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Recent Proofs of the Riemannian Penrose Conjecture

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In these notes we summarize two recent proofs, by Huisken and Ilmanen [**19**] and by the first author [**6**], of the Riemannian Penrose conjecture in general relativity. The conjecture, which is a statement about the mass of black holes in a three dimensional space-like slice of a spacetime, was first proposed by Penrose [**27**] in 1973 as a test of the cosmic censor conjecture.

In the seventies, Geroch [**11**] and Jang and Wald [**26**] noticed that the Hawking mass of a surface in a 3-manifold was nondecreasing if the surface was flowed out at a rate equal to the inverse of its mean curvature at each point. In addition, they observed that this would be a proof of the Riemannian Penrose conjecture except for the fact that singularites in the flow could develop if the mean curvature ever became zero, preventing the flow from continuing. In 1997, Huisken and Ilmanen found a very nice way to generalize inverse mean curvature flow in such a way that their generalized inverse mean curvature flow always exists, thereby proving the Riemannian Penrose conjecture for a single black hole. (The Huisken-Ilmanen proof allows for multiple black holes, but the bound on the mass is in terms of the area of the largest one.) Then in 1998, using a completely different technique which relies on the positive mass theorem and conformal deformations of metrics, the first author found a proof of the Riemannian Penrose conjecture which works for any number of black holes.

In section 1 we define the important terms and give the statements of the Riemannian positive mass theorem and the Riemannian Penrose conjecture. In section 2, we summarize the first author's proof of the Riemannian Penrose conjecture and provide many of the details. In section 3, we summarize the Huisken-Ilmanen proof, and in section 4 we define the new quasi-local mass quantities which come naturally out of the two approaches.

1. Introduction

General relativity is a theory of gravity which asserts that matter causes four dimensional space-time to be curved, and that our perception of gravity is a consequence of this curvature. Let $(N^4, \bar{g})$ be the space-time manifold with metric $\bar{g}$ of signature $(- + ++)$. Then the central formula of general relativity is Einstein's equation,

$$G = 8\pi T, \tag{1}$$

where T is the stress-energy tensor, $G = Ric(\bar{g}) - \frac{1}{2}R(\bar{g})\cdot\bar{g}$ is the Einstein curvature tensor, $Ric(\bar{g})$ is the Ricci curvature tensor, and $R(\bar{g})$ is the scalar curvature of $\bar{g}$. The beauty of general relativity is that this simple formula explains gravity more accurately than Newtonian physics and is entirely consistent with large scale observations.

However, the nature of the behavior of matter in general relativity is still not well understood. It is not even well understood how to define how much energy and momentum exist in a given region, except in special cases. There does exist a well defined notion of local energy and momentum density which is simply given by the stress-energy tensor which, by equation 1, can be computed in terms of the curvature of N^4. Also, if we assume that the matter of the space-time manifold N^4 is concentrated in some central region of the universe, then N^4 becomes flatter as we get farther away from this central region. If the curvature of N^4 decays quickly enough, then N^4 is said to be asymptotically flat (definition 13 in section 2.4), and with these assumptions it is then possible to define the total mass of the space-time N^4. Interestingly enough, though, the definition of local energy-momentum density, which involves curvature terms of N^4, bears no obvious resemblance to the definition of the total mass of N^4, which is a parameter related to how fast the metric becomes flat at infinity.

The Penrose conjecture [**27**], [**19**], [**6**] and the positive mass theorem ([**31**], [**32**], [**33**], [**34**], [**37**]) can both be thought of as basic attempts at understanding the relationship between the local energy density of a space-time N^4 and the total mass of N^4. In physical terms, the positive mass theorem states that an isolated gravitational system with nonnegative local energy density must have nonnegative total energy. The idea is that nonnegative energy densities must "add up" to something nonnegative. The Penrose conjecture, on the other hand, states that if an isolated gravitational system with nonnegative local energy density contains black holes contributing a mass m, then the total energy of the system must be at least m.

1.1. Statement of the Riemannian Penrose Conjecture. Important cases of the positive mass theorem and the Penrose conjecture can be translated into statements about complete, asymptotically flat Riemannian 3-manifolds (M^3, g) with nonnegative scalar curvature. If we consider (M^3, g, h) as a space-like hypersurface of $(N^4, \bar{g})$ with metric g_{ij} and second fundamental form h_{ij} in N^4, then equation 1 implies that

$$\mu = G_0^0 = \frac{1}{16\pi}[R - \sum_{i,j} h^{ij}h_{ij} + (\sum_i h_i^i)^2], \tag{2}$$

$$J^i = G_0^i = \frac{1}{8\pi}\sum_j \nabla_j[h^{ij} - (\sum_k h_k^k)g^{ij}], \tag{3}$$

where R is the scalar curvature of the metric g, μ is the local energy density, and J^i is the local current density. The assumption of nonnegative energy density everywhere in N^4 implies that we must have

$$\mu \geq \left(\sum_i J^i J_i\right)^{\frac{1}{2}} \tag{4}$$

at all points on M^3. Equations 2, 3, and 4 are called the constraint equations for (M^3, g, h) in $(N^4, \bar{g})$. Thus we see that if we restrict our attention to 3-manifolds which have zero second fundamental form h in N^3, the constraint equations are equivalent to the condition that the Riemannian manifold (M^3, g) has nonnegative scalar curvature everywhere.

A Riemannian manifold (M^3, g) is said to be asymptotically flat if, outside a compact set, it is the disjoint union of one or more regions (called ends) diffeomorphic to $(\mathbf{R}^3 \backslash B_1(0), \delta)$, where the metric g in each of these $\mathbf{R}^3$ coordinate charts approaches the standard metric δ on $\mathbf{R}^3$ at infinity (with certain asymptotic decay conditions - see definition 13, **[33]**, **[2]**). The positive mass theorem and the Penrose conjecture are both statements which refer to a particular **chosen end** of (M^3, g). The total mass of (M^3, g) is then a parameter related to how fast this chosen end of (M^3, g) becomes flat at infinity. The most general definition of the total mass (also called the ADM mass) is given in equation 59 in section 2.4. In addition, we give an alternative definition of total mass in the next subsection.

The positive mass theorem was first proved by Schoen and Yau **[31]** in 1979 using minimal surfaces. The Riemannian positive mass theorem is a special case of the positive mass theorem which comes from considering the space-like hypersurfaces which have zero second fundamental form in the spacetime.

The Riemannian Positive Mass Theorem
Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature and total mass m. Then

$$m \geq 0, \tag{5}$$

with equality if and only if (M^3, g) is isometric to $\mathbf{R}^3$ with the standard flat metric.

Apparent horizons of black holes in N^4 correspond to outermost minimal surfaces of M^3 if we assume M^3 has zero second fundamental form in N^4. A minimal surface is a surface which has zero mean curvature (and hence is a critical point for the area functional). An outermost minimal surface is a minimal surface which is not contained entirely inside another minimal surface. Again, there is a chosen end of M^3, and "contained entirely inside" is defined with respect to this end. Interestingly, it follows from a stability argument **[16]** that outermost minimal surfaces are always spheres. There could be more than one outermost sphere, with each minimal sphere corresponding to a different black hole, and we note that outermost minimal spheres never intersect.

As an example, consider the Schwarzschild manifolds $(\mathbf{R}^3 \backslash \{0\}, s)$ where $s_{ij} = (1 + m/2r)^4 \, \delta_{ij}$ and m is a positive constant and equals the total mass of the manifold. This manifold has zero scalar curvature everywhere, is spherically symmetric, and it can be checked that it has an outermost minimal sphere at $r = m/2$.

We define the horizon of (M^3, g) to be the union of all of the outermost minimal spheres in M^3, so that the horizon of a manifold can have multiple connected components. We note that it is usually more common to call each outermost minimal sphere a horizon, so that their union is referred to as "horizons", but it turns out to be mathematically more convenient for our purposes to refer to the union of all of the outermost minimal spheres as one object, which we will call the horizon of (M^3, g).

There is a very convincing (but not rigorous) physical motivation to define the mass that a collection of black holes contributes to be $\sqrt{\frac{A}{16\pi}}$, where A is the total surface area of the horizon of (M^3, g). Then the physical statement that a system with nonnegative energy density containing black holes contributing a mass m must have total mass at least m can be translated into the following geometric statement [**27**], [**26**].

The Riemannian Penrose Conjecture
Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature and total mass m whose outermost minimal spheres have total surface area A. Then

$$m \geq \sqrt{\frac{A}{16\pi}}, \tag{6}$$

with equality if and only if (M^3, g) is isometric to the Schwarzschild metric $(\mathbf{R}^3 \backslash \{0\}, s)$ of mass m outside their respective horizons.

The first breakthrough on the Riemannian Penrose conjecture was made by Huisken and Ilmanen who proved the above theorem in the case that the horizon of (M^3, g) has only one component [**19**]. Specifically, they proved that $m \geq \sqrt{\frac{A_{max}}{16\pi}}$, where A_{max} is the area of the largest component of the horizon of (M^3, g). Their proof is as interesting as the result itself. In the seventies, Geroch [**11**] observed that in a manifold with nonnegative scalar curvature, the Hawking mass of a sphere (but not surfaces with multiple components) was monotone increasing under a $1/H$ flow, where H is the mean curvature of the sphere. Jang and Wald [**26**] proposed using this to attack the Riemannian Penrose conjecture by flowing the horizon of the manifold out to infinity. However, it is not hard to concoct situations in which the $1/H$ flow of a sphere develops singularities, preventing the idea from working much of the time. Huisken and Ilmanen's approach then was to define a generalized $1/H$ flow which sometimes "jumps" in order to prevent singularities from developing. Their techniques are described in section 3.

In section 2, we sketch the proof of the Riemannian Penrose conjecture due to the first author. The overview of the proof of this result is given in the introduction to section 2. The basic idea of the approach to the problem is to flow the original metric continuously to a Schwarzschild metric (outside the horizon). The particular flow we define has the important property that the area of the horizon stays constant while the total mass of the manifold is non-increasing. Then since the Schwarzschild metric gives equality in the Penrose inequality, the inequality follows for the original metric.

Other contributions have also been made by Herzlich [**18**] using the Dirac operator which Witten [**37**] used to prove the positive mass theorem, by Gibbons [**12**] in the special case of collapsing shells, by Tod [**36**], by Bartnik [**4**] for quasi-spherical metrics, and by the first author [**5**] using isoperimetric surfaces.

1.2. Definitions and Setup. Without loss of generality, we will be able to assume that an asymptotically flat metric (see definition 13) has an even nicer form at infinity in each end because of the following lemma and definition.

LEMMA 1. **(Schoen, Yau [33])** *Let (M^3, g) be any asymptotically flat metric with nonnegative scalar curvature. Then given any $\epsilon > 0$, there exists a metric g_0*

with nonnegative scalar curvature which is harmonically flat at infinity (defined in the next definition) such that

$$1 - \epsilon \le \frac{g_0(\vec{v}, \vec{v})}{g(\vec{v}, \vec{v})} \le 1 + \epsilon \tag{7}$$

for all nonzero vectors $\vec{v}$ in the tangent space at every point in M and

$$|\bar{m}_k - m_k| \le \epsilon \tag{8}$$

where $\bar{m}_k$ and m_k are respectively the total masses of (M^3, g_0) and (M^3, g) in the kth end.

Notice that because of equation 7, the percentage difference in areas as well as lengths between the two metrics is arbitrarily small. Hence, since the mass changes arbitrarily little also and since inequality 6 is a closed condition, it follows that the Riemannian Penrose inequality for asymptotically flat manifolds follows from proving the inequality for manifolds which are harmonically flat at infinity.

