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Mirror Principle, A Survey

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Abstract. We give an exposition of our recent work on computing intersection numbers on stable map moduli spaces for a projective space. We also discuss a generalization of our approach to balloon manifolds, including projective toric manifolds.

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1. Introduction

1.1. Background

Let M be a compact manifold. Given k homology cycles $\gamma_1, .., \gamma_k$ on M, their intersection number is defined to be the number

$$\#\gamma_1 \cap \cdots \cap \gamma_k.$$

A problem in intersection theory is to compute this number for some given choices of cycles. Dualizing this problem, we let $\omega_1, ..., \omega_k$ be the Poincaré dual cohomology classes of the γ. Then the same intersection number is given by the integration

$$\int_M \omega_1 \wedge \cdots \wedge \omega_k.$$

The problem of computing intersection numbers often arise in moduli theory. In this case, M is typically a kind of moduli space, and hence can be both noncompact and singular. Thus defining intersection theory becomes even more interesting and nontrivial. The pioneering work of Donaldson in gauge theory has led to major breakthroughs in this direction.

Let X be a projective manifold, and $d \in H_2(X, \mathbf{Z})$. Let $M(d, X)$ be the set of holomorphic maps $f : \mathbf{P}^1 \to X$ with $f_*[\mathbf{P}^1] = d$, modulo the automorphisms of the projective line $\mathbf{P}^1$ (ie. $f \sim f \circ \sigma$ if σ is an automorphism of $\mathbf{P}^1$). We would like to compute intersection numbers on the space $M(d, X)$. There are two immediate problems. First, $M(d, X)$ is not compact. Thus one must find a suitable compactification $\bar{M}$ of $M(d, X)$. Second there may well have singularities which render the usual rules of integration (hence intersection theory) invalid on $\bar{M}$. Both problems have now been solved, in two different approaches. We discuss them briefly here.

Historically, the first approach is to view X as a symplectic almost complex manifold, and to replace $M(d, X)$ by a moduli space of pseudo-holomorphic maps. Following the work of Sacks-Uhlenbeck and Gromov, Parker-Wolfson and others show that there is a satisfactory compactification $\bar{M}(d, X)$, on which intersection theory can be defined. This can also be done for higher genus curves with mark points as well.

What classes do we integrate on $\bar{M}$? In the pioneering work of physicists, they consider intersection numbers of cohomology classes which arise in the ambient space X. To do this we must consider the moduli space of maps $f : C \to X$ of curves C into X with smooth mark points $x_1, .., x_k \in C$ (which we continue to denote by $\bar{M}$ for now). This $\bar{M}$ is now equipped with evaluation maps $ev_i : (f, C; x_1, .., x_k) \mapsto f(x_i)$. If ω_i are cohomology classes on X, then the pullbacks $ev_i^* \omega_i$ are classes on $\bar{M}$. And so one can attempt to compute

$$\int_{\bar{M}} ev_1^* \omega_1 \cdots ev_k^* \omega_k.$$

In the language of sigma models physics, Dijkgraaf-Verlinde-Verlinde-Witten have shown that these intersection numbers can be organized into a single generating function, known as the prepotential. Moreover this function satisfies a system of third order nonlinear PDEs, called the WDVV equations. Vafa has proposed a deformation of the cohomology ring $H^*(X, \mathbf{Z})$, and the associativity condition of the new ring is given by the WDVV equations. The deformed ring is now known as the quantum cohomology ring of X. For a suitable manifold X, the intersection numbers encoded in its quantum cohomology can be interpreted as enumerating rational curves in X (with appropriate incidence conditions). In a foundational work of Ruan-Tian, they give a mathematical construction of the intersection numbers above, now known as Gromov-Witten invariants, culminating in the first mathematical formulation of quantum cohomology theory for symplectic manifolds.

In a second approach, Kontsevich gives an alternative definition by restricting to algebraic manifolds, and by proposing an algebraic (ie. stable map) compactification of the space of holomorphic maps. Kontsevich-Manin then formulate an axiomatic framework for quantum cohomology of algebraic manifolds. In the case of genus zero and for a special class of algebraic manifolds (such as homogeneous manifolds), it is known that the stable map compactification is sufficient for defining a good intersection theory. In general however the stable map compactification alone is inadequate for defining a good intersection theory, and hence poses a nontrivial problem. This problem has recently been solved by Li-Tian. The problem has also been studied in the works of Behrend-Fantechi, Fukaya-Ono, Ruan and Siebert. The problem is solved by constructing a so-called virtual fundamental cycle, to play the role of the ordinary fundamental class of a space in topology. This construction ultimately makes integration, hence an algebraic construction of GW invariants, on the stable map compactification possible. Li-Tian and Siebert have recently shown that the intersection theories in the symplectic and the algebraic approaches are essentially the same in the projective category.

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1.2. Problem and outline

Throughout this paper, *we shall restrict ourselves to the genus zero case,* ie. our moduli space $\bar{M}(d, X)$ consists of maps $f : C \to X$ where the curve C has arithmetic genus zero. *From now on, we will drop the notation without the bar, and adopt the notation* $M(d,X) = \bar{M}(d,X)$.

Aside from the intersection numbers for those cohomology classes of the type $ev_i^*\omega$ discussed earlier, there are *other* interesting and important natural cohomology classes to consider. One such case is to consider ***characteristic classes of vector bundles*** on the moduli space of maps. In this paper, we study vector bundles which are induced on the moduli spaces $M(d,X)$ by an appropriate vector bundle V on the ambient space X. The cohomology classes we consider here will be a class b of multiplicative type on vector bundles. We call b multiplicative if for every exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, we have $b(E) = b(E')b(E'')$. Examples of multiplicative classes [31] are the top Chern class (Euler class) and the total Chern class.

A vector bundle V on X is called convex if $H^1(\mathbf{P}^1, f^*V) = 0$ for any map $f : \mathbf{P}^1 \to X$. The bundle is called concave if $H^0(\mathbf{P}^1, f^*V) = 0$ for any map $f : \mathbf{P}^1 \to X$. If V is a convex bundle, then there is a naturally induced bundle U_d on every moduli space $M(d,X)$. The fiber at the point (f,C) is the vector space of sections $H^0(C, f^*V)$. The primary objects we study are the intersection numbers

$$K_d := \int_{LT(d,X)]} b(U_d). \tag{1.1}$$

and their generating function (prepotential):

$$\Phi := \sum_{d>0} K_d \, e^{dt}.$$

Here $LT(d,X)$ denotes the virtual fundamental class of the stable map moduli space $M(d,X)$ [42][8]. We also study the analogous problem for concave vector bundles.

We begin with an example $X = \mathbf{P}^n$. For simplicity, assume for now that V is a direct sum of line bundles $\oplus_a \mathcal{O}(l_a) \oplus \; \oplus_b \mathcal{O}(-k_b)$, with $l_a, k_b > 0$, on $\mathbf{P}^n$, and that $b = e$ is the Euler class. To get a nontrivial answer for the number K_d, we require that the degree of the class $e(U_d)$ agrees with the degree of $LT(d,X)$. That just means that $\sum_a l_a + \sum_b k_b = n+1$ and that $\#l_a - \#k_b = n-3$.

Introduce the formal (HyperGeometric) series:

$$HG(t) = e^{-Ht/\alpha} \left(\Omega + \sum_{d>0} \frac{\prod_a \prod_{m=0}^{l_a d} (l_a H - m\alpha) \times \prod_b \prod_{m=1}^{k_b d - 1} (-k_b H + m\alpha)}{\prod_{m=1}^{d} (H - m\alpha)^{n+1}} e^{dt} \right).$$

where $\Omega := \prod_a l_a H / \prod_b (-k_b H)$, and α, H are formal variables.

Theorem 1.1. *There are two power series $f(e^t), g(e^t)$ (given below), such that*

$$2\Phi(T) - T\Phi'(T) = \alpha^3 \int_{\mathbf{P}^n} \left(e^{f/\alpha}\ HG(t) - e^{-H(t+g)/\alpha}\Omega\right)$$

where $T(t) = t + g(e^t)$ and H is the positive generator of $H^2(\mathbf{P}^n, \mathbf{Z})$.

The power series f, g above are uniquely and explicitly determined by the condition that $e^{f/\alpha}\ HG(t) - e^{-H(t+g)/\alpha}\Omega$ is of order $O(\alpha^{-2})$ when expanded in powers of α^{-1}.

Note that in the theorem, the left hand side gives all the intersection numbers K_d on the moduli spaces $M(d, \mathbf{P}^n)$, while the right hand side is an elementary integral, over the ambient space $\mathbf{P}^n$, of a cohomology valued function. This theorem is an example of what we called a *Mirror Principle.* It says that the prepotential Φ is computable in terms of an *explicit* function of hypergeometric type $HG(t)$ via a scaling and a change of variable. The last two operations are known as mirror transformations [14]. When V is a negative line bundle, the theorem above yields as special case a formula in local mirror symmetry recently studied by Vafa, Katz and others.

The theorem above is proved in [45]. In this paper, we discuss the ideas of the proof and many generalizations of the theorem. According to our set-up, there are three directions we can generalize: by replacing $\mathbf{P}^n$ by a balloon manifold X (defined below), by allowing a bundle V which need not split on X, and by allowing b to be other characteristic classes of multiplicative type [31]. To keep this exposition simple, we will only discuss the case when V is a direct sum of line bundles, b is the Euler class, and that TX is a convex bundle. More general cases will be discussed elsewhere.

A balloon manifold X is a compact projective manifold equipped with a Hamiltonian torus T-action such that the fixed point set X^T is finite and that every codimension one subtorus $U \subset T$ has $dim\ X^U \leq 1$ [29]. This property is (equivalent to) what is known as the GKM property [27]. The projective space $\mathbf{P}^n$ is the simplest balloon manifold. Other examples include projective toric manifolds and homogeneous manifolds.

Outline: We begin in Section 2 with a discussion of our proof for the above theorem for $\mathbf{P}^n$. Details of this can be found in [45]. We then discuss in Sections 3-7 a generalization of this theorem to to balloon manifolds. Details appear in [46]. We then conclude with a few research directions.

1.3. The mirror formula

Specializing the theorem above to $X = \mathbf{P}^4$ and $V = \mathcal{O}(5)$, our theorem yields the celebrated formula of Candelas-de la Ossa-Green-Parkes [14](see [45] theorem 3.4). It is also known as the mirror formula. This formula has been derived using the conjectural physical phenomenon known as mirror symmetry. The latter was first conjectured by physicists including Dixon-Lerche and Vafa-Warner on the basis of Gepner's earlier work. Mirror symmetry took a dramatic turn upon the appearance of the papers of Greene-Plesser [26] and of Candelas et al [14]. In [26] they have established the existence of mirror manifolds for a particular class of Calabi-Yau manifolds, including the quintic threefold.

In this special case, the first number $K_1 = 2875$ is classical, and goes back perhaps to Schubert. This is the number of lines in a generic quintic threefold in $\mathbf{P}^4$. From the point of view of enumerative geometry, K_2 has been computed by S. Katz [36]; and K_3 has been computed by Ellingsrud-Stromme [19] using torus action and the Atiyah-Bott formula. Using a similar technique, Kontsevich [38] has derived a recursive relation for the K_d and computed K_4. In **[24]** Givental announced a proof of Candelas' mirror formula. However, the proof is incomplete as was noted by experts (e.g, **[47]**) at the time of its publication. Finally, based on the works of Witten, Kontsevich, Li-Tian, and some new ideas on the concept of Euler data, Lian-Liu-Yau **[45]** gave the first complete proof of Candelas' formula in 1997. By mid 1998, two papers by Procesi et al. **[11]** and Pandharipande **[50]** appeared. Using **[45]** **[11]** **[50]**, the gaps in **[24]** can be fixed. For a more detailed description of these approaches see **[60]**. While the case of the quintic is historically important, recent development in string duality shows that mirror principle in the multi-moduli cases is essenital (see e.g. **[61]**). Mirror principle has been generalized in **[46]** to the case of balloon manifolds (and some results to general projective manifolds). In **[25]**, the special case of toric manifolds has been studied based on a series of axioms which need to be proven.

1.4. An analogy with elliptic genus

Mirror principle is a principle about computing characteristic classes of multiplicative type on certain moduli spaces in terms of hypergeometric series. It is rather interesting to note that, through such a principle, we can interpret the coefficients of many classical hypergeometric series as ratios of various equivariant characteristic classes. An example is the equivariant Euler classes on the linear sigma models to be introduced below. This is very similar to the work in elliptic genus which give geometric meaning to the classical Jacobi theta functions. In fact we see that both works are basically giving geometric flesh to purely classical algebraic objects. Even the calculations in both theories are similar in spirit. Compare for example the localization to the original manifold inside the loop space in the elliptic genus case, and inside the linear sigma model in the mirror case. In both cases, the original manifold is realized as a fixed submanifold of certain natural S^1-action on the corresponding sigma model.

