Quantum Chaos, Symmetry and Zeta Functions[1]
Lecture I: Quantum Chaos

Peter Sarnak

Department of Mathematics,
Princeton University,
Princeton, NJ 08544.
sarnak@math.princeton.edu

1 Introduction

It is perhaps a little premature to give a mathematical lecture such as this on a subject in which the proven results are rather modest. However given the attention the subject has received in the Physics literature (the results being primarily computational) and given the interesting phenomena and conjectures that have emerged, it seems worthwhile. Quantum Chaos is concerned with the study of the semiclassical limit of the quantization of a classically

[1] Parts of these lectures were presented at the following conferences: ICMP Brisbane 1997, Joint AMS-SAMS 1997, Supersymmetry and trace formulae (Newton Institute 1997), and Journée Arithmetique (Limoges 1997).

chaotic Hamiltonian. More precisely let $H(q,p)$ be a Hamiltonian on a $2n$-dimensional symplectic manifold. We assume that the constant energy sets $H(q,p)$ = constant are compact. Let $\widehat{H}(\hbar)$ be a quantization of the classical flow defined by H, with $\hbar \to 0$ corresponding to the semi-classical limit (we give numerous explicit examples below). The corresponding Schrödinger eigenvalue equation is

$$\widehat{H}(\hbar)\psi(h) + \lambda(\hbar)\psi(\hbar) = 0 \tag{1.1}$$

It has a discrete set of eigenvalues

$$\lambda_1(\hbar) \le \lambda_2(\hbar) \le \lambda_3(\hbar) \ldots \longrightarrow \infty \tag{1.2}$$

with a corresponding orthonormal basis of eigenfunctions

$$\psi_1(\hbar), \psi_2(\hbar), \ldots \tag{1.3}$$

The problem mentioned above is to investigate the fine structure of the spectrum ($\lambda_j(\hbar)$'s) or eigenfunctions ($\psi_j(\hbar)$), either as $\hbar \to 0$ (semi-classical limit) or as $j \to \infty$ (large energy limit). In most of the examples below these are equivalent. We consider two extreme cases for the classical motion:

(I) The classical flow is completely integrable — that is, there are n Poisson commuting invariants of the motion $H_1 = H, H_2, \ldots, H_n$ [1].

(II) The classical flow is chaotic by which we mean that it is ergodic on the invariant surface H = constant and that it has positive Liapunov exponents, etc. [1, 2].

A rich family of Hamiltonians are those coming form a Riemannian metric on a manifold X. That is, H defined on $T^*(X)$ is given in the form

$$H(x,\xi) = \frac{1}{2}\sum_{i,j} g_{ij}(x)\xi^i\xi^j \tag{1.4}$$

where $g(x)$ is the metric on X and ξ is a cotangent vector. In this case the Hamilton flow is just the geodesic flow. The standard quantization of H is $\widehat{H} = \Delta := \operatorname{div}\operatorname{grad}$, the Laplace-Beltrami operator on functions on X. In these cases we suppress $\hbar$, since the $\hbar \to 0$ limit coincides with the $\lambda_j \to \infty$ limit. Among these geodesic flows, examples of (I) above are surfaces of revolution in $\mathbb{R}^3$, for which the Clairaut Integral [2] provides a second integral of the motion. A compact manifold X of negative sectional curvature provides an archetypical example of (II) above (Hopf, Anosov [1]).

We do not discuss at all the very interesting questions about the behavior of $\psi_j(\hbar)$ as $\hbar \to 0$. See Sarnak [3], Zelditch [4] and Hurt [5] for surveys of the mathematical results concerning the eigenfunctions and Heller [6] for the numerical phenomenon of enhancement of such eigenstates on the unstable periodic orbits. Our aim here is to review the developments concerning the spectrum. In order to describe the basic Conjecture, we need to digress into the topic of random matrix theory.

2 Random Matrix Models

In the early 50's Wigner [7] suggested that the resonance lines of a heavy nucleus (their determination by analytic means being intractable) might be modeled by the spectrum of a large random matrix. He introduced the ensembles: Gaussian Orthogonal Ensemble "GOE" and Gaussian Unitary Ensemble "GUE" which are probability measures on $N \times N$ real symmetric and $N \times N$ Hermitian matrices, respectively. In the first case the measure is invariant under the action of the orthogonal group while in the second it is invariant by the unitary group. He raised the question of the local (scaled) spacings between the eigenvalues (see below for a precise definition) of members of these ensembles as $N \to \infty$. The answer was provided by Gaudin [8] and Gaudin-Mehta [9], by an ingenious use of orthogonal polynomials. Later Dyson [10] introduced his closely related circular ensembles COE, CUE and CSE which he termed the 3-fold way. These are the Riemannian Symmetric spaces (with their volume forms) $U(N)/O(N)$, $U(N)$ and $U(2N)/U\mathrm{Sp}(2N)$, respectively. He investigated their local spacing statistics and showed that those of the first two agree with GOE and GUE. Recent works by Altland and Zirnbauer [11] on some quantum problems (e.g. "Chaotic Andreev Quantum Dot") and by Katz and Sarnak [12,13] on zeros of zeta functions — see Lecture 2 — show that in some finer analysis these three symmetry classes do not suffice. It appears that each of the infinite families of irreducible symmetric spaces of Cartan (see [14] and [11]) should be considered. We describe five such families which play a role in these lectures.

