TRACE FORMULA ON THE ADELE CLASS SPACE AND WEIL POSITIVITY

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Abstract
We shall first show that the classification of factors, as seen in the unusual light of André Weil's Basic Number Theory, is a natural substitute for the Brauer theory of central simple algebras in local class field theory at Archimedian places. Passing to the global case provides a natural geometric framework in which the Frobenius, its eigenvalues and the Lefschetz formula interpretation of the explicit formulas continue to hold even for number fields. The geometric space involved is the Adele class space, i.e. the quotient of Adeles by the multiplicative group of the global field. We shall then explain that this leads to a natural spectral interpretation of the zeros of the Riemann zeta function and then prove the positivity of the Weil distribution assuming the validity of the analogue of the Selberg trace formula. The latter remains unproved and is equivalent to RH for all L-functions with Grössencharakter.

Introduction

Global fields k provide a natural context for the Riemann Hypothesis on the zeros of the zeta function and its generalization to Hecke L-functions. When the characteristic of k is non zero this conjecture was proved by A. Weil. His proof relies on the following dictionary (put in modern language) which gives a geometric meaning in terms of algebraic geometry over finite fields, to the function theoretic properties of the zeta functions. Recall that k is a function field over a curve Σ defined over $\mathbb{F}_q$,

Algebraic Geometry	*Function Theory*
Eigenvalues of action of Frobenius on $H^*_{\mathrm{et}}(\overline{\Sigma}, \mathbb{Q}_\ell)$	Zeros and poles of ζ
Poincaré duality in ℓ-adic cohomology	Functional equation
Lefschetz formula for the Frobenius	Explicit formulas
Castelnuovo positivity	Riemann Hypothesis

We shall describe a third column in this dictionary, which will make sense for any global field. It is based on the geometry of the Adele class space,

$$X = A/k^*\,, \ A = \text{Adeles of } k\,. \tag{1}$$

This space is of the same nature as the space of leaves of the horocycle foliation of a Riemann surface (section I) and the same geometry will be used to analyse it.

Our spectral interpretation of the zeros of zeta involves Hilbert space. The reasons why Hilbert space (apparently invented by Hilbert for this purpose) should be involved are manifold, let us mention three,

(A) The discovery of Hugh Montgomery ([M]) about the statistical fluctuations of the spacings of zeros of zeta. Numerical tests by Odlyzko ([O]) and further theoretical work by Katz-Sarnak ([KS]) give overwhelming evidence that zeros of zeta should be the eigenvalues of a hermitian matrix.

(B) The equivalence between RH and the positivity of the Weil distribution on the Idele class group C_k which shows that Hilbert space is implicitly present.

(C) The deep arithmetic significance of the work of A. Selberg on the spectral analysis of the Laplacian on $L^2(G/\Gamma)$ where Γ is an arithmetic subgroup of a semi simple Lie group G.

Direct atempts (cf. [B]) to construct the Polya-Hilbert space giving a spectral realization of the zeros of ζ using quantum mechanics, meet a serious minus sign problem explained in [B].

The very same $-$ sign appears in the Riemann-Weil explicit formula,

$$\sum_{L(\chi,\rho)=0} \widehat{h}(\chi,\rho) - \widehat{h}(0) - \widehat{h}(1) = -\sum_v \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u \,, \tag{2}$$

where h is a test function on the Idele class group C_k, $\widehat{h}$ is its Fourier transform,

$$\widehat{h}(\chi,z) = \int_{C_k} h(u)\,\chi(u)\,|u|^z \, d^*u \,, \tag{3}$$

and the finite values $\int'$ are suitably normalized. If we use the above dictionary when $\mathrm{char}(k) \neq 0$, the geometric origin of this $-$ sign becomes clear, the formula (2) is the Lefschetz formula,

$$\#\text{ of fixed points of } \varphi = \mathrm{Trace}\,\varphi/H^0 - \mathrm{Trace}\,\varphi/H^1 + \mathrm{Trace}\,\varphi/H^2 \tag{4}$$

in which the space $H^1_{\mathrm{et}}(\overline{\Sigma},\mathbb{Q}_\ell)$ which provides the spectral realization of the zeros appears with a $-$ sign. This indicates that the spectral realization of zeros of zeta should be of cohomological nature or to be more specific, that the Polya-Hilbert space providing this realization should appear as the last term of an exact sequence of Hilbert spaces,

$$0 \to \mathcal{H}_0 \overset{T}{\to} \mathcal{H}_1 \to \mathcal{H} \to 0 \,. \tag{5}$$

Let $X = A/k^*$ be the Adele class space. Our basic idea is to take for $\mathcal{H}_0$ a suitable completion of the codimension 2 subspace of functions on X such that,

$$f(0) = 0 \,, \quad \int f\,dx = 0 \,, \tag{6}$$

while $\mathcal{H}_1 = L^2(C_k)$ and T is the restriction map coming from the inclusion $C_k \to X$, multiplied by $|a|^{1/2}$,

$$(Tf)(a) = |a|^{1/2} f(a) \,. \tag{7}$$

The action of the Idele class group C_k which gives the spectral realization is then the obvious one, for $\mathcal{H}_0$

$$(U(g)f)(x) = f(g^{-1}x) \qquad \forall\, g \in C_k \tag{8}$$

using the action by multiplication of C_k on X, and similarly the regular representation V for $\mathcal{H}_1$.

This idea works but there are two subtle points; first since X is a delicate quotient space the function spaces for X are naturally obtained by starting with function spaces on A and moding out by the "gauge transformations"

$$f \to f_q\,, \ f_q(x) = f(xq)\,, \quad \forall\, q \in k^*\,. \tag{9}$$

Here the natural function space is the Bruhat-Schwarz space $\mathcal{S}(A)$ and by (6) the codimension 2 subspace,

$$\mathcal{S}(A)_0 = \left\{ f \in \mathcal{S}(A)\,;\ f(0) = 0\,,\ \int f\,dx = 0 \right\}\,. \tag{10}$$

The restriction map T is then given by,

$$T(f)(a) = |a|^{1/2} \sum_{q\,\in\,k^*} f(aq) \qquad \forall\, a \in C_k\,. \tag{11}$$

The corresponding function $T(f)$ belongs to $\mathcal{S}(C_k)$ and all functions $f - f_q$ are in the kernel of T.

The second subtle point is that since C_k is abelian and non compact, its regular representation does not contain any finite dimensional subrepresentation so that the Polya-Hilbert space cannot be a subrepresentation (or unitary quotient) of V. There is an easy way out (which will be improved later) which is to replace $L^2(C_k)$ by $L^2_\delta(C_k)$ using the polynomial weight $(\log^2|a|)^{\delta/2}$, i.e. the norm,

$$\|\xi\|^2_\delta = \int_{C_k} |\xi(a)|^2\,(1 + \log^2|a|)^{\delta/2}\,d^*a\,. \tag{12}$$

Let $\mathrm{char}(k) = 0$ so that $\mathrm{Mod}\,k = \mathbb{R}^*_+$ and $C_k = K \times \mathbb{R}^*_+$ where K is the compact group $C_{k,1} = \{a \in C_k\,;\ |a| = 1\}$.

Theorem. ([Co]) *Let $\delta > 1$, $\mathcal{H}$ be the cokernel of T in $L^2_\delta(C_k)$ and W the quotient representation of C_k. Let χ be a character of K, $\widetilde{\chi} = \chi \times 1$ the corresponding character of C_k. Let $\mathcal{H}_\chi = \{\xi \in \mathcal{H}\,;\ W(g)\,\xi = \chi(g)\,\xi \quad \forall\, g \in K\}$ and $D_\chi = \lim\limits_{\epsilon \to 0} \frac{1}{\epsilon}\,(W(e^\epsilon) - 1)$. Then D_χ is an unbounded closed operator with discrete spectrum, $\mathrm{Sp}\,D_\chi \subset i\,\mathbb{R}$ is the set of imaginary parts of zeros of the L function with Grössencharakter $\widetilde{\chi}$ which have real part $1/2$. Moreover the spectral multiplicity of ρ is the largest integer $n < \frac{1+\delta}{2}$ in $\{1, \ldots,$ multiplicity as a zero of $L\}$.*

A similar result holds for $\mathrm{char}(k) > 0$. This allows to compute the character of the representation W as,

$$\mathrm{Trace}(W(h)) = \sum_{\substack{L\left(\chi,\frac{1}{2}+\rho\right) = 0 \\ \rho\,\in\, i\,\mathbb{R}/N^\perp}} \widehat{h}(\chi, \rho) \tag{13}$$

where $N = \mathrm{Mod}(k)$, $W(h) = \int W(g)\,h(g)\,d^*g$, $h \in \mathcal{S}(C_k)$, $\widehat{h}$ is defined in (3) and the multiplicity is counted as in the theorem.

This result is only preliminary because of the unwanted parameter δ which artificially restricts the multiplicities. The analogue of the Hodge $*$ operation is given on $\mathcal{H}_0$ by the Fourier transform,

$$(14) \qquad (Ff)(x) = \int_A f(y)\,\alpha(xy)\,dy \qquad \forall\, f \in \mathcal{S}(A)_0$$

which, because we take the quotient by (9), is independent of the choice of additive character α of A such that $\alpha \neq 1$ and $\alpha(q) = 1 \quad \forall\, q \in k$. Note also that $F^2 = 1$ on the quotient. On $\mathcal{H}_1$ the Hodge $*$ is given by,

$$(15) \qquad (*\,\xi)(a) = \xi(a^{-1}) \qquad \forall\, a \in C_k\,.$$

The Poisson formula means exactly that T commutes with the $*$ operation. (cf. section V for signs and powers of i). This is just a reformulation of the work of Tate and Iwasawa on the proof of the functional equation, but we shall now see that if we follow the proof by Atiyah-Bott ([AB]) of the Lefschetz formula we do obtain a clear geometric meaning for the Weil distribution. One can of course as in ([G]) define inner products on function spaces on C_k using the Weil distribution, but as long as the latter is put by hands and does not appear naturally one has very little chance to understand why it should be positive. Now, let φ be a diffeomorphism of a smooth manifold Σ and assume that the graph of φ is transverse to the diagonal, one can then easily define and compute (cf. [AB]) the distribution theoretic trace of the permutation U of functions on Σ associated to φ,

$$(16) \qquad (U\xi)(x) = \xi(\varphi(x)) \qquad \forall\, x \in \Sigma\,.$$

One has "Trace" $(U) = \int k(x,x)\,dx$, where $k(x,y)\,dy$ is the Schwarz kernel associated to U, i.e. the distribution on $\Sigma \times \Sigma$ such that,

$$(17) \qquad (U\xi)(x) = \int k(x,y)\,\xi(y)\,dy\,.$$

Now near the diagonal and in local coordinates one has,

$$(18) \qquad k(x,y) = \delta(y - \varphi(x))\,,$$

where δ is the Dirac distribution. One then obtains,

$$(19) \qquad \text{“Trace”}\ (U) = \sum_{\varphi(x)=x} \frac{1}{|1 - \varphi'(x)|}\,,$$

where φ' is the Jacobian of φ and $|\ |$ stands for the absolute value of the determinant.

With more work ([GS]) one obtains a similar formula for the distributional trace of the action of a flow,

$$(20) \qquad (U_t\,\xi)(x) = \xi(F_t(x)) \qquad \forall\, x \in \Sigma\,,\ t \in \mathbb{R}.$$

It is given, under suitable transversality hypothesis, by

$$\text{(21)} \qquad \text{“Trace”}\ (U(h)) = \sum_{\gamma} \int_{I_\gamma} \frac{h(u)}{|1-(F_u)_*|}\, d^*u\,,$$

where $U(h) = \int h(t)\,U(t)\,dt$, h is a test function on $\mathbb{R}$, the γ labels the periodic orbits of the flow, including the fixed points, I_γ is the corresponding isotropy subgroup, and $(F_u)_*$ is the tangent map to F_u on the transverse space to the orbits, and finally d^*u is the unique Haar measure on I_γ which is of covolume 1 in $(\mathbb{R}, dt)$.

Now it is truly remarkable that when one analyzes the periodic orbits of the action of C_k on X one finds that (21) becomes,

$$\text{(22)} \qquad \text{“Trace”}\ (U(h)) = \sum_{v} \int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u\,.$$

Thus, the isotropy subgroups I_γ are parametrized by the places v of k and coincide with the natural cocompact inclusion $k_v^* \subset C_k$ which relates local to global in class field theory. The denominator $|1-u|$ is for the module of the local field k_v and the u^{-1} in $h(u^{-1})$ comes from the discrepancy between notations (8) and (16). It turns out that if one normalizes the Haar measure d^*u of modulated groups as in Weil ([W3]) , by,

$$\text{(23)} \qquad \int_{1\leq|u|\leq\Lambda} d^*u \sim \log\Lambda \qquad \text{for } \Lambda \to \infty\,,$$

one gets the same covolume 1 condition as in (21).

The transversality condition imposes the condition $h(1) = 0$. The distributional trace for the action of C_k on C_k by translations vanishes under the condition $h(1) = 0$.

Thus equating the alternate sum of traces on $\mathcal{H}_0$, $\mathcal{H}_1$ with the trace on the cohomology should thus provide the geometric understanding of the Riemann-Weil explicit formula (2) and in fact of RH using (13) if it could be justified for some value of δ.

The trace of permutation matrices is positive and this explains the Hadamard positivity,

$$\text{(24)} \qquad \text{“Trace”}\ (U(h)) \geq 0 \qquad \forall h\,,\ h(1)=0\,,\ h(u)\geq 0 \quad \forall u \in C$$

(not to be confused with Weil postivity).

To eliminate the artificial parameter δ and give rigorous meaning, as a Hilbert space trace, to the distribution “trace”, one proceeds as in the Selberg trace formula and introduces a cutoff. In physics terminology the divergence of the trace is both infrared and ultraviolet as is seen in the simplest case of the action of K^* on $L^2(K)$ for a local field K. In this local case one lets,

$$\text{(25)} \qquad R_\Lambda = \widehat{P}_\Lambda\, P_\Lambda\,,\ \Lambda \in \mathbb{R}_+\,,$$

where P_Λ is the orthogonal projection on the subspace,

$$\{\xi \in L^2(K)\,;\; \xi(x) = 0 \qquad \forall\, x, |x| > \Lambda\}\,, \tag{26}$$

while $\widehat{P}_\Lambda = F\,P_\Lambda\,F^{-1}$, F the Fourier transform.

One proves ([Co]) in this local case the following analogue of the Selberg trace formula,

$$\operatorname{Trace}(R_\Lambda\,U(h)) = 2\,h(1)\log'(\Lambda) + \int' \frac{h(u^{-1})}{|1-u|}\,d^*u + o(1) \tag{27}$$

where $h \in \mathcal{S}(K^*)$ has compact support, $2\log'(\Lambda) = \int_{\lambda\,\in\,K^*,\,|\lambda|\,\in\,[\Lambda^{-1},\Lambda]} d^*\lambda$, and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on K which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform vanishes at 1.

As it turns out this principal value agrees with that of Weil for the choice of Fourier transform F associated to the standard character of K.

Let k be a global field and let first S be a finite set of places of k containing all the infinite places. To S corresponds the following localized version of the action of C_k on X. One replaces C_k by

$$C_S = \prod_{v\in S} k_v^*/O_S^*\,, \tag{28}$$

where $O_S^* \subset k^*$ is the group of S-units. One replaces X by

$$X_S = \prod_{v\in S} k_v/O_S^*\,. \tag{29}$$

The Hilbert space $L^2(X_S)$, its Fourier transform F and the orthogonal projections P_Λ, $\widehat{P}_\Lambda = F\,P_\Lambda\,F^{-1}$ continue to make sense, with

$$\operatorname{Im} P_\Lambda = \{\xi \in L^2(X_S)\,;\; \xi(x) = 0 \quad \forall\, x\,,\; |x| > \Lambda\}\,. \tag{30}$$

As soon as S contains more than 3 elements, (e.g. $\{2,3,\infty\}$ for $k = \mathbb{Q}$) the space X_S is an extremely delicate quotient space. It is thus quite remarkable that the *trace formula* holds,

Theorem. ([Co]) *For any $h \in \mathcal{S}_c(C_S)$ one has, with $R_\Lambda = \widehat{P}_\Lambda\,P_\Lambda$,*

$$\mathit{Trace}\,(R_\Lambda\,U(h)) = 2\log'(\Lambda)\,h(1) + \sum_{v\in S} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\,d^*u + o(1)$$

where the notations are as above and the finite values $\int'$ depend on the additive character of Πk_v defining the Fourier transform F. When $\operatorname{Char}(k) = 0$ the projectors P_Λ, $\widehat{P}_\Lambda$ commute on L^2_χ for Λ large enough so that one can replace R_Λ by the orthogonal projection Q_Λ on $\operatorname{Im} P_\Lambda \cap \operatorname{Im} \widehat{P}_\Lambda$. The situation for $\operatorname{Char}(k) = 0$ is more delicate since P_Λ and $\widehat{P}_\Lambda$ do not commute (for Λ large) even in the local Archimedean case. But fortunately these operators commute ([LPS]) with a specific second order differential operator, whose eigenfunctions, the Prolate Spheroidal Wave functions provide the right filtration Q_Λ. This allows to replace R_Λ by Q_Λ and to state the global trace formula

$$\mathrm{Trace}\,(Q_\Lambda\, U(h)) = 2\log'(\Lambda)\, h(1) + \sum_v \int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)\,.$$

Our final result is that the validity of this trace formula implies (in fact is equivalent to) the positivity of the Weil distribution, i.e. *RH* for all L-functions with Grössencharakter. Moreover the filtration by Q_Λ allows to define the Adelic cohomology and to complete the dictionary between the function theory and the geometry of the Adele class space X.