More precisely, for elliptic genus, the theta function

$$\theta(z,\tau) = 2sinh\, z/2 \prod_{n=1} (1-q^n)(1-q^n e^{2\pi iz})(1-q^n e^{-2\pi iz})$$

where $q = e^{2\pi i\tau}$, now shows up in the $\hat{A}$-class of the loop space. Of course it also appears as characters of Kac-Moody algebra, or loop group, in the normalized equivariant Euler class of the normal bundle of the manifold embedded in its double loop space as the fixed point set of the torus action. The same function also appears in the K-theory equivariant Euler class of the manifold embedded in its loop space as the fixed point set of the circle action induced by the rotation of the loop.

String theory helps us find the deep geometric meaning behind these classical functions. Now recall that the classical hypergeometric series has coeffients of the form

$$\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1).$$

One notes that each $(a)_n$ can be considered as an equivariant Euler class of an S^1-equivariant bundle

$$V = \bigoplus_{m=1}^{n-1} L_m$$

with where L_m denotes a line bundle L with the weight of the S^1-action given by e^{mt}, we will write L_m as Le^{mt} in the following discussion to indicate the S^1-action. This kind of bundles always shows up in the equivariant localization formulas. In fact it shows up as the normal bundle of the fixed point component inside the ambient manifold [3]. In this case, by taking a to be a multiple of the Chern class of L, we can view $(a)_n$, more or less, as the equivariant Euler class of V.

Our mirror principle shows that it is indeed a general phenomenon that many classical hypergeometric series arises from our geometric computations of equivariant characteristic classes on moduli spaces of stable maps. It is also amazing that the hypergeometric series in this case also arise as the periods of certain Calabi-Yau manifolds, the so-called mirror manifold. A proper understanding of this phenomenon will shed light on the mystery of mirror symmetry predicted by string theorists.

1.5. Main Ideas

We now sketch our main ideas for computing the classes $b(V_d)$.

Step 1. Localization on the linear sigma model. Consider the moduli spaces $M_d(X) := M_{0,0}((1,d), \mathbf{P}^1 \times X)$. The projection $\mathbf{P}^1 \times X \to X$ induces a map $\pi : M_d(X) \to M_{0,0}(d, X)$. Moreover, the standard action of S^1 on $\mathbf{P}^1$ induces an S^1 action on $M_d(X)$. We first study a slightly different problem. Namely consider the classes $\pi^* b(V_d)$ on $M_d(X)$, instead of $b(V_d)$ on $M_{0,0}(d, X)$. First, there is a canonical way to embed fiber products (see below)

$$F_r = M_{0,1}(r, X) \times_X M_{0,1}(d-r, X)$$

each as an S^1 fixed point component into $M_d(X)$. Let $i_r : F_r \to M_d(X)$ be the inclusion map. Second, there is an evaluation map $e : F_r \to X$ for each r. Third, suppose that there is a projective manifold W_d with S^1 action, that there is an equivariant map $\varphi : M_d(X) \to W_d$, and embeddings $j_r : X \to W_d$, such that the diagram

$$\begin{array}{ccc} F_r & \xrightarrow{i_r} & M_d(X) \\ e \downarrow & & \downarrow \varphi \\ X & \xrightarrow{j_r} & W_d \end{array}$$

commutes. Let α denotes the weight of the standard S^1 action on $\mathbf{P}^1$. Then applying the localization formula [3], this diagram allows us to recast our problem to one of studying the S^1-equivariant classes

$$Q_d := \varphi_! \pi^* b(V_d)$$

defined on W_d. Moreover we can expand the class

$$A_d := \frac{j_0^* Q_d}{e_{S^1}(X_0/W_d)}$$

on X in powers of α^{-1}, and find that it is of order α^{-2}.

The spaces W_d in the commutative diagram above are called the linear sigma model of X. They have been introduced in [48] following [58] when X is a toric manifold,

Step 2. Gluing identity. Consider the vector bundle $\mathcal{U}_d := \pi^* V_d \to M_d(X)$, restricted to the fixed point components F_r. A point in (C, f) in F_r is a pair $(C_1, f_1, x_1) \times (C_2, f_2, x_2)$ of 1-pointed stable maps glued together at the marked points, ie. $f_1(x_1) = f_2(x_2)$. From this, we get an exact sequence of bundles on F_r:

$$0 \to i_r^* \mathcal{U}_d \to U_r' \oplus U_{d-r}' \to e^* V \to 0.$$

Here $i_r^* \mathcal{U}_d$ is the restriction to F_r of the bundle $\mathcal{U}_d \to M_d(X)$. And U_r' is the pullback of the bundle $U_r \to M_{0,1}(d, X)$ induced by V, and similarly for U_{d-r}'. Taking the multiplicative characteristic class b, we get the identity on F_r:

$$e^* b(V) b(i_r^* \mathcal{U}_d) = b(U_r') b(U_{d-r}').$$

This is what we call the *gluing identity.* This may be translated to a similar quadratic identity, via Step 1, for Q_d in the equivariant cohomology groups $H_{S^1}^*(W_d)$. The new identity is called the Euler data identity.

Step 3. Linking theorem. The construction above is functorial, so that if X comes equipped with a torus T action, then the entire construction becomes $G = S^1 \times T$ equivariant and not just S^1 equivariant. In particular, the Euler data identity is an identity of G-equivariant classes on W_d. Our problem is to first compute the G-equivariant classes Q_d on W_d satisfying the Euler data identity, and with the property that $A_d \sim \alpha^{-2}$. Note that the restrictions $Q_d|_p$ to the T fixed points p in $X_0 \subset W_d$ are polynomial functions on the Lie algebra of G. Suppose that X is a balloon manifold. Then it can be shown that (with a nondegeneracy assumption on $e_G(X_0/W_d)$) the classes Q_d are uniquely determined by the values of the $Q_d|_p$, when α is some scalar multiple of a weight on the tangent space $T_p X$. These values of $Q_d|_p$ can be computed explicitly by exploiting the structure of a balloon manifold.

Once these values are known, it is often easy to manufacture explicit G-equivariant classes $\tilde{Q}_d$ with the restrictions $\tilde{Q}_d|_p$ having the above same values, and satisfying the Euler data identity. In this case, we say that the data $\tilde{Q}_d$ are linked to the data Q_d. By a suitable change of variables, one can also arrange that $\frac{j_0^* \tilde{Q}_d}{e_{S^1}(X_0/W_d)} \sim \alpha^{-2}$. By the preceding discussion, we get $Q_d = \tilde{Q}_d$.

Step 4. Computing $\Phi(t)$. Once the classes $Q_d = \varphi_! \pi^* b(V_d)$ are determined, we can unwind the many maps used in Step 1. The preceding computations can be done simply in the form of power series. This finally computes the generating function $\Phi(t)$.

The answer for $\Phi(t)$ is given in the form of Conjecture 9.1. In this paper, for clarity, we restrict ourselves to the case when the tangent bundle of X is convex. We prove that Conjecture 9.1 holds whenever X is a balloon manifold having a linear sigma model W_d such that $e_G(X_0/W_d)$ satisfies a nondegeneracy condition.

In the nonconvex case, we must replace $M_{0,k}(d,X)$ by Li-Tian's virtual fundamental cycle [42] for the purpose of localization and integration. The work devoted entirely to dealing with the added technicality arising from this replacement will be discussed elsewhere. All the results in this paper will generalize with only slight modifications as a result of this replacement, but with no change to the overall conceptual framework.

By the equivalence, established in [43], of symplectic GW theory and algebraic GW theory for projective manifolds, we also expect that the results in this paper can be readily generalized to the symplectic case [54][55].

2. Mirror Principle for $\mathbf{P}^n$

Let X be a projective manifold. We denote by $M_{g,k}(d,X)$ the degree d stable map moduli space of genus g with k marked points [38]. Throughout this paper, we restrict ourselves to $g=0$ and $k=0,1$. Later we shall denote the Li-Tian virtual fundamental cycle of $M_{0,k}(d,X)$ by the same notation. We now introduce a few notations. There are two natural morphisms:

$$\rho : M_{0,1}(d,X) \to M_{0,0}(d,X), \quad (f,C;x) \mapsto (f,C)$$
$$ev : M_{0,1}(d,X) \to X, \quad (f,C;x) \mapsto f(x).$$

We will also consider stable maps into the space $\mathbf{P}^1 \times X$, and the projection map

$$\pi : M_d(X) := M_{0,0}((1,d), \mathbf{P}^1 \times X) \to M_{0,0}(d,X), \quad (f,C) \mapsto (\pi_2 \circ f, C_{stab}).$$

where $\pi_2 : \mathbf{P}^1 \times \mathbf{P}^n \to \mathbf{P}^n$ is the projection onto the second factor. Here C_{stab} is the stabilization of C, ie. C_{stab} is obtained from C by contracting all unstable components C' such that $\pi_2 \circ f(C')$ is a point. The moduli space $M_d(X)$ is called the degree d *nonlinear sigma model of* X.

Throughout this section, we restrict ourselves to the case $X = \mathbf{P}^n$, and we will write $M_d = M_d(\mathbf{P}^n)$. A smooth point in M_d can be represented by a map

$id \times f : \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^n$ where $f = [f_0, .., f_n]$ is an $(n+1)$-tuple of degree d polynomials in two variables w_0, w_1. Let N_d be the space of such $(n+1)$-tuples of degree d polynomials modulo scalars. Then obviously $N_d \cong \mathbf{P}^{(n+1)d+n}$. Moreover we have a map from the smooth points of M_d into N_d defined by $(id \times f, \mathbf{P}^1) \mapsto [f_0, .., f_n]$. It can be shown that this map has a regular extension to all of M_d [57] (see [45] and references therein). It is known as a collapsing map, and is denoted by

$$\varphi : M_d \to N_d.$$

The space N_d is called the degree d *linear sigma model for* $\mathbf{P}^n$.

Let V be a convex bundle, as defined in the introduction. Then for each d, we have an induced vector bundle U_d on $M_{0,0}(d, \mathbf{P}^n)$, whose fiber at (f, C) is the space of sections $H^0(C, f^*V)$. Note that $rk\ U_d = \langle c_1(V), d\rangle + rk\ V$. Similarly if V is a concave bundle, then we have a bundle U_d with fiber at (f, C) given by $H^1(C, f^*V)$, and has $rk\ U_d = -\langle c_1(V), d\rangle - rk\ V$. For the purpose of exposition, we restrict ourselves to the case when V is a direct sum of positive line bundles. Our theory extends readily to a general class of equivariant convex and concave bundles as discussed in [45].

To summarize the objects introduced above, we have the following diagram:

$$\begin{array}{ccccccccc} & & V_d = \pi^* U_d & & U_d & & \rho^* U_d & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ N_d & \xleftarrow{\varphi} & M_d & \xrightarrow{\pi} & M_{0,0}(d, \mathbf{P}^n) & \xleftarrow{\rho} & M_{0,1}(d, \mathbf{P}^n) & \xrightarrow{ev} & \mathbf{P}^n \end{array}.$$

Now comes an important idea: the manifold $\mathbf{P}^n$ has a natural torus action given by the maximal torus T of $Aut(\mathbf{P}^n)$. Since the construction of the objects are natural, all the objects in the diagram above are T-equivariant. Let S^1 act on $\mathbf{P}^1$ by rotation with weight α. Then the natural $S^1 \times T$ action on $\mathbf{P}^1 \times \mathbf{P}^n$ makes the maps φ and π become $S^1 \times T$-equivariant. Techniques of localizations in equivariant cohomology can therefore be applied here.

2.1. Localization formulas

The key technique of our proof is the equivariant localization formula, due to Bott, Atiyah-Bott and Berline-Vergne. The spirit of localization was in the Bott residue formula. We first explain this formula.

Let X be a compact smooth manifold of complex dimension $2n$ with a torus T-action. Let $\{F\}$ be the components of the fixed point set. Let $H_T^*(X)$ denote the equivariant cohomology group of X, and $i_F : F \to X$ the inclusion map. There are two natural homomorphisms:

$$i_F^*: \quad H_T^*(X) \to H_T^*(F), \quad i_{F!}: \quad H_T^*(F) \to \ H_T^*(X).$$

which are respectively the pull-back, and the Gysin homomorphism. The following well-known formula will be used

$$i_F^* i_{F!}(\omega) = e_T(F/M)\omega$$

for a class ω in $H_T^*(F)$.

We will use a variation of the Bott residue formula, or the localization formula,

Lemma 2.1. *Given any equivariant cohomology class ω on X, we have*

$$\omega = \sum_F i_{F!}[i_F^*\omega / e_T(F/X)]$$

in $H_T^*(X)$.

This formula is equivalent to the integral version of the localization formula

$$\int_X \omega = \int_X \omega_{2n} = \sum_F \int_F [i_F^*\omega / e_T(F/X)].$$

Another important fact is that, if V is an equivariant vector bundle on X, then any characteristic class of V has an equivariant extension. More precisely, say if c_{2k} is a characteristic class of degree $2k$, then we can find

$$c = c_{2k} + c_{2k-2} + \cdots + c_0$$

such that c is an equivariant cohomology class.