- $U(N)$, the compact group of $N \times N$ unitary matrices in its standard realization. With Haar measure, this is Dyson's CUE.
- $U\mathrm{Sp}(2N)$, the compact group of $2N \times 2N$ unitary matrices which preserve the standard symplectic form, that is, all $A \in U(2N)$ satisfying ${}^tAJA = J$, where $J = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$.
- $SO(2N)$, the compact group of $2N \times 2N$ unitary matrices preserving the standard Euclidean inner product, that is, $A \in U(2N)$ satisfying ${}^tAA = I$ and with $\det A = 1$.
- $SO(2N+1)$, same as above but odd dimensional.

All of the above compact groups become Riemannian symmetric spaces when equipped with a bi-invariant metric and they comprise the symmetric spaces of type II, see [14].

- $U(N)/O(N)$ (this is Dyson's COE), which we realize as the symmetric, unitary $N \times N$ matrices A. The map $B \to B^tB$ identifies $U(N)/O(N)$ with these matrices and turns this into a compact symmetric space for which the corresponding volume form gives the COE. Note that this symmetric space is the compact dual of $SL_N(\mathbb{R})/SO(N)$ (or more precisely $GL_N(\mathbb{R})/O(N)$) which is Wigner's GOE.

We denote by $G(N)$ any one of the above ensembles whose members A are $N \times N$ unitary matrices and whose invariant probability measure is denoted by dA. Denote the eigenvalues of an $A \in G(N)$ by $e^{i\theta_1(A)}, e^{i\theta_2(A)}, \ldots, e^{i\theta_N(A)}$ where

$$0 \leq \theta_1(A) \leq \theta_2(A) \ldots \leq \theta_N(A) < 2\pi \tag{2.1}$$

We turn to the definitions of the local scaling spacing distributions. For $k \geq 1$ the k-th (scaling) consecutive spacing measure $\mu_k(A)$ on $[0,\infty)$ is defined to be

$$\mu_k(A)([a,b]) = \frac{\#\left\{1 \leq j \leq N \middle| \frac{N}{2\pi}(\theta_{j+k} - \theta_j) \in [a,b]\right\}}{N} \tag{2.2}$$

Notice the factor $\frac{N}{2\pi}$ which scales the spacings so that the mean of $\mu_k(A)$ is equal to k. The pair correlation $R_2(A)$ measures the distribution between all pairs of (scaled) eigenvalues of A. For $[a,b] \subset \mathbb{R}$,

$$R_2(A)[a,b] = \frac{\#\left\{j \neq k \middle| \frac{N}{2\pi}(\theta_j - \theta_k) \in [a,b]\right\}}{N} . \tag{2.3}$$

Higher correlations may be defined similarly (see [12]).

The behavior of the measures above as $N \to \infty$ may be studied by the method of Gaudin mentioned above. Katz and Sarnak [12] show that there are measures $\mu_k(\mathrm{II})$ and $R_2(\mathrm{II})$, such that for any of the four families $G(N)$ of the type II symmetric spaces above, we have

$$\lim_{N\to\infty} \int_{G(N)} \mu_k(A)dA = \mu_k(\mathrm{II}) . \tag{2.4}$$

A "Law of large numbers" which ensures that for the typical (in measure) $A \in G(N)$, $\mu_k(A)$ and $R_2(A)$ approach $\mu_k(\mathrm{II})$ and $R_2(\mathrm{II})$ as $N \to \infty$:

$$\lim_{N\to\infty} \int_{G(N)} D(\mu_k(A), \mu_k(\mathrm{II}))dA = 0 \tag{2.5}$$

where

$$D(\nu_1, \nu_2) = \sup\{|\nu_1(\mathrm{I}) - \nu_2(\mathrm{I})|: \ I \subset R \text{ is an interval}\} \tag{2.6}$$

is the Kolmogoroff-Smirnov distance between ν_1 and ν_2.
For $[a,b] \subset \mathbb{R}$,

$$\lim_{N\to\infty} \int_{G(N)} |R_2(A)[a,b] - R_2(\mathrm{II})[a,b]| dA = 0 \tag{2.7}$$

Since one of the ensembles of type II is $U(N)$ that is, CUE, it follows that $\mu_k(\mathrm{II})$ and $R_2(\mathrm{II})$ coincide with the corresponding measures for CUE (or GUE), $\mu_k(\mathrm{GUE})$ and $R_2(\mathrm{GUE})$. These were determined by Gaudin [8] and Dyson [10], respectively. Dyson shows that

$$R_2(\mathrm{CUE})[a,b] = \int_a^b \left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx \tag{2.8}$$

while Gaudin finds an expression for the densities of $\mu_k(\mathrm{CUE})$ in terms of Fredholm determinants of operators on $L^2[-1,1]$, see [8,9]. The latter allows for a numerical computation of these densities and in particular their graphs are given in [9].

For $G(N) = U(N)/O(N)$, that is, COE, the corresponding limits $\mu_k(\mathrm{COE})$ $(= \mu_k(\mathrm{GOE}))$ were determined similarly, see Mehta [9]. They are quite different to the CUE measures. For example

$$R_2(\mathrm{GOE})[a,b] = \int_a^b r_{\mathrm{GOE}}(x) dx \tag{2.9}$$

where

$$r_{\mathrm{GOE}}(x) = 1 - \left(\frac{\sin \pi x}{\pi x}\right)^2 - \left(\frac{\sin \pi x}{\pi x}\right)' \int_x^\infty \left(\frac{\sin \pi t}{\pi t}\right) dt \tag{2.10}$$