DEFINITION 1. *A Riemannian manifold is defined to be* **harmonically flat at infinity** *if, outside a compact set, it is the disjoint union of regions (which we will again call ends) with zero scalar curvature which are conformal to $(\mathbf{R}^3 \backslash B_1(0), \delta)$ with the conformal factor approaching a positive constant at infinity in each region.*

Now it is fairly easy to define the total mass of an end of a manifold (M^3, g_0) which is harmonically flat at infinity. Define g_{flat} to be a smooth metric on M^3 conformal to g_0 such that in each end of M^3 in the above definition (M^3, g_{flat}) is isometric to $(\mathbf{R}^3 \backslash B_1(0), \delta)$. Define $\mathcal{U}_0(x)$ such that

$$g_0 = \mathcal{U}_0(x)^4 g_{flat}. \tag{9}$$

Since (M^3, g_0) has zero scalar curvature in each end, $(\mathbf{R}^3 \backslash B_1(0), \mathcal{U}_0(x)^4 \delta)$ must have zero scalar curvature. This implies that $\mathcal{U}_0(x)$ is harmonic in $(\mathbf{R}^3 \backslash B_1(0), \delta)$ (see equation 95 in appendix E). Since $\mathcal{U}_0(x)$ is a harmonic function going to a constant at infinity, we may expand it in terms of spherical harmonics to get

$$\mathcal{U}_0(x) = a + \frac{b}{|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right), \tag{10}$$

where a and b are constants.

DEFINITION 2. *The* **total mass** *(of an end) of a Riemannian 3-manifold which is harmonically flat at infinity is defined to be $2ab$ in the above equation.*

While the constants a and b scale depending on how we represent (M^3, g_0) as the disjoint union of a compact set and ends in definition 1, it can be checked that $2ab$ does not. Furthermore, this definition agrees with the standard definition of the total mass of an asymptotically flat manifold (defined in equation 59) in the case that the manifold is harmonically flat at infinity. In section 2 we will choose to work with definition 2 because it is more convenient for the calculations we will be doing.

Now we turn our attention to the definition and properties of horizons. For convenience, we modify the topology of M^3 by compactifying all of the ends of M^3 except for the chosen end by adding the points $\{\infty_k\}$.

DEFINITION 3. *Define $\mathcal{S}$ to be the collection of surfaces which are smooth compact boundaries of open sets in M^3 containing the points $\{\infty_k\}$.*

All of the surfaces that we will be dealing with in section 2 will be in $\mathcal{S}$. Also, we see that all of the surfaces in $\mathcal{S}$ divide M^3 into two regions, an inside (the open set) and an outside (the complement of the open set). Thus, the notion of one surface in $\mathcal{S}$ (entirely) enclosing another surface in $\mathcal{S}$ is well defined.

DEFINITION 4. *A* **horizon** *of (M^3, g) is any zero mean curvature surface in $\mathcal{S}$.*

A horizon may have multiple components. Furthermore, by minimizing area over surfaces in $\mathcal{S}$, a horizon is guaranteed to exist when M^3 has more than one end.

DEFINITION 5. *A horizon is defined to be* **outermost** *if it is not enclosed by another horizon.*

We note that when at least one horizon exists, there is always a unique outermost horizon, with respect to the chosen end.

DEFINITION 6. *A surface $\Sigma \in \mathcal{S}$ is definedto be* **outer-minimizing** *if every other surface $\tilde{\Sigma} \in \mathcal{S}$ which encloses it has $|\tilde{\Sigma}| \geq |\Sigma|$.*

An outer-minimizing surface must have nonnegative mean curvature since otherwise the first variation formula would imply that an outward variation would yield a surface with less area. Also, in the case that Σ is a horizon, it is an outer-minimizing horizon if and only if it is not enclosed by a horizon with less area. We also note that, as with outermost horizons, it follows from a second variation argument that each component of an outer-minimizing horizon must be a sphere [**30**]. We will not use outermost horizons in section 2, but simply point out that outermost horizons are always outer-minimizing. Hence, the following theorem, which we will prove in section 2, is a slight generalization of the Riemannian Penrose conjecture.

THEOREM 1. *Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature, total mass m, and an outer-minimizing horizon (with one or more components) of total area A. Then*

$$m \geq \sqrt{\frac{A}{16\pi}} \tag{11}$$

with equality if and only if (M^3, g) is isometric to a Schwarzschild manifold outside their respective outermost horizons.

Besides the flat metric on $\mathbf{R}^3$, the Schwarzschild manifolds are the only other complete spherically symmetric 3-manifolds with zero scalar curvature, and as previously mentioned can be described explicitly as $(\mathbf{R}^3 \backslash \{0\}, s)$ where

$$s_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij},$$

r is the distance from the origin in $\mathbf{R}^3$, and m is a positive constant and equals the total mass of the manifold. Then since the Schwarzschild manifolds have a single minimal sphere which is the coordinate sphere of radius $m/2$, we can verify they give equality in the above theorem.

2. Proof of the Riemannian Penrose Conjecture Using the Positive Mass Theorem

In this section we give the overview of the proof of theorem 1 which is a slight generalization of the Riemannian Penrose conjecture [6]. As discussed in the previous section, without loss of generality for proving the Riemannian Penrose inequality we may make the following convenient, but still quite general, assumption.

Assumptions: For the rest of this section, we will assume that $(M^3,\, g_0)$ has nonnegative scalar curvature and is harmonically flat at infinity, unless otherwise stated.

We will then generalize our results to the asymptotically flat case and handle the case of equality of theorem 1 in section 2.4.

We define a continuous family of conformal metrics $\{g_t\}$ on M^3, where

$$g_t = u_t(x)^4 g_0 \tag{12}$$

and $u_0(x) \equiv 1$. Given the metric g_t, define

$$\Sigma(t) = \text{the outermost minimal area enclosure of } \Sigma_0 \text{ in } (M^3, g_t) \tag{13}$$

where Σ_0 is the original outer-minimizing horizon in (M^3, g_0) and we stay inside the collection of surfaces $\mathcal{S}$ defined in the previous section. In the cases in which we are interested, $\Sigma(t)$ will not touch Σ_0, from which it follows that $\Sigma(t)$ is actually an outer-minimizing horizon of (M^3, g_t). Then given the horizon $\Sigma(t)$, define $v_t(x)$ such that

$$\begin{cases} \Delta_{g_0} v_t(x) & \equiv & 0 & \text{outside } \Sigma(t) \\ v_t(x) & = & 0 & \text{on } \Sigma(t) \\ \lim_{x\to\infty} v_t(x) & = & -e^{-t} & \end{cases} \tag{14}$$

and $v_t(x) \equiv 0$ inside $\Sigma(t)$. Finally, given $v_t(x)$, define

$$u_t(x) = 1 + \int_0^t v_s(x)\,ds \tag{15}$$

so that $u_t(x)$ is continuous in t and has $u_0(x) \equiv 1$.

THEOREM 2. *Taken together, equations 12, 13, 14, and 15 define a first order o.d.e. in t for $u_t(x)$ which has a solution which is Lipschitz in the t variable, C^1 in the x variable everywhere, and smooth in the x variable outside $\Sigma(t)$. Furthermore, $\Sigma(t)$ is a smooth, outer-minimizing horizon in (M^3, g_t) for all $t \geq 0$, and $\Sigma(t_2)$ encloses but does not touch $\Sigma(t_1)$ for all $t_2 > t_1 \geq 0$.*

Proof. See [6].

Since $v_t(x)$ is a superharmonic function in (M^3, g_0), it follows that $u_t(x)$ is superharmonic as well, and from equation 15 we see that $\lim_{x\to\infty} u_t(x) = e^{-t}$ and consequently that $u_t(x) > 0$ for all t. Then since

$$R(g_t) = u_t(x)^{-5}(-8\Delta_g + R(g))u_t(x) \tag{16}$$

it follows that (M^3, g_t) is an asymptotically flat manifold with nonnegative scalar curvature.

Even so, it still may not seem like g_t is particularly naturally defined since the rate of change of g_t appears to depend on t and the original metric g_0 in equation 14. We would prefer a flow where the rate of change of g_t is only a function of g_t

(and M^3 and Σ_0 perhaps), and interestingly enough this actually does turn out to be the case. In appendix E we prove this very important fact and provide a very natural motivation for defining this conformal flow of metrics.

DEFINITION 7. *The function $A(t)$ is defined to be the total area of the horizon $\Sigma(t)$ in (M^3, g_t).*

DEFINITION 8. *The function $m(t)$ is defined to be the total mass of (M^3, g_t) in the chosen end.*

The next theorem is the key property of the conformal flow of metrics.

THEOREM 3. *The function $A(t)$ is constant in t and $m(t)$ is non-increasing in t, for all $t \geq 0$.*

Proof. The fact that $A'(t) = 0$ follows from the fact that to first order the metric is not changing on $\Sigma(t)$ (since $v_t(x) = 0$ there) and from the fact that to first order the area of $\Sigma(t)$ does not change as it moves outward since $\Sigma(t)$ has zero mean curvature in (M^3, g_t). Hence, $A(t)$ is constant by construction really. We refer the reader to [**6**] for the rigorous argument. The interesting part of theorem 3 is proving that $m'(t) \leq 0$. Curiously, this follows from a nice trick using the Riemannian positive mass theorem and is explained in detail in sections 2.2 and 2.3. $\square$

Another important aspect of this conformal flow of the metric (which we do not have room to prove in these notes) is that outside the horizon $\Sigma(t)$, the manifold (M^3, g_t) becomes more and more spherically symmetric and "approaches" a Schwarzschild manifold $(\mathbf{R}^3 \backslash \{0\}, s)$ in the limit as t goes to ∞. More precisely,

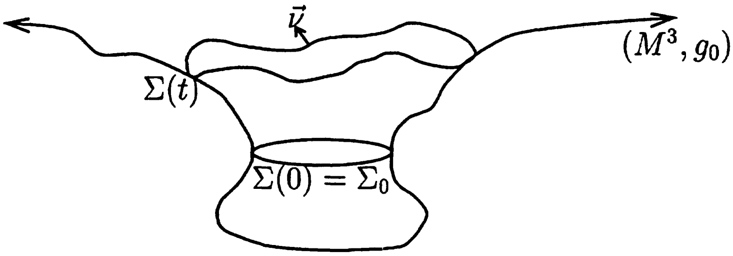
THEOREM 4. *For sufficiently large t, there exists a diffeomorphism ϕ_t between (M^3, g_t) outside the horizon $\Sigma(t)$ and a fixed Schwarzschild manifold $(\mathbf{R}^3 \backslash \{0\}, s)$ outside its horizon. Furthermore, for all $\epsilon > 0$, there exists a T such that for all $t > T$, the metrics g_t and $\phi_t^*(s)$ (when determining the lengths of unit vectors of (M^3, g_t)) are within ϵ of each other and the total masses of the two manifolds are within ϵ of each other. Hence,*

$$\lim_{t \to \infty} \frac{m(t)}{\sqrt{A(t)}} = \sqrt{\frac{1}{16\pi}}. \tag{17}$$

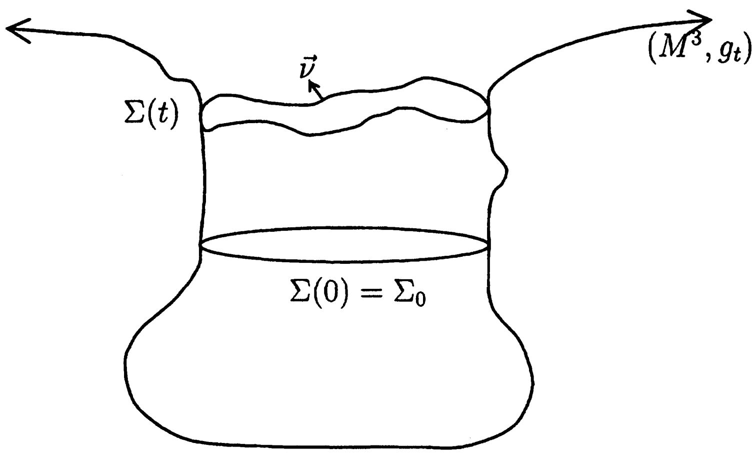
Proof. See [**6**].

Inequality 11 of theorem 1 then follows from theorems 2, 3 and 4, for harmonically flat manifolds. In addition, in section 2.4 we will see that the case of equality in theorem 1 follows from the fact that $m'(0) = 0$ if and only if (M^3, g_0) is isometric to a Schwarzschild manifold outside their respective outermost horizons. We will also generalize our results to the asymptotically flat case in section 2.4.