Function Theory	*Geometry*
Zeros and poles of Zeta	Action of C_k on Adelic cohomology
Functional Equation	$*$ operation
Explicit formulas	Lefschetz formula for action of C_k on X
Riemann Hypothesis	Trace formula

The contents of this survey paper are organized as follows,

I Local class field theory and the classification of injective factors.

In this section I shall first look back at my early work on the classification of von Neumann algebras and cast it in the unusual light of André Weil's Basic Number Theory ([W1]). It then appears as a clear substitute for the missing Brauer theory of central simple algebras for Archimedian local fields.

Let K be a *local* field, i.e. a nondiscrete locally compact field. The action of $K^* = GL_1(K)$ on the additive group K by multiplication,

$$(\lambda, x) \to \lambda x \qquad \forall \lambda \in K^*, \; x \in K, \tag{1}$$

together with the uniqueness, up to scale, of the haar measure of the additive group K, yield a homomorphism,

$$a \in K^* \to |a| \in \mathbb{R}_+^*, \tag{2}$$

from K^* to $\mathbb{R}_+^*$, called the *module* of K. Its range

$$\mathrm{Mod}(K) = \{|\lambda| \in \mathbb{R}_+^* \; ; \; \lambda \in K^*\} \tag{3}$$

is a closed subgroup of $\mathbb{R}_+^*$.

The fields $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (of quaternions) are the only one with $\mathrm{Mod}(K) = \mathbb{R}_+^*$, they are called Archimedian local fields.

Let K be a non Archimedian local field, then

$$R = \{x \in K \; ; \; |x| \leq 1\}, \tag{4}$$

is the unique maximal compact subring of K and the quotient R/P of R by its unique maximal ideal is a finite field $\mathbb{F}_q$ (with $q = p^\ell$ a prime power). One has,

$$\mathrm{Mod}(K) = q^{\mathbb{Z}} \subset \mathbb{R}_+^* \,. \tag{5}$$

Let K be commutative. An extension $K \subset K'$ of finite degree of K is called *unramified* iff the dimension of K' over K is the order of $\mathrm{Mod}(K')$ as a subgroup of $\mathrm{Mod}(K)$. When this is so, the field K' is commutative, is generated over K by roots of unity of order prime to q, and is a cyclic Galois extension of K with Galois group generated by the automorphism $\theta \in \mathrm{Aut}_K(K')$ such that,

$$\theta(\mu) = \mu^q \,, \tag{6}$$

for any root of unity of order prime to q in K'.

The unramified extensions of finite degree of K are classified by the subgroups,

$$\Gamma \subset \mathrm{Mod}(K) \,, \; \Gamma \neq \{1\} \,. \tag{7}$$

Let then $\overline{K}$ be an algebraic closure of K, $K_{\mathrm{sep}} \subset \overline{K}$ the separable algebraic closure, $K_{\mathrm{ab}} \subset K_{\mathrm{sep}}$ the maximal abelian extension of K and $K_{\mathrm{un}} \subset K_{\mathrm{ab}}$ the maximal unramified extension of K, i.e. the union of all unramified extensions of finite degree. One has,

$$K \subset K_{\mathrm{un}} \subset K_{\mathrm{ab}} \subset K_{\mathrm{sep}} \subset \overline{K} \,, \tag{8}$$

and the Galois group $\mathrm{Gal}(K_{\mathrm{un}} : K)$ is topologically generated by θ called the Frobenius automorphism.

The correspondence (7) is given by,

$$K' = \{x \in K_{\mathrm{un}} \,;\; \theta_\lambda(x) = x \quad \forall \lambda \in \Gamma\} \,, \tag{9}$$

with rather obvious notations so that θ_q is the θ of (6). Let then W_K be the subgroup of $\mathrm{Gal}(K_{\mathrm{ab}} : K)$ whose elements induce on K_{un} an integral power of the Frobenius automorphism. One endows W_K with the locally compact topology dictated by the exact sequence of groups,

$$1 \to \mathrm{Gal}(K_{\mathrm{ab}} : K_{\mathrm{un}}) \to W_K \to \mathrm{Mod}(K) \to 1 \,, \tag{10}$$

and the main result of local class field theory asserts the existence of a canonical isomorphism,

$$W_K \xrightarrow{\sim} K^* \,, \tag{11}$$

compatible with the module.

The basic step in the construction of the isomorphism (11) is the classification of finite dimensional central simple algebras A over K. Any such algebra is of the form,

$$A = M_n(D) \,, \tag{12}$$

where D is a (central) division algebra over K and the symbol M_n stands for $n \times n$ matrices.

Moreover D is the crossed product of an unramified extension K' of K by a 2-cocycle on its cyclic Galois group. Elementary group cohomology then yields the isomorphism,

$$\mathrm{Br}(K) \overset{\eta}{\to} \mathbb{Q}/\mathbb{Z}\,, \tag{13}$$

of the Brauer group of classes of central simple algebras over K (with tensor product as the group law), with the group $\mathbb{Q}/\mathbb{Z}$ of roots of 1 in $\mathbb{C}$.

All the above discussion was under the assumption that K is non Archimedean. For Archimedean fields $\mathbb{R}$ and $\mathbb{C}$ the same questions have an idiotically simple answer. Since $\mathbb{C}$ is algebraically closed one has $K = \overline{K}$ and the whole picture collapses. For $K = \mathbb{R}$ the only non trivial value of the Hasse invariant η is

$$\eta(\mathbb{H}) = -1\,. \tag{14}$$

A Galois group G is by construction totally disconnected so that a morphism from K^* to G is necessarily trivial on the connected component of $1 \in K^*$.

Let k be a *global* field, i.e. a discrete cocompact subfield of a (non discrete) locally compact semi-simple commutative ring A. (Cf. Iwasawa *Ann. of Math.* **57** (1953).) The topological ring A is canonically associated to k and called the Adele ring of k, one has,

$$A = \prod_{\mathrm{res}} k_v\,, \tag{15}$$

where the product is the restricted product of the local fields k_v labelled by the places of k.

When the characteristic of k is $p > 1$ so that k is a function field over $\mathbb{F}_q$, one has

$$k \subset k_{\mathrm{un}} \subset k_{\mathrm{ab}} \subset k_{\mathrm{sep}} \subset \overline{k}\,, \tag{16}$$

where, as above $\overline{k}$ is an algebraic closure of k, k_{sep} the separable algebraic closure, k_{ab} the maximal abelian extension and k_{un} is obtained by adjoining to k all roots of unity of order prime to p.

One defines the Weil group W_k as above as the subgroup of $\mathrm{Gal}(k_{\mathrm{ab}} : k)$ of those automorphisms which induce on k_{un} an integral power of θ,

$$\theta(\mu) = \mu^q \qquad \forall\, \mu \text{ root of 1 of order prime to } p\,. \tag{17}$$

The main theorem of global class field theory asserts the existence of a canonical isomorphism,

$$W_k \simeq C_k = GL_1(A)/GL_1(k)\,, \tag{18}$$

of locally compact groups.

When k is of characteristic 0, i.e. is a number field, one has a canonical isomorphism,

$$\mathrm{Gal}(k_{\mathrm{ab}} : k) \simeq C_k / D_k \,, \tag{19}$$

where D_k is the connected component of identity in the Idele class group $C_k = GL_1(A)/GL_1(k)$, but because of the Archimedian places of k there is no interpretation of C_k analogous to the Galois group interpretation for function fields. According to A. Weil ([W4]),

"La recherche d'une interprétation pour C_k si k est un corps de nombres, analogue en quelque manière à l'interprétation par un groupe de Galois quand k est un corps de fonctions, me semble constituer l'un des problèmes fondamentaux de la théorie des nombres à l'heure actuelle ; il se peut qu'une telle interprétation renferme la clef de l'hypothèse de Riemann ...".

Galois groups are by construction projective limits of the finite groups attached to finite extensions. To get connected groups one clearly needs to relax this finiteness condition which is the same as the finite dimensionality of the central simple algebras. Since Archimedian places of k are responsible for the non triviality of D_k it is natural to ask the following preliminary question,

"Is there a non trivial Brauer theory of central simple algebras over $\mathbb{C}$."

As we shall see shortly the *approximately finite dimensional* simple central algebras over $\mathbb{C}$ provide a satisfactory answer to this question. They are classified by their module,

$$\mathrm{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^* \,, \tag{20}$$

which is a virtual closed subgroup of $\mathbb{R}_+^*$.

Let us now explain this statement with more care. First we exclude the trivial case $M = M_n(\mathbb{C})$ of matrix algebras. Next $\mathrm{Mod}(M)$ is a virtual subgroup of $\mathbb{R}_+^*$, in the sense of G. Mackey, i.e. an ergodic action of $\mathbb{R}_+^*$. All ergodic flows appear and M_1 is isomorphic to M_2 iff $\mathrm{Mod}(M_1) \cong \mathrm{Mod}(M_2)$.

The birth place of central simple algebras is as the commutant of isotypic representations. When one works over $\mathbb{C}$ it is natural to consider unitary representations in Hilbert space so that we shall restrict our attention to algebras M which appear as commutants of unitary representations. They are called von Neumann algebras. The terms central and simple keep their usual algebraic meaning.

The classification involves three independent parts,

(A) The definition of the invariant $\mathrm{Mod}(M)$ for arbitrary factors (central von Neumann algebras).

(B) The equivalence of all possible notions of approximate finite dimensionality.

(C) The proof that Mod is a complete invariant and that all virtual subgroups are obtained.

The module of a factor M was first defined ([Co2]) as a closed subgroup of $\mathbb{R}_+^*$ by the equality

(21) $$S(M) = \bigcap_{\varphi} \operatorname{Spec}(\Delta_\varphi) \subset \mathbb{R}_+$$

where φ varies among (faithful, normal) states on M, i.e. linear forms $\varphi : M \to \mathbb{C}$ such that,

(22) $$\varphi(x^*x) \geq 0 \qquad \forall\, x \in M\,,\ \varphi(1) = 1\,,$$

while the operator Δ_φ is the *modular operator* ([T])

(23) $$\Delta_\varphi = S_\varphi^*\, S_\varphi\,,$$

which is the *module* of the involution $x \to x^*$ in the Hilbert space attached to the sesquilinear form,

(24) $$\langle x, y\rangle = \varphi(y^*x)\,,\ x, y \in M\,.$$

In the case of local fields the module was a group homomorphism ((2)) from K^* to $\mathbb{R}_+^*$. The counterpart for factors is the group homomorphism, ([Co2])

(25) $$\delta : \mathbb{R} \to \operatorname{Out}(M) = \operatorname{Aut}(M)/\operatorname{Int}(M)\,,$$

from the additive group $\mathbb{R}$ viewed as the dual of $\mathbb{R}_+^*$ for the pairing,

(26) $$(\lambda, t) \to \lambda^{it} \qquad \forall\,\lambda \in \mathbb{R}_+^*\,,\ t \in \mathbb{R},$$

to the group of automorphism classes of M modulo inner automorphisms.

The virtual subgroup,

(27) $$\operatorname{Mod}(M) \underset{\sim}{\subset} \mathbb{R}_+^*\,,$$

is the *flow of weights* ([Ta],[K],[CT]) of M. It is obtained from the module δ as the dual action of $\mathbb{R}_+^*$ on the abelian algebra,

(28) $$C = \text{Center of } M \rtimes_\delta \mathbb{R},$$

where $M \rtimes_\delta \mathbb{R}$ is the crossed product of M by the modular automorphism group δ.

This takes care of (A), to describe (B) let us simply state the equivalence ([Co1]) of the following conditions

(29) M is the closure of the union of an increasing sequence of finite dimensional algebras.

(30) M is complemented as a subspace of the normed space of all operators in a Hilbert space.

The condition (29) is obviously what one would expect for an approximately finite dimensional algebra. Condition (30) is similar to *amenability* for discrete groups and the implication (30) $\Rightarrow$ (29) is a very powerful tool.

We refer to [Co1],[K],[Ha] for (C) and we just describe the actual construction of the central simple algebra M associated to a given virtual subgroup,

$$\Gamma \underset{\sim}{\subset} \mathbb{R}_+^* \,. \tag{31}$$

Among the approximately finite dimensional factors (central von Neumann algebras), only two are not simple. The first is the algebra

$$M_\infty(\mathbb{C})\,, \tag{32}$$

of all operators in Hilbert space. The second factor is the unique approximately finite dimensional factor of type II_∞. It is

$$R_{0,1} = R \otimes M_\infty(\mathbb{C})\,, \tag{33}$$

where R is the unique approximately finite dimensional factor with a finite trace τ_0, i.e. a state such that,

$$\tau_0(xy) = \tau_0(yx) \qquad \forall\, x,y \in R\,. \tag{34}$$

The tensor product of τ_0 by the standard semifinite trace on $M_\infty(\mathbb{C})$ yields a semifinite trace τ on $R_{0,1}$. There exists, up to conjugacy, a unique one parameter group of automorphisms $\theta_\lambda \in \mathrm{Aut}(R_{0,1})$, $\lambda \in \mathbb{R}_+^*$ such that,

$$\tau(\theta_\lambda(a)) = \lambda\tau(a) \qquad \forall\, a \in \mathrm{Domain}\,\tau\,,\ \lambda \in \mathbb{R}_+^*\,. \tag{35}$$

Let first $\Gamma \subset \mathbb{R}_+^*$ be an ordinary closed subgroup of $\mathbb{R}_+^*$. Then the corresponding factor R_Γ with module Γ is given by the equality:

$$R_\Gamma = \{x \in R_{0,1}\,;\ \theta_\lambda(x) = x \quad \forall\, \lambda \in \Gamma\}\,, \tag{36}$$

in perfect analogy with (9).

A virtual subgroup $\Gamma \underset{\sim}{\subset} \mathbb{R}_+^*$ is by definition an ergodic action of $\mathbb{R}_+^*$ on an abelian von Neumann algebra A, and the formula (36) easily extends to,

$$R_\Gamma = \{x \in R_{0,1} \otimes A\,;\ (\theta_\lambda \otimes \alpha_\lambda)\,x = x \quad \forall\, \lambda \in \mathbb{R}_+^*\}\,. \tag{37}$$

(This reduces to (36) for the action of $\mathbb{R}_+^*$ on the algebra $A = L^\infty(X)$ where X is the homogeneous space $X = \mathbb{R}_+^*/\Gamma$.)

The pair $(R_{0,1}, \theta_\lambda)$ arises very naturally in geometry from the geodesic flow of a compact Riemann surface (of genus > 1). Let $V = S^*\Sigma$ be the unit cosphere bundle of such a surface Σ, and F be the stable foliation of the geodesic flow. The latter defines a one parameter group of automorphisms of the foliated manifold (V, F)

and thus a one parameter group of automorphisms θ_λ of the von Neumann algebra $L^\infty(V,F)$.

This algebra is easy to describe, its elements are random operators $T = (T_f)$, i.e. bounded measurable families of operators T_f parametrized by the leaves f of the foliation. For each leaf f the operator T_f acts in the Hilbert space $L^2(f)$ of square integrable densities on the manifold f. Two random operators are identified if they are equal for almost all leaves f (i.e. a set of leaves whose union in V is negligible). The algebraic operators of sum and product are given by,

(38) $$(T_1 + T_2)_f = (T_1)_f + (T_2)_f \, , \ (T_1 T_2)_f = (T_1)_f \, (T_2)_f \, ,$$

i.e. are effected pointwise.

One proves that,

(39) $$L^\infty(V,F) \simeq R_{0,1} \, ,$$

and that the geodesic flow θ_λ satisfies (35). Indeed the foliation (V,F) admits up to scale a unique transverse measure Λ and the trace τ is given (cf. [C]) by the formal expression,

(40) $$\tau(T) = \int \mathrm{Trace}(T_f) \, d\Lambda(f) \, ,$$

since the geodesic flow satisfies $\theta_\lambda(\Lambda) = \lambda\Lambda$ one obtains (35) from simple geometric considerations. The formula (37) shows that most approximately finite dimensional factors already arise from foliations, for instance the unique approximately finite dimensional factor R_∞ such that,

(41) $$\mathrm{Mod}(R_\infty) = \mathbb{R}_+^* \, ,$$

arises from the codimension 1 foliation of $V = S^*\Sigma$ generated by F and the geodesic flow.

In fact this relation between the classification of central simple algebras over $\mathbb{C}$ and the geometry of foliations goes much deeper. For instance using cyclic cohomology together with the following simple fact,

(42) "A connected group can only act trivially on a homotopy invariant cohomology theory",

one proves (cf. [C]) that for any codimension are foliation F of a compact manifold V with non vanishing Godbillon-Vey class one has,

(43) $$\mathrm{Mod}(M) \text{ has finite covolume in } \mathbb{R}_+^* \, ,$$

where $M = L^\infty(V,F)$ and a virtual subgroup of finite covolume is a flow with a finite invariant measure.

II Global class field theory and spontaneous symmetry breaking.

In the above discussion of approximately finite dimensional central simple algebras, we have been working locally over $\mathbb{C}$. We shall now describe a particularly interesting example ([BC]) of Hecke algebra intimately related to arithmetic, and defined over $\mathbb{Q}$.