The results above for type II $G(N)$'s show that the spacings between **all** the eigenvalues of a typical A in such a $G(N)$ is universal as $N \to \infty$. This is in sharp contrast to the distribution of the eigenvalue closest to 1. For $k \geq 1$ let $\nu_k(G(N))$ be the distribution on $[0,\infty)$ of the k-th eigenvalue of A, as A varies over $G(N)$. That is,

$$\nu_k(G(N))[a,b] = \left|\left\{A \in G(N) \ \middle| \ \frac{\theta_k(A)N}{2\pi} \in [a,b]\right\}\right| \tag{2.11}$$

Similarly, one defines the 1-level scaling densities (or, more generally, n-level densities) of eigenvalues near 1. For $A \in G(N)$ and $[a,b] \subset \mathbb{R}$, let

$$D_1(A)[a,b] = \#\left\{\theta(A) \ \middle| \ e^{i\theta(A)} \text{ is an eigenvalue of } A \text{ and } \tfrac{\theta(A)N}{2\pi} \in [a,b]\right\} \tag{2.12}$$

The averages of D are denoted W;

$$W_1(G(N)) = \int_{G(N)} D_1(A)dA \tag{2.13}$$

In [12] it is shown that there are measures $\nu_k(G)$ on $[0,\infty)$ (which depend on the ensemble G — see [12] for graphs of their densities) such that

$$\lim_{N\to\infty} \nu_k(G(N)) = \nu_k(G) \tag{2.14}$$

and

$$\lim_{N\to\infty} W_1(G(N))[a,b] = \int_a^b w_1(G)(x)dx \tag{2.15}$$

where

$$w_1(G)(x) = \begin{cases} 1 & \text{if } G = U \text{ (or SU)} \\ 1 - \frac{\sin 2\pi x}{2\pi x} & \text{if } G = \text{Sp} \\ 1 + \frac{\sin 2\pi x}{2\pi x} & \text{if } G = \text{SO (even)} \\ \delta_0 + 1 - \frac{\sin 2\pi x}{2\pi x} & \text{if } G = \text{SO (odd)} \end{cases} \tag{2.16}$$

The above models $G(N)$ are all (essentially) irreducible symmetric spaces. The antithesis is the case of a completely reducible space such as the group (or symmetric space) $T^N = U(1)\times U(1)\ldots\times U(1)$, that is, the N-torus. Haar measure on T^N is the product measure $\frac{dx_1}{2\pi}\cdot\ldots\cdot\frac{dx_N}{2\pi}$, which says that the x_j's are uniformly distributed independent random variables. The question of the local scaling statistics for these is well studied in the probability theory literature. It is well known [15] that the local spacings approximate a Poisson Process as $N\to\infty$. For this reason the spacings for this T^N model, as $N\to\infty$, are known as "Poisson statistics" in the physics literature. The scaled k-th consecutive spacings $\mu_k(x_1,\ldots,x_N)$ are easily shown (in the sense of (2.4) and (2.5)) to converge to $\mu_k(\text{Poisson}) := \frac{x^k e^{-x}}{k!}\,dx$. The limiting pair correlation $R_2(\text{Poisson})$ is simply dx on $\mathbb{R}$. The measures $\nu_1(T^N)$ converges to $e^{-x}dx$ as $N\to\infty$.

3 The Basic Conjecture

The general belief is that the local (scaled) spacing statistics of the eigenvalues of a system should be dictated by symmetry and given the symmetry type, it is universal. For the semi-classical problems as in §1.1, we must in general first scale the spectrum (or, as it sometimes is called, "unfold") before considering the spacings. For example, in the case of a compact Riemannian manifold X of dimension n, Weyl's law asserts that

$$\lambda_j \sim C_n(\text{Vol}(X)j^{2/n}, \quad \text{as} \quad j\to\infty, \tag{3.1}$$

C_n a constant depending on n.
Hence if we set

$$\hat{\lambda}_j = \frac{\lambda_j^{n/2}}{(C_n \mathrm{Vol}(X))^{n/2}}. \tag{3.2}$$

Then $\hat{\lambda}_j \sim j$.

We examine the local spacings for the numbers $\hat{\lambda}_j$. That is, as in §1.2 set

$$\mu_k(X,N)[a,b] = \frac{\#\{j \le N \mid \hat{\lambda}_{j+k} - \hat{\lambda}_j \in [a,b]\}}{N} \tag{3.3}$$

and similarly for the other statistics. Note that $n = \dim X = 2$ is particularly pleasant in that no unfolding is necessary.

Another technical point is that if there are symmetries of the system they should be taken into account. Thus, for example, in the case of a Riemannian X, if there are discrete isometries of X one should decompose the spectrum according to these symmetries — that is, "desymmetrize".

The following is the basic conjecture of the subject. It appears by now to be a well accepted and established phenomenon in the physics literature.

Basic Conjecture

(A) (Berry-Taylor [16]). If H is completely integrable then the local spacing statistics for H in the semi-classical limit $\hbar \to 0$ or the large energy limit is Poissonian (i.e., behaves like the typical $x \in T^N$ as $N \to \infty$).

(B) (Bohigas-Giannoni-Schmit [17]). If H is chaotic then the local spacings statistics for the eigenvalues of $\widehat{H}$ in the above limits follow

(i) GOE (COE) statistics if H has time reversal symmetry[2]

(ii) GUE (= CUE) statistics if H does not have time reversal symmetry.

Put another way, (A) asserts that the eigenvalues in the classically integrable case behave like random numbers while (B) asserts that the eigenvalues of a chaotic H behave like specific random matrix models, in particular they are "rigid" (an examination of the measures $\mu_1(\mathrm{GOE})$ and $\mu_1(\mathrm{GUE})$ show that their densities vanish to first and second order at $x = 0$, respectively. This signifies that for these models the eigenvalues "repel" each other).