Here is a general way to calculate the terms in the localization formula. If c_{2k} is a Chern class, then by using splitting principle, it can be expressed as a symmetric functions of the Chern roots $P(x_1, \cdots, x_l)$ where $l = dim\, V$. Then at the fixed point component F, V can be decomposed into direct sums of line bundles

$$V|_F = L_1 \oplus \cdots, \oplus L_l$$

with the action on L_j given by, say $e^{2\pi\sqrt{-1}n_j t}$. Then the restriction of its equivariant counterpart c to F is

$$i_F^* c = P(c_1(L_1) + n_1 t, \cdots, c_1(L_l) + n_l t).$$

The computation of the equivariant Euler class of F in X is similar. The restriction of TX to F also has an decompostion into line bundles

$$TX|_F = E_1 \oplus, \cdots, \oplus E_n$$

where S^1 acts on E_j by $e^{2\pi\sqrt{-1}m_j t}$, then

$$e_T(F/TX) = \prod_j (c_1(E_j) + m_j t).$$

The following functorial formula is oftenly used in our work. Assume we have another manifold Y with T-action, and an equivariant map

$$f: \quad X \to Y.$$

Let E be a fixed component in Y and $f^{-1}(E) = F$ is a fixed component of X. Let f_0 be the restriction of f to F. That is we have

$$f_0: \quad F \to E.$$

Lemma 2.2. *Given any class $\omega \in H_T^*(X)$, we have the formula:*

$$f_{0!}[i_F^*\omega/e_T(F/X)] = j_E^*(f_!\omega)/e_T(E/Y)$$

where j_E is the inclusion of E inside Y.

Proof: Let us consider localization of $\omega f^*(i_{E!}(1))$ on X,

$$\omega f^*(j_{E!}(1)) = i_{F!}[i_F^*(\omega f^*(j_{E!}(1)))/e_T(F/X)].$$

Note the contribution from fixed components other than F vanish. Apply the push-forward $f_!$ to both sides, we get

$$f_!(\omega) j_{E!}(1) = f_! i_{F!}[i_F^*(\omega f^*(j_{E!}(1)))/e_T(F/X)].$$

Now $f \circ i_F = j_E \circ f_0$ which implies

$$f_! i_{F!} = j_{E!} f_{0!}, \quad i_F^* f^* = f_0^* j_E^*.$$

Plug into the above equality, we get

$$f_!(\omega)j_{E!}(1) = j_{E!}f_{0!}[i_F^*(\omega)f_0^*e_T(E/Y)/e_T(F/X)].$$

Apply j_E^* to both sides, we then arrive at

$$j_E^*f_!(\omega)e_T(E/Y) = e_T(E/Y)f_{0!}[i_F^*(\omega)f_0^*e_T(E/Y)/e_T(F/X)].$$

Since $e_T(E/Y)$ is invertible, we get

$$j_E^*f_!(\omega)/e_T(E/Y) = f_{0!}[i_F^*(\omega)/e_T(F/X)]$$

which is precisely the wanted identity.

Note that the above argument only uses the very basics of localization, so it can be easily extended to the cases like orbifolds. By using the virtual localization formula in [21], we can see that this formula can also be extended to the cases of virtual fundamental cycles.

2.2. Euler data

As before V will be a direct sum of positive line bundles $\oplus_a \mathcal{O}(l_a)$ on $\mathbf{P}^n$. We now apply the functorial localization formula to case of the equivariant map $\varphi : M_d \to N_d$. For convenience, we will allow $S^1 \times T$ equivariant cohomology classes to be rational (rather than just polynomial) in the Lie algebra of $S^1 \times T$.

From the explicit definition of N_d it is easy to see that the S^1 fixed point components in N_d are

$$X_r := \{[a_0 w_0^r w_1^{d-r}, .., a_n w_0^r w_1^{d-r}] \mid [a_0, .., a_n] \in \mathbf{P}^n\} \cong \mathbf{P}^n$$

for $r = 0, .., d$. We denote by $j_r : X_r \to N_d$ the natural inclusion map, and identify X_r with $\mathbf{P}^n$ as above. Note that the T-fixed points in X_r are identified with the points $p_i = [0, .., 1, .., 0] \in \mathbf{P}^n$ where 1 appears in the i slot. The corresponding S^1 fixed point component $\varphi^{-1}(X_r)$ is given by

$$F_r := M_{0,1}(r, \mathbf{P}^n) \times_{\mathbf{P}^n} M_{0,1}(d-r, \mathbf{P}^n)$$

Here a point (f, C) is obtained from a degree r stable map (f_1, C_1, x_1) and a degree $d-r$ stable map (f_2, C_2, x_2) into X with $f(x_1) = f(x_2)$, by gluing C_1, C_2 to $\mathbf{P}^1$ with x_1, x_2 identified with $0, \infty \in \mathbf{P}^1$ (see [45] for details).

Let $e_T(U_d)$ be the equivariant Euler class of U_d. Then $\pi^* e_T(U_d)$ is an equivariant class on M_d. Set

$$Q_d := \varphi_!(\pi^* e_T(U_d))$$

which is now an equivariant class on N_d. We note that the restriction $\varphi|_{F_r} : F_r \to X_r = \mathbf{P}^n$ coincides with the map $e := ev \times ev : F_r \to \mathbf{P}^n$, $(f_1, C_1, x_1) \times (C_2, f_2, x_2) \mapsto f(x_1) = f(x_2)$. Applying the functorial localization formula of the preceding subsection to this case, we get

Lemma 2.3. *For each $r = 0, .., d$, we have the formula*

$$\frac{i_r^* Q_d}{e_G(X_r/N_d)} = e_![\frac{\pi^* e_T(U_d)}{e_G(F_r/M_d)}].$$

Recall that a point (f, C) in $F_r \subset M_d$ comes from gluing together a pair of stable maps $(f_1, C_1, x_1), (f_2, C_2, x_2)$ with $f_1(x_1) = f_2(x_2) = x \in \mathbf{P}^n$. From this, we get an sequence over C:

$$0 \to f^* V \to f_1^* V \oplus f_2^* V \to V|_x \to 0.$$

Passing to cohomology, we have

$$0 \to H^0(C, f^* V) \to H^0(C_1, f_1^* V) \oplus H^0(C_2, f_2^* V) \to V|_x \to 0.$$

Hence we obtain an exact sequence for the U_d restricted to F_r:

$$0 \to U_d \to U_r \oplus U_{d-r} \to e^* V \to 0.$$

Taking Euler class, we get an identity, which we call the *gluing identity*:

$$e^* e_T(V) e_T(U_d) = e_T(U_r) e_T(U_{d-r})$$

Let L_r be the universal line bundle on $M_{0,1}(r, \mathbf{P}^n)$ whose fiber at $(f, C; x)$ is the tangent line at $x \in C$. Then we have [45]

$$\begin{aligned} e_G(F_r/M_d) &= -\alpha(-\alpha + c_1(L_{d-r}))\alpha(\alpha + c_1(L_r)) \\ e_G(F_0/M_r) &= \alpha(\alpha + c_1(L_r)) \\ e_G(F_{d-r}/M_{d-r}) &= -\alpha(-\alpha + c_1(L_{d-r})). \end{aligned}$$

From this we get

$$e_G(F_r/M_d) = e_G(F_0/M_r)\overline{e_G(F_0/M_{d-r})}$$

where $\bar{}$ means replacing α be $-\alpha$. Plugging this into the gluing identity, and applying the preceding lemma, we get

$$e_T(V)\ \frac{j_r^*(Q_d)}{e_G(X_r/N_d)} = \overline{\left(\frac{j_0^*(Q_r)}{e_G(X_0/N_r)}\right)}\ \frac{j_0^*(Q_{d-r})}{e_G(X_0/N_{d-r})}. \tag{2.1}$$

Given an $S^1 \times T$ equivariant class ω on N_d we introduce the notation

$$i_r^* \omega^v := \frac{j_r^*(\omega)}{e_G(X_r/N_d)}.$$

Definition 2.4. *A sequence $P = \{P_d\}$ of $S^1 \times T$ equivariant classes on N_d is called an Euler data if the following identity holds on $\mathbf{P}^n$:*

$$P_0 \; i_r^* P_d^v = \overline{i_0^* P_r^v} \; i_0^* P_{d-r}^v.$$

for $r = 0, .., d$, and $\int_{N_d} P_d \kappa^s$ is a polynomial in α for each s.

Note that this definition reduces to the definition introduced in [45] if we use the fact that

$$e_G(X_r/N_d) = \overline{e_G(X_0/N_r)} \; e_G(X_0/N_{d-r}).$$

Explicitly this identity is

$$\prod_i \prod_{\substack{m=0 \\ m \neq r}}^{d} (H - \lambda_i + r\alpha - m\alpha) = \prod_i \prod_{m=1}^{r} (H - \lambda_i + m\alpha) \times \prod_i \prod_{m=1}^{d-r} (H - \lambda_i - m\alpha)$$

where $\lambda_0, .., \lambda_n$ are the weights for the standard representation of T on $\mathbf{C}^{n+1}$, and H is the T-equivariant hyperplane class on $\mathbf{P}^n$. The reason for rewriting the Euler data identity in the definition above will become clear when we generalize to the case of balloon manifolds.

Now from eqn. (2.1), we conclude that

Lemma 2.5. *The sequence $Q = \{Q_d\}$ is an Euler data.*

We now assume the Euler class $e(U_d)$ has degree equal to the dimension of $M_{0,0}(d, X)$. Applying the localization formula in Lemma 2.3, we can deduce that

Lemma 2.6. *The following formula holds:*

$$\int_X e^{-H \cdot t/\alpha} i_0^* Q_d^v = \alpha^{-3}(2 - d \cdot t) K_d, \quad \textit{where} \;\; K_d = \int_{M_{0,0}(d,\mathbf{P}^n)} e(U_d).$$

Moreover $deg_\alpha i_0^* Q_d^v \leq -2$.

2.3. Linking, uniqueness, and mirror transformations

Definition 2.7. *We say that two Euler data P, Q are linked if for any distinct $i, j = 0, .., n$ and $\delta > 0$,*

$$j_0^*(P_d - Q_d)|_{p_i} = 0, \;\; \textit{at} \;\; \alpha = (\lambda_i - \lambda_j)/\delta.$$

The condition in the definition corresponds to the restriction to the smooth fixed point in M_d: a degree δ cover of the T-invariant projective line (balloon) passing through p_i, p_j. For $V = \oplus_a \mathcal{O}(l_a)$, a computation at such a smooth fixed point shows that the Euler data $Q_d := \varphi_!(\pi^* e_T(U_d))$ is linked to

$$P_d = \prod_a \prod_{m=0}^{l_a d} (l_a \kappa - m\alpha) \tag{2.2}$$

where κ is the $S^1 \times T$ equivariant hyperplane class of $N_d = \mathbf{P}^{(n+1)d+n}$. It is trivial to check that the sequence P_d is an Euler data. We also have the explicit formula:

$$i_0^* P_d^v = \frac{\prod_a \prod_{m=0}^{l_a d} (l_a H - m\alpha)}{\prod_i \prod_{m=1}^{d} (H - \lambda_i - m\alpha)}. \tag{2.3}$$

Lemma 2.8. *Let $P = \{P_d\}$ and $Q = \{Q_d\}$ be two linked Euler data. If*

$$deg_\alpha(i_0^* P_d^v - i_0^* Q_d^v) \le -2$$

for each d, then $P = Q$.

The main idea of the proof is applying the residue formula on the projective space N_d, and doing induction on the degree d. The degree bound in α together with the linking condition forces $j_0^*(P_d - Q_d)$ in the residue formula to be zero. See [45].

We now consider mirror transformations. For a given sequence B_d of T-equivariant classes on $\mathbf{P}^n$, we define the corresponding HyperGeometric series:

$$HG[B](t) = e^{-Ht/\alpha} \sum_{d \ge 0} B_d \, e^{dt}.$$

Let $R := \mathbf{Q}(\lambda)[\alpha]$ where $\mathbf{Q}(\lambda)$ denotes the quotient field of $H_T^*(pt)$. Given the formal power series f, g in e^t coefficients in R and without constant term, we let $\tilde{B}_d$ be the sequence of equivariant cohomology classes defined by the relation

$$e^{f/\alpha} HG[B](t + g) = HG[\tilde{B}](t). \tag{2.4}$$

From the sequence $\tilde{B}$, by using localization on N_d, we can lift each $\tilde{B}_d$ to an equivariant cohomology class $\tilde{P}_d$ on N_d which is meromorphic in α, such that the resulting sequence $\tilde{P} = \{\tilde{P}_d\}$ satisfies the Euler data identity and $i_0^*\tilde{P}_d^v = \tilde{B}_d$. We call the sequence $\tilde{P}$ the Lagrange lift of $\tilde{B}$ [45].