2.1. Some Qualitative Features of the Conformal Flow of Metrics.



The diagrams above and below are meant to help illustrate some of the properties of the conformal flow of the metric. The above picture is the original metric which has an outer-minimizing horizon Σ_0. As t increases, $\Sigma(t)$ moves outwards, but never inwards. In the diagram below, we can observe one of the consequences of the fact that $A(t) = A_0$ is constant in t. Since the metric is not changing inside $\Sigma(t)$, all of the horizons $\Sigma(s)$, $0 \leq s \leq t$ have area A_0 in (M^3, g_t). Hence, inside $\Sigma(t)$, the manifold (M^3, g_t) becomes cylinder-like in the sense that it is laminated by all of the previous horizons which all have the same area A_0 with respect to the metric g_t.



Now let us suppose that the original horizon Σ_0 of (M^3, g) had two components, for example. Then each of the components of the horizon will move outwards as t increases, and at some point before they touch they will suddenly jump outwards to form a horizon with a single component enclosing the previous horizons with two components. Even horizons with only one component will sometimes jump outwards, and it is interesting that this phenomenon of surfaces jumping is also found in the Huisken-Ilmanen approach to the Penrose conjecture using their generalized $1/H$ flow.

We end this subsection with two definitions which will be needed later.

DEFINITION 9. *Define*

$$\Sigma^+(t) = \lim_{s\to t^+} \Sigma(s), \qquad \Sigma^-(t) = \lim_{s\to t^-} \Sigma(s), \tag{18}$$

We note that since $\Sigma(t_2)$ encloses $\Sigma(t_1)$ for all $t_2 > t_1 \geq 0$ by theorem 2, the above limits always exist and are unique. Then we can define the jump times for the horizons $\Sigma(t)$ very naturally.

DEFINITION 10. *Define the jump times J to be the set of all $t \geq 0$ with $\Sigma^+(t) \neq \Sigma^-(t)$, where we define $\Sigma^-(0) = \Sigma_0$.*

2.2. Green's Functions on Asy. Flat 3-Manifolds with Nonnegative Scalar Curvature and the Riemannian Positive Mass Theorem. In this subsection we will prove a few theorems about certain Green's functions on asymptotically flat 3-manifolds with nonnegative scalar curvature which will be needed in the next subsection to prove that $m(t)$ is nonincreasing in t. However, the theorems in this subsection, which follow from and generalize the Riemannian positive mass theorem, are also of independent interest.

The results of this subsection are closely related to the beautiful ideas used by Bunting and Masood-ul-Alam in [8] to prove the non-existence of multiple black holes in asymptotically flat, static, vacuum space-times. Hence, while theorems 5 and 6 do not appear in their paper, these two theorems follow from a natural extension of their techniques.

DEFINITION 11. *Given a complete, asymptotically flat manifold $(M^3, \bar{g})$ with multiple asymptotically flat ends (with one chosen end), define*

$$\mathcal{E}(\bar{g}) = \inf_{\phi} \left\{ \frac{1}{2\pi} \int_{(M^3,\bar{g})} |\nabla\phi|^2 \, dV \right\} \tag{19}$$

where the infimum is taken over all smooth $\phi(x)$ which go to one in the chosen end and zero in the other ends.

Without loss of generality, we may assume that $(M^3, \bar{g})$ is actually harmonically flat at infinity (as defined in section 1.2). Then since such a modification can be done so as to change the metric uniformly pointwise as small as one likes (by lemma 1), it follows that $\mathcal{E}(\bar{g})$ changes as small as one likes as well. We remind the reader that the total mass of $(M^3, \bar{g})$ also changes arbitrarily little with such a deformation.

From standard theory it follows that the infimum in the above definition is achieved by the Green's function $\phi(x)$ which satisifies

$$\begin{cases} \lim_{x\to\infty_0} \phi(x) &= 1 \\ \Delta\phi &= 0 \\ \lim_{x\to\infty_k} \phi(x) &= 0 \text{ for all } k \neq 0 \end{cases} \tag{20}$$

where ∞_k are the points at infinity of the various asymptotically flat ends and ∞_0 is infinity in the chosen end. Define the level sets of $\phi(x)$ to be

$$\Sigma_l = \{x \mid \phi(x) = l\} \tag{21}$$

for $0 < l < 1$. Then it follows from Sard's theorem and the smoothness of $\phi(x)$ that $\Sigma(l)$ is a smooth surface for almost every l. Then by the co-area formula it follows that

$$\mathcal{E}(\bar{g}) = \frac{1}{2\pi}\int_0^1 dl \int_{\Sigma_l} |\nabla\phi|\, dA = \frac{1}{2\pi}\int_0^1 dl \int_{\Sigma_l} \frac{d\phi}{d\eta}\, dA \tag{22}$$

since $\nabla\phi$ is orthogonal to the unit normal vector η of Σ_l. But by the divergence theorem (and since $\phi(x)$ is harmonic), $\int_\Sigma \frac{d\phi}{d\eta}\, dA$ is constant for all homologous Σ. Hence,

$$\mathcal{E}(\bar{g}) = \frac{1}{2\pi}\int_\Sigma \frac{d\phi}{d\eta}\, dA \tag{23}$$

where Σ is any surface in M^3 in $\mathcal{S}$ (which is the set of smooth, compact boundaries of open regions which contains the points at infinity $\{\infty_k\}$ in all of the ends except the chosen one). Then since $(M^3, \bar{g})$ is harmonically flat at infinity, we know that in the chosen end $\phi(x) = 1 - c/|x| + \mathcal{O}(1/|x|^2)$, so that from equation 23 it follows that

$$\phi(x) = 1 - \frac{\mathcal{E}(\bar{g})}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right) \tag{24}$$

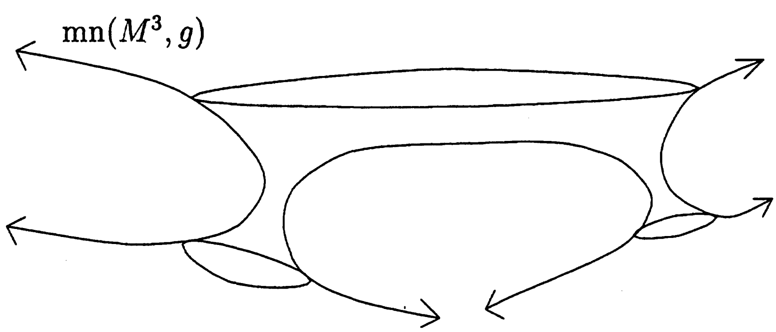
by letting Σ in equation 23 be a large sphere in the chosen end.

THEOREM 5. *Let $(M^3, \bar{g})$ be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature which has multiple asymptotically flat ends and total mass $\bar{m}$ in the chosen end. Then*

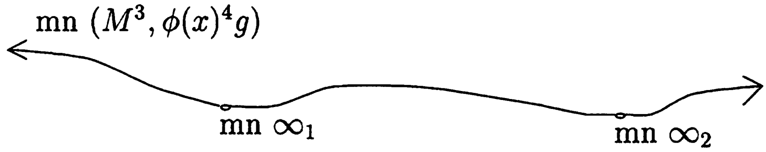
$$\bar{m} \geq \mathcal{E}(\bar{g}) \tag{25}$$

with equality if and only if $(M^3, \bar{g})$ has zero scalar curvature and is conformal to $(\mathbf{R}^3, \delta)$ minus a finite number of points.

Proof. Again, without loss of generality we will assume that $(M^3, \bar{g})$ is harmonically flat at infinity, which by definition means that all of the ends are harmonically flat at infinity. In the picture below, $(M^3, \bar{g})$ has three ends, the chosen end (at the top of the picture) and two other ends.



Then we consider the metric $(M^3, \tilde{g})$, with $\tilde{g} = \phi(x)^4 \bar{g}$, drawn below. Since $\phi(x)$ goes to zero (and is bounded above by $C/|x|$) in all of the harmonically flat ends other than the chosen one, the metric $\tilde{g} = \phi(x)^4 \bar{g}$ in each end is conformal to a punctured ball with the conformal factor being a bounded harmonic function to the fourth power in the punctured ball. Hence, by the removable singularity theorem, this harmonic function can be extended to the whole ball, which proves that the metric $\tilde{g}$ can be extended smoothly over all of the points at infinity in the compactified ends.



Furthermore, $(M^3, \tilde{g})$ has nonnegative scalar curvature since $(M^3, \bar{g})$ has nonnegative scalar curvature and $\phi(x)$ is harmonic with respect to $\bar{g}$ (see equation 95 in appendix E). Moreover, $(M^3 \cup \{\infty_k\}, \tilde{g})$ has nonnegative scalar curvature too since in a neighborhood of each ∞_k the manifold is conformal to a ball, with the conformal factor being a positive harmonic function to the fourth power. Hence, since $(M^3 \cup \{\infty_k\}, \tilde{g})$ is a complete 3-manifold with nonnegative scalar curvature with a single harmonically flat end, we may apply the Riemannian positive mass theorem to this manifold to conclude that the total mass of this manifold, which we will call $\tilde{m}$, is nonnegative.

Now we will compute $\tilde{m}$ in terms of $\bar{m}$ and $\mathcal{E}(\bar{g})$. Since $\bar{g}$ is harmonically flat at infinity, we know that by definition 2 we have $\bar{g} = \bar{\mathcal{U}}(x)^4 \bar{g}_{flat}$, where $(M^3, \bar{g}_{flat})$ is isometric to $(\mathbf{R}^3 \backslash B_r(0), \delta)$ in the harmonically flat end of $(M^3, \bar{g})$,

$$\bar{\mathcal{U}}(x) = 1 + \frac{\bar{m}}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right), \tag{26}$$

and the scale of the harmonically flat coordinate chart $\mathbf{R}^3 \backslash B_r(0)$ has been chosen so that $\bar{\mathcal{U}}(x)$ goes to one at infinity in the chosen end. Furthermore, since $\tilde{g} = \phi(x)^4 \bar{g}$ is also harmonically flat at infinity (which follows from equation 94 in appendix E), we have $\tilde{g} = \tilde{\mathcal{U}}(x)^4 \bar{g}_{flat} = \phi(x)^4 \bar{\mathcal{U}}(x)^4 \bar{g}_{flat}$ where

$$\tilde{\mathcal{U}}(x) = 1 + \frac{\tilde{m}}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{27}$$

Then comparing equations 24, 26, and 27 with $\tilde{\mathcal{U}}(x) = \bar{\mathcal{U}}(x)\phi(x)$ yields

$$\tilde{m} = \bar{m} - \mathcal{E}(\bar{g}) \geq 0 \tag{28}$$

by the Riemannian positive mass theorem, which proves inequality 25 for harmonically flat manifolds. Then since asymptotically flat manifolds can be arbitrarily well approximated by harmonically flat manifolds by lemma 1, inequality 25 follows for asymptotically flat manifolds as well.

To prove the case of equality, we require a generalization of the case of equality of the positive mass theorem given in [7] as theorem 5.3. In that paper, we say that a singular manifold has *generalized nonnegative scalar curvature* if it is the limit (in the sense given in [7]) of smooth manifolds with nonnegative scalar curvature.

Note that if $(M^3, \bar{g})$ is only asymptotically flat and not harmonically flat, then we have not shown that $(M^3, \tilde{g})$ can be extended smoothly over the missing points $\{\infty_k\}$. However, as a possibly singular manifold, it does have generalized nonnegative scalar curvature since it is the limit of smooth manifolds with nonnegative scalar curvature (since $(M^3, \bar{g})$ can be arbitrarily well approximated by harmonically flat manifolds).

Then theorem 5.3 in [7] states that if $(M^3, \bar{g})$ has generalized nonnegative scalar curvature, zero mass, and positive isoperimetric constant, then $(M^3, \bar{g})$ is flat (outside the singular set). Hence, it follows that if we have equality in inequality 25, then $\bar{g}$ is flat. Hence, $(M^3, \bar{g})$ is isometric to $(\mathbf{R}^3, \delta)$ minus a finite number of points, and since the harmonic conformal factor $\phi(x)$ preserves the sign of the scalar curvature by equation 95, the case of equality of the theorem follows. □

DEFINITION 12. *Given a complete, asymptotically flat manifold (M^3, g) with horizon $\Sigma \in \mathcal{S}$ (defined in section 1.2), define*

$$\mathcal{E}(\Sigma, g) = \inf_{\varphi}\left\{\frac{1}{2\pi}\int_{M^3} |\nabla\varphi|^2 \, dV\right\} \tag{29}$$

where the infimum is taken over all smooth $\varphi(x)$ which go to one at infinity and equal zero on the horizon Σ (and are zero inside Σ). (By definition of $\mathcal{S}$, all of the ends other than the chosen end are contained inside Σ.)