The primary and most compelling evidence for the above are a host of numerical experiments[3], all for 2 degrees of freedom. Typical of these are the following, for which a few thousand eigenvalues can be computed. The first Hamiltonian is a billiard ball in a quarter ellipse (desymmetrized), Figure 1, $\widehat{H}_E$ is the Laplacian on E with (say) Dirichlet boundary conditions. H_E is integrable (Euclid) and the spectrum is Poissonian [19]. The second is a

[2]That is to say, it behaves like the typical member of GOE... .

[3]and even physical experiments [18]

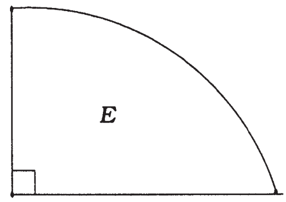


Figure 1:

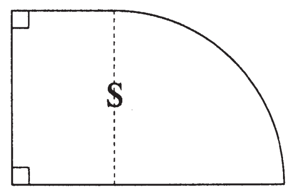


Figure 2:

billiard ball in a (quarter) Stadium S, Figure 2, considered by Bunimovich [20], who shows this classical Hamiltonian is chaotic. This H_S has time reversal symmetry (as do H_E, H_B, H_T, and H_A) and the spectrum follows GOE statistics [18,19]. Next is H_B, the billiard ball in a box with side lengths

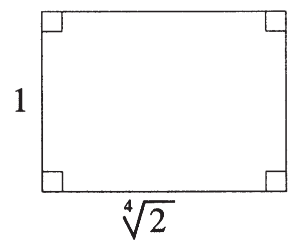


Figure 3:

shown in Figure 3. H_B is integrable and the spacings are Poissonian [16]. The next Hamiltonian H_T is the billiard motion along geodesics in a hyperbolic triangle T, Figure 4 (i.e., a triangle in the hyperbolic plane with the billiard, as always, obeying the law that angle of incidence equals angle of reflection). This is essentially the same as the geodesic flow on a surface of constant negative curvature, in particular H_T is chaotic. For the angles $\alpha = \pi/8$, $\beta = \pi/2$, $\gamma = \frac{67}{200}\pi$ and pretty much any other choice, the spectrum of $\widehat{H}_T$ (which we take to be the hyperbolic Laplacian on T with Dirichlet boundary conditions) is GOE [21].

Finally, H_A is the Hamiltonian in a triangle T as in Figure 4 except that the angles are chosen to be $\alpha = \pi/8$, $\beta = \pi/2$, $\gamma = \pi/3$. In this case very

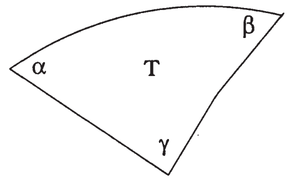


Figure 4:

surprisingly the spacings for $\widehat{H}_A$ are Poissonian! [21, 22].

All of the above, save for H_A, are phenomenological confirmations of the Basic Conjecture (we will return to this exception later). We now review the analytical results that have been established towards the Basic Conjecture.

4 Completely Integrable Hamiltonians

The basis of Conjecture A above of Berry-Tabor is their numerical experimentation with some specific examples. Recently there has been some progress in analyzing these, we discuss this next.

4.1 Particles in a box([16])

H_B of Section 3 is an example of such a Hamiltonian. These are essentially geodesic motion on a flat two dimensional torus $X_L = \mathbb{R}^2/L$. Here L is a lattice in $\mathbb{R}^2$. The angle that a geodesic makes with a given geodesic will remain constant during the geodesic flow. Thus the corresponding Hamiltonian is completely integrable. The eigenfunctions of X are $e(\langle\gamma, x\rangle)$ with $\gamma \in L^*$ the lattice dual to L. Hence the spectrum of X (or rather the Laplacian on X) is the set

$$\{4\pi^2|\gamma|^2 \colon \gamma \in L^*\} \ . \tag{4.1}$$

If w_1, w_2 is a $\mathbb{Z}$-basis of L^* and $F_L(x_1, x_2) = 4\pi^2|x_1w_1+x_2w|^2$, then $\mathrm{Spec}(X_L)$ is the set of values of the quadratic form F_L at the integers $\mathbb{Z}^2$. We write $F_L(x_1, x_2) = \alpha x_1^2 + \beta x_1x_2 + \gamma x_2^2$, $(\alpha, \beta, \gamma) \in \mathbb{R}^3$. If $(\alpha, \beta, \gamma) = \lambda(\alpha_1, \beta_1, \gamma_1)$ with $\alpha_1, \beta_1, \gamma_1 \in Q$ we say that F_L, or L, is rational. This is a singular case both classically and quantum mechanically, since the set of lengths of periodic orbits of the classical flow is highly degenerate and so is the spectrum. [16] therefore avoid these as far as Conjecture A goes, that is, they (and we) assume that L is irrational.

The pair-correlation for the spectrum of X_L is concerned with the values at integers of an irrational quadratic form of signature $(2, 2)$. In fact it is easily seen that the issue of the pair correlation being Poissonian is equivalent to a quantitative form of the classical Oppenheim Conjecture. See [23] for a review of this classical problem and its solution using the machinery of unipotent flows (Margulis [24], Ratner [25]). In [26] Eskin, Margulis, and Moses establish such a quantitative Oppenheim for irrational quadratic forms of signature $(3,1)$. However, signature $(2,2)$ is more subtle and the behavior depends on the Diophantine properties of the coefficients.

