely, the existence and uniqueness of solutions of Navier-Stokes equations when the initial condition $u_0(x)$ belongs to a given functional space E. We then want to compute the velocity field $u(x,t)$ for $t \geq 0$ and to prove estimates of the form

$$\sup_{t\geq 0} \|u(\cdot,t)\|_E \leq C\|u_0\|_E . \tag{1.1}$$

This problem was treated by Jean Leray when $E = L^2(\mathbb{R}^3)$. Indeed Leray proved the existence of weak solutions to Navier-Stokes equations (section 4). The uniqueness of such solutions for a given $u_0(x)$ is still an open problem. The same problem was treated by T. Kato in 1984 when $E = L^3(\mathbb{R}^3)$. The existence of a global mild solution was obtained by a beautiful argument which will be unveiled in section 19. However the uniqueness problem was open until now and recently solved by P.G. Lemarié-Rieusset. This issue concerns solutions to Navier-Stokes equations $u(x,t)$, $x \in \mathbb{R}^3$, $t \geq 0$, which are continuous in the time variable with values in $L^3(\mathbb{R}^3)$ as functions of x.

On the way the relevance of wavelet analysis for Navier-Stokes equations will be carefully studied and the results are more subtle than expected.

2 Notations

Navier-Stokes equations describe the motion of incompressible and homogeneous fluids. It is natural to assume that this fluid is contained in a bounded domain Ω. However we will restrict our attention to the opposite situation where there is no boundary. In other words the fluid is filling the space $\mathbb{R}^3$.

The velocity at a given point $x \in \mathbb{R}^3$ and at a given time $t \geq 0$ of the fluid is denoted by $u(x,t) = \big(u_1(x,t), u_2(x,t), u_3(x,t)\big)$ where u_1, u_2 and u_3 are real valued.

Similarly the pressure is denoted by $p(x,t)$ and is a real valued function.

In the absence of external forces, the Navier-Stokes equations read

$$(2.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nu \Delta u - (u_1\partial_1 + u_2\partial_2 + u_3\partial_3)u - \nabla p \\ \partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0 \\ u(x,0) = u_0(x)\,. \end{cases}$$

Here and in what follows, $\partial_j = \partial/\partial x_j$, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$, ∇p is the gradient of the pressure and $u_0(x) = \big(u_{0,1}(x), u_{0,2}(x), u_{0,3}(x)\big)$ is the initial condition.

The viscosity ν is a positive parameter which can be eliminated by a convenient rescaling (section 5).

The system (2.1) contains four unknown functions u_1, u_2, u_3 and p but consists of four equations. The balance is correct.

The pressure does not show up in the initial data. It will soon be explained how the pressure can be eliminated from (2.1). Therefore (2.1) will reduce into a system of three equations governing the evolution of the velocity field.

We say that $u(\cdot,t)$ is a classical solution of (2.1) if the regularity of u with respect to x and t and the decay at infinity permit all the calculations which will now be performed.

We say that a solution $u(\cdot,t)$ of (2.1) is global in time if it is defined for $t \in [0,\infty)$ with some other required properties. These properties concern smoothness and decay at infinity. A solution $u(\cdot,t)$ of (2.1) is local in time if it is defined for $0 \le t \le T$ for some positive T.

In these notes $u(x,t)$ will be studied as a vector-valued function of the time variable. More precisely we will start with a functional Banach space $E \subset \mathcal{S}'(\mathbb{R}^3)$. This Banach space is used for describing size or regularity properties of functions $f(x)$ of the x variable. Then $u(x,t)$ will always be viewed as an E-valued function of the time variable.

Indeed we will be looking for solutions $u(x,t)$ belonging to $C([0,\infty);E)$. Here and in what follows we incorrectly write $u(\cdot,t) \in E$ instead of $u(\cdot,t) \in E^3$.

A solution u in $C([0,\infty);E)$ is certainly not a classical solution. Smoothness in t or x is missing. This issue will be addressed in section 9 where the concept of a mild solution will be defined.

Before defining these mild solutions, let us start with Leray's weak solutions which correspond to the Banach space $E = L^2(\mathbb{R}^3)$.

The following two sections do not contain original results. Their goal is to pave the way to Federbush's program. This program is a wavelet-based approach to Navier-Stokes equations.

3 Weak solutions of Navier Stokes equations

The definition of a weak solution relies on a conservation law which is satisfied by the solutions of Navier-Stokes equations : some energy is non increasing.

We start with an elementary remark

Lemma 1. *Let* $a_1 = a_1(x)$, a_2 *and* a_3 *be three real-valued functions belonging to* $L^1_{\text{loc}}(\mathbb{R}^3)$. *Let us assume*

$$\partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3 \;=\; 0 \tag{3.1}$$

in the distributional sense and consider the differential operator

$$X \;=\; a_1\partial_1 + a_2\partial_2 + a_3\partial_3\,. \tag{3.2}$$

Then

$$X^* \;=\; -X \tag{3.3}$$

where X^* *is defined by* $<X^* f, g>=<f, Xg>$ *and* $<f,g>= \int_{\mathbb{R}^3} f(x)\,\overline{g(x)}\,dx$.

This follows from a trivial integration by parts.

Let us now assume that $u(x,t)$ is a classical solution of Navier-Stokes equations. We furthermore assume that $u(x,t)$ satisfies together with its derivatives in x or t an estimate of the form $0(|x|^{-2})$ as $|x| \to +\infty$. A similar assumption is made on the pressure $p(x,t)$ and its gradient ∇p. We then have

Lemma 2. *For any such classical solution of the Navier-Stokes equations and for every* $t \geq 0$, *we have*

$$\|u(\cdot,t)\|_2^2 + 2\int_0^t \|\nabla u(\cdot,s)\|_2^2\,ds \;=\; \|u(\cdot,0)\|_2^2\,. \tag{3.4}$$

Here and in what follows we are assuming that the viscosity ν equals 1. We will return to this issue when the scaling properties of the Navier-Stokes equations will be described. It will be shown that an obvious rescaling permits to reduce the general case to this special situation. Let us prove lemma 2.

One writes $X = u_1\partial_1 + u_2\partial_2 + u_3\partial_3$ and the Navier-Stokes equations read

$$\text{(3.5)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - Xu - \nabla p \\ \operatorname{div} u = 0 \\ u(x,0) = u_0(x)\,. \end{cases}$$

The next lemma to be used is the following remark.

Lemma 3. *If $u = u(x) = (u_1(x), u_2(x), u_3(x))$ with $u_j \in L^2(\mathbb{R}^3)$ and $\operatorname{div} u = 0$ in the distributional sense, then for any real valued function $p(x)$ such that $\nabla p \in L^2(\mathbb{R}^3)$, we have*

$$\text{(3.6)} \qquad <\nabla p, u> = 0\,.$$

It suffices to approximate p by a sequence p_j of functions in the Schwartz class such that

$$\|\nabla p - \nabla p_j\|_2 \to 0 \qquad (j \to +\infty)\,.$$

Proving $<\nabla p_j, u> = 0$ is a trivial check since $\int \partial_1 p_j\, u_1\, dx = -\int p_j\, \partial_1 u_1\, dx$ and so on.

Returning to lemma 2, we compute the three integrals

$$\int_{\mathbb{R}^3} u \cdot \Delta u\, dx = I_1(t) \quad , \quad \int_{\mathbb{R}^3} Xu \cdot u\, dx = I_2(t)$$

and

$$\int_{\mathbb{R}^3} \nabla p \cdot u\, dx = I_3(t)\,.$$

We then obtain $I_1(t) = -\int_{\mathbb{R}^3} |\nabla u|^2\, dx$. The integrations by parts are justified by the decay properties we imposed on u and its derivatives. Lemma 1 yields $I_2(t) = 0$ and lemma 3 implies $I_3(t) = 0$. Moreover

$$<\frac{\partial u}{\partial t}, u> = \frac{1}{2}\frac{d}{dt}\|u\|_2^2\,.$$

These computations yield

$$\text{(3.7)} \qquad \frac{1}{2}\frac{d}{dt}\|u\|_2^2 = -\int |\nabla u|^2\, dx = -\|\nabla u\|_2^2$$

which immediately implies (3.6).

As usual in mathematical physics, this conservation law leads to a natural functional space F. The solutions to Navier-Stokes equations should belong to this functional space and the problem of existence and uniqueness should be solved inside this framework. A solution belonging to this functional space F will be named a weak solution.

From (3.9), we shall demand the following two properties to a weak solution

$$\sup_{t\geq 0} \|u(\cdot,t)\|_2 \qquad \text{is finite} \tag{3.8}$$

$$\int_0^\infty \|\nabla u(\cdot,t)\|_2^2\,dt \qquad \text{is finite} \tag{3.9}$$

and we immediately warn the reader that a third condition should be imposed since (3.10) and (3.11) do not permit to define $u(\cdot,0)$.

Let us remind the reader with the definition of the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^3)$.

Definition 4. *A function $f(x)$ belongs to $\dot{H}^1(\mathbb{R}^3)$ if and only if $f(x)$ belongs to $L^6(\mathbb{R}^3)$ and ∇f belongs to $L^2(\mathbb{R}^3)$.*
The norm of f in $\dot{H}^1(\mathbb{R}^3)$ is $\|\nabla f\|_2$.

This definition looks surprising since the L^6-norm does not enter into the definition of the $\dot{H}^1(\mathbb{R}^3)$ norm. Indeed the assumption $f \in L^6$ can be replaced by a much weaker one as

$$\int_{|x|\leq R} |f(x)|\,dx = o(R^3)\,, \qquad R \to +\infty \tag{3.10}$$

which prevents constant functions to belong to $\dot{H}^1(\mathbb{R}^3)$.

The ordinary Sobolev space $H^1(\mathbb{R}^3)$ is defined by the two conditions $f \in L^2(\mathbb{R}^3)$ and $\nabla f \in L^2(\mathbb{R}^3)$.

Using these notations, we observe that a weak solution should belong to $L^\infty([0,\infty); L^2(\mathbb{R}^3))$ and to $L^2([0,\infty); \dot{H}^1(\mathbb{R}^3))$.

As it was said before, a third condition is needed for defining weak solutions. For unveiling this third condition Sobolev spaces with negative indices are needed.

Definition 5. *The Sobolev space $H^{-1}(\mathbb{R}^3)$ is the space of tempered distributions f whose Fourier transforms $\hat{f}(\xi)$ locally belong to $L^2(\mathbb{R}^3)$ and satisfy*

$$\int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 (1+|\xi|^2)^{-1} \, d\xi < \infty . \tag{3.11}$$

From this definition, it is trivially checked that $H^{-1}(\mathbb{R}^3)$ is the dual space of $H^1(\mathbb{R}^3)$. More precisely $H^1(\mathbb{R}^3)$ is a substitute for the space of testing functions. When $\mathcal{S}(\mathbb{R}^3)$ is replaced by $H^1(\mathbb{R}^3)$, then $\mathcal{S}'(\mathbb{R}^3)$ is replaced by $H^{-1}(\mathbb{R}^3)$.

We now arrive to the definition of a weak solution.

Definition 6. *If $T > 0$, a weak solution $u(x,t)$ of the Navier-Stokes equations is defined by the following three properties*

$$u(x,t) \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \tag{3.12}$$

$$u(x,t) \in L^2([0,T]; H^1(\mathbb{R}^3)) \tag{3.13}$$

and

$$\frac{\partial u}{\partial t} \in L^1([0,T]; H^{-1}(\mathbb{R}^3)) . \tag{3.14}$$

Similarly the pressure should belong to $L^1([0,T]; L^2(\mathbb{R}^3))$ and the Navier-Stokes equations should read

$$\frac{\partial u}{\partial t} = \Delta u - \partial_1(u_1 u) - \partial_2(u_2 u) - \partial_3(u_3 u) - \nabla p . \tag{3.15}$$

The meaning of (3.15) needs to be clarified as well as the meaning of the initial condition $u(x,0) = u_0(x)$.

We now want to motivate the definition of such weak solutions by proving some *a priori* estimates.

Lemma 4. *If $u(x,t)$ is a classical solution of the Navier-Stokes equations, we have*

$$\sup_{0\le t\le T} \|u(\cdot,t)\|_2 \le \|u_0\|_2 \tag{3.16}$$

$$\int_0^T \|u(\cdot,t)\|^2_{H^1(\mathbb{R}^3)}\,dt \;\le\; \Big(T+\frac{1}{2}\Big)\,\|u_0\|_2^2 \tag{3.17}$$

$$\int_0^T \|p(\cdot,t)\|_2\,dt \;\le\; C(T)\,\|u_0\|_2 \tag{3.18}$$

and finally

$$\int_0^T \Big\|\frac{\partial u}{\partial t}(\cdot,t)\Big\|_{H^{-1}(\mathbb{R}^3)}\,dt \;\le\; C'(T)\,\|u_0\|_2\,. \tag{3.19}$$

Let us first show that the pressure can be easily computed from the velocity field $u(x,t)$ whenever both p and u are smooth and tend to 0 in a suitable way.

In order to make life easier, we introduce the Riesz transformations R_1, R_2, R_3. They are defined by

$$R_j \;=\; -i\,\partial_j(-\Delta)^{-1/2} \qquad \text{where} \quad \partial_j = \frac{\partial}{\partial x_j}\,. \tag{3.20}$$

If $\hat{f}(\xi)$ denotes the Fourier transform of f, we have

$$(R_j f)\hat{}\,(\xi) \;=\; \frac{\xi_j}{|\xi|}\,\hat{f}(\xi)\,. \tag{3.21}$$

Finally R_j is a Calderón-Zygmund operator and we have

$$R_j f(x) \;=\; c \quad p.v. \int \frac{x_j - y_j}{|x-y|^4}\, f(y)\,dy \tag{3.22}$$

where c is a constant.

The Riesz transformations continuously map $L^p(\mathbb{R}^3)$ into itself when $1 < p < \infty$ and, when properly defined, continuously map $L^\infty(\mathbb{R}^3)$ into $BMO(\mathbb{R}^3)$ [32].

Lemma 5. *If (u,p) is a classical solution of the Navier-Stokes equations, we have*

$$p \;=\; R_j R_k(u_j u_k)\,. \tag{3.23}$$

Here and in what follows, $\alpha_j\beta_j$ means $\sum_1^3 \alpha_j\beta_j$. For proving this lemma, we compute the divergence of all terms in (2.1). Since $\operatorname{div} u = 0$, we also have $\operatorname{div} \frac{\partial u}{\partial t} = 0$ and $\operatorname{div} \Delta u = 0$. We then obtain

$$\Delta p + \partial_j \partial_k (u_j u_k) = 0. \tag{3.24}$$

Since we assumed that both p and u_j tend to zero at infinity, this implies (3.23).

Let us stress that R_j is not a local operator. This means that the calculation of the pressure at a given point x needs a global knowledge of the velocity field.

If $u(x,t)$ is a classical solution of Navier-Stokes equations (3.16) follows from (3.4). Similarly $\|u(\cdot,t)\|^2_{H^1(\mathbb{R}^3)} = \|u(\cdot,t)\|_2^2 + \|\nabla u(\cdot,t)\|_2^2$ and (3.17) also follows from (3.4).

The pressure estimate is slightly deeper. Indeed one uses $\|f\|_6 \le C\|\nabla f\|_2$ for $f \in \dot{H}^1(\mathbb{R}^3)$. Therefore $\int_0^T \|u(\cdot,t)\|_q^2\, dt \le C\|u_0\|_2^2$ for $2 \le q \le 6$. Hlder's inequality yields

$$\int_0^T \big\|u_j(\cdot,t)\, u_k(\cdot,t)\big\|_r\, dt \le C\|u_0\|_2^2 \tag{3.25}$$

for $1 \le r \le 3$. Here $u = (u_1, u_2, u_3)$.

Finally the pressure can be computed by (3.23). Since the Riesz transformations are bounded on $L^r(\mathbb{R}^3)$ for $1 < r < \infty$, we obtain

$$\int_0^T \|p(\cdot,t)\|_r\, dt \le C\|u_0\|_2^2 . \tag{3.26}$$

We finally turn to the proof of (3.19). Returning to the Navier-Stokes equations, we scan Δu, ∇p and $u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u = \partial_1(u_1 u) + \partial_1(u_2 u) + \partial_3(u_3 u)$.

Since $\Delta : H^1(\mathbb{R}^3) \to H^{-1}(\mathbb{R}^3)$ is a contraction, we have

$$\int_0^T \|\Delta u\|^2_{H^{-1}(\mathbb{R}^3)}\, dt \le \int_0^T \|u\|^2_{H^1}\, dt \le \frac{\|u_0\|_2^2}{2}.$$

Concerning the pressure, (3.26) is used with $r = 2$ and yields the required estimate.

Finally $\partial_1(u_1 u) + \partial_2(u_2 u) + \partial_3(u_3 u)$ is estimated in $L^1([0,T]; H^{-1}(\mathbb{R}^3))$ using (3.25) with $r = 2$. Lemma 4 is now proved.

We return to definition and specify the meaning of (3.17).

Both sides of (3.17) belong to $L^1([0,T]; H^{-1}(\mathbb{R}^3))$. Indeed the left-hand side does belong by (3.16). Concerning the right-hand side, it suffices to repeat the above discussion and to use (3.14) or (3.15).

Finally $u(\cdot,t)$ belongs to $C([0,T]; H^{-1}(\mathbb{R}^3))$ (indeed $u(\cdot,t)$ is absolutely continuous with respect to the time variable). Therefore $u(\cdot,0)$ can be given a meaning in the distributional sense.

Let us insist on the fact that the right-hand side of (3.17) cannot be written $u_1\partial_1 u + \cdots + u_3\partial_3 u$. Indeed one cannot multiply a function in $H^{-1}(\mathbb{R}^3)$ by a function in $L^2(\mathbb{R}^3)$. This type of problems will be answered by the paraproduct algorithm of section 14.

The existence of weak solutions will be proved in the next section in relation with the variational formulation of the Navier-Stokes equations.

4 A variational formulation of the Navier-Stokes equations

Following notations which were introduced by J.L. Lions, V will denote the closed subspace of $(H^1(\mathbb{R}^3))^3$ consisting of all vector fields $v(x) = \big(v_1(x), v_2(x), v_3(x)\big)$ such that

$$\operatorname{div} v \;=\; \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 \;=\; 0\,. \tag{4.1}$$

The variational formulation of the Navier-Stokes equations consists in making (3.17) more explicit by writing all scalar products with all testing functions. These testing functions are functions of the x variable since we always single out the dependance in the time variable and consider $u(x,t)$ as a vector valued function of t (and not a vector valued distribution with respect to t).

But instead of using all testing functions in $\mathcal{S}(\mathbb{R}^3)$ we want instead to use testing functions in $H^1(\mathbb{R}^3)$. In other words we replace the duality between $\mathcal{S}(\mathbb{R}^3)$ and $\mathcal{S}'(\mathbb{R}^3)$ by the duality between $H^1(\mathbb{R}^3)$ and $H^{-1}(\mathbb{R}^3)$.

Moreover we indeed use $v \in V$ as testing functions. The variational formulation of the Navier-Stokes equations relies on the following lemma.

Lemma 6. *If $p(x,t) \in L^1([0,T]; L^2(\mathbb{R}^3))$ and $v \in V$, we then have*

$$<\nabla p, v> = 0 \qquad \text{on} \quad [0,T]\,. \tag{4.2}$$

Let us first observe that $< \nabla p, v > \in L^1[0,T]$. Indeed ∇p belongs to $L^1([0,T]; H^{-1}(\mathbb{R}^3))$ while v belongs to $H^1(\mathbb{R}^3)$. Since H^{-1} is the dual space of $H^1(\mathbb{R}^3)$, we can conclude.

For proving (4.2) it then suffices to use an approximation $p_j(x,t)$ to $p(x,t)$. We can assume p_j, $p_j(x,t)$ is a continuous function of the time variable with values in $\mathcal{S}(\mathbb{R}^3)$ together with

$$\int_0^T \|p(\cdot,t) - p_j(\cdot,t)\|_2 \, dt \to 0 \qquad (j \to +\infty) . \tag{4.3}$$

Then

$$<\nabla p_j, v> = - <p_j, \operatorname{div} v> = 0 .$$

It suffices to pass to the limit as j tends to infinity to obtain (4.2).

Definition 7. *Let $u = u(x,t)$, $x \in \mathbb{R}^3$, $0 \le t \le T$, be a weak solution of Navier-Stokes equations. Then the variational formulation of* (3.15) *reads*

$$\begin{cases} \dfrac{d}{dt} <u,v> = <u, \Delta v> + <u, Xv> \\ \operatorname{div} u = 0 \\ u(x,0) = u_0(x) . \end{cases} \tag{4.4}$$

Here and in what follows $<u,v> = \sum_1^3 \int u_j(x) \overline{v_j(x)} \, dx$ and $X = u_1 \frac{\partial}{\partial x_1} + u_2 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3}$.

Let us be more specific. All terms in the first equation belong to $L^1[0,T]$.

Indeed $\frac{\partial u}{\partial t}$ belongs to $L^1([0,T]; H^{-1}(\mathbb{R}^3))$ while v belongs to $H^1(\mathbb{R}^3)$. Then $< \frac{\partial u}{\partial t}, v > \in L^1[0,T]$ and a simple limiting argument yields

$$< \frac{\partial u}{\partial t}, v > = \frac{d}{dt} <u,v> . \tag{4.5}$$

Concerning $<u, \Delta v>$, we observe that $u \in L^2([0,T]; H^1(\mathbb{R}^3))$ while $\Delta v \in H^{-1}(\mathbb{R}^3)$. Therefore $<u, \Delta v> \in L^2[0,T]$.

Finally $u \in L^2([0,T]; L^4(\mathbb{R}^3))$ which implies $Xv \in L^2([0,T]; L^{4/3}(\mathbb{R}^3))$. Since $u \in L^2([0,T]; L^4)$ we obtain $<u, Xv> \in L^1([0,T])$.

Let us conversely assume (3.12), (3.13), (3.14) and (4.4). We would like to obtain (3.15). Indeed $R(x,t) = \frac{\partial u}{\partial t} - \Delta u + Xu$ satisfies $<R, v> = 0$ for each $v \in V$. This implies $R = \nabla p$ for some scalar valued function p and Navier-Stokes equations are recovered.

For us be more explicit and consider a distribution $S = (S_1, S_2, S_3)$, $S_j \in \mathcal{S}'(\mathbb{R}^3)$, such that for any $v = (v_1, v_2, v_3)$ with $v_j \in \mathcal{S}(\mathbb{R}^3)$ and $\operatorname{div} v = 0$ we have $< S, v >= 0$. Then $S = \nabla p$. Indeed we check $< S, v >= 0$ on $v_1 = -\partial_2 f$, $v_2 = \partial_1 f$, $v_3 = 0$, $f \in \mathcal{S}(\mathbb{R}^3)$ and obtain $\partial_1 S_2 - \partial_2 S_1 = 0$ in the distributional sense. Similarly $\partial_2 S_3 - \partial_3 S_2 = 0$ and $\partial_3 S_1 - \partial_1 S_3 = 0$. This implies $S = \nabla p$ as announced.

Once the variational formulation of the Navier-Stokes equations is written, one would like to move one step further.

Instead of writing (4.32) for every $v \in V$, one is using a sequence $v_0, v_1, \dots, v_m, \dots$ of vectors of V with the following two properties

(4.6) $\qquad v_0, v_1, \dots, v_m, \dots$ are linearly independant

(4.7) $\qquad$ the linear span of $v_0, v_1, \dots, v_m, \dots$ is dense in V.

We then denote by V_m the linear span of $v_0, v_1, \dots, v_{m-1}$. We then have $V_m \subset V_{m+1}$ and $\bigcup_{m \geq 0} V_m$ is dense in V.

We now arrive to the definition of a Galerkin scheme.

We want to approximate the weak solution $u(x,t)$ by a function $u_N(x,t)$ which is no longer a solution but belongs to $C^\infty([0,T], V_N)$.

Moreover this approximation $u_N(x,t)$ is not the orthogonal projection of $u(x,t)$ onto V_N. This implies that the convergence of u_N to u will be a delicate issue.

Instead we define $u_{0,N}(x) \in V_N$ as the orthogonal projection of u_0 onto V_N. This refers to the standard inner product in $L^2(\mathbb{R}^3)$.

We now define $u_N(x,t)$ by

$$\frac{d}{dt} < u_N, v_m > = < u_N, \Delta v_m > + < u_N, (u_N \cdot \nabla) v_m > \tag{4.8}$$

where $0 \leq m \leq N-1$ and

$$u_N(x,0) = u_{0,N}(x)\,. \tag{4.9}$$

Let us simplify the discussion by assuming the following. If H_0 denotes the closed subspace of $(L^2(\mathbb{R}^3))^3$ defined by $\partial_1 f_1(x) + \partial_2 f_2(x) + \partial_3 f_3(x) = 0$, then v_m, $m \geq 0$, is a Hilbert space basis of H_0. We then write

$$u_N(x,t) = \alpha_{0,N}(t)\, v_0(x) + \cdots + \alpha_{N-1,N}(t)\, v_{N-1}(x) \tag{4.10}$$

and (4.8) reads

$$\frac{d}{dt}\alpha_N(t) = A[\alpha_N] + B[\alpha_N, \alpha_N] \tag{4.11}$$

where $\alpha_N(t) = \big(\alpha_{0,N}(t), \dots, \alpha_{N-1,N}(t)\big)$, A being an ordinary $N \times N$ matrix and $B[X, X]$ and ordinary quadratic form. Obviously A and B depend on N.

This ordinary differential equation should be completed with $\alpha_N(0) = \gamma_N$ where $\gamma_N = \big(\gamma_0, \dots, \gamma_{N-1}\big)$ when $u_0(x) = \sum_0^\infty \gamma_m v_m(x)$.

Since (4.11) is non linear, the only issue concerns a possible blow up in finite time.

However the proof of the energy estimate can be rewritten inside the approximation space V_N and yields

$$\|u_N(\cdot, t)\|_2 \leq \|u_{0,N}\|_2 \leq \|u_0\|_2 \,. \tag{4.12}$$

It implies that (4.8) has a unique global solution and $u_N(x, t)$ belongs to $C^\infty([0, T]; V_N)$ for all positive T's.

This being achieved, one would like to build a true solution from this sequence $u_N(x, t)$, $N \geq 1$. Since u_N uniformly satisfies the estimates listed in lemma 4, one is tempted to use a compactness argument. From $u_N(x, t)$ one is extracting a subsequence $u_{N_j}(x, t)$ which weakly converges to $u(x, t)$. For proving that $u(x, t)$ is a solution, one should intertwin weak limits and non linear mappings. We all know that it is impossible and our naive approach fails. We then follow J.L. Lions and construct a subsequence $u_{N_j}(x, t)$ such that

$$\lim_{j \to +\infty} \int_0^T \int_{|x| \leq R} \big|u_{N_j}(x, t) - u(x, t)\big|^2 dx\, dt = 0 \tag{4.13}$$

for every positive R.

For proving this strong convergence, a new estimate is needed.

J.L. Lions proves the existence of a positive exponent γ and of a constant $C(\gamma)$ such that, for any $u_0(x)$ in $L^2(\mathbb{R}^3)$ and $N \geq 1$, one has

$$\int_{-\infty}^{\infty} |\tau|^{2\gamma}\, \|\hat{u}_N(\cdot, \tau)\|_2^2\, d\tau \leq C(\gamma)\, \|u_0\|_2^2 \,. \tag{4.14}$$

Here $\hat{u}_N(x, \tau) = \int_{-\infty}^{\infty} e^{-it\tau}\, u_N(x, t)\, dt$ and the L^2 norm is computed with respect to the x variable. This new estimate, once combined with

$$\sup_{t \geq 0} \|u_N(\cdot, t)\|_2 \leq C_0 \quad \text{and} \quad \int_0^T \|u_N(\cdot, t)\|_{H^1}^2\, dt \leq C_1$$

is used for showing that the weak convergence in $L^2([0,T]; H^1(\mathbb{R}^3))$ of u_{N_j} implies (4.41).

The reader is referred to [65] and J. Leray's theorem will be stated.

Theorem 4.1. *Let us assume that* $u_0(x) \in L^2(\mathbb{R}^3)$ *and* $\operatorname{div} u_0(x) = 0$. *Then for every* $T > 0$, *there exists a weak solution* $u(x,t)$, $0 \le t \le T$, *of the Navier-Stokes equations such that* $u(x,0) = u_0(x)$.

The uniqueness of such a weak solution is still an open problem.

5 The affine group action

The Navier-Stokes equations are invariant under a group action. More precisely if $(u(x,t), p(x,t))$ is a solution defined on $\mathbb{R}^3 \times (0,\infty)$, the same is true for (u_λ, p_λ) where $0 < \lambda < \infty$ and

$$(5.1) \qquad u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t) \quad , \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t) .$$

If instead one would consider $u(\lambda x, \lambda t)$ and $p(\lambda x, \lambda t)$, there are a solution of a distinct Navier-Stokes equation where ν is being replaced by $\lambda\nu$. We can fix $\lambda > 0$ such that $\lambda\nu = 1$ and assume that the viscosity is 1.

Returning to the invariance of the Navier-Stokes equation, if $(u(x,t), p(x,t))$ is a solution, the same is true for $\big(u(x-x_0,t), p(x-x_0,t)\big)$ for $x_0 \in \mathbb{R}^3$.

Finally for any $\tau \ge 0$, $\big(u(x,t+\tau), p(x,t+\tau)\big)$ is still a solution.