The infimum in the above definition is achieved by the Green's function $\varphi(x)$ which satisifies

$$\begin{cases} \lim_{x\to\infty_0} \varphi(x) &= 1 \\ \Delta\varphi &= 0 \\ \varphi(x) &= 0 \text{ on } \Sigma \end{cases} \tag{30}$$

and as before,

$$\varphi(x) = 1 - \frac{\mathcal{E}(\Sigma, g)}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right) \tag{31}$$

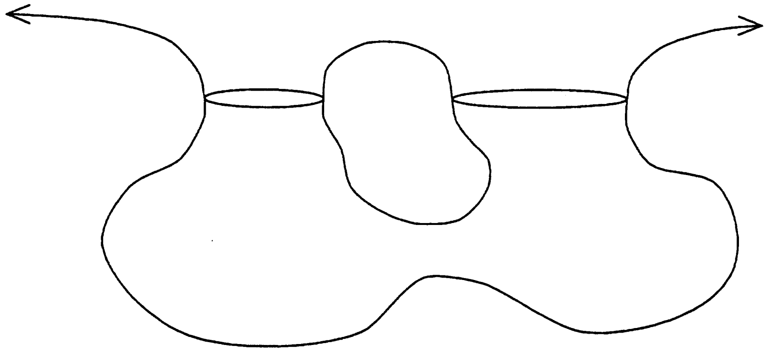
in the chosen end.

THEOREM 6. *Let (M^3, g) be a complete, smooth, asymptotically flat 3-manifold with nonnegative scalar curvature with a horizon $\Sigma \in \mathcal{S}$ and total mass m (in the chosen end). Then*

$$m \geq \frac{1}{2}\mathcal{E}(\Sigma, g) \tag{32}$$

with equality if and only if (M^3, g) is a Schwarzschild manifold outside the horizon Σ.

Proof. Let M^3_{Σ} be the closed region of M^3 which is outside (or on) Σ. Since $\Sigma \in \mathcal{S}$, (M^3_{Σ}, g) has only one end, and we recall that Σ could have multiple components. For example, in the picture below Σ has two components.



Then the basic idea is to reflect (M^3_Σ, g) through Σ to get a manifold $(\bar{M}^3_\Sigma, \bar{g})$ with two asymptotically flat ends. Then define $\phi(x)$ on $(\bar{M}^3_\Sigma, \bar{g})$ using equation 20 and $\varphi(x)$ on (M^3_Σ, g) using equation 30. It follows from symmetry that $\phi(x) = \frac{1}{2}$ on Σ, so that

$$\phi(x) = \frac{1}{2}(\varphi(x) + 1) \tag{33}$$

on (M^3_Σ, g). Then

$$\mathcal{E}(\bar{g}) = \frac{1}{2}\mathcal{E}(\Sigma, g) \tag{34}$$

so that theorem 6 follows from theorem 5.

The only technicality is that theorem 5 applies to smooth manifolds with nonnegative scalar curvature, and $(\bar{M}^3_\Sigma, \bar{g})$ is typically not smooth along Σ, which also makes it unclear how to define the scalar curvature there. However, it happens that because Σ has zero mean curvature, these issues can be resolved.

This idea of reflecting a manifold through its horizon is used by Bunting and Masood-ul-Alam in [**8**], and the issue of the smoothness of the reflected manifold appears in their paper as well. However, in their setting they have the simpler case in which the horizon not only has zero mean curvature but also has zero second fundamental form. Hence, the reflected manifold is $C^{1,1}$, which apparently is sufficient for their purposes.

However, in our setting we can not assume that the horizon Σ has zero second fundamental form, so that $(\bar{M}^3_\Sigma, \bar{g})$ is only Lipschitz. To solve this problem, given $\delta > 0$ we will define a smooth manifold $(\tilde{M}^3_{\Sigma,\delta}, \tilde{g}_\delta)$ with nonnegative scalar curvature which, in the limit as δ approaches zero, approaches $(\bar{M}^3_\Sigma, \bar{g})$ (meaning that there exists a diffeomorphism under which the metrics are arbitrarily uniformly close to each other and the total masses are arbitrarily close). Then by definition 11 it follows that $\mathcal{E}(\tilde{g}_\delta)$ is close to $\mathcal{E}(\bar{g})$, from which we will be able to conclude

$$m \approx \tilde{m}_\delta \geq \mathcal{E}(\tilde{g}_\delta) \approx \mathcal{E}(\bar{g}) = \frac{1}{2}\mathcal{E}(\Sigma, g), \tag{35}$$

where $\tilde{m}_\delta$ is the mass of $(\tilde{M}^3_{\Sigma,\delta}, \tilde{g}_\delta)$ and the approximations in the above inequality can be made to be arbitrarily accurate by choosing δ small, thereby proving inequality 32.

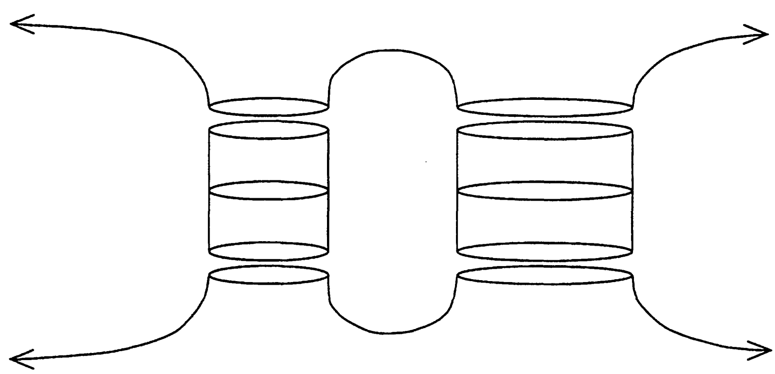
The first step is to constuct the smooth manifolds

$$(\tilde{M}^3_{\Sigma,\delta}, \tilde{g}_\delta) \approx (M^3_\Sigma, g) \cup (\Sigma \times [0, 2\delta], G) \cup (M^3_\Sigma, g), \tag{36}$$

where idenifications are made along the boundaries of these three manifolds as drawn below. (To be precise, the second (M^3_Σ, g) in the above union is meant to be a copy of the first (M^3_Σ, g) and therefore distinct.) We will define the metric G such that the metric $\bar{g}_\delta$ is smooth, although it will not have nonnegative scalar curvature. Then we will define

$$\tilde{g}_\delta = u_\delta(x)^4 \bar{g}_\delta \tag{37}$$

so that $\tilde{g}_\delta$ is not only smooth but also has nonnegative scalar curvature, and we will show that because of our choice of the metric G, $u_\delta(x)$ approaches one in the limit as δ approaches zero.



We will use the local coordinates (z,t) to describe points on $\Sigma \times [0, 2\delta]$, where $z = (z_1, z_2) \in$ a local coordinate chart for Σ and $t \in [0, 2\delta]$. Then we define $G(\partial_t, \partial_t) = 1$, $G(\partial_t, \partial_{z_1}) = 0$, and $G(\partial_t, \partial_{z_2}) = 0$. Then it follows that $\Sigma \times t$ is obtained by flowing $\Sigma \times 0$ in the unit normal direction for a time t, and that ∂_t is orthogonal to $\Sigma \times t$. Hence, all that remains to fully define the metric G is to define it smoothly on the tangent planes of $\Sigma \times t$ for $0 \le t \le 2\delta$.

Let $\bar{G}(z,t)$ be the metric G restricted to $\Sigma \times t$. Then

$$\frac{d}{dt}\bar{G}_{ij}(z,t) = 2\bar{G}_{ik}(z,t) h^k_j(z,t) \tag{38}$$

where $h^k_j(z,t)$ is the second fundamental form of $\Sigma \times t$ in $(\Sigma \times [0, 2\delta], G)$ with respect to the normal vector ∂_t. Furthermore, since $\Sigma \times 0$ is identified with $\Sigma \in (M^3_\Sigma, g)$, we can extend the coordinates (z,t) for t slightly less than zero into (M^3_Σ, g), thereby giving us smooth initial data for $\bar{G}_{ij}(z,t)$ and $h^k_j(z,t)$ for $-\epsilon < t \le 0$, for some positive ϵ.

Now we extend $h^k_j(z,t)$ smoothly for $0 \le t \le 2\delta$ in such a way that $h^k_j(z,t)$ is an odd function about $t = \delta$, meaning that $h^k_j(z,t) = -h^k_j(z, 2\delta - t)$. Naturally there are many ways to accomplish this smooth extension.

Then we define $\bar{G}_{ij}(z,t)$ to be the smooth solution to the o.d.e. given in equation 38 using the initial data for $\bar{G}_{ij}(z,t)$ at $t = 0$. By the oddness of $h^k_j(z,t)$ about $t = \delta$ it follows that $\bar{G}_{ij}(z,t)$ is symmetric about $t = \delta$, that is, $\bar{G}_{ij}(z,t) = \bar{G}_{ij}(z, 2\delta - t)$. Hence, the identification of $\Sigma \times (2\delta)$ with $\Sigma \in$ the second copy of (M^3_Σ, g) is smooth by symmetry. This completes the smooth construction of the metric $(\bar{M}^3_{\Sigma,\delta}, \bar{g}_\delta)$.

Now define $H(z,t) = \Sigma_j h_j^j(z,t)$ to be the mean curvature of $\Sigma \times t$ in $(\Sigma \times [0,2\delta], G)$, and let $\dot{H}(z,t) = \frac{d}{dt}H(z,t)$. We note that

$$H(z,0) = 0 = H(z,2\delta) \tag{39}$$

since Σ is a horizon (and hence has zero mean curvature) in (M_Σ^3, g). Let $\alpha = \sup_z \dot{H}(z,0)$ and let $\beta = \sup_z \sum_{jk} h_j^k(z,0)h_k^j(z,0)$, which we note are functions of the metric g on M_Σ^3 and are independent of δ. Then we require that the smooth extension we choose for h_j^k satisfies

$$\dot{H}(z,t) \le 2|\alpha| + 1, \tag{40}$$

(which is possible because of equation 39) and

$$\sum_{jk} h_j^k(z,t)h_k^j(z,t) \le 2\beta + 1. \tag{41}$$

Then combining equations 39 and 40 also yields

$$|H(z,t)| \le (2|\alpha| + 1)\delta. \tag{42}$$

These estimates allow us to bound the scalar curvature of $(\tilde{M}_{\Sigma,\delta}^3, \bar{g}_\delta)$ from below since by the second variation formula and the Gauss equation we have that

$$R = -2\dot{H} + 2K - |h|^2 - H^2 \tag{43}$$

where $R(z,t)$ is scalar curvature and $K(z,t)$ is the Gauss curvature of $\Sigma \times t$. At this point we realize that we also need a lower bound K_0 for $K(z,t)$ which is independent of δ, which follows from imposing an upper bound on the C^2 norm (in the z variable) of our smooth choice of $h_j^k(z,t)$. Then using this combined with inequalities 40, 41, and 42 we get

$$R(z,t) \ge R_0 \tag{44}$$

where R_0 is independent of δ (for $\delta < 1$).

Now we are ready to define $\tilde{g}_\delta$ using equation 37. We already know that $(\tilde{M}_{\Sigma,\delta}^3, \bar{g}_\delta)$ is smooth and has nonnegative scalar curvature everywhere except possibly in $\Sigma \times [0,2\delta]$ where it has $R \ge R_0$. If $R_0 \ge 0$, then we just let $u_\delta(x) = 1$ so that $\tilde{g} = \bar{g}$. Otherwise, we define $u_\delta(x)$ such that

$$(-8\Delta_{\bar{g}} + \mathcal{R}_\delta(x))u_\delta(x) = 0 \tag{45}$$

and $u_\delta(x)$ goes to one in both asymptotically flat ends, where $\mathcal{R}_\delta(x)$ equals R_0 in $\Sigma \times [0,2\delta]$, equals zero for x more than a distance δ from $\Sigma \times [0,2\delta]$, is smooth, and takes values in $[R_0, 0]$ everywhere. Then it follows that for sufficiently small δ, $u_\delta(x)$ is a smooth superharmonic function. Furthermore, since $\mathcal{R}_\delta$ is zero everywhere except on an open set whose volume is going to zero as δ goes to zero, and since $\mathcal{R}_\delta$ is uniformly bounded from below on this small set, it follows from bounding Green's functions from above that

$$1 \le u_\delta(x) \le 1 + \epsilon(\delta) \tag{46}$$

where ϵ goes to zero as δ approaches zero.