In [27] we were recently able to resolve this pair-correlation problem, at least for almost all X_L's, by a direct analysis. We show that for almost all X_L in the sense of Lebesgue measure on (α, β, γ), the pair correlation for $\mathrm{Spec}(X_L)$ is Poissonian. The proof involves a reduction of the problem to a delicate question about bounding integer points in $\mathbb{Z}^8$ satisfying a pair of inequalities defined by integral homogeneous forms of degree 4. Using the result about almost all X_L's one can show quite easily (see [27]) that for the topologically generic X_L (i.e., in the sense of Baire category), the consecutive spacing measures $\mu_1(X_L, N)$ do not converge as $N \to \infty$. In fact they oscillate between at least two distinct probability measures. The proof of the almost all X_L result above offers no explicit example of an X_L for which the pair-correlation is Poissonian. I learned recently from Eskin that he, Margulis, and Moses have extended the machinery used in the quantitative Oppenheim theory to include such a form as $x_1^2 + \sqrt{2}\,x_2^2 - x_3^2 - \sqrt{2}\,x_4^2$. In particular this shows that H_B of Section 3 has Poisson pair-correlation. This then is the first *explicit* example of an H for which some local spacing statistic has been shown to be what is predicted by the Basic Conjecture A.

The almost everywhere technique above generalizes. VanderKam [28] has shown that the generic, in measure, flat torus in $\mathbb{R}^4$ has Poisson pair-correlation while again for the topologically generic such X the consecutive spacings don't exist. We note that in this case one must unfold ($\dim X = 4$) so that the pair-correlation is concerned with values at integers of a quartic form in eight variables.

We emphasize that the analytic results above all concern pair-correlation only.

4.2 Boxed Oscillator ([16])

The second example investigated numerically in Berry-Tabor is the Hamiltonian H_α on $T(\mathbb{R} \times S^1)$ given by

$$H_\alpha(x, \xi) = \xi_1^2 + x_1^2 + \alpha \xi_2^2 \tag{4.2}$$

This corresponds to a harmonic oscillator in one direction and a boxed (or periodic) motion in the other. Fixing $\hbar$ we have

$$\widehat{H}_\alpha = -\frac{\partial^2}{\partial x_1^2} + x_1^2 - \alpha \frac{\partial^2}{\partial x_2^2} \,. \tag{4.3}$$

Here $\alpha > 0$ is a parameter. H_α is completely integrable and the spectrum of $\widehat{H}_\alpha$ may be computed explicitly by separation of variables:

$$\operatorname{Spec}(\widehat{H}_\alpha) = \{\lambda_{m,n} = \alpha n^2 + m \mid m \geq 1, n \in \mathbb{Z}\} \tag{4.4}$$

To avoid obvious degeneracy we assume as we did in the last example that α is irrational. In this example we are interested in the large energy limit,$\lambda \to \infty$. The number of eigenvalues of H_α in the interval $[M, M+1]$ is approximately $\sqrt{M}$. In forming the local scaling spacing distributions such as $\mu_1(H_\alpha, M)$, we do so with the eigenvalues in $[M, M+1]$. Hence these questions reduce to the local spacing statistics for the sequence $\alpha n^2(\bmod 1)$, $n \leq N$ as $N \to \infty$. While the question of the equidistribution of $\alpha n^2(\bmod 1)$ and variations thereof have been studied since Weyl [29], the issue of local spacings has not been considered. (Note that $\alpha n(\bmod 1)$ is not random in our sense, since the consecutive spacings assume at most three values [30]).

The analogue of the pair-correlation result of (4.1) for almost all $\widehat{H}_\alpha$'s is quite easy to carry out here and was done by Rudnick and Sarnak [31]. In another work Rudnick-Zaharescu and Sarnak [32] have recently been able to go a lot further. Using Diophantine approximations to α by rationals a/q (depending on N) one can reduce the above questions to ones about the spacings between the numbers $an^2(\bmod q)$, $1 \leq n \leq N$ and $q^{1/2+\varepsilon} \ll N \ll q^{1-\varepsilon}$, $\varepsilon > 0$ as $q \to \infty$ (note the ranges on N which are crucial). To calculate the k-level correlations for the last sequence one is led to estimating the number of solutions to

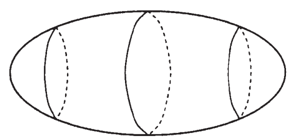
$$\left.\begin{aligned} &a(n_1^2 - n_2^2) \equiv b_1(\bmod q) \\ &\vdots \\ &a(n_k^2 - n_{k+1}^2) \equiv b_k(\bmod q) \\ &1 \leq n_j \leq N \end{aligned}\right\} \tag{4.5}$$

These equations, at least if q is prime, define a curve in affine $k+1$ dimensional space over the finite field $\mathbb{F}_q$. One may use the Riemann Hypothesis for curves over finite fields, established by Weil [33], to estimate the number of solutions to (4.6). For N in the range $\left[q^{1-\frac{1}{2k}}, q^{1-\varepsilon}\right]$ this procedure works well and yields the answer that one would obtain if $an^2(\bmod q)$ were random. This analysis leads to the Conjecture that for any irrational α, there is a subsequence $N_j \to \infty$ such that $n^2\alpha(\bmod 1)$, $n \leq N_j$, has all its local spacings Poissonian. Moreover we [32] conjecture that if α is of type $\varkappa < 3$ (we say α is of type $\varkappa$ if there is a $C_\alpha > 0$ such that for all integers a, q, $(a,q) = 1$, $\left|\alpha - \frac{a}{q}\right| \geq C_\alpha q^{-\varkappa}$) then $n^2\alpha(\bmod 1)$ is Poissonian along the full sequence of integers N. With the above analysis it is proven in [32] that if α is not of type 3 then, as predicted by the above Conjecture, there is a subsequence N_j of N going to infinity such that the scaled spacings of

$n^2\alpha(\bmod 1)$, $n \leq N_j$ are Poissonian (this is established for all the k-level correlations and hence for the consecutive spacings as well). In particular for such α (which are topologically generic but measure theoretically null) we have that $\mathrm{Spec}(\widehat{H}_\alpha)$ is fully Poissonian along a subsequence of M's that is the basic conjecture is true along a subsequence of energies. In any case the Conjectures above give precise predictions about the spacings of $\mathrm{Spec}(\widehat{H}_\alpha)$ and its dependence on the Diophantine properties of α.