Lemma 2.9. *Let P be an Euler data and B be the sequence $B_d = i_0^*P_d^v$. Let $\tilde{B}$ be defined by (2.4), and $\tilde{P}$ be its Lagrange lift. Then $\tilde{P}$ is an Euler data which is linked to P.*

We call the transformation of Euler data $P \mapsto \tilde{P}$ defined above a *mirror transformation.*

2.4. Computing the prepotential

We now return to the Euler data $Q_d := \varphi_!(\pi^* e_T(U_d))$. Put $A_d := i_0^* Q_d^v$. Then

$$HG[A](t) = e^{-Ht/\alpha} \sum_{d\geq 0} i_0^* Q_d^v \, e^{dt} = e^{-Ht/\alpha} \sum_{d\geq 0} \frac{j_0^* Q_d}{e_G(X_0/N_d)} e^{dt}. \tag{2.5}$$

Introduce the prepotential

$$\Phi = \sum_{d>0} K_d e^{dt}.$$

Then by Lemma 2.6, we get

$$\int_X \left(HG[A](t) - e^{-Ht/\alpha} e(V)\right) = \alpha^{-3}(2\Phi - t\frac{d\Phi}{dt}). \tag{2.6}$$

We have seen that the Euler data Q above is linked to the Euler data P in eqn. (2.2). Let B be the sequence $B_d := i_0^* P_d^v$. Expanding in powers of α^{-1} using the expression (2.3), we get the asymptotic form

$$HG[B](t) = P_0(F_0 - \alpha^{-1}H(F_0 t + F) + \alpha^{-1}G) + O(\alpha^{-2}) \tag{2.7}$$

where F_0, F and G are power series with coefficients in R and are explicitly determined by this equation. Put $f := \alpha \log F_0 - \frac{G}{F_0}$, and $g := \frac{F}{F_0}$. Using Lemma 2.6 and eqns. (2.5), (2.7), we conclude that

$$e^{f/\alpha} HG[B](t) - HG[A](t+g) = O(\alpha^{-2}).$$

Combining this with Lemmas 2.9 and 2.8, we conclude that

$$e^{f/\alpha} HG[B](t) = HG[A](t+g).$$

Thus eqn. (2.6) becomes

$$2\Phi(T) - T\frac{d\Phi(T)}{dT} = \alpha^3 \int_X \left(e^{f/\alpha} HG[B](t) - e^{-HT/\alpha} e(V)\right)$$

where $T := t + g$. This is mirror principle for the convex bundle V on $\mathbf{P}^n$ (cf. Theorem 1.1).

3. Generalizations

3.1. Balloon manifolds

By a *balloon manifold*, we mean a complex projective T-manifold X with the following properties. There are only finite number of T-fixed points. At each fixed point p, the T-weights on the isotropic representation T_pX are pairwise linearly independent. This class of manifolds were introduced by Goresky-Kottwitz-MacPherson [27]. (We refer the reader to [29] for an excellent exposition.) Throughout this paper, we assume that X *is convex*, ie. $H^1(\mathbf{P}^1, f^*TX) = 0$ for any holomorphic map $f : \mathbf{P}^1 \to X$.

One important property of a balloon n-fold is that at each fixed point p, there are exactly n balloons, ie. T-invariant $\mathbf{P}^1$, each balloon connecting p to one other fixed point q. The induced action on each balloon is the standard rotation with two fixed points p and q. (see [27][30]). We *denote by pq the balloon connecting the fixed points p, q.* Toric manifolds, complex C-spaces and spherical manifolds are examples of balloon manifolds.

We fix a T equivariant embedding of X into the product of projective spaces

$$\mathbf{P}(n) := \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_m}$$

such that the pull-backs of the hyperplane classes $H = (H_1, \cdots, H_m)$ generate $H^2(X, \mathbf{Q})$. We use the same notations for the corresponding equivariant classes of the H's, and their restrictions to X. For $\omega \in H^2(X)$ and $d \in H_2(X)$, we denote their pairing by $\langle \omega, d \rangle$.

For convenience, we introduce the following notations:

$$\begin{aligned} H &= (H_1, .., H_m) \\ H \cdot \zeta = H_\zeta &= H_1\zeta_1 + \cdots + H_m\zeta_m \\ H(p) &= (H_1(p), .., H_m(p)) \\ H_\zeta(p) &= H_1(p)\zeta_1 + \cdots + H_m(p)\zeta_m. \end{aligned}$$

Here $\zeta = (\zeta_1, .., \zeta_m)$ are formal variables. We denote by $K^\vee \subset H_2(X)$ the set of points in $H_2(X, \mathbf{Z})_{free}$ in the dual of the closure of the Kähler cone of X. Since $K^\vee$ is a semigroup in $H_2(X)$, it defines a partial ordering $\succ$ on the lattice $H_2(X, \mathbf{Z})_{free}$. That is, $d \succeq r$ iff $d - r \in K^\vee$. Let $\{H_j^\vee\}$ be the basis dual to the $\{H_j\}$ in $H_2(X)$. If $d \succeq r$ for two classes d, $r \in H_2(X)$, then $d - r = d_1 H_1^\vee + \cdots + d_m H_m^\vee$ for nonnegative integers $d_1, \cdots, d_m$.

We also consider a balloon manifold as a symplectic manifold with a symplectic structure given by $\omega = H_\zeta$ for some generic ζ. By the convexity theorem of Atiyah [2] and Guillemin-Sternberg [28], the image of the moment map μ_ζ in the dual Lie algebra $\mathcal{T}^*$ is a convex polytope, known as the moment polytope. When X is a toric manifold, the moment polytope is known as a Delzant polytope [17]. In this case, it is well-known that the normal fan of this polytope is the defining fan of X.

We say X a multiplicity-free manifold, if for each point p in $\mathcal{T}^*$, the inverse image $\mu_\zeta^{-1}(p)$ is connected.

Lemma 3.1. *Let X be a multiplicity-free balloon manifold, then $H(p) \neq H(q)$ for any two distinct fixed points p and q in X.*

We shall assume throughout this paper that $H(p) \neq H(q)$ *for all distinct fixed points p, q in X*. Equivalently, if $c(p) = c(q)$ for all $c \in H_T^2(X)$, then $p = q$. This condition is also equivalent to the statement that the moment map with respect to $\omega = H_\zeta$ and the T action is injective to the set of vertices of the moment polytope, when restricted to the fixed points X^T. By the above lemma we know that toric manifolds and compact homogeneous manifolds all satisfy this condition.

Suppose that X is a balloon manifold, and that we have equivariant classes $\{D_a\}$ in $H_T^2(X)$ with the following property. At every fixed point p, $D_a(p)$ is either zero or it is a weight on T_pX. Let pq be a balloon in X. The induced T-action on pq is the standard rotation with fixed points p, q. By applying the localization formula on $pq \simeq \mathbf{P}^1$ and the integral $\langle c, [pq]\rangle$, we have

$$c(q) = c(p) + \langle c, [pq]\rangle\lambda$$

for all $c \in H_T^*(X)$, where λ is the weight on the tangent line $T_q(pq)$. Let $\lambda = D_a(q)$. Specializing to $c = D_a$, we get $D_a(q) = D_a(p) + \langle D_a, [pq]\rangle D_a(q)$. This shows that $\langle D_a, [pq]\rangle \neq 0$. For otherwise we would have $D_a(q) = D_a(p) \neq 0$, and this would mean that $D_a(p)$ is a weight on $T_p(po)$ for some edge po running *in the direction* of $D_a(q) = D_a(p)$ from p to o. So we had three vertices lying joined in a line from q to p to o in the moment graph. This would mean that there is a pair of linearly dependent weights on the tangent space T_pX, which can't happen in a balloon manifold. A similar argument shows that $\langle D_a, [pq]\rangle = 1$.

Lemma 3.2. *Let $\omega = H_\zeta$ and $p, q \in X^T$, $r \succ 0$ and λ be a weight on T_qX. If $\omega(q) = \omega(p) + \langle \omega, r \rangle \lambda$ for generic ζ, then p, q are joined by a balloon, $r = [pq]$, and λ is the weight on the tangent line $T_q(pq)$.*

3.2. Sigma models

Let X be balloon manifold with a fixed T-equivariant embedding $X \to \mathbf{P}(n)$, as discussed above. We write

$$M_d(X) := M_{0,0}((1,d), \mathbf{P}^1 \times X).$$

Since X is assumed to be convex, $M_d(X)$ is an orbifold. The standard S^1 action on $\mathbf{P}^1$ together with the T action on X induce a $G = S^1 \times T$ action on $M_d(X)$.

Here is a description of some S^1 fixed point components F_r, labelled by $0 \preceq r \preceq d$, inside of $M_d(X)$. Let F_r be the fiber product

$$F_r := M_{0,1}(r, X) \times_X M_{0,1}(d-r, X)$$

More precisely, consider the map

$$ev_r \times ev_{d-r} : M_{0,1}(r, X) \times M_{0,1}(d-r, X) \to X \times X$$

given by evaluations at the marked points; and

$$\Delta : \; X \to X \times X$$

the diagonal map. Then

$$F_r = (ev_r \times ev_{d-r})^{-1} \Delta(X).$$

Note that $F_d = M_{0,1}(d, X)$ by convention. The set F_r can be identified with an S^1 fixed point component of $M_d(X)$ as follows. Consider the case $r \neq 0, d$ first. Given a point $(C_1, f_1, x_1) \times (C_2, f_2, x_2)$ in F_r, we get a new curve C by gluing C_1, C_2 to $\mathbf{P}^1$ with x_1, x_2 glued to $0, \infty \in \mathbf{P}^1$ respectively. The new curve C is mapped into $\mathbf{P}^1 \times X$ as follows. Map $\mathbf{P}^1 \subset C$ identically onto $\mathbf{P}^1$, and collapse C_1, C_2 to $0, \infty$ respectively; then map C_1, C_2 into X with f_1, f_2 respectively, and collapse the $\mathbf{P}^1$ to $f(x_1) = f(x_2)$. This defines a point (C, f) in $M_d(X)$. For $r = 0$, we glue (C_1, f_1, x_1) to $\mathbf{P}^1$ at x_1 and 0. For $r = d$, we glue (C_2, f_2, x_2) to $\mathbf{P}^1$ at x_2 and ∞. We will identify F_r as a subset of $M_d(X)$ as above, and let

$$i_r : F_r \to M_d(X)$$

denotes the inclusion map. Clearly, we also have an evaluation map

$$e_r : F_r \to X$$

which sends a pair in F_r to the value at the marked point. In the following, we will simply write e_r as e without causing any coonfusion.

We call a compact manifold or orbifold W_d with $G = S^1 \times T$ action a *linear sigma model* of degree d for X, if the following conditions are satisfied:

1. The S^1 action on W_d has fixed point components given by X_r, labelled by $0 \preceq r \preceq d$, and each X_r is T-equivariantly isomorphic to X.
2. There is a G-equivariant birational map φ from $M_d(X)$ to W_d, such that $\varphi|_{F_r} = e$, and $\varphi^{-1}(X_r) = F_r$.
3. All equivariant cohomology classes in $H^2_G(W_d)$ are lifted from $H^2_T(X)$, and the lift $\hat{D} \in H^2_G(W_d)$ of $D \in H^2_T(X)$ restricts to $D + \langle D, r\rangle\alpha$ on X_r.
4. The G-equivariant Euler class of the normal bundle of X_0 in W_d has the form
$$e_G(X_0/W_d) = \prod_a \prod_{m_a} (D_a - m_a\alpha)$$
where the m_a's are positive integers and the D_a's are classes in $H^2_T(X)$, such that at a given T fixed point p in X, the nonzero $D_a(p)$'s are multiples of distinct weights of T_pX.

Here a birational map, in algebraic geometry language, is a regular morphism which is an isomorphism when restricted to a Zariski open set in $M_d(X)$.

Note W_d need not be unique. We identify X_r with X by assumption 1, and denote by

$$j_r : X_r \to W_d$$

the inclusion map.

We call a balloon manifold X *admissible* if it has a linear sigma model W_d for each d, and that $H_\zeta(p) \neq H_\zeta(q)$ for any two distinct fixed points p, q in X. The main result in this paper is to show that the mirror principle holds for any admissible balloon manifold.

As noted earlier, we restrict ourselves to the case when TX is convex. The definition above must be slightly weakened when dealing with the nonconvex case. Details will be discussed in a future paper.

Remark 3.3. *Condition 4 is actually assuming more than what we need. This condition can be replaced by the following weaker, but more technical condition. For* *each fixed point p and for any d, as a function of α, $e_G(X_0/W_d)|_p$ has possible zero only at either 0 or a multiple of a weight λ on T_pX. In addition if $[pq]$ is a balloon and $d=\delta[pq]$, then λ/δ is at worse a simple zero. For example, the following form would meet this criterion:*

$$e_G(X_0/W_d) = \frac{\prod_a \prod_{m_a}(D_a - m_a\alpha)}{\prod_b \prod_{n_b}(D_b - n_b\alpha)}$$

where the m_a, n_b are nonzero scalars.

For examples of admissible balloon manifolds, see [46].

4. The Gluing Identity

Let X be an admissible balloon manifold from now on. In this section, we apply the functorial localization formula to the linear sigma model. The argument used here is modelled on the one used in [45], except that the T-action is *not* used here. Thus all the results in this section hold for manifolds without T action. We will have more to say about the mirror principle without T action later.

