Questions related to the dimension of Kakeya sets

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The problems we will mention are all fairly well-known and none of them originate with the author. Roughly speaking they come from an area where harmonic analysis interacts with certain aspects of geometric measure theory or combinatorial geometry. They are discussed in more detail in several expository articles in the literature, most recently in [9] which contains proofs or references for most of the statements made below. Relevant general references are [3] and [8].

Perhaps the most basic is the following form of the Kakeya maximal conjecture: define a maximal operator from functions on $\mathbb{R}^n$ to functions on the sphere depending on a parameter $\delta > 0$ in the folowing way: if $e \in S^{n-1}$, $a \in \mathbb{R}^n$ then let

$$T_e^\delta(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot e| \le \frac{1}{2}, |(x-a)^\perp| \le \delta\}$$

where $x^\perp = x - (x \cdot e)e$. Thus $T_e^\delta(a)$ is essentially the δ-neighborhood of the unit line segment in the e direction centered at a. If $f : \mathbb{R}^n \to \mathbb{R}$ then (see Bourgain [1]) we define its Kakeya maximal function $f_\delta^* : S^{n-1} \to \mathbb{R}$ via

$$f_\delta^*(e) = \sup_{a\in\mathbb{R}^n} \frac{1}{|T_e^\delta(a)|} \int_{T_e^\delta(a)} |f|$$

It follows from existence of Kakeya sets (compact sets with measure zero containing a unit line segment in each direction) that it is impossible to make an $L^p \to L^p$ estimate for this maximal function with a bound independent of δ, and the question is whether one can obtain less than power dependence on δ:

Question 1: Is there an estimate

$$\forall \epsilon > 0 \exists C_\epsilon : \|f_\delta^*\|_{L^p(S^{n-1})} \leq C_\epsilon \delta^{-\epsilon} \|f\|_{L^p(\mathbb{R}^n)} \forall f, \text{ where } p = n \qquad (1)$$

A somewhat weaker but morally equivalent version of this question asks whether a Kakeya set in $\mathbb{R}^n$ must have Hausdorff dimension n. This would follow ([1], Lemma 2.15) from an affirmative answer to question 1.

Many techniques are known [8] for proving maximal inequalities like (1) - they typically involve either combinatorial- geometric techniques, Fourier analysis techniques based on the Plancherel theorem or some combination thereof. It is well known that no estimate like (1) can hold on L^p with $p < n$ (consider f = characteristic function of a ball of radius δ). This makes (1) seem hard to approach via Fourier analysis techniques with the exception of the case $n = 2$ where (1) is known and can be proved in a page using either Fourier analysis [1] or the geometrical argument of A. Cordoba. In higher dimensions it is not hard to show using geometrical or semi geometrical arguments that Kakeya sets have dimension $\geq \frac{n+1}{2}$ and there has been a certain amount of progress past this point, beginning with [1]. The current best result implies Kakeya sets have dimension $\geq \frac{n+2}{2}$, see [9] and the references there. These results are still based on fairly straightforward ad hoc geometrical arguments and it appears hard to go further with such an approach. Further remarks on some of the issues involved are in [9] and at the end of [10]. Schlag's paper [6] is also relevant here.

Let us point out that there are a large number of unsolved problems in several areas whose solution seems dependent on being able to answer question 1.

Many of them are questions in Fourier analysis which are connected to (1) via the "disc multiplier" argument of C. Fefferman [4] and related considerations. The simplest problem to state and the easiest to connect to (1) is the well known

Restriction conjecture of Stein: let μ be a measure on the unit sphere in $\mathbb{R}^n$ which has a bounded density with respect to surface measure and let $\hat{\mu}$ be its Fourier transform. Then is there an estimate $\hat{\mu} \in L^p \ \forall p > \frac{2n}{n-1}$?

It is known (e.g. [2]) that this estimate would imply (1) by a variant on the argument of [4].

On the other hand it is unclear whether (1) would imply the restriction conjecture; a proof of such a conditional result would presumably be a very significant advance. It is known that (1) (or improved partial results on (1)) leads to improved partial results on e.g. the restriction problem. This is due to Bourgain [1] and [2] and there has been some further recent related work by Moyua-Vargas-Vega and by Tao.

In addition, there are a number of questions of a geometrical nature which are closely related to (1). A classical problem motivated by Besicovitch's construction of Kakeya sets with measure zero is the following:

Problem of $(n, 2)$ sets: does there exist a set in $\mathbb{R}^n$ with measure zero which contains a translate of every 2-plane?

Such a set would be called a Besicovitch $(n, 2)$-set. One can also formulate this question more quantitatively in terms of the maximal 2-plane transform. Further, one can ask the analogous question for k-planes, $k \geq 3$ and there is no value of k for which a negative answer is known for all n. For example, for every $n \geq 5$ it is unknown whether $(n, 2)$-sets exist. An argument in [1] shows that if the answer to question 1 is affirmative then there is no $(n, 2)$ set for any n.

Next we mention a question in $\mathbb{R}^2$; so far as the author knows there is no implication between this question and (1), but it is natural to expect that they are related.

Suppose that $\alpha \in (0, 1]$ and that E is a compact set in $\mathbb{R}^2$ with the following property: for every direction e there is a line segment in the e-direction which intersects E in Hausdorff dimension $\geq \alpha$. Then what is the optimal lower bound for the Hausdorff dimension of E?

This question arose from considerations in [5] (the author is grateful to Yuval Peres for this reference). It is of course possible to prove various partial results: so far as the author is aware, what is known is that $\dim E \geq \max(2\alpha, \frac{1}{2} + \alpha)$ and that $\dim E$ can be as small as $\frac{1}{2} + \frac{3}{2}\alpha$.

The latter fact is essentially an example of Erdos (of an elementary number theoretic nature) which also shows sharpness of some related combinatorial results.

Finally let us mention a question in $\mathbb{R}^2$ of a different nature, somewhat analogous to the question of whether (1) would imply the restriction conjecture. This question originates in [7]. Namely, consider the initial value problem for the wave equation in n space variables, where in principle $n \geq 2$ is arbitrary: $\Box u = 0, u(\cdot, 0) = f, \frac{\partial u}{\partial t}(\cdot, 0) = 0$.

Then is there an estimate

$$\forall \epsilon > 0 \exists C_\epsilon : \|u\|_{L^p(\mathbb{R}^n \times [1,2])} \leq C_\epsilon \|f\|_{p,\epsilon} \tag{2}$$

when $p \in [2, \frac{2n}{n-1}]$? Here $\|\cdot\|_{p,\epsilon}$ is the inhomogeneous L^p Sobolev norm with ϵ derivatives.

This can be regarded as a souped up version of the restriction conjecture and is known to imply the restriction conjecture, hence (1), and also the well-known Bochner-Riesz conjecture. Hence it is presumably quite difficult in three or more dimensions. On the other hand, when $n = 2$ the restriction and Bochner-Riesz conjectures are known to hold [8] but (2) is nevertheless open. We note that (2) with $n = 2$ encodes some geometric-combinatorial information (beyond the easy two dimensional case of (1)). In particular it would imply the following fact: let E be a set in $\mathbb{R}^2$ which contains a circle of every radius. Then E must have Hausdorff dimension 2. This fact can be given an independent proof, which is nontrivial but is based on known combinatorial arguments. But the latter arguments do not readily give a proof of the $n = 2$ case of (2), which could be a good testing ground for new techniques in this area.

References

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