This affine group action should be incorporated into a Galerkin scheme. Guy Battle and Paul Federbush achieved this program and their results will now be described.

Let us denote by H_0 the subspace of $H = (L^2(\mathbb{R}^3))^3$ which is defined by $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$. Since this condition can be rewritten as $R_1 u_1 + R_2 u_2 + R_3 u_3 = 0$ where R_j are the Riesz transformations, H_0 is a closed subspace of H. G. Battle and P. Federbush succeeded in constructing a basis for H_0 of the form

$$(5.2) \qquad 2^{3j/2}\, \psi(2^j x - k) , \qquad \psi \in A ,\ j \in \mathbf{Z} ,\ k \in \mathbf{Z}^3$$

where A is a finite collection of 14 mother wavelets belonging to the Schwartz class.

Before moving further, let us sketch the construction of Federbush's basis. We closely follow the modification proposed by P.G. Lemarié-Rieusset in [50] of Federbush's original construction.

6 A divergence-free wavelet basis

The starting point is the orthonormal wavelet basis of $L^2(\mathbb{R})$ which is presented in [55]. We begin with the so-called scaling function $\varphi(t)$ which is a smooth version of the indicator function of $[0,1]$. Here $\varphi(t)$ belongs to the Schwartz class, its Fourier transform $\hat{\varphi}(\xi)$ satisfies $\hat{\varphi}(\xi) = 1$ on $[-2\pi/3\,,\,2\pi/3]$, $\hat{\varphi}(\xi) = 0$ if $|\xi| \geq 4\pi/3$, $\hat{\varphi}(\xi) \in [0,1]$, $\hat{\varphi}(-\xi) = \hat{\varphi}(\xi)$ and finally

$$(6.1) \qquad |\hat{\varphi}(\pi+s)|^2 + |\hat{\varphi}(\pi-s)|^2 = 1 \quad , \quad |s| \leq \pi/3\,.$$

These assumptions imply $\sum_{-\infty}^{\infty} |\hat{\varphi}(2k\pi+\xi)|^2 = 1$ everywhere. Therefore $\varphi(t-k)$, $k \in \mathbf{Z}$, is an orthonormal sequence in $L^2(\mathbb{R})$. We denote by V_0 the closed linear span of this sequence $\varphi(t-k)$, $k \in \mathbf{Z}$. Next V_j is defined by

$$(6.2) \qquad f(t) \in V_0 \iff 2^{j/2} f(2^j t) \in V_j$$

where $f \in L^2(\mathbb{R})$ and $j \in \mathbf{Z}$.

A crucial observation is the inclusion $V_j \subset V_{j+1}$ which relies on the specific properties of φ.

The next step consists in studying the orthogonal complement W_j of V_j inside V_{j+1}. Property (6.2) implies

$$(6.3) \qquad f(t) \in W_0 \iff 2^{j/2} f(2^j t) \in W_j\,.$$

There exists a function $\psi(t)$ in the Schwartz class such that $\psi(t-k)$, $k \in \mathbf{Z}$, is an orthonormal basis of W_0.

The Fourier transform $\hat{\psi}(\xi)$ of ψ is supported by $\frac{2\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}$ and

$$(6.4) \qquad 2^{j/2}\,\psi(2^j t-k)\,, \qquad j \in \mathbf{Z}\,,\ k \in \mathbf{Z}$$

is an orthonormal basis for $L^2(\mathbb{R})$. This function $\psi(t)$ is the mother wavelet.

Both the scaling function $\varphi(t)$ and the mother wavelet $\psi(t)$ are needed to construct an orthonormal wavelet basis for $L^2(\mathbb{R}^3)$.

Indeed one writes $\varphi_0(t) = \varphi(t)$, $\varphi_1(t) = \psi(t)$ and the three-dimensional scaling function is

$$(6.5) \qquad \varphi(x) = \varphi_0(x_1)\,\varphi_0(x_2)\,\varphi_0(x_3)$$

while the 7 three-dimensional wavelets are given by

$$\psi_\varepsilon(x) = \varphi_{\varepsilon_1}(x_1)\,\varphi_{\varepsilon_2}(x_2)\,\varphi_{\varepsilon_3}(x_3) \quad , \quad \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \tag{6.6}$$

and $\varepsilon \in \{0,1\}^3$, $\varepsilon \neq (0,0,0)$.

With these notations, we obtain the three-dimensional wavelet basis as

$$2^{3j/2}\,\psi_\varepsilon(2^j x - k)\,, \qquad j \in \mathbf{Z}\,,\ k \in \mathbf{Z}^3\,, \quad \varepsilon \in \{0,1\}^3 \quad , \quad \varepsilon \neq (0,0,0)\,. \tag{6.7}$$

We now denote by H_0 the closed subspace of $H = (L^2(\mathrm{I\!R}^3))^3$ defined by $R_1u_1 + R_2u_2 + R_3u_3 = 0$ where R_1, R_2, R_3 are the three Riesz transformations. Each component u_1, u_2 or u_3 is then expanded into a wavelet expansion using the orthonormal basis given by (6.51).

Roughly speaking, the full equation $R_1u_1 + R_2u_2 + R_3u_3 = 0$ can be decoupled into

$$R_1\,u_1^{(j,\varepsilon)} + R_2\,u_2^{(j,\varepsilon)} + R_3\,u_3^{(j,\varepsilon)} = 0 \tag{6.8}$$

where $u_1^{(j,\varepsilon)}$ belongs to the closed linear span of $2^{3j/2}\,\psi_\varepsilon(2^j x - k)$, $k \in \mathbf{Z}^3$.

Then if $\varepsilon_1 = 1$, (6.52) permits to compute $u_1^{(j,\varepsilon)}$ as a linear combination of $u_2^{(j,\varepsilon)}$ and $u_3^{(j,\varepsilon)}$. Since one of the three indices $\varepsilon_1, \varepsilon_2$ or ε_3 is 1, (6.52) can always be solved.

This construction yields a basis for H_0 which is no longer an orthogonal one. Let us clarify this point.

Definition 8. *A Riesz basis $(e_\lambda)_{\lambda\in\Lambda}$ in a Hilbert space H is a collection of vectors of H for which there exists an orthonormal basis $(f_\lambda)_{\lambda\in\Lambda}$ of H and an isomorphism $U : H \to H$ such that*

$$U(f_\lambda) = e_\lambda\,. \tag{6.9}$$

It implies that each vector $x \in H$ can uniquely be written as

$$x = \sum_{\lambda\in\Lambda} \alpha_\lambda\, e_\lambda \tag{6.10}$$

where

$$C_1\|x\| \leq \left(\sum |\alpha_\lambda|^2\right)^{1/2} \leq C_2\|x\| \tag{6.11}$$

for two constants $C_2 \geq C_1 > 0$.

If $C_2 = C_1 = 1$, $(e_\lambda)_{\lambda\in\Lambda}$ is an orthonormal basis. In general, one defines the dual basis $\tilde{e}_\lambda$ by

$$(U^*)^{-1}(f_\lambda) = \tilde{e}_\lambda . \tag{6.12}$$

This dual basis is still a Riesz basis and one has

$$\alpha_\lambda = < x, \tilde{e}_\lambda > . \tag{6.13}$$

With this definition in mind, the Lemarié-Rieusset's version of the Battle-Federbush theorem reads

Theorem 6.1. *There exists a finite collection A of 14 divergence free vector fields* $\vec{\psi}(x) = \big(\psi_1(x), \psi_2(x), \psi_3(x)\big)$ *with the following properties*

$$\psi_j \in \mathcal{S}(\mathbb{R}^3) , \qquad 1 \leq j \leq 3 \tag{6.14}$$

$$\hat{\psi}_j(\xi) = 0 \quad \text{if} \quad |\xi|_\infty \leq \frac{2\pi}{3} \quad \text{or} \quad |\xi|_\infty \geq \frac{8\pi}{3} \tag{6.15}$$

where $|\xi|_\infty = \sup\big(|\xi_1|, |\xi_2|, |\xi_3|\big)$

$$2^{3j/2}\, \psi(2^j x - k) , \qquad \psi \in A , \ j \in \mathbf{Z} , \ k \in \mathbf{Z}^3 , \tag{6.16}$$

is a Riesz basis of H_0

(6.17) *the dual basis is given by*

$$2^{3j/2}\, \tilde{\psi}(2^j x - k) , \qquad \tilde{\psi} \in \tilde{A} , \ j \in \mathbf{Z} , \ k \in \mathbf{Z}^3$$

where $\tilde{A}$ *is a second collection of 14 wavelets satisfying* (6.14) *and* (6.15).

This remarkable theorem will be proved in the appendix and we are now ready for describing P. Federbush's program.

7 Federbush's program

Federbush's program is quite natural and extremely appealing. It consists in using a Galerkin scheme for solving Navier-Stokes equations and in incorporating the affine group action into this Galerkin scheme. We then respect the invariance of the Navier-Stokes equations with respect to this affine group action. This approach is consistent with our scientific knowledge about turbulence. We will return to this point.

A Galerkin scheme where affine group action is incorporated is the Battle-Federbush basis. This basis reads $2^{3j/2}\,\vec{\psi}(2^j x-k)$, $j\in\mathbf{Z}$, $k\in\mathbf{Z}^3$, $\vec{\psi}\in A$ where A is a collection of 14 divergence free mother wavelets $\vec{\psi}=(\psi_1,\psi_2,\psi_3)$ for which ψ_1,ψ_2 and ψ_3 belong to the Schwartz class. Such an approach is aimed to decouple the Navier-Stokes equation into a sequence of equations of the general form

$$\frac{\partial}{\partial t}u_j = L_j(u_j)+\sum_{j'}\sum_{j''}B_{j,j',j''}(u_{j'},u_{j''})\,. \tag{7.1}$$

Here $u_j(x,t)=\sum_k \alpha(j,k,t)\,2^{3j/2}\,\vec{\psi}(2^j x-k)$ (the sum over A will be omitted for keeping the notations as short as possible).

Since the Fourier transform of $2^{3j/2}\,\vec{\psi}(2^j x-k)$ is supported by the dyadic annulus

$$\frac{2\pi}{3}\,2^j \le |\xi|_\infty \le \frac{8\pi}{3}\,2^j \tag{7.2}$$

the closed linear span of $2^{3j/2}\,\vec{\psi}(2^j x-k)$, $\vec{\psi}\in A$, $k\in\mathbf{Z}^3$, will be named a frequency channel and denoted by $\vec{W}_j$, $j\in\mathbf{Z}$.

Returning to Federbush's program, the goal consists in decoupling the Navier-Stokes equations into a sequence of equations. This decoupling only applies to the linear term (Δu) while the non-linear terms $\big(\partial_1(u_1u)+\cdots+\partial_3(u_3u)\big)$ are coupled by non-linear interactions. These non-linear interactions are described by the bilinear operators $B_{j,j',j''}(u_{j'},u_{j''})$.

The success of this approach depends on the following property which is expected from $B_{j,j',j''}$. These interactions should become negligible when $|j'-j|+|j''-j|$ tends to infinity.

In other words, what is happening inside a frequency channel should only affect the neighboring frequency channels.

Before entering into a more precise criticism of this program, let us observe that it is backed on some previous work by P. Frick and V. Zimin.

In a superb paper entitled *Hierarchical models of turbulence* [33] they write

> "Ideas, like the ones used to create wavelet analysis were proposed by Zimin (1981) for construction of a hierarchical model of turbulence...
>
> In a paper by Zimin (1981) a special functional basis has been presented. Functions of this basis are related to a hierarchical

system of vortices of different sizes. The number of vortices in a unit volume increases with decreasing size and each function is well localized both in Fourier and physical spaces. The product of the characteristic scales of localization in Fourier and coordinate spaces satisfies the uncertainty relation...

The cascade equations, written for the quantities A_i, each define the velocity oscillations in some interval of wave numbers and describe the principal characteristics of energy redistribution processes between different scales. The cascade equations minimize the dimensionality of systems which describe the turbulent flows in wide range of wave numbers, and have a form

$$\frac{d}{dt} A_i = Y_i A_i + \sum_{j,k}\sum X_{ijk} A_j A_k \ldots \tag{7.3}$$

The hierarchical model of turbulence is based on the natural assumption that turbulence is an ensemble of vortices of progressively diminishing scales. The hierarchical basis for two-dimensional turbulence describes the ensemble of the vortices, in which any vortex of the given size consists of four vortices of half size and so on. The ensemble of vortices of the same size forms a "level". The functions of the hierarchical basis are constructed in such a way that Fourier-images of vortices of single level occupy only single octave in the wave-number space and regions of localization of different levels in the Fourier space do not overlap.

The wave-number space is divided at ring zones such that $\pi 2^N < |k| < \pi 2^{N+1} \ldots$"

This quotation from Frick and Zimin implies that these authors are using the Shannon wavelet basis. This remark is made explicit in their paper. The scaling function of the one-dimensional Shannon basis is the standard sinc function $\varphi(t) = \frac{\sin \pi t}{\pi t}$ while the corresponding mother wavelet is given by

$$\psi(t) = 2\varphi(2t) - \varphi(t)\,. \tag{7.4}$$

However these functions have a poor localization in the coordinate space. They do have an ideal localization in the frequency domain. Shannon wavelets correspond to ideal filters in signal processing and these ideal filters are unrealistic ones since their numerical support is infinite. For the same reason, Shannon wavelets cannot be used in numerical analysis.

We now want to use "modern technology" and further analyze Federbush's program.

Let us try to decouple Navier-Stokes equations. We denote by $\psi_\lambda(x)$, $\lambda \in \Lambda$, the divergence-free wavelet basis which is described in theorem 6.1. We have $\Lambda = A \times \mathbf{Z} \times \mathbf{Z}^3$ and $\psi_\lambda(x) = 2^{3j/2}\psi(2^j x - k)$, $\psi \in A$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}^3$, $\lambda = (\psi, j, k) \in \Lambda$.

Using the variational formulation we are led to writing

$$u(x,t) = \sum_{\lambda \in \Lambda} \alpha_\lambda(t)\,\psi_\lambda(x)\,. \tag{7.5}$$

The corresponding Galerkin scheme reads

$$\frac{d}{dt}\alpha_\lambda(t) = \sum_{\lambda'} \omega(\lambda,\lambda')\,\alpha_{\lambda'}(t) + \sum_{\lambda',\lambda''}\sum \beta(\lambda,\lambda',\lambda'')\,\alpha_{\lambda'}(t)\,\alpha_{\lambda''}(t) \tag{7.6}$$

where

$$\begin{aligned} \omega(\lambda,\lambda') &= (\Delta\psi_{\lambda'}, \tilde{\psi}_\lambda) \\ \beta(\lambda,\lambda',\lambda'') &= b(\psi_{\lambda'}, \psi_{\lambda''}, \tilde{\psi}_\lambda) \end{aligned}$$

and

$$b(u,v,w) = \sum_1^3 \sum_1^3 \int u_k(x)\big(\partial_k v_\ell\big)(x)\,\overline{w}_\ell(x)\,dx\,.$$

Concerning $\omega(\lambda,\lambda')$, we first observe that $\omega(\lambda,\lambda') = 0$ whenever $|j'-j| \geq 2$. Since the mother wavelets belong to the Schwartz class, we also obtain a rapid decay when $|j'-j| \leq 1$ and when the distance between λ' and λ tends to infinity. This distance is defined as $(2^j + 2^{j'})|k2^{-j} - k'2^{-j'}|$ if $k2^{-j} \neq k'2^{-j'}$ and $2^{|j'-j|}$ if $k2^{-j} = k'2^{-j'}$, $\psi \neq \psi'$.

It should be stressed that one cannot obtain a rapid decay at infinity if Shannon's wavelets are being used. Returning to the one-dimensional case, it is not true that $\int \varphi''(x)\,\varphi(x-k)\,dx$ has a rapid decay at infinity as $|k| \to +\infty$. Indeed this integral is $0(k^{-2})$ and this estimate is sharp.

This means that the Frick-Zimin program cannot be completed the way it is stated.

Let us forget this remark and return to Federbush's version of the Frick-Zimin approach. There is unexpected bad news given by the following claim.

Claim. *There exists a vector $\ell \in \mathbf{Z}^3$ with the following property : if $j' = j''$, $k' - k'' = \ell$ and if j tends to $-\infty$, then the trilinear coefficients $\beta(\lambda, \lambda', \lambda'')$ are large.*

More precisely, these coefficients are large whenever the dyadic cube $Q(\lambda)$ contains $Q(\lambda') \cup Q(\lambda'')$. Here $Q(\lambda)$ denotes the dyadic cube defined by $2^j x - k \in [0,1)^3$ when $\psi_\lambda(x) = 2^{3j/2}\,\psi(2^j x - k)$.

This unpleasant fact contradicts big claims made by some scientists ([28], page 664).

Let us now prove our claim. We are assuming $j' = j''$ and first want to prove that the function

$$(\psi_{\lambda'} \cdot \nabla)\psi_{\lambda''} \;=\; \psi_{\lambda'}^{(1)}\partial_1\psi_{\lambda''} + \psi_{\lambda'}^{(2)}\partial_2\psi_{\lambda''} + \psi_{\lambda'}^{(3)}\partial_3\psi_{\lambda''}$$

has a non vanishing integral. Indeed

$$I(\lambda', \lambda'') \;=\; \int (\psi_{\lambda'} \cdot \nabla)\psi_{\lambda''}\, dx \;=\; 2^{j'}\gamma(k' - k'')\,.$$

Let us show that γ does not vanish identically. Otherwise one would have

$$\text{(7.7)} \qquad \int (\psi \cdot \nabla)\,\psi(x-k)\, dx \;=\; 0\,, \qquad k \in \mathbf{Z}^3\,.$$

But (7.7) is equivalent to

$$\text{(7.8)} \qquad \sum_{k \in \mathbf{Z}^3} \sigma(\xi - 2k\pi) \;=\; 0$$

where

$$\text{(7.9)} \qquad \sigma(\xi) \;=\; [\xi_1\hat{\psi}_1(\xi) + \xi_2\hat{\psi}_2(\xi) + \xi_3\hat{\psi}_3(\xi)]\,\overline{\hat{\psi}}(\xi)\,.$$

Here we need to compute $\hat{\psi}(\xi)$, $\sigma(\xi)$ and to check that (7.9) does not hold. These computations are based on the explicit calculations made in the appendix and will be omitted. Therefore the product $(\psi_{\lambda'} \cdot \nabla)\psi_{\lambda''}$ is given by

$$\text{(7.10)} \qquad (\psi_{\lambda'} \cdot \nabla)\psi_{\lambda''} \;=\; 2^{4j'}\,\omega(2^{j'}x - k')$$

where $\omega \in \mathcal{S}(\mathbb{R}^3)$, $\omega = \omega_\ell$ and $\int \omega(x)\, dx \neq 0$. Then

$$\begin{aligned}\beta(\lambda, \lambda', \lambda'') \;&=\; 2^{4j'} \int \omega(2^{j'}x - k')\,\tilde{\psi}_\lambda(x)\, dx \\ &=\; 2^{j'}\,\tilde{\psi}_\lambda(k' 2^{-j'}) \cdot (\gamma(k' - k'') + 0(2^{j-j'}))\end{aligned}$$

as $j - j'$ tends to $-\infty$. This proves our claim.

This discussion will be elaborated in section 14 where the paraproduct algorithm will be studied. Indeed the paraproduct algorithm consists in writing the ordinary product $f(x)\,g(x)$ between two arbitrary functions as a sum of three series. Two of them only contain oscillatory terms. The third one is the most difficult. It reads $\sum_{-\infty}^{\infty} \Delta_j(f)\,\Delta_j(g)$ and these products do not have any cancellation. They are similar to $(\psi_{\lambda'} \cdot \nabla)\psi_{\lambda''}$ when $j' = j''$.

Let us now return to P. Federbush's program.

In a paper entitled *Navier and Stokes meet the wavelet*, P. Federbush was much hoping that our understanding of the Navier-Stokes equations could benefit from wavelet bases.

Let me first stress that I fully believe that P. Federbush's claim is correct. However Federbush's paper is disappointing.

First of all, Federbush is not proposing a new numerical scheme for solving Navier-Stokes equations. Instead he is using his wavelet basis for proving new estimates. These estimates are defined by functional norms. The corresponding functional spaces which are used in Federbush's paper are the Morrey-Campanato spaces.

Definition 9. *If $s \in (0,3)$, the Banach space $M^{2,s}$ is defined by the following property. A function $f(x)$ belongs to $M^{2,s}$ if and only if f is locally square integrable and if there exists a constant C such that*

$$\text{(7.11)} \qquad \sup_{x_0 \in \mathbb{R}^3} \sup_{0<R<\infty} \left(R^{-s} \int_{|x-x_0| \le R} |f(x)|^2 \, dx \right)^{1/2} \le C\,.$$

If $s = 0$, this would define L^2 and if $s = 3$, (7.72) would define L^∞.

P. Federbush restricted his attention to the case $1 < s < 3$ and proved the following statement. If $u_0(x)$ belongs to $M^{2,s}$ and $\operatorname{div} u_0(x) = 0$, then there exists a positive T and a solution

$$\text{(7.12)} \qquad u(x,t) \in C\big([0,T], M^{2,s}\big)$$

of the Navier-Stokes equations such that $u(x,0) = u_0(x)$.

These solutions $u(x,t) \in C([0,T], M^{2,s})$ are no longer classical solutions of Navier-Stokes equations. They are not weak solutions in Leray's sense. The solutions constructed by P. Federbush in his paper are the so-called "mild solutions" of Navier-Stokes equations. This concept will be defined in section 12.

With my student Marco Cannone, we wanted to better understand Federbush's paper and the role played by the hierarchical wavelet basis in the proof. We soon observed that Federbush's theorem is becoming a trivial statement if a Littlewood-Paley analysis will be defined in section 14 and wavelet analysis can be defined as an improved version of Littlewood-Paley analysis.

The second piece of bad news is the following. The case $s = 1$ in the definition of the Morrey-Campanato spaces $M^{2,s}$ cannot be treated by Federbush's approach. This exponent plays a crucial role since homogeneous functions of degree -1 belong to $M^{2,1}$. If the initial condition $u_0(x)$ exhibits this homogeneity, then the corresponding solution of the Navier-Stokes equation will be self-similar. This means that

$$\lambda u(\lambda x, \lambda^2 t) = u(x,t) , \qquad 0 < \lambda < \infty . \tag{7.13}$$

In other words solving Navier-Stokes equations when $u_0 \in M^{2,1}$ paves the way to the construction of self-similar solutions to the Navier-Stokes equations. In the next section, the special role played by this Morrey-Campanato space $M^{2,1}$ will be clarified. Here we want to stress that Tosio Kato found an algorithm for solving Navier-Stokes equations when $u_0 \in M^{2,1}$. This remarkable algorithm is not based on Fourier analysis or wavelet analysis. Kato's approach will be presented in section 21.

These remarks may lead to the conclusion that wavelet analysis is the worst tool for studying Navier-Stokes equations. If Federbush's paper were the test for ranking all available tools, the conclusion would be obvious. Wavelet analysis is the worst tool, Littlewood-Paley analysis and paraproduct algorithms are doing a much better job and T. Kato's algorithm is number one.

This bad news does not end the discussion. Indeed Littlewood-Paley analysis or wavelet analysis were used in the first proof of the uniqueness of mild solutions $u \in C[0, \infty)$, of Navier-Stokes equations. The existence of such solutions was proved by T. Kato but Kato's algorithm left the uniqueness problem open. Unfortunately a much simpler proof was soon obtained. This simpler proof does not use any spectral method and the usefulness of Littlewood-Paley methods cannot be proved in this uniqueness issue.

8 Banach spaces adapted to Navier-Stokes equations

Our goal is to describe an improved version of Kato's program. T. Kato treated an important example. He proved the existence of a positive constant α such that the following implication holds : if $u_0(x)$ belongs to $L^3(\mathbb{R}^3)$, if $\|u_0\|_3 < \alpha$ and $\operatorname{div} u_0(x) = 0$, then there exists a solution

$$u(x,t) \in C\big([0,\infty); L^3(\mathbb{R}^3)\big) \tag{8.1}$$

of the Navier-Stokes equations such that

$$u(x,0) = u_0(x) . \tag{8.2}$$

The uniqueness of such a solution will be proved in section 20 and a second proof can be found in section 26. Let us observe that $u_0(x)$ and $\lambda u_0(x)(\lambda x)$, $\lambda > 0$, are sharing the same L^3 norm. In other words the exponent 3 cannot be replaced by an other one in Kato's theorem. Indeed if the initial condition $u_0(x)$ does not satisfy $\|u_0\|_q < \alpha$, then $\lambda u_0(\lambda x)$ will certainly do it for some value of λ. This means that either the condition $\|u_0\|_q < \alpha$ is not needed and can be replaced by $u_0 \in L^q(\mathbb{R}^3)$ or it is needed and $q = 3$. The first option applies to the case $q = 2$ and nobody knows what is happening for other values of q.

In our attempt to generalize Kato's program, $L^3(\mathbb{R}^3)$ will be replaced by other Banach spaces satisfying some properties which are given in definition 10 and 11.

Definition 10. *A Banach space E is a functional Banach space if*

(8.3) $\mathcal{S}(\mathbb{R}^n) \subset E \subset \mathcal{S}'(\mathbb{R}^n)$

(8.4) *these two canonical embeddings are continuous ones*

(8.5) *either these two embeddings have a dense range*
or E is the dual space F^* of a functional Banach space F
for which these two embeddings have a dense range.

For example $L^p(\mathbb{R}^n)$, $1 \le p < \infty$ is fulfilling these requirements while $L^\infty(\mathbb{R}^n)$ is the dual space of $L^1(\mathbb{R}^n)$. It is clear that $\mathcal{S}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ but $\mathcal{S}(\mathbb{R}^n)$ is not dense inside $L^\infty(\mathbb{R}^n)$.

Definition 10 paves the way to the following one.

Definition 11. *A functional Banach space E is adapted to the Navier-Stokes equations if*

$$\|f(x)\|_E = \|\lambda f(\lambda x)\|_E\,, \qquad 0<\lambda<\infty \tag{8.6}$$

$$\|f(x-x_0)\|_E = \|f(x)\|_E \tag{8.7}$$

(8.8) *there is a constant C such that*

$$\left(\int_{|x|\le 1} |f(x)|^2\,dx\right)^{1/2} \le C\|f\|_E\,.$$

If E is adapted to the Navier-Stokes equations, our program consists in generalizing Kato's theorem. More precisely we would like to prove the following conjecture :

Conjecture. *Let us assume that a functional Banach space E is adapted to the Navier-Stokes equations. Then there exists a positive number η such that*

$$\|u_0\|_E < \eta \quad \textit{and} \quad \operatorname{div} u_0(x) = 0 \tag{8.9}$$

implies that a solution to the Navier-Stokes equations $u(x,t)$ exists with the following properties

$$u(x,t) \in C([0,\infty);E) \tag{8.10}$$

$$u(x,0) = u_0(x)\,. \tag{8.11}$$

Some results in this direction will be given in section 19 and 22.

When $\mathcal{S}(\mathbb{R}^n)$ is dense in E, $C([0,\infty);E)$ will denote the Banach space consisting of all continuous and bounded functions $f : [0,\infty) \to E$. This continuity refers to the norm topology on E. The norm of f in $C([0,\infty);E)$ is defined as

$$\sup_{t\ge 0} \|f(\cdot,t)\|_E\,. \tag{8.12}$$

When $\mathcal{S}(\mathbb{R}^n)$ is not dense in E but instead E is the dual space F^* of a functional Banach space F in which $\mathcal{S}(\mathbb{R}^n)$ is dense, it would be natural to define $C([0,\infty);E)$ as being the Banach space of continuous and bounded functions from $[0,\infty)$ into E when E is given the $\sigma(E,F)$ topology.

However we will not adopt this definition. Indeed some non-linear operations will be performed on such functions f and the $\sigma(E,F)$ topology is not consistent with such nonlinearities.

The definition of $u \in C([0,\infty);E)$ is motivated by the simple example of the heat equation

$$(8.13) \qquad \frac{\partial u}{\partial t} = \Delta u \quad , \quad u = u(x,t) \quad , \quad u(x,0) = u_0(x)$$

where $u_0 \in E$ and $\sup\left\{\|u(\cdot,t)\|_E\,;\, t \geq 0\right\}$ is finite. Then we have

$$(8.14) \qquad u(x,t) \;=\; S(t)u_0 \quad , \quad S(t) = \exp(t\Delta) \quad , \quad t \geq 0\,.$$

At $t_0 = 0$, we have $\lim_{t\downarrow 0} u(x,t) = u_0(x)$ when E is given its $\sigma(E,F)$ topology and this result is optimal when E is not a separable Banach space. However when $t_0 > 0$, $\lim_{t\to t_0} \|u(x,t) - u(x,t_0)\|_E = 0$. This leads to the following definition :

Definition 12. *When the functional Banach space E is not separable but instead is the dual F^* of a separable Banach space F, we will write*

$$(8.15) \qquad u(x,t) \;\in\; C([0,\infty);E)$$

if and only if

$$(8.16) \qquad \lim_{t\downarrow 0} u(x,t) \;=\; u(x,0) \quad \textit{when } E \textit{ is equipped}$$

with its $\sigma(E,F)$ topology and

$$(8.17) \qquad \lim_{t\to t_0} \|u(x,t) - u(x,t_0)\|_E \;=\; 0 \quad \textit{for} \quad t_0 > 0\,.$$

This definition will receive an *a posteriori* vindication in section 23. Indeed for a large family of adapted Banach spaces E, (8.91) will be improved into

$$(8.18) \qquad \|u(x,t') - u(x,t)\|_E \;\leq\; C\left|\frac{t'}{t} - 1\right|^{1/2}$$

for $0 < t < t' < 2t$ with $C = C(u_0)$.