Furthermore, by equations 44, 37, and 95, $(\tilde{M}_{\Sigma,\delta}^3, \tilde{g}_\delta)$ has nonnegative scalar curvature, and since $u_\delta(x)$ and and $(\tilde{M}_{\Sigma,\delta}^3, \bar{g}_\delta)$ are smooth, $(\tilde{M}_{\Sigma,\delta}^3, \tilde{g}_\delta)$ is too. In addition, it follows from the construction of $(\tilde{M}_{\Sigma,\delta}^3,\ \tilde{g}_\delta)$ that there exists a diffeomorphism into $(\bar{M}^3, \bar{g})$ with respect to which the metrics are arbitrarily uniformly close to each other in the limit as δ goes to zero. Hence, by equation 46, we see that the same statement is true for $(\tilde{M}^3_{\Sigma,\delta}, \tilde{g}_\delta)$. Finally, it follows from equation 45 that $\tilde{m}_\delta$, the mass of $(\tilde{M}^3_{\Sigma,\delta}, \tilde{g}_\delta)$, converges to m, the mass of $(\bar{M}^3_\Sigma, \bar{g})$, in the limit as δ goes to zero. Hence, inequality 35 follows, proving inequality 32.

To prove the case of equality, we note that we can view the above proof in a different way. Since the singular manifold $(\bar{M}^3_\Sigma, \bar{g})$ is the limit of the smooth manifolds $(\tilde{M}^3_{\Sigma,\delta}, \tilde{g}_\delta)$ which have nonnegative scalar curvature, it follows that $(\bar{M}^3_\Sigma, \bar{g})$ has generalized nonnegative scalar curvature as defined in [**7**]. Then if we reexamine the proof of theorem 5, we see that the theorem, including the case of equality, is also true for singular manifolds like $(\bar{M}^3_\Sigma, \bar{g})$ which have generalized nonnegative scalar curvature (see the discussion at the end of the proof of theorem 5). Hence, by equation 34 and theorem 5, we get equality in inequality 32 if and only if $(\bar{M}^3_\Sigma, \bar{g})$ has zero scalar curvature and is conformal to $(\mathbf{R}^3, \delta)$ minus a finite number of points.

Since $(\bar{M}^3_\Sigma, \bar{g})$ has two ends, it must be conformal to $(\mathbf{R}^3\backslash\{0\}, \delta)$, and since it has zero scalar curvature, it follows from equation 95 that it must be a Schwarzschild metric. Hence, in the case of equality for inequality 32, (M^3, g) must be a Schwarzschild manifold outside Σ. □

2.3. Proof That $m(t)$ Is Nonincreasing. In this subsection we will prove that $m(t)$, the total mass of (M^3, g_t), is non-increasing in t, as claimed in theorem 3. The fact that $m(t)$ is nonincreasing is of course central to the argument presented in this section for proving the Riemannian Penrose conjecture and is perhaps the most important property of the conformal flow of metrics $\{g_t\}$.

Recalling definition 9 from section 2.1, we begin by stating a lemma proved in [**6**].

LEMMA 2. *The left and right hand derivatives $\frac{d}{dt^\pm}$ of $u_t(x)$ exist for all $t > 0$ and are equal except at a countable number of t-values. Furthermore,*

$$\frac{d}{dt^+} u_t(x) = v_t^+(x) \tag{47}$$

and

$$\frac{d}{dt^-} u_t(x) = v_t^-(x) \tag{48}$$

where $v_t^\pm(x)$ equals zero inside $\Sigma^\pm(t)$ (see definition 9) and outside $\Sigma^\pm(t)$ is the harmonic function which equals 0 on $\Sigma^\pm(t)$ and goes to $-e^{-t}$ at infinity.

We will use this lemma to compute the left and right hand derivatives of $m(t)$. As proven at the end of appendix E, the flow of metrics $\{g_t\}$ we are considering has the property that the rate of change of the metric g_t is just a function of g_t and not of t or g_0. Hence, we will just prove that $m'(0) \leq 0$, from which it will follow that $m'(t) \leq 0$. So without loss of generality, we will assume that the flow begins at some time $-t_0 < 0$, and then compute the left and right hand derivatives of $m(t)$ at $t = 0$.

Also, in [**6**] we prove that $\Sigma^+(t)$ and $\Sigma^-(t)$ are horizons in (M^3, g_t). Furthermore, $v_0^\pm(x)$ is harmonic in (M^3, g_0), equals 0 on $\Sigma^\pm(0)$, and goes to -1 at infinity. Hence, by equation 31 and theorem 6 of the previous subsection,

$$v_0^\pm(x) = -1 + \frac{\mathcal{E}(\Sigma^\pm(0), g_0)}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right) \tag{49}$$

where

$$m(0) \geq \frac{1}{2}\mathcal{E}(\Sigma^{\pm}(0), g_0). \tag{50}$$

Now we are ready to compute $m'(t)$. As in section 1.2, let $g_0 = \mathcal{U}_0(x)^4 g_{flat}$, where (M^3, g_{flat}) is isometric to $(\mathbf{R}^3 \backslash B_r(0), \delta)$ in the harmonically flat end, where we have chosen r and scaled the harmonically flat coordinate chart such that $\mathcal{U}_0(x)$ goes to one at infinity. Then by definition 2 for the total mass, we have that

$$\mathcal{U}_0(x) = 1 + \frac{m(0)}{2|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{51}$$

We will also let $g_t = \mathcal{U}_t(x)^4 g_{flat}$ in the harmonically flat end. Then since $g_t = u_t(x)^4 g_0$, it follows that

$$\mathcal{U}_t(x) = u_t(x)\,\mathcal{U}_0(x). \tag{52}$$

Now we define $\alpha(t)$ and $\beta(t)$ such that

$$u_t(x) = \alpha(t) + \frac{\beta(t)}{|x|} + \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{53}$$

Then since $u_0(x) \equiv 1$ and $\frac{d}{dt^\pm}\, u_t(x)|_{t=0} = v_0^{\pm}(x)$, it follows from equation 49 that

$$\begin{aligned} \alpha(0) &= 1, & \tfrac{d}{dt^\pm}\, \alpha(t)|_{t=0} &= -1, \\ \beta(0) &= 0, & \tfrac{d}{dt^\pm}\, \beta(t)|_{t=0} &= \tfrac{1}{2}\mathcal{E}(\Sigma^{\pm}(0), g_0). \end{aligned} \tag{54}$$

Thus, by equation 52,

$$\mathcal{U}_t(x) = \alpha(t) + \frac{1}{|x|}\left(\beta(t) + \frac{m(0)}{2}\alpha(t)\right) + \mathcal{O}\left(\frac{1}{|x|^2}\right) \tag{55}$$

so that by definition 2

$$m(t) = 2\alpha(t)\left(\beta(t) + \frac{m(0)}{2}\alpha(t)\right). \tag{56}$$

Hence, by equation 54

$$\frac{d}{dt^\pm}\, m(t)|_{t=0} \;=\; \mathcal{E}(\Sigma^{\pm}(0), g_0) - 2m(0) \;\leq\; 0 \tag{57}$$

by equation 50. Then since we were able to choose $t = 0$ without loss of generality as previously discussed, we have proven the following theorem.

THEOREM 7. *The left and right hand derivatives $\frac{d}{dt^\pm}$ of $m(t)$ exist for all $t > 0$ and are equal except at a countable number of t-values. Furthermore,*

$$\frac{d}{dt^\pm}\, m(t) \leq 0 \tag{58}$$

for all $t > 0$ (and the right hand derivative of $m(t)$ at $t = 0$ exists and is nonpositive as well).

Hence, $m'(t)$ exists almost everywhere, and since $m(t)$ is continuous and its left and right hand derivatives are all nonpositive, $m(t)$ is nonincreasing.

2.4. Generalization to Asymptotically Flat Manifolds and the Case of Equality. Up to this point we have assumed that (M^3, g_0) was harmonically flat at infinity. In particular, theorems 2, 3, and 4 only apply to harmonically flat manifolds as stated. In this subsection, we will extend theorems 2 and 3 and elements of theorem 4 to asymptotically flat manifolds. This will prove the main theorem, theorem 1, except for the case of equality, which we will see follows from the case of equality of theorem 6.

It is worth noting that the main reason for initially considering only harmonically flat manifolds was convenience. Alternatively, we could have ignored harmonically flat manifolds and dealt only with asymptotically flat manifolds. However, this would have complicated some of the arguments unnecessariy, so we chose to delay these considerations until now.

DEFINITION 13. *(M^n, g) is said to be* **asymptotically flat** *if there is a compact set $K \subset M$ such that $M \backslash K$ is the disjoint union of ends $\{E_k\}$, such that for each end there exists a diffeomorphism $\Phi_k : E_k \to \mathbf{R}^n \backslash\ B_1(0)$ such that, in the coordinate chart defined by Φ_k,*

$$g = \sum_{i,j} g_{ij}(x) dx^i dx^j$$

where

$$g_{ij}(x) = \delta_{ij} + O(|x|^{-p})$$

$$|x||g_{ij,k}(x)| + |x|^2 |g_{ij,kl(x)}| = O(|x|^{-p})$$

$$|R(g)| = O(|x|^{-q})$$

for some $p > \frac{n-2}{2}$ and some $q > n$, where we have used commas to denote partial derivatives in the coordinate chart, and $R(g)$ is the scalar curvature of (M^n, g).

These assumptions on the asymptotic behavior of (M^n, g) at infinity imply the existence of the limit

$$M_{ADM}(g) = (4\omega_{n-1})^{-1} \lim_{\sigma \to \infty} \int_{S_\sigma} \sum_{i,j} (g_{ij,i}\nu_j - g_{ii,j}\nu_j)\, d\mu \tag{59}$$

where $\omega_{n-1} = Vol(S^{n-1}(1))$, S_σ is the coordinate sphere of radius σ, ν is the unit normal to S_σ, and $d\mu$ is the area element of S_σ in the coordinate chart. The quantity M_{ADM} is called the **total mass** of (M^n, g) (see **[1]**, **[2]**, **[29]**, and **[33]**), and agrees with the definition of total mass for harmonically flat 3-manifolds given in definition 2.

It turns out that the arguments in the proof given in **[6]** of the existence theorem, theorem 2, do not use harmonic flatness anywhere, so we immediately get existence of the conformal flow of metrics for asymptotically flat manifolds. Similarly, the arguments used to prove that $A(t)$ is constant still hold. Next we reexaming the proof of theorem 7 which proved that $m(t)$ was nonincreasing. The only modification we need to make is to use the more general definition for the total mass of an asymptotically flat manifold given by equation 59. It is then straight forward to check that equation 57 and hence theorem 7 are still true. Hence,

THEOREM 8. *Theorems 2 and 3 are true for asymptotically flat manifolds as well as harmonically flat manifolds.*

We choose not to extend theorem 4 to asymptotically flat manifolds, but conjecture that it is still true. Instead, we observe that we must still have

$$m(t) \geq \sqrt{\frac{A(t)}{16\pi}} \tag{60}$$

for asymptotically flat manifolds. Otherwise, given an asymptotically flat counterexample, we could use lemma 1 to perturb the manifold slightly making it harmonically flat at infinity such that it still violated equation 60. Then applying the conformal flow of metrics to this harmonically flat manifold would violate theorems 3 and 4, which is a contradiction. Setting $t = 0$ in inequality 60 then proves the Riemannian Penrose inequality for asymptotically flat manifolds.