4.3 Surfaces of Revolution

The above examples are special in that the spectrum could be given by an explicit formula. However in the integrable case the eigenvalues obtained by the Bohr-Sommerfeld quantization conditions [34] are sufficiently good approximations in the semiclassical limit, to yield similar working expressions. We illustrate this with X a surface of revolution of the following shape: Colin



X

Figure 5:

de Verdiere [34] has shown that for such an X there are piecewise analytic functions $F_2(x_1, x_2)$ and $F_0(x_1, x_2)$, the first homogeneous of degree two and the second of degree zero, such that to an accuracy which is good enough to study scaled spacings;

$$\mathrm{Spec}(\Delta_X) = \left\{ F\left(\ell + \frac{1}{2}, k\right) : |\ell| \leq k,\ k = 1, 2, \ldots \right\} \tag{4.6}$$

where $F = F_2 + F_0$.

In the case of a flat torus F_2 is a quadratic form, $F_0 = 0$ and the above is exact. We see that more generally the Basic Conjecture A is a question about whether the values of such an F at the integers are random? Along these lines Sinai [35] and Major [36] have shown that if F_2 is constructed from a generic (in measure) Brownian path (in particular is nowhere differentiable) then its values at the integer lattice are Poissonian. This does not apply to the above but we believe it does point to Basic Conjecture A being true in some generic sense.

Among these surfaces of revolution are some singular examples called Zoll surfaces Z [37]. These are surfaces all of whose geodesics are closed and are of the same length. They form an infinite dimensional family [37]. For these $F_2(x_1, x_2) = \frac{x_1^2}{2}$, which is degenerate and which results in the spectrum looking like a small perturbation of that of the round sphere (which is the most familiar Zoll surface). That is, the spectrum consists of clusters of $2k+1$ eigenvalues within $O(1/k)$ of $\frac{k(k+1)}{2}$, $k \geq 1$. Thus it is clear that the consecutive spacing measures, as defined in (3.3), will approach a unit delta mass at the origin. However, given the structure of the spectrum of a Zoll surface, it is natural to form the spacing distributions out of each cluster of size $2k+1$ and to unfold appropriately. This was done by Uribe and Zelditch [38] who determine the limiting behavior of the corresponding pair-correlation $R_2(k, Z)$ as $k \to \infty$. The result is a density on $\mathbb{R}$ determined by the geometry of the space of geodesics on Z and this density very much depends on Z. In particular it is non-universal.

This concludes our review of Case A of the Basic Conjecture. Examples (4.1) and (4.2) show that the problem is very subtle in its dependence on parameters. The Basic Conjecture A is true at best in some generic sense. The examples indicate that generic should be in the sense of almost-all with respect to a measure on the space of Hamiltonians under consideration. A precise formulation, however, is missing at this time.

5 Chaotic Case

An analytic attack on the Basic Conjecture B for chaotic H's is much more difficult. There are no examples with explicitly computable spectra. Moreover there is no semi-classical approximation of the Bohr-Sommerfeld type (this being the reason that Bohr was unsuccessful at quantizing anything but completely integrable H's). The only rigorous tools available are the trace formula (Selberg [39], Gutzwiller [40], Balian-Bloch [41], and Duistermaat-Guillemin [42]). These give expressions for sums over the spectrum of $\widehat{H}$ in terms of sums over the periodic orbits of the classical flow. The simplest case and the only one for which there is an exact formula is that of Selberg which in the case of X being a hyperbolic surface (i.e., constant curvature $K \equiv -1$) of genus $b \geq 2$ reads as follows [39]:

For any $g \in C_0^\infty(\mathbb{R})$, $g(x) = g(-x)$ and $h(t) = \hat{g}(t)$ its Fourier transform

$$\sum_{j=0}^{\infty} h(t_j) = (b-1)\int_{-\infty}^{\infty} th(t)\tanh(\pi t)dt + \sum_{p\in P}\sum_{k=1}^{\infty} \frac{\ell(p)}{e^{k\ell(p)/2} - e^{-k\ell(p)/2}} g(k\ell(p)) \tag{5.1}$$

where $\lambda_j = t_j^2 + \frac{1}{4}$ are the eigenvalues of Δ_X, and P is the set of prime closed geodesics on X and for $p \in P$, $\ell(p)$ is its length.

For the analysis of sums over the spectrum λ_j in intervals $[\lambda, \lambda + R]$ where $R \geq \sqrt{\lambda}$, the above trace formula is very powerful since on the right hand side there are essentially no contributions from the periodic orbits. However for $R = \lambda^\alpha$ with $\alpha < \frac{1}{2}$, the right hand side involves sums over exponentially many periodic orbits and is very difficult to analyze. In particular one can express the pair correlation for spec(Δ_X) in terms of such sums over periodic orbits. However the problem of establishing cancellations over these long sums over periodic orbits is beyond the range of present technology. Consequently there are no proven results concerning the local spacings for the spectrum of such X's.