Let us end this section with a simple but striking fact. There exist two adapted functional Banach spaces E_0 and E_1 such that any adapted functional space necessarily satisfies

$$(8.19) \qquad E_0 \;\subset\; E \;\subset\; E_1\,.$$

In other words, among adapted functional Banach spaces, there exists a maximal space E_1 and a minimal space E_0.

Let us stress that the other adapted Banach spaces are not ordered by inclusion. Simple examples will be given in section 24.

Let us first concentrate on the maximal space E_1. Let us consider the Morrey-Campanato space $M^2 = M^{2,1}$. We then have

Lemma 7. *The Morrey-Campanato space M^2 is adapted to the Navier-Stokes equations. Conversely if E is adapated to the Navier-Stokes equations, then E is continuously embedded into M^2.*

The second assertion is almost obvious. If $f \in E$ and $\|f\|_E \leq 1$ then $f_{\lambda,x_0}(x) = \lambda f(\lambda(x - x_0))$ also belongs to the unit ball of E. We then apply (8.8) to f_{λ,x_0} and immediately obtain $f \in M^2$.

The proof of the first assertion is straightforward. However the Morrey-Campanato space M^2 is not separable : $\mathcal{S}(\mathbb{R}^3)$ is not dense in M^2.

Instead M^2 is the dual space of a separable Banach space F. This Banach space F admits an atomic decomposition and we first descrribe the corresponding atoms. These atoms $a_B(x)$ are labelled by balls $B \subset \mathbb{R}^3$. More precisely B is an arbitrary ball in $\mathbb{R}^3$ and $a_B(x)$ should satisfy the following two properties

$$\|a_B\|_2 \leq R^{-1/2} \qquad \text{when} \quad B = \{x\,;\, |x-x_0| \leq R\} \tag{8.20}$$

$$a_B(x) = 0 \qquad \text{if} \quad x \notin B\,. \tag{8.21}$$

Then a function $f(x)$ belongs to F if and only if $f(x) = \sum_0^\infty \lambda_j a_{B_j}(x)$ where a_{B_j} satisfies (8.17) and (8.18) and

$$\sum_0^\infty |\lambda_j| < \infty\,. \tag{8.22}$$

The norm of f in F is the infimum of $\sum_0^\infty |\lambda_j|$ computed over all atomic decompositions of $f(x)$. We obviously have $\mathcal{S}(\mathbb{R}^3) \subset F$ and this continuous embedding is dense. moreover the dual space F^* coincides with the Morrey-Campanato space M^1.

In the opposite direction there exists a smallest Banach space which is adapted to the Navier-Stokes equations. This example will be revisited in section 24 and we now announce the following statement.

Lemma 8. *If E is adapted to the Navier-Stokes equations, then the homogeneous Besov space $\dot{B}_1^{2,1}(\mathbb{R}^3)$ is contained in E, this embedding is continuous and $\dot{B}_1^{2,1}(\mathbb{R}^3)$ is also adapted to the Navier-Stokes equations.*

The simplest description of $\dot{B}_1^{2,1}$ is the following one. If the orthonormal wavelet basis of section 6 is being used, then $f \in \dot{B}_1^{2,1}$ if and only if

$$f(x) = \sum_j \sum_k \alpha(j,k)\, 2^j \psi(2^j x - k)$$

with

$$\sum_j \sum_k |\alpha(j,k)| < \infty .$$

For the sake of simplicity, the sum over $\psi \in A$ (with $\sharp A = 7$) is ommited.

Before ending this section, let us state and prove an important lemma.

Lemma 9. *If E is adapted to the Navier-Stokes equations, for every φ in the Schwartz class $\mathcal{S}(\mathbb{R}^3)$, there exists a constant $C(\varphi)$ such that*

$$\|f * \varphi_t\|_\infty \le C(\varphi)\, t^{-1} \|f\|_E , \qquad t > 0 , \ f \in E , \tag{8.23}$$

with $\varphi_t(x) = t^{-3}\varphi(x/t)$.

Indeed $(f * \varphi_t)(x) = \int f(x+ty)\,\varphi(-y)\,dy$. We define a new function $f_{t,x}(y)$ by $f_{t,x}(y) = t\,f(ty+x)$ and observe that $\|f_{t,x}\|_E = \|f\|_E$. Moreover each function φ belonging to the Schwartz class yields a continuous linear form on E since E is continuously embedded in $\mathcal{S}'(\mathbb{R}^3)$. Therefore

$$| < f_{t,x}, \varphi > | \le C(\varphi)\, \|f_{t,x}\|_E = C(\varphi)\, \|f\|_E . \tag{8.24}$$

Now the scalar product $< f_{t,x}, \varphi >$ is precisely the integral we want to estimate, up to some trivial modifications.

The next section is a first step towards defining the concept of a mild solution of the Navier-Stokes equations. We begin with a simplified model which is the heat equation.

9 Mild solutions to the heat equation

We consider the heat equation with a forcing term $g = g(x,t)$. It reads

$$\begin{cases} \dfrac{\partial f}{\partial t} = \Delta f + g \\ f(x,0) = f_0(x) . \end{cases} \tag{9.1}$$

We are looking for a solution f of (9.1) as being continuous in the time variable with values in a functional Banach space E. For the time being, E is assumed to be separable and $f \in C([0,\infty); E)$ has the usual meaning.

Let us now consider a second functional Banach space E' with the following two properties

(9.2) $\Delta : E \to E'$ is continuous

(9.3) if φ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ then the convolution operator $f \to f * \varphi$ continuously maps E' into E

Definition 13. *With the preceding notations, a mild solution f to the heat equation* (9.1) *is a solution belonging to* $C([0,\infty), E)$ *when g belongs to* $C([0,\infty), E')$.

Let us denote by $S(t) = \exp(t\Delta)$, $t \geq 0$, the heat semi-group. We then have

Theorem 9.1. *If such a mild solution exists, it is uniquely given by*

$$f(x,t) = S(t)f_0 + \int_0^t S(t-s)\,g(s)\,ds\,. \tag{9.4}$$

This statement is trivial if $g = 0$ and $S(t)f_0$ obviously satisfies the required continuity with respect to the time variable if $f_0 \in E$. We can turn to the case where $f_0 = 0$. We then observe that the integral $\int_0^t S(t-s)\,g(s)\,ds$ belongs to $C([0,\infty), E')$. This integral does not belong to $C([0,\infty), E)$ in general and a counter-example is given at the end of this section. In other words, mild solutions do not exist in general when g belongs to $C([0,\infty), E')$.

It is interesting to have a sharper look at the integrand. A partial answer is given by the following lemma

Lemma 10. *If $\Delta : E \to E'$ is an isomorphism between the Banach spaces E and E', there exists a constant C such that*

$$\|S(t)f\|_E \leq Ct^{-1}\|f\|_{E'}\,. \tag{9.5}$$

For proving (9.5), one writes $f = \Delta u$, $u \in E$ with $\|u\|_E \leq C_0\|f\|_E$. Then $t\Delta S(t) = Q_t$ is a convolution operator with $\psi_t(x) = t^{-3}\psi(x/t)$ where ψ is the celebrated D. Marr's wavelet. Therefore $Q_t : E \to E$ is continuous and the operator norm does not depend on t.

Once (9.103) is obtained, we have $\|S(t-s)\, g(s)\|_E \le C'(t-s)^{-1}$ which diverges when s approaches t.

We now return to the proof of (9.102). We denote by φ a function in the Schwartz class whose Fourier transform $\hat{\varphi}(\xi)$ satisfies

$$\hat{\varphi}(\xi) = 1 \quad \text{on } |\xi| \le 1 \quad , \quad \hat{\varphi}(\xi) = 0 \quad \text{on } |\xi| \ge 2\,. \tag{9.6}$$

For $\varepsilon > 0$, we write $\varphi_\varepsilon(x) = \varepsilon^{-3}\,\varphi(x/\varepsilon)$ and denote by G_ε the convolution operator with φ_ε. We then obviously have

$$G_\varepsilon\, G_{2\varepsilon} \;=\; G_{2\varepsilon}\, G_\varepsilon \;=\; G_{2\varepsilon}\,. \tag{9.7}$$

Now $\frac{\partial f}{\partial t} = \Delta f + g$ implies

$$\frac{\partial f_\varepsilon}{\partial t} \;=\; A_\varepsilon f_\varepsilon + g_\varepsilon \tag{9.8}$$

where $f_\varepsilon = G_{2\varepsilon}(f)$, $g_\varepsilon = G_{2\varepsilon}(g)$ and $A_\varepsilon = \Delta G_\varepsilon$. This operator A_ε acts boundedly on E and (9.106) together with $f_\varepsilon(x,0) = 0$ obviously implies

$$f_\varepsilon(t) \;=\; \int_0^t \exp\big((t-s)A_\varepsilon\big)\, g_\varepsilon(s)\, ds\,. \tag{9.9}$$

The Fourier transform of the integrand reads

$$\exp\big(-(t-s)\,|\xi|^2\,\hat{\varphi}(\varepsilon\xi)\big)\,\hat{\varphi}(2\varepsilon\xi)\,\hat{g}(s,\xi)\,. \tag{9.10}$$

Since $\hat{\varphi}(\xi) = 1$ on the support of $\hat{\varphi}(2\xi)$, this can be simplified into

$$\exp\big(-(t-s)\,|\xi|^2\big)\,\hat{\varphi}(2\varepsilon\xi)\,\hat{g}(s,\xi) \tag{9.11}$$

and (9.107) can be rewritten as

$$f_\varepsilon(t) \;=\; \int_0^t s(t-s)\, g_\varepsilon(s)\, ds\,. \tag{9.12}$$

Finally $g_\varepsilon \to g$ in E' and $\|g_\varepsilon\|_{E'} \le C\|g\|_{E'}$. Lebesgue dominated convergence theorem implies the convergence in E' of f_ε to $\int_0^t S(t-s)\, g(s)\, ds$. But f_ε converges to f in E. It implies

$$f(t) \;=\; \int_0^t S(t-s)\, g(s)\, ds\,. \tag{9.13}$$

Indeed if a sequence u_j belongs to $E \cap E'$ and converges to u in E and to v in E', then for any function φ in the Schwartz class, $u_j * \varphi$ tends to $u * \varphi$ in E and also to $v * \varphi$ in E. It yields $u * \varphi = v * \varphi$ for every φ and $u = v$.

The unpleasant fact that mild solutions do not always exist in the framework of theorem is one of the main difficulties we will have to face for solving Navier-Stokes equations.

In the simplest case when $E' = L^2(\mathbb{R}^3)$ and E is the Sobolev space $H^2(\mathbb{R}^3)$, it is not true that $g \in C([0,\infty); L^2(\mathbb{R}^3))$ implies $f \in C([0,\infty); H^2(\mathbb{R}^3))$ as the most trivial counter-examples show.

There is however a simple and interesting situation where everything smoothly works.

Let us assume that $E = E_\alpha$, $\alpha > -3$, is defined by the following pointwise condition on the Fourier transform $\hat{f}(\xi)$ of f

$$|\hat{f}(\xi)| \le C|\xi|^\alpha . \tag{9.14}$$

We then define $E' = E_{\alpha+2}$ and it is now a trivial check that $g \in C([0,\infty); E')$ implies $f \in C([0,\infty); E)$.

10 Mild solutions to Navier-Stokes equations : the L^3-case

Let us start with the simplest example which is provided by the adapted space $E = L^3(\mathbb{R}^3)$. We will then move to more involved cases which will culminate with the Morrey-Campanato space $M^2(\mathbb{R}^3)$.

Definition 14. *A mild solution of Navier-Stokes equations is a velocity field $u(x,t) \in C([0,\infty), L^3(\mathbb{R}^3))$ together with a pressure $p(x,t) \in C([0,\infty), L^{3/2}(\mathbb{R}^3))$ such that*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \partial_j(u_j u) - \nabla p \\ \operatorname{div} u = 0 \\ u(x,0) = u_0(x) . \end{cases} \tag{10.1}$$

This equation needs to be given a meaning. First of all the Laplace operator Δ is an isomorphism from $E = L^3(\mathbb{R}^3)$ onto $E' = \{\Delta f\,;\, f \in L^3(\mathbb{R}^3)\}$. Such a statement is a tautology. But it is more interesting to observe that $\partial_j(u_j u) \in C([0,\infty); E')$ and so does ∇p.

Indeed $u \in C([0,\infty), L^3)$ implies $u_j u \in C([0,\infty); L^{3/2})$ and each function f in $L^{3/2}(\mathbb{R}^3)$ can uniquely be written as Λg where $\Lambda = (-\Delta)^{1/2}$ is the Calderón operator and $g \in L^3(\mathbb{R}^3)$. Applying this remark to $u_j u$ and p yields the required remark. Therefore $\frac{\partial u}{\partial t}$ belongs to $C([0,\infty); E')$.

Everything is now prepared to apply theorem 9.1. We write $w(x,t) = \partial_j(u_j u) + \nabla p$ and obtain

$$u(x,t) = S(t)u_0 + \int_0^t S(t-s)\, w(\cdot,s)\, ds\,. \tag{10.2}$$

This equation will be solved using Picard's fixed point theorem. However the main difficulty is the one we already mentioned in the linear case. Indeed the bilinear mapping which shows up in $\int_0^t S(t-s)\, w(\cdot,s)\, ds$ is not bounded for the natural norm we are using which is $\sup\{\|u(\cdot,t)\|_3\, ;\, 0 \le t < \infty\}$.

This first example can be generalized. We begin with a definition

Definition 15. *Let E be a functional Banach space. We say that E is fully adapted to the Navier-Stokes equations if the following two properties are satisfied*

(10.3) *the Riesz transformations R_1, R_2 and R_3 act boundedly on E*

(10.4) *there exists a constant C such that for any two functions f and g belonging to E there exists a unique function h in E such that*

$$\Lambda h = fg \tag{10.4.a}$$

and

$$\|h\|_E \le C\|f\|_E\, \|g\|_E\,. \tag{10.4.b}$$

The bilinear operator defined by (10.4.a) is in some sense induced by the fundamental bilinear operator that governs the Navier-Stokes equations. This bilinear operator will be presented in Definition 18. A simpler scalar version is defined as

$$B(f,g) = \int_0^t \Lambda S(t-s)\, f(s)\, g(s)\, ds \tag{10.5}$$

where $f(s) = f(\cdot,s)$, $g(s) = g(\cdot,s)$ and $\Lambda = \sqrt{-\Delta}$.

We want to know whether B boundedly maps $C([0,\infty); E) \times C([0,\infty); E)$ into $C([0,\infty); E)$ when the norm in $C([0,\infty); E)$ is defined as

$$\sup_{t\ge 0} \|f(\cdot,t)\|_E\,. \tag{10.6}$$

We then have

Lemma 11. *If B boundedly maps $C([0,\infty);E) \times C([0,\infty);E)$ into $C([0,\infty);E)$, then* (10.4.a) *and* (10.4.b) *are satisfied.*

This is trivially checked. Indeed let $f(x,s) = f(x)$, $g(x,s) = g(x)$, $s \geq 0$, and let us compute $B(f,g)$. We obtain $B(f,g) = [I - S(t)]\,\Lambda^{-1}(fg)$ and $\|B(f,g)\|_E \leq C\|f\|_E\,\|g\|_E$ implies (10.4).

The converse statement is not true and the Lebesgue space $L^3(\mathbb{R}^3)$ is a counter-example. However the corresponding Lorentz space $L^{3,\infty}(\mathbb{R}^3)$ is an interesting example where B is bounded (theorem 18.2 of section 18).

The Besov spaces $\dot{B}_q^{-(1-3/q),\infty}(\mathbb{R}^3)$ are fully adapted to the Navier-Stokes equations as it will be proved in section 24. An other example is given by

$$E = \left\{f \in \mathcal{S}'(\mathbb{R}^3)\,;\, |\hat{f}(\xi)| \leq C|\xi|^{-2}\right\}. \tag{10.7}$$

If f and g both belong to E, then $(f,g)\hat{}(\xi) = \hat{f} * \hat{g}$ satisfies $|(fg)\hat{}(\xi)| \leq C|\xi|^{-1}$ as a trivial computation shows. Then h is defined by $\hat{h}(\xi) = |\xi|^{-1}(fg)\hat{}(\xi)$ and we then have $|\hat{h}(\xi)| \leq C|\xi|^{-1}$.

An important observation is the following

Theorem 10.1. *The Morrey-Campanato space M^2 is not fully adapted to the Navier-Stokes equations.*

We begin with an obvious remark :

Lemma 12. *A function f of the two variables x_2 and x_3 belongs to $M^2(\mathbb{R}^3)$ if and only if $f(x_2,x_3) \in L^2(\mathbb{R}^2)$.*

Indeed one writes

$$\int_{|x|\leq R} |f(x_2,x_3)|^2\,dx \;\leq\; CR \tag{10.8}$$

and it suffices to let R tend to infinity to obtain the required equivalence.

If both f and g belong to M^2, then their pointwise product $f(x)\,g(x) = u(x)$ satisfies

$$\int_{B(x_0,R)} |u(x)|\,dx \;\leq\; C\,R \tag{10.9}$$

where C does not depend on x_0 or R.

If both f and g only depend on x_2 and x_3, so does u and $u(x_2, x_3)$ belongs to $L^1(\mathbb{R}^2)$.

If $u = \Lambda h$ in $\mathcal{S}'(\mathbb{R}^3)$, the same relation holds in $\mathcal{S}'(\mathbb{R}^2)$.

Finally u is an arbitrary function in $L^1(\mathbb{R}^2)$ and if $\iint u(x_2, x_3)\, dx_2\, dx_3 \neq 0$, then h cannot belong to $L^2(\mathbb{R}^2)$ since $\iint (\xi_2^2 + \xi_3^2)^{-1} |\hat{u}(\xi_2, \xi_3)|^2\, d\xi_2\, d\xi_3 = +\infty$.

We now return to the definition of mild solutions $u \in C([0,\infty); E)$ to the Navier-Stokes equations when E is a functional Banach space which is fully adapted to the Navier-Stokes equations.

This definition relies on the Leray projector we now want to define.

11 The Leray projection

The Leray projector is aimed to get rid of the pressure in the Navier-Stokes equations.

Keeping the same notations as above, let us denote by H the Hilbert space $(L^2(\mathbb{R}^3))^3$ and by $H_0 \subset H$ the closed linear subspace defined by $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$. This condition will be rewritten as

$$R_1 u_1 + R_2 u_2 + R_3 u_3 = 0 \tag{11.1}$$

where R_1, R_2 and R_3 are the Riesz transformations.

The Leray projector $\mathbb{P}$ is defined as the orthogonal projection from H onto H_0.

The computation of $\mathbb{P}$ is straightforward and one obtains

$$\mathbb{P}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 - R_1(\sigma) \\ u_2 - R_2(\sigma) \\ u_3 - R_3(\sigma) \end{pmatrix} \tag{11.2}$$

where

$$\sigma = R_1 u_1 + R_2 u_2 + R_3 u_3\,. \tag{11.3}$$

The Leray projector $\mathbb{P}$ is acting on $(L^2(\mathbb{R}^3))^3$ but this action can be extended to every functional Banach space on which the Riesz transformations act boundedly.

12 Mild solutions of Navier-Stokes equations : the general case

We now consider a Banach space E which is fully adapted to Navier-Stokes equations and define the two functional spaces E' and Π by the following conditions :

$$\Delta : E \to E' \quad \text{is an isometrical isomorphism} \tag{12.1}$$

$$\Lambda : E \to \Pi \quad \text{is an isometrical isomorphism.} \tag{12.2}$$

It will be proved (lemma 13) that the kernel of Δ is reduced to $\{0\}$ and that the same property holds for Λ.

Definition 16. *Let the functional Banach space E be fully adapted to the Navier-Stokes equations. Let $E' = \Delta E$ and $\Pi = \Lambda E$ as above.*

A mild solution solution $(u(x,t), p(x,t))$ to the Navier-Stokes equations is defined by the following conditions

$$u(x,t) \in C([0,\infty); E) \tag{12.3}$$

$$\frac{\partial u}{\partial t}(x,t) \in C([0,\infty); E') \tag{12.4}$$

$$p(x,t) \in C([0,\infty); \Pi) \tag{12.5}$$

$$\frac{\partial u}{\partial t} = \Delta u - \partial_j(u_j u) - \nabla p \tag{12.6}$$

$$\operatorname{div} u = 0 \tag{12.7}$$

$$u(x,0) = u_0(x)\,. \tag{12.8}$$

Let us make a few remarks about this definition.

If E is a separable functional Banach space, then $C([0,\infty); E)$ has the usual meaning. Otherwise E is assumed to be the dual space F^* of a separable Banach space F and $u \in C([0,\infty); E)$ always means

$$u \in C((0,\infty); E) \quad \text{when } E \text{ is given its norm topology} \tag{12.9}$$

$$u \in C([0,\infty); E) \quad \text{when } E \text{ is given its weak-star topology.} \tag{12.10}$$

Concerning the non-linearity in the Navier-Stokes equations, we have $u_j u_k = \sum_1^3 \partial_\ell(v_{j,k,\ell})$ by (10.4). Then $v_{j,k,\ell}$ belong to E and $\sum_1^3 \partial_j(u_j u_k) = \sum_1^3 \sum_1^3 \partial_j \partial_\ell(v_{j,k,\ell})$ belongs to E'. Similarly Δu and ∇p belong to E'.

These remarks lead to the following

Lemma 13. *If the functional Banach space E is fully adapted to the Navier-Stokes equations, then each of the terms which appear in the left-hand side or in the right-hand side of* (12.130) *belongs to* $C([0,\infty);E')$.

We now reach our main goal

Theorem 12.1. *If $\big(u(x,t),p(x,t)\big)$ is a mild solution of the Navier-Stokes equations, we then have*

$$\frac{\partial u}{\partial t} = \Delta u - \mathbb{P}\big(\partial_j(u_j u)\big) \tag{12.11}$$

while the pressure $p(x,t)$ is given by

$$p = R_j\, R_k(u_j u_k)\,. \tag{12.12}$$

If conversely $u \in C([0,\infty);E)$, $\frac{\partial u}{\partial t} \in C([0,\infty);E')$ and $\operatorname{div} u_0(x) = 0$, then (12.135) *and* (12.136) *imply that $\big(u(x,t),p(x,t)\big)$ is a mild solution to the Navier-Stokes equations.*

This fundamental theorem is an easy consequence of the following remark.

Lemma 14. *If v belongs to F and satisfies $\mathbb{P}(v) = 0$, then $v = \nabla p$ where p is a scalar function and p belongs to Π. Moreover p is uniquely defined by this equation.*

For proving this remark, we return to the definition of $\mathbb{P}$. Then $\mathbb{P}(v) = 0$ reads $v_1 = R_1(\sigma)$, $v_2 = R_2(\sigma)$, $v_3 = R_3(\sigma)$ where $\sigma = R_1 v_1 + R_2 v_2 + R_3 v_3$. We have $\sigma \in E'$ since the Riesz transformations act boundedly on E'. Since $\Lambda : \Pi \to E'$ is onto, σ can be written $\sigma = \Lambda p$ and this is the required conclusion.

For proving that $\Lambda : \Pi \to E'$ is $1-1$, we assume $\Lambda p = 0$. Since p belongs to the range of Λ, we write $p = \Lambda f$ and obtain $\Delta f = 0$. Therefore f is a harmonic function. Since E is contained in $\mathcal{S}'(\mathbb{R}^3)$, f is a harmonic polynomial. Let φ be a function in the Schwartz class and $\varphi_t(x) = t^{-3}\,\varphi(x/t)$.

Then $\|f * \varphi_t\|_\infty \le Ct^{-1}$ (lemma 10). But $(f*\varphi_t)(x) = \int f(x-yt)\,\varphi(y)\,dy$ is a polynomial in t. Therefore this polynomial identically vanishes and $f = 0$.

We now return to theorem 12.1. If (u,p) is a mild solution, we have $\mathbb{P}(u) = u$, $\mathbb{P}(\Delta u) = \Delta u$, $\mathbb{P}\big(\frac{\partial u}{\partial t}\big) = \frac{\partial u}{\partial t}$ and finally $\mathbb{P}(\nabla p) = 0$. These identities yield (12.135). Moreover applying the divergence operator to (12.130) identities yield (12.11). Moreover applying the divergence operator to (12.6) we obtain $\Delta p + \partial_j \partial_k (u_j u_k) = 0$. But the functional space Π cannot contain polynomials.

If p is such a polynomial, we have $p = \Lambda f$ where $f \in E$. Returning to the Fourier transforms, we obtain $|\xi| \hat{f}(\xi) = 0$ on $\mathbb{R}^3 \setminus \{0\}$. Therefore f is a polynomial. Since $f \in E$, we have $f = 0$. We then obtain $p = R_j R_k (u_j u_k)$ as announced.

Let us once more stress that $u_j u_k = \sum_1^3 \partial_\ell (v_{j,k,\ell})$ where $v_{j,k,\ell} \in E$ which yields $p \in \Pi$.

We now prove the converse statement in theorem 12.1. We begin with a simple observation.

Lemma 15. *If* $u(x,t) \in C([0,\infty); E)$, $\operatorname{div} u_0 = 0$ *and* (12.11) *is satisfied, then* $\operatorname{div} u(\cdot, t) = 0$ *for* $t \geq 0$.

For proving lemma 15, we apply the divergence operator to (12.11) and obtain $\frac{\partial w}{\partial t} = \Delta w$ where $w = \operatorname{div} u$. We have $w \in C([0,\infty), \Pi)$ and $w(\cdot, 0) = \operatorname{div} u_0 = 0$.

Theorem 9.1 applies and yields $w = 0$. Next we consider the difference

$$\varepsilon(x,t) = \frac{\partial u}{\partial t} - \Delta u + \partial_j (u_j u) . \tag{12.13}$$

Then (12.11) and $\operatorname{div} u = 0$ imply $\mathbb{P}[\varepsilon(\cdot, t)] = 0$ in $C([0,\infty); E')$. This yields $\varepsilon(\cdot, t) = -\nabla p$ and $p \in C([0,\infty), \Pi)$. Our goal is now achieved and $\big(u(x,t), p(x,t)\big)$ is the mild solution we looking for.

This proof paves the way to our second definition of a mild solution. In this definition, the pressure has disappeared.

Definition 17. *Let E be a Banach space which is fully adapted to the Navier-Stokes equations. Let us assume as we did above that $\Delta : E \to E'$ is an isometrical isomorphism. Then a mild solution to the Navier-Stokes equations is defined by the following three conditions*

$$u \in C([0,\infty); E) \tag{12.14}$$

$$\frac{\partial u}{\partial t} \in C([0,\infty); E') \tag{12.15}$$

$$\frac{\partial u}{\partial t} = \Delta u - \mathbb{P}\big(\partial_j (u_j u)\big) . \tag{12.16}$$

13 More about mild solutions

A fundamental observation concerning definition 17 is the following : if $u(x,t)$ is a mild solution to the Navier-Stokes equations, the same is true for $u(x, t+\tau)$, $\tau \geq 0$.

This remark will be used in section 20.

We now want to drop the condition (12.139) in the definition of a mild solution to the Navier-Stokes equations.

Following T. Kato, we apply theorem 9.1 to (12.140) and reach the third definition of a mild solution

Definition 18. *With the same notations as above, a mild solution of the Navier-Stokes equations is defined by the following two conditions*

$$(13.1) \qquad u(\cdot,t) \in C([0,\infty);E)$$

$$(13.2) \qquad u(\cdot,t) = S(t)u_0 - \mathbb{P}\int_0^t S(t-s)\,[\partial_j(u_j u)(s)]\,ds\,.$$

As was already mentioned when theorem 9.1 was proved, the status of this integral is far from being obvious. Indeed $\partial_j(u_j u)$ belongs to $C([0,\infty);E')$ and our integral is an E' valued Bochner integral. However it is not true that this integral should belong to E. If $u(\cdot,t)$ is a mild solution to the Navier-Stokes equations, it is the case and our integral belongs to E. What we want to stress is the following fact. There is no hope to solve (13.142) by a genuine application of Picard's fixed point theorem.

This issue will be fixed by a remarkable approach dued to T. Kato and his collaborators. This approach will be presented in section 19. For the time being, we will impose a digression to the reader. This digression is aimed to study the paraproduct algorithm. This algorithm is needed to proving that $\dot{B}_q^{-(1-3/q),\infty}$ is fully adapted to the Navier-Stokes equations.

As was already stated, the Morrey-Campanato space M^2 is not fully adapted to the Navier-Stokes equations. Indeed (10.116) is not satisfied while the Riesz transformations are bounded on M^2.

For defining the concept of a mild solution to the Navier-Stokes equations when $E = M^2$, we return to theorem 9.1 and define a suitable functional Banach space E'. This Banach space should contain all functions or distributions of the form Δf where $f \in M^2$. Then the mapping $\Delta : E \to E'$ will be continuous.

This Banach space should also contains all functions of the form $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3$ when u_1, u_2 and u_3 satisfy the estimate $\sup_{x_0} \sup_{0<R<\infty} \frac{1}{R} \cdot \int_{|x-x_0|\le R} |f(x)|\, dx < \infty$. Finally E' should be invariant under the Riesz transformation.