The case of equality of theorem 1 then follows from equation 57 and theorem 6. If we have equality in the Riemannian Penrose inequality, then applying the conformal flow of metrics to this initial metric must also give equality in inequality 60 for all $t \geq 0$. Hence, the right hand derivative of $m(t)$ at $t = 0$ equals zero, so by equation 57,

$$\mathcal{E}(\Sigma^+(0), g_0) \;=\; 2m(0). \tag{61}$$

By definition 9 and equation 13, $\Sigma^+(0)$ is the outermost minimal area enclosure of Σ_0 in (M^3, g). Furthermore, by the case of equality of theorem 6, (M^3, g) is a Schwarzschild manifold outside $\Sigma^+(0)$. Hence, $\Sigma^+(0)$ is the outermost horizon of (M^3, g), so (M^3, g) is isometric to a Schwarzschild manifold outside their respective outermost horizons. This proves the case of equality for theorem 1 and consequently the Riemannian Penrose inequality as well.

The reader might also have noticed that none of the arguments in this section have used anything about the original manifold inside the original horizon Σ_0. Hence, we can generalize theorem 1 to the following.

THEOREM 9. *Let (M^3, g) be a complete, smooth 3-manifold with boundary which has nonnegative scalar curvature and a single asymptotically flat end with total mass m. Then if the boundary is an outer-minimizing horizon (with one or more components) of total area A,*

$$m \geq \sqrt{\frac{A}{16\pi}} \tag{62}$$

with equality if and only if (M^3, g) is isometric to a Schwarzschild manifold outside their respective outermost horizons.

3. The Huisken-Ilmanen Proof of the Penrose Conjecture Using Inverse Mean Curvature Flows

We will now describe the approach taken by Huisken and Ilmanen[**19**] to establish the Penrose conjecture in the slightly weaker form.

THEOREM 10. *Let M be a complete, connected, asymptotically flat 3-manifold with nonnegative scalar curvature. Assume that the boundary is compact and consists of minimal surfaces, and that M contains no other compact minimal surfaces. Then $m \geq 0$, and*

$$16\pi m^2 \geq |N| \tag{63}$$

where $|N|$ is the area of any connected component of ∂M. Equality holds if and only if M is one-half of the spatial Schwarzschild manifold.

The main difference between this theorem and the full Riemannian Penrose conjecture is that here the lower bound on the mass is given in terms of the largest of the areas of the boundary components, while in the full conjecture it is given in terms of the total area of the boundary components. On the other hand, **[19]** does establish the conjecture in great generality, and it precedes the work of Bray **[6]** which proves the full conjecture. Furthermore the strategy employed in the Huisken/Ilmanen proof is totally different from that of Bray, and it is likely that their approach will shed light on other related questions concerning scalar curvature in three dimensions; indeed we describe an application to the positivity of a quasi-local mass defined by Bartnik**[3]**. Their approach involves the study of an evolution equation for compact 2-surfaces in M. This approach has a long history, and we now proceed to describe this history and the work of **[19]**.

3.1. Quasi-Local Mass. A central problem in general relativity is to understand how to quantify the amount of mass inside a given region. In appendix C, we define, in the spherically symmetric case, a function $m(V)$ for a 2-sphere containing a volume V which is increasing as a function of V beyond the outermost minimal sphere. Furthermore, for large V, $m(V)$ equals the total mass of the manifold. Hence, it seems reasonable to say that the spherically symmetric sphere $\Sigma(V)$ defined in appendix C contains a mass $m(V)$. The function $m(V)$ should be called a quasi-local mass function.

Naturally we would like to define a quasi-local mass function which would measure the amount of mass inside any surface Σ which is the boundary of a region in any 3-manifold M^3 (with $R \geq 0$). We refer the reader to **[10]**, **[3]** ,**[11]**, and **[13]** for more discussion on this topic.

A simple definition of quasi-local mass was proposed by Hawking and is called the Hawking mass. Referring to appendix C, we recall that in the spherically symmetric case,

$$m(V) = \left(\frac{A(V)}{16\pi}\right)^{\frac{1}{2}} \left(1 - \frac{1}{16\pi} A(V) A'(V)^2\right) \tag{64}$$

It so happens that $A'(V) = H(V)$, where $H(V)$ is the mean curvature of the spherically symmetric sphere $\Sigma(V)$. Hence, one way to generalize equation 64 is to define

$$m(\Sigma) = \left(\frac{A}{16\pi}\right)^{\frac{1}{2}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2\right) \tag{65}$$

where A is the area of Σ and H is the mean curvature of Σ in M^3. It turns out that this definition of quasi-local mass, first defined by Hawking, has some very important properties, and seems to be especially natural in case the surface Σ is of least area for the volume which it bounds. As we shall see, the Huisken/Ilmanen approach employs this definition for surfaces in their evolution, and shows that it has the correct monotonicity properties.

Another definition of quasi-local mass which has the potential to be meaningful for more general surfaces was given by Bartnik**[3]**. The Bartnik mass is defined for any compact 3-manifold Ω with boundary and with nonnegative scalar curvature which can be embedded as a domain in an asymptotically flat manifold M with $R \geq 0$ in such a way that $M \setminus \Omega$ contains no compact minimal surfaces. The Bartnik mass $m_B(\Omega)$ is then defined to be the infimum of the ADM masses of all possible such M which contain an isometric embedding of Ω as a subdomain. The fact that $m_B(\Omega) \geq 0$ follows from the positive mass theorem, but it is not clear the this quantity is positive since it is an infimum taken over a very large set of manifolds. The work of Huisken/Ilmanen implies the following result.

THEOREM 11. *For any Ω for which the Batnik mass is defined, it is true that $m_B(\Omega) > 0$ unless Ω is flat (hence locally isometric to a domain in $\mathbf{R}^3$).*

3.2. Inverse Mean Curvature Flow and Geroch Monotonicity. In [**11**], Geroch showed that if M^3 has nonnegative scalar curvature, then the Hawking mass is nondecreasing when the surface Σ is flowed outward at a speed equal to the inverse of its mean curvature. In view of this, Jang and Wald proposed using the Hawking mass function to prove the Penrose inequality [**26**]. They suggested that we should let $\Sigma(0)$ be an outermost horizon and then to flow out using an inverse mean curvature flow to create a family of surfaces $\Sigma(t)$ flowing out to infinity. Since the Hawking mass function $m(t) = m(\Sigma(t))$ is nondecreasing as a function of t, we have

$$\lim_{t\to\infty} m(t) \geq m(0) \tag{66}$$

Furthermore $m(0) = \sqrt{\frac{A}{16\pi}}$, where A is the area of the outermost horizon $\Sigma(0)$, since horizons have zero mean curvature. Hence, Jang and Wald proposed a proof of the Penrose inequality which is naturally a generalization of the proof which works in the spherically symmetric case.

The main difficulty in making this formal argument rigorous is to prove the existence and regularity for the inverse mean curvature flow. Naturally, if the mean curvature of the surface ever went to zero or became negative, the flow could not exist, at least in this form. Indeed, in [**19**] it is observed that if the initial surface Σ_0 in $\mathbf{R}^3$ is taken to be the boundary of a small tubular neighborhood of a standard circle, then Σ_0 will flow outward for a short time resulting in a fattening of the torus. At some time the mean curvature must go to zero, and the flow must become singular. The main difficulty in this approach is how to define the flow after times at which it becomes singular. In the next section we discuss how Huisken and Ilmanen solved this problem.

3.3. The Huisken-Ilmanen Proof. Recall that the smooth flow which we are studying is the inverse mean curvature flow given by

$$\frac{\partial x}{\partial t} = H^{-1}\nu(x), \quad x \in \Sigma_t \tag{67}$$

with initial condition Σ_0 specified to be a slight inward perturbation of the boundary surface with $H > 0$. (The condition that M contains no other compact minimal surfaces implies that the boundary surfaces are local minima for area, and guarantees that it is possible to find nearby surfaces in M with $H > 0$.) In (67) ν denotes the inward normal vector to M (opposite to the direction of the mean curvature vector). As we have seen, there is little hope that solutions of this flow will remain smooth as surfaces in M and continuous in t. To resolve this difficulty (and others) [**19**] rewrite the flow as an equation for the level sets of a function $u(x)$. First if we assume that the level sets of u are smooth surfaces and $|\nabla u| \neq 0$, the equation becomes

$$\operatorname{div}_M \left(\frac{\nabla u}{|\nabla u|} \right) = |\nabla u|. \tag{68}$$

Instead of being an evolution equation, (68) is a degenerate elliptic equation. It is degenerate because the leading order part is invariant under the replacement of u by $f(u)$ since it measures the mean curvature of the level surfaces of u. The advantage of looking at a level set formulation of the flow is that it becomes possible for the level sets to "fatten", and in this way to jump in space. This is an essential feature of any weak formulation of the flow, as examples such as the torus example suggest.

Since we cannot expect solutions to be smooth, we may formulate a notion of weak solution. Since the left hand side of (68) is the Euler-Lagrange operator obtained from the functional $\int_M |\nabla u|\, dx$, it is natural to consider freezing the right hand side as $|\nabla u|$, and considering the functional $J_u(v) := \int_K (|\nabla v| + v|\nabla u|)\, dx$ where K is a compact subset of M. We may then seek minima of this functional for a given u and hope that for a suitable choice of u, the function u will achieve the minimum. In any event, we make the following definition.

DEFINITION 14. *A locally Lipschitz function u on an open set Ω is called a weak solution of (68) on Ω if*

$$J_u^K(u) \leq J_u^K(v) \tag{69}$$

for every locally Lipschitz function v such that $u - v$ has compact support in Ω and K is any compact subset of Ω containing the support of $u - v$.

We can think of the level sets $E_t = \{x : u(x) < t\}$ as evolving, and one can impose initial conditions on the solutions by choosing an open set E_0. We say that u is a solution of the initial value problem with initial condition E_0 if u is a locally Lipschitz solution of (69) in $M \setminus E_0$, and $E_0 = \{u < 0\}$.

In order to come to a better geometric understanding of weak solutions, it is useful to reformulate the condition strictly in terms of level sets. This can be done in the following way. Given a locally Lipschitz function u, we may define the functional $J_u(\cdot)$ on measurable sets F by

$$J_u^K(F) = |\partial^* F \cap K| - \int_{F \cap K} |\nabla u| \tag{70}$$

where K is a compact subset of M and $|\partial^* F|$ denotes the 2-dimensional Hausdorff measure of the reduced boundary of F (assuming that F is a set of finite perimeter). We then say that a set E minimizes J_u in a set A if

$$J_u^K(E) \leq J_u^K(F)$$

for any set F with symmetric difference $F \Delta E \subseteq K$, and K compact in A. The following result holds ([**19**]).

LEMMA 3. *A locally Lipschitz function u satisfies (69) in an open set Ω if and only if for each t, E_t minimizes J_u in Ω.*

The above lemma allows us to use known regularity results to obtain the following regularity results for solutions of (69) with intitial condition E_0. Precisely, if ∂E_0 is $C^{1,\alpha}$ for some $\alpha > 0$, then for each $t \geq 0$ we have ∂E_t is also $C^{1,\alpha}$; in fact, we also have $\partial\{u > t\}$ is $C^{1,\alpha}$. This will be important for establishing monotonicity of the Hawking mass.

Another important geometric property of weak solutions which is essential for proving the Penrose conjecture is that the sets E_t are *area minimizing hulls.* This should be thought of as a mean curvature analogue of convexity. If E is a set and Ω is an open set, we say that E is a *minimizing hull* in Ω if E minimizes boundary area for sets containing E. Precisely, if

$$|\partial^* E \cap K| \leq |\partial^* F \cap K|$$

for any F containing E such that $F \setminus E \subset K$ and K is a compact set of Ω.

The existence and uniqueness theorem for the initial value problem can be stated.