One way to proceed is to impose some assumptions about the statistical nature of the long periods $l(p)$, $p \in P$. While it appears difficult to verify such assumptions (for any given X or even on average X's) it is interesting to see what these lead to. The results of Berry [43] and the more recent deeper investigation of Bogomolny and Keating [44] lead to results about the local spacing for spec(Δ_X) which are consistent with GOE (note that geodesic flow always has time reversal symmetry). However it remains problematic, even with these assumptions, to recover the GOE pair correlation, (2.10).

One aspect that is clarified by an analysis using the trace formula is that if the periodic orbit spectrum is highly degenerate then the local spacing will not be GOE. Indeed if X is a hyperbolic surface as above then X may be realized as $\Gamma \setminus \mathbb{H}^2$ where $\mathbb{H}^2$ is the hyperbolic plane and Γ is a discrete co-compact subgroup of $SL_2(R)$. Among such Γ's are such groups as $SL_2(Z)$ (which has finite volume but not compact quotient) which are defined through integers and are called arithmetic (see Borel [45] for definition). There are 85 hyperbolic triangles T in H^2 for which the reflections in their sides generate such a discrete arithmetic Γ (Takeuchi [46]). Example H_A of Section 3 is one of these. If $X = X_\Gamma$ is arithmetic then the length spectrum is very highly degenerate [LS]. For these one shows that:

$$\sum_{l(p)\leq V,\ \text{distinct } l(p)'s, p\in P} 1 = O(e^{V/2}) \tag{5.2}$$

as $V \to \infty$.

On the other hand it follows easily from (5.1) that for any hyperbolic X

$$\sum_{l(p)\leq V, p\in P} 1 \sim \frac{e^V}{V} \text{as } V \to \infty \tag{5.3}$$

Hence in the arithmetic case the average degeneracy of a length of a closed geodesic of length l is about $e^{l/2}$. As was first observed in [47] and [48], this degeneracy manifests itself in the local spacing statistics of the spectrum of Δ_{X_Γ} being non GOE (actually, Selberg had noticed long before that this degeneracy forces the remainder term in Weyl's law to be very large - see Hejhal [49]). Numerical experiments with a large subset (H_A being one such) of the 85 arithmetic triangles show that they have Poisson spacing statistics! Various levels of analytic explanations of this phenomenon have been given using the trace formula (5.1) (see [22, 47, 48, 50, 51]). We note these triangles have no extra symmetries. However there is a large family of geometrically defined operators on $L^2(X_\Gamma)$ known as Hecke operators, which commute with Δ_{x_Γ} [3]. Moreover there are many such Hecke operators iff Γ is arithmetic [52].

A completely different approach to the basic conjecture B has been put forth by Agam, Altshuler, Andreev and Simons [53]. Their ideas come from the theory of "disordered systems" such as the spectral properties of $H = \Delta + V$ in $L^2(R^3)$, where V is a random potential. For these, when averaging over the potential V, Efetov [54] has shown how one directly arrives at the random matrix correlations in the extended state regime, by using a functional integral representation for the Greens function of $\hat{H}$. These integrals are analyzed using supersymmetric methods and in particular nonlinear σ-models [54]. [53] adapt these methods to the case of $\hat{H}$ with H chaotic. Unlike the case above where averages over V are performed, they attempt to deal with an individual Hamiltonian H. At present their conclusions appear to be the strong - for example they assert that for a triangle such as H_A of Section 3 (which satisfies their assumptions about chaotic H and about location of poles of the corresponding Ruelle zeta function) the spacings are GOE. Nevertheless their approach is interesting and in particular its strength may lie in studying ensembles of H's with the idea of establishing that the GOE is valid for the random H (like the case of random V) or for a typical H in the sense of measure, from the ensemble. In fact, very recently Zirnbauer [55] has pursued this possibility for the quantization of chaotic symplectic transformations and his approach looks promising. In passing we note that there is a growing literature concerning the Basic Conjecture for symplectomorphisms instead of Hamiltonians, see Zelditch [4] for review of mathematical results in this direction.

We conclude Lecture 1 with some comments about the Basic Conjecture B. Our understanding of this case is a lot poorer than that for A. In part, this is due to the lack of examples that can be studied analytically. The known exceptions to GOE, that is the arithmetic X's, are sparse. In particular consider the analogue of the spaces of flat tori, which are the moduli spaces $\mathcal{M}_b$ of hyperbolic surfaces of genus b. $\mathcal{M}_b$ is a $3b - 3$ complex manifold [56] and the set of arithmetic X's in a given $\mathcal{M}_b$ is finite [45]. One certainly expects that, measure-wise, the generic X in $\mathcal{M}_b$ satisfies GOE. With our present minimal understanding it is even possible that Basic Conjecture B is valid for all but finitely many X's in $\mathcal{M}_b$ though this seems to me to be unlikely.

In the second lecture we examine similar questions for the spacing and densities of zeros of zeta functions. Here much more progress on the analytic side is possible and a solid picture has emerged.

References

[1] V. Arnold and A. Avez, *Ergodic problems of Mechanics* (W. A. Benjamin, New York, 1968).

[2] Y. Sinai, *Math. Notes.* **18**, (1976), princeton University Press, Princeton.

[3] P. Sarnak, *The Schur Lectures*, Vol. 8 of *Israel Math. Conf. Proc.* (AMS, Tel Aviv 1992, 1995), pp. 183–236.

[4] S. Zelditch, Quantum dynamics from a semi-classical point of view, 1997, preprint.