Let $\Gamma_0 \subset \Gamma$ be an almost normal subgroup of a discrete group Γ, i.e. one assumes,

$$\Gamma_0 \cap s\,\Gamma_0\, s^{-1} \text{ has finite index in } \Gamma_0 \qquad \forall\, s \in \Gamma\,. \tag{1}$$

Equivalently the orbits of the left action of Γ_0 on Γ/Γ_0 are all finite. One defines the Hecke algebra,

$$\mathcal{H}(\Gamma, \Gamma_0)\,, \tag{2}$$

as the convolution algebra of integer valued Γ_0 biinvariant functions with finite support. For any field k one lets,

$$\mathcal{H}_k(\Gamma, \Gamma_0) = \mathcal{H}(\Gamma, \Gamma_0) \otimes_{\mathbb{Z}} k\,, \tag{3}$$

be obtained by extending the coefficient ring from $\mathbb{Z}$ to k. We let $\Gamma = P_{\mathbb{Q}}^+$ be the group of 2×2 rational matrices,

$$\Gamma = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} ;\ a \in \mathbb{Q}^+\,,\ b \in \mathbb{Q} \right\}\,, \tag{4}$$

and $\Gamma_0 = P_{\mathbb{Z}}^+$ be the subgroup of integral matrices,

$$\Gamma_0 = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} ;\ n \in \mathbb{Z} \right\}\,. \tag{5}$$

One checks that Γ_0 is almost normal in Γ.

To obtain a central simple algebra over $\mathbb{C}$ in the sense of the previous section we just take the commutant of the right regular representation of Γ on $\Gamma_0 \backslash \Gamma$, i.e. the weak closure of $\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ in the Hilbert space,

$$\ell^2(\Gamma_0 \backslash \Gamma)\,, \tag{6}$$

of Γ_0 left invariant function on Γ with norm square,

$$\|\xi\|^2 = \sum_{\gamma \in \Gamma_0 \backslash \Gamma} |\xi(\gamma)|^2 . \tag{7}$$

This central simple algebra over $\mathbb{C}$ is approximately finite dimensional and its module is $\mathbb{R}_+^*$ so that it is the same as R_∞ of (I.41).

In particular its modular automorphism group is highly non trivial and one can compute it explicitly for the state φ associated to the vector $\xi_0 \in \ell^2(\Gamma_0 \backslash \Gamma)$ corresponding to the left coset Γ_0.

The modular automorphism group σ_t^φ leaves the dense subalgebra $\mathcal{H}_\mathbb{C}(\Gamma, \Gamma_0) \subset R_\infty$ globally invariant and is given by the formula,

$$\sigma_t^\varphi(f)(\gamma) = L(\gamma)^{-it} R(\gamma)^{it} f(\gamma) \qquad \forall \gamma \in \Gamma_0 \backslash \Gamma / \Gamma_0 \tag{8}$$

for any $f \in \mathcal{H}_\mathbb{C}(\Gamma, \Gamma_0)$. Here we let,

$$\begin{aligned} L(\gamma) &= \text{ Cardinality of the image of } \Gamma_0 \gamma \Gamma_0 \text{ in } \Gamma / \Gamma_0 \\ R(\gamma) &= \text{ Cardinality of the image of } \Gamma_0 \gamma \Gamma_0 \text{ in } \Gamma_0 \backslash \Gamma . \end{aligned} \tag{9}$$

This is enough to make contact with the formalism of quantum statistical mechanics which we now briefly describe.

A quantum statistical system is given by,

1) The C^* algebra of observables A,

2) The time evolution $(\sigma_t)_{t \in \mathbb{R}}$ which is a one parameter group of automorphisms of A.

An equilibrium or KMS (for Kubo-Martin and Schwinger) state, at inverse temperature β is a state φ on A which fulfills the following condition,

(10) For any $x, y \in A$ there exists a bounded holomorphic function (continuous on the closed strip), $F_{x,y}(z)$, $0 \leq \operatorname{Im} z \leq \beta$ such that

$$\begin{aligned} F_{x,y}(t) &= \varphi(x\, \sigma_t(y)) && \forall\, t \in \mathbb{R} \\ F_{x,y}(t + i\beta) &= \varphi(\sigma_t(y) x) && \forall\, t \in \mathbb{R} . \end{aligned}$$

For fixed β the KMS$_\beta$ states form a Choquet simplex and thus decompose uniquely as a statistical superposition from the pure phases given by the extreme points. For interesting systems with nontrivial interaction, one expects in general that for large temperature T, (i.e. small β since $\beta = \frac{1}{T}$ up to a conversion factor) the disorder will be predominant so that there will exist only one KMS$_\beta$ state. For low enough temperatures some order should set in and allow for the coexistence of distinct thermodynamical phases so that the simplex K_β of KMS$_\beta$ states should be non trivial. A given symmetry group G of the system will necessarily act trivially on K_β for large T since K_β is a point, but acts in general non trivially on K_β for small T so that it is no longer a symmetry of a given pure phase. This phenomenon of *spontaneous symmetry breaking* as well as the very particular properties of the critical temperature T_c at the boundary of the two regions are corner stones of statistical mechanics.

In our case we just let A be the C^* algebra which is the *norm* closure of $\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ in the algebra of operators in $\ell^2(\Gamma_0 \backslash \Gamma)$. We let $\sigma_t \in \mathrm{Aut}(A)$ be the unique extension of the automorphisms σ_t^φ of (8).

For $\beta = 1$ it is tautological that φ is a KMS_β state since we obtained σ_t^φ precisely this way ([T]). One proves ([BC]) that for any $\beta \leq 1$ (i.e. for $T = 1$) there exists one and only one KMS_β state.

The compact group G,

$$G = C_{\mathbb{Q}} / D_{\mathbb{Q}}\,, \tag{11}$$

quotient of the Idele class group $C_{\mathbb{Q}}$ by the connected component of identity $D_{\mathbb{Q}} \simeq \mathbb{R}_+^*$, acts in a very simple and natural manner as symmetries of the system (A, σ_t). (To see this one notes that the right action of Γ on $\Gamma_0 \backslash \Gamma$ extends to the action of $P_{\mathcal{A}}$ on the restricted product of the trees of $SL(2, \mathbb{Q}_p)$ where $\mathcal{A}$ is the ring of finite Adeles (cf. [BC]).

For $\beta > 1$ this symmetry group G of our system, is spontaneously broken, the compact convex sets K_β are non trivial and have the same structure as K_∞, which we now describe. First some terminology, a KMS_β state for $\beta = \infty$ is called a *ground state* and the KMS_∞ condition is equivalent to *positivity of energy* in the corresponding Hilbert space representation.

Remember that $\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ contains $\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ so,

$$\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0) \subset A\,. \tag{12}$$

By [BC] theorem 5 and proposition 24 one has,

Theorem. *Let $\mathcal{E}(K_\infty)$ be the set of extremal KMS_∞ states.*

a) *The group G acts freely and transitively on $\mathcal{E}(K_\infty)$ by composition, $\varphi \to \varphi \circ g^{-1}$, $\forall\, g \in G$.*

b) *For any $\varphi \in \mathcal{E}(K_\infty)$ one has,*

$$\varphi(\mathcal{H}_{\mathbb{Q}}) = \mathbb{Q}_{\mathrm{ab}}\,,$$

and for any element $\alpha \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}} : \mathbb{Q})$ there exists a unique extension of $\alpha \circ \varphi$, by continuity, as a state of A. One has $\alpha \circ \varphi \in \mathcal{E}(K_\infty)$.

c) *The map $\alpha \to \varphi^{-1}(\alpha \circ \varphi) \in G = C_k / D_k$ defined for $\alpha \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}} : \mathbb{Q})$ is the isomorphism of global class field theory (I.19).*

This last map is independent of the choice of φ. What is quite remarkable in this result is that the existence of the subalgebra $\mathcal{H}_{\mathbb{Q}} \subset \mathcal{H}_{\mathbb{C}}$ allows to bring into action the Galois group of $\mathbb{C}$ on the *values of states*. Since the Galois group of $\mathbb{C} : \mathbb{Q}$ is (except for $z \to \bar{z}$) formed of *discontinuous* automorphisms it is quite surprising that its action can actually be compatible with the characteristic *positivity* of states.

It is by no means clear how to extend the above construction to arbitrary number fields k while preserving the 3 results of the theorem. The ideas of G. Moore ([Mo]) could well be relevant. There is however an easy computation which relates the above construction to an object which makes sense for any global field k. Indeed if we let as above R_∞ be the weak closure of $\mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0)$ in $\ell^2(\Gamma_0\backslash\Gamma)$, we can compute the associated pair $(R_{0,1}, \theta_\lambda)$ of section I.

By the result of [Laca] the C^* algebra closure of $\mathcal{H}_{\mathbb{C}}$ is a full corner of the crossed product C^* algebra,

$$C_0(\mathcal{A}) \rtimes \mathbb{Q}_+^* \,, \tag{13}$$

where $\mathcal{A}$ is the locally compact space of finite Adeles. It follows immediately that,

$$R_{0,1} = L^\infty(\mathbb{Q}_A) \rtimes \mathbb{Q}^* \,, \tag{14}$$

i.e. the von Neumann algebra crossed product of the L^∞ functions on Adeles of $\mathbb{Q}$ by the action of $\mathbb{Q}^*$ by multiplication.

The one parameter group of automorphisms, $\theta_\lambda \in \mathrm{Aut}(R_{0,1})$, is obtained as the restriction to,

$$D_{\mathbb{Q}} = \mathbb{R}_+^* \,, \tag{15}$$

of the obvious action of the Idele class group $C_{\mathbb{Q}}$,

$$(g, x) \to g\,x \qquad \forall\, g \in C_{\mathbb{Q}} \,,\; x \in A_{\mathbb{Q}}/\mathbb{Q}^* \,, \tag{16}$$

on the space $X = A_{\mathbb{Q}}/\mathbb{Q}^*$ of Adele classes.

Our next goal will be to show that the latter space is intimately related to the *zeros* of the Hecke L-functions with Grössencharakter.

(We showed in [BC] that the partition function of the above system is the Riemann zeta function.)

III Zeros of zeta and random matrices.

It is an old idea, due to Polya and Hilbert that in order to understand the location of the zeros of the Riemann zeta function, one should find a Hilbert space $\mathcal{H}$ and an operator D in $\mathcal{H}$ whose spectrum is given by the non trivial zeros of the zeta function. The hope then is that suitable selfadjointness properties of D (of $i\left(D - \frac{1}{2}\right)$ more precisely) or positivity properties of $\Delta = D(1-D)$ will be easier to handle than the original conjecture. The main reasons why this idea should be taken seriously are first the work of A. Selberg ([Se]) in which a suitable Laplacian Δ is related in the above way to an analogue of the zeta function, and secondly the theoretical ([M][B][KS]) and experimental evidence ([O][BG]) on the fluctuations of the spacing between consecutive zeros of zeta. The number of zeros of zeta whose imaginary part is less than $E > 0$,

$$N(E) = \#\text{ of zeros } \rho\ ,\ 0 < \operatorname{Im}\rho < E \tag{1}$$

has an asymptotic expression ([R]) given by

$$N(E) = \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + \frac{7}{8} + o(1) + N_{\rm osc}(E) \tag{2}$$

where the oscillatory part of this step function is

$$N_{\rm osc}(E) = \frac{1}{\pi}\operatorname{Im}\log\zeta\left(\frac{1}{2} + iE\right) \tag{3}$$

assuming that E is not the imaginary part of a zero and taking for the logarithm the branch which is 0 at $+\infty$ (connected to $\frac{1}{2} + iE$ by a straight horizontal line).

One shows (cf. [Pat]) that $N_{\rm osc}(E)$ is $O(\log E)$. In the decomposition (2) the two terms $\langle N(E)\rangle = N(E) - N_{\rm osc}(E)$ and $N_{\rm osc}(E)$ play an independent role. The first one $\langle N(E)\rangle$ which gives the average density of zeros is computed as follows. Let $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Then $\xi(s) = \xi(1-s)$ and $\xi(\bar{s}) = \overline{\xi(s)}$ so that ξ is real on the line $\operatorname{Re}(s) = 1/2$ and $\operatorname{Im}\log\xi(s) = \operatorname{Im}(-s/2\log\pi + \log\Gamma(s/2) + \log\zeta(s)) \in \mathbb{Z}\pi$. This shows that,

$$N(E) = 1 + \frac{1}{\pi}\left(-\frac{E}{2}\log\pi + \operatorname{Im}\log\Gamma\left(\frac{1}{4} + i\frac{E}{2}\right)\right) + \frac{1}{\pi}\operatorname{Im}\log\zeta\left(\frac{1}{2} + iE\right)$$

The asymptotic expansion of $\langle N(E)\rangle$ is thus given by the Stirling formula

$$\Gamma(z) = e^{-z}z^{z-1/2}\sqrt{2\pi}(1 + O(1/z))$$

valid for $-\pi < argz < \pi$. Hence the global behavior of the function $N(E)$ is described by the smooth part

$$< N(E) >= \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right) + \frac{7}{8} + o(1)$$

The second $N_{\rm osc}(E)$ is a manifestation of the randomness of the actual location of the zeros, and to eliminate the role of the density one returns to the situation of uniform density by the transformation

$$x_j = \langle N(E_j)\rangle \quad (E_j \text{ the } j^{\rm th} \text{ imaginary part of zero of zeta})\,. \tag{4}$$

Thus the spacing between two consecutive x_j is now 1 in average and the only information that remains is in the statistical fluctuation. As it turns out ([M][O]) these fluctuations are the same as the fluctuations of the eigenvalues of a random hermitian matrix of very large size.

H. Montgomery [M] proved (assuming RH) a weakening of the following conjecture (with $\alpha, \beta > 0$),

$$\begin{aligned} &\mathrm{Card}\,\{(i,j)\,;\, i,j \in 1,\ldots,M\,;\, x_i - x_j \in [\alpha,\beta]\} \\ &\qquad \sim M\int_\alpha^\beta \left(1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2\right) du \end{aligned} \tag{5}$$

This law (5) is precisely the same as the correlation between eigenvalues of hermitian matrices of the Gaussian Unitary Ensemble ([Me]) which we shall now describe. Moreover, numerical tests due to A. Odlyzko ([O][BG]) have confirmed with great precision the behaviour (5) as well as the analogous behaviour for more than two zeros. In [KS], N. Katz and P. Sarnak proved an analogue of the Montgomery-Odlyzko law for zeta and L-functions of function fields over curves.

The Gaussian Unitary Ensemble is the probability measure on the vector space $M_N(\mathbb{C})_{sa}$ of $N \times N$ Hermitian matrices, given, up to normalization, by the density $e^{-TraceA^2}dA$, $A \in M_N(\mathbb{C})_{sa}$, where dA denotes the Haar measure with respect to the additive structure. This measure is unitarily invariant.

We let $p(E_1, E_2, \ldots E_N)dE_1 \ldots dE_N$ be the corresponding measure on the eigenvalues $E_1, E_2, \ldots E_N$ of the matrix, i.e. the probability measure induced on the collections of eigenvalues by the Gaussian measure.

One has ([Me]),

$$p(E_1,\ldots,E_N) = 1/N!\left(\det_{\substack{0\le k\le N-1\\ 1\le l\le N}} (\Phi_k(E_l))\right)^2, \tag{6}$$

where Φ_n denotes the n-th Hermite function,

$$\Phi_n(x) = \frac{(-1)^n\pi^{-1/4}}{\sqrt{2^n n!}}(\partial^n e^{-x^2})e^{x^2/2}. \tag{7}$$

One has $\Phi_n(x) = P_n(x)e^{-x^2/2}$, where P_n is the Hermite polynomial of degree n. The probability density of eigenvalues is by definition,

$$R_N(E) = \int p(E, E_2, \ldots E_N) dE_2 \ldots dE_N. \tag{8}$$

Using (6) it is given, up to normalization, by the following expression,

$$\sum_{\sigma,\pi \in S_N} \epsilon(\sigma)\epsilon(\pi) \int \Phi_{\sigma(1)}(E)\Phi_{\sigma(2)}(E_2)\ldots\Phi_{\sigma(N)}(E_N)$$
$$\Phi_{\pi(1)}(E)\Phi_{\pi(2)}(E_2)\ldots\Phi_{\pi(N)}(E_N)dE_2\ldots dE_N$$

The integral is nonzero only if $\sigma(k) = \pi(k)$, $k = 2, \ldots, N$, and hence only for $\sigma = \pi$. Since $\sigma(1) = \pi(1)$ can be any number between 0 and $N-1$, one gets $R_N = \frac{1}{N}K_N(E)$, where

$$K_N(E) = \sum_{j=0}^{N-1} \Phi_j(E)^2. \tag{9}$$

One has $K_N(E) = K_N(E, E)$, where $K_N = \sum_{j=0}^{N-1} |\Phi_j >< \Phi_j|$ is the orthogonal projection on the spectral subspace $H \leq 2N-1$ of the harmonic oscillator $H = -\partial^2 + x^2 = p^2 + q^2$.

