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Is there finite-time blow-up in 3-D Euler flow ?

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According to Richardson's ideas on high Reynolds number turbulence, energy introduced at the scale ℓ_0, cascades down to the scale $\eta \ll \ell_0$ where it is dissipated. Consider the total time $T_\star$ which is the sum of the eddy turnover times associated with all the intermediate steps of the cascade. From standard phenomenology à la Kolmogorov 1941 (K41), the eddy turnover time varies as $\ell^{2/3}$. If we let the viscosity ν, and thus η, tend to zero, $T_\star$ is the sum of an infinite *convergent* geometric series. Thus it takes a *finite time* for energy to cascade to infinitesimal scales. We also know that in the limit $\nu \to 0$, the enstrophy, the mean square vorticity, goes to infinity as ν^{-1} (to ensure a finite energy dissipation).

From such observations, it is tempting to conjecture that ideal flow, the solution of the (incompressible) 3-D Euler equation

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p \qquad (1)$$

$$\nabla \cdot \mathbf{v} = 0, \qquad (2)$$

when initially regular,[1] will spontaneously develop a singularity in a finite time (finite-time blow-up).

[1]For example, by having only large-scale motion initially, so that the flow is very smooth, actually analytic.

This would be incorrect for at least two reasons. Firstly, the kind of phenomenology assumed above is meant only to describe the (statistically) steady state in which energy input and energy dissipation balance each other. The inviscid ($\nu = 0$) initial-value problem is not within its scope. Secondly, a basic assumption needed for K41 is that the symmetries of the Navier–Stokes or Euler equation are recovered in a statistical sense. This requires the flow to be *highly disorganized.* Kolmogorov himself was clearly aware of this, since in a footnote to his first 1941 paper he wrote:

> ... *In virtue of the chaotic*[2] *mechanism of translation of motion from the pulsations of lower orders to the pulsations of higher orders, ... the fine pulsations of higher orders are subjected to approximately space-isotropy statistical régime ...*

Complex spatial structures have never been observed in numerical simulations of inviscid flow with smooth initial conditions. Note that inviscid flow has frozen-in vortex lines the topology of which cannot change since no viscous reconnection can take place.

There is yet another phenomenological argument, not requiring K41, which suggests finite-time blow-up. Consider the equation for the vorticity $\omega \equiv \nabla \wedge \mathbf{v}$ for inviscid flow, written as

$$D_t \omega = \omega \cdot \nabla \mathbf{v}, \tag{3}$$

where $D_t \equiv \partial_t + \mathbf{v} \cdot \nabla$ denotes the Lagrangian derivative. Observe that $\nabla \mathbf{v}$ has the same dimensions as ω and can be related to it by an operator involving Poisson-type integrals. (For this use the fact that $\nabla^2 \mathbf{v} = -\nabla \wedge \omega$.) It is then tempting to predict that the solutions of (3) will behave as the solution of the scalar nonlinear equation

$$D_t s = s^2, \tag{4}$$

which blows up in a time $1/s(0)$ when $s(0) > 0$. Actually, (4) is just the sort of equation one obtains in trying to find rigorous *upper bounds* to various norms when studying the well-posedness of the Euler problem. This is precisely why the well-posedness 'in the large' (i.e. for arbitrary $t > 0$) is an open problem in three dimensions.[3] This problem has been singled out by Saffman (1981) as 'one of the most challenging of the present time for both the mathematician and the numerical analyst'.

The evidence is that the solutions of the Euler equation behave in a way much tamer than predicted by (4). Since such evidence cannot be obtained by experimental means, one has to resort to numerical simulations. For example, Brachet *et al.* (1983) studied the Taylor-Green vortex for which

[2]'khaotitsheskogo' in the original

[3]In two dimensions, the absence of singularities for any $t > 0$ has been proven by Hölder (1933) and Wolibner (1933).

the initial conditions are simple trigonometric polynomials in the x, y and z variables. Using a spectral method on a 256^3 grid they found that, as long as the simulation does not run out of resolution, the width $\delta(t)$ of the complex-space analyticity strip, as a function of real time decreases exponentially. If this result can be safely extrapolated to later times, it follows that the Taylor–Green vortex will *never* develop a real-space real-time singularity: there is no inviscid blow-up. When this result was obtained in 1981, it came as a rather big surprise. Indeed, based on the kind of phenomenology described above and also on results from closure, there was a widespread belief that finite-time blow-up would take place.[4]

But is it safe to extrapolate the behavior of $\delta(t)$? About ten years later it became possible to extend the Brachet *et al.* (1983) calculation, using a grid of 864^3 points and also to study flows with random initial conditions without the somewhat special symmetries of the Taylor–Green vortex which helped in reducing computational work (Brachet, Meneguzzi, Vincent, Politano and Sulem 1992). Again, exponential decrease of $\delta(t)$ was observed.

However formidable a 864^3 simulation may look, it can only explore a span of scales of about 300, because it uses a *uniform* grid. Pumir and Siggia (1990) developed a different approach using grid-refinement 'where needed' and were thereby able to explore a span of scales of up to 10^5. Still, no blow-up was observed. Somewhat paradoxically, simulations by Grauer and Sideris (1991) and Pumir and Siggia (1992) of two-dimensional axisymmetric flow with a poloidal component of the velocity (an instance for which there is no regularity theorem) have given some evidence of finite-time blow-up. This is, however, a controversial issue (see, e.g., E and Shu 1994).

All these simulations have also given us a qualitative explanation for why ideal flow is much more regular then predicted by naive phenomenology: the exponential decrease of $\delta(t)$ corresponds to an exponential flattening of vorticity 'pancakes'. The vorticity in such structures has a very fast dependence on the spatial coordinate transverse to the pancake, so that the flow is to leading order one-dimensional. If the flow were exactly one-dimensional, the nonlinearity would vanish (as a consequence of the incompressibility condition). This *depletion of nonlinearity* explains why the growth of the vorticity is much slower than predicted by (4) which ignores this phenomenon. The problem of finite-time blow-up remains thus completely open.

The material above is mostly taken from Section 7.8 of "Turbulence, the Legacy of A.N. Kolmogorov" by U. Frisch, CUP (1995). There, the reader will find all the references and additional material on depletion of nonlinearity (in Section 9.3).

[4]I shared such a belief, but G.I. Taylor did not, as appears from a brief statement made to S.A. Orszag in 1969 which was communicated to me privately. As for A.N. Kolmogorov I am not aware of anything he has said on this matter.