The explicit construction of E' is not trivial but will not be needed in what follows.

Right now a digression is needed. Apart from two trivial examples of fully adapted Banach spaces, all the interesting examples are based upon the paraproduct algorithm. This algorithm is also playing a crucial role in the proof of uniqueness when $E = L^3(\mathbb{R}^3)$. The next section contains a review on the paraproduct algorithms.

14 The paraproduct algorithm

There are two versions of the paraproduct algorithm. The first one is implicit in [20] and was given its full strength and flexibility by J.M. Bony. The second one is a wavelet based algorithm and is related to P. Federbush's program.

The Bony's paraproduct is using the Littlewood-Paley analysis.

For defining a Littlewood-Paley analysis we fix a function $\varphi(x)$ in the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ such that $\hat{\varphi}(\xi) = 1$ on the ball $|\xi| \le 1/2$ while $\hat{\varphi}(\xi) = 0$ outside the ball $|\xi| < 1$.

We also may assume $0 \le \hat{\varphi}(\xi) \le 1$ everywhere and that φ (or $\hat{\varphi}$) is a radial function. But these two properties are not needed in what follows.

For $j \in \mathbf{Z}$, we write $\varphi_j(x) = 2^{3j}\, \varphi(2^j x)$ and we denote by S_j the convolution operator defined by $S_j(f) = f * \varphi_j$.

If E is a functional Banach space, we have $\lim_{j\to+\infty} \|S_j(f) - f\|_E = 0$ when $\mathcal{S}(\mathbb{R}^3)$ is dense in E. If $E = F^*$ where F is such a functional Banach space, we have $S_j(f) \rightharpoonup f$ $(j \to +\infty)$ where here the limit exists for the $\sigma(E, F)$ topology.

A functional Banach space E is adapted to the Littlewood-Paley decomposition if

$$\lim_{j\to-\infty} \|S_j(f)\|_E = 0 \tag{14.1}$$

for every $f \in E$. This condition concerns the case where $\mathcal{S}(\mathbb{R}^3)$ is dense in E. If $E = F^*$, we instead impose $S_j(f) \rightharpoonup 0$ in $\sigma(E, F)$ as $j \to -\infty$.

We next write $\Delta_j = S_{j+1} - S_j$. If E is adapted to the Littlewood-Paley decomposition, we have

$$f = \sum_{-\infty}^{\infty} \Delta_j(f) \tag{14.2}$$

where this series converges to f in E when $\mathcal{S}(\mathbb{R}^3)$ is dense in E and converges to f in $\sigma(E, F)$ when $E = F^*$.

Among adapted Banach spaces, one lists the Lebesgue spaces $L^p(\mathbb{R}^3)$ when $1 < p < \infty$. However L^1 is not adapted. Indeed if $f \in L^1$ and $\int f\,dx = 1$, then the right-hand side of (14.144) cannot converge to f for this L^1-norm since $\int \Delta_j(f)\,dx = 0$.

Similarly L^∞ is not adapted to the Littlewood-Paley decomposition. An obvious counter-example is given by $f = 1$ where $\Delta_j(f) = 0$ for all j's.

We have $\Delta_j(f) = f * \psi_j$ where $\psi_j(x) = 2^{3j}\,\psi(2^j x)$ and $\psi(x) = 8\varphi(2x) - \varphi(x)$. The Fourier transform $\hat{\psi}(\xi)$ of ψ is compactly supported and vanishes on $|\xi| \le 1/2$ and outside the ball $|\xi| \le 2$. Therefore ψ is similar to a mother wavelet and a Littlewood-Paley analysis is similar to a wavelet analysis [55].

The paraproduct algorithm was first designed for studying some involved bi-linear operators $B(f, g)$ which appear in Calderón's program [56]. Later on J.M. Bony observed that this algorithm was also relevant for studying the ordinary product $f(x)\,g(x)$ whenever this product cannot be given an ordinary meaning. This situation occurs when f and g are both badly behaved (e.g. $f = g = \delta_0$ where δ_0 is the Dirac mass at 0). In this situation the paraproduct algorithm yields an "additive renormalization" or "regularization" of a divergent expansion.

Nowadays para-product algorithms are expected to play a role in numerical analysis [54].

It is now time to unveil the paraproduct algorithm which reads

$$f(x)\,g(x) = \sum_{-\infty}^{\infty} A_j(x) + \sum_{-\infty}^{\infty} B_j(x) + \sum_{-\infty}^{\infty} C_j(x) \tag{14.3}$$

where

(14.4)
$$A_j(x) = S_{j-2}(f)\,\Delta_j(g) \quad , \quad B_j(x) = S_{j-2}(g)\,\Delta_j(f)$$

while

$$C_j(x) = \big(\Delta_{j-2}(f) + \cdots + \Delta_{j+2}(f)\big)\,\Delta_j(g)\,. \tag{14.5}$$

The relevance of this splitting of the product $f(x)\,g(x)$ will be clarified by the following remark.

Lemma 16. *Let A and B be two compact sets in $\mathbb{R}^3$ and $A+B$ denote the compact set of all sums $\xi+\eta$, $\xi \in A$, $\eta \in B$.*

If $f(x)$ and $g(x)$ are two functions such that the Fourier transform $\hat{f}$ of f is supported by A and the Fourier transform $\hat{g}$ of g is supported by B, then the Fourier transform of the pointwise product $f(x)\,g(x)$ is supported by $A+B$.

If this obvious remark is applied to $S_{j-2}(f)\,\Delta_j(g) = A_j$, we find that $\hat{A}_j(\xi)$ is supported by the dyadic annulus $\frac{1}{4}2^j \le |\xi| \le \frac{9}{4}2^j$. Therefore the series $\sum_{-\infty}^{\infty} A_j(x)$ is behaving like a Littlewood-Paley series. The same applies to $\sum_{-\infty}^{\infty} B_j(x)$. If, for instance, both f and g belong to $L^1(\mathbb{R}^3)$, then $\sum_{-\infty}^{\infty} A_j(x)$ and $\sum_{-\infty}^{\infty} B_j(x)$ are convergent in the Besov space $\dot{B}_1^{-3,\infty}$. We will return to this point.

The devil is hidden in the third series $\sum_{-\infty}^{\infty} C_j(x)$. Indeed $C_j(x)$ contains the square terms $\Delta_j(f)\,\Delta_j(g)$ and a few other terms. If, for example, $g = \overline{f}$ and if φ is real valued, then $\Delta_j(f)\,\Delta_j(g) = |\Delta_j(f)|^2$ and there are no cancellations. If both f and g belong to $L^1(\mathbb{R}^3)$ the series $\sum_{-\infty}^{\infty} C_j(x)$ diverges. Indeed one cannot multiply two L^1 functions. Using the paraproduct algorithm, we isolate the few terms which are responsible for this issue.

Using a metaphore, the paraproduct algorithm can be compared to the celebrated Richardson's cascade. Indeed high frequencies cascade to low frequencies in each product $\Delta_j(f)\,\Delta_j(g)$.

There is a second version of the paraproduct algorithm which is based on a wavelet series expansion.

Let ψ_λ, $\lambda \in \Lambda$, be the orthonormal wavelet basis which is constructed in section 6. We have $\Lambda = \mathbf{Z} \times \mathbf{Z}^3 \times E$ where E is the finite set of $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \{0,1\}^3$ with $\varepsilon \ne (0,0,0)$. if $\lambda(j,k,\varepsilon)$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}^3$, $\varepsilon \in E$, we then have $\psi_\lambda(x) = 2^{3j/2}\,\psi_\varepsilon(2^j x - k)$. Finally $\psi_\varepsilon(x)$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^3)$ and $\hat{\psi}_\varepsilon(\xi) = 0$ if either $|\xi| \le 2\pi/3$ or $|\xi| \ge 8\pi/3$. Here $|\xi| = \sup\big(|\xi_1|, |\xi_2|, |\xi_3|\big)$.

We consider two functions f and g in $L^2(\mathbb{R}^3)$. Then $f(x) = \sum \alpha_\lambda \psi_\lambda(x)$ and $g(x) = \sum \beta_\lambda \psi_\lambda(x)$. This leads to

$$\begin{aligned}(14.6)\qquad f(x)\,g(x) &= \sum_{\lambda\in\Lambda} \alpha_\lambda\beta_\lambda\psi_\lambda^2(x) + \sum\sum_{\lambda\neq\lambda'} \alpha_\lambda\beta_{\lambda'}\psi_\lambda(x)\psi_{\lambda'}(x)\\ &= U(x) + V(x)\,.\end{aligned}$$

If for simplifying the discussion, ψ_E, $\varepsilon \in E$, are real valued, the square terms $\psi_\lambda^2(x)$ satisfy $\int \psi_\lambda^2(x)\,dx = 1$. In contrast the rectangle terms satisfy $\int \psi_\lambda(x)\,\psi_{\lambda'}(x)\,dx = 0$. Therefore the series $U(x)$ is likely to diverge while the series $V(x)$ is like to converge. This convergence is provided by the cancellations which are given by the orthogonality.

In order to extend the definition of the product $f(x)\,g(x)$ to some functional settings where this product has no meaning, it suffices to eliminate the divergent series $\sum \alpha_\lambda\,\beta_\lambda\,\psi_\lambda^2(x)$.

However there are many examples where the cancellation provided by $\int \psi_\lambda(x)\,\psi_{\lambda'}(x)\,dx = 0$ is not sufficient. In order to prove the convergence of $V(x)$ we often need a stronger cancellation such as $\int x^\alpha\,\psi_\lambda(x)\,\psi_{\lambda'}(x)\,dx = 0$ for $|\alpha| \le N$ where the integer N is related to the functional setting. But orthonormal wavelet bases do not have this strong orthogonality. That is why the wavelet based paraproduct was given up [54].

15 Examples of Banach spaces which are fully adapted to the Navier-Stokes equations

We first list two trivial examples for which the paraproduct algorithm is not needed.

The first one is the Lebesgue space $L^3(\mathbb{R}^3)$. Its role was discovered by T. Kato [43] and we will return to the uniqueness issue in Kato's theorem in section 20.

For proving that $L^3(\mathbb{R}^3)$ is fully adapted we concentrate on (10.116). Indeed singular integral operators are bounded on $L^3(\mathbb{R}^3)$.

If both $f(x)$ and $g(x)$ belong to $L^3(\mathbb{R}^3)$, then $f(x)\,g(x)$ belongs to $L^{3/2}(\mathbb{R}^3)$ and $h = \Lambda^{-1}(fg)$ is back in $L^3(\mathbb{R}^3)$ as Sobolev embedding theorem tells us.

The same approach is valid for the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$. Instead of writing the proof, let us concentrate on an interesting example. The function $f(x) = |x|^{-1}$ obviously belongs to $L^{3,\infty}(\mathbb{R}^3)$ and the special case $f(x) = g(x) = |x|^{-1}$ yields $h(x) = c|x|^{-1}$ where c is a constant.

This example will introduce the third example where E is defined by a pointwise estimate on the Fourier transform $\hat{f}(\xi)$ of f. This estimate reads

$$|\hat{f}(\xi)| \le C|\xi|^{-2}. \tag{15.1}$$

If both f and g belong to E, then

$$|(fg)\hat{}(\xi)| \le |\hat{f}| * |\hat{g}| \le C|\xi|^{-1}$$

which obviously implies $fg = \Lambda(h)$ with $h \in E$.

The next examples we want to treat are the Besov spaces $B_q = \dot{B}_q^{\alpha,\infty}$, $\alpha = 3/q - 1$. The paraproduct algorithm will be used to show that B_q is fully adapted to the Navier-Stokes equations when $1 \le q < 3$.

The discussion will be clarified if the general definition of homogeneous Besov spaces is presented.

Definition 19. *If $1 \le p \le \infty$, $1 \le q \le \infty$ and if s is a real number, the homogeneous Besov space $\dot{B}_p^{s,q}(\mathbb{R}^n)$ is defined by*

$$\|\Delta_j(f)\|_p \le \varepsilon_j 2^{-js} \qquad j \in \mathbf{Z} \tag{15.2}$$

where

$$\left(\sum_{-\infty}^{\infty} \varepsilon_j^q\right)^{1/q} < \infty. \tag{15.3}$$

This definition should be completed with

$$\|S_j(f)\|_\infty \le \overline{\varepsilon}_j 2^{-j(s-n/p)} \tag{15.4}$$

if $s < n/p$. We also impose

$$\left(\sum_{-\infty}^{\infty} \overline{\varepsilon}_j^q\right)^{1/q} < \infty. \tag{15.5}$$

Let us explain why (15.4) is a natural *a priori* assumption on a function f in $\dot{B}_p^{s,q}$.

For this purpose, the celebrated Bernstein's inequalities will be used. They are given in a slightly more general context.

Lemma 17. *There exists a constant C such that, for $1 \le p \le q \le \infty$, one has*

$$\|f\|_q \le C\, R^{n(1/p-1/q)}\, \|f\|_p \tag{15.6}$$

whenever f belongs to $L^p(\mathbb{R})$ and its Fourier transform $\hat{f}(\xi)$ is supported by $|\xi| \le R$.

Returning to (15.152) we have

$$\|\Delta_j(f)\|_\infty \le C\,\varepsilon_j\, 2^{-j(s-n/p)}\,. \tag{15.7}$$

If the Littlewood-Paley expansion of f converges to f in the distributional sense, we can write

$$S_j(f) = \sum_{j'<j} \Delta_{j'}(f) \tag{15.8}$$

and (15.155) clearly implies (15.152) and $\bar{\varepsilon}_j \in \ell^q(\mathbf{Z})$.

If conversely (15.152) holds, then $\|S_j(f)\|_\infty \to 0$ ($j \to -\infty$) and the Littlewood-Paley expansion of f converges to f. This means that (15.152) can be replaced by the seemingly weaker condition

$$S_j(f) \rightharpoonup 0 \quad \text{in} \quad \sigma(\mathcal{S}', \mathcal{S})\,. \tag{15.9}$$

We now concentrate on the specific homogeneous Besov space $\dot{B}_q^{\alpha,\infty}$ where $\alpha = 3/q - 1$. It is defined by

$$\|\Delta_j(f)\|_q \le C\, 2^{-j\alpha}\,, \quad j \in \mathbf{Z}\,,\ \alpha = 3/q - 1 \tag{15.10}$$

together with

$$\|S_j(f)\|_\infty \le C\, 2^j\,. \tag{15.11}$$

We then have

Theorem 15.1. *The Banach space B_q is fully adapted to the Navier-Stokes equations if $1 \le q < 3$.*

Let us begin with the dilation invariance of the norm. The norm of f in B_q is defined as

$$\|f\| = \sup_{j \in \mathbf{Z}} 2^{j(3/q-1)}\, \|\Delta_j(f)\|_q \tag{15.12}$$

which obviously implies

$$\|2^m f(2^m x)\| = \|f\| \quad \text{if} \quad m \in \mathbf{Z}. \tag{15.13}$$

On the other hand there exist two constants $C_2 \geq C_1 > 0$ such that

$$C_1\|f\| \leq \|f(\lambda x)\| \leq C_2\|f\| \tag{15.14}$$

if $1 \leq \lambda \leq 2$.

We then introduce a new norm which is defined by

$$\sup\{\lambda\|f(\lambda x)\|\,;\, 0 < \lambda < \infty\} = |||f|||. \tag{15.15}$$

This new norm satisfies

$$C_1\|f\| \leq |||f||| \leq C_2\|f\|, \qquad f \in B_q \tag{15.16}$$

and we have

$$|||f(x)||| = |||\lambda f(\lambda x)|||, \qquad 0 < \lambda < \infty. \tag{15.17}$$

This being checked, we turn to the difficult statement in theorem 15.1. It is based on the following lemma

Lemma 18. *Let f_j, $j \in \mathbf{Z}$, be a sequence of functions in $L^p(\mathbb{R}^n)$. Let us assume the following two conditions*

(15.18) *the Fourier transform $\hat{f}_j(\xi)$ of f_j is supported by the ball $|\xi| \leq 2^j$*

$$\|f_j\|_p \leq \varepsilon_j 2^{-js} \quad \textit{where} \quad s > 0 \quad \textit{and} \quad \sum_{-\infty}^{\infty} \varepsilon_j^q < \infty. \tag{15.19}$$

Then $\sum_{-\infty}^{\infty} f_j$ belongs to $\dot{B}_p^{s,q}$.

Similarly if the Fourier transform $\hat{f}_j(\xi)$ of f_j vanishes on the ball $|\xi| \leq 2^j$ and if (15.19) *holds with $s < 0$ and $\sum_{-\infty}^{\infty} \varepsilon_j^q < \infty$, then $\sum_{-\infty}^{\infty} f_j$ belongs to $\dot{B}_p^{s,q}$.*

Let us prove this statement when $s > 0$. We have $\Delta_m\left(\sum_{-\infty}^{\infty} f_j\right) = \Delta_m\left(\sum_{m-2}^{\infty} f_j\right) = w_m$ by (15.18). Then

$$\|w_m\|_p \leq C \sum_{m-2}^{\infty} \|f_j\|_p \leq C \sum_{m-2}^{\infty} \varepsilon_j 2^{-js} = \eta_m 2^{-ms} \quad \text{where} \quad \eta_m \in \ell^q(\mathbf{Z}).$$

It should be observed that (15.18) can be replaced by a much weaker statement. If indeed $0 < s < m$, $m \in \mathbb{N}$, it suffices to assume the following

$$\|\partial^\alpha f_j\|_p \leq \eta_j \, 2^{j(m-s)}\,, \qquad |\alpha| = m \tag{15.20}$$

with

$$\sum_{-\infty}^{\infty} \eta_j^q < \infty\,. \tag{15.21}$$

Then $\sum_{-\infty}^{\infty} f_j$ belongs to $\dot{B}_p^{s,q}$.

We now return to $B^q = \dot{B}_q^{-(1-3/q),\infty}$ when $1 \leq q < 3$. Writing $f = S_0(f)+\Delta_0(f)+\cdots+\Delta_j(f)+\cdots$ we have $S_0(f) \in L^\infty(\mathbb{R}^3)$ while $\|\Delta_j(f)\|_q \leq \varepsilon_j \, 2^{j(1-3/q)}$. If $1 \leq q \leq 2$, this implies $\|\Delta_j(f)\|_2 \leq \varepsilon_j \, 2^{-j/2}$ by Bernstein's inequalities and $\sum_0^\infty \Delta_j(f) \in L^2$. If $2 < q < 3$, we obviously have $\sum_0^\infty \Delta_j(f) \in L^q$. Finally f always belongs to L^2_{loc}.

For proving (10.4) the paraproduct algorithm will be used. Then $f(x)\, g(x) = A(x) + B(x) + C(x)$ where

$$A(x) = \sum_{-\infty}^{\infty} S_{j-2}(f)\, \Delta_j(g) \tag{15.22}$$

$$B(x) = \sum_{-\infty}^{\infty} S_{j-2}(g)\, \Delta_j(f) \tag{15.23}$$

and

$$C(x) = \sum\sum_{|j'-j|\leq 2} \Delta_j(g)\, \Delta_{j'}(f)\,. \tag{15.24}$$

Both A and B are Littlewood-Paley expansions. We observe that $\|S_{j-2}(f) \cdot \Delta_j(g)\|_q \leq C\, 2^{j(2-3/q)}$ and therefore obtain $A \in \dot{B}_q^{\beta,\infty}$ where $\beta = \alpha - 1$. Therefore $A = \Lambda a$ with $a \in \dot{B}_q^{\alpha,\infty} = B_q$. The same remark applies to B.

As always in the paraproduct algorithm, the devil is hidden inside the third series $C(x)$.

Indeed we have $\|\Delta_j(g)\, \Delta_{j'}(f)\|_q \leq C\, 2^{j(2-3/q)}$ which only permits to apply lemma 18 if $1 \leq q < 3/2$. If $3/2 \leq q < 3$, the idea consists in fixing r such that $\sup(2,q) \leq r < 3$ and in applying Bernstein's inequalities. We obtain

$$\|\Delta_j(f)\|_r \leq C\, 2^{-3\varepsilon j} \qquad \text{where} \quad \frac{1}{r} = \frac{1}{3} + \varepsilon\,, \quad \varepsilon > 0\,,$$

which yields $\|\Delta_j(f)\,\Delta_{j'}(g)\|_{r/2} \le C\,2^{-6\varepsilon j}$.

Now lemma 18 can be applied and we obtain $C \in \dot{B}^{6\varepsilon,\infty}_{r/2}$. Therefore $\|\Delta_j(C)\|_{r/2} \le C\,2^{-6\varepsilon j}$ which implies by Bernstein's inequalities $\|\Delta_j(C)\|_q \le C\,2^{j(2-3/q)}$ and the proof ends as before.

When $q = 1$, $B_q = \dot{B}^{2,\infty}_1$. The proof we just gave works as well for the space $\dot{B}^{2,1}_1$ we want to study now.

This Banach space has a fascinating property. Among all the functional Banach spaces E which are adapted to the Navier-Stokes equations, $\dot{B}^{2,1}_1$ is the smallest one. This minimality property is easily explained. Let us consider the orthonormal wavelet basis $2^{3j/2}\,\psi(2^j x - k)$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}^3$, $\psi \in A$ (with $\sharp\, A = 7$, $A \subset \mathcal{S}(\mathbb{R}^3)$).

We then have $\|2^j\psi(2^j x - k)\|_E = \|\psi\|_E$. Since E is a Banach space, this implies $\sum\sum \alpha(j,k)\,2^j\psi(2^j x - k) \in E$ whenever $\sum\sum |\alpha(j,k)| \le 1$. But any $f \in \dot{B}^{2,1}_1$ is given by a wavelet expansion $\sum\sum \alpha(j,k)\,2^j\psi(2^j x - k)$ where $\sum\sum |\alpha(j,k)| < \infty$. This implies the minimality of $\dot{B}^{2,1}_1$.

There is a second approach to $\dot{B}^{2,1}_1$ which relates this Banach space to the bump algebra.

The bump algebra is defined by the following properties : if $g_{(\alpha,a)}(x) = \exp(-\alpha|x-a|^2)$, $\alpha > 0$, $a \in \mathbb{R}^3$, then a function in the bump algebra can be written as a series

$$f(x) = \sum_0^\infty \lambda_j\, g_{(\alpha_j,a_j)}(x) \tag{15.25}$$

where $\alpha_j > 0$, $a_j \in \mathbb{R}^3$ and $\sum_0^\infty |\lambda_j| < \infty$.

Such an expansion is not unique and the norm of $f(x)$ in the bump algebra is defined as the infimum of $\sum_0^\infty |\lambda_j|$ over all expansions of $f(x)$. Finally the bump algebra is closed under pointwise multiplication. This is obvious since $g_{(\alpha,a)}\,g_{(\beta,b)} = \lambda g_{(\gamma,c)}$ where $0 < \lambda \le 1$, $\gamma = \alpha + \beta$ and $c = (\alpha a + \beta b)/\alpha + \beta$.

The bump algebra is identical to the Besov space $\dot{B}^{3,1}_1$ and each $f \in \dot{B}^{2,1}_1$ can be written as $f = \Lambda f$ where $g \in \dot{B}^{3,1}_1$.

Before ending this section, a variant on the preceding proof will be given. The following result will be needed in the proof of theorem 26.1.

Theorem 15.2. *If $3/2 < q < 6$, $f \in L^q(\mathbb{R}^3)$ and $g \in \dot{B}^{1/2,\infty}_2$, then* *the pointwise product $f(x)\,g(x)$ belongs to the homogeneous Besov space* $\dot{B}_2^{1/2-3/q,\infty}$.

For proving this fact, the paraproduct algorithm is again used.

The first series is $A(x) = \sum_{-\infty}^{\infty} A_j(x)$ where $A_j = S_{j-2}(f)\,\Delta_j(g)$. Since f belongs to $L^q(\mathbb{R}^3)$, we have

$$\|S_{j-2}(f)\|_\infty \le C\,2^{3j/q}\,\|f\|_q \tag{15.26}$$

by Bernstein's inequalities. This yields

$$\|A_j\|_2 \le C\,2^{-j(1/2-3/q)}\,\|f\|_q\,\|g\|_B \tag{15.27}$$

where $\|g\|_B$ denotes the norm of g in the homogeneous Besov space $\dot{B}_2^{1/2,\infty}$.

This estimates implies $\sum_{-\infty}^{\infty} A_j \in \dot{B}_2^{1/2-3/q,\infty}$ and the hypothesis $q \in (3/2, 6)$ is not needed.

We now turn to the second series $B(x) = \sum_{-\infty}^{\infty} B_j(x)$. Here $q < 6$ is needed. This implies that two exponents $s > 3$ and $r \ge q$ exist with $\frac{1}{s} + \frac{1}{r} = \frac{1}{2}$. We then have

$$\|\Delta_j(g)\|_s \le C\,2^{j(1-3/s)}\,\|g\|_B \tag{15.28}$$

which implies

$$\|S_j(g)\|_s \le C'\,2^{j(1-3/s)}\,\|g\|_B\,. \tag{15.29}$$

Bernstein's inequalities yield

$$\|\Delta_j(f)\|_r \le C\,2^{3j(1/q-1/r)}\,\|f\|_q\,. \tag{15.30}$$

All together these estimates give

$$\big\|S_{j-2}(g)\,\Delta_j(f)\big\|_2 \le C\,2^{-j(1/2-3/q)}\,\|f\|_q\,\|g\|_B \tag{15.31}$$

as expected.

The last series in the paraproduct algorithm are $\sum_{-\infty}^{\infty} \Delta_j(f)\,\Delta_{j'}(f)$ where $|j'-j| \le 2$. We consider $\sum_{-\infty}^{\infty} \Delta_j(f)\,\Delta_j(g)$ since the four other series can receive a similar treatment. Since we cannot expect any cancellation in $\Delta_j(f)\,\Delta_j(g)$, lemma 18 should be used with $s > 0$.

We separately treat the two cases $3/2 < q \le 2$ and $2 \le q < 6$.

In the first case, the L^1 norm of $C_j = \Delta_j(f)\,\Delta_j(g)$ is estimated by Hlder's inequality and one obtains $\|C_j\|_1 \leq C\,2^{j(3/q-2)}$. Then $\sum_{-\infty}^{\infty} C_j(x)$ belongs to $\dot{B}_2^{1/2-3/q,\infty}$. Since $\dot{B}_1^{2-3/q,\infty}$ is contained in $\dot{B}_2^{1/2-3/q,\infty}$, this ends the proof.

When $2 \leq q < 6$, an exponent r is defined by $\frac{1}{2} + \frac{1}{q} = \frac{1}{r}$ and Hlder's inequality yields

$$\|\Delta_j(f)\,\Delta_j(g)\|_r \leq C\,2^{-j/2}\,\|f\|_q\,\|g\|_B \tag{15.32}$$

which implies $\sum_{-\infty}^{\infty} C_j \in \dot{B}_r^{1/2,\infty}$. Here again $\dot{B}_r^{1/2,\infty}$ is contained in $\dot{B}_2^{1/2-3/q,\infty}$ and the proof is completed.

It should be observed that theorem 15.2 fails if $q = 3/2$ or $q = 6$.

In both cases, we start a simple observation. The function $g(x) = \frac{1}{|x|}$ belongs to $\dot{B}_2^{1/2,\infty}$. Indeed this function belongs to $\dot{B}_q^{-(1-1/3),\infty}$ for $q \geq 1$. We then study the linear operator defined by the pointwise multiplication by $\frac{1}{|x|}$.

For studying this operator, the following lemma will be used.

Lemma 19. *There exists a constant C_0 such that*

$$\|f\|_2 \leq C_0 \sup\left\{\|e^{i\omega\cdot x} f(x)\|_{\dot{B}_2^{0,\infty}}\,;\ \omega \in \mathbb{R}^3\right\} \tag{15.33}$$

for $f \in L^2(\mathbb{R}^3)$.

The proof of (15.33) is almost obvious. If $\hat{f}$ is compactly supported and vanishes when $|\xi| > R$, then we have

$$c_1\|f\|_2 \leq \left\|e^{i\omega\cdot x} f(x)\right\|_{\dot{B}_2^{0,\infty}} \leq c_2\|f\|_2 \tag{15.34}$$

whenever $2R = |\omega|$. Here c_2 and c_1 are two positive constants. Then (15.34) implies (15.33). When $\hat{f}$ is no longer compactly supported, it suffices to approach f by such functions.

Let us assume for a while that the pointwise product between a function g in $L^6(\mathbb{R}^3)$ with $\frac{1}{|x|}$ belongs to $\dot{B}_2^{0,\infty}$. Since $g(x)$ and $e^{i\omega\cdot x}\,g(x)$ have the same L^6 norms, this would imply the following

$$\left\|f(x)\,|x|^{-1}\right\|_2 \leq C\|f\|_6\,. \tag{15.35}$$

This is obviously wrong since $|x|^{-1}$ does not belong to $L^3(\mathbb{R}^3)$.

The other limiting case is $q = 3/2$.

Let us denote by $\theta(x)$ a cut-off function such that $\theta(x) = 1$ on $|x| \leq 1/4$ and $\theta(x) = 0$ if $|x| \geq 1/2$. We then consider $f(x) = |x|^{-2} \, |\log |x||^{-1} \, \theta(x)$ which belongs to $L^{3/2}$. The pointwise product between $f(x)$ and $\frac{1}{|x|}$ is $|x|^{-3} \, |\log |x||^{-1} \, \theta(x)$. This product is a pointwise function which is not locally integrable at 0. Therefore it cannot belong to $\dot{B}_2^{-3/2,\infty}$ and it is not a tempered distribution.

