The Cauchy problem for Nonlinear Schrödinger Equation (NLS) with critical nonlinearity

J. Bourgain
IAS at Princeton

Consider the initial value problem

$$\begin{cases} iu_t - \Delta u + u|u|^{p-2} = 0 \\ u(0) = \phi \in H^s(\mathbb{R}^D) \end{cases} \tag{1}$$

corresponding to defocusing NLS with Hamiltonian

$$H(\phi) = \int |\nabla\phi|^2 + \int |\phi|^p. \tag{2}$$

If we fix $s \geq 0$, then the critical nonlinearity is given by

$$p - 2 = \frac{4}{D - 2s}. \tag{3}$$

In this case, there is local wellposedness on a time interval $[0, T^*[$ and global wellposedness ($T^* = \infty$) for small data (these facts hold also in the focusing case). The question is whether in the defocusing case, the smallness assumption may be dropped in the global existence result. Two cases are particularly significant because of conservation laws.

(I) $\underline{s = 0}$: The L^2-critical case

Global wellposedness holds if $\phi \in H^1$ or $(1 + |x|)\phi \in L^2$.

The question remains for $\phi \in L^2$

(II) $\underline{s=1}$: The H^1-critical case.

In particular, for 3D, the global existence of classical solutions for

$$iu_t - \Delta u + u|u|^4 = 0 \tag{4}$$

is open (one also expects scattering).

This is perhaps the most important question in this area.

Comments

Questions (I), (II) are related. In both cases, the issue consists in disproving certain concentrations (of L^2 or H^1-norm) on small regions in space-time by means of apriori inequalities. For the 3D wave equation

$$y_{tt} - \Delta y + y^5 = 0 \tag{5}$$

the problem was solved by Struwe (radial case) [S] and Grillakis [G]. It is basically an interplay between Strichartz's theory and the apriori inequality of Morawetz. On an heuristic level, the main difference between (9) and (5) is infinite propagation speed. The question is open for (4) also in the radial case. In fact, no global solutions that are not small in some sense seem known. See [G-V], [L-S], [C] for the NLS background.

References

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