Recall that we have the commutative diagram:

$$\begin{array}{ccc} F_r & \xrightarrow{i_r} & M_d(X) \\ e\downarrow & & \downarrow\varphi \\ X_r & \xrightarrow{j_r} & W_d. \end{array}$$

We also have the natural forgetting map $\rho : M_{0,1}(d,X) \to M_{0,0}(d,X)$, and the projection map $\pi : M_d(X) \to M_{0,0}(d,X)$. Note that we have a commutative diagram

$$\begin{array}{ccc} M_d(X) & & \\ \pi\downarrow & \nwarrow i_0 & \\ M_{0,0}(d,X) & \xleftarrow{\rho} & M_{0,1}(d,X). \end{array}$$

Let $\varphi : M_d(X) \to W_d$, $e : F_r \to X_r$ play the respective roles of $f : X \to Y$, $g : F \to E$ in the functorial localization formula. Then it follows that

Lemma 4.1. *Given any G-equivariant cohomology class ω on $M_d(X)$, we have the following equality on X_r for $0 \preceq r \preceq d$:*

$$\frac{j_r^*\varphi_!(\omega)}{e_G(X_r/W_d)} = e_!\left(\frac{i_r^*(\omega)}{e_G(F_r/M_d(X))}\right).$$

Actually this lemma may be viewed as an equivariant version of the so-called excess intersection formula of [21], Theorem 6.3.

Let L_r denote the line bundle on $M_{0,1}(r,X)$ whose fiber at $(f,C;x)$ is the tangent line at the marked point $x \in C$. Let π_1 denote the projection from $\mathbf{P}^1 \times X$ to $\mathbf{P}^1$.

The normal bundle of F_r in $M_d(X)$ can be computed just as in [45]. For $r \neq 0, d$, we have

$$N(F_r/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_1}C_0 \otimes L_r + T_{x_2}C_0 \otimes L_{d-r} - A_{C_0}.$$

Here we have used the notations as in [45]: a point (f_1, C_1, x_1) in $M_{0,1}(r,X)$ and a point (f_2, C_2, x_2) in $M_{0,1}(d-r,X)$ is glued to $C_0 \simeq \mathbf{P}^1$ at 0 and ∞ respectively to get the point (f,C) in $M_d(X)$ with $C \simeq C_1 \cup C_0 \cup C_2$. Since x_1 and x_2 are mapped to the same point in X under the projection $\pi_2 : \mathbf{P}^1 \times X \to X$, so this point can be considered as a point in F_r by gluing together (f_1, C_1, x_1) and (f_2, C_2, x_2) at the marked points. Similarly, for $r = 0, d$, we have

$$N(F_0/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_1}C_0 \otimes L_d - A_{C_0}$$

and

$$N(F_d/M_d(X)) = H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1) + T_{x_2}C_0 \otimes L_d - A_{C_0}.$$

In the above $H^0(C_0, (\pi_1 \circ f)^* T\mathbf{P}^1)$ corresponds to the deformation of C_0; $T_{x_1}C_0 \otimes L_r$ and $T_{x_2}C_0 \otimes L_{d-r}$ correspond respectively to the deformations of the nodal points x_1 and x_2; A_{C_0} denotes the automorphism group to be quotiented out.

The equivariant Euler classes of the normal bundles above are computed as in [45], to which we refer the readers for details. For $r \neq 0, d$, the equivariant Euler classes are:

$$e_G(F_r/M_d(X)) = -\alpha(-\alpha + c_1(L_{d-r})) \cdot \alpha(\alpha + c_1(L_r))$$

where the two factors on the right hand side are pullbacked to F_r from $M_{0,1}(d-r,X)$, $M_{0,1}(r,X)$ respectively. For $r = 0, d$, we have

$$e_G(F_0/M_d(X)) = -\alpha(-\alpha + c_1(L_d)), \; e_G(F_d/M_d(X)) = \alpha(\alpha + c_1(L_d))$$

respectively. Combining this with the preceding lemma, we get the following equality on $X = X_0$:

$$\frac{j_0^* \varphi_!(\omega)}{e_G(X_0/W_d)} = ev_! \left(\frac{i_0^*(\omega)}{\alpha(\alpha - c_1(L_d))} \right).$$

Here we have dropped the subscript from ev_d. In particular, if ψ is a class on $M_{0,0}(d,X)$, then for $\omega = \pi^*\psi$, we get $i_0^*(\omega) = i_0^*(\pi^*\psi) = \rho^*\psi$. This yields

Lemma 4.2. *Given any T-equivariant cohomology class ψ on $M_{0,0}(d, X)$, we have the following equality on X:*

$$\frac{j_0^* \varphi_!(\pi^* \psi)}{e_G(X_0/W_d)} = ev_! \left(\frac{\rho^* \psi}{\alpha(\alpha - c_1(L_d))} \right).$$

Lemma 4.3. *For $0 \preceq r \preceq d$, we have the following equality on X:*

$$e_G(X_r/W_d) = \overline{e_G(X_0/W_r)} e_G(X_0/W_{d-r}).$$

In particular, we have

$$e_G(X_d/W_d) = \overline{e_G(X_0/W_d)}.$$

Fix a T-equivariant multiplicative class b_T. Fix a T-equivariant bundle of the form $V = V^+ \oplus V^-$, where $V^\pm$ are respectively the convex/concave bundles. (cf. [45].) We call such a V a mixed bundle. We assume that

$$\Omega := \frac{b_T(V^+)}{b_T(V^-)}$$

is a well-defined invertible class on X. By convention, if $V = V^\pm$ is purely convex/concave, then $\Omega = b_T(V^\pm)^{\pm 1}$. Recall that the bundle $V \to X$ induces the bundles

$$V_d \to M_{0,0}(d, X), \quad U_d \to M_{0,1}(d, X), \quad \mathcal{U}_d \to M_d(X).$$

Moreover, they are related by $U_d = \rho^* V_d$, $\mathcal{U}_d = \pi^* V_d$, Throughout this section, we denote

$$Q: \quad Q_d := \varphi_!(\pi^* b_T(V_d)).$$

If ω is a class on W_d, we write

$$i_r^* \omega^v := \frac{j_r^* \omega}{e_G(X_r/W_d)}$$

which is a class on $X = X_r$.

Lemma 4.4. *For $0 \preceq r \preceq d$,*

$$\Omega\ i_r^* Q_d^v = \overline{i_0^* Q_r^v}\ i_0^* Q_{d-r}^v.$$

This is what we call the *gluing identity*. For a proof of this key lemma, see [46].

Remark 4.5.

(a) If we take V to be the trivial line bundle, and b_T to be the total Chern class, then the preceding lemma reduces to Lemma 4.3.

(b) All the lemmas in this section, in fact, holds for a general projective manifold X without T-action, provided that we still have the S^1-equivariant map $\varphi : M_d(X) \to W_d$, with properties 1.-2. stated in section 3. All G-equivariant classes above are then replaced by their S^1-equivariant counterparts.

5. Euler Data

Notations: We denote by κ_i the G-equivariant class on W_d with the property that $j_r^*\kappa_i = H_i + \langle H_i, r\rangle\alpha$. By the localization theorem, κ_i is determined by these restriction conditions, and is a class in the localized equivariant cohomology of W_d. More generally a class $\phi \in H_T^2(X)$ has a G-equivariant lift $\hat{\phi} \in H_G^2(W_d)$ determined by $j_r^*\hat{\phi} = \phi + \langle \phi, r\rangle\alpha$. We denote by $\langle H_T^2(X)\rangle$ the ring generated by $H_T^2(X)$, and by R_d the ring generated by their lifts $\hat{\phi}$. We put $\mathcal{R} = \mathbf{Q}(\mathcal{T}^*)[\alpha]$, where $\mathbf{Q}(\mathcal{T}^*)$ is the rational function field on the Lie algebra of T. For convenience, we introduce the notations

$$\kappa \cdot \zeta = \kappa_\zeta := \kappa_1\zeta_1 + \cdots + \kappa_m\zeta_m$$

$$i_r^*\omega^v := \frac{j_r^*\omega}{e_G(X_r/W_d)}$$

where ω is a class on W_d.

It is often necessary to work over a larger field than $\mathbf{C}$ for coefficients of cohomology groups. For example when we consider the case of the equivariant Chern polynomial c_T, a formal variable x is introduced. In this case we replace everywhere the scalars $\mathbf{C}$ by $\mathbf{C}(x)$. This will be implicit in all of the discussion below.

Recall the localization formula:

$$\int_{W_d} \omega = \sum_{0 \preceq r \preceq d} \int_X \frac{j_r^*(\omega)}{e_G(X_r/W_d)}.$$

We shall often apply the following version:

$$\int_{W_d} \omega\ e^{\kappa_\zeta} = \sum_{0 \preceq r \preceq d} \int_X i_r^*\omega^v\ e^{H_\zeta + \langle H_\zeta, r\rangle\alpha}.$$

Definition 5.1. *Fix an invertible class $\Omega \in H_T^*(X)^{-1}$. A list $P: P_d \in H_G^*(W_d)^{-1}$, $d \succ 0$, is a Ω-Euler data if on X,*

$$\Omega\ i_r^* P_d^v = \overline{i_0^* P_r^v}\ i_0^* P_{d-r}^v$$

(called Euler data identity) for all $r \preceq d$, and the $\int_{W_d} P_d \cdot \omega$ are polynomial in α for all $\omega \in R_d$. By convention we set $P_0 = \Omega$.

For examples of Euler data, see [46].

5.1. An algebraic property

Let $\mathcal{S}$ denotes the set of sequences $B: \ B_d \in H_G^*(X)^{-1},\ d \succeq 0$. By convention, we set $B_0 = \Omega$.

Definition 5.2. *Given any $B \in \mathcal{S}$, define the formal series*

$$HG[B](t) := e^{-H\cdot t/\alpha}\left(\Omega + \sum_{d\succ 0} B_d\ e^{d\cdot t}\right).$$

Note that $e^{H\cdot t/\alpha} HG[B](t)$ takes value in the ring $H_G(X)^{-1}[[K^\vee]]$. (Notations: if R is a ring, hen $\mathcal{R}[[K^\vee]] := \{\sum_{d\in\Lambda} a_d e^{d\cdot t} | a_d \in \mathcal{R}\}$. We use the notations $e^{d\cdot t} = e^{\langle H_t, d\rangle}$ interchangeably.)

Let P be an Euler data, and let B be the list with $B_d := i_0^* P_d^v$. By the localization formula and the Euler data identity, we have

$$\begin{aligned}\int_{W_d} P_d\ e^{\kappa_\zeta} &= \sum_{r\preceq d} \int_X i_r^* P_d^v\ e^{H_\zeta + \langle H_\zeta, r\rangle \alpha} \\ &= \sum_{r\preceq d} e^{-d\cdot\tau} \int_X \Omega^{-1} \overline{\left[e^{-H_t/\alpha} i_0^* P_r^v\ e^{r\cdot t}\right]}\ \left[e^{-H_\tau/\alpha} i_0^* P_{d-r}^v\ e^{(d-r)\cdot\tau}\right].\end{aligned}$$

Here $t = \zeta\alpha + \tau$. Note that $\bar{\zeta} = -\zeta$, $\bar{\alpha} = -\alpha$, and all other variables are invariant under the "bar" operation. Now multiply both sides by $e^{d\cdot\tau}$ and sum over $d \in K^\vee$, we get the formula:

$$\sum_d e^{d\cdot\tau} \int_{W_d} P_d\ e^{\kappa_\zeta} = \int_X \Omega^{-1}\ \overline{HG[B](\zeta\alpha + \tau)}\ HG[B](\tau). \tag{5.1}$$

By definition, the coefficient of $e^{d\cdot\tau}$ on the right hand side is a power series in ζ with coefficients which are polynomial in α, ie. the series lies in $\mathcal{R}[[e^\tau, \zeta]]$.

Conversely, given $B \in \mathcal{S}$ such that

$$\int_X \Omega^{-1}\, \overline{HG[B](\zeta\alpha+\tau)}\; HG[B](\tau) \in \mathcal{R}[[e^\tau, \zeta]],$$

there exists a unique Euler data $P:\ P_d$ satisfying (5.1). Namely, P_d is defined by the conditions

$$j_r^* P_d = \Omega^{-1} e_G(X_r/W_d)\; \overline{B_r}\; B_{d-r}.$$

Thus an Euler data P gives rise to a list $B \in \mathcal{S}$ in a canonical way. Abusing the terminology, *we shall call such a B an Euler data.*

5.2. Linking and Uniqueness

Lemma 5.3. *Suppose A, B are Euler data with $A_r = B_r$ for all $r \prec d$. Suppose that the $(A_d - B_d)|_q$, $q \in X^T$, are Laurent polynomial in α. Suppose also that $deg_\alpha(A_d - B_d) \le -2$. Then $A_d = B_d$.*

Definition 5.4. *Two Euler data A, B are linked if for every balloon pq in X and every $d = \delta[pq] \succ 0$,*

$$(A_d - B_d)|_q$$

is regular at $\alpha = \frac{\lambda}{\delta}$ where λ is the weight on the tangent line $T_q(pq)$.

Suppose A, B both come from Euler data Q, P respectively, ie. $A_d = i_0^* Q_d^v$ and $B_d = i_0^* P_d^v$. Suppose also that

$$j_0^*(P_d)|_q = j_0^*(Q_d)|_q \quad \text{at } \alpha = \lambda/\delta. \tag{5.2}$$

whenever $d = \delta[pq] \succ 0$ as above. Recall that $\alpha = \lambda/\delta$ is at worst a simple pole of $1/e_G(X_0/W_d)|_q$. It follows that $(A_d - B_d)|_q$ is regular at this value. This shows that the conditions (5.2) guarantee that A, B are linked.