16 T. Kato's algorithm : an abstract lemma

The goal of this algorithm is to construct mild solutions $u \in C([0,\infty); E)$ to the Navier-Stokes equations when the functional Banach space E is adapted to the Navier-Stokes equations.

More precisely we would like to prove the following conjecture :

(16.1) there exists a positive number α and a constant C such that for $u_0 \in E$ satisfying $\|u_0\|_E < \alpha$ and $\operatorname{div} u_0 = 0$, there exists a mild solution $u \in C([0,\infty); E)$ to the Navier-Stokes equations such that $u(\cdot, 0) = u_0$ and

$$\sup_{t \geq 0} \|u(\cdot, t)\|_E \leq C \|u_0\|_E \,. \tag{16.2}$$

As it was earlier mentioned, the condition $\|u_0\|_E < \alpha$ cannot be given a meaning if the functional space E is not adapted to the Navier-Stokes equations. Indeed the Navier-Stokes equations are invariant under a certain action of the affine group and the sufficient condition $\|u_0\|_E < \alpha$ should reflect this invariance.

T. Kato's algorithm relies on the following abstract lemma.

Lemma 20. *Let Y be a Banach space and let $B : Y \times Y \to Y$ be a continuous bilinear operator : there exists a constant C_0 such that*

$$\|B(y,z)\| \leq C_0 \, \|y\| \, \|z\| \tag{16.3}$$

for $y, z \in Y$, $\| \cdot \|$ denoting the norm in Y.

Then for $a \in Y$ such that $\|a\| < \frac{1}{4C_0}$, there exists at least a solution $x \in Y$ to the equation

$$x = a + B(x,x) \,. \tag{16.4}$$

For proving this lemma we inductively define a sequence x_k, $k \in \mathbb{N}$ (and $k = 0$) by $x_0 = 0$ and $x_{k+1} = a + B(x_k, x_k)$. We then obtain

(16.5)
$$x_{k+1} - x_k = B(x_k - x_{k-1}, x_k) + B(x_{k-1}, x_k - x_{k-1})$$

which implies

$$\|x_{k+1} - x_k\| \leq C_0 \|x_k - x_{k-1}\| \big(\|x_{k-1}\| + \|x_k\|\big) . \tag{16.6}$$

For estimating $\|x_{k+1} - x_k\|$, we construct an increasing sequence r_k of real numbers by $r_0 = 0$, $r_1 = \|a\|$ and $r_{k+1} = \|a\| + r_k^2$. We then have

$$r_{k+1} - r_k = C_0 \big(r_k - r_{k-1}\big)\big(r_k + r_{k-1}\big) \tag{16.7}$$

and r_k is actually increasing. This sequence r_k converges to r if and only if $4C_0\|a\| \leq 1$. This limit r is the smallest solution of the scalar equation $t = \|a\| + C_0 t^2$. This equation is a special case of (16.4) where $Y = \mathbb{R}$ and $B(t,t) = C_0 t^2$. An obvious induction yields $\|x_{k+1} - x_k\| \leq r_{k+1} - r_k$. Therefore $\sum_0^\infty \|x_{k+1} - x_k\| \leq r$ and x_k, $k \in \mathbb{N}$ is a Cauchy sequence which tends to a limit x. We have $\|x\| \leq r = \frac{1 - \sqrt{1 - 4\|a\| C_0}}{2C_0} \leq 2\|a\|$.

This well-known proof is interesting since it is sharp. Indeed $\|a\| < \frac{1}{4C_0}$ cannot be replaced by $\|a\| < \frac{1+\varepsilon}{4C_0}$ where ε is positive. A counter-example is precisely given by the scalar equation $t = \|a\| + C_0 t^2$.

Returning to lemma 20, it is completed by the following remark.

Lemma 21. *We keep the same assumptions and notations as in lemma 20. Then the equation* (16.4) *has a unique solution x such that $\|x\| < \frac{1}{2C_0}$.*

We first observe that the solution given by the iterative scheme satisfies $\|x\| \leq 2\|a\| < \frac{1}{2C_0}$.

If y is another such solution and $y = x + z$, we then have $z = B(x, z) + B(z, y)$ which implies $\|z\| \leq C_0 \|x\|\, \|z\| + C_0 \|y\|\, \|z\|$. If $z \neq 0$, this reads $1 \leq C_0\big(\|x\| + \|y\|\big) < 1$.

17 Kato's algorithm applied to Navier-Stokes equations : a straightforward example

We want to apply lemma 20 to Navier-Stokes equations. More precisely we want Y to be the Banach space $C([0, \infty); E)$ where E is a functional space which is adapted to the Navier-Stokes equations. We want a to be $S(t)u_0 \in C([0,\infty);E)$. Finally the Navier-Stokes equations read

$$u = u(\cdot,t) = S(t)u_0 + B(u,u) = a + B(u,u) \tag{17.1}$$

where

$$B(u,v) = \mathbb{P}\int_0^t S(t-s)\,\partial_j(u_j v)(s)\,ds\,. \tag{17.2}$$

Some explanations may be useful. Here and in what follows, $a_j b_j$ means $a_1b_1 + a_2b_2 + a_3b_3$. We will write $u \in C([0,\infty);E)$ instead of the unpleasant $u \in C([0,\infty);E^3)$. The Banach space $Y = C([0,\infty);E)$ will be given the norm

$$\sup_{t\geq 0} \|u(\cdot,t)\|_E\,. \tag{17.3}$$

More important are the following remarks about the bilinear operator B. The Banach spaces E which will be used are invariant under the Riesz transformations. We can therefore forget the Leray projector $\mathbb{P}$ which acts boundedly on E.

In a more sophisticated version of Kato's algorithm (section 19) an L^∞ estimate will also be needed. Since the Leray projector $\mathbb{P}$ is not bounded on $L^\infty(\mathbb{R}^n)$, it should be incorporated inside the integral sign and the three operators $\mathbb{P}$, $S(t-s)$ and ∂_j should be glued together. The resulting operator $\Gamma_j(t-s)$, $j = 1, 2$ or 3, is a convolution with $(t-s)^{-2}\,\omega_j\big(x(t-s)^{-1/2}\big)$, $j = 1, 2$ or 3. Elementary calculations show that $\omega_j \in C^\infty(\mathbb{R}^3)$ with $\omega_j(x) = 0(|x|^{-4})$ at infinity. Therefore ω_j belongs to $L^1(\mathbb{R}^3)$ and the operator norm of

$$\mathbb{P}\,S(t-s)\partial_j \,:\, L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3) \tag{17.4}$$

is $C(t-s)^{-1/2}$ where C is an absolute constant. This remark applies as well if $L^\infty(\mathbb{R}^3)$ is replaced by any functional Banach space X whose norm is translation invariant.

In (17.192) u and v are vector fields satisfying

$$\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 = 0\,.$$

However it can be proved that for most of the estimates we have in mind, this fact is useless [57]. That is why at the proof level the true bilinear operator $B(u,v)$ will be replaced by their scalar counterparts defined as

(17.5)

$$B_j(f,g) = \int_0^t S(t-s)\,\partial_j(fg)\,(s)\,ds\,, \qquad 1 \leq j \leq 3\,,$$

where $f = f(x,t)$ and $g = g(x,t)$ both belong to $C([0,\infty); E)$.

We can even move further and consider

$$B(f,g) = \int_0^t S(t-s)\,\Lambda(fg)\,(s)\,ds \tag{17.6}$$

where $\Lambda = \sqrt{-\Delta}$ is the celebrated Calderón's operator. Then $B_j = -iR_jB$ where R_j denotes the Riesz transformation. Since E is assumed to be invariant under the action of R_j, any estimate for $B(f,g)$ will imply the corresponding estimate for $B_j(f,g)$.

In our first example of Kato's algorithm, the functional Banach space E is defined by a pointwise estimate on the Fourier transform $\hat{f}$ of f. It reads

$$|\hat{f}(\xi)| \le C|\xi|^{-2}. \tag{17.7}$$

In other words $\Delta f \in PM(\mathbb{R}^3)$ where following Kahane's notations, a distribution S is a pseudo-measure if and only if its Fourier transform $\hat{S}$ belongs to $L^\infty(\mathbb{R}^3)$.

Therefore E is not a separable Banach space. Instead it is a dual space F^* where $g \in F$ if and only if $g = \Delta h$ and h belongs to the Wiener algebra [41]. Therefore $u \in C([0,\infty); E)$ means the following two properties

(17.8) u is continuous from $(0,\infty)$ to E when E is given its norm topology

(17.9) $u(\cdot,t) \rightharpoonup u(\cdot,0)$ with respect to $\sigma(E,F)$ as t tends to 0.

We then write $Y = C([0,\infty); E)$ and have

Lemma 22. *The bilinear operator B defined by* (17.6) *is continuous from $Y \times Y$ into Y.*

We will only prove the bilinear estimate

$$\|B(f,g)\|_Y \le C\,\|f\|_Y\,\|g\|_Y \tag{17.10}$$

when

$$\|f\|_Y = \sup_{t\ge 0} \|f(\cdot,t)\|_E\,. \tag{17.11}$$

The proof of the required continuity with respect to the time variable is identical to the one which will be given when $E = L^{3,\infty}$. For that reason, this part will be omitted.

Returning to (17.200), we observe that $|\hat{f}(\xi)| \leq C|\xi|^{-2}$ and $|\hat{g}(\xi)| \leq C|\xi|^{-2}$ imply $|(fg)\hat{\ }(\xi)| \leq C|\xi|^{-1}$. On the Fourier transform side,

$$[B(f,g)]\hat{\ }(\xi) = \int_0^t |\xi|\, e^{-(t-s)|\xi|^2}\, (fg)\hat{\ }(\xi)\, ds$$

which implies

$$\left|[B(f,g)]\hat{\ }(\xi)\right| \leq C \int_0^t |\xi|\, e^{-(t-s)|\xi|^2} |\xi|^{-1}\, ds \leq C|\xi|^{-2}.$$

We have proved the following result

Theorem 17.1. *If the adapted Banach space E is defined by* (17.197), *there exists a positive number η such that for every initial condition satisfying $\|u_0\|_E < \eta$, there exists a mild solution*

$$u \in C([0,\infty); E)$$

to the Navier-Stokes equations such that $u(x,0) = u_0(x)$ and

$$\sup_{t\geq 0} \|u(x,t)\|_E \leq 2\|u_0\|_E. \tag{17.12}$$

Moreover this solution is uniquely defined by the weaker condition

$$\sup_{t\geq 0} \|u(x,t)\|_E < 2\eta. \tag{17.13}$$

18 Kato's algorithm applied to the Navier-Stokes equations : the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$

Let us remind the reader with the general definition of the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$. If f is a measurable function defined on $\mathbb{R}^n$, we let E_j, $j \in \mathbf{Z}$, be the set of points $x \in \mathbb{R}^n$ for which $2^j \leq |f(x)| < 2^{j+1}$. If $|E_j|$ denotes the Lebesgue measure of E_j, then $f \in L^{p,q}(\mathbb{R}^n)$ means

$$|E_j|^{1/p}\, 2^j \in \ell^q(\mathbf{Z}). \tag{18.1}$$

Let us observe some formal similarities between this definition and the definition of the Besov spaces. Writing $f_j = \mathbf{1}_{E_j} f$ where $\mathbf{1}_E$ is the indicator function of E, (18.1) reads

$$\|f_j\|_p \in \ell^q(\mathbf{Z}) \tag{18.2}$$

and we have $f = \sum_{-\infty}^{\infty} f_j$.

But in contrast with what is happening in the Besov space case, the mapping $f \to (f_j)_{j\in\mathbf{Z}}$ is not a linear one.

We now assume $1 < p < \infty$ and $q = \infty$. Then (18.1) can be rewritten as

$$\left|\{x\,;\,|f(x)| > \lambda\}\right| \le C\,\lambda^{-p}\,. \tag{18.3}$$

If E is any measurable set with a finite measure, (18.3) implies

$$\int_E |f(x)|\,dx \le C'|E|^{1/p'} \tag{18.4}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

This fails when $p = 1$. Obviously (18.4) implies $f \in L^{p,\infty}(\mathbb{R}^n)$.

Then (18.4) can easily be used for defining a norm on $L^{p,\infty}$. It suffices to write

$$\|f\|_{(p,\infty)} = \sup\left\{|E|^{-1/p'}\int_E |f(x)|\,dx\,;\,E \in \mathcal{B}\right\} \tag{18.5}$$

where $\mathcal{B}$ is the collection of all Borel sets with a finite and positive measure.

An other access to the norm is given by the following observation. A function $a(x)$ is an atom if $a(x) = |E|^{-1/p'}\,\mathbf{1}_E(x)$ for some E in $\mathcal{B}$. We then consider the Banach space consisting of all $f(x) \in L^{p'}(\mathbb{R}^n)$ which admit a decomposition

$$f(x) = \sum_0^{\infty} \lambda_j\,a_j(x) \tag{18.6}$$

where $a_j(x)$ is a sequence of atoms and where $\sum_0^{\infty} |\lambda_j| < \infty$.

The norm of f being the obvious quotient norm given by the infimum of $\sum_0^{\infty} |\lambda_j|$ computed over all possible expansions of f.

It is not difficult to check that this Banach space is identical to $L^{p',1}(\mathbb{R}^n)$. Then $L^{p,\infty} = E$ is the dual space of $L^{p',1} = F$. This Banach space F is separable while E is not.

We now return to $L^{3,\infty}(\mathbb{R}^3)$. If $f \in L^{3,\infty}(\mathbb{R}^3)$, we have

$$\int_E |f(x)|^2\,dx \le C|E|^{1/3}\,, \qquad E \in \mathcal{B} \tag{18.7}$$

and it is interesting to compare this property to the definition of the Morrey-Campanato space where E is a ball.

As it was stated many times, $u(x,t) \in C([0,\infty);E)$, $E = L^{3,\infty}(\mathbb{R}^3)$ means $u(x,t) \rightharpoonup u(x,0)$ in $\sigma(E,F)$ as t tends to 0 together with

$$\lim_{t\to t_0} \left\|u(\cdot,t) - u(\cdot,t_0)\right\|_E = 0\,, \qquad t_0 > 0\,. \tag{18.8}$$

We then have

Lemma 23. *If $E = L^{3,\infty}(\mathbb{R}^3)$ and $Y = C([0,\infty);E)$ then the bilinear operator $\mathcal{B}$ is continuous from $Y \times Y$ into Y.*

Let us begin with a general fact.

We start with a kernel $K_t(x,y)$, $t > 0$, $x \in \mathbb{R}^3$, $y \in \mathbb{R}^3$, fulfilling the following estimate

$$|K_t(x,y)| \le Ct^{-3}\left(1 + |x-y|/t\right)^{-4} \tag{18.9}$$

and denote by $P_t : L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)$, $1 \le p \le \infty$, the operator defined by

$$P_t f(x) = \int K_t(x,y)\, f(y)\, dy\,. \tag{18.10}$$

We then have a pointwise inequality

$$|P_t f|(x) \le C\left(\varphi_t * |f|\right)(x) \tag{18.11}$$

where $\varphi(x) = (1+|x|)^{-4}$. Therefore

$$\|P_t f\|_E \le C\|f\|_E \tag{18.12}$$

whenever the functional Banach space E is translation invariant and is a lattice. That means that $f_2 \in E$ and $|f_1(x)| \le |f_2(x)|$ implies $f_1 \in E$. This remark and (18.215) apply to $L^{p,\infty}(\mathbb{R}^3)$ when $1 < p < \infty$.

Next we consider a real number $\alpha \in (0,3)$ and an exponent p belonging to $(1, 3/\alpha)$. We then study a linear operator T which maps continuous functions $f \in C([0,\infty); E)$, $E = L^{p,\infty}$, into functions of $x \in \mathbb{R}^3$ by

$$g(x) = \int_0^\infty P_t\, f(\cdot, t)\, t^{\alpha-1}\, dt\,. \tag{18.13}$$

Keeping these notations we have

Theorem 18.1. *If $\alpha \in (0,3)$ and $p \in (1, 3/\alpha)$, then*

$$\|g\|_{(q,\infty)} \le C \sup_{t\ge 0} \|f(\cdot,t)\|_{(p,\infty)} \tag{18.14}$$

where $q = \frac{3p}{3-\alpha p}$ and $\|\cdot\|_{(p,\infty)}$ is the weak-L^p norm.

The proof is trivial but will be given for the reader's convenience. Let $\lambda \in (0,\infty)$ be a threshold. We consider the set E of points x for which $|g(x)| > \lambda$ and we want to estimate the Lebesgue measure $|E|$ of E. We then split $\int_0^\infty P_t f(\cdot,t) t^{\alpha-1}\, dt$ into $\int_0^\tau + \int_\tau^\infty = u(x) + v(x)$. The relation between τ and λ will soon be clarified.

We first estimate $\|v\|_\infty$. Indeed as a function of y, $K_t(x,\cdot)$ belongs to $L^{p',1}$ where $1/p' + 1/p = 1$. Moreover $\|K_t(x,\cdot)\|_{p',1} \le Ct^{-3/p}$ and this implies

$$\|P_t f(\cdot,t)\|_\infty \le Ct^{-3/p} \sup_{t\ge 0} \|f(\cdot,t)\|_{(p,\infty)}\,. \tag{18.15}$$

Therefore $\|v\|_\infty \le C\int_\tau^\infty t^{\alpha-3/p-1}\, dt = C'\tau^{\alpha-3/p}$. We now define τ by $C'\tau^{\alpha-3p} = \lambda/2$. If $|g(x)| > \lambda$, that forces $|u(x)| > \lambda/2$. But

$$\|u\|_{(p,\infty)} \le C\int_0^\tau \|f(\cdot,t)\|_{(p,\infty)}\, t^{\alpha-1}\, dt = C'\tau^\alpha$$

(here $p > 1$ is crucially needed). Finally

$$\big|\{x\,;\, |u(x)| > \lambda/2\}\big| \le C''\Big(\frac{\lambda}{\tau^\alpha}\Big)^{-p} = C_0\lambda^{-q}$$

where $q = \frac{3p}{3-\alpha p}$ as announced.

We now treat a very specific example. We consider

$$g(x,t) = \int_0^t S(t-s)\, \Lambda\, f(\cdot, s)\, ds$$

where $\sup_{s\geq 0} \|f(\cdot,s)\|_{(3/2,\infty)} \leq 1$ and want to prove that a constant C exists for which

$$\sup_{t\geq 0} \|g(\cdot,t)\|_{(3,\infty)} \leq C\,.$$

For reducing this example to the framework given in theorem 18.1, we first change s into $t-s$ and write

$$g(x,t) = \int_0^\infty S(s)\,\Lambda\, F(\cdot,s)\,ds$$

with $F(\cdot,s) = f(\cdot,t-s)$ if $0 \leq s \leq t$ and $F(\cdot,s) = 0$ if $s > t$.

Finally $\Lambda\, S(s)$ is a convolution operator. The corresponding kernel $K_s(x,y)$ is $s^{-2}\,\omega\big(\frac{x-y}{\sqrt{s}}\big)$ where $|\omega(x)| \leq C(1+|x|)^{-4}$. This leads to a final change of variables $s = \tau^2$ and theorem 18.1 applies with $\alpha = 1$.

We now investigate continuity with respect to the t-variable.

Lemma 24. *Let us assume that $f(\cdot,s) \in C([0,\infty);E)$, $E = L^{3/2,\infty}$ where this continuity with respect to the t variable is defined by* (17.198) *and* (17.199).

Then $g(\cdot,t) \in C([0,\infty);F)$ where $F = L^{3,\infty}$ and the continuity is similarly defined.

We begin with the required weak continuity at $t_0 = 0$. Our first observation is a trivial one. Indeed

$$\begin{aligned}\|g(\cdot,t)\|_{3/2,\infty} &\leq C\int_0^t (t-s)^{-1/2}\,\|f(\cdot,s)\|_{(3/2,\infty)}\,ds\\ &\leq 2Ct^{1/2}\sup_{0\leq s\leq t}\|f(\cdot,s)\|_{(3/2,\infty)}\,.\end{aligned}$$

Therefore $\lim_{t\downarrow 0}\|g(\cdot,t)\|_{3/2,\infty} = 0$ which suffices to imply the required weak continuity.

For proving the required continuity at $t_0 > 0$, we compute $\|g(\cdot,t') - g(\cdot,t)\|_E$ when $0 < a < t < t' < 2a$. This positive a exists if $|t-t_0| < t_0/3$ and $|t'-t_0| < t_0/3$.

For symplifying the notations, we assume

$$\|f(\cdot,t)\|_E \leq 1 \qquad \text{for} \quad t \geq 0\,.$$

We then split $f(\cdot,t)$ into the sum $f_1(\cdot,t) + f_2(\cdot,t)$ where $f_1(\cdot,t) = 0$ for $t \geq a/2$, $f_2(\cdot,t) = 0$ for $t \leq a/4$, $f_2(\cdot,t)$ is continuous from $[0,\infty)$ into

E when E is given its strong norm topology and finally $\|f_1(\cdot,t)\|_E \le 1$, $\|f_2(\cdot,t)\| \le 1$ for $t \ge 0$. Then

$$\begin{aligned} g(\cdot,t) &= \int_0^t S(t-s)\,\Lambda\, f(\cdot,s)\,ds \\ &= \int_0^t S(t-s)\,\Lambda\, f_1(\cdot,s)\,ds + \int_0^t S(t-s)\,\Lambda\, f_2(\cdot,s)\,ds \\ &= u(\cdot,t) + v(\cdot,t)\,. \end{aligned}$$

We first treat $u(\cdot,t) = \int_0^t S(t-s)\,\Lambda f_1(\cdot,s)\,ds$. We then have

$$\begin{aligned} \left\| \frac{\partial}{\partial t} u(\cdot,t) \right\|_{(3,\infty)} &\le \int_0^\infty \left\| \Delta^2 S(t-s)\,(\Lambda^{-1} f_1)(\cdot,s) \right\|_{(3,\infty)} \\ &\le C \int_0^{a/2} (t-s)^{-2}\,ds\,. \end{aligned}$$

This last estimate is coming from the boundedness of $\Lambda^{-1} : L^{3/2,\infty} \to L^{3,\infty}$. Concerning v, we write

$$\begin{aligned} v(\cdot,t') - v(\cdot,t) &= \int_0^{t'} S(\tau)\,\Lambda f_2(\cdot,t'-\tau)d\tau - \int_0^t S(\tau)\,\Lambda f_2(\cdot,t-\tau)d\tau \\ &= \int_t^{t'} S(\tau)\,\Lambda\, f_2(\cdot,t'-\tau)\,d\tau\,+ \\ &\quad + \int_0^t S(\tau)\,\Lambda\big[f_2(\cdot,t'-\tau) - f_2(\cdot,t-\tau)\big]\,d\tau \;=\; U + V\,. \end{aligned}$$

Concerning U, we once move factor out the Calderón operator which reads

$$U = \int_t^{t'} [S(\tau)\Delta]\,\Lambda^{-1}\, f_2(\cdot,t'-\tau)\,d\tau$$

and implies

$$\|U\|_{3,\infty} \le \int_t^{t'} \left\| Q(\tau)\,\Lambda^{-1} f_2(\cdot,t'-\tau) \right\|_{3,\infty} \frac{d\tau}{\tau}\,.$$

The operator $Q(\tau) = \tau\,\Delta\,\exp(\tau\Delta)$, $\tau \ge 0$, is uniformly bounded on $L^{3,\infty}$. We finally obtain

$$\|U\|_{3,\infty} \le C \log \frac{t'}{t}\,.$$

Concerning V, it suffices to use theorem 18.1 together with the continuity with respect to the t variable of f_2. We then have $\big\| f_2(\cdot,t'-\tau) - f_2(\cdot,t-$

$\tau)\|_{3/2,\infty} \le \varepsilon$ if $|t'-t| \le \eta$ and $0 \le \tau \le t \le t' \le 2a$. It implies $\|V\|_{3,\infty} \le C\varepsilon$ as expected.

We proved the following statement.

Theorem 18.2. *There exists a positive number* η *such that for every* $u_0(x)$ *in* $L^{3,\infty}(\mathbb{R}^3)$ *fulfilling*

$$\|u_0\|_{3,\infty} < \eta \quad \text{and} \quad \operatorname{div} u_0(x) = 0 \tag{18.16}$$

there is a solution $u(x,t)$ *of the Navier-Stokes equations with the following properties*

$$u \in C([0,\infty)\,;\, L^{3,\infty}(\mathbb{R}^3)) \tag{18.17}$$

$$u(x,0) = u_0(x) \tag{18.18}$$

$$\sup_{t\ge 0} \|u(x,t)\|_{(3,\infty)} \le 2\|u_0\|_{(3,\infty)}\,. \tag{18.19}$$

Moreover this solution is uniquely defined by (18.220), (18.221) *and*

$$\sup_{t\ge 0} \|u(x,t)\|_{(3,\infty)} < 2\eta\,. \tag{18.20}$$

19 Kato's algorithm : the Lebesgue space $L^3(\mathbb{R}^3)$

Theorem 18.2 is surprising since the continuity of $B : Y \times Y \to Y$ is failing in a more natural setting given by $Y = C([0,\infty); L^3(\mathbb{R}^3))$.

For treating this example, T. Kato made the following crucial observations. First the linear evolution $S(t)u_0$ belongs to a much narrower space $Z \subset Y$ and secondly the iterative scheme which is being used for solving (17.191) is indeed confined inside Z.

The definition of Z heavily depends on the adapted functional space E we are working with.

When $E = L^3(\mathbb{R}^3)$ or the Morrey-Campanato space M^2, the definition of Z relies on the L^∞-norm of $u(\cdot,t)$. In other cases, this definition is much more involved and will be unveiled in section 22.

We begin with a simple observation which was proved in section 8 (lemma 9).

Lemma 25. *If the functional Banach space* E *is adapted to the Navier-Stokes equations, there exists a constant* C *such that for* $f \in E$ *and* $t > 0$, *we have*

$$\|S(t)f\|_\infty \le C t^{-1/2} \|f\|_E\,. \tag{19.1}$$

This estimate leads to the following definition of Z as proposed by T. Kato : Z is the subspace of $Y = C([0,\infty); E)$ consisting of all $u(x,t) \in Y$ such that

$$\sup_{t\geq 0} t^{1/2}\,\|u(\cdot,t)\|_\infty \;=\; \gamma \;<\; \infty \tag{19.2}$$

and $\|u\|_Z = \gamma + \sup_{t\geq 0}\,\|u(\cdot,t)\|_E$. Lemma 25 implies the following

$$\|S(t)u_0\|_Z \;\leq\; C_1\|u_0\|_E \tag{19.3}$$

and Kato's program will depend on the following crucial estimate

$$\|B(u,v)\|_Z \;\leq\; C_0\,\|u\|_Z\,\|v\|_Z\,. \tag{19.4}$$

We illustrate this approach when $E = L^3(\mathbb{R}^3)$ and prove (19.4). We begin with the first half of the Z-norm and prove

$$\sup_{t\geq 0}\,\|B(u,v)\|_3 \;\leq\; C\,\|u\|_Z\,\|v\|_Z\,. \tag{19.5}$$

Indeed $B(u,v) = w(\cdot,t) = \int_0^t \mathbb{P}\,S(t-s)\,\partial_j(u_j v)\,ds$ and $\mathbb{P}\,S(t-s)\partial_j$ is a convolution operator with a (matrix valued) function $(t-s)^{-2}\,\omega((t-s)^{-1/2}x)$. As it was observed, the L^1-norm of this function is $(t-s)^{-1/2}\,\|\omega\|_1$. Therefore

$$\|w(\cdot,t)\|_3 \;\leq\; \|\omega\|_1\,\int_0^t (t-s)^{-1/2}\,\|uv\|_3\,ds\,.$$

We then observe that $\|uv\|_3 \leq \|u\|_\infty\|v\|_3 \leq s^{-1/2}\,\|u\|_Z\,\|v\|_Z$ and the computation ends with the following trivial remark

$$\int_0^t (t-s)^{-1/2}\,s^{-1/2}\,ds \;=\; \pi\,. \tag{19.6}$$

This simple approach fails if one tries to estimate the L^∞ norm. Indeed $\|u\|_\infty \leq s^{-1/2}\,\|u\|_Z$, $\|v\|_\infty \leq s^{-1/2}\,\|v\|_Z$ lead to a divergent integral.

But this problem can be easily fixed. Indeed one uses the duality between L^4 and $L^{4/3}$ which yields

$$\|w(\cdot,t)\|_\infty \;\leq\; C\int_0^t (t-s)^{-7/8}\,\|uv\|_4\,ds$$

since the $L^{4/3}$ norm of $(t-s)^{-2}\,\omega\big((x(t-s)^{-1/2}\big)$ is $C(t-s)^{-7/8}$. Next $\|uv\|_4 \leq \|u\|_8\,\|v\|_8$ and these L^8 norms are trivialy estimated. Indeed $\|u\|_3 \leq \|u\|_Z$ while $\|u\|_\infty \leq s^{-1/2}\|u\|_Z$ which yields $\|u\|_8 \leq s^{-5/16}\,\|u\|_Z$.