[5] N. Hurt, *MIA* **397**, (1997), kluwe Academic Publishers, Boston.

[6] R. Heller, *Phys. Rev. Lett.* **53**, 1515 (1984).

[7] E. Wigner, *SIAM Review* **9**, 1 (1967).

[8] M. Gaudin, *Nuclear Phys.* **25**, 447 (1961).

[9] M. L. Mehta, *Random Matrix Theory* (Academic Press, N.Y., 1991).

[10] F. Dyson, *J. Math. Phys.* **3**, 166 (1962).

[11] A. Altland and M. Zirnbauer, *cond-mat/9602137* (1996).

[12] N. Katz and P. Sarnak, *Random matrices, Frobenius eigenvalues and monodromy* (AMS, Rhode Island, 1998), to appear.

[13] N. Katz and P. Sarnak, Zeros of zeta functions, their spacings and their spectral nature, 1997, preprint.

[14] S. Helgason, *Differential geometry, Lie Groups, and symmetric spaces* (Academic Press, New York, 1978).

[15] W. Feller, *An introduction to probability theory and its applications. Vol. II* (John Wiley & Sons Inc., New York, 1966).

[16] M. Berry and M. Tabor, *Proc. Roy. Soc. London A* **356**, 375 (1977).

[17] O. Bohigas and M.Giannoni, *Phys. Rev. Lett.* **52**, 1 (1984).

[18] H. D. Gräf et al, *Phys. Rev. Lett.* **69**, 1296 (1992).

[19] O. Bohigas, M. Giannoni, and C. Schmit, *J. Physique Lett.* **45**, (1984).

[20] L. Bunimovich, *Functional Analysis and Applications* **8**, 254 (1974).

[21] C. Schmit, *Chaos and Quantum Physics (in Les Houches, conference)* (North Holland, Amsterdam, 1991), pp. 333–369.

[22] R. Aurich, E. Bogomolny, and F. Steiner, *Physica D* **48**, 76 (1991).

[23] A. Borel, *BAMS* **32**, 184 (1995).

[24] G. Margulis, *C. R. Acad. Sci. Paris SerI Math.* **304**, 247 (1987).

[25] M. Ratner, *Ann. Math.* **134**, 545 (1991).

[26] A. Eskin, G. Margulis, and S. Moses, *Electronic Res. Announcements AMS* **1**, 124 (1995).

[27] P. Sarnak, Values at integers of Binary Quadratic Forms, to appear in a volume in memory of C. Herz, edited by S. Drury.

[28] J. Vanderkam, Pair correlation of four-dimensional flat tori, 1997, preprint.

[29] H. Weyl, *MAth. Ann.* **77**, 313 (1916).

[30] S. Swierczowski, *Fund. Math.* **46**, 187 (1958).

[31] Z. Rudnick and P. Sarnak, *Duke Math. J.* **81**, 269 (1996).

[32] Z. Rudnick, P. Sarnak, and A. Zahareschu, in preparation.

[33] A. Weil, *Proc. Nat. Acad. Sci. U.S.A.* **27**, 345 (1941).

[34] Y. Colin de Verdiere, *Invent. Math.* **43**, 15 (1977).

[35] Y. Sinai, *Dynamical Systems and Stat. Mech., Adv. Soviet Math.* **3**, 199 (AMS 1991).

[36] P. Major, *Probab. Theory Related Fields* **92**, 423 (1992).

[37] A. Besse, *Manifolds all of whose geodesics are closed* (Springer-Verlag, N.Y., 1978).

[38] A. Uribe and S. Zelditch, *Com. Math. Phys.* **154**, 313 (1993).

[39] A. Selberg, *J. Ind. Math. Soc.* **20**, 47 (1956).

[40] M. Gutzwiller, *J. Math. Phys.* **8**, 1979 (1967).

[41] B. Balian and C. Bloch, *Ann. Phys.* **69**, 76 .

[42] H. Duistermaat and V. Guillemin, *Invent. Math.* **29**, 39 (1975).

[43] M. Berry, *Proc. Royal. Soc. London* **A 400**, 229 (1985).

[44] E. Bogomolny and J. Keating, *Phys. Rev. Lett.* **77**, 1472 (1996).

[45] A. Borel, *Ann. Sc. Norm. Super. Pisa Sci(4)* **8**, 1 (1981).

[46] K. Takeuchi, *J. Fac. Sci. Univ. Tokyo* **24**, 201 (1977).

[47] E. Bogomolny, B. Georgeot, M. Giannoni, and C. Schmit, *Phys. Rev. Lett.* **69**, 1477 (1992).

[48] F. S. J. Bolte, G. Steil, *Phys. Rev. Lett.* **69**, 288 (1992).

[49] D. Hejhal, *S.L.N.* **548**, (1976), springer, N.Y.

[50] W. Luo and P. Sarnak, *Com. Math. Phys.* **161**, 423 (1994).

[51] E. Bogomolny, F. Leyoraz, and C. Schmit, *Comm. Math. Phys.* **176**, 577 (1996).

[52] G. Margulis, in *Proc. Int. Congr. Math.* (Canadian Math. Congress, Vancouver, 1975), p. 21.

[53] O. Agam, B. Simons, A. Andreev, and B. Altshuler, *Nucl. Phys. B* **482**, 536 (1996).

[54] K. B. Efetov, *Advances in Physics* **32**, 53 (1983).

[55] S. Zirnbauer, The color-flavor transformation and a new approach to quantum chaotic maps, 1997, preprint.

[56] A. Weil, *Collected Works* (Springer, N.Y., 1980), Vol. II, pp. 381–391.