Theorem 5.5. *Suppose A, B are linked Euler data satisfying the following properties: for $d \succ 0$,*

(i) If $q \in X^T$, the only possible poles of $(A_d - B_d)|_q$ are scalar multiples of a weight on T_qX.

(ii) $deg_\alpha(A_d - B_d) \le -2$.

Then $A = B$.

Remark 5.6. *In our applications later, the situation is better then the conditions (i)-(ii) demand. We will have two Euler data A, B such that A_d, B_d separately, rather than just $A_d - B_d$, will satisfy both conditions (i)-(ii) at the outset. In this situation, to prove that $A = B$, it suffices to prove that they are linked.*

5.3. Mirror Transformations

Throughout this section, we fix an invertible class Ω on X, and will denote by $\mathcal{A}$ the set of Ω-Euler data.

Definition 5.7. *A map $\mu : \mathcal{A} \to \mathcal{A}$ is called a mirror transformation if it preserves linking. In other words, $\mu(A)$ and A are linked for any $A \in \mathcal{A}$. We call $\mu(A)$ a mirror transform of A.*

We now consider a construction of mirror transformations, as motivated by the classic example of [14]. Consider a transformation $\mu : \mathcal{S} \to \mathcal{S}$, $B \to \tilde{B}$, of the type

$$\tilde{B}_d = B_d + \sum_{r \prec d} a_{d,r} B_r \tag{5.3}$$

where the $a_{d,r} \in H_G^*(X)^{-1}$ are a given set of coefficients. This transformation is obviously invertible, and preserves $B_0 = \Omega$.

Definition 5.8. *The transformation (5.3) is said to have the regularity property if for every balloon pq in X and $d = \delta[pq]$, the coefficients are such that their restrictions $a_{d,r}(q)$, $r \prec d$, are regular at $\alpha = \lambda/\delta$ where λ is the weight on $T_q(pq)$.*

It can be easily shown [46]that transformation (5.3) having the regularity property preserves linking.

Again, motivated by [14] and [33], we consider the following special types of transformations. Given a power series $f \in \mathcal{R}[[K^\vee]]$ with no constant term, we have an invertible transformation $\mu_f : \mathcal{S} \to \mathcal{S}$, $B \mapsto \tilde{B}$, such that

$$e^{f/\alpha}\, HG[B](t) = HG[\tilde{B}](t).$$

In fact, we have

$$\tilde{B}_d = B_d + \sum_{r \prec d} f_{d-r} B_r$$

where $e^{f/\alpha} = \sum_{s\succeq 0} f_s e^{s\cdot t}$, $f_s \in \mathcal{R}[\alpha^{-1}]$. This is clearly a transformation of type (5.3) having the regularity property. (In fact, all the coefficients f_{d-r} are regular away from $\alpha = 0$.)

Given power series $g = (g_1, .., g_m)$, $g_j \in \mathcal{R}[[K^\vee]]$ with no constant term, we have an invertible transformation $\nu_g : \mathcal{S} \to \mathcal{S}$, $B \mapsto \tilde{B}$, such that

$$HG[B](t+g) = HG[\tilde{B}](t).$$

In fact since

$$HG[B](t+g) = e^{-H\cdot t/\alpha} e^{-H\cdot g/\alpha} \sum_{d\succeq 0} B_d \, e^{d\cdot t} e^{d\cdot g},$$

if we write $e^{d\cdot g} = \sum_{s\succeq 0} g_{d,s} e^{s\cdot t}$, $g_{d,s} \in \mathcal{R}$ and $e^{-H\cdot g/\alpha} = \sum_{s\succeq 0} \hat{g}_s e^{s\cdot t}$, $\hat{g}_s \in \mathcal{R}[H/\alpha]$, then

$$\tilde{B}_d = B_d + \sum_{r\prec d} a_{d,r} B_r$$

where the $a_{d,r} \in H_G^*(X)^{-1}$ are quadratic expressions in the $g, \hat{g}$. Thus we obtain another transformation $\mathcal{S} \to \mathcal{S}$ of type (5.3), again having the regularity property.

Theorem 5.9. *The transformations $\mu_f, \nu_g : B \mapsto \tilde{B}$ above each defines a mirror transformation. That is, if B is a Euler data then $\mu_f(B)$ and $\nu_g(B)$ are both Euler data linked to B.*

Remark 5.10. *All mirror transformations we will use later will be of the type μ_f, ν_g as above. Moreover, all Euler data we will encounter will have property (i) of Theorem 5.5. The transformations μ_f, ν_g clearly preserve this property.*

Theorem 5.11. *Suppose that A, B have property (i) of Theorem 5.5, and that A, B are linked. Suppose that A is an Euler data with $deg_\alpha A_d \le -2$ for all $d \prec 0$, and that there exists power series $f \in \mathcal{R}[[K^\vee]]$, $g = (g_1, .., g_m)$, $g_j \in \mathcal{R}[[K^\vee]]$, all without constant term, such that*

$$e^{f/\alpha} HG[B](t) = \Omega - \Omega \frac{H\cdot(t+g)}{\alpha} + O(\alpha^{-2}) \tag{5.4}$$

when expanded in powers of α^{-1}. Then

$$HG[A](t+g) = e^{f/\alpha}\, HG[B](t).$$

Proof: By Theorem 5.9, f, g define two mirror transformations μ_f, ν_g, with

$$\begin{aligned} HG[\tilde{B}](t) &= e^{f/\alpha} HG[B](t) \\ HG[\tilde{A}](t) &= HG[A](t+g) \end{aligned} \tag{5.5}$$

where $\tilde{B} = \mu_f(B)$, $\tilde{A} = \nu_g(A)$. Now both $\tilde{B}, \tilde{A}$ have property (i) of Theorem 5.5. (See remark after Theorem 5.9.)

Since $deg_\alpha A_d \leq -2$, $HG[\tilde{A}](t)$ has the same asymtotic form as $HG[\tilde{B}](t)$ in eqn. (5.4) $mod\ O(\alpha^{-2})$. It follows that

$$e^{H\cdot t/\alpha}\ HG[\tilde{A} - \tilde{B}](t) \equiv O(\alpha^{-2}),$$

or equivalently $deg_\alpha(\tilde{A}_d - \tilde{B}_d) \leq -2$. Thus $\tilde{A}, \tilde{B}$ satisfy condition (ii) of Theorem 5.5. Since A is linked to B, it follows that $\tilde{A}$ is linked to $\tilde{B}$. By Theorem 5.5, we conclude that $\tilde{A} = \tilde{B}$. Now our assertion follows from eqns. (5.5). □

Remark 5.12. *The preceding theorem says that one way to compute A (or Q) is by first finding an explicit Euler data B linked to A, and then relate A and B via mirror transformations.*

6. From stable map moduli to Euler data

Fix an admissible balloon manifold with $c_1(X) \geq 0$. Fix a T-equivariant multiplicative class b_T. Its nonequivariant limit is denoted by b. Fix a T-equivariant bundle of the form $V = V^+ \oplus V^-$, where $V^\pm$ are respectively the convex/concave bundles. As before, we write

$$\Omega = \frac{b_T(V^+)}{b_T(V^-)}.$$

Let V_d be the bundle induced by V on the 0-pointed degree d stable map moduli of X. Throughout this section, we denote

$$\begin{aligned} Q: \quad Q_d &:= \varphi_!(\pi^* b_T(V_d)) \\ K_d &:= \int_{M_{0,0}(d,X)} b(V_d) \\ \Phi &:= \sum K_d\, e^{d\cdot t} \\ A: \quad A_d &:= i_0^* Q_d^v. \end{aligned}$$

Note that all these objects depend on the choice of b_T and V, though the notations do not reflect this.

6.1. The Euler data Q

Theorem 6.1. *(i)* $deg_\alpha A_d \leq -2$.
(ii) If for each d the class $b_T(V_d)$ *has homogeneous degree the same as the degree of* $M_{0,0}(d,X)$*, then in the nonequivariant limit we have*

$$\int_X e^{-H\cdot t/\alpha} A_d = \alpha^{-3}(2 - d\cdot t)K_d$$

$$\int_X \left(HG[A](t) - e^{-H\cdot t/\alpha}\Omega\right) = \alpha^{-3}(2\Phi - \sum t_i \frac{\partial\Phi}{\partial t_i}).$$

Proof: Earlier we have proved that

$$A_d = i_0^* Q_d^v = ev_! \left(\frac{\rho^* b_T(V_d)}{\alpha(\alpha - c_1(L))}\right),$$

where $L = L_d$ is the line bundle on $\mathcal{M}_{0,1}(d,X)$ whose fiber at a point $(f, C; x)$ is the tangent line at x.

Assertion (i) now follows immediately from this formula. The second equality in assertion (ii) follows from the first equality. By the above formula again,

$$\begin{aligned} I &:= \int_X e^{-H\cdot t/\alpha} A_d \\ &= \int_{M_{0,1}(d,X)} e^{-ev^* H\cdot t/\alpha} \frac{\rho^* b(V_d)}{\alpha(\alpha - c_1(L))} \\ &= \int_{M_{0,0}(d,X)} b(V_d)\ \rho_! \left(\frac{e^{-ev^* H\cdot t/\alpha}}{\alpha(\alpha - c_1(L))}\right). \end{aligned}$$

Now $b(V_d)$ has homogeneous degree the same as the dimension $M_{0,0}(d,X)$. The second factor in the last integrand contributes a scalar factor given by integration over a generic fiber E (which is a $\mathbf{P}^1$) of ρ. So we pick out the degree 1 term in $\frac{e^{-ev^* H\cdot t/\alpha}}{\alpha(\alpha - c_1(L))}$, which is just $\frac{-ev^* H\cdot t}{\alpha^3} + \frac{c_1(L)}{\alpha^3}$. Restricting to the generic fiber E, say over $(f, C) \in M_{0,0}(d,X)$, the evaluation map ev is equal to f, which is a degree d map $E \cong \mathbf{P}^1 \to X$. It follows that

$$\int_E ev^* H = d.$$

Moreover, since $c_1(L)$ restricted to E is just the first Chern class of the tangent bundle to E, it follows that

$$\int_E c_1(L) = 2.$$

So we have

$$I = \left(-\frac{d \cdot t}{\alpha^3} + \frac{2}{\alpha^3}\right) K_d. \quad \square$$

Theorem 6.2. *More generally suppose b_T is an equivariant multiplicative class of the form*

$$b_T(V) = x^r + x^{r-1} b_1(V) + \cdots + b_r(V), \quad rk\ V = r$$

where x is a formal variable, b_i is a characteristic class of degree i. Suppose $s := rk\ V_d - dim\ M_{0,0}(d, X) \geq 0$ is independent of $d \succ 0$. Then

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X e^{-H \cdot t/\alpha} A_d = \alpha^{-3} x^{-s} (2 - d \cdot t) K_d$$

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(HG[A](t) - e^{-H \cdot t/\alpha}\Omega\right) = \alpha^{-3} x^{-s} \Big(2\Phi - \sum t_i \frac{\partial \Phi}{\partial t_i}\Big).$$

Proof: The proof is entirely analogous to (ii) above. $\square$

6.2. Linking theorem for A

Now consider a mixed bundle $V = V^+ \oplus V^-$ on X. Fixed a choice of equivariant multiplicative class b_T. We assume that V has the following property: there exists nontrivial T-equivariant line bundles $L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-$ on X with $c_1(L_i^+) \geq 0$ and $c_1(L_j^-) < 0$, such that for any balloon $pq \cong \mathbf{P}^1$ in X we have

$$V^\pm|_{pq} = \oplus_{i=1}^{N^\pm} L_i^\pm|_{pq}.$$

Note that $N^\pm = rk\ V^\pm$. We also require that

$$b_T(V^+)/b_T(V^-) = \prod_i b_T(L_i^+) / \prod_j b_T(L_j^-).$$

In this case we call the list $(L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-)$ the splitting type of V. Note that V is not assumed to split over X. Given such a bundle V and a choice of multiplicative class b_T, we obtain an Euler data $Q: \ Q_d = \varphi_!(\pi^* b_T(V_d))$ (or A) as before.

Theorem 6.3. *Let $b_T = e_T$ be the equivariant Euler class. Let pq be a balloon, $d = \delta[pq] \succ 0$, and λ be the weight on the tangent line $T_q(pq)$. Then at $\alpha = \lambda/\delta$, we have*

$$j_0^*(Q_d)|_q = \prod_i \prod_{k=0}^{\langle c_1(L_i^+), d\rangle} \left(c_1(L_i^+)|_q - k\lambda/\delta\right) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-), d\rangle - 1} \left(c_1(L_j^-)|_q + k\lambda/\delta\right).$$

In particular Q is linked to

$$P: \quad P_d = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (\hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (\hat{L}_j^- + k\alpha).$$

Proof: We first consider one positive line bundle L. As in [45], we consider a point $(f, C) \in M_d(X)$ where f is δ-cover from $C = \mathbf{P}^1$ to the balloon $pq \simeq \mathbf{P}^1$. For $\alpha = \lambda/\delta$, this map can be written as

$$f: \ C \to \mathbf{P}^1 \times pq \subset \mathbf{P}^1 \times X$$

where the second map is the inclusion. In terms of coordinates we can write the first map as

$$f: \ [w_0, w_1] \to [w_1, w_0] \times [w_0^\delta, w_1^\delta].$$

Note that the T-action induces standard rotation on $pq \simeq \mathbf{P}^1$ with the weights λ_1, λ_2 and $\lambda = \lambda_1 - \lambda_2$. It is now easy to see that this point (f, C) is fixed by the subgroup of G with $\alpha = \lambda/\delta$. On the other hand as argued in [45], $(\pi_2 \circ f, C)$ is then a smooth fixed point in $M_{0,0}(d, X)$ under the T-action. The restriction $j_0^* Q_d|_p$ with $\alpha = \lambda/\delta$ is equal to the value of $e_T(\mathcal{U}_d)$ at (f, C). This, in turn, is equal to the restriction of $e_T(V_d)$ at $(\pi_2 \circ f, C)$ in $M_{0,0}(d, X)$.