We now need to show that $B(u,v)$ belongs to $C([0,\infty); L^3)$ if u and v belong to Z. The main issue is the continuity at 0. For proving the fundamental property

$$\lim_{t\downarrow 0} \|B(u,v)\|_3 = 0\,, \qquad u \in Z\,,\ v \in Z \tag{19.7}$$

we need to start with a simple observation. If $u_0 \in L^3(\mathbb{R}^3)$, we have

$$\lim_{t\downarrow 0} t^{1/2}\,\|S(t)u_0\|_\infty = 0\,. \tag{19.8}$$

This allows us to modify and sharpen the definition of Z. We impose for $u \in Z$

$$\lim_{t\downarrow 0} t^{1/2}\,\|u(\cdot,t)\|_\infty = 0\,. \tag{19.9}$$

The continuity of the bilinear operator $B : Z \times Z \to Z$ is proved with the same argument we previously used and (19.230) follows from the definition of Z.

We can state

Theorem 19.1. *There exists a positive constant η such that for any initial condition $u_0(x) \in L^3(\mathbb{R}^3)$ such that*

$$\|u_0\|_3 < \eta \quad \textit{and} \quad \operatorname{div} u_0(x) = 0 \tag{19.10}$$

there exists a mild solution

$$u(x,t) \in C\big([0,\infty); L^3(\mathbb{R}^3)\big) \tag{19.11}$$

to the Navier-Stokes equations such that $u(x,0) = u_0(x)$.

This solution satisfies

$$\sup_{t\geq 0}\big(\|u(\cdot,t)\|_3 + t^{1/2}\,\|u(\cdot,t)\|_\infty\big) \leq C\|u_0\|_3 \tag{19.12}$$

where C is an absolute constant.

Finally there exists a second positive constant $\beta < 2\eta$ such that this solution is uniquely defined by

$$\sup_{t>0} \|u(\cdot,t)\|_3 + t^{1/2}\,\|u(\cdot,t)\|_\infty < \beta\,. \tag{19.13}$$

Let us observe that Kato's algorithm is here applied to the Z-norm and that β is no longer 2η since we need to take in account (19.3).

Before leaving this example, let us make a last remark. We already know that our solution $u(x,t)$ satisfies

$$\lim_{t\to 0} t^{1/2}\,\|u(\cdot,t)\|_\infty = 0\,. \tag{19.14}$$

Indeed uniqueness can be proved with (19.14) instead of (19.13). Let us sketch the proof of this observation.

Instead of studying solutions u in $C([0,\infty);E)$ we could as well consider solutions in $C([0,T];E)$ where T is a positive constant. All the estimates we proved in the first setting can be proved in this second framework and the constants remain unchanged : they do not depend on T.

If (19.14) is satisfied, we can chose T small enough in such a way that

$$\sup_{0\le t\le T}\left(\|u(\cdot,t)\|_3 + t^{1/2}\,\|u(\cdot,t)\|_\infty\right) < \beta\,.$$

This implies uniqueness on $[0,T]$.

There is a much deeper statement which will be unveiled in the next section.

20 Uniqueness of L^3 valued mild solutions

In this section a simplified proof of Lemarié-Rieusset's theorem will be given. The original and more involved proof will be described in section 26.

Theorem 20.1. *Let $u(x,t)\in C([0,T];L^3(\mathbb{R}^3))$ and $v(x,t)\in C([0,T];L^3(\mathbb{R}^3))$ be two mild solutions of Navier-Stokes equations such that $u(x,0)=v(x,0)=u_0(x)$.*

Then $u=v$ on $[0,T]$.

Let us denote by τ the supremum of the set of $\sigma\in[0,T]$ such that $u=v$ on $[0,\sigma]$. By continuity, $u=v$ on $[0,\tau]$. If $\tau<T$ we will prove that $u=v$ on $[0\,,\,\tau+\varepsilon]$ for some positive ε. This forces $\tau=T$ as announced.

We now consider $\overline{u}(\cdot,t)=u(\cdot\,,\,t+\tau)$ and $\overline{v}(\cdot,t)=v(\cdot\,,\,t+\tau)$. These two functions are defined on $[0\,,\,T-\tau]$. Both are mild solutions to the Navier-Stokes equations (13.2) and they coincide when $t=0$.

Changing the notations, we conclude that theorem 20.1 will be proved as soons as we can prove the following statement

$$(20.1) \qquad u = v \qquad \text{on} \quad [0, \varepsilon]$$

for some positive ε.

For proving (20.238) one writes

$$(20.2) \qquad u \;=\; S(t)u_0 + B(u,u) \quad , \quad v \;=\; S(t)u_0 + B(v,v)$$

and $v = u + w$. Then

$$w \;=\; B(u{+}w, u{+}w) - B(u,u) \;=\; B(v,w) + B(w,u)\,.$$

Our goal is to estimate

$$(20.3) \qquad \eta(t) \;=\; \sup_{0 \le s \le t} \|w(\cdot, s)\|_{(3,\infty)}$$

and prove

$$(20.4) \qquad \eta(t) \;\le\; \frac{1}{2}\,\eta(t) \qquad \text{if} \quad 0 \le t \le \varepsilon$$

which obviously implies $\eta(t) = 0$ if $0 \le t \le \varepsilon$. For estimating $\eta(t)$, we write

$$(20.5) \qquad \begin{aligned} w \;&=\; B(S(t)u_0, w) + B(v{-}S(t)u_0, w) + \\ &\quad + B(w, u{-}S(t)u_0) + B(w, S(t)u_0) \\ &=\; A + B + C + D\,. \end{aligned}$$

For estimating $\|D(t)\|_{3,\infty}$ we return to the definition of the bilinear operator and bound $\big\|S(t-s)\,\partial_j[w_j(s)\,S(s)u_0]\big\|_{3,\infty}$ by $C(t-s)^{-1/2}\,\|w_j\|_{3,\infty}\cdot \|s^{1/2}S(s)u_0\|_\infty\, s^{-1/2}$. This yields the trivial estimate

(20.6)

$$\|B(w, S(t)u_0)\|_{(3,\infty)} \;\le\; C\,\pi\,\eta(t) \sup_{0 \le s \le t} \|s^{1/2}\,S(s)u_0\|_\infty\,.$$

A similar estimate holds for $\|A(t)\|_{(3,\infty)}$. Concerning $\|B(t)\|_{(3,\infty)}$ and $\|C(t)\|_{(3,\infty)}$, we apply lemma 23 and observe that

$$\|u{-}S(t)u_0\|_{(3,\infty)} \;\le\; \|u{-}S(t)u_0\|_3 \;=\; \alpha(t)$$

where $\alpha(t)$ tends to 0 with t.

All together we obtain

$$\|w(\cdot,t)\|_{(3,\infty)} \le \gamma(t) \sup_{0\le s\le t} \|w(\cdot,s)\|_{(3,\infty)} \tag{20.7}$$

where $\gamma(t)$ tends to 0 with t. This immediately implies (20.1) and theorem 20.1 follows.

21 Kato's program : the Morrey-Campanato spaces

We now turn to another application of T. Kato's program : the Morrey-Campanato space $E = M^2(\mathbb{R}^3)$ which is defined by

$$\sup_{x_0\in\mathbb{R}^3} \sup_{R>0} R^{-1} \int_{|x-x_0|\le R} |f(x)|^2\,dx < \infty\,. \tag{21.1}$$

Keeping the same notations and definitions as above, we observe that $E = F^*$ where F is a separable Banach space. Then $Y = C([0,\infty), E)$ is given the same meaning as in section 12 and $Z \subset Y$ is the subspace of Y consisting of all functions $u(\cdot,t)$ for which

$$\sup_{t\ge 0} \left\{t^{1/2}\,\|u(\cdot,t)\|_\infty + \|u(\cdot,t)\|_E\right\} = \|u\|_Z \tag{21.2}$$

is finite.

We already know from theorem that the expected estimate

$$\|B(f,g)\|_Y \le C\,\|f\|_Y\,\|g\|_Y \tag{21.3}$$

cannot hold. Indeed E is not fully adapted to Navier-Stokes equations.

However, as it was the case when $E = L^3(\mathbb{R}^3)$, it is not difficult to obtain

$$\|B(f,g)\|_Z \le C\,\|f\|_Z\,\|g\|_Z\,. \tag{21.4}$$

Once more the easy part in proving (21.4) concerns the E component of the Z-norm. Indeed we have

$$\|fg\|_E \le \|f\|_\infty\,\|g\|_E \le s^{-1/2}\,\|f\|_Z\,\|g\|_Z \tag{21.5}$$

if $f = f(\cdot, s)$, $g = g(\cdot, s)$. Then (19.224) implies

(21.6)

$$\|B(u,v)\|_E \;\le\; \|\omega\|_1 \left(\int_0^t (t-s)^{-1/2}\, s^{-1/2}\, ds \right) \|u\|_Z \, \|v\|_Z$$

where the notations are the sames as in section 19.

We need to treat the L^∞-component of $w = B(u,v)$. Let us denote by $W(s,t,x)$ the integrand $S(t-s)\,\partial_j(u_j(s)v(s))$. We then have

$$\|W(s,t,x)\|_\infty \;\le\; C(t-s)^{-1/2}\, s^{-1}\, \|u\|_Z \, \|v\|_Z$$

since $\|u(s)\|_\infty \le s^{-1/2}\,\|u\|_Z$ and the same for v. On the other hand $\|uv\|_E \le \|u\|_\infty \, \|v\|_E$. Finally lemma 9 is again used and yields

$$\|W(s,t,x)\|_\infty \;\le\; C(t-s)^{-1}\, s^{-1/2}\, \|u\|_Z \, \|v\|_Z\,.$$

We have obtained two extimates on $\|W(s,t,x)\|_\infty$ and their geometrical mean yields

$$\|W(s,t,x)\|_\infty \;\le\; C(t-s)^{-3/4}\, s^{-3/4}\, \|u\|_Z \, \|v\|_Z\,. \tag{21.7}$$

This can be integrated over $[0,t]$ and we therefore obtain $\|w(\cdot,t)\|_\infty \le Ct^{-1/2}\,\|u\|_Z\,\|v\|_Z$.

The continuity of $B(u,v)$ with respect to the time variable is treated as in section 18 and we obtain

Theorem 21.1. *If $E = M^2(\mathbb{R}^3)$, there exists a positive number η such that for every $u_0 \in E$ with $\|u_0\|_E < \eta$, there exists a solution*

$$u(x,t) \;\in\; C([0,\infty); E) \tag{21.8}$$

of Navier-Stokes equations which satisfies

$$u(x,0) \;=\; u_0(x) \tag{21.9}$$

$$\sup_{t\ge 0} \left(\|u(\cdot,t)\|_E + t^{1/2}\, \|u(\cdot,t)\|_\infty \right) \;\le\; C\, \|u_0\|_E\,. \tag{21.10}$$

Moreover there exists a second positive constant $\beta > 2\eta$ such that this solution is uniquely defined by

$$\sup_{t\ge 0} \left(\|u(\cdot,t)\|_E + t^{1/2}\, \|u(\cdot,t)\|_\infty \right) \;<\; \beta\,.$$

This section will be concluded with a remark concerning the preceding examples. For defining mild solutions, we needed to give a specific meaning to the pointwise product $u_j(s)\, u(s)$ between two components of the velocity field.

Either we were assuming that the Banach space E was fully adapted to Navier-Stokes equations. Then we could write $fg = \Lambda h$ for any two functions f, g in E and we knew that h belongs to E. This was the basic property on which the continuity of the bilinear operator B was grounded.

Or we were assuming that E was defined by size conditions on f. This implied $\|fg\|_E \le \|f\|_E\, \|g\|_\infty$ and this observation was used when $E = L^3(\mathbb{R}^3)$ or $E = M^2(\mathbb{R}^3)$.

We now consider a third option and this leads us to the next section.

22 Kato's algorithm revisited

In a joint work with M.A. Muschietti, we planned to apply T. Kato's algorithm to Banach spaces which are defined by regularity conditions instead of size conditions. In such contexts it is no longer possible to multiply a function in E with a function in L^∞. In other words, L^∞ should be replaced by a Banach algebra A of pointwise multipliers of functions of E.

This Banach algebra is given the operator norm of the corresponding multiplier. In other words

$$\|m\|_A = \sup\left\{\|m(x)\, f(x)\|_E\, ;\ \|f\|_E \le 1\right\}. \tag{22.1}$$

The relation between A and E is specified by the following assumption.

Definition 20. *Let E be a functional Banach space. Let us assume that E is adapted to the Navier-Stokes equations. Let us assume that for $f \in E$ and $g \in E$ we have*

$$\|fg\|_E \le C\big(\|f\|_\infty + \|\Lambda f\|_E\big)\, \|g\|_E\, . \tag{22.2}$$

We finally assume that the Riesz transformations are bounded on E. We then say that E is adapted to Kato's algorithm.

Property (22.2) can be rephrased into

$$\|f\|_A \le C\big(\|f\|_\infty + \|\Lambda f\|_E\big) \tag{22.3}$$

where $\|f\|_A$ is the pointwise multiplier norm. We then have

Theorem 22.1. *Let us assume that the Banach space E is adapted to Kato's algorithm.*

Then there exists a positive number η such that for every $u_0 \in E$ with

$$\|u_0\|_E < \eta \quad , \quad \operatorname{div} u_0 = 0 \tag{22.4}$$

there exists a mild solution $u \in C([0,\infty); E)$ to the Navier-Stokes equations such that $u(x,0) = u_0(x)$.

This solution satisfies

$$\sup_{t\geq 0} \left(\|u(\cdot,t)\|_E + t^{1/2}\,\|u(\cdot,t)\|_A\right) \leq C\,\|u_0\|_E \tag{22.5}$$

where $C = C(E)$ is a constant.

Finally there exists a second positive number $\beta > 2\eta$ such that this solution is uniquely defined by

$$\sup_{t\geq 0} \left(\|u(\cdot,t)\|_E + t^{1/2}\,\|u(\cdot,t)\|_A\right) < \beta\,.$$

The proof of theorem 22.1 is similar in spirit to Kato's approach.

Let us define the Z-norm of $u(\cdot,t)$ as

$$\|u\|_Z = \sup_{t\geq 0} \left\{\|u(\cdot,t)\|_E + t^{1/2}\,\|u(\cdot,t)\|_A\right\} \tag{22.6}$$

and show the following properties

$$\|B(u,v)\|_Z \leq C\,\|u\|_Z\,\|v\|_Z \tag{22.7}$$

$$\|S(t)u_0\|_Z \leq C\,\|u_0\|_E\,. \tag{22.8}$$

Then lemma 20 will be applied and theorem 22.1 will follow.

The proof of (22.261) mimics the one we gave when $E = L^3(\mathbb{R}^3)$. Indeed

$$\begin{aligned}\|B(u,v)\|_E &\leq C\int_0^t \|S(t-s)\,\partial_j(u_jv)\|_E\,ds\\ &\leq C'\left(\int_0^t (t-s)^{-1/2}s^{-1/2}\,ds\right)\|u\|_Z\,\|v\|_Z = C'\pi\,\|u\|_Z\,\|v\|_Z\,.\end{aligned}$$

On the other hand $\|S(t-s)\,\partial_j(u_jv)\|_A \leq C(t-s)^{-1/2}\,s^{-1}\,\|u\|_Z\,\|v\|_Z$ since A is a Banach algebra.

For obtaining an other estimate, we apply (22.257). We have $\|S(t-s)\,\partial_j(u_ju)\|_\infty \leq C(t-s)^{-1}\,\|u_jv\|_E \leq C(t-s)^{-1}\,s^{-1/2}\,\|u\|_Z\,\|v\|_Z$. We obtained the first estimate by lemma 9 and the second by $\|u_j v\|_E \leq \|u\|_A \|v\|_E$. On the other hand $\|\Lambda S(t-s)\,\partial_j(u_j u)\|_E \leq C(t-s)^{-1}\|u_j v\|_E \leq C(t-s)^{-1}$ $s^{-1/2}\|u\|_Z\|v\|_Z$.

Then (22.3) yields $\|S(t-s)\,\partial_j(u_j v)\|_A \leq C(t-s)^{-1}s^{-1/2}\|u\|_Z\|v\|_Z$. This, together with the first estimate, implies

$$\|S(t-s)\,\partial_j(u_j v)\|_A \leq C(t-s)^{-3/4}s^{-3/4}\|u\|_Z\|v\|_Z \tag{22.9}$$

which yields $\|B(u,v)\|_A \leq Ct^{-1/2}\|u\|_Z\|v\|_Z$.

Everything runs smoothly and it suffices to study the linear evolution $S(t)u_0$.

We apply (22.3) to bound $\|S(t)u_0\|_A$. We need to bound $\|S(t)u_0\|_\infty$ together with $\|\Lambda S(t)u_0\|_E$. Everything works as above and we obtain the required estimate.

For later use we state the following remark.

Lemma 26. *If $m(x)$ belongs to the multiplier algebra A, so does $m(\lambda x)$ for $\lambda > 0$ and we have*

$$\|m(\lambda x)\|_A = \|m(x)\|_A . \tag{22.10}$$

Indeed $\|m(\lambda x)\|_A = \sup\left\{\|m(\lambda x)\,f(x)\|_E\,;\ \|f\|_E \leq 1\right\}$. We write $f(x) = \lambda g(\lambda x)$ and $\|f\|_E \leq 1$ is equivalent to $\|g\|_E \leq 1$. Then

$$m(\lambda x)\,f(x) = \lambda m(\lambda x)\,g(\lambda x)$$

implies $\|m(\lambda x)f(x)\|_E = \|m(x)\,g(x)\|_E$. This immediately implies (22.10).

23 Improved regularity of solutions of Navier-Stokes equations

We now answer the problem of obtaining an improved regularity for mild solutions to Navier-Stokes equations. This improved regularity only concerns the behavior of $u(x,t)$ for $t > 0$. We return to the definition of the Banach space Z which is used to prove the convergence of the iterative scheme. We now modify the definition of the Z-norm and write

(23.1)
$$\|u(\cdot,t)\|_Z = \sup_{t\geq 0}\left\{\|u(\cdot,t)\|_E + t^{1/2}\,\|\Lambda u(\cdot,t)\|_E + t^{1/2}\,\|u(\cdot,t)\|_A\right\}.$$

We already proved that the bilinear operator B is continuous for this modified Z-norm. Indeed $t^{1/2}\,\|\Lambda u(\cdot,t)\|_E$ was used for controlling $t^{1/2}\,\|u(\cdot,t)\|_A$. Then the proof of theorem 22.1 tells us the following : the solution which is built through the iterative scheme satisfies $t^{1/2}\,\|\Lambda u(\cdot,t)\|_E \leq C < \infty$.

We now check that this improved regularity in the x variable implies an improved regularity in the t variable. Indeed we have

Lemma 27. *For $0 \leq \alpha \leq 1$ and $t \geq 0$ we have*

$$\|(S(t)-I)f\|_E \leq C\,t^\alpha\,\|\Lambda^{2\alpha} f\|_E \tag{23.2}$$

for every f in E.

On the Fourier transform side, it amounts to checking that $\frac{e^{-t|\xi|^2}-1}{t^\alpha\,|\xi|^{2\alpha}}$ is the Fourier transform of $\frac{1}{t^{3/2}}\,g_\alpha\big(\frac{x}{\sqrt{t}}\big)$ with $g_\alpha \in L^1(\mathbb{R}^3)$. This is trivial when $0 < \alpha \leq 1$ and lemma is trivial if $\alpha = 0$.

Let us study the regularity in t of the bilinear operator. If $t' \geq t$, we need to estimate the difference

$$\begin{aligned}
&\int_0^{t'} S(t'-s)\,\partial_j(u_j u)\,ds - \int_0^t S(t-s)\,\partial_j(u_j u)\,ds \\
&\quad = \int_t^{t'} S(t'-s)\,\partial_j(u_j u)\,ds + \int_0^t \{S(t'-s) - S(t-s)\}\,\partial_j(u_j u)\,ds \\
&\quad = U + V\,.
\end{aligned}$$

We obviously have $\|U\|_E \leq C\int_t^{t'}(t'-s)^{-1/2}\,\|u_j u\|_E\,ds \leq C'\big(\int_t^{t'}(t'-s)^{-1/2}\cdot s^{-1/2}\,ds\big)\,|u|_Z^2$. Finally we obtain

$$\|U\|_E \leq C'(t'-t)^{1/2}\,t^{-1/2}\,\|u\|_Z^2\,.$$

For controlling $\|V\|_E$, we write

$$S(t'-s) - S(t-s) = [S(t'-t) - I]\,S(t-s)\,. \tag{23.3}$$

We want to apply lemma 27 with $\alpha = 1/2$. We are led to estimating

$$\left\|\int_0^t \Lambda S(t-s)\,\partial_j(u_j u)\,ds\right\|_E\,. \tag{23.4}$$

A key observation is the following estimate

$$\|\partial_j(u_j u)\|_E \leq \frac{C}{s}\,\|u\|_Z^2\,. \tag{23.5}$$

Indeed Leibniz' rule is applied and $\|u_j \partial_j u_k\|_E$ is estimated by $\|u_j\|_A \|\partial_j u_k\|_E$. We then apply our new definition of the Z-norm and we use the identity $-i\partial_j = R_j \Lambda$ where R_j is the Riesz transformation. All together we obtain (23.5).

Returning to $\|\Lambda S(t-s)\,\partial_j(u_j u)\|_E$ we either use (23.5) and obtain $C(t-s)^{-1/2} s^{-1} \|u\|_Z^2$ or group together $\Lambda S(t-s)\partial_j$. This second option yields $C(t-s)^{-1} \|u_j u\|_E \le C'(t-s)^{-1} s^{-1/2} \|u\|_Z^2$. Altogether these two estimates yield $\|\Lambda S(t-s)\,\partial_j(u_j u)\|_E \le C(t-s)^{-3/4} s^{-3/4} \|u\|_Z^2$ and

$$\left\| \int_0^t \Lambda S(t-s)\,\partial_j(u_j u)\,ds \right\|_E \le C t^{-1/2} \|u\|_Z^2 . \tag{23.6}$$

We have proved the following theorem

Theorem 23.1. *Let us assume that the functional Banach space E is adapted to Kato's algorithm. Then the solution $u(x,t)$ to the Navier-Stokes equations which is constructed in theorem 22.1 satisfies*

$$\|u(x,t') - u(x,t)\|_E \le C(u_0)(t'-t)^{1/2}\, t^{-1/2} \tag{23.7}$$

if $t' \ge t$.

This estimate is only interesting if $t \le t' \le 2t$.

We now combine theorem 23.1 to theorem 20.1. They imply the following

Theorem 23.2. *There exists a positive number η with the following property. For every $u_0(x)$ in $L^3(\mathbb{R}^3)$ with $\|u_0\|_3 < \eta$, there exists a unique mild solution $u(x,t) \in C([0,\infty); L^3(\mathbb{R}^3))$ of Navier-Stokes equations such that $u(x,0) = u_0(x)$. Moreover this solution satisfies the improved regularity property*

$$\|u(x,t') - u(x,t)\|_3 \le C \left| \frac{t'}{t} - 1 \right|^{1/2} \tag{23.8}$$

if $0 < t < t' < 2t$.

The constant C which appears in (23.8) only depends on $\|u_0\|_3$.

24 Examples of Banach spaces which are adapted to Kato's algorithm

The first group of examples is trivial. If E is a functional Banach space and if any function $m(x)$ which belongs to $L^\infty(\mathbb{R}^3)$ is a pointwise multiplier of E, then E is adapted to Kato's algorithm.

This remark applies to $L^3(\mathbb{R}^3)$, $L^{3,\infty}(\mathbb{R}^3)$ and more generally to the Lorentz spaces $L^{3,q}(\mathbb{R}^3)$ for $1 \le q \le \infty$. Another example is the Morrey-Campanato space M^2.

The second group consists of the Besov spaces $B_q = \dot{B}_q^{\alpha,\infty}$ where $\alpha = 3/q - 1$. But here we will not demand the condition $1 \le q < 3$. Indeed we have

Theorem 24.1. *If $1 \le q < 6$, the homogeneous Besov space $\dot{B}_q^{\alpha,\infty}$ is adapted to Kato's algorithm.*

We need to show that $m(x) \in L^\infty(\mathbb{R}^3)$ and $\Lambda m \in B_q$ imply that $m(x)$ is a pointwise multiplier for B_q. The paraproduct algorithm is used and we obtain

$$m(x)\, f(x) \;=\; A(x) + B(x) + C(x)$$

where

$$\begin{aligned} A(x) &= \sum_{-\infty}^{\infty} S_{j-2}(m)\, \Delta_j(f) \\ B(x) &= \sum_{-\infty}^{\infty} S_{j-2}(f)\, \Delta_j(m) \end{aligned}$$

and

$$C(x) \;=\; \sum\sum_{|j'-j|\le 2} \Delta_j(m)\, \Delta_{j'}(f)\,.$$

Concerning $A(x)$ everything is trivial since $\|S_j(m)\|_\infty \le C\|m\|_\infty$. The series is a Littlewood-Paley expansion and A belongs to B_q.

The second series is similar. We have $\|S_j(f)\|_\infty \le C2^j\|f\|_{B_q}$ while $\|\Delta_j(m)\|_q \le C\, 2^{-3j/q}$ follows from $\Lambda m \in B_q$.

As always the devil is hidden in the third series. We first begin with the trivial case when $1 \le q < 3$. Then $\|\Delta_j(m)\, \Delta_{j'}(f)\|_q \le C\|m\|_\infty\, \|\Delta_{j'}(f)\|_q \le C\|m\|_\infty\, 2^{j(1-3/q)}$ since $|j' - j| \le 2$. Finally lemma 18 applies and yields $C \in B_q$.

When $3 \le q < 6$, this approach fails and $\Lambda m \in B_q$ is needed. Indeed we have $\|\Delta_j(m)\|_q \le C\, 2^{-3j/q}$ which implies $\|\Delta_j(m)\, \Delta_{j'}(f)\|_{q/2} \le C\, 2^{j(1-6/q)}$.

Since $q < 6$, lemma 18 can be used and yields $C(x) \in \dot{B}_{q/2}^{6/q-1,\infty}$. Now it is trivial to check that $\dot{B}_{q/2}^{6/q-1,\infty}$ is contained in $\dot{B}_q^{3/q-1,\infty}$ for $1 \le q \le \infty$. The same proof works as well if $\dot{B}_1^{2,\infty}$ is replaced by the minimal space $\dot{B}_1^{2,1}$.

Let us now observe that theorem 23.2 is sharp. Indeed $\dot{B}_6^{-1/2,\infty}$ is not adapted to Kato's algorithm.

Here is a simple counter-example. We define $m(x) = \left(\sum_0^\infty 2^{-j}\, e^{i4^j x_1}\right)\varphi(x)$ where the Fourier transform of φ is supported by $|\xi| \le 1/2$. We then have $m(x) \in L^\infty(\mathbb{R}^3)$ and $\Lambda m \in \dot{B}_q^{-1/2,\infty}$ for $1 \le q \le \infty$. On the other hand $f(x) = \left(\sum_0^\infty 2^j\, e^{-i4^j x_1}\right)\varphi(x)$ belongs to $\dot{B}_6^{-1/2,\infty}$ (indeed to $\dot{B}_q^{-1/2,\infty}$ for $1 \le q \le \infty$) while $m(x)\, f(x)$ does not belong to $\mathcal{S}'(\mathbb{R}^3)$. In fact the paraproduct algorithm applies to $m(x)\, f(x)$. The series which were denoted by $A(x)$ and $B(x)$ trivially belong to $\dot{B}_6^{-1/2,\infty}$ while $C(x) = \sum_0^\infty \varphi^2(x)$ does not converge in the distributional sense.

Does it mean that Navier-Stokes equations cannot be solved through Kato's algorithm when $E = \dot{B}_6^{-1/2,\infty}$? M. Cannone addressed this issue [12] and proposed the following solution.

The Banach space Z is now defined by the following two conditions

$$\sup_{t\ge 0} \|u(\cdot,t)\|_E < \infty \tag{24.1}$$

together with

$$\sup_{t\ge 0} t^{1/4}\, \|u(\cdot,t)\|_6 < \infty\,. \tag{24.2}$$

It is then an easy exercise to check the following two estimates on the bilinear term. We do have

$$\|B(u,v)\|_3 \le C\, \|u\|_Z\, \|v\|_Z \tag{24.3}$$

and

$$t^{1/4}\, \|B(u,v)\|_6 \le C\, \|u\|_Z\, \|v\|_Z\,. \tag{24.4}$$

We then observe that $L^3(\mathbb{R}^3)$ is continuously embedded in $\dot{B}_6^{-1/2,\infty}$ and lemma 20 can be applied.

Returning to theorem 22.1, an interesting application is given by the homogeneous Sobolev space $\dot{H}^{1/2}$ which coincides with the homogeneous Besov space $\dot{B}_2^{1/2,2}$.

If $m(x)$ belongs to $L^\infty(\mathbb{R}^3)$, the series of the series $A(x) = \sum_{-\infty}^\infty S_{j-2}(m)\cdot \Delta_j(f)$ belongs to $\dot{H}^s(\mathbb{R}^3)$ whenever f does. The value of s is irrelevant.

On the other hand if Λm belongs to $\dot{H}^{1/2}$, then m belongs to $\dot{H}^{3/2}(\mathbb{R}^3)$ and

$$B(x) = \int_{-\infty}^{\infty} S_{j-2}(f)\,\Delta_j(m) \qquad \text{belongs to} \quad H^{1/2}.$$

Indeed we have $\|S_j(f)\|_\infty \leq C\,2^j$ and $\|\Delta_j(m)\|_2 \leq \varepsilon_j\,2^{-3/2j}$ with $\sum_{-\infty}^{\infty} \varepsilon_j^2 < \infty$.