The asymptotic behaviour of $K_N(E)$ is then given by the semiclassical approximation. To the part of the spectrum $H \leq 2N-1$ corresponds the disk of radius $\sqrt{2N-1}$ in the (p,q) plane with measure $\frac{1}{2\pi}dp \wedge dq$. Then the asymptotic behavior of $\int f(E)K_N(E)dE$ =Trace$(f(q)K_N)$, $N \to \infty$, is given by,

$$\frac{1}{2\pi}\int_{p^2+q^2 \leq 2N} f(q)dp \wedge dq = \frac{1}{\pi}\int_{-\sqrt{2N}}^{\sqrt{2N}} f(E)\sqrt{2N - E^2}dE \tag{10}$$

and for R_N one gets the asymptotic behavior

$$R_N(E) \sim \frac{1}{\pi N}\sqrt{2N - E^2} \tag{11}$$

i.e. the semicircle law.

This gives the density of eigenvalues and the analogue of the transformation (4) near $E = 0$ is $x_j = \frac{\sqrt{2N}}{\pi} E_j$. To study the local fluctuations one considers the two point correlation function,

$$R_N(E_1, E_2) = \int p(E_1, E_2, E_3 \ldots E_N)dE_3 \ldots dE_N \tag{12}$$

and its limit behavior when $N \to \infty$, with $E_j = \frac{\pi x_j}{\sqrt{2N}}$, $j = 1, 2$. As before, one needs to compute,

$$\sum_{\sigma,\pi \in S_N} \epsilon(\sigma)\epsilon(\pi) \int \Phi_{\sigma(1)}(E)\Phi_{\sigma(2)}(E_2)\ldots\Phi_{\sigma(N)}(E_N)$$
$$\Phi_{\pi(1)}(E)\Phi_{\pi(2)}(E_2)\ldots\Phi_{\pi(N)}(E_N)dE_3\ldots dE_N$$

One gets nonzero terms only if $\sigma(k) = \pi(k)$, $k = 3, \ldots, N$. It means that possible nonzero terms are obtained when $\sigma(1) = \pi(1)$, $\sigma(2) = \pi(2)$, and in this case $\epsilon(\sigma) = \epsilon(\pi)$ and when $\sigma(1) = \pi(2)$, $\sigma(2) = \pi(1)$, and in this case $\epsilon(\sigma) = -\epsilon(\pi)$. The value of the integral is thus,

$$\frac{1}{N(N-1)} \sum_{k,l} \Phi_k(E_1)^2 \Phi_l(E_2)^2 - \Phi_k(E_1)\Phi_k(E_2)\Phi_l(E_1)\Phi_l(E_2) =$$
$$\frac{1}{N(N-1)} \begin{vmatrix} K_N(E_1,E_1) & K_N(E_1,E_2) \\ K_N(E_1,E_2) & K_N(E_2,E_2) \end{vmatrix}$$

The asymptotic behavior of $K_N(x, y)$ is obtained using ([Me]),

$$K_N(x,y) = \sum_{n=0}^{N-1} \Phi_n(x)\Phi_n(y) = \left(\frac{N}{2}\right)^{1/2} \frac{\Phi_N(x)\Phi_{N-1}(y) - \Phi_N(y)\Phi_{N-1}(x)}{x-y} \tag{13}$$

from which one easily gets ([Me]),

$$K_N\left(\frac{\pi x_1}{\sqrt{2N}}, \frac{\pi x_2}{\sqrt{2N}}\right) \sim \frac{\sqrt{2N}}{\pi} \frac{\sin \pi x_1 \cos \pi x_2 - \sin \pi x_2 \cos \pi x_1}{\pi(x_1 - x_2)} = \frac{\sqrt{2N}}{\pi} \frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)} \tag{14}$$

One then obtains,

$$R_N(\frac{\pi x_1}{\sqrt{2N}}, \frac{\pi x_2}{\sqrt{2N}}) \sim \frac{2}{\pi^2 N}\left(1 - \left(\frac{\sin \pi(x_1 - x_2)}{\pi(x_1 - x_2)}\right)^2\right), \tag{15}$$

which is identical to the Montgomery-Odlyzko law.

This is thus an excellent motivation to try and find a natural pair $(\mathcal{H}, D)$ where naturality should mean for instance that one should not even have to define the zeta function, let alone its analytic continuation, in order to obtain the pair (in order for instance to avoid the joke of defining $\mathcal{H}$ as the ℓ^2 space built on the zeros of zeta).

IV Quantum chaos and the minus sign problem.

We shall first describe following [B] the direct atempt to construct the Polya-Hilbert space from quantization of a classical dynamical system. The original motivation for the theory of random matrices comes from quantum mechanics. In this theory the quantization of the classical dynamical system given by the phase space X and hamiltonian h gives rise to a Hilbert space $\mathcal{H}$ and a selfadjoint operator H whose spectrum is the essential physical observable of the system. For complicated systems the only useful information about this spectrum is that, while the average part of the counting function,

$$N(E) = \#\text{ eigenvalues of } H \text{ in } [0,E] \tag{1}$$

is computed by a semiclassical approximation mainly as a volume in phase space, the oscillatory part,

$$N_{\text{osc}}(E) = N(E) - \langle N(E) \rangle \tag{2}$$

is the same as for a random matrix, governed by the statistic dictated by the symmetries of the system.

In the absence of a magnetic field, i.e. for a classical hamiltonian of the form,

$$h = \frac{1}{2m} p^2 + V(q) \tag{3}$$

where V is a real-valued potential on configuration space, there is a natural symmetry of classical phase space, called time reversal symmetry,

$$T(p,q) = (-p,q) \tag{4}$$

which preserves h, and entails that the correct ensemble on the random matrices is not the above GUE but rather the gaussian orthogonal ensemble: GOE. Thus the oscillatory part $N_{\text{osc}}(E)$ behaves in the same way as for a random *real symmetric* matrix.

Of course H is just a specific operator in $\mathcal{H}$ and, in order that it behaves *generically* it is necessary (cf. [B]) that the classical hamiltonian system (X,h) be *chaotic* with isolated *periodic orbits* whose unstability exponents (i.e. the logarithm of the eigenvalues of the Poincaré return map acting on the transverse space to the orbits) are different from 0.

One can then ([B]) write down an asymptotic semiclassical approximation to the oscillatory function $N_{\rm osc}(E)$

$$N_{\rm osc}(E) = \frac{1}{\pi}\,\mathrm{Im}\int_0^\infty \mathrm{Trace}(H-(E+i\eta))^{-1}\,id\eta \tag{5}$$

using the stationary phase approximation of the corresponding functional integral. For a system whose configuration space is 2-dimensional, this gives ([B] (15)),

$$N_{\rm osc}(E) \simeq \frac{1}{\pi}\sum_{\gamma_p}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{1}{2\mathrm{sh}\left(\frac{m\lambda_p}{2}\right)}\,\sin(S_{\rm pm}(E)) \tag{6}$$

where the γ_p are the primitive periodic orbits, the label m corresponds to the number of traversals of this orbit, while the corresponding unstability exponents are $\pm\lambda_p$. The phase $S_{\rm pm}(E)$ is up to a constant equal to $m\,E\,T_\gamma^\#$ where $T_\gamma^\#$ is the period of the primitive orbit γ_p.

The formula (6) gives very precious information ([B]) on the hypothetical "Riemann flow" whose quantization should produce the Polya-Hilbert space. The point is that the Euler product formula for the zeta function yields (cf. [B]) a similar asymptotic formula for $N_{\rm osc}(E)$ (3),

$$N_{\rm osc}(E) \simeq \frac{-1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{1}{p^{m/2}}\,\sin\left(m\,E\,\log p\right). \tag{7}$$

Comparing (6) and (7) gives the following information,

(A) The periodic primitive orbits should be labelled by the prime numbers $p = 2,3,5,7,\ldots$, their periods should be the $\log p$ and their unstability exponents $\lambda_p = \pm\log p$.

Moreover, since each orbit is only counted once, the Riemann flow should not possess the symmetry T of (4) whose effect would be to duplicate the count of orbits. This last point excludes in particular the geodesic flows since they have the time reversal symmetry T. Thus we get

(B) The Riemann flow cannot satisfy time reversal symmetry.

However there are two important mismatches (cf. [B]) between the two formulas (6) and (7). The first one is the overall *minus sign* in front of formula (7), the second one is that though $2\mathrm{sh}\left(\frac{m\lambda_p}{2}\right) \sim p^{m/2}$ when $m\to\infty$, we do not have an equality for finite values of m.

We shall see in the next section how to overcome this $-$ sign problem. To put the solution in physics terminology, the spectral interpretation will appear in a natural manner as an absorption spectrum. Recall that spectral lines which are observed in spectroscopy (e.g. from the light coming from distant stars) are of two kinds: on the one hand one observes emission lines which are bright lines on a dark background, on the other hand one observes absorption lines which are dark lines on a bright background. It is the latter which will serve as a model for our spectral realization of zeros of zeta.

V Spectral interpretation of critical zeros.

The very same $-$ sign appears in the Riemann-Weil explicit formula ([W3]) which we now briefly describe. One lets k be a global field. One identifies the quotient $C_k/C_{k,1}$ with the range of the module,

$$N = \{|g|\,;\, g \in C_k\} \subset \mathbb{R}_+^* \,. \tag{1}$$

One endows N with its normalized Haar measure d^*x where for modulated groups the normalization is as in Weil ([W3]),

$$\int_{1\leq|u|\leq\Lambda} d^*u \sim \log\Lambda \qquad \text{for } \Lambda \to \infty\,, \tag{2}$$

Given a function F on N such that, for some $b > \frac{1}{2}$,

$$|F(\nu)| = 0(\nu^b) \quad \nu \to 0\,,\; |F(\nu)| = 0(\nu^{-b})\,,\; \nu \to \infty\,,$$

one lets,

$$\Phi(s) = \int_N F(\nu)\,\nu^{1/2-s}\,d^*\nu\,. \tag{3}$$

Given a Grössencharakter $\mathcal{X}$, i.e. a character of C_k and any ρ in the strip $0 < \mathrm{Re}(\rho) < 1$, one lets $N(\mathcal{X},\rho)$ be the order of $L(\mathcal{X},s)$ at $s=\rho$. One lets,

$$S(\mathcal{X},F) = \sum_\rho N(\mathcal{X},\rho)\,\Phi(\rho) \tag{4}$$

where the sum takes place over ρ's in the above open strip. One then defines a distribution Δ on C_k by,

$$\Delta = \log|d^{-1}|\,\delta_1 + D - \sum_v D_v\,, \tag{5}$$

where δ_1 is the Dirac mass at $1 \in C_k$, where d is a differental idele of k so that $|d|^{-1}$ is up to sign the discriminant of k when char $(k) = 0$ and is q^{2g-2} when k is a function field over a curve of genus g with coefficients in the finite field $\mathbb{F}_q$.

The distribution D is given by,

$$D(f) = \int_{C_k} f(w)\,(|w|^{1/2} + |w|^{-1/2})\,d^*w \tag{6}$$

where the Haar measure d^*w is normalized. The distributions D_v are labeled by the places v of k and are obtained as follows. For each v one considers the natural proper homomorphism,

$$k_v^* \to C_k \ , \ x \to \text{class of } (1, \dots, x, 1 \dots) \tag{7}$$

of the multiplicative group of the local field k_v in the idele class group C_k.

One then has,

$$D_v(f) = Pfw \int_{k_v^*} \frac{f(u)}{|1-u|} \, |u|^{1/2} \, d^*u \tag{8}$$

where the Haar measure d^*u is normalized, and where the Weil Principal value Pfw of the integral is obtained as follows, for a local field $K = k_v$,

$$Pfw \int_{k_v^*} 1_{R_v^*} \frac{1}{|1-u|} \, d^*u = 0 \,, \tag{9}$$

if the local field k_v is non Archimedean, and otherwise:

$$Pfw \int_{k_v^*} \varphi(u) \, d^*u = PF_0 \int_{\mathbb{R}_+^*} \psi(\nu) \, d^*\nu \,, \tag{10}$$

where $\psi(\nu) = \int_{|u|=\nu} \varphi(u) \, d_\nu u$ is obtained by integrating φ over the fibers, while

$$PF_0 \int \psi(\nu) d^*\nu = 2\log(2\pi)\, c + \lim_{t\to\infty} \left(\int (1 - f_0^{2t}) \, \psi(\nu) \, d^*\nu - 2c \log t \right) \,, \tag{11}$$

where one assumes that $\psi - c \ f_1^{-1}$ is integrable on $\mathbb{R}_+^*$, and

$$f_0(\nu) = \inf(\nu^{1/2}, \nu^{-1/2}) \qquad \forall \, \nu \in \mathbb{R}_+^* \,, \qquad f_1 = f_0^{-1} - f_0 \,.$$

The Weil explicit formula is then,

Theorem 1. ([W3]) *With the above notations one has* $S(\mathcal{X}, F) = \Delta(F(|w|) \, \mathcal{X}(w))$.

Let us make the following change of variables,

$$|g|^{-1/2} \, h(g^{-1}) = F(|g|) \, \mathcal{X}_0(g) \,, \tag{12}$$

and rewrite the above equality in terms of h.

By (3) one has,

$$\Phi\left(\frac{1}{2} + is\right) = \int_{C_k} F(|g|) \, |g|^{-is} \, d^*g \,, \tag{13}$$

thus, in terms of h,

$$\int h(g) \, \mathcal{X}_1(g) \, |g|^{1/2+is} \, d^*g = \int F(|g^{-1}|) \, \mathcal{X}_0(g^{-1}) \, \mathcal{X}_1(g) \, |g|^{is} \, d^*g \,, \tag{14}$$

which is equal to 0 if $\mathcal{X}_1/C_{k,1} \neq \mathcal{X}_0/C_{k,1}$ and for $\mathcal{X}_1 = \mathcal{X}_0$,

$$\int h(g)\,\mathcal{X}_0(g)\,|g|^{1/2+is}\,d^*g = \Phi\left(\frac{1}{2}+is\right). \tag{15}$$

We define the Fourier transform on C_k by,

$$\widehat{h}(\chi, z) = \int_{C^k} h(u)\,\chi(u)\,|u|^z\,d^*u\,. \tag{16}$$

Thus,

$$\operatorname{Supp}\widehat{h} \subset \mathcal{X}_0 \times \mathbb{R}\,,\ \widehat{h}(\mathcal{X}_0, \rho) = \Phi(\rho)\,, \tag{17}$$

and

$$S(\mathcal{X}_0, F) = \sum_{\substack{L(\mathcal{X},\rho)=0,\mathcal{X}\in\widehat{C}_{k,1}\\ 0<\operatorname{Re}\rho<1}} \widehat{h}(\mathcal{X}, \rho) \tag{18}$$

using a fixed decomposition $C_k = C_{k,1} \times N$.

Let us now evaluate each term in (5).

The first gives $(\log|d^{-1}|)\,h(1)$. One has, using (6) and (12),

$$\begin{aligned}\langle D, F(|g|)\,\mathcal{X}_0(g)\rangle &= \int_{C_k} |g|^{-1/2}\,h(g^{-1})\,(|g|^{1/2}+|g|^{-1/2})\,d^*g\\ &= \int_{C_k} h(u)\,(1+|u|)\,d^*u = \widehat{h}(0)+\widehat{h}(1)\,,\end{aligned}$$

where for the trivial character of $C_{k,1}$ one uses the notation

$$\widehat{h}(z) = \widehat{h}(1, z) \qquad \forall\, z \in \mathbb{C}.$$

Thus the first two terms of (5) give

$$(\log|d^{-1}|)\,h(1) + \widehat{h}(0) + \widehat{h}(1)\,. \tag{19}$$

Let then v be a place of k, one has by (8) and (12),

$$\langle D_v, F(|g|)\,\mathcal{X}_0(g)\rangle = Pfw\int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\,d^*u\,.$$

We can thus write the contribution of the last term of (5) as,

$$-\sum_v Pfw\int_{k_v^*} \frac{h(u^{-1})}{|1-u|}\,d^*u\,. \tag{20}$$

Thus the equality of Weil can be rewritten as,

$$\widehat{h}(0)+\widehat{h}(1) - \sum_{\substack{L(\mathcal{X},\rho)=0,\mathcal{X}\in\widehat{C}_{k,1}\\ 0<\operatorname{Re}\rho<1}} \widehat{h}(\mathcal{X}, \rho) = (\log|d|)\,h(1)+ \tag{21}$$

$$\sum_v Pfw \int_{k_v^*} \frac{h(u^{-1})}{|1-u|} d^*u\,.$$

One can slightly improve on this formula and write it in the form,

$$\sum_{L(\chi,\rho)=0} \widehat{h}(\chi,\rho) - \widehat{h}(0) - \widehat{h}(1) = -\sum_v \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|} d^*u\,, \tag{22}$$

where the principal values $\int'$ depend upon the global Fourier transform. This point was noticed by S. Haran ([H]) and it is crucial for us that these principal values actually coincide ([Co]) with those dictated by the explicit form of the trace formula (cf. sections VIII and IX).