THEOREM 12. *Assume that M is an asymptotically flat 3-manifold without boundary. For any nonempty bounded smooth open set E_0, there exists a proper, locally Lipschitz solution of (69) satisfying the initial condition E_0. Furthermore this solution is unique in $M \setminus E_0$.*

This solution is constructed by an elliptic regularization procedure. The idea is to change the equation (68) to the following

$$\operatorname{div}_M \left(\frac{\nabla u^\epsilon}{\sqrt{|\nabla u^\epsilon|^2 + \epsilon^2}} \right) - \sqrt{|\nabla u^\epsilon|^2 + \epsilon^2} = 0. \tag{71}$$

This equation has the following geometric meaning. It is simply the equation which says that the family of hypersurfaces $N_t = \text{graph}(\epsilon^{-1}\ (u^\epsilon - t))$ is a solution of the inverse mean curvature flow in $M \times \mathbf{R}$. The approach of [**19**] is then to solve (71) with $\epsilon > 0$ small on $B_R \setminus E_0$ where R is chosen large, and the boundary conditions $u^\epsilon = 0$ on ∂E_0 and $u^\epsilon = R$ on ∂B_R are imposed. A uniform gradient estimate on such solutions is then derived and it is shown that the u^ϵ converge to a proper solution of (69) with initial conditions E_0.

Another important property of the flow is that if ∂E_0 is connected, then it is true that ∂E_t is connected for all $t > 0$ as well. This follows from the maximum principle as a disconnected level set could only occur is u had an interior maximum point. Furthermore, the large time asymptotics of the flow can be analyzed, and for large t it is shown that ∂E_t is smooth and sufficiently close to a large coordinate sphere that the Hawking mass converges to the ADM mass.

Finally, to prove the monotonicity of the Hawking mass under this weak flow, an analagous formula is derived for the smooth graphical flow given by the graph of u^ϵ, and a delicate limiting argument is used to show the monotonicity for the actual limiting flow. The formal argument used by Geroch, Jang, and Wald can then be applied for the Huisken/Ilmanen flow, and this establishes the Penrose conjecture in case there is a single horizon (boundary component of M).

To handle the case of several horizons, Huisken and Ilmanen run the flow starting from one chosen horizon, leaving the others fixed. They then show that there is no loss of Hawking mass at the times at which this flow jumps across the other horizons. This is done by flowing until E_t almost touches another horizon and then taking the minimizing hull of E_t and that horizon. They show that the Hawking mass does not decrease in this process, and the minimizing hull has connected boundary. In this way they engulf all of the horizons, and show that the Penrose inequality holds where the area is the maximum area of the horizons.

4. New Quasi-Local Mass Functions

In this section we define several new quasi-local mass functions following [**6**]. The idea of a quasi-local mass function is that it should define how much mass is in a given region (or, equivalently, inside a given surface) of (M^3, g). When the region is very large, the quasi-local mass function should approach the total mass of (M^3, g). In addition, a reasonable quasi-local mass function should have some kind of monotonicity property, meaning that if one region contains another region, then it should have larger mass.

In [**19**], Huisken and Ilmanen used the Hawking mass to prove the Riemannian Penrose inequality for a single horizon. In fact, they proved that the Hawking mass of a surface is nondecreasing with respect to their generalized inverse mean curvature flow. With this in mind, it is not too surprising that the other proof the Riemannian Penrose inequality given in section 2 motivates yet another definition of quasi-local mass $m_g(\Sigma)$ (definition 15). This quasi-local mass function is also nondecreasing with respect to a certain family of surfaces, which we will describe. We then end the section by defining two more new quasi-local mass functions, $\tilde{m}_g(\Sigma)$ and $m_{inner}(\Sigma)$, which have very good monotonicity properties but are typically impractical to compute.

Let (M^3, g) be a complete asymptotically flat manifold with nonnegative scalar curvature and total mass m. Let Σ be any surface in M^3 which is in the class of surfaces $\mathcal{S}$ defined in section 1.2.

DEFINITION 15. *Suppose $u(x)$ is a positive harmonic function in (M^3, g) outside Σ going to a constant at infinity scaled such that $(M^3, u(x)^4 g)$ has the same total mass as (M^3, g).*

Then if Σ is an outer-minimizing horizon with area A in $(M^3, u(x)^4 g)$, we define the quasi-local mass of Σ in (M^3, g) to be

$$m_g(\Sigma) = \sqrt{\frac{A}{16\pi}}. \tag{72}$$

DEFINITION 16. *We define S to be the subset of $\mathcal{S}$ of surfaces Σ for which such a conformal factor $u(x)$ exists, and we note that (by equation 94 mostly) this conformal factor is unique for each Σ when it exists.*

As usual Σ could have multiple components. It is also interesting that

$$m_g(\tilde{\Sigma}) = m_{u(x)^4 g}(\tilde{\Sigma}) \tag{73}$$

for all surfaces $\tilde{\Sigma} \in S$ where the conformal factor $u(x)$ is any harmonic function in (M^3, g) defined outside $\tilde{\Sigma}$ which goes to a constant at infinity scaled such that $(M^3, u(x)^4 g)$ has the same total mass as (M^3, g).

LEMMA 4. *The quasi-local mass function $m_g(\Sigma)$ defined for (M^3, g) is monotone increasing for the family of surfaces $\Sigma(t)$ defined by equation 13. That is, $m_g(\Sigma(t))$ is nondecreasing in t. Furthermore,*

$$m_g(\Sigma(0)) = \sqrt{\frac{A}{16\pi}} \tag{74}$$

where A is the area of the original outer-minimizing horizon Σ_0 in (M^3, g), and

$$\lim_{t \to \infty} m_g(\Sigma(t)) = m, \tag{75}$$

the total mass of (M^3, g).

Proof. We consider the conformal flow of metrics g_t beginning with (M^3, g) defined in section 2. Then we note that by equation 13, $\Sigma(t)$ is an outer-minimizing horizon in $(M^3, u_t(x)^4 g)$ with area $A(t)$. Hence, $u_t(x)$ satisfies the conditions in definition 15 except that it is not scaled to have the correct mass. Hence, since mass has units of length, it follows that

$$m_g(\Sigma(t)) = \frac{m}{m(t)}\sqrt{\frac{A(t)}{16\pi}} \tag{76}$$

where again m is the total mass of (M^3, g). Then the lemma follows from theorems 3 and 4 and the fact that $m(0) = m$.

There is a trick which allows us to extend the definition of this quasi-local mass function to all surfaces in $\mathcal{S}$.

DEFINITION 17. *Define*

$$\tilde{m}_g(\Sigma) = \sup\left\{m_g(\tilde{\Sigma}) \mid \Sigma \text{ (entirely) encloses } \tilde{\Sigma} \in \mathcal{S}\right\} \tag{77}$$

where Σ is any surface in $\mathcal{S}$.

It follows trivially that $\tilde{m}_g(\Sigma)$ is monotone with respect to enclosure, which is a desirable property for quasi-local mass functions to have since larger regions should contain more mass in the nonnegative energy density setting which we are in. We also note that this same construction can be used with the Hawking mass to make it monotone with respect to enclosure too, where the original Hawking mass should only be defined to exist for surfaces which equal their own outermost minimal area enclosures, as motivated by the results of Huisken and Ilmanen in [**19**].

We also notice that the existence of the Penrose inequality allows us to define another new quasi-local mass function which is similar in nature to the Bartnik mass [**3**]. In fact, whereas the Bartnik mass could also be called an outer quasi-local mass function, it makes sense to call the new quasi-local mass function defined below the inner quasi-local mass function, which is clear from the definition.

DEFINITION 18. *Given a surface $\Sigma \in \mathcal{S}$ in (M^3, g), consider all other asymptotically flat, complete, Riemannian manifolds $(\tilde{M}^3, \tilde{g})$ with nonnegative scalar curvature which are isometric to (M^3, g) outside Σ. Then we define*

$$m_{inner}(\Sigma) = \sup\sqrt{\frac{\tilde{A}}{16\pi}} \tag{78}$$

where $\tilde{A}$ is the infimum of the areas of all of the surfaces in $(\tilde{M}^3, \tilde{g})$ in $\tilde{\mathcal{S}}$.

We note here that $\tilde{\mathcal{S}}$ is defined the same way as $\mathcal{S}$ in definition 3 and that the surface in $\tilde{\mathcal{S}}$ with minimum area may have multiple components. We also note that for $\tilde{A}$ to be nonzero that $(\tilde{M}^3, \tilde{g})$ must have more than one asymptotically flat end.

LEMMA 5. *Let (M^3, g) be an asymptotically flat, complete, Riemannian manifold with nonnegative scalar curvature, and let $\Sigma_1, \Sigma_2 \in \mathcal{S}$ such that Σ_2 (entirely) encloses Σ_1. Then*

$$m \geq m_{inner}(\Sigma_2) \geq m_{inner}(\Sigma_1) \tag{79}$$

where m is the total mass of (M^3, g).

Proof. Follows directly from the Penrose inequality and definition 18.

Also, if Σ is outer-minimizing (see definition 6), then

$$m_{outer}(\Sigma) \geq m_{inner}(\Sigma) \tag{80}$$

where $m_{outer}(\Sigma)$ is basically the Bartnik mass [**3**] except that we only consider extensions of the metric in which Σ continues to be outer-minimizing. The proof of this inequality and related discussions will be included in a paper on quasi-local mass which is currently in progress.

Appendix A. Motivation behind the Penrose Conjecture

In 1973, Roger Penrose proposed the Penrose inequality as a test of the cosmic censor hypothesis [**27**]. The cosmic censor hypothesis states that naked singularities do not develop starting with physically reasonable nonsingular generic initial conditions for the Cauchy problem in general relativity. (However, it has been shown by Christodoulou [**9**] that naked singularities can develop from nongeneric initial conditions.) If naked singularities did typically develop from generic initial conditions, then this could be a serious problem for general relativity as a theory since it might not be possible to solve the Einstein equations uniquely past these singularities. Singularities such as black holes do develop but are shielded from observers at infinity by their horizons so that the Einstein equations can still be solved from the point of view of an observer at infinity.

A summary of Penrose's argument can be found in [**26**]. The main idea is to consider a space-time $(N^4, \bar{g})$ with given initial conditions for the Cauchy problem (M^3, g, h) which satisfy the constraint equations given in equations 2, 3, and 4. To get the Riemannian Penrose conjecture, we make the additional assumption that the second fundamental form $h \equiv 0$, which by the constraint equations implies that (M^3, g) has nonnegative scalar curvature. We also suppose that (M^3, g) has outermost apparent horizons of total area A, event horizons of total area A_i, and total mass m_i (see [**16**], [**17**] for the definitions of these horizons). As long as a singularity does not form, then it is assumed that eventually the space-time should converge on some stationary final state, presumably a Kerr solution ([**21**], [**15**], [**28**]), which satisfies

$$A_f = 8\pi[m_f^2 + (m_f^4 - J^2)^{\frac{1}{2}}] \leq 16\pi m_f^2, \tag{81}$$

where A_f is the area of the horizon of the Kerr black hole, m_f is the mass at infinity, and J is the angular momentum.

However, by the Hawking area theorem [**14**], the total area of the event horizons of the black holes is nondecreasing. Thus, $A_f \geq A_i$. Also, presumably some energy radiates off to infinity, so we expect to have $m_i \geq m_f$.