The third series is $C(x) = \sum\sum_{|j'-j|\leq 2} \Delta_j(f)\,\Delta_{j'}(m)$. We have $\|\Delta_j(f)\cdot \Delta_{j'}(m)\|_1 \leq \eta_j\,2^{-2j}$ where $\eta_j \in \ell^1(\mathbf{Z})$. Lemma 18 can therefore be applied and yields $C(x) \in \dot{B}_1^{2,1}$ which is the minimal Banach space described in section 8.

Since $\dot{H}^{1/2}$ is embedded into $L^3(\mathbb{R}^3)$ the proof we gave for the uniqueness of mild solutions applies as well. We therefore obtain the following

Theorem 24.2. *There exists a positive number η with the following property:*

> *for each $u_0(x)$ in the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3)$ fulfilling $\|u_0\|_{\dot{H}^{1/2}} < \eta$ there exists a unique global solution $u(x,t) \in C\big([0,\infty); \dot{H}^{1/2}(\mathbb{R}^3)\big)$ of Navier-Stokes equations such that $u(x,0) = u_0(x)$.*

We obtain a similar statement if $\dot{H}^{1/2}(\mathbb{R}^3)$ is replaced by the minimal Banach space $\dot{B}_1^{2,1}$.

This minimal space is interesting since it answers a natural question concerning regularity. The largest amount of regularity which can be discussed inside the framework of adapted Banach spaces is precisely defined by the $\dot{B}_1^{2,1}$ norm. We cannot go further and deal with three derivatives. The corresponding Banach space would not be adapted to Navier-Stokes equations.

On the other hand it is fortunate that our modification of Kato's algorithm applies to this minimal space $\dot{B}_1^{2,1}$.

It is also clear that a wavelet analysis yields the best understanding of this minimal space $\dot{B}_1^{2,1}$. Indeed wavelets provide us with the canonical isomorphism between $\dot{B}_1^{2,1}$ and ℓ^1.

From these remarks one can argue that wavelets still have some links with Navier-Stokes equations.

25 Self-similar solutions of Navier-Stokes equations

The existence of self-similar solutions of the Navier-Stokes equations is a remarkable application of the results which were presented in the preceding section.

A Banach space E is assumed to be adapted to Kato's algorithm and is also assumed to contain non-trivial homogeneous functions of degree -1. An example is given by $E = L^{3,\infty}$ or $E = \dot{B}_1^{2,\infty}$ or by the Morrey-Campanato space M^2.

A self-similar solution of Navier-Stokes equations is a solution $u \in C([0,\infty); E)$ which is invariant under the canonical rescaling. In other words

$$u(x,t) = \lambda u(\lambda x, \lambda^2 t)\,, \qquad 0 < \lambda < \infty\,. \tag{25.1}$$

It implies

$$u(x,t) = \frac{1}{\sqrt{t}} U\Big(\frac{x}{\sqrt{t}}\Big) \quad , \quad U(x) = u(x,1)\,. \tag{25.2}$$

We then necessarily have

$$\lambda u_0(\lambda x) = \lim_{t \downarrow 0} \lambda u(\lambda x, \lambda^2 t) = u_0(x)\,. \tag{25.3}$$

Let us conversely assume $u_0 \in E$, $\lambda u_0(\lambda x) = u_0(x)$, $0 < \lambda < \infty$ and $\|u_0\|_E < \eta$ where η is a positive constant whose value will be given in the proof.

We want to show that the corresponding solution of Navier-Stokes equations which is given by theorem 22.1 is indeed a self-similar solution.

Let us denote by $u_\lambda(x,t)$ the function $\lambda u(\lambda x, \lambda^2 t)$. We already know that $u_\lambda(x,t)$ is a solution of the Navier-Stokes equations. Next we observe that $u_\lambda(x,0) = u(x,0) = u_0(x)$.

Finally we need to show that u_λ satisfies (22.5) when u does. Indeed $\sup_{t\geq 0} \|u_\lambda(\cdot,t)\|_E = \sup_{t\geq 0} \|u(\cdot,t)\|_E$ while, for every $m \in A$

$$\|m(\lambda x)\|_A = \|m(x)\|_A\,. \tag{25.4}$$

This immediately yields

$$\begin{aligned}\sup_{t\geq 0} t^{1/2} \|u_\lambda(\cdot,t)\|_A &= \sup_{t\geq 0} \lambda t^{1/2} \|u(\lambda x, \lambda^2 t)\|_A \\ &= \sup_{t\geq 0} t^{1/2} \|u(\lambda x, t)\|_A = \sup_{t\geq 0} t^{1/2} \|u(x,t)\|_A\,.\end{aligned}$$

Our claim is now proved and the uniqueness of the solution constructed in theorem 22.1 yields $u_\lambda = u$. We just proved the following result.

Theorem 25.1. *Let us assume that a functional Banach space E is adapted to Kato's algorithm as indicated in definition 20.*

If an initial condition $u_0(x)$ belongs to E with $\|u_0\|_E < \eta$ and satisfies the homogeneity condition

$$\lambda u_0(\lambda x) = u_0(x)\,, \qquad 0 < \lambda < \infty\,, \tag{25.5}$$

then the corresponding solution $u(x,t)$ of Navier-Stokes equations is a self-similar solution.

As it was specified, $\eta > 0$ is the constant that appears in theorem 22.1 and the solution which is here refered to is the unique solution of Navier-Stokes equation described in theorem 22.1.

This general theorem covers and extends some of the *ad-hoc* results which were obtained in [12] or [13].

Let us be more specific about a special case. If E is the Lorentz space $L^{3,\infty}(\mathbb{R}^3)$, then the sophistication of Kato's algorithm which is given in section 22 is not needed. Theorem 18.2 implies the existence of self-similar solutions f in $L^{3,\infty}(\mathbb{R}^3)$.

Moreover the subspace of $L^{3,\infty}(\mathbb{R}^3)$ which is defined by $f(\lambda x) = \lambda^{-1} \cdot f(\lambda x)$, $0 < \lambda < \infty$, is extremely simple. Indeed $f(x)$ belongs to this subspace if and only if the restriction of $f(x)$ to the unit sphere S^2 belongs to $L^3(S^2)$.

This means that the search for self-similar solutions belonging to $C([0,\infty); L^{3,\infty}(\mathbb{R}^3))$ is a simple and natural problem which can be treated in a self-consistent approach.

There is however an interesting construction of self-similar solutions which is not covered by our general approach.

Indeed in [13] we fix a large integer m and consider the homogeneous space $\dot{E}_m$ defined by $|\partial^\alpha f(x)| \le C|x|^{-1-|\alpha|}$, $|\alpha| \le m$. This Banach space is not translation invariant and cannot be incorporated inside our general approach. However if $u_0(x)$ satisfies $\lambda u_0(\lambda x) = u_0(x)$, $0 < \lambda < \infty$, and if the norm of $u_0(x)$ in $\dot{E}_m$ is small, then the corresponding solution $u(x,t)$ of Navier-Stokes equations belongs to $C([0,\infty); \dot{E}_m)$.

This theorem which is proved in [13] cannot be covered by our general theory. Indeed inside this general theory, the regularity cannot exceed 2 since $\dot{B}_1^{2,1}(\mathbb{R}^3)$ is the minimal adapted space.

26 Uniqueness of mild solutions to Navier-Stokes equations

We return to the remarkable theorem by G. Furioli, P.G. Lemarié-Rieusset and E. Terraneo.

In section 20 was given a simple proof of this discovery (theorem 20.1).

Here we would like to present the original proof which is based on paraproduct estimates or spectral methods. In contrast our new proof depends on real variable estimates and we cannot state as we did before that Littlewood-Paley analysis is actually needed for proving uniqueness.

In this section, all the results of section 20 are forgotten and the proof of uniqueness starts from scratch.

Therefore section 20 does not exist and this section is viewed as a continuation of section 19. We begin with recalling what was proved in section 19.

We return to the case where the adapted Banach space E is $L^3(\mathbb{R}^3)$ and study mild solutions $u(x,t) \in C([0,\infty);E)$ to Navier-Stokes equations. Existence of such solutions was proved under the condition $\|u_0\|_3 < \eta$ and uniqueness under the condition

$$\sup_{t\geq 0}\left\{\|u(\cdot,t)\|_3 + t^{1/2}\,\|u(\cdot,t)\|_\infty\right\} < \beta \tag{26.1}$$

where $\beta > \eta$ is a small constant.

We still do not know if the assumption $\|u_0\|_3 < \eta$ is actually needed. But P.G. Lemarié-Rieusset proved uniqueness without assuming (26.1). We present here this remarkable theorem [35].

Theorem 26.1. *For $T > 0$, let $u(x,t)$ and $v(x,t)$ belong to $C([0,T];L^3)$, $L^3 = L^3(\mathbb{R}^3)$.*

If u and v are two mild solutions of Navier-Stokes equations such that $u(\cdot,0) = v(\cdot,0) = u_0(x)$, then $u(\cdot,t) = v(\cdot,t)$ for $0 \leq t \leq T$.

The proof relies on some new estimates on the bilinear operator $B(f,g)$.

Let E be the homogeneous Besov space $\dot{B}_2^{1/2,\infty}(\mathbb{R}^3)$. Then E is not embedded in $L^3(\mathbb{R}^3)$ and conversely $L^3(\mathbb{R}^3)$ is not embedded in E.

As it was stated before, the bilinear operator

$$B(f,g) = \int_0^t S(t-s)\,\Lambda\big[f(\cdot,s)\,g(\cdot,s)\big]\,ds$$

is not bounded from $Y \times Y$ into Y when $Y = C([0,\infty); L^3(\mathbb{R}^3))$.

However we have

Proposition 1. *There exists a constant C such that for any $f(x,t)$, any $g(x,t)$ and $t \geq 0$*

$$\|B(f,g)\|_E(t) \leq C \sup_{0\leq s\leq t} \|f(\cdot,s)\|_3 \sup_{0\leq s\leq t} \|g(\cdot,s)\|_3 \,. \tag{26.2}$$

This result will be completed with the following ones

Proposition 2. *With the same notations, we have*

$$\|B(f,g)\|_E(t) \leq C \sup_{0\leq s\leq t} \|f(\cdot,s)\|_3 \sup_{0\leq s\leq t} \|g(\cdot,s)\|_E \,. \tag{26.3}$$

Proposition 3. *If $3 \leq q < 6$, there exists a constant C_q such that*

$$\|B(f,g)\|_E(t) \leq C_q \sup_{0\leq s\leq t} \left\{ s^{1/2(1-3/q)} \|f(\cdot,s)\|_q \right\} \cdot \sup_{0\leq s\leq t} \|g(\cdot,s)\|_E \,. \tag{26.4}$$

Let us begin with proposition 1.

We first sketch the proof and then give a complete proof.

The main observation about the bilinear operator $B(f,g)$ concerns the nature of the divergence. As it was already checked many times, the integrand blows up as s reaches t. To get a better understanding of this divergence, one splits this integral into a series

$$\sum B_j(f,g) \quad \text{where} \quad B_j(f,g) = \int_{\{4^{-j-1}\leq t-s\leq 4^{-j}\}} S(t-s)\,\Lambda(fg)\,ds\,.$$

We now mimic $\Lambda S(t-s)$ by $2^j\Delta_j$, we pretend that $f(\cdot,s)$ and $g(\cdot,s)$ are constant functions as $4^{-j-1} \leq t-s \leq 4^{-j}$ and finally replace ds by the step size 4^{-j}.

With such claims, the bilinear operator is modelled by $\sum_{-\infty}^{\infty} 2^{-j}\,\Delta_j(f_j g_j)$ where $\sup_j \|f_j\|_3 < \infty$, $\sup_j \|g_j\|_3 < \infty$ and Δ_j is defined in the section devoted to the Littlewood-Paley analysis. We then have

Lemma 28. *If* $\|f_j\|_3 \le C_0$ *and* $\|g_j\|_3 \le C_0$, *then* $\sum_{-\infty}^{\infty} 2^{-j}\, \Delta_j(f_j g_j)$ *belongs to the homogeneous Besov space* $\dot{B}_{3/2}^{1,\infty}$.

The proof is trivial by lemma 18, section 15. Finally $\dot{B}_{3/2}^{1,\infty}$ is contained in $\dot{B}_2^{1/2,\infty}$.

The actual proof is following the same strategy. The integral which is defining $B(f,g)$ is split into a series of terms where $s \in [t-4^{-j}\,,\, t-4^{-j-1}]$ which yields $w = B(f,g) = \sum_{-\infty}^{\infty} w_j$. It is now easy to control the $L^{3/2}$ norm of w_j as well as the $L^{3/2}$-norm of the second derivatives $\partial^\alpha w_j$, $|\alpha| = 2$. Then one applies lemma 18 or one of its variants.

We would like to give an other proof which was found by M. Cannone. We start with the following observation.

Lemma 29. *On the homogeneous Besov space* $\dot{B}_3^{-1,1}$, *the norm* $\sum_{-\infty}^{\infty} 2^{-j} \cdot \|\Delta_j(f)\|_3$ *and the continuous counterpart* $\int_0^\infty \|\Lambda S(t) f\|_3\, dt$ *are equivalent ones.*

Indeed one writes $f = \Lambda g$ where $g \in \dot{B}_3^{0,1}$ and $\Lambda = (-\Delta)^{1/2}$. Then $\Lambda\, S(t) f = -t\Delta \exp(t\Delta)\, \frac{1}{t}$ and $\int_0^\infty \|\Lambda S(t) f\|_3\, dt = \int_0^\infty \|Q(t) g\|_3 \frac{dt}{t}$ where $Q(t) = -t\Delta \exp(t\Delta)$. We therefore obtain a classical definition of the norm of g in the homogeneous Besov space $\dot{B}_3^{0,1}$.

For computing the norm of $B(f,g)$ in $\dot{B}_{3/2}^{1,\infty}$, it suffices to estimate

$$\sup\left\{ < B(f,s), h > ;\ \|h\|_{\dot{B}_3^{-1,1}} \le 1 \right\}.$$

One writes $h(x,t) = \Lambda\, S(t) h$ and obtains

$$< B(f,g), h > = \int_0^t \int_{\mathbb{R}^3} h(x, t-s)\, f(x,s)\, g(x,s)\, dx\, ds\,.$$

One is then using lemma 29 for obtaining

$$\begin{aligned} |< B(f,g), h >| &\le \sup_{0\le s\le t} \|f(\cdot,s)\|_3 \sup_{0\le s\le t} \|g(\cdot,s)\|_3 \int_0^\infty \|h(\cdot,s)\|_3\, ds \\ &\le \sup_{0\le s\le t} \|f(\cdot,s)\|_3 \sup_{0\le s\le t} \|g(\cdot,s)\|_3 \end{aligned}$$

as announced.

The proof of proposition 2 is slightly more involved. Let us observe that proposition 2 is a special case of proposition 3 when $q = 3$. We now concentrate on proposition 3. We begin with rephrasing theorem 15.2 as a lemma.

Lemma 30. *If f and g are two functions defined on* $\mathbb{R}^3$, *we then have when* $3/2 < q < 6$

$$\|\Delta_j(fg)\|_2 \leq C_q 2^{j(3/q-1/2)} \|f\|_q \|g\|_{\dot{B}_2^{1/2,\infty}}.$$

For proving (26.4) we write $\Gamma(t) = \Lambda S(t)$ with $\Lambda = \sqrt{-\Delta}$ and we want to estimate

$$\sup_{j\in\mathbf{Z}} \left\{ 2^{j/2} \left\| \Delta_j \int_0^t \Gamma(t-s)\,[f(s)\,g(s)]\,ds \right\|_2 \right\}$$

where $f(s) = f(\cdot, s)$, $g(s) = g(\cdot, s)$.

We begin with a trivial observation. One trivially checks that $\Delta_j = \Delta_j(\Delta_{j-1} + \Delta_j + \Delta_{j+1})$.

We concentrate on Δ_j^2 since the two other terms are similar to this one.

This leads to writing $\Delta_j\Gamma(t) = \Gamma_j(t)\Delta_j$ where $\Gamma_j(t) = \Gamma(t)\Delta_j$. We need a trivial remark.

Lemma 31. *The operator norm of* $\Gamma_j(t) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ *does not exceed* $C\,2^j(1 + 4^j t)^{-2}$, $j \in \mathbf{Z}$, $t \geq 0$ *where C is a constant.*

This remark is trivial if we move to the Fourier transform side. The operator norm of $\Gamma_j(t)$ is the L^∞ norm of the corresponding multiplier which reads

$$|\xi| \exp\left(-t|\xi|^2\right) \hat{\psi}(2^{-j}\xi). \tag{26.5}$$

This L^∞ norm is better computed after performing a dilation $\xi \to 2^j\xi$ and the computation is trivial.

Returning to (26.4) we assume

$$\sup_{0\leq s\leq t} s^{1/2(1-3/q)} \|f(\cdot, s)\|_q \leq 1 \tag{26.6}$$

together with

$$\sup_{0\leq s\leq t} \|g(\cdot, s)\|_E \leq 1.$$

Then lemma 31 and proposition 3 yield

$$(26.7)\qquad \left\| \Delta_j \int_0^t \Gamma(t-s)\,[f(s)\,g(s)]\,ds \right\|_2 \leq C_q\, 2^{j(1/2+3/q)} \int_0^t [1+4^j(t-s)]^{-2}\, s^{-1/2(1-3/q)}\, ds\,.$$

Since $\alpha = \frac{1}{2}(1-\frac{3}{q}) \in [0,1)$, we have

$$\sup_{T\geq 0} \int_0^T [1+(T-\tau)^2]^{-1}\,\tau^{-\alpha}\,d\tau \;=\; C_\alpha \;<\; \infty\,.$$

This estimate is applied to (26.7) with $T = 4^j t$ and $\tau = 4^j s$. Then (26.4) is proved.

We now return to theorem 26.1.

The initial condition $u_0(x)$ belongs to $L^3(\mathbb{R}^3)$ but we do not assume $\|u_0\|_3 < \eta$ where $\eta > 0$ is the small positive number which played a crucial role in theorem 19.1 or in theorem 22.1.

We denote by $u(\cdot,t)$ and $v(\cdot,t)$ two mild solutions of Navier-Stokes equations. Both u and v belong to $C([0,T];L^3(\mathbb{R}^3))$ and $u(\cdot,0) = v(\cdot,0) = u_0(x)$.

We want to show that $u(\cdot,t) = v(\cdot,t)$ if $0 \leq t \leq T$. For proving this fact it suffices to show that there exists a positive τ such that $u(\cdot,t) = v(\cdot,t)$ if $0 \leq t \leq \tau$.

Indeed let T_0 be the upper bound of the set of $\tau > 0$ for which $u = v$ on $[0,\tau]$. Since u and v belong to $C([0,T];L^3(\mathbb{R}^3))$ we have $u = v$ on $[0,T_0]$.

We then consider $\tilde{u}(x,t) = u(x\,,\,T_0+t)$ and $\tilde{v}(x,t) = v(x\,,\,T_0+t)$. These new functions are still mild solutions of Navier-Stokes equations (13.2) if $T_0 < T$. We have $\tilde{v}(x,0) = \tilde{u}(x,0)$. Therefore there exists a positive τ such that $\tilde{u}(x,t) = \tilde{v}(x,t)$ if $0 \leq t \leq \tau$ which contradicts the definition of T_0. Therefore $T_0 = T$. For proving that such a positive τ exists, we write $v = u + w$ which yields

$$(26.8)\qquad w \;=\; B(u+w\,,\,u+w) - B(u,u) \;=\; B(v,w) + B(w,u)\,.$$

First we observe that $w(\cdot,t)$ belongs to $C([0,T];B)$ where $B = \dot{B}_2^{1/2,\infty}$. This follows from proposition 2. Let us warm the reader that $u(\cdot,t)$ does not belong to $C([0,T];B)$. Indeed the linear evolution $S(t)u_0$ cannot have this property since $L^3(\mathbb{R}^3)$ is not contained in B. This is a striking difference with our new proof where $L^{3,\infty}$ is used instead of B.

We then write $\alpha(t) = \|w(\cdot,t)\|_B$ and want to prove $\alpha(t) = 0$ on $[0,\tau]$.

Returning to (26.8), we further split the right-hand side into

$$(26.9) \qquad w(\cdot,t) = B\big(v-S(t)v_0, w\big) + B\big(s(t)v_0, w\big) + \\ +B\big(w, u-S(t)u_0\big) + B\big(w, S(t)u_0\big)$$

with $u_0 = v_0$.

We then define

$$(26.10) \qquad \varepsilon(t) = \|u-S(t)u_0\|_3 + \|v-S(t)v_0\|_3$$

and we know that $\lim_{t\downarrow 0}\varepsilon(t) = 0$ since u and v both belong to $C([0,T];L^3(\mathbb{R}^3))$. We similarly define

$$(26.11) \qquad \eta(t) = t^{1/8}\,\|S(t)u_0\|_4$$

and we know that $\eta(t) \le C\|u_0\|_3$ together with

$$(26.12) \qquad \lim_{t\downarrow 0}\eta(t) = 0\,.$$

Writing in a systematic way $\overline{f}(t) = \sup\{f(s)\,;\,0 \le s \le t\}$ when f is a non negative function, we want to prove

$$(26.13) \qquad \overline{\alpha}(t) \le C\,\overline{\alpha}(t)\,[\overline{\varepsilon}(t) + \overline{\eta}(t)]\,.$$

This obviously implies

$$(26.14) \qquad \alpha(t) = 0 \quad \text{on} \quad [0,\tau]$$

since $\alpha(t)$ is bounded and $\lim_{t\downarrow 0}\big(\overline{\varepsilon}(t) + \overline{\eta}(t)\big) = 0$. Returning to (26.9) it suffices to apply proposition 3 to each one of the four terms in (26.9).

27 Appendix : construction of a divergence-free wavelet basis

As it was explicitely stated in section 6, the first construction of a divergence-free orthonormal wavelet basis was achieved by G. Battle and P. Federbush. Then P.G. Lemarié-Rieusset built a new basis with better spectral properties. We closely follow Lemarié-Rieusset's paper [50] in this appendix.

We start with the classical construction of the wavelet basis of the form $2^{j/2}\psi(2^j t-k)$, $j\in\mathbf{Z}$, $k\in\mathbf{Z}$, $\psi\in\mathcal{S}(\mathbb{R})$.

For building this basis, one begins with a scaling function $\varphi\in\mathcal{S}(\mathbb{R})$ such that

$$\hat{\varphi}(\xi)=1 \quad \text{on} \quad \left[-\frac{2\pi}{3},\frac{2\pi}{3}\right] \tag{27.1}$$

$$\hat{\varphi}(\xi)=0 \quad \text{if} \quad |\xi|\geq\frac{4\pi}{3} \tag{27.2}$$

$$0\leq\hat{\varphi}(\xi)\leq 1 \quad , \quad \hat{\varphi}(\xi)=\hat{\varphi}(-\xi) \tag{27.3}$$

$$|\hat{\varphi}(\pi+\eta)|^2+|\hat{\varphi}(\pi-\eta)|^2=1\,, \qquad |\eta|\leq\frac{\pi}{3}\,. \tag{27.4}$$

These properties obviously imply

$$\sum_{-\infty}^{\infty}|\hat{\varphi}(\xi+2k\pi)|^2=1 \tag{27.5}$$

and $\varphi(t-k)$, $k\in\mathbf{Z}$, is an orthonormal sequence.

Moreover if V_0 denotes the closed linear span of this sequence $\varphi(t-k)$, $k\in\mathbf{Z}$, and if V_j is defined by

$$f(t)\in V_0 \iff f(2^j t)\in V_j \tag{27.6}$$

we have $V_0\subset V_1$ as (27.1) and (27.2) show.

Then $\bigcap_{-\infty}^{\infty}V_j=\{0\}$, $\bigcup_{-\infty}^{\infty}V_j$ is dense in $L^2(\mathbb{R})$ and one can define W_j as being the orthogonal complement of V_j inside V_{j+1}. We then have

$$f\in W_0 \iff f(2^j t)\in W_j \tag{27.7}$$

and there exists an orthonormal basis of W_0 of the form $\psi(t-k)$, $k\in\mathbf{Z}$. Here ψ belongs to the Schwartz class. Moreover $\hat{\psi}(\xi)=0$ if $|\xi|\leq 2\pi/3$ or $|\xi|\geq 8\pi/3$.

These properties immediately imply the following

$$2^{j/2}\psi(2^j t-k)\,, \quad j\in\mathbf{Z}\,,\ k\in\mathbf{Z} \tag{27.8}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

The first step in the construction is the following lemma

Lemma 32. *There exists an increasing sequence $\tilde V_j$ of closed subspaces of $L^2(\mathbb{R})$ such that*

$$(27.9) \qquad \bigcap_{-\infty}^{\infty} \tilde V_j = \{0\} \quad , \quad \bigcup_{-\infty}^{\infty} \tilde V_j \quad \textit{is dense in } L^2(\mathbb{R})$$

$$(27.10) \qquad f(x) \in \tilde V_j \iff f(2x) \in \tilde V_{j+1}$$

(27.11) *there exists $\tilde\varphi \in \mathcal{S}(\mathbb{R})$ such that $\tilde\varphi(x-k)$, $k \in \mathbf{Z}$, is a Riesz basis of $\tilde V_0$*

$$(27.12) \qquad \frac{d}{dx} : \tilde V_j \to V_j \quad \textit{is continuous with a dense range.}$$

This lemma will be completed with the following information.

Lemma 33. *There exists a closed subspace $\tilde W_j$ of $\tilde V_{j+1}$ such that $\tilde V_j + \tilde W_j = \tilde V_{j+1}$ and*

$$(27.13) \qquad \frac{d}{dx} : \tilde W_j \to W_j$$

is an isomorphism.

This construction is indeed simpler than expected. We firt define $\tilde\varphi(x) = \int_x^{x+1} \varphi(t)\,dt$ and lemma 32 is obviously checked.

Similarly $\tilde\psi(x)$ is defined by $\int_{-\infty}^{\infty} \psi(t)\,dt$ and lemma 33 follows.

For analyzing $L^2(\mathbb{R}^3)$ we have three options at our disposal. Option 1 is given by the multiresolution analysis $\tilde V_j \otimes V_j \otimes V_j$, $j \in \mathbf{Z}$. Option 2 is given by $V_j \otimes \tilde V_j \otimes V_j$, $j \in \mathbf{Z}$. Similarly option 3 is provided by $V_j \otimes V_j \otimes \tilde V_j$, $j \in \mathbf{Z}$.

When a vector field $u(x) = (u_1(x), u_2(x), u_3(x))$ will be analyzed, option 1 will be used for u_1, option 2 for u_2 and option 3 for u_3.

The frequency channels corresponding to option 1 are denoted bu $W_{j,\alpha}^{(1)}$. Similarly $W_{j,\alpha}^{(2)}$ corresponds to option 2 and $W_{j,\alpha}^{(3)}$ to option 3.

We need to explain the meaning of this index α. Here $\alpha \in A$ where A is the set $\{0,1\}^3 \setminus \{(0,0,0)\}$. In other words $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_j \in \{0,1\}$ and where $\alpha = (0,0,0)$ is omitted.

Finally $\alpha_1 = 0$ means that $W_{j,\alpha}^{(1)} = \tilde V_j \otimes \cdot \otimes \cdot$ while $\alpha_1 = 1$ means $W_{j,\alpha}^{(1)} = \tilde W_j \otimes \cdot \otimes \cdot$ and similarly for α_2 and α_3. For instance, if $\alpha = (0,1,1)$, then $W_{j,\alpha}^{(1)} = \tilde V_j \otimes W_j \otimes W_j$.

The same notations are used for $W_{j,\alpha}^{(2)}$ and $W_{j,\alpha}^{(3)}$.

Once these notations are being fixed, we return to a vector field $u(x) = (u_1(x), u_2(x), u_3(x))$ and decompose each component into the corresponding wavelet expansion. The first component is treated with the first option and so on. We then have

$$\begin{aligned} u_1(x) &= \sum_\alpha \sum_j w_{j,\alpha}^{(1)}(x) \qquad & w_{j,\alpha}^{(1)} \in W_{j,\alpha}^{(1)} \\ u_2(x) &= \sum_\alpha \sum_j w_{j,\alpha}^{(2)}(x) \qquad & w_{j,\alpha}^{(2)} \in W_{j,\alpha}^{(2)} \\ u_3(x) &= \sum_\alpha \sum_j w_{j,\alpha}^{(3)}(x) \qquad & w_{j,\alpha}^{(3)} \in W_{j,\alpha}^{(3)}. \end{aligned}$$

We now reach the crucial lemma.

Lemma 34. *If $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$ in the distributional sense, then*

(27.14)
$$\partial_1 w_{j,\alpha}^{(1)} + \partial_2 w_{j,\alpha}^{(2)} + \partial_3 w_{j,\alpha}^{(3)} = 0\,, \qquad j \in \mathbf{Z}\,,\ \alpha \in A\,.$$

In other words $\operatorname{div} u = 0$ is fully decoupled into frequency channels.

Let us prove this striking property. We denote by $W_{j,\alpha}$ the standard multiresolution frequency channels where the standard multiresolution is $V_j \otimes V_j \otimes V_j$.