Let us use the geometric dictionary (cf. Introduction) when $\mathrm{char}(k) \neq 0$, namely,

Algebraic Geometry	*Function Theory*
Eigenvalues of action of Frobenius on $H^*_{\mathrm{et}}(\overline{\Sigma}, \mathbb{Q}_\ell)$	Zeros and poles of ζ
Poincaré duality in ℓ-adic cohomology	Functional equation
Lefschetz formula for the Frobenius	Explicit formulas
Castelnuovo positivity	Riemann Hypothesis

The geometric origin of the $-$ sign in (22) becomes clear, (22) is the Lefschetz formula,

$$\#\text{ of fixed points of } \varphi = \mathrm{Trace}\,\varphi/H^0 - \mathrm{Trace}\,\varphi/H^1 + \mathrm{Trace}\,\varphi/H^2 \tag{23}$$

in which the space $H^1_{\mathrm{et}}(\overline{\Sigma}, \mathbb{Q}_\ell)$ which provides the spectral realization of the zeros appears with a $-$ sign. This indicates that the spectral realization of zeros of zeta should be of cohomological nature or to be more specific, that the Polya-Hilbert space should appear as the last term of an exact sequence of Hilbert spaces,

$$0 \to \mathcal{H}_0 \xrightarrow{T} \mathcal{H}_1 \to \mathcal{H} \to 0\,. \tag{24}$$

The example one can keep in mind for (24) is the assembled Euler complex for a Riemann surface, where $\mathcal{H}_0$ is the *codimension* 2 *subspace* of the space of differential forms of even degree orthogonal to harmonic forms, where $\mathcal{H}_1$ is the space of 1-forms and where $T = d + d^*$ is the sum of the de Rham coboundary with its adjoint d^*.

Since we want to obtain the spectral interpretation not only for zeta functions but for all L-functions with Grössencharakter we do not expect to have only an action of $\mathbb{Z}$ for $\mathrm{char}(k) > 0$ corresponding to the Frobenius, or of the group $\mathbb{R}_+^*$ if $\mathrm{char}(k) = 0$, but to have the equivariance of (24) with respect to a natural action of the Idele class group $C_k = GL_1(A)/k^*$.

Let $X = A/k^*$ be the Adele class space. Our basic idea is to take for $\mathcal{H}_0$ a suitable completion of the codimension 2 subspace of functions on X such that,

$$f(0) = 0\,, \quad \int f\,dx = 0\,, \tag{25}$$

while $\mathcal{H}_1 = L^2(C_k)$ and T is the restriction map E coming from the inclusion $C_k \to X$, multiplied by $|a|^{1/2}$,

$$(Ef)(a) = |a|^{1/2}\, f(a)\,. \tag{26}$$

The action of C_k is then the obvious one, for $\mathcal{H}_0$

$$(U(g)f)(x) = f(g^{-1}\,x) \qquad \forall\, g \in C_k \tag{27}$$

using the action of C_k on X by multiplication,

$$(j, a) \to ja \qquad \forall\, j \in C_k\ ,\ a \in X \tag{28}$$

and similarly the regular representation V for $\mathcal{H}_1$.

There is a subtle point however which is that since C_k is abelian and non compact, its regular representation does not contain any finite dimensional subrepresentation so that the Polya-Hilbert space cannot be a subrepresentation (or unitary quotient) of V. There is an easy way out (which we shall improve shortly) which is to replace the regular representation $L^2(C_k)$ by $L^2_\delta(C_k)$ using the polynomial weight $(\log^2 |a|)^{\delta/2}$, i.e. the norm,

$$\|\xi\|_\delta^2 = \int_{C_k} |\xi(a)|^2\,(1 + \log^2 |a|)^{\delta/2}\, d^*a\,. \tag{29}$$

The left regular representation V of C_k on $L^2_\delta(C_k)$ is

$$(V(a)\,\xi)\,(g) = \xi(a^{-1}\,g) \qquad \forall\, g, a \in C_k\,. \tag{30}$$

Note that because of the weight $(1 + \log^2 |x|)^{\delta/2}$, this representation is *not* unitary but it satisfies the growth estimate

$$\|V(g)\| = 0\,(\log |g|)^{\delta/2} \quad \text{when} \quad |g| \to \infty \tag{31}$$

Similarly, we shall construct the Hilbert space L^2_δ of functions on X with growth indexed by $\delta > 1$. Since X is a quotient space we shall first learn in the usual manifold case how to obtain the Hilbert space $L^2(M)$ of square integrable functions on a manifold M by working only on the universal cover $\widetilde{M}$ with the action of $\Gamma = \pi_1(M)$. Every function $f \in C_c^\infty(\widetilde{M})$ gives rise to a function $\widetilde{f}$ on M by

$$\widetilde{f}(x) = \sum_{\pi(\widetilde{x})=x} f(\widetilde{x}) \tag{32}$$

and all $g \in C^\infty(M)$ appear in this way. Moreover, one can write the Hilbert space inner product $\int_M \widetilde{f_1}(x)\, \widetilde{f_2}(x)\, dx$, in terms of f_1 and f_2 alone. Thus $\|\widetilde{f}\|^2 = \int \left|\sum_{\gamma\in\Gamma} f(\gamma x)\right|^2 dx$ where the integral is performed on a fundamental domain for Γ acting on $\widetilde{M}$. This formula defines a prehilbert space norm on $C_c^\infty(\widetilde{M})$ and $L^2(M)$ is just the completion of $C_c^\infty(\widetilde{M})$ for that norm. Note that any function of the form $f - f_\gamma$ has vanishing norm and hence disappears in the process of completion.

We can now define the Hilbert space $L_\delta^2(X)_0$ as the completion of the codimension 2 subspace

$$\mathcal{S}(A)_0 = \{f \in \mathcal{S}(A) \; ; \; f(0) = 0 \, , \; \int f\, dx = 0\} \tag{33}$$

for the norm $\| \; \|_\delta$ given by

$$\|f\|_\delta^2 = \int \Big|\sum_{q\in k^*} f(qx)\Big|^2 (1 + \log^2 |x|)^{\delta/2}\, |x|\, d^*x \tag{34}$$

where the integral is performed on A^*/k^* and d^*x is the multiplicative Haar measure on A^*/k^*. Note that $|qx| = |x|$ for any $q \in k^*$.

The key point is that we use the measure $|x|\, d^*x$ instead of the additive Haar measure dx. Of course for a local field K one has $dx = |x|\, d^*x$ but this fails in the above global situation. Instead one has,

$$dx = \lim_{\varepsilon\to 0} \varepsilon\, |x|^{1+\varepsilon}\, d^*x \, , \tag{35}$$

but the corresponding divergent normalization coefficient plays no role in computations of adjoints or of traces of operators.

One has a natural representation of C_k on $L_\delta^2(X)_0$ given by (27), and the result is independent of the choice of a lift of j in $J_k = \mathrm{GL}_1(A)$ because the functions $f - f_q$ are in the kernel of the norm. The conditions (33) which define $\mathcal{S}(A)_0$ are invariant under the action of C_k and give the following action of C_k on the 2-dimensional supplement of $\mathcal{S}(A)_0 \subset \mathcal{S}(A)$; this supplement is $\mathbb{C} \oplus \mathbb{C}(1)$ where $\mathbb{C}$ is the trivial C_k module (corresponding to $f(0)$) while the Tate twist $\mathbb{C}(1)$ is the module

$$(j, \lambda) \to |j|\, \lambda \tag{36}$$

coming from the equality

$$\int f(j^{-1} x)\, dx = |j| \int f(x)\, dx \, . \tag{37}$$

We let E be the linear isometry from $L^2_\delta(X)_0$ into $L^2_\delta(C_k)$ given by the equality,

$$E(f)\,(g) = |g|^{1/2} \sum_{q \in k^*} f(qg) \qquad \forall\, g \in C_k\,. \tag{38}$$

By comparing (29) with (34) we see that E is an isometry and the factor $|g|^{1/2}$ is dictated by comparing the measures $|g|\,d^*g$ of (34) with d^*g of (29).

One has $E(U(a)\,f)\,(g) = |g|^{1/2} \sum_{k^*} (U(a)\,f)\,(qg) = |g|^{1/2} \sum_{k^*} f(a^{-1}\,qg) = |a|^{1/2}\,|a^{-1}\,g|^{1/2} \sum_{k^*} f(q\,a^{-1}\,g) = |a|^{1/2}\,(V(a)\,E(f))\,(g)$. Thus,

$$E\,U(a) = |a|^{1/2}\,V(a)\,E\,. \tag{39}$$

This equivariance shows that the range of E in $L^2_\delta(C_k)$ is a closed invariant subspace for the representation V.

The following theorem and its corollary show that the cokernel $\mathcal{H} = L^2_\delta(C_k)/\operatorname{Im}(E)$ of the isometry E plays the role of the Polya-Hilbert space. Since $\operatorname{Im} E$ is invariant under the representation V we let W be the corresponding representation of C_k on $\mathcal{H}$.

Let $\operatorname{char}(k) = 0$ so that $\operatorname{Mod} k = \mathbb{R}^*_+$ and $C_k = K \times \mathbb{R}^*_+$ where K is the compact group $C_{k,1} = \{a \in C_k\,;\ |a| = 1\}$.

Theorem 2.([Co]) *Let $\delta > 1$, $\mathcal{H}$ be the cokernel of T in $L^2_\delta(C)$ and W the quotient representation of C_k. Let χ be a character of K, $\widetilde{\chi} = \chi \times 1$ the corresponding character of C_k. Let $\mathcal{H}_\chi = \{\xi \in \mathcal{H}\,;\ W(g)\,\xi = \chi(g)\,\xi \quad \forall\, g \in K\}$ and $D_\chi = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\,(W(e^\epsilon) - 1)$. Then D_χ is an unbounded closed operator with discrete spectrum,* $\operatorname{Sp} D_\chi \subset i\,\mathbb{R}$ *is the set of imaginary parts of zeros of the L function with Grössencharakter $\widetilde{\chi}$ which have real part $1/2$. Moreover the spectral multiplicity of ρ is the largest integer $n < \frac{1+\delta}{2}$ in $\{1, \ldots,$ multiplicity as a zero of $L\}$.*

Theorem 2 has a similar formulation when the characteristic of k is non zero. The following corollary is valid for global fields k of arbitrary characteristic.

Corollary 3. *For any Schwartz function $h \in \mathcal{S}(C_k)$ the operator $W(h) = \int W(g)\,h(g)\,d^*\,g$ in $\mathcal{H}$ is of trace class, and its trace is given by*

$$\mathit{Trace}\,W(h) = \sum_{\substack{L\left(\widetilde{\chi},\frac{1}{2}+\rho\right)=0 \\ \rho \in i\,\mathbb{R}/N^\perp}} \widehat{h}(\widetilde{\chi}, \rho)$$

where the multiplicity is counted as in Theorem 2 and where the Fourier transform $\widehat{h}$ of h is defined by,

$$\widehat{h}(\widetilde{\chi}, \rho) = \int_{C_k} h(u)\,\widetilde{\chi}(u)\,|u|^\rho\,d^*\,u\,.$$

This result is only preliminary because of the unwanted parameter δ which artificially restricts the multiplicities. Let us pursue a little further the analogy between the exact sequence,

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(C_k) \to \mathcal{H} \to 0, \tag{40}$$

and the exact sequence,

$$0 \to \mathcal{H}_0 \xrightarrow{T} \mathcal{H}_1 \to \mathcal{H} \to 0\,, \tag{41}$$

coming from the assembled Euler complex for a Riemann surface. Thus, here $\mathcal{H}_0$ is the *codimension* 2 *subspace* of the space of differential forms of even degree orthogonal to harmonic forms, and $\mathcal{H}_1$ is the space of 1-forms while $T = d + d^*$ is the sum of the de Rham coboundary with its adjoint d^*.

In this case Poincaré duality is given by the Hodge $*$ operation, which when multiplied by suitable powers of i satisfies $\tau^2 = 1$ and anticommutes with $T = d{+}d^*$.

In our case, the analogue of the Hodge $*$ operation is given on $\mathcal{H}_0 = L^2_\delta(X)_0$ by the Fourier transform,

$$(Ff)(x) = \int_A f(y)\,\alpha(xy)\,dy \qquad \forall\, f \in \mathcal{S}(A)_0. \tag{42}$$

Here, we identified the Abelian group A of Adeles of k with its Pontrjagin dual by means of the pairing $\langle a, b\rangle = \alpha(ab)$, where $\alpha : A \to U(1)$ is a nontrivial character which vanishes on $k \subset A$. Note that such a character is *not canonical*, but that any two such characters α and α' are related by k^*,

$$\alpha'(a) = \alpha(qa) \quad \forall a \in A\,. \tag{43}$$

It follows that the corresponding Fourier transformations on A are related by

$$\hat{f}' = \hat{f}_q\,. \tag{44}$$

which, because we take the quotient by k^*, is independent of the choice of additive character α of A. Note also that $F^2 = 1$ on the quotient.

On $\mathcal{H}_1 = L^2_\delta(C_k)$ the Hodge $*$ is given by,

$$(\tau\,\xi)(a) = -\xi(a^{-1}) \qquad \forall\, a \in C_k\,. \tag{45}$$

The Poisson formula means exactly that E anticommutes with the $*$ operation.

If we modify the choice of non canonical isomorphism $C_k = K \times \mathbb{R}^*_+$ where K is the compact group $C_{k,1} = \{a \in C_k\,;\ |a| = 1\}$, this modifies the operator D by

$$D' = D - i\,s \tag{46}$$

where $s \in \mathbb{R}$ is determined by the equality

$$\widetilde{\chi}'(g) = \widetilde{\chi}(g)\,|g|^{is} \qquad \forall\, g \in C_k\,. \tag{47}$$

The coherence of the statement of the theorem is insured by the equality

$$L(\widetilde{\chi}', z) = L(\widetilde{\chi}, z + i\,s) \qquad \forall\, z \in \mathbb{C}. \tag{48}$$

When the zeros of L have multiplicity and δ is large enough the operator D is *not* semisimple and has a non trivial Jordan form.

The proof of theorem 2 ([Co]) is based on the distribution theoretic interpretation by A. Weil [W2] of the idea of Tate and Iwasawa on the functional equation. Our construction should be compared with [Bg] and [Z].

Since we obtain the Hilbert space $L^2_\delta(X)_0$ by imposing two linear conditions on $\mathcal{S}(A)$,

$$0 \to \mathcal{S}(A)_0 \to \mathcal{S}(A) \overset{L}{\to} \mathbb{C} \oplus \mathbb{C}(1) \to 0 \tag{49}$$

we shall define $L^2_\delta(X)$ so that it fits in an exact sequence of C_k-modules

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(X) \to \mathbb{C} \oplus \mathbb{C}(1) \to 0\,. \tag{50}$$

(Note that, as in the case of Riemann surfaces, the $*$ operation (42) does exchange the two modules $\mathbb{C}$ and $\mathbb{C}(1)$). We can then use the exact sequence of C_k-modules

$$0 \to L^2_\delta(X)_0 \to L^2_\delta(C_k) \to \mathcal{H} \to 0 \tag{51}$$

together with Corollary 3 to compute in a formal manner what the character of the module $L^2_\delta(X)$ should be. We obtain,

$$\text{“Trace”}\ (U(h)) = \widehat{h}(0) + \widehat{h}(1) - \sum_{\substack{L(\chi,\rho)=0 \\ \mathrm{Re}\rho=\frac{1}{2}}} \widehat{h}(\chi,\rho) + \infty\, h(1) \tag{52}$$

where $\widehat{h}(\chi,\rho)$ is defined by Corollary 3 and

$$U(h) = \int_{C_k} U(g)\, h(g)\, d^*g \tag{53}$$

while the test function h is in a suitable function space. Note that the trace on the left hand side of (52) only makes sense after a suitable regularisation since the left regular representation of C_k is not traçable. This situation is similar to the one encountered by Atiyah and Bott ([AB]) in their proof of the Lefschetz formula. We shall first learn how to compute in a formal manner the above trace from the fixed points of the action of C_k on X. In sections VIII and IX, we shall show how to regularize the trace and completely eliminate the parameter δ.

VI The distribution trace formula for flows on manifolds

In order to understand how the left hand side of V(52) should be computed we shall first give an account of the proof of the usual Lefschetz formula by Atiyah-Bott ([AB]) and describe the computation of the distribution theoretic trace for flows on manifolds, which is a variation on the theme of [AB] and is due to Guillemin-Sternberg [GS]. We first recall for the convenience of the reader the coordinate free treatment of distributions of [GS].

Given a vector space E over $\mathbb{R}$, $\dim E = n$, a density is a map, $\rho \in |E^*|$,

$$\rho : \wedge^n E \to \mathbb{C} \tag{1}$$

such that $\rho(\lambda v) = |\lambda| \, \rho(v) \qquad \forall \, \lambda \in \mathbb{R}, \qquad \forall \, v \in \wedge^n E.$

Given a linear map $T : E \to F$ we let $|T^*| : |F^*| \to |E^*|$ be the corresponding linear map, it depends contravariantly on T.

A smooth compactly supported density $\rho \in C_c^\infty(M, |T^*M|)$ on an arbitrary manifold M has a canonical integral,

$$\int \rho \in \mathbb{C}. \tag{2}$$

One defines the generalized sections of a vector bundle L on M as the dual space of $C_c^\infty(M, L^* \otimes |T^*M|)$

$$C^{-\infty}(M, L) = \text{ dual of } C_c^\infty(M, L^* \otimes |T^*M|) \tag{3}$$

where L^* is the dual bundle. One has a natural inclusion,

$$C^\infty(M, L) \subset C^{-\infty}(M, L) \tag{4}$$

given by the pairing

$$\sigma \in C^\infty(M, L) \ , \ s \in C_c^\infty(M, L^* \otimes |T^*M|) \to \int \langle s, \sigma \rangle \tag{5}$$

where $\langle s, \sigma \rangle$ is viewed as a density, $\langle s, \sigma \rangle \in C_c^\infty(M, |T^*M|)$.

One has a similar notion of generalized section with compact support.

