GEOMETRY OF NODAL SETS AND MULTIPLICITY OF EIGENVALUES

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This note is concerned with several problems on the geometry of nodal sets of harmonic functions and related problems on multiplicities of eigenvalues of the Laplace operator on surfaces. Partly these questions are based on the problems raised by S.T. Yau in [16] and [17] (see also [15]). Let us denote by $H_k(*)$ the k-dimensional Hausdorff measure. Let $B_r \subset \mathbb{R}^n$ be the ball $|x| < r$. Let u be a positive harmonic function defined in B_1. Denote by $n_u = u^{(-1)}(0)$ the nodal set of function u. In dimensions greater than 2 little is known about the geometry of n_u. Does the bound

$$H_{n-1}(n_u) \leq K \tag{1}$$

imply any further "regularity" of the set n_u? For a precise setting of the last question let us denote by N_K the set of all nodal sets of harmonic functions satisfying (1).

Is N_K closed in the Hausdorff topology?

We can pose the last question in a different form:

Conjecture 1. *There are constants C and N depending only on n and K such that if u is a harmonic function in the ball $B_1 \subset \mathbb{R}^n$ and $H_{n-1}(n_u) \leq K$ then*

$$\max_{B_{1/2}} |u| \leq C \sum_{|\alpha|=0}^{N} |D^\alpha u(0)| \,.$$

In dimension 2 the last inequality holds with the constants C and N depending only on the number of nodal domains of the function u [11].

As a weak form of the Conjecture 1 we formulate the next one.

Conjecture 2. *There exists a universal constant $\delta > 0$ such that if u is a harmonic function in $B_1 \subset \mathbb{R}^3$ and $u(0) = 0$ then*

$$H_2(n_u) > \delta\,.$$

One can also ask if the conclusion of Conjecture 2 holds for solutions of second order elliptic equations with smooth or measurable coefficients in dimensions ≥ 3. In dimension 2, Conjecture 2 is obviously true.

Conjecture 2 implies the following one

Conjecture 3. *Let $G \in \mathbb{R}^3$ be a bounded domain with smooth boundary ∂G. Let u be a Dirichlet eigenfunction:*

$$\Delta u = \lambda u \quad on \quad G,$$

$$u = 0 \quad on \quad \partial G.$$

Then

$$H_2(n_u) > C\lambda^{1/2}$$

Donnelly and Fefferman proved the last inequality for domains bounded by real analytic surfaces, [5].

Let $u \in C^\infty(\overline{B_1})$, be a non-zero solution of a second order elliptic equation with smooth coefficients. Let us denote by E the set of second order zeros of function u. The metric properties of E are of the special interest. Let M be a compact subdomain of B_1. Then $H_{n-2}(E \cap M) < \infty$, [6], [7]. But the structure of the set $E \cap \partial B_1$ is far being understood. The following conjecture is a longstanding problem

Conjecture 4. *Let u be a harmonic function in $B_1 \subset \mathbb{R}^3, u \in C^\infty(\overline{B_1})$,*

$$F = \Big\{x \in \partial B_1 : u(x) = \nabla u(x) = 0\Big\}$$

Then $H_2(F) = 0$.

A C^1 counterexample to the last conjecture was given by Bourgain and Wolff, [2].

We now pass to a discussion of eigenfunctions. Let M be a two-dimensional compact manifold and g be a smooth Riemannian metric on M. Let $u_1, u_2, \ldots$ be eigenfunctions and $0 = \lambda_0 \leq \lambda_1 \leq \ldots$ be the eigenvalues of the Laplace operator on (M, g). Denote by $\mu_k(M, g)$ the multiplicity of λ_k. Set

$$m_k(M) = \sup_g \mu_k(M, g)\,,$$

where sup is taken over all smooth metrics g on M.

In the important paper [3] Cheng proved that for any k and M, $m_k(M) < \infty$. Let M be a two-dimensional sphere $\mathcal{S}^2$. Then

$$\begin{aligned} m_1(\mathcal{S}^2) &= 3\,, \quad [3]\,, \\ m_2(\mathcal{S}^2) &= 3\,, \quad [8]\,, \\ 2\sqrt{k} - 1 \le m_k(\mathcal{S}^2) &\le 2k - 1 \end{aligned}$$

for $k \geq 2$. The left-hand side inequality is given by a standard sphere.

Question. What is the right rate of growth of $m_k(\mathcal{S}^2)$ when $k \to \infty$?

Let M_γ be a surface of genus γ. Then for small γ the number $m_1(M_\gamma)$ is known explicitly, [1], [3], [4], [10], [14]. But for big γ we know only the inequalities

$$c\sqrt{\gamma} < m_1(M_\gamma) < C\gamma\,,$$

There is again an interesting question: what is the right rate of growth of $m_1(M_\gamma)$ when $\gamma \to \infty$?

Inverse spectral problems are discussed in the conjectures of [16]. (See the appendix, below.) Since the time of that publication, there has been much progress on the problem of describing metrics on the torus whose Laplacian has the same spectrum as the square torus by Zelditch [18], [19]. But the problem remains open.

Estimating L_p norms of eigenfunctions u_k is an important problem. (See [13] and the discussion of the problem in [12].) The case of L_∞ is mentioned in one of the problems in the appendix. Zygmund [20] has shown that on the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ the following inequality holds:

$$||u_k||_4 \leq 5^{1/4}||u_k||_2$$

Question. Can one prove on a surface of negative curvature

$$||u_k||_4 \leq C\lambda_k^{\epsilon}||u_k||_2,$$

for any $\epsilon > 0$?

Appendix to GEOMETRY OF NODAL SETS AND MULTIPLICITY OF EIGENVALUES [1]

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The spectrum for the Laplacian on the two-dimensional sphere is very rigid. There is a good chance that it may determine the metric completely. Recently S. Y. Cheng proved a beautful theorem that if the multiplicity of the spectrum of a metric is the same as the multiplicity of the spectrum of the sphere, then all the geodesics of the metric are closed. In other words, the surface must be a Zoll surface. Can the spectrum of a Zoll surface have the same multiplicity as the spectrum of S^2? Can one say something about the multiplicity of surfaces of higher genus? For example, if the multiplicity of the spectrum of a metric on the torus is the same as the one on the square torus, what can we say about the metric?

The study of the geometry of the eigenfunctions of the Laplacian or the Schrödinger operator is very interesting. The size of the level sets and the critical sets of the eigenfunctions needs to be estimated. For the former quantity, there are excellent works due to Donnelly-Fefferman [5] , Hardt-Simon and Dong. For the latter quantity, essentially nothing is known. Another very important related question is the estimate of the maximum norm of the eigenfunctions when we normalize the L^2 norm to be one. For what types of manifolds can the maximum norm be estimated independent of the size of the eigenvalues?

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