Then $f \in W_{j,\alpha}^{(1)}$ implies $\partial_1 f \in W_{j,\alpha}$ by (27.12) or (27.13). Similarly $f \in W_{j,\alpha}^{(2)}$ implies $\partial_2 f \in W_{j,\alpha}$ and the same for the third option. If $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$ we therefore obtain

$$\sum_j \sum_\alpha \left(\partial_1 w_{j,\alpha}^{(1)} + \partial_2 w_{j,\alpha}^{(2)} + \partial_3 w_{j,\alpha}^{(3)}\right) = 0\,. \tag{27.15}$$

We then observe that $L^2(\mathbb{R}^3) = \bigoplus_{j,\alpha} W_{j,\alpha}$ where this sum is orthogonal. Therefore (27.15) is decoupled into (27.14).

The second remark we need is the following.

Lemma 35. *If $\alpha_1 = 1$, then*

$$\partial_1 = W_{j,\alpha}^{(1)} \to W_{j,\alpha} \tag{27.16}$$

is an isomorphism.

This obviously follows from (27.13).

We now construct a basis for the closed vector space $H_{(j,\alpha)} \subset W^{(j)}_{j,\alpha} \times W^{(2)}_{j,\alpha} \times W^{(3)}_{j,\alpha}$ defined by

$$\partial_1 w^{(1)}_{j,\alpha} + \partial_2 w^{(2)}_{j,\alpha} + \partial_3 w^{(3)}_{j,\alpha} = 0 . \tag{27.17}$$

Indeed $\alpha \neq (0,0,0)$. Therefore one of the three indices α_1, α_2 or α_3 is 1. We only treat the case $\alpha_1 = 1$ since the other two are fully similar ones.

In the first case (27.17) can be written

$$w^{(1)}_{j,\alpha} = T_1\left(\partial_2 w^{(2)}_{j,\alpha} + \partial_3 w^{(3)}_{j,\alpha}\right) . \tag{27.18}$$

The basis we are looking for is given by the following obvious lemma.

Lemma 36. *Let H_1 and H_2 be two Hilbert spaces and $T : H_1 \to H_2$ be a continuous linear mapping.*

Let $V \subset H_1 \times H_2$ be the graph of this mapping T.

Then for each Riesz basis e_j, $j \in J$, of H_1, the collection $\big(e_j, T(e_j)\big)_{j\in J}$ is a Riesz basis of V.

Coming back to (27.18) it suffices to treat the case $j = 0$ since everything is dilation invariant. If $j = 0$, everything is invariant under the $\mathbf{Z}^3$ group action. We are therefore led to constructing the "mother divergence-free wavelets". If, for instance, $\alpha = (1,0,0)$, we then either obtain

$$\left(-\tilde{\psi}(x_1)\left[\varphi(x_2+1) - \varphi(x_2)\right]\varphi(x_3)\, ,\, \psi(x_1)\,\tilde{\varphi}(x_2)\,\varphi(x_3)\, ,\, 0\right)$$

and

$$\left(-\tilde{\psi}(x_1)\,\varphi(x_2)\left[\varphi(x_3+1) - \varphi(x_3)\right] ,\, 0\, ,\, \psi(x_1)\,\varphi(x_2)\,\tilde{\varphi}(x_3)\right) .$$

Let us explain why. Since $\alpha_1 = 1$, $\partial_1 : W^{(1)}_{j,\alpha} \to W_{j,\alpha}$ is an isomorphism. This means that we can choose a basis for the second component $u_2(x)$ together with a basis for the third component $u_3(x)$ and then compute $u_1(x)$. The basis which is chosen for the second component is $\psi(x_1 - k_1)\,\tilde{\varphi}(x_2 - k_2)\,\varphi(x_3 - k_3)$, $k_1 \in \mathbf{Z}$, $k_2 \in \mathbf{Z}$, $k_3 \in \mathbf{Z}$. This agrees with lemma 32 and with the conventions we made about V_j vs $\tilde{V}_j$. Similarly for the third one.

We now need to solve $\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0$. Let us begin with $u_2(x) = \psi(x_1)\,\tilde{\varphi}(x_2)\,\varphi(x_3)$ and $u_3(x) = 0$. Then $\frac{d}{dx_2}\,\tilde{\varphi}(x_2) = \varphi(x_2+1) - \varphi(x_2)$ while

$\tilde{\psi}(x_1)$ is precisely defined by $\frac{d}{dx_1}\tilde{\psi}(x_1) = \psi(x_1)$. That is explaining the first row.

The explanation of the second row is fully similar. If $\alpha = (1,1,0)$, the construction rules allow two prossibilities. We can either compute $u_1(x)$ when u_2 and u_3 are given or compute $u_2(x)$ in terms of u_1 and u_3.

We stick to the first option. It yields

$$(-\tilde{\psi}(x_1)\,\psi(x_2)\,\varphi(x_3)\,,\,\psi(x_1)\,\tilde{\psi}(x_2)\,\varphi(x_3)\,,\,0)$$

and

$$\Big(-\tilde{\psi}(x_1)\,\psi(x_2)\big(\varphi(x_3+1)-\varphi(x_3)\big)\,,\,0\,,\,\psi(x_1)\,\psi(x_2)\,\tilde{\varphi}(x_3)\Big)\,.$$

If $\alpha = (1,1,1)$ we also compute $u_1(x)$. It yields

$$(-\tilde{\psi}(x_1)\,\psi(x_2)\,\psi(x_3)\,,\,\psi(x_1)\,\tilde{\psi}(x_2)\,\psi(x_3)\,,\,0)$$

and

$$(-\tilde{\psi}(x_1)\,\psi(x_2)\,\psi(x_3)\,,\,0\,,\,\psi(x_1)\,\psi(x_2)\,\tilde{\psi}(x_3))\,.$$

The other cases are fully similar to these ones. We then have two mother wavelets inside each frequency channel. These frequency channels are indexed by $\alpha \in A$. Since $\sharp A = 7$, it yields 14 mother wavelets as announced.

28 Appendix 2. Wavelets and the div-curl lemma

P.L. Lions conjectured the following :

Let $E(x) = (E_1(x), E_2(x), \ldots, E_n(x))$ and $B(x) = (B_1(x), B_2(x), \ldots, B_n(x))$ be two vector fields satisfying the following three conditions

(28.1) $\quad E_j \in L^2(\mathbb{R}^n) \;,\quad B_j \in L^2(\mathbb{R}^n) \;,\quad 1 \le j \le n$

(28.2) $\quad \operatorname{div} E(x) = 0$ in the distributional sense

(28.3) $\quad \operatorname{curl} B(x) = 0$ in the distributional sense.

Then $(E\cdot B)(x) = E_1(x)B_1(x) + \cdots + E_n(x)B_n(x)$ belongs to the Stein & Weiss space $\mathcal{H}^1(\mathbb{R}^n)$.

My first reaction to P.L. Lions' conjecture was : this cannot be true ! It is so great that it should have been known if it were true.

But the following night I proved this conjecture and I acknowledged I was too pessimistic.

Here I want to provide the reader with a much better proof together with a systematic treatment of bilinear operators generalizing this specific example. This new approach is a joint work with Sylvia Dobyinsky [18].

We start with the heat semi-group $P_t = \exp(t\Delta)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. We then write $Q_t = -t\frac{\partial}{\partial t}P_t = -t\Delta P_t$. Then to any $f \in L^2(\mathbb{R}^n)$, we associate the function $g(x)$ defined as

$$g(f)(x) = 2\left(\int_0^\infty |Q_t f(x)|^2 \frac{dt}{t}\right)^{1/2} \tag{28.4}$$

in full similarity with the celebrated Littlewood-Paley function.

A trivial calculation yields

$$\int_{\mathbb{R}^n} |f(x)|^2\, dx = \int_{\mathbb{R}^n} g^2(x)\, dx\,. \tag{28.5}$$

We then consider the difference $h(x) = |f(x)|^2 - g^2(x)$. We obviously have $\int h(x)\, dx = 0$.

But much more is true. Indeed $h(x)$ belongs to the Stein & Weiss space $\mathcal{H}^1(\mathbb{R}^n)$ whenever $f \in L^2(\mathbb{R}^n)$. We need the bilinear version of this fact. The pseudo-product $u\sharp v$ between $u \in L^2(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}^n)$ is defined as

$$u\sharp v(x) = 4\int_0^\infty Q_t(u)\, Q_t(v)\, \frac{dt}{t}\,. \tag{28.6}$$

We then have

Theorem 28.1. *There exists a constant $C(n)$ such that the difference between the product $u(x)\, v(x)$ and the pseudo-product $u\sharp v$ between two functions in $L^2(\mathbb{R}^n)$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$ and satisfies*

$$\|uv - u\sharp v\|_{\mathcal{H}^1} \le C(n)\, \|u\|_2\, \|v\|_2\,. \tag{28.7}$$

The proof of this fact is straightforward. A simple calculation yields

$$\begin{aligned} w(x) &= uv - u\sharp v \\ &= (2\pi)^{-2n} \iint \exp\big(ix\cdot(\xi+\eta)\big)\, \pi(\xi,\eta)\, \hat{u}(\xi)\, \hat{v}(\eta)\, d\xi\, d\eta \end{aligned} \tag{28.8}$$

where $\pi(\xi,\eta) = \left(\frac{|\xi|^2-|\eta|^2}{|\xi|^2+|\eta|^2}\right)^2$.

This bi-linear symbol π belongs to $C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0,0\})$. Moreover π is homogeneous of degree 0 and vanishes on $\xi + \eta = 0$. These three properties imply that the bilinear operator defined by (28.8) maps $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $\mathcal{H}^1(\mathbb{R}^n)$. A reference is [20] or [56].

We now return to the proof of P.L. Lions' conjecture. We need to prove the following

$$E_1(x)B_1(x) + \cdots + E_n(x)B_n(x) \in \mathcal{H}^1(\mathbb{R}^n) \tag{28.9}$$

whenever (28.1), (28.2) and (28.3) are satisfied.

We first replace each $E_j(x)B_j(x)$ by $E_j \sharp B_j$. The corresponding error term belongs to $\mathcal{H}^1(\mathbb{R}^n)$. We then treat the main terms by the following theorem.

Theorem 28.2. *If* (28.1), (28.2) *and* (28.3) *are satisfied, then the sum*

(28.10)

$$\sigma(x) = E_1 \sharp B_1 + \cdots + E_n \sharp B_n \quad \textit{belongs to} \quad \dot{B}_1^{0,1}.$$

This homogeneous Besov space is defined by the following three equivalent properties

$$\sum_{-\infty}^{\infty} \|\Delta_j(f)\|_1 = \|f\|_{\dot{B}_1^{0,1}} < \infty \tag{28.11}$$

where $\Delta_j(f)$, $j \in \mathbf{Z}$, are the dyadic blocks of a Littlewood-Paley decomposition

$$f(x) = \sum\sum \alpha(j,k)\, 2^{nj}\, \psi(2^j x - k) \tag{28.12}$$

where $\sum\sum |\alpha(j,k)| < \infty$, $2^{nj/2}\,\psi(2^j x - k)$ being an orthonormal wavelet basis with enough smoothness and vanishing moments. Indeed $2^n - 1$ mother wavelets are needed and the sum over ψ is omitted.

The third definition of $\dot{B}_1^{0,1}$ is given by

$$\int_0^\infty \|Q_t(f)\|_1 \frac{dt}{t} < \infty. \tag{28.13}$$

For simplifying the notations, the norm in $\dot{B}_1^{0,1}$ will be denoted by $\|\cdot\|$. All other norms will carry indices. We have

$$\sigma(x) = 4 \int_0^\infty Q_t(E) \cdot Q_t(B) \frac{dt}{t} \tag{28.14}$$

and by convexity of the ball of $\dot{B}_1^{0,1}$, it suffices to show the following

$$\int_0^\infty \|Q_t(E)\cdot Q_t(B)\|\,\frac{dt}{t} \le C\,\|E\|_2\,\|B\|_2 \tag{28.15}$$

when (28.1), (28.2) and (28.3) are satisfied.

For estimating $\|Q_t(E)\cdot Q_t(B)\|$, the following lemma will be used.

Lemma 37. *Let $f(x)$, $x\in\mathbb{R}^n$, be a function with the following two properties*

$$f=\operatorname{div}F \quad \textit{where} \quad F(x)=\big(F_1(x),\dots,F_n(x)\big) \tag{28.16}$$

and $\|F_j\|_1\le 1$, $1\le j\le n$

$$\partial_j f\in L^1(\mathbb{R}^n) \quad \textit{and} \quad \|\partial_j f\|_1\le 1\,,\ 1\le j\le n\,. \tag{28.17}$$

Then $\|f\|\le C(n)$.

The proof is straightforward. If $\sum_{-\infty}^{\infty}\Delta_j(f)$ is the Littlewood-Paley expansion of f, we estimate $\|\Delta_j(f)\|_1$ by $2^j\|F\|_1$. This first estimate is obtained by the Bernstein's inequality applied to $\Delta_j(F)$. Bernstein's inequality reads

$$\|\partial_j u\|_1 \le R\,\|u\|_1\,, \qquad 1\le j\le n \tag{28.18}$$

when the Fourier transform of u is carried by $|\xi|\le R$.

The second estimate is

$$\|\Delta_j(f)\|_1 \le 2^{-j}\big(\|\partial_1 f\|_1+\cdots+\|\partial_n f\|_1\big)\,. \tag{28.19}$$

It is obtained by writing

$$g = \Delta^{-1}\big(\partial_1(\partial_1 g)+\cdots+\partial_n(\partial_n g)\big) \tag{28.20}$$

and observing that

$$\|\Delta^{-1}(h)\|_1 \le R^{-2}\,\|h\|_1 \tag{28.21}$$

if the Fourier transform of h vanishes on $|\xi|\le R$.

Then Bernstein's inequality, (28.17) and (28.21) imply (28.19).

The conclusion of lemma can obviously be rewritten as

$$\|f\| \le C(n)\,\big[\|\nabla f\|_1+\|F\|_1\big]\,. \tag{28.22}$$

We now apply (28.22) to $f_t(x) = t^n f(tx)$ and observe that $\|f_t\| = \|f\|$. But $\|\nabla f_t\|_1 = t\,\|\nabla f\|_1$ while $\|F_t\|_1 = t^{-1}\,\|F\|_1$. By optimizing on $t > 0$, we obtain

$$\|f\| \le C(n)\big(\|\nabla f\|_1\,\|F\|_1\big)^{1/2}. \tag{28.23}$$

We now return to (28.15). The first remark is the following. Since $\operatorname{curl} B = 0$, we obtain

$$B(x) = \nabla U(x) \tag{28.24}$$

where U is scalar valued. Moreover $\|B_1\|_2 + \cdots + \|B_n\|_2$ and

$$\left(\int_0^\infty t^{-1}\|Q_t\,U\|_2^2\,\frac{dt}{t}\right)^{1/2} \tag{28.25}$$

are equivalent norms. Finally

$$Q_t(E)\cdot Q_t(B) = \operatorname{div} Q_t(E)\,Q_t(U). \tag{28.26}$$

After writing (28.26), we can drop out the fundamental assumptions (28.2) and (28.3). Only (28.1) is retained.

Using the obvious inequality $2\sqrt{ab} \le t^{-1/2}a + t^{1/2}b$, $a, b > 0$, together with Cauchy-Schwarz, (28.23) implies

$$\|Q_t(E)\cdot Q_t(B)\| \le A(t) + B(t) + C(t) \tag{28.27}$$

where

$$\begin{aligned} A(t) &= C\,t^{-1/2}\,\|Q_t(U)\|_2\,\|Q_t(E)\|_2 \\ B(t) &= C\,t^{1/2}\sum_1^n\sum_1^n \|\partial_j Q_t E_k\|_2\,\|Q_t B_k\|_2 \\ \text{and} \qquad C(t) &= C\,t^{1/2}\sum_1^n\sum_1^n \|Q_t E_k\|_2\,\|\partial_j Q_t B_k\|_2\,. \end{aligned}$$

We want to prove that both $A(t)$, $B(t)$ and $C(t)$ can be written as $p(t)\,q(t)$ where

$$\left(\int_0^\infty p^2(t)\,\frac{dt}{t}\right)^{1/2} \le C\,\|E\|_2 \tag{28.28}$$

and

$$\left(\int_0^\infty q^2(t)\,\frac{dt}{t}\right)^{1/2} \le C\,\|B\|_2\,. \tag{28.29}$$

This will suffice to end our proof.

Concerning $A(t)$, we have $p(t) = t^{-1/2}\,\|Q_t(U)\|_2$, $q(t) = \|Q_t(E)\|_2$ and we simply use (28.25). The treatment of $B(t)$ is not deeper. We have $p(t) = t^{1/2}\|\partial_j Q_t E_k\|_2$ and $q(t) = \|Q_t B_k\|_2$. Indeed $t^{1/2}\,\partial_j\,Q_t\,f = \tilde{Q}_t f$ where

$$\int_0^\infty \|\tilde{Q}_t\|_2^2\,\frac{dt}{t} \le C\,\|f\|_2^2 \tag{28.30}$$

as Plancherel identity shows.

The treatment of $C(t)$ is similar and left to the reader.

Conclusion.

We return to our fundamental issue. What would we benefit from using a divergence-free wavelet basis ?

The proof would begin the same way. The vector field $E(x)$ is expanded into a divergence free wavelet basis

$$E(x) = \sum\sum\sum \alpha(j,k)\,2^{nj/2}\,\psi(2^j x - k) \tag{28.31}$$

where the first sum runs over the 14 divergence free mother wavelets, the second over all scales and the third over $k \in \mathbf{Z}^3$.

To expand the second vector field $B(x)$ we need to use a dual wavelet basis. This means that instead of $\tilde{V}_j$, $\tilde{W}_j$ we need to use $V_j^\sharp$, $W_j^\sharp$ where $W_0^\sharp$ is defined by the condition

$$\frac{d}{dt} : W_0 \to W_0^\sharp$$

is an isomorphism.

Similarly $\Phi(t-k)$, $k \in \mathbf{Z}$, is a Riesz basis of $V_0^\sharp$ where $\Phi(t-1) - \Phi(t) = \varphi'(t)$.

With these notations, $B(x)$ is expanded into the same type of basis that was used for $E(x)$. We then have

$$B(x) = \sum\sum\sum \beta(j,k)\,2^{nj/2}\,\psi^\sharp(2^j x - k) \tag{28.32}$$

where here there are only 7 mother wavelets. Indeed $B = \nabla U$ where $U(x)$ is a scalar valued function. Then (28.31) and (28.32) will be simplified into

$$E(x) = \sum_j \sum_\alpha E_{j,\alpha}(x) \tag{28.33}$$

$$B(x) = \sum_j \sum_\alpha B_{j,\alpha}(x) \tag{28.34}$$

and we write

$$E(x) \cdot B(x) = \sum_j \sum_\alpha E_{j,\alpha}(x) \cdot B_{j,\alpha}(x) + \Gamma(x) \,. \tag{28.35}$$

It can be proved that $\Gamma(x)$ belongs to $\mathcal{H}^1(\mathbb{R}^n)$. This is not related to $\operatorname{div} E(x) = 0$, $\operatorname{curl} B(x) = 0$. We now turn to the main term

$$\sigma(x) = \sum_j \sum_\alpha E_{j,\alpha}(x) \cdot B_{j,\alpha}(x) \,. \tag{28.36}$$

We know from Lemarié-Rieusset's clever construction that

$$\operatorname{div} E_{j,\alpha}(x) = 0 \quad , \quad \operatorname{curl} B_{j,\alpha}(x) = 0 \,. \tag{28.37}$$

Then the proof we gave with a continuous formalism applies to this discrete formalism.

The conclusion is clear. Divergence-free wavelet bases do not pay.

29 Conclusion

We wanted to address a fundamental issue concerning the role played by wavelets or Littlewood-Paley analysis and paraproduct algorithms in solving Navier-Stokes equations.

There are indeed several issues. The first one concerns the mathematical theory of Navier-Stokes equations. The second issue which will be addressed is the role played by wavelets in the numerical simulation of Navier-Stokes equations.

The last problem addresses the role played by wavelets as a visualisation tool when one is working on experimental turbulence.

Let us first consider existence theorems. To the best of our understanding, the most powerful algorithm is the improvement on Kato's algorithm we developed with M.A. Muschietti.

This algorithm does not rely on wavelets or on Littlewood-Paley analysis. However if we want to apply this algorithm when the initial condition $u_0(x)$ is smooth, we are immediately entering the wavelet world. Indeed the largest amount of smoothness that can be imposed on $u_0(x)$ is defined by the minimal adapted Banach space E. This minimal Banach space is the homogeneous Besov space $\dot{B}_1^{2,1}(\mathbb{R}^3)$. This Banach space cannot be understood unless wavelet analysis is being used.

To be even more precise, let us assume that the initial condition $u_0(x)$ is a sum of a few normalized divergence-free wavelets. These wavelets are $2^j\psi(2^jx - k)$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}^3$, $\psi \in A$. Here A is a finite set consisting of 14 mother wavelets $\psi = (\psi_1, \psi_2, \psi_3)$ where ψ_1, ψ_2 and ψ_3 belong to the Schwartz class.

The coefficients of this expansion of $u_0(x)$ are denoted by $\alpha(j,k)$, $j \in \mathbf{Z}$, $k \in \mathbf{Z}^3$, and are assumed to belong to $\ell^1(\mathbf{Z}^4)$. Then this property of u_0 is preserved by the Navier-Stokes equations : uniformly in $t \geq 0$, $x \to u(x,t)$ is enjoying the same regularity whenever $\sum_j \sum_k |\alpha(j,k)| \leq \eta$.

These remarks imply that P. Federbush was right. Indeed "Navier and Stokes meet the wavelet" as it was announced by P. Federbush. But this meeting did not take place in the room which was arranged by P. Federbush. The Morrey-Campanato space $M^2(\mathbb{R}^3)$ is the largest space which is adapated to Navier-Stokes equations and P. Federbush failed in using a wavelet based Galerkin algorithm to construct mild solutions when $u_0(x) \in M^2(\mathbb{R}^3)$. Such a construction was given in these notes using Kato's approach to the problem.

We should remark that functions in $M^2(\mathbb{R}^3)$ admit a simple characterization by size properties of wavelet coefficients. However this characterization is not the best approach to $M^2(\mathbb{R}^3)$ which is more naturally defined by plain size estimates. This could be the reason why Kato's algorithm is so efficient in this situation while P. Federbush failed. It is possible that Federbush's approach would work in the context of the minimal space $\dot{B}_1^{2,1}(\mathbb{R}^3)$ for which a wavelet analysis is so natural.

Let us then turn to uniqueness of mild solutions. Here also we have to face bad news. It is clear that the original proof by P.G. Lemarié-Rieusset and his collaborators is based on Besov spaces and paraproduct algorithms. But this proof nowadays appears as awkward as compared to the proof based on the "real variable methods" developed by A. Calderón and A. Zygmund.

Does it mean that real variable methods are always winning against spectral methods when applied to Navier-Stokes equations ?

This is certainly not the case. Indeed mathematicians are incorporating wavelet analysis inside a broader context which is named "micro-local analysis".

A.P. Calderón announced in 1965 that a better understanding of algebras of pseudo-differential coefficients with minimal smoothness assumptions should have a tremendous impact on non-linear PDE's.

Then Calderón proposed a collection of entirely new operators. It was a challenging task to prove the boundedness of these operators. Today the best proofs are based on a wavelet analysis.

This means that wavelet analysis is obviously playing an important role in Calderón's program.

J.M. Bony reshaped Calderón's program and launched the celebrated para-differential calculus. The goal is the same as in Calderón's approach. It consists in including in the calculus the pointwise multiplications by non-smooth functions.

As indicated in [56], there exists a version of the paraproduct operator $\pi(a, f)$ which is diagonal in a wavelet basis.

This does not mean that wavelets are playing a key role in Bony's program. However Bony's program is not antagonistic to wavelet analysis.

Bony's para-differential operators have important applications to many problems concerning Navier-Stokes equations. The best references are J.Y. Chemin [16], T. Kato and G. Ponce and finally Michael Taylor [61], [62].

The main ingredients in these approaches are non trivial commutator estimates. These estimates are similar to the ones Calderón, Coifman and myself were able to prove. Let us stress that the commutators which are involved lie beyond pseudo-differential operators.

Returning to the main issue, should one bet on wavelets for a deeper insight into Navier-Stokes equations ? Let us confess that we do not know. In contrast we can already state that paradifferential operators or microlocal analysis are already playing a key role. The first person who tried to convince me that microlocal analysis and non-linear PDE's had strong ties was Luc Tartar. But at that time my knowledge and my interest in non-linear PDE's was too limited to understand his deep views.

We now turn to a distinct issue which is the role that wavelets might play in numerical simulations of Navier-Stokes equations.

Many scientists are hoping that wavelets will be a key ingredient in the next century codes. A description of this research area is extracted from a research program by Nicholas Kevlahan, LMD, ENS, 24 rue Lhomond, 75231 Paris Cedex 05.

> "The wavelet transform was first introduced as an analysis technique, but numerical methods have been developed recently which use wavelet bases to actually solve partial differential equations (Frölich & Schneider 1996 ; Charton & Perrier 1996). These methods are particularly well-suited to equations, such as the Navier-Stokes equations at high Reynolds number, whose solutions contain isolated multiscale structures or quasi-singularities. In collaboration with Kai Schneider (ICT, Universität Karlsruhe) I compared simulations using these wavelet techniques with standard spectral simulations and nonlinearly filtered spectral simulations (Schneider, Kevlahan & Farge 1997). The results showed that the wavelet methods are very accurate, and require fewer active modes than spectral methods. Furthermore, the number of active wavelet modes is approximately constant in time, even during intense nonlinear interactions, whereas the number of active spectral modes peaks when the interactions are most intense..."

In his Ph. D. dissertation, Mats Holmström also advocates wavelets for solving PDEs.

> "In fluid dynamics we have shocks, boundary layers and turbulence. For these examples the solution can be smooth in most of the solution domain, with small areas where the solution changes quickly. When solving such problems numerically we would like to adjust the discretization to the solution. In terms of finite difference methods, we want to have many points in areas where the solution has strong variation, and few points where the solution is smooth. If we use a Galerkin method this corresponds to the representation of the solution having fewer basis functions in the smooth areas. ... The most common way of compressing such a representation is thresholding. We delete all wavelet coefficients of magnitude less than some threshold ε ... Note that by thresholding a wavelet representation we have a way to automatically find a sparse representation, and we can

also use this representation to compute function values at any point...

The method can be viewed as an adaptive mesh method, where the mesh is automatically refined around sharp variations of the solution. An advantage of the method is that we never have to care about where and how to refine the mesh. All this is handled by thresholding the wavelet coefficients. The method is well suited for large problems with a solution that is well compressed in a wavelet basis... The constants in the estimates are large. They can be reduced by using other wavelets than Daubechies wavelets but the problem size has to be large before the wavelet method outperforms classical methods..."

We strongly believe in wavelet based algorithms for solving non linear PDEs. When more powerful computers are available, wavelet based algorithms might win. Indeed the issue will then be a sharper analysis of the shocks or singularities that might develop in the non-linear evolution.

Let me however qualify this remark. In my opinion, wavelet analysis belongs to the same group of tools as refinements of meshes or multipanel processing do. We also would like to include the celebrated Oslo algorithm ("knot removal") in this group. All these algorithms are addressing the same issue : one should adapt the grid to the solution which is being computed. In the case of an evolution equation, the grid should also evolve in time.

The equivalence between wavelet analysis, refinements of meshes and multipanel processing is grounded on the pioneering work on non-linear approximation which was achieved by De Vore and inspired by Peller and Peetre.

Returning to Navier-Stokes equations, my belief concerns algorithms that mimic wavelet methods. These algorithms automatically introduce more segmentation or grid points when a singularity is developing.

A last issue concerns the role played by wavelets as a visualisation tool.

Let us return to N. Kevlahan. He is advocating a wavelet analysis as a post-processing :

"The wavelet transform is a new harmonic analysis technique developed in France during the 1980s (Grossmann & Morlet). Marie Farge showed that this technique is appropriate for the analysis of turbulence because it permits a localization that is

> both spatial and spectral ([27], [30]). This means that one can link the physical characteristics of turbulence (e.g. the presence of vortices) to its pectral or statistical properties (e.g. energy spectrum slope) [28]."

This means that here wavelets are used as a visualisation tool.

Then N. Kevlahan confirms M. Farge's hopes :

> "A wavelet analysis permitted me to cleanly separate the vorticity filaments from the coherent vortices..."

This was achieved on a numerical simulation of two-dimensional turbulence. The simulation was obtained with a conventional algorithm, but wavelets were used as a discrimination tool in a pattern analysis. The success of the work of Farge and Kevlahan responds to a concern expressed by R. Azencott that detecting and isolating coherent vortices is a highly non trivial task and that wavelet people might be using the wrong tools.

While M. Farge and his group are using a wavelet analysis for detecting and isolating vertices on numerical simulations of 2D turbulence, A. Arneodo and his team used a wavelet analysis on the turbulence signal which was measured by Y. Gagne inside the Modane wind tunnel.

A. Arneodo and his team made two important discoveries. He could prove the multifractal structure of the velocity field as a function of the time variable [3] and he was able to detect huge transients which were interpreted by A. Arneodo as vorticity filaments crossing the hot wire. This hot wire is used to measure the velocity inside the wind tunnel.

Arneodo's hypothesis was eventually confirmed by Y. Couder and his group [26].

It is time to conclude.

It is now obvious that (1) wavelet analysis is the best available tool for scrutinizing the intricate behavior of experimental turbulence and (2) there are some hopes that wavelet type algorithms will be increasingly used in numerical simulations. I am including adaptive refinement of meshes and multipanel processing inside this group of algorithms.

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