Given a smooth map $\varphi : X \to Y$, then if φ is *proper*, it gives a (contravariantly) associated map

$$\varphi^* : C_c^\infty(Y, L) \to C_c^\infty(X, \varphi^*(L)) \ , \ (\varphi^* \, \xi)(x) = \xi(\varphi(x)) \tag{6}$$

where $\varphi^*(L)$ is the pull back of the vector bundle L.

Thus, given a linear form on $C_c^\infty(X, \varphi^*(L))$ one has a (covariantly) associated linear form on $C_c^\infty(Y, L)$. In particular with L trivial we see that generalized densities $\rho \in C^{-\infty}(X, |T^*X|)$ pushforward,

$$\varphi_*(\rho) \in C^{-\infty}(Y, |T^*Y|) \tag{7}$$

with $\langle \varphi_*(\rho), \xi \rangle = \langle \rho, \varphi^* \xi \rangle \qquad \forall \xi \in C_c^\infty(X)$.

This gives the natural functoriality of generalized sections, they pushforward under proper maps. However under suitable transversality conditions which are automatic for submersions, generalized sections also pull back. For instance, if φ is a fibration and $\rho \in C_c^\infty(X, |T^*X|)$ is a density then one can integrate ρ along the fibers, the obtained density on Y, $\varphi_*(\rho)$ is given as in (7) by

$$\langle \varphi_*(\rho), f \rangle = \langle \rho, \varphi^* f \rangle \qquad \forall f \in C^\infty(Y). \tag{8}$$

The point is that the result is not only a generalized section but a smooth section $\varphi_*(\rho) \in C_c^\infty(Y, |T^*Y|)$.

It follows that if $f \in C^{-\infty}(Y)$ is a generalized function, then one obtains a generalized function $\varphi^*(f)$ on X by,

$$\langle \varphi^*(f), \rho \rangle = \langle f, \varphi_*(\rho) \rangle \qquad \forall \rho \in C_c^\infty(X, |T^*X|)\,. \tag{9}$$

In general, the pullback $\varphi^*(f)$ of a generalized function f, continues to make sense provided the following transversality condition holds,

$$d(\varphi^*(l)) \neq 0 \quad \forall l \in WF(f)\,. \tag{10}$$

where $WF(f)$ is the wave front set of f ([GS]).

Next, let us recall the construction ([GS]) of the generalized section of a vector bundle L on a manifold X associated to a submanifold $Z \subset X$ and a symbol,

$$\sigma \in C^\infty(Z, L \otimes |N_Z|)\,. \tag{11}$$

where N_Z is the normal bundle of Z. The construction is the same as that of the current of integration on a cycle. Given $\xi \in C_c^\infty(X, L^* \otimes |T^*X|)$, the product $\sigma\, \xi/Z$ is a density on Z, since it is a section of $|T_Z^*| = |T_X^*| \otimes |N_Z|$. One can thus integrate it over Z.

When $Z = X$ one has $N_Z = \{0\}$ and $|N_Z|$ has a canonical section, so that the current associated to σ is just given by (5).

Now let $\varphi : X \to Y$ with Z a submanifold of Y and σ as in (11).

Let us assume that φ is transverse to Z, so that for each $x \in X$ with $y = \varphi(x) \in Z$ one has

$$\varphi_*(T_x) + T_{\varphi(x)}(Z) = T_y Y\,. \tag{12}$$

Let

$$\tau_x = \{X \in T_x \ , \ \varphi_*(X) \in T_y(Z)\} . \tag{13}$$

Then φ_* gives a canonical isomorphism,

$$\varphi_* : T_x(X)/\tau_x \simeq T_y(Y)/T_y(Z) = N_y(Z) . \tag{14}$$

And $\varphi^{-1}(Z)$ is a submanifold of X of the same codimension as Z with a natural isomorphism of normal bundles

$$\varphi_* : N_{\varphi^{-1}(Z)} \simeq \varphi^* N_Z . \tag{15}$$

In particular, given a (generalized) δ-section of a bundle L with support Z and symbol $\sigma \in C^\infty(Z, L \otimes |N_Z|)$ one has a corresponding symbol on $\varphi^{-1}(Z)$ given by

$$\varphi^* \sigma(x) = |(\varphi_*)^{-1}| \sigma(\varphi(x)) \in (\varphi^* L)_x \otimes |N_x| \tag{16}$$

using the inverse of the isomorphism (15), which requires the transversality condition.

Now for any δ-section associated to Z, σ, the wave front set is contained in the conormal bundle of the submanifold Z which shows that if φ is transverse to Z the pull back $\varphi^* \delta_{Z,\sigma}$ of the distribution on Y associated to Z, σ makes sense, it is equal to $\delta_{\varphi^{-1}(Z), \varphi^*(\sigma)}$.

Let us now recall the formulation ([GS]) of the Schwartz kernel theorem. One considers a continuous linear map,

$$T : C_c^\infty(Y) \to C^{-\infty}(X) , \tag{17}$$

the statement is that one can write it as

$$(T\,\xi)\,(x) = \int k(x,y)\,\xi(y)\,dy \tag{18}$$

where $k(x,y)\,dy$ is a generalized section,

$$k \in C^{-\infty}(X \times Y \ , \ \mathrm{pr}_Y^*(|T^*Y|)) . \tag{19}$$

Let $f : X \to Y$ be a smooth map, and $T = f^*$ the operator

$$(T\,\xi)\,(x) = \xi\,(f(x)) \qquad \forall\, \xi \in C_c^\infty(Y) . \tag{20}$$

The corresponding k is the δ-section associated to the submanifold of $X \times Y$ given by

$$\mathrm{Graph}(f) = \{(x, f(x)) \ ; \ x \in X\} = Z \tag{21}$$

and its symbol, $\sigma \in C^\infty(Z, \mathrm{pr}_Y^*(|T^*Y|) \otimes |N_Z|)$ is obtained as follows.

Given $\xi \in T_x^*(X)$, $\eta \in T_y^*(Y)$ one has $(\xi,\eta) \in N_Z^*$ iff it is orthogonal to $(v, f_* v)$ for any $v \in T_x(X)$, i.e. $\langle v,\xi\rangle + \langle f_* v, \eta\rangle = 0$ so that

$$\xi = -f_*^t \eta\,. \tag{22}$$

Thus one has a canonical isomorphism $j : T_y^*(Y) \simeq N_Z^*$, $\eta \overset{j}{\to} (-f_*^t\eta,\eta)$. The transposed $(j^{-1})^t$ is given by $(j^{-1})^t(Y) =$ class of $(0,Y)$ in N_Z, $\forall Y \in T_y(Y)$. One has,

$$\sigma = |j^{-1}| \in C^\infty(Z, \mathrm{pr}_Y^*(|T^*Y|) \otimes |N_Z|)\,. \tag{23}$$

We denote the corresponding δ-distribution by

$$k(x,y)\,dy = \delta(y - f(x))\,dy\,. \tag{24}$$

One then checks the formula,

$$\int \delta(y - f(x))\,\xi(y)\,dy = \xi(f(x)) \qquad \forall \xi \in C_c^\infty(Y)\,. \tag{25}$$

Let us now consider a manifold M with a flow F_t

$$F_t(x) = \exp(t\,v)\,x \qquad v \in C^\infty(M, T_M) \tag{26}$$

and the corresponding map f,

$$f : M \times \mathbb{R} \to M\ ,\ f(x,t) = F_t(x)\,. \tag{27}$$

We apply the above discussion with $X = M \times \mathbb{R}$, $Y = M$. The graph of f is the submanifold Z of $X \times Y$,

$$Z = \{(x,t,y)\ ;\ y = F_t(x)\}\,. \tag{28}$$

One lets φ be the diagonal map,

$$\varphi(x,t) = (x,t,x)\ ,\ \varphi : M \times \mathbb{R} \to X \times Y \tag{29}$$

and the first issue is the transversality $\varphi \pitchfork Z$.

We thus need to consider (12) for each (x,t) such that $\varphi(x,t) \in Z$, i.e. such that $x = F_t(x)$. One looks at the image by φ_* of the tangent space $T_x\,M \times \mathbb{R}$ to $M \times \mathbb{R}$ at (x,t). One lets ∂_t be the natural vector field on $\mathbb{R}$. The image of $(X, \lambda\,\partial_t)$ is $(X, \lambda\,\partial_t, X)$ for $X \in T_x\,M, \lambda \in \mathbb{R}$. Dividing the tangent space of $M \times \mathbb{R} \times M$ by the image of φ_* one gets an isomorphism,

$$(X, \lambda\,\partial_t, Y) \to Y - X \tag{30}$$

with $T_x M$. The tangent space to Z is $\{(X', \mu\,\partial_t, (F_t)_* X' + \mu\, v_{F_t(x)});\ X' \in T_x M, \mu \in \mathbb{R}\}$. Thus the transversality condition means that every element of $T_x M$ is of the form

$$(F_t)_* X - X + \mu\, v_x \qquad X \in T_x M\ ,\ \mu \in \mathbb{R}. \tag{31}$$

One has

$$(F_t)_*\, \mu\, v_x = \mu\, v_x \tag{32}$$

so that $(F_t)_*$ defines a quotient map, the Poincaré return map

$$P : T_x/\mathbb{R} v_x \to T_x/\mathbb{R} v_x = N_x \tag{33}$$

and the transversality condition (31) means exactly,

$$1 - P \qquad \text{is invertible.} \tag{34}$$

Let us make this hypothesis and compute the symbol σ of the distribution,

$$\tau = \varphi^*(\delta(y - F_t(x))\, dy)\,. \tag{35}$$

First, as above, let $W = \varphi^{-1}(Z) = \{(x,t)\ ;\ F_t(x) = x\}$. The codimension of $\varphi^{-1}(Z)$ in $M \times \mathbb{R}$ is the same as the codimension of Z in $M \times \mathbb{R} \times M$ so it is $\dim M$ which shows that $\varphi^{-1}(Z)$ is 1-dimensional. If $(x,t) \in \varphi^{-1}(Z)$ then $(F_s(x), t) \in \varphi^{-1}(Z)$. Thus, if we assume that v does not vanish at x, the map,

$$(x,t) \overset{q}{\to} t \tag{36}$$

is locally constant on the connected component of $\varphi^{-1}(Z)$ containing (x,t).
This allows to identify the transverse space to $W = \varphi^{-1}(Z)$ as the product,

$$N_{x,t}^W \simeq N_x \times \mathbb{R} \tag{37}$$

where to $(X, \lambda\,\partial_t) \in T_{x,t}(M \times \mathbb{R})$ we associate the pair $(\widetilde{X}, \lambda)$ given by the class of X in $N_x = T_x/\mathbb{R} v_x$ and $\lambda \in \mathbb{R}$.
The symbol σ of the distribution (35) is a smooth section of $|N^W|$ tensored by the pull back $\varphi^*(L)$ where $L = \mathrm{pr}_Y^*\, |T_M^*|$, and one has

$$\varphi^*(L) \simeq |p^*\, T_M^*| \tag{38}$$

where

$$p(x,t) = x \qquad \forall\, (x,t) \in M \times \mathbb{R}. \tag{39}$$

To compute σ one needs the isomorphism,

$$N_{(x,t)}^W \overset{\varphi_*}{\to} T_{\varphi(x,t)}(M \times \mathbb{R} \times M)/T_{\varphi(x,t)}(Z) = N^Z\,. \tag{40}$$

The map $\varphi_* : N^W_{x,t} \to N^Z$ is given by

$$\varphi_*(X, \lambda\,\partial_t) = (1 - (F_t)_*)\,X - \lambda\,v \qquad X \in N_x\ ,\ \lambda \in \mathbb{R} \tag{41}$$

and the symbol σ is just

$$\sigma = |\varphi_*^{-1}| \in |p^*\,T^*_M| \otimes |N^W|\,. \tag{42}$$

Let us now consider the second projection,

$$q(x,t) = t \in \mathbb{R} \tag{43}$$

and compute the pushforward $q_*(\tau)$ of the distribution τ.

By construction $q_*(\tau)$ is a generalized function.

We first look at the contribution of a periodic orbit, the corresponding part of $\varphi^{-1}(Z)$ is of the form,

$$\varphi^{-1}(Z) = V \times \Gamma \subset M \times \mathbb{R} \tag{44}$$

where Γ is a discrete cocompact subgroup of $\mathbb{R}$, while $V \subset M$ is a one dimensional compact submanifold of M.

To compute $q_*(\tau)$, we let $h(t)\,|dt|$ be a 1-density on $\mathbb{R}$ and pull it back by q as the section on $M \times \mathbb{R}$ of the bundle $q^*\,|T^*|$,

$$\xi(x,t) = h(t)\,|dt|\,. \tag{45}$$

We now need to compute $\int_{\varphi^{-1}(Z)} \xi\,\sigma$. We can look at the contribution of each component: $V \times \{T\}$, $T \in \Gamma$.

One gets ([GS]),

$$T^{\#}\,\frac{1}{|1 - P_T|}\;h(T)\,, \tag{46}$$

where $T^{\#}$ is the length of the primitive orbit or equivalently the covolume of Γ in $\mathbb{R}$ for the Haar measure $|dt|$. We can thus write the contributions of the periodic orbits as

$$\sum_{\gamma_p}\sum_{\Gamma} \mathrm{Covol}(\Gamma)\,\frac{1}{|1 - P_T|}\;h(T)\,, \tag{47}$$

where the test function h vanishes at 0.

The next case to consider is when the vector field v_x has an isolated 0, $v_{x_0} = 0$. In that case, the transversality condition (31) becomes

$$1 - (F_t)_* \quad \text{invertible (at } x_0)\,. \tag{48}$$

One has $F_t(x_0) = x_0$ for all $t \in \mathbb{R}$ and now the relevant component of $\varphi^{-1}(Z)$ is $\{x_0\} \times \mathbb{R}$. The transverse space N^W is identified with T_x and the map $\varphi_* : N^W \simeq N^Z$ is given by:

$$\varphi_* = 1 - (F_t)_* \,. \tag{49}$$

Thus the symbol σ is the scalar function $|1 - (F_t)_*|^{-1}$. The generalized section $q_* \, \varphi^*(\delta(y - F_t(x))\, dy)$ is the function, $t \to |1 - (F_t)_*|^{-1}$. We can thus write the contribution of the zeros of the flow as ([GS]),

$$\sum_{zeros} \int \frac{h(t)}{|1 - (F_t)_*|} \, dt \tag{50}$$

where h is a test function vanishing at 0.

We can thus collect the contributions 47 and 50 as

$$\sum_{\gamma} \int_{I_\gamma} \frac{h(u)}{|1 - (F_u)_*|} \, d^*u \tag{51}$$

where h is as above, I_γ is the isotropy group of the periodic orbit γ, the haar measure d^*u on I_γ is normalised so that the covolume of I_γ is equal to one and we still write $(F_u)_*$ for its restriction to the transverse space of γ.

VII The global case, and the formal trace computation.

We shall consider the action of C_k on X and write down the analogue of VI (51) for the distribution trace formula.

Both X and C_k are defined as quotients and we let

$$\pi : A \to X\,,\ c : \mathrm{GL}_1(A) \to C_k \tag{1}$$

be the corresponding quotient maps.

As above we consider the graph Z of the action

$$f : X \times C_k \to X\,,\ f(x,\lambda) = \lambda x \tag{2}$$

and the diagonal map

$$\varphi : X \times C_k \to X \times C_k \times X \qquad \varphi(x,\lambda) = (x,\lambda,x)\,. \tag{3}$$

We first investigate the fixed points, $\varphi^{-1}(Z)$, i.e. the pairs $(x,\lambda) \in X \times C_k$ such that $\lambda x = x$. Let $x = \pi(\tilde{x})$ and $\lambda = c(j)$. Then the equality $\lambda x = x$ means that $\pi(j\tilde{x}) = \pi(\tilde{x})$ thus there exists $q \in k^*$ such that with $\tilde{j} = qj$, one has

$$\tilde{j}\tilde{x} = \tilde{x}\,. \tag{4}$$

Recall now that A is the restricted direct product $A = \prod_{\mathrm{res}} k_v$ of the local fields k_v obtained by completion of k with respect to the place v. The equality (4) means that $\tilde{j}_v \tilde{x}_v = \tilde{x}_v$, thus, if $\tilde{x}_v \neq 0$ for all v it follows that $\tilde{j}_v = 1\ \forall v$ and $\tilde{j} = 1$. This shows that the projection of $\varphi^{-1}(Z) \cap C_k \backslash \{1\}$ on X is the union of the hyperplanes

$$\cup H_v\,;\ H_v = \pi(\tilde{H}_v)\,,\ \tilde{H}_v = \{x\,;\ x_v = 0\}\,. \tag{5}$$

Each $\tilde{H}_v$ is closed in A and is invariant under multiplication by elements of k^*. Thus each H_v is a closed subset of X and one checks that it is the closure of the orbit under C_k of any of its generic points

$$x\,,\ x_u = 0 \quad \Longleftrightarrow \quad u = v\,. \tag{6}$$

For any such point x, the isotropy group I_x is the image in C_k of the multiplicative group k_v^*,

$$I_x = k_v^* \tag{7}$$

by the map $\lambda \in k_v^* \to (1, \ldots, 1, \lambda, 1, \ldots)$. This map already occurs in class field theory (cf [W1]) to relate the local theory to the global one.