Assume the restriction of L to $pq \simeq \mathbf{P}^1$ is $\mathcal{O}(l)$ with $l = \langle c_1(L), [pq]\rangle$. We compute that the equivariant Euler class restricted to this point $(\pi_2 \circ f, C)$. As in [45], we get

$$e_T(U_d) = \prod_{m=0}^{l\delta} (l\lambda_1 - m\frac{\lambda}{\delta}).$$

Also note that $c_1(L)(p) = l\lambda_1$ and $d = \delta[pq]$, this implies that $Q_d = \varphi_!(\pi^* e_T(V_d))$ is linked to

$$P_d = \prod_{m=0}^{\langle c_1(L),d\rangle} (c_1(\hat{L}) - m\alpha).$$

Similarly for a concave line bundle L, if its restriction to the balloon pq is $\mathcal{O}(-l)$ with $-l = \langle c_1(L), [pq]\rangle$, then

$$e_T(U_d) = \prod_{m=-}^{l\delta-1} (-l\lambda_1 + m\frac{\lambda}{\delta})$$

which implies the formula that in this case Q_d is linked to

$$P_d = \prod_{m=1}^{-\langle c_1(L),d\rangle-1} (c_1(\hat{L}) + m\alpha).$$

The general case is just a product of these cases. □

Similarly we can prove the following formula for the Chern polynomial.

Theorem 6.4. *Let $b_T = c_T$ be the equivariant Chern polynomial. Let pq be a balloon, $d = \delta[pq] \succ 0$, and λ be the weight on the tangent line $T_q(pq)$. Then at $\alpha = \lambda/\delta$, we have*

$$j_0^*(Q_d)|_q = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+)|_q - k\lambda/\delta) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle-1} (x + c_1(L_j^-)|_q + k\lambda/\delta)\,.$$

In particular Q is linked to

$$P: \;\; P_d = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + \hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle-1} (x + \hat{L}_j^- + k\alpha).$$

7. Applications

7.1. Toric manifolds

We call a toric manifold X *reflexive* if its defining fan satisfies the following combinatorial condition: the convex hull of the primitive generators of the 1-cones in the fan is a reflexive polytope. It has been shown [4][52] that a pair of polar reflexive polytopes gives rise to a pair of mirror (in the sense of Hodge numbers) Calabi-Yau varieties, by taking anti-canonical hypersurfaces in the corresponding reflexive toric manifolds. It has been conjectured that [5] a similar statement holds for complete intersections in toric manifolds. It is known that [34] a toric manifold X is reflexive iff $c_1(X) \geq 0$. We shall assume that X is reflexive. Recall that for a (convex) toric manifold X, we have

$$e_G(X_0/W_d) = \prod_a \prod_{k=1}^{\langle D_a,d\rangle} (D_a - k\alpha)$$

where each D_a is the T-equivariant first Chern classes of the line bundles corresponding to a T-invariant hypersurfaces in X.

7.2. Chern polynomials for mixed bundles

To proceed, we make two further choices: let b_T be the T-equivariant Chern polynomial c_T, and let $V = V^+ \oplus V^-$ be a mixed bundle with splitting type $(L_1^+, .., L_{N^+}^+; L_1^-, .., L_{N^-}^-)$. Here the L's are T-equivariant line bundles on X with

$$c_1(L_i^+) \geq 0, \quad c_1(L_j^-) < 0,$$
$$\Omega := c_T(V^+)/c_T(V^-) = \prod_i (x + c_1(L_i^+)) / \prod_j (x + c_1(L_j^-))$$
$$\sum_i c_1(L_i^+) - \sum_j c_1(L_j^-) = c_1(X).$$

From this, we get an Ω-Euler data $Q: \quad Q_d = \varphi_!(\pi^* c_T(V_d))$ as before. By the Linking Theorem, Q is linked to the Euler data

$$P: \quad P_d = \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + \hat{L}_i^+ - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (x + \hat{L}_j^- + k\alpha).$$

As before, we set

$$B: \quad B_d = i_0^* P_d^v, \qquad A: \quad A_d = i_0^* Q_d^v.$$

We consider three separate cases. We will be using the elementary formula

$$\prod_{k=1}^{M} (\frac{\omega}{\alpha} - k) \equiv (-1)^M M! (1 - \frac{\omega}{\alpha} \sum_{k=1}^{M} \frac{1}{k}) \tag{7.1}$$

where " $\equiv$" here means equal *mod* $O(\alpha^{-2})$, to compute the leading terms of

$$\begin{aligned} B_d &= \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_i^-),d\rangle - 1} (x + c_1(L_i^-) + k\alpha) \\ &\times \frac{1}{\prod_a \prod_{k=1}^{\langle D_a,d\rangle} (D_a - k\alpha)} \\ &= \Omega \; c_T(V^-) \; \alpha^{-N^-} \prod_i \prod_{k=1}^{\langle c_1(L_i^+),d\rangle} (\frac{x + c_1(L_i^+)}{\alpha} - k) \times \prod_j \prod_{k=1}^{-\langle c_1(L_i^-),d\rangle - 1} (\frac{x + c_1(L_i^-)}{\alpha} + k) \\ &\times \frac{1}{\prod_a \prod_{k=1}^{\langle D_a,d\rangle} (\frac{D_a}{\alpha} - k)}. \end{aligned} \tag{7.2}$$

First suppose that $rk\ V^- = N^- \geq 2$. In this case we have

$$deg_\alpha B_d = -rk\ V^- \leq -2$$

and hence

$$HG[B](t) \equiv \Omega - \Omega\frac{H \cdot t}{\alpha}.$$

By Theorem 5.11, we conclude that $A = B$ and $Q = P$. This completes the computation of A and Q in this case.

Now consider the case $rk\ V^- = N^- = 1$, hence V^- is a line bundle. In this case we have

$$\begin{aligned} B_d &\equiv \alpha^{-1}\ \Omega\ (x + c_1(V^-))\ (-1)^{\langle c_1(V^-),d\rangle}(-\langle c_1(V^-),d\rangle - 1)!\frac{\prod_i \langle c_1(L_i^+),d\rangle!}{\prod_a \langle D_a, d\rangle!} \\ &=: \alpha^{-1}\ \Omega\ (\sum_i H_i \phi_{d,i} + \psi_d) \end{aligned}$$

where the $\phi_{d,i} \in \mathbf{Q}$, $\psi_d \in \mathbf{Q}[\mathcal{T}^*, x]$, are determined uniquely by the writing $c_1(V^-) \in H^2_{\mathcal{T}}(X)$ in the last equality, according to the decomposition $H^2_{\mathcal{T}}(X) = \oplus_{i=1}^m \mathbf{Q}H_i \oplus \mathcal{T}^*$. Hence we get

$$e^{-H\cdot t/\alpha}\ B_d \equiv \Omega\ (\alpha^{-1} H \cdot \phi_d + \alpha^{-1}\psi_d).$$

Summing over $d \in K^\vee$, we get

$$\begin{aligned} HG[B](t) &\equiv \Omega(1 - \alpha^{-1}H \cdot (t + F) + \alpha^{-1}G) \\ F &:= -\sum \phi_d\ e^{d\cdot t} \\ G &:= \sum \psi_d\ e^{d\cdot t}. \end{aligned}$$

From this we get

$$e^{-G/\alpha}\ HG[B](t) \equiv \Omega - \Omega\frac{H \cdot (t + F)}{\alpha}$$

By Theorem 5.11, we conclude that

$$e^{-G/\alpha}HG[B](t) = HG[A](t + F). \tag{7.3}$$

This completes our computation of A and Q in this case.

Recall that

$$\begin{aligned} dim\ M_{0,0}(d, X) &= \langle c_1(X), d\rangle + n - 3 \\ rk\ V_d &= \sum_i \langle c_1(L_i^+), d\rangle - \sum_j \langle c_1(L_j^-), d\rangle + N^+ - N_- \\ &= \langle c_1(X), d\rangle + rk\ V^+ - rk\ V^-. \end{aligned}$$

To applied Theorem 6.2, we assume that $rk\ V^+ - rk\ V^- \geq n - 3$, and we determine all K_d immediately. Explicitly (in the nonequivariant limit $\mathcal{T}^* \to 0$):

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(e^{-G/\alpha}HG[B](t) - e^{-H\cdot\tilde{t}/\alpha}\Omega\right) = \alpha^{-3}x^{-s}(2\Phi(\tilde{t}) - \sum_i \tilde{t}_i\frac{\partial\Phi(\tilde{t})}{\partial\tilde{t}_i}). \tag{7.4}$$

where $s := rk\ V^+ - rk\ V^- - (n-3)$, $\tilde{t} := t + F(t)$. Note that this same formula applies also when $rk\ V^- \geq 2$, whereby we put $G, F = 0$.

We now consider the case when V is purely convex: $N^- = 0$.

7.3. Convex bundle

We will denote the L_i^+ simply by L_i. Using formulas (7.1) and (7.2), we get

$$B_d \equiv \Omega \frac{\prod_i \langle c_1(L_i), d\rangle!}{\prod_a \langle D_a, d\rangle!}(1 + \alpha^{-1}\sum_a D_a \sum_{k=1}^{\langle D_a,d\rangle} \frac{1}{k} - \alpha^{-1}\sum_i (x + c_1(L_i)) \sum_{k=1}^{\langle c_1(L_i),d\rangle} \frac{1}{k})$$
$$=: \Omega\lambda_d + \alpha^{-1}\sum_i H_i\phi_{d,i} + \alpha^{-1}\psi_d$$

Here the $\lambda_d, \phi_{d,i} \in \mathbf{Q}$, $\psi_d \in \mathbf{Q}[\mathcal{T}^*, x]$ are determined uniquely by the writing each $D_a, c_1(L_i) \in H_T^2(X)$ in the last equality, according to the decomposition $H_T^2(X) = \oplus_{i=1}^m \mathbf{Q}H_i \oplus \mathcal{T}^*$. Since $e^{-H\cdot t/\alpha} \equiv 1 - \alpha^{-1}H \cdot t$, we get

$$e^{-H\cdot t/\alpha}\, B_d \equiv \Omega(\lambda_d - \alpha^{-1}H \cdot (\lambda_d t - \phi_d) + \alpha^{-1}\psi_d).$$

Summing over $d \in K^\vee$, we get

$$\begin{aligned} HG[B](t) &\equiv \Omega\left(F_0 - \alpha^{-1}H \cdot (F_0 t + F) + \alpha^{-1}G\right) \\ F_0 &:= 1 + \sum \lambda_d\, e^{d\cdot t} \\ F &:= -\sum \phi_d\, e^{d\cdot t} \\ G &:= \sum \psi_d\, e^{d\cdot t}. \end{aligned}$$

Put $f := \alpha\, log\, F_0 - \frac{G}{F_0}$. Then we get

$$e^{f/\alpha}\, HG[B](t) \equiv \Omega - \Omega\frac{H \cdot (t + \frac{F}{F_0})}{\alpha}$$

By Theorem 5.11, we conclude that

$$e^{f/\alpha}\, HG[B](t) = HG[A](t + \frac{F}{F_0}). \tag{7.5}$$

This completes our computation of A and Q in this case.

Again to apply Theorem 6.2, we assume that $rk\ V \geq n - 3$, and determine all K_d immediately. Explicitly:

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(e^{f/\alpha}\, HG[B](t) - e^{-H\cdot\tilde{t}/\alpha}\Omega\right) = \alpha^{-3}x^{-s}(2\Phi(\tilde{t}) - \sum_i \tilde{t}_i \frac{\partial\Phi(\tilde{t})}{\partial\tilde{t}_i}). \tag{7.6}$$

where $s := rk\ V - (n-3)$, $\tilde{t} := t + \frac{F(t)}{F_0(t)}$.

Let us now specialize to the case $rk\ V = n - 3$ (ie. $s = 0$), and $V = \oplus_i L_i$. We can then set $x = 0$, so that $b_T = c_T$ becomes the equivariant Euler class e_T, and the

K_d is just the intersection numbers for $e(V_d)$. Then the formula (7.6) yields the general formula derived in [33] and in [32], on the basis of the conjectural mirror correspondence. Note that

$$F_0 = \sum \frac{\prod_i \langle c_1(L_i), d\rangle!}{\prod_a \langle D_a, d\rangle!} e^{d \cdot t}$$

is an example of a hypergeometric function [23]. It has been proved in [34] that F_0 is the unique holomorphic period of Calabi-Yau hypersurfaces near the so-called large radius limit. For the purpose of comparison, we should mention that the definition of Φ here differs from the prepotential in [33][32] by a degree three polynomial in $\tilde{t}$, and the definition of the hypergeometric series $HG[B](t)$ here differs from that denoted by $w_0(x, \rho)$ in [33][32] by an irrelevant overall constant factor.