Both groups k_v^* and C_k are commensurable to $\mathbb{R}_+^*$ by the module homomorphism, which is proper with cocompact range,

$$G \xrightarrow{|\ |} \mathbb{R}_+^* . \tag{8}$$

Since the restriction to k_v^* of the module of C_k is the module of k_v^*, it follows that

$$I_x \text{ is a cocompact subgroup of } C_k . \tag{9}$$

This allows to normalize the respective Haar measures in such a way that the covolume of I_x is 1. This is in fact insured by the canonical normalization of the Haar measures of modulated groups ([W3]),

$$\int_{|g|\in[1,\Lambda]} d^*g \sim \log \Lambda \quad \text{when} \quad \Lambda \to +\infty . \tag{10}$$

It is important to note that though I_x is cocompact in C_k, the orbit of x is not closed and one needs to close it, the result being H_v. We shall learn how to justify this point later in section VIII, in the similar situation of the action of C_S on X_S. We can now in view of the results of the two preceding sections, write down the contribution of each H_v to the distributional trace;

Since $\tilde{H}_v$ is a hyperplane, we can identify the transverse space N_x to H_v at x with the quotient

$$N_x = A/\tilde{H}_v = k_v, \tag{11}$$

namely the additive group of the local field k_v. Given $j \in I_x$ one has $j_u = 1 \ \forall u \neq v$, and $j_v = \lambda \in k_v^*$. The action of j on A is linear and fixes x, thus the action on the transverse space N_x is given by

$$(\lambda, a) \to \lambda a \quad \forall a \in k_v. \tag{12}$$

We can thus proceed with some faith and write down the contribution of H_v to the distributional trace in the form,

$$\int_{k_v^*} \frac{h(\lambda)}{|1-\lambda|} d^*\lambda, \tag{13}$$

where h is a test function on C_k which vanishes at 1. We now have to take care of a discrepancy in notation with the fifth section, where we used the symbol $U(j)$ for the operation

$$(U(j)f)(x) = f(j^{-1}x) \tag{14}$$

whereas we use j in the above discussion. This amounts to replace the test function $h(u)$ by $h(u^{-1})$ and we thus obtain as a formal analogue of VI(51) the following expression for the distributional trace

$$\text{“Trace” } (U(h)) = \sum_v \int_{k_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u \,. \tag{15}$$

Now the right-hand side of (15) is, when restricted to the hyperplane $h(1) = 0$, the distribution obtained by André Weil ([W3]) (cf. Theorem V.1) as the synthesis of the explicit formulas of number theory for all L-functions with Grössencharakter. In particular we can rewrite it as

$$\hat{h}(0) + \hat{h}(1) - \sum_{L(\chi,\rho)=0} \hat{h}(\chi,\rho) + \infty \; h(1) \tag{16}$$

where this time the restriction $\mathrm{Re}(\rho) = \frac{1}{2}$ has been eliminated.

Thus, equating (52) of section V and (16) for $h(1) = 0$ would yield the desired information on the zeros. Of course, this does require first eliminating the role of δ, and (as in [AB]) to prove that the distributional trace coincides with the ordinary operator theoretic trace on the cokernel of E. This is achieved for the usual set-up of the Lefschetz fixed point theorem by the use of families.

A very important property of the right hand side of (15) is the following "Hadamard positivity": If the test function $h, h(1) = 0$ is positive,

$$h(u) \geq 0 \quad \forall \, u \in C_k \tag{17}$$

then the right-hand side is *positive*. This indicated from the very start that in order to obtain the Polya-Hilbert space from the Riemann flow, it is *not* quantization that should be involved but simply the passage to the L^2 space, $X \to L^2(X)$. Indeed the positivity of (17) is typical of *permutation matrices* rather than of quantization. This distinction plays a crucial role in the above discussion of the trace formula, in particular the expected trace formula is not a semi-classical formula but a Lefschetz formula in the spirit of [AB].

The above discussion is *not* a rigorous justification of this formula. The first obvious obstacle is that the distributional trace is only formal and to give it a rigorous meaning tied up to Hilbert space operators, one needs as we shall see in section VIII, to perform a cutoff. The second difficulty comes from the presence of the parameter δ as a label for the Hilbert space, while δ does not appear in the trace formula. As we shall see in the next two sections the cutoff will completely eliminate the role of δ, and we shall nevertheless show (by proving positivity of the Weil distribution) that the validity of the (δ independent) trace formula is equivalent to the Riemann Hypothesis for all Grössencharakters of k.

VIII The trace formula in the S-local case.

In the formal trace computation of section VII, we skiped over the difficulties inherent to the tricky structure of the space X. In order to understand how to handle trace formulas on such spaces we shall consider the slightly simpler situation which arises when one only considers a finite set S of places of k. As soon as the cardinality of S is larger than 3, the corresponding space X_S does share most of the tricky features of the space X. In particular it is no longer of type I in the sense of Noncommutative Geometry.

We shall nevertheless describe a precise general result ([Co] theorem 4) which shows that the above handling of periodic orbits and of their contribution to the trace is the correct one. It will in particular show why the orbit of the fixed point 0, or of elements $x \in A$, such that x_v vanishes for at least two places do not contribute to the trace formula. At the same time, we shall handle the lack of transversality when $h(1) \neq 0$.

Let us begin by the local case. let K be a local field. We deal directly with the following operator in $L^2(K)$,

$$(1) \qquad U(h) = \int h(\lambda)\, U(\lambda)\, d^*\lambda\,,$$

where the scaling operator $U(\lambda)$ is defined by

$$(2) \qquad (U(\lambda)\,\xi)(x) = \xi(\lambda^{-1}\,x) \qquad \forall\, x \in K$$

and where the multiplicative Haar measure $d^*\lambda$ is normalized by,

$$(3) \qquad \int_{|\lambda|\in[1,\Lambda]} d^*\lambda \sim \log \Lambda \qquad \text{when } \Lambda \to \infty\,.$$

To understand the "trace" of $U(h)$ we shall proceed as in the Selberg trace formula ([Se]) and use a cutoff. In physics terminology the divergence of the trace is both infrared and ultraviolet. To perform an infrared cutoff, we use the orthogonal projection P_Λ onto the subspace,

$$(4) \qquad P_\Lambda = \{\xi \in L^2(K)\,;\ \xi(x) = 0 \qquad \forall x\ ,\ |x| > \Lambda\}\,.$$

Thus, P_Λ is the multiplication operator by the function ρ_Λ, where $\rho_\Lambda(x) = 1$ if $|x| \leq \Lambda$, and $\rho(x) = 0$ for $|x| > \Lambda$. This gives an infrared cutoff and to get an ultraviolet cutoff we use $\widehat{P}_\Lambda = F P_\Lambda F^{-1}$ where F is the Fourier transform (which depends upon the basic character α). We let

$$R_\Lambda = \widehat{P}_\Lambda \, P_\Lambda \,. \tag{5}$$

One then obtains ([Co]),

Theorem 1. *Let K be a local field with basic character α. Let $h \in \mathcal{S}(K^*)$ have compact support. Then $R_\Lambda \, U(h)$ is a trace class operator and when $\Lambda \to \infty$, one has*

$$\mathrm{Trace}\,(R_\Lambda \, U(h)) = 2h(1)\log' \Lambda + \int' \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

where $2\log' \Lambda = \int_{\lambda \in K^*, \, |\lambda| \in [\Lambda^{-1}, \Lambda]} d^*\lambda$, and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on K which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform vanishes at 1.

As it turns out ([Co]), this principal value agrees with that of Weil (cf. section V) for the choice of F associated to the standard character of K.

Let us now describe the reduced framework for the trace formula. We now let k be a global field and S a finite set of places of k containing all infinite places. The group O_S^* of S-units is defined as the subgroup of k^*,

$$O_S^* = \{q \in k^*, |q_v| = 1, v \notin S\} \tag{6}$$

It is cocompact in J_S^1 where,

$$J_S = \prod_{v \in S} k_v^* \tag{7}$$

and,

$$J_S^1 = \{j \in J_S, |j| = 1\}. \tag{8}$$

Thus the quotient group $C_S = J_S / O_S^*$ plays the same role as C_k, and acts on the quotient X_S of $A_S = \prod_{v \in S} k_v$ by O_S^*.

To keep in mind a simple example, one can take $k = \mathbb{Q}$, while S consists of the three places 2, 3, and ∞. One checks in this example that the topology of X_S is not of type I since for instance the group $O_S^* = \{\pm 2^n 3^m; \, n, m \in \mathbb{Z}\}$ acts ergodically on $\{0\} \times \mathbb{R} \subset A_S$.

We normalize the multiplicative Haar measure $d^*\lambda$ of C_S by,

$$\int_{|\lambda| \in [1, \Lambda]} d^*\lambda \sim \log \Lambda \qquad \text{when } \Lambda \to \infty \,, \tag{9}$$

and normalize the multiplicative Haar measure $d^*\lambda$ of J_S so that it agrees with the above on a fundamental domain D for the action of O_S^* on J_S.

There is no difficulty in defining the Hilbert space $L^2(X_S)$ of square integrable functions on X_S. We proceed as in section V (without the δ), and complete (and separate) the Schwartz space $\mathcal{S}(A_S)$ for the pre-Hilbert structure given by,

$$\|f\|^2 = \int \Big| \sum_{q\in O_S^*} f(qx)\Big|^2 |x|\, d^*x \tag{10}$$

where the integral is performed on C_S or equivalently on a fundamental domain D for the action of O_S^* on J_S. To show that (10) makes sense, one proves that for $f \in \mathcal{S}(A_S)$, the function $E_0(f)(x) = \sum_{q\in O_S^*} f(qx)$ is bounded above by a power of $Log|x|$ when $|x|$ tends to zero. To see this when f is the characteristic function of $\{x \in A_S, |x_v| \leq 1, \forall v \in S\}$, one uses the cocompactness of O_S^* in J_S^1, to replace the sum by an integral. The latter is then comparable to,

$$\int_{u_i\geq 0, \sum u_i = -Log|x|} \prod du_i, \tag{11}$$

where the index i varies in S. The general case follows.

The scaling operator $U(\lambda)$ is defined by,

$$(U(\lambda)\,\xi)(x) = \xi(\lambda^{-1}\,x) \qquad \forall x \in A_S \tag{12}$$

and the same formula, with $x \in X_S$ defines its action on $L^2(X_S)$. Given a smooth compactly supported function h on C_S, $U(h) = \int h(g)U(g)d^*g$ makes sense as an operator acting on $L^2(X_S)$.

We shall first see that the Fourier transform F on $\mathcal{S}(A_S)$ does extend to a unitary operator on the Hilbert space $L^2(X_S)$.

Lemma 2. ([Co]) *a) For any $f_i \in \mathcal{S}(A_S)$ the series $\sum_{O_S^*} \langle f_1, U(q)\, f_2\rangle_A$ of inner products in $L^2(A_S)$ converges geometrically on the abelian finitely generated group O_S^*. Moreover its sum is equal to the inner product of f_1 and f_2 in the Hilbert space $L^2(X_S)$.*

b) Let $\alpha = \prod \alpha_v$ be a basic character of the additive group A_S and F the corresponding Fourier transformation. The map $f \to F(f)$, $f \in \mathcal{S}(A_S)$ extends uniquely to a unitary operator in the Hilbert space $L^2(X_S)$.

Now exactly as above for the case of local fields (theorem 1), we need to use a cutoff. For this we use the orthogonal projection P_Λ onto the subspace,

$$P_\Lambda = \{\xi \in L^2(X_S)\,;\ \xi(x) = 0 \qquad \forall x\ ,\ |x| > \Lambda\}\,. \tag{13}$$

Thus, P_Λ is the multiplication operator by the function ρ_Λ, where $\rho_\Lambda(x) = 1$ if $|x| \leq \Lambda$, and $\rho(x) = 0$ for $|x| > \Lambda$. This gives an infrared cutoff and to get an ultraviolet cutoff we use $\widehat{P}_\Lambda = FP_\Lambda F^{-1}$ where F is the Fourier transform (lemma 1) which depends upon the choice of the basic character $\alpha = \prod \alpha_v$. We let

$$R_\Lambda = \widehat{P}_\Lambda\, P_\Lambda\,. \tag{14}$$

One then gets ([Co]),

Theorem 3. *Let A_S be as above, with basic character $\alpha = \prod \alpha_v$. Let $h \in \mathcal{S}(C_S)$ have compact support. Then when $\Lambda \to \infty$, one has*

$$\mathrm{Trace}\,(R_\Lambda\, U(h)) = 2h(1)\log' \Lambda + \sum_{v \in S} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

where $2\log' \Lambda = \int_{\lambda \in C_S,\, |\lambda| \in [\Lambda^{-1}, \Lambda]} d^*\lambda$, each k_v^* is embedded in C_S by the map $u \to (1, 1, ..., u, ..., 1)$ and the principal value $\int'$ is uniquely determined by the pairing with the unique distribution on k_v which agrees with $\frac{du}{|1-u|}$ for $u \neq 1$ and whose Fourier transform relative to α_v vanishes at 1.

Let us now discuss the global trace formula.

IX The global trace formula and the geometric dictionary.

The main difficulty created by the parameter δ in Theorem V.2 is that the formal trace computation of section VII is independent of δ, and thus cannot give in general the expected value of the trace of corollary V.3, since in the latter each critical zero ρ is counted with a multiplicity equal to the largest integer $n < \frac{1+\delta}{2}$, $n \leq$ multiplicity of ρ as a zero of L. In particular for L functions with multiple zeros, the δ-dependence of the spectral side is nontrivial. It is also clear that the function space $L^2_\delta(X)$ artificially eliminates the non-critical zeros by the introduction of the δ.

As we shall see, all these problems are eliminated by the cutoff. The latter will be performed directly on the Hilbert space $L^2(X)$ so that the only value of δ that we shall use is $\delta = 0$. All zeros will play a role in the spectral side of the trace formula, but while the critical zeros will appear per-se, the non critical ones will appear as resonances and enter in the trace formula through their harmonic potential with respect to the critical line. Thus the spectral side is now entirely canonical and independent of δ, and by proving positivity of the Weil distribution, we shall show that its equality with the geometric side, i.e. the global analogue of Theorem VIII.3, is equivalent to the Riemann Hypothesis for all L-functions with Grössencharakter.

The Abelian group A of Adeles of k is its own Pontrjagin dual by means of the pairing

$$\langle a,b\rangle = \alpha(ab) \tag{1}$$

where $\alpha : A \to U(1)$ is a nontrivial character which vanishes on $k \subset A$.

We fix the additive character α as above, $\alpha = \prod \alpha_v$ and let d be a differental idele,

$$\alpha(x) = \alpha_0(d\,x) \quad \forall x \in A\,, \tag{2}$$

where $\alpha_0 = \prod \alpha_{0,v}$ is the product of the local normalized additive characters (cf [W1]). We let S_0 be the finite set of places where α_v is ramified.

We shall first concentrate on the case of positive characteristic, i.e. of function fields, both because it is technically simpler and also because it allows to keep track of the geometric significance of the construction.

In order to understand how to perform in the global case, the cutoff $R_\Lambda = \widehat{P}_\Lambda\, P_\Lambda$ of section VIII, we shall first analyze the relative position of the pair of projections

$\widehat{P}_\Lambda$, P_Λ when $\Lambda \to \infty$. Thus, we let $S \supset S_0$ be a finite set of places of k, large enough so that $mod(C_S) = mod(C_k) = q^{\mathbb{Z}}$ and that for any fundamental domain D for the action of O_S^* on J_S, the product $D \times \prod R_v^*$ is a fundamental domain for the action of k^* on J_k.

Both $\widehat{P}_\Lambda$ and P_Λ commute with the decomposition of $L^2(X_S)$ as the direct sum of the subspaces, indexed by characters χ_0 of $C_{S,1}$,

$$(3) \qquad L^2_{\chi_0} = \{\xi \in L^2(X_S)\,;\, \xi(a^{-1}x) = \chi_0(a)\,\xi(x),\ \forall x \in X_S\,, a \in C_{S,1}\}$$

which corresponds to the projections $P_{\chi_0} = \int \overline{\chi_0}(a)\, U(a)\, d_1\, a$, where $d_1\, a$ is the Haar measure of total mass 1 on $C_{S,1}$.

Lemma 1. *Let χ_0 be a character of $C_{S,1}$, then for Λ large enough $\widehat{P}_\Lambda$ and P_Λ commute on the Hilbert space $L^2_{\chi_0}$.*

We can thus rewrite Theorem VIII.3 in the case of positive characteristic as,

Corollary 2. *Let Q_Λ be the orthogonal projection on the subspace of $L^2(X_S)$ spanned by the $f \in \mathcal{S}(A_S)$ which vanish as well as their Fourier transform for $|x| > \Lambda$. Let $h \in \mathcal{S}(C_S)$ have compact support. Then when $\Lambda \to \infty$, one has*

$$\text{Trace}\,(Q_\Lambda\, U(h)) = 2h(1)\log' \Lambda + \sum_{v \in S} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

where $2\log' \Lambda = \int_{\lambda \in C_S,\, |\lambda| \in [\Lambda^{-1}, \Lambda]} d^*\lambda$, and the other notations are as in Theorem VIII.3

In fact the proof of lemma 1 ([Co]) shows that the subspaces B_Λ stabilize very quickly, so that the natural map $\xi \to \xi \otimes 1_R$ from $L^2(X_S)$ to $L^2(X'_S)$ for $S \subset S'$ maps B_Λ^S onto $B_\Lambda^{S'}$.

We thus get from corollary 2 an S-independent global formulation of the cutoff and of the trace formula. We let $L^2(X)$ be the Hilbert space $L^2_\delta(X)$ of section V for the trivial value $\delta = 0$ which of course eliminates the unpleasant term from the inner product, and we let Q_Λ be the orthogonal projection on the subspace B_Λ of $L^2(X)$ spanned by the $f \in \mathcal{S}(A)$ which vanish as well as their Fourier transform for $|x| > \Lambda$. As we mentionned earlier, the proof of lemma 1 shows that for S and Λ large enough (and fixed character χ), the natural map $\xi \to \xi \otimes 1_R$ from $L^2(X_S)_\chi$ to $L^2(X)_\chi$ maps B_Λ^S onto B_Λ.

It is thus natural to expect that the following global analogue of the trace formula of corollary 2 actually holds, i.e. that when $\Lambda \to \infty$, one has,

$$(4) \qquad \text{Trace}\,(Q_\Lambda\, U(h)) = 2h(1)\log' \Lambda + \sum_{v} \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

where $2\log' \Lambda = \int_{\lambda \in C_k,\, |\lambda| \in [\Lambda^{-1}, \Lambda]} d^*\lambda$, and the other notations are as in Theorem VIII.3.

We can prove directly that (4) holds when h is supported by $C_{k,1}$ but are not able to prove (4) directly for arbitrary h (even though the right hand side of the formula only contains finitely many nonzero terms since $h \in \mathcal{S}(C_k)$ has compact support). What we shall show however is that the trace formula (4) implies the positivity of the Weil distribution, and hence the validity of RH for k. Remember that we are still in positive characteristic where RH is actually a theorem of A.Weil. It will thus be important to check the actual equivalence between the validity of RH and the formula (4). This is achieved by,

Theorem 3.([Co]) *Let k be a global field of positive characteritic and Q_Λ be the orthogonal projection on the subspace of $L^2(X)$ spanned by the $f \in \mathcal{S}(A)$ such that $f(x)$ and $\hat{f}(x)$ vanish for $|x| > \Lambda$. Let $h \in \mathcal{S}(C_k)$ have compact support. Then the following conditions are equivalent,*

a) When $\Lambda \to \infty$, one has

$$\text{Trace}\,(Q_\Lambda\, U(h)) = 2h(1)\log' \Lambda + \sum_v \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

b) All L functions with Grössencharakter on k satisfy the Riemann Hypothesis.

To prove that a) implies b), one proves (assuming a)) the positivity of the Weil distribution

(5) $$\Delta = \log |d^{-1}|\, \delta_1 + D - \sum_v D_v\,.$$

First, by theorem V.2 applied for $\delta = 0$, the map E,

(6) $$E(f)\,(g) = |g|^{1/2} \sum_{q \in k^*} f(qg) \qquad \forall\, g \in C_k\,,$$

defines a surjective isometry from $L^2(X)_0$ to $L^2(C_k)$ such that,

(7) $$E\, U(a) = |a|^{1/2}\, V(a)\, E\,,$$

where the left regular representation V of C_k on $L^2(C_k)$ is given by,

(8) $$(V(a)\,\xi)\,(g) = \xi(a^{-1}\, g) \qquad \forall\, g, a \in C_k\,.$$

Let S_Λ be the subspace of $L^2(C_k)$ given by,

(9) $$S_\Lambda = \{\xi \in L^2(C_k)\,;\; \xi(g) = 0,\; \forall g\,,\; |g| \notin [\Lambda^{-1}, \Lambda]\}\,.$$

We shall denote by the same letter the corresponding orthogonal projection.

Let $B_{\Lambda,0}$ be the subspace of $L^2(X)_0$ spanned by the $f \in \mathcal{S}(A)_0$ such that $f(x)$ and $\hat{f}(x)$ vanish for $|x| > \Lambda$ and $Q_{\Lambda,0}$ be the corresponding orthogonal projection.

Let $f \in \mathcal{S}(A)_0$ be such that $f(x)$ and $\widehat{f}(x)$ vanish for $|x| > \Lambda$, then $E(f)\,(g)$ vanishes for $|g| > \Lambda$, and the equality,

$$E(f)(g) = E(\widehat{f})\left(\frac{1}{g}\right) \qquad f \in \mathcal{S}(A)_0, \tag{10}$$

shows that $E(f)\,(g)$ vanishes for $|g| < \Lambda^{-1}$.

This shows that $E(B_{\Lambda,0}) \subset S_\Lambda$, so that if we let $Q'_{\Lambda,0} = E\,Q_{\Lambda,0}\,E^{-1}$, we get the inequality,

$$Q'_{\Lambda,0} \leq S_\Lambda \tag{11}$$

and for any Λ the following distribution on C_k is of positive type,

$$\Delta_\Lambda(f) = \text{Trace}\left((S_\Lambda - Q'_{\Lambda,0})\,V(f)\right), \tag{12}$$

i.e. one has,

$$\Delta_\Lambda(f * f^*) \geq 0, \tag{13}$$

where $f^*(g) = \overline{f}(g^{-1})$ for all $g \in C_k$.

Let then $f(g) = |g|^{-1/2}\,h(g^{-1})$, so that one has $E\,U(h) = V(\tilde{f})E$ where $\tilde{f}(g) = f(g^{-1})$ for all $g \in C_k$. One has,

$$\sum_v D_v(f) - \log|d^{-1}| = \sum_v \int'_{k_v^*} \frac{h(u^{-1})}{|1-u|}\, d^*u. \tag{14}$$

One has Trace $(S_\Lambda\, V(f)) = 2f(1)\log'\Lambda$, thus using a) we see that the limit of Δ_Λ when $\Lambda \to \infty$ is the Weil distribution Δ (cf. section V). The term D in the latter comes from the nuance between the subspaces B_Λ and $B_{\Lambda,0}$. This shows using (13), that the distribution Δ is of positive type so that b) holds (cf. [W3]).

To show that b) implies a), one computes from the zeros of L-functions and independently of any hypothesis the limit of the distributions Δ_Λ when $\Lambda \to \infty$. We choose (non canonically) an isomorphism

$$C_k \simeq C_{k,1} \times N\,. \tag{15}$$

where $N = \text{range}\,|\ | \subset \mathbb{R}_+^*$, $N \simeq \mathbb{Z}$ is the subgroup $q^{\mathbb{Z}} \subset \mathbb{R}_+^*$.

For $\rho \in \mathbb{C}$ we let $d\mu_\rho(z)$ be the harmonic measure of ρ with respect to the line $i\,\mathbb{R} \subset \mathbb{C}$. It is a probability measure on the line $i\,\mathbb{R}$ and coincides with the Dirac mass at ρ when ρ is on the line.

The implication b)$\Rightarrow$a) follows immediately from the explicit formulas and the following lemma,

Lemma 4. *The limit of the distributions Δ_Λ when $\Lambda \to \infty$ is given by,*

$$\Delta_\infty(f) = \sum_{\substack{L\left(\widetilde{\chi},\frac{1}{2}+\rho\right)=0 \\ \rho \in B/N^\perp}} N(\widetilde{\chi}, \frac{1}{2}+\rho) \int_{z\in i\mathbb{R}} \widehat{f}(\widetilde{\chi}, z)\,d\mu_\rho(z)$$

where B is the open strip $B = \{\rho \in \mathbb{C}; Re(\rho) \in]\frac{-1}{2}, \frac{1}{2}[\}$, $N(\widetilde{\chi}, \frac{1}{2} + \rho)$ is the multiplicity of the zero, $d\mu_\rho(z)$ is the harmonic measure of ρ with respect to the line $i\,\mathbb{R} \subset \mathbb{C}$, and the Fourier transform $\widehat{f}$ of f is defined by,

$$\widehat{f}(\widetilde{\chi}, \rho) = \int_{C_k} f(u)\,\widetilde{\chi}(u)\,|u|^\rho\, d^*u\,.$$

One should compare this lemma with Corollary 3 of section V. In the latter only the critical zeros were coming into play and with a multiplicity controlled by δ. In the above lemma, all zeros do appear and with their full multiplicity, but while the critical zeros appear per-se, the non-critical ones play the role of resonances as in the Fermi theory.

Let us now explain how the above results extend to number fields k. We first need to analyze, as above, the relative position of the projections P_Λ and $\widehat{P}_\Lambda$. Let us first remind the reader of the well known geometry of pairs of projectors. Recall that a pair of orthogonal projections P_i in Hilbert space is the same thing as a unitary representation of the dihedral group $\Gamma = \mathbb{Z}/2 * \mathbb{Z}/2$. To the generators U_i of Γ correspond the operators $2P_i - 1$. The group Γ is the semidirect product of the subgroup generated by $U = U_1 U_2$ by the group $\mathbb{Z}/2$, acting by $U \mapsto U^{-1}$. Its irreducible unitary representations are parametrized by an angle $\theta \in [0, \frac{\pi}{2}]$, the corresponding orthogonal projections P_i being associated to the one dimensional subspaces $y = 0$ and $y = x\,tg(\theta)$ in the Euclidean x, y plane. In particular these representations are at most two dimensional. A general unitary representation is characterized by the operator Θ whose value is the above angle θ in the irreducible case. It is uniquely defined by the equality,

$$\text{Sin}(\Theta) = |P_1 - P_2|, \tag{16}$$

and commutes with P_i.

The first obvious difficulty is that when v is an Archimedian place there exists no non-zero function on k_v which vanishes as well as its Fourier transform for $|x| > \Lambda$. This would be a difficult obstacle were it not for the work of Landau, Pollak and Slepian ([LPS]) in the early sixties, motivated by problems of electrical engineering, which allows to overcome it by showing that though the projections P_Λ and $\widehat{P}_\Lambda$ do not commute exactly even for large Λ, their angle is sufficiently well behaved so that the subspace B_Λ makes good sense.

For simplicity we shall take $k = \mathbb{Q}$, so that the only infinite place is real. Let P_Λ be the orthogonal projection onto the subspace,

$$P_\Lambda = \{\xi \in L^2(\mathbb{R})\,;\ \xi(x) = 0,\ \forall x\ ,\ |x| > \Lambda\}\,. \tag{17}$$

and $\widehat{P}_\Lambda = F P_\Lambda F^{-1}$ where F is the Fourier transform associated to the basic character $\alpha(x) = e^{-2\pi i x}$. What the above authors have done is to analyze the relative position of the projections P_Λ, $\widehat{P}_\Lambda$ for $\Lambda \to \infty$ in order to account for the obvious existence of signals (a recorded music piece for instance) which for all practical purposes have finite support both in the time variable and the dual frequency variable.

The key observation of ([LPS]) is that the following second order differential operator on $\mathbb{R}$ actually commutes with the projections P_Λ, $\widehat{P}_\Lambda$,

$$H_\Lambda \psi(x) = -\partial((\Lambda^2 - x^2)\,\partial)\psi(x) + (2\pi\Lambda x)^2\,\psi(x), \tag{18}$$

where ∂ is ordinary differentiation in one variable. Exactly as the generator $x\,\partial$ of scaling commutes with the orthogonal projection on the space of functions with positive support, the operator $\partial((\Lambda^2 - x^2)\,\partial)$ commutes with P_Λ. Moreover H_Λ commutes with Fourier transform F, and the commutativity of H_Λ with $\widehat{P}_\Lambda$ thus follows.

If one sticks to functions with support in $[-\Lambda, \Lambda]$, the operator H_Λ has discrete simple spectrum, and was studied long before the work of [LPS]. It appears from the factorization of the Helmoltz equation $\Delta\,\psi + k^2\,\psi = 0$ in one of the few separable coordinate systems in Euclidean 3-space, called the prolate spheroidal coordinates. Its eigenvalues $\chi_n(\Lambda), n \geq 0$ are simple and positive. The corresponding eigenfunctions ψ_n are called the prolate spheroidal wave functions and since $P_\Lambda\,\widehat{P}_\Lambda\,P_\Lambda$ commutes with H_Λ, they are the eigenfunctions of $P_\Lambda\,\widehat{P}_\Lambda\,P_\Lambda$. A lot is known about them, in particular one can take them to be real valued, and they are even for n even and odd for n odd. The key result of [LPS] is that the corresponding eigenvalues λ_n of the operator $P_\Lambda\,\widehat{P}_\Lambda\,P_\Lambda$ are decreasing very slowly from $\lambda_0 \simeq 1$ until the value $n \simeq 4\Lambda^2$ of the index n, they then decrease from $\simeq 1$ to $\simeq 0$ in an interval of length $\simeq \log(\Lambda)$ and then stay close to 0. Of course this gives the eigenvalues of Θ, it dictates the analogue of the subspace B_Λ of lemma 1, as the linear span of the ψ_n, $n \leq 4\Lambda^2$, and it gives the justification of the semi-classical counting of the number of quantum mechanical states which are localized in the interval $[-\Lambda, \Lambda]$ as well as their Fourier transform as the area of the corresponding square in phase space.

We now know what is the subspace B_Λ for the single place ∞, and to obtain it for an arbitrary set of places (containing the infinite one), we just use the same rule as in the case of function fields, i.e. we consider the map,

$$\psi \mapsto \psi \otimes 1_R, \tag{19}$$

which suffices when we deal with the Riemann zeta function. Note also that in that case we restrict ourselves to even functions on $\mathbb{R}$. This gives the analogue of Corollary 2, Theorem 3, and Lemma 4.

We refer to [Co] to see how the formula for the number of zeros

$$N(E) \sim (E/2\pi)(\log(E/2\pi) - 1) + 7/8 + o(1) + N_{osc}(E) \tag{20}$$

appears from our spectral interpretation.

The filtration Q_Λ, S_Λ of the short complex,

$$0 \to L^2(X)_0 \to L^2(C_k) \to 0, \tag{21}$$

allows to define Adelic cohomology in which all nontrivial zeros of L-functions do appear and to complete the following dictionary with unproved last line, between the function theory and the geometry of the Adele class space X,

Function Theory	*Geometry*
Zeros and poles of Zeta	Action of C_k on Adelic cohomology
Functional Equation	$*$ operation
Explicit formula	Lefschetz formula for action of C_k on X
Riemann Hypothesis	Trace formula

General remarks.

a) There is a close analogy between the construction of the Hilbert space $L^2(X)$ in section V, and the construction of the physical Hilbert space ([S] theorem 2.1) in constructive quantum field theory, in the case of gauge theories. In both cases the action of the invariance group (the group $k^* = GL_1(k)$ in our case, the gauge group in the case of gauge theories) is wiped out by the very definition of the inner product. Compare with ([S]) top of page 17.

b) It is quite remarkable that the eigenvalues of the angle operator Θ which we discussed above, also play a key role in the theory of random hermitian matrices. To be more specific, let $E(n,s)$ be the large N limit of the probability that there are exactly n eigenvalues of a random Hermitian $N \times N$ matrix in the interval $[-\frac{\pi}{\sqrt{2N}}t, \frac{\pi}{\sqrt{2N}}t]$, $t = s/2$.

Let us compute this probability $E(n,s)$ (cf.[Me]). One has by construction $\sum\limits_n E(n,s) = 1$. We will do the computation for $n = 0$, for other values of n the computation is similar.

With the notations of section III, $E(0,s)$ is clearly given by the large N limit of,

$$\int\limits_{|E_j|\geq\theta} p_N(E_1,\dots E_N)dE_1\dots dE_N,$$

where $\theta = \frac{\pi}{\sqrt{2N}}t$. This equals

$$\frac{1}{N!}\int\limits_{|E_j|\geq\theta}\sum_{\sigma,\pi\in S_N}\epsilon(\sigma)\epsilon(\pi)\Phi_{\sigma(1)}(E_1)\Phi_{\pi(1)}(E_1)\Phi_{\sigma(2)}(E_2)\Phi_{\pi(2)}(E_2)\dots$$

$$\Phi_{\sigma(N)}(E_N)\Phi_{\pi(N)}(E_N)dE_1dE_2\dots dE_N = ,$$

$$\sum_\sigma \epsilon(\sigma)\prod_1^N (\Phi_1,(1-P_\theta)\Phi_{\sigma(1)})\dots(\Phi_N,(1-P_\theta)\Phi_{\sigma(N)})$$

where P_θ is the operator of multiplication by $1_{[-\theta,\theta]}$, the characteristic function of the interval $[-\theta,\theta]$.

We rewrite this as $\det(K_N(1-P_\theta)K_N)|_{\text{Range of } K_N} = \prod\limits_1^N(1-\lambda_{j,N})$, where $\lambda_{j,N}$ are the nonzero eigenvalues of $K_N P_\theta$.

For $N \to \infty$, the equality III.14 allows to replace K_N by the operator given by the kernel,

$$k(x,y) = \frac{\sin\pi(x-y)}{\pi(x-y)}$$

Hence we get as $N \to \infty$,

$$E(0,s) = \prod_{1}^{\infty}(1 - \lambda_j(s)),$$

where $s = 2t$, and the $\lambda_j(s)$ are the eigenvalues of the operator $\widehat{P_\pi} P_t$. Here we let, as above, $\widehat{P_\lambda} = \mathcal{F} P_\lambda \mathcal{F}^{-1}$, and $\mathcal{F}$ denotes the Fourier transform, $\mathcal{F}\xi(u) = \int e^{ixu}\xi(x)dx$. Note finally that the eigenvalues of $\widehat{P_a} P_b$ only depend upon the product ab so that the relation with the eigenvalues of Θ should be clear.

c) This paper was finalized during my visit to O.S.U. in October - November, 1998 and I am grateful to this University for its warm hospitality and to A. Gorokhovsky for taking careful notes in my class.

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