Precursors to the above general formula have been many examples [32][6][15][9]. We now specialize to a few numerical examples which have been frequently studied by both physicists and mathematicians alike.

7.4. A complete intersection in $\mathbf{P}^1 \times \mathbf{P}^2 \times \mathbf{P}^2$

The complete intersection of degrees $(1,3,0)$, $(1,0,3)$ in this 5-dimensional toric manifold X has been studied in [32] using mirror symmetry, and in [35] computing some of the intersection numbers K_d for the Euler class $b = e$ in terms of modular forms.

From our point of view, that complete intersection correspond to the following choice of convex bundle:

$$V = \mathcal{O}_1(1) \otimes \mathcal{O}_2(3) \oplus \mathcal{O}_1(1) \otimes \mathcal{O}_3(3)$$

where $\mathcal{O}_i(l)$ denotes the pullback of $\mathcal{O}(l)$ from the ith factor. The Kähler cone of X is abviously generated by the hyperplanes H_1, H_2, H_3 from the three factors of X, and hence $K^\vee$ can be identified with the set of $d = (d_1, d_2, d_3) \in \mathbf{Z}^3_{\geq 0}$. We consider intersection numbers K_d for the Euler class $b = e$ as before. Thus we set $\Omega = e(V) = (H_1 + 3H_2)(H_1 + 3H_3)$. The Euler data P we need in eqn. (7.6) is given by

$$P_d = \prod_{k=0}^{d_1+3d_2} (\kappa_1 + 3\kappa_2 - k\alpha) \times \prod_{k=0}^{d_1+3d_3} (\kappa_1 + 3\kappa_3 - k\alpha)$$

$$j_0^*(P_d) = \prod_{k=0}^{d_1+3d_2} (H_1 + 3H_2 - k\alpha) \times \prod_{k=0}^{d_1+3d_3} (H_1 + 3H_3 - k\alpha).$$

The linear sigma model is $W_d = N_d(\mathbf{P}(n)) = N_{d_1,1} \times N_{d_2,2} \times N_{d_3,2}$. The equivariant Euler class, after taking nonequivariant limit with respect to the T action, is given by

$$e_G(X_0/W_d) = \prod_{m=1}^{d_1}(H_1 - m\alpha)^2 \prod_{m=1}^{d_2}(H_2 - m\alpha)^3 \prod_{m=1}^{d_3}(H_3 - m\alpha)^3.$$

Now we can easily write down the hypregeometric series and all the K_d can be computed by our formula (7.6) at once using the obvious intersection form on X, given by the relations:

$$\int_X H_1 H_2^2 H_3^2 = 1, \quad H_1^2 = 1, \quad H_2^3 = 1, \quad H_3^3 = 1.$$

Once we have the hypergeometric series, the corresponding Picard-Fuchs equation can be easily written down as given in [32].

7.5. $V = \mathcal{O}_1(-2) \otimes \mathcal{O}_2(-2)$ *on* $\mathbf{P}^1 \times \mathbf{P}^1$

Here we denote by $\mathcal{O}_i(l)$ the pullback of $\mathcal{O}(l)$ from the ith factor of $X = \mathbf{P}^1 \times \mathbf{P}^1$. Our bundle V has $rk\ V^+ - rk\ V^- = n - 3 = -1$. Thus we can apply our formula (7.4) with $x = 0$. We put $\Omega = \frac{1}{H_1 H_2}$. The Euler data P in eqn. (7.4) that compute the K_d is now given by:

$$P_d = \prod_{k=1}^{2d_1-1}(-2\kappa_1 + k\alpha) \times \prod_{k=1}^{2d_2-1}(-2\kappa_2 + k\alpha).$$

The corresponding equivariant Euler class, after taking the nonequivariant limit with respext to the T-action is

$$e_G(X_0/W_d) = \prod_{m=1}^{d_1}(H_1 - m\alpha)^2 \prod_{m=1}^{d_2}(H_2 - m\alpha)^2.$$

Again one can immediately write down the hypergeometric series as well as the corresponding Picard-Fuchs equation by using our mirror principle.

8. Generalizations and Concluding Remarks

8.1. A weighted projective space

Consider the following example: the concave bundle $V = O(-6)$ over $\mathbf{P}_{3,2,1}$, $\Omega = \frac{1}{6H}$. This example will be studied in our subsequent paper by using resolution of singularities. This is an example of "local mirror symmetry" studied in physics [39]. The mirror formula there can derived as a special case of our general result. In fact, the Euler data which computes the K_d in this case is determined by

$$j_0^* P_d = \prod_{m=1}^{6d-1} (-6H + m\alpha).$$

The corresponding equivariant Euler class, after taking nonequivariant limit with respect to the T action, is:

$$e_G(X_0/W_d) = \prod_{m=1}^{d} (H - m\alpha) \prod_{m=1}^{2d} (2H - m\alpha) \prod_{m=1}^{3d} (3H - m\alpha).$$

The corresponding hypergeometric series and Picard-Fuchs equation can be immediately written down. It turns out that the hypergeometric series gives the periods of a meromorphic 1-form for a family of elliptic curves [39].

8.2. General projective balloon manifolds

Let X be a projective manifold embedded in $\mathbf{P}(n)$, with a system of homogeneous polynomial defining equations $P(z^1, \cdots, z^n) = 0$, where $z^j = (z_1^j, \cdots, z_{n_j}^j)$. For each P, by taking the coefficients of each monomial $w_0^a w_1^b$ in $P(f^1, \cdots, f^n) = 0$, where $f^j = [f_1^j(w_0, w_1), \cdots, f_{n_j}^j(w_0, w_1)]$ for $j = 1, \cdots, k$ is the tuple of polynomials that define the coordinates of $N_d(\mathbf{P}(n))$, we get several equations of the same degree as P. These equations together define a projective variety, which we denote by $N_d(X)$, in $N_d(\mathbf{P}(n))$.

As discussed earlier, we see that the S^1 fixed point components in $N_d(X)$ are given by the X_r's which are copies of X. We do not know whether the localization formula holds on $N_d(X)$. The localization formula holds if the fixed point components embedded into W_d as local complete intersection subvarieties. It is likely that this is the case for any convex projective manifold. If this is true, then we can take $N_d(X)$ to be the linear sigma model W_d for X. Then our mirror principle may apply readily to compute multiplicative characteristic numbers on $M_{0,0}(d, X)$ in terms of the hypergeometric series.

8.3. A General Mirror Formula

Many of our results so far are proved for projective manifolds without T-action. Here we first discuss a formula for computing the numbers

$$K_d = \int_{M_{0,0}(d,X)} b(V_d)$$

for a general convex projective n-fold X *without T-action.* For simplicity, let's focus on the case when the multiplicative class b is the Chern polynomial c, and V is a direct sum of line bundles on X. There is a similar formulation in the general case. We fix a projective embedding $X \to \mathbf{P}(n)$, as before. Note that the map $\varphi : M_d(X) \to N_d(\mathbf{P}(n))$ is now only S^1-equivariant. Recall that the subvariety $W_d := \varphi(M_d(X)) \subset N_d(\mathbf{P}(n))$ contains as S^1 fixed point components copies of X: X_r, $0 \preceq r \preceq d$. We assume that the localization formula holds on it.

We denote by $e_{S^1}(X_0/W_d)$ the equivariant Euler class of the normal bundle of X_0 in W_d. Let

$$V = V^+ \oplus V^-, \quad V^+ := \oplus L_i^+, \quad V^- := \oplus L_j^-$$

satisfying $c_1(V^+) - c_1(V^-) = c_1(X)$ and $rk(V^+) - rk(V^-) - (n-3) \geq 0$, where the $L_i^{\pm}$ are respectively convex/concave line bundles on X. Let

$$\Omega = B_0 := c(V^+)/c(V^-) = \prod_i (x + c_1(L_i^+))/\prod_j (x + c_1(L_j^-))$$

$$B_d := \frac{1}{e_{S^1}(X_0/W_d)} \times \prod_i \prod_{k=0}^{\langle c_1(L_i^+),d\rangle} (x + c_1(L_i^+) - k\alpha) \times \prod_j \prod_{k=1}^{-\langle c_1(L_j^-),d\rangle - 1} (x + c_1(L_j^-) + k\alpha).$$

$$HG[B](t) := \sum B_d e^{d\cdot t}$$

$$\Phi(t) := \sum K_d e^{d\cdot t}.$$

Conjecture 8.1. *There exist unique power series $G(t), F(t)$ such that the following formula holds:*

$$\frac{1}{s!}\left(\frac{d}{dx}\right)^s |_{x=0} \int_X \left(e^{-G/\alpha} HG[B](t) - e^{-H\cdot \tilde{t}/\alpha}\Omega\right) = \alpha^{-3}x^{-s}(2\Phi(\tilde{t}) - \sum_i \tilde{t}_i \frac{\partial \Phi(\tilde{t})}{\partial \tilde{t}_i}).$$

where $s := rk\ V^+ - rk\ V^- - (n-3)$, $\tilde{t} := t + F(t)$. Moreover G, F are determined by the condition that the integrand on the left hand side is of order $O(\alpha^{-2})$.

8.4. Formulas without T-action

One of our key ingredient, the functorial localization formula plays an important role in relating the data on $M_d(X)$ and those on W_d. It turns out that similar formula holds in K-theory. It holds even when X has no group action. This indicates that our method may be extended to compute K-theory multiplicative type characteristic classes on $M_d(X)$ (and ultimately on $M_{0,0}(d,X)$), in terms of certain q-hypergeometric series, even for projective manifolds without group action.

We now write down the relevant localization formulas for convex X without torus action, both in cohomology and in K-theory. The notations and proofs are basically the same as before. Given a manifold X, let's assume that there is a linear sigma model W_d.

Lemma 8.2. *For any equivariant cohomology class ω on $M_d(X)$, the following equality holds on X_r for any $0 \preceq r \preceq d$:*

$$\frac{j_r^*\varphi_!(\omega)}{e_{S^1}(X_r/W_d)} = e_![\frac{i_r^*\omega}{e_{S^1}(F_r/M_d(X))}].$$

Here $e_{S^1}(\cdot)$ denotes the S^1-equivariant Euler class. As in the cases we have studied earlier, the left hand side of the above formula indicates that when $V = L$ is a line bundle, we should compare the Euler data $Q_d = \varphi_!\pi^*e(V_d)$, to the Euler data given by

$$P_d = \prod_{m=0}^{\langle c_1(L),d\rangle} (c_1(L) - m\alpha).$$

What is left is to develope uniqueness and mirror transformations, which we are unable to achieve at this moment, though they can be easily axiomized.

Now let us look at K-theory formula, which can be proved by using equivariant localization in K-theory. First following the same idea, we get the explicit formula as follows: given any equivariant element V in $K(W_d)$, we have

$$V = \sum_r j_{r!}\frac{j_r^*V}{E_G(X_r/W_d)}$$

where $E_G(X_r/W_d)$ is the equivariant Euler class of the normal bundle of X_r in $N_d(X)$. Here the push-forward and pull-backs by j_r denote the corresponding operations in K-theory. By taking $V = 1$, we get

$$e_![\frac{1}{E_G(F_r/M_d(X))}] = \frac{j_r^*\varphi_!(1)}{E_G(X_r/W_d)}.$$

Second, we have the following lemma; If $\varphi_!(1) = 1$, which is the case if $X = \mathbf{P}^n$, this formula gives explicit formulas for some K-theoretic characteristic numbers of the moduli spaces.

Lemma 8.3. *Given any equivariant element V in $K_G(M_d(X))$, then we have formula*

$$e_![\frac{i_r^* V}{E_G(F_r/M_d(X))}] = \frac{j_r^*(\varphi_! V)}{E_G(X_r/W_d)}$$

where $E_G(\cdot)$ denotes the equivariant Euler class of the corresponding normal bundle in K-group.

In particular we have explicit expressions from the decomposition of the normal bundles:

$$E_G(F_r/M_d(X)) = (1 - e^{\alpha})(1 - e^{-\alpha})(1 - e^{\alpha} L_r)(1 - e^{-\alpha} L_{d-r})$$

and similarly

$$E_G(F_0/M_d(X)) = (1 - e^{\alpha})(1 - e^{\alpha} L_d), \ E_G(F_d/M_d(X)) = (1 - e^{-\alpha})(1 - e^{-\alpha} L_d).$$

For a toric manifold X, we also have the explicit class in K-group,

$$E_G(X_0/W_d) = \prod_a \prod_{m=1}^{\langle D_a, d\rangle} (1 - e^{m\alpha}[D_a])$$

where $[D_a]$ are the equivariant line bundle corresponding to the T divisors D_a.

If V is a multipicative type K-theory characteristic class, then we can develope a similar theory of Euler data and uniqueness. These result can also be extended to the nonconvex case without a hitch.

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