Apparent horizons are defined to be the outer boundaries of regions in M^3 which contain trapped or marginally trapped surfaces [**16**]. Apparent horizons themselves must then be marginally trapped surfaces, and hence satisfy

$$H + h^{ij}(g_{ij} - r_i r_j) = 0 \tag{82}$$

where H is the mean curvature of the apparent horizons in M^3, h is the second fundamental form of (M^3, g) in $(N^4, \bar{g})$, and r is the outward unit normal to the apparent horizons in M^3. Hence, since we chose M^3 to have zero second fundamental form, the apparent horizons are zero mean curvature surfaces in M^3. Furthermore, if we consider the surface (possibly with multiple components) of smallest area which encloses the apparent horizons, it too must have zero mean curvature and hence is a marginally trapped surface in M^3. Thus, the union of the apparent horizons in M^3 form an outermost minimal surface of M^3. Furthermore, we note that it follows from a stability argument [**16**] that each component of this outermost minimal surface must be a sphere. Since event horizons always contain the apparent horizons, $A_i \geq A$, so putting all the inequalities together we conclude that

$$m_i \geq m_f \geq \sqrt{\frac{A_f}{16\pi}} \geq \sqrt{\frac{A_i}{16\pi}} \geq \sqrt{\frac{A}{16\pi}} \tag{83}$$

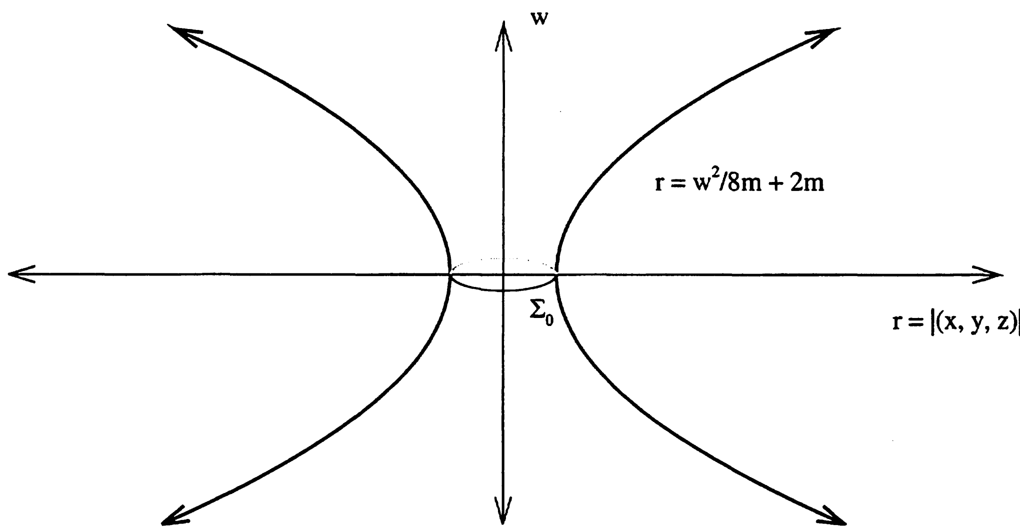


FIGURE 1. The space-like Schwarzschild metric of mass m, $(\mathbf{R}^3 - \{0\}, h)$, viewed as a submanifold of four-dimensional Euclidean space.

Thus, Penrose argued, assuming the cosmic censor hypothesis and a few reasonable sounding assumptions about the nature of gravitational collapse, given a complete asymptotically flat 3-manifold M^3 of total mass m_i with nonnegative scalar curvature which has an outermost minimal surface of total area A, then

$$m_i \geq \sqrt{\frac{A}{16\pi}} \tag{84}$$

Conversely, he argued, if one could find an M^3 which was a counterexample to the above inequality, then it would be likely that the counterexample, when used as initial conditions in the Cauchy problem for Einstein's equation, would produce a naked singularity. Hence, the two proofs of the above inequality given in sections 2 and 3 (for a single horizon) rule out one possibility for forcing counterexamples to the cosmic censor hypothesis.

Appendix B. The Schwarzschild Metric

The space-like Schwarzschild metric is a particularly important example to consider when discussing the Penrose inequality. First of all, it is the only 3-manifold which gives equality in the Penrose inequality. Also, if a 3-manifold is assumed to be complete, spherically symmetric, and have zero scalar curvature, then it must be isometric to either a Schwarzschild metric of mass $m > 0$ or $\mathbf{R}^3$, which can be viewed as the Schwarzschild metric when $m = 0$.

The space-like Schwarzschild metric, $(\mathbf{R}^3 - \{0\}, h)$, is a time symmetric, asymptotically flat, three-dimensional maximal slice (chosen to have zero momentum at infinity) of the four-dimensional Schwarzschild space-time metric.The space-like Schwarzschild metric is conformal to $\mathbf{R}^3 - \{0\}$ with $h_{ij} = \left(1 + \frac{m}{2r}\right)^4 \delta_{ij}$. The Schwarzschild metric of mass m, $(\mathbf{R}^3 - \{0\}, h)$, can also be isometrically embedded into four-dimensional Euclidean space as the three-dimensional set of points in $\mathbf{R}^4 =$

$\{(x, y, z, w)\}$ satisfying $|(x, y, z)| = \frac{w^2}{8m} + 2m$, seen in figure 1. Hence, Σ_0 is a minimal sphere of area $16\pi m^2$, so we have equality in the Penrose inequality.

Appendix C. The Spherically Symmetric Case

In this section we sketch a proof of the Penrose inequality in the case that M^3 is spherically symmetric. The proof is very easy conceptually, but what is more important is that some of the ideas generalize. In particular, we will see why the minimal sphere in the Penrose inequality must be outermost.

Let (M^3, g) be a complete, spherically symmetric 3-manifold with nonnegative scalar curvature which has a single, asymptotically flat end. For convenience, we also assume that (M^3, g) is isometric to the Schwarzschild metric of some mass m outside a large compact set. Then the total mass of (M^3, g) is m. Let $\Sigma(V)$ be the spherically symmetric sphere containing a volume V in M^3. Let $A(V)$ be the area of this sphere. It turns out that the function $A(V)$, $V \geq 0$, captures all the information about M^3 since M^3 is spherically symmetric.

Let $R(V)$ be the scalar curvature of M^3 on $\Sigma(V)$. It happens that

$$R(V) = \frac{8\pi}{A} - 2A(V)A''(V) - \frac{3}{2}A'(V)^2. \tag{85}$$

Define

$$m(V) = \left(\frac{A(V)}{16\pi}\right)^{\frac{1}{2}} \left(1 - \frac{1}{16\pi}A(V)A'(V)^2\right). \tag{86}$$

It turns out that $m'(V) \geq 0$ whenever $A'(V) \geq 0$ since

$$m'(V) = \frac{A'(V)}{16\pi} \left(\frac{A(V)}{16\pi}\right)^{\frac{1}{2}} R(V) \tag{87}$$

and $R(V) \geq 0$.

Let $\Sigma(V_0)$ be the outermost minimal sphere. It follows that $A'(V) \geq 0$ for all $V \geq V_0$. Hence, $m(V)$ is increasing in this range as well, so

$$\lim_{V \to \infty} m(V) \geq m(V_0) \tag{88}$$

Furthermore $m(V_0) = \sqrt{\frac{A(V_0)}{16\pi}}$ since $A'(V_0) = 0$. Also, we assumed that M^3 was isometric to the Schwarzschild metric outside a large compact set, and we claim that $m(V) = m$, the mass parameter of the Schwarzschild metric, in this region, or equivalently, for $V > V_{LARGE}$ for some $V_{LARGE} > 0$. To see this, consider the mass function $m(V)$ defined on the Schwarzschild metric, where now V refers to the volume contained by the spherically symmetric spheres of the Schwarzschild metric which is outside the horizon. Then by equation 87, $m(V)$ is constant for all V on the Schwarzschild metric since the Schwarzschild metric has zero scalar curvature. Furthermore, setting $V = 0$ and considering $m(V)$ at the horizon yields

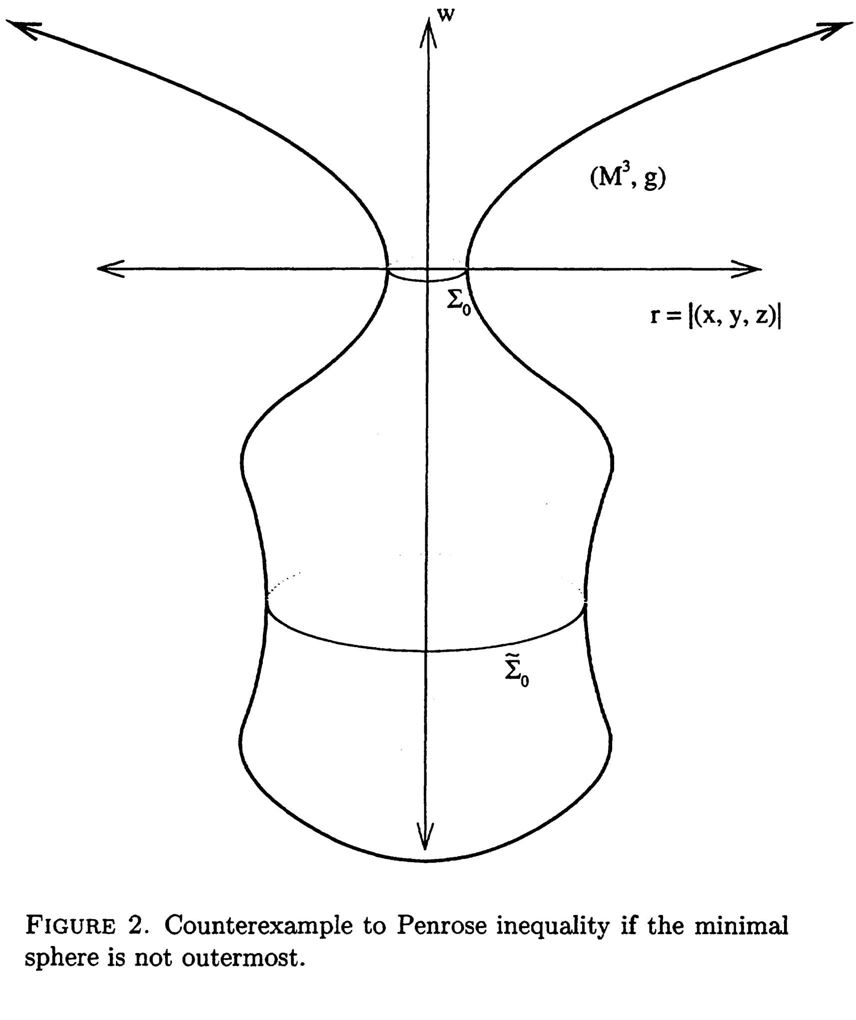


FIGURE 2. Counterexample to Penrose inequality if the minimal sphere is not outermost.

$m(0) = \sqrt{\frac{A(0)}{16\pi}} = m$, the mass parameter of the Schwarzschild metric, since the Schwarzschild metric gives equality in the Penrose inequality. Thus, $m(V) = m$ for all V in the Schwarzschild metric, so going back to (M^3, g), we see that $m(V) = m$, the mass parameter of the Schwarzschild metric, for $V > V_{LARGE}$. Thus, it follows from inequality 88 that

$$m \geq \sqrt{\frac{A(V_0)}{16\pi}} \tag{89}$$

which proves the Penrose inequality for spherically symmetric manifolds.

Conversely, we see that equation 87 can be used to construct spherically symmetric manifolds which do not satisfy the Penrose inequality if we do not require the minimal sphere to be outermost. In figure 2, we are viewing (M^3, g) as an isometrically embedded submanifold of $\mathbf{R}^4$ with the standard Euclidean metric. (M^3, g) is spherically symmetric and is constructed by rotating the curve shown above around the w-axis in $\mathbf{R}^4$. Hence, Σ_0 and $\tilde{\Sigma}_0$ are both 2-spheres, and we can choose the curve shown above so that the scalar curvature of (M^3, g), $R(g)$, is non-negative.

The Penrose inequality, $m \geq \sqrt{\frac{|\Sigma_0|}{16\pi}}$, is true for Σ_0, but is not true for $\tilde{\Sigma}_0$. However, $\tilde{\Sigma}_0$ is not an outermost minimal sphere since it is contained by another minimal sphere, namely, Σ_0. In fact, since $\tilde{\Sigma}_0$ is not outermost, we can construct a spherically symmetric manifold like the one shown above so that the area of $\tilde{\Sigma}_0$ is as large as we like and the total mass of (M^3, g) is still one.

Appendix D. An Example Solution to the Conformal Flow of Metrics

In this appendix we give the simplest example of a solution to the first order o.d.e. conformal flow of metrics defined by equations 12, 13, 14, and 15. The initial metric in this example is the three dimensional, space-like Schwarzschild metric which represents a single, non-rotating black hole in vacuum. The Schwarzschild metrics are also very natural from a geometric standpoint as well since they are spherically symmetric and have zero scalar curvature.

Since the flow does not change the metric inside the horizon, we will define this metric to have its horizon as a boundary, which is always allowable. Then (M^3, g_0) will be defined to be isometric to $(\mathbf{R}^3 - B_{m/2}(0), \mathcal{U}_0(x)^4\delta_{ij})$, where

$$\mathcal{U}_0(x) = 1 + \frac{m}{2r} \tag{90}$$

where r is the distance from the origin in $(\mathbf{R}^3, \delta_{ij})$ and m is a positive constant equal to the mass of the black hole.

Next we define (M^3, g_t) to be isometric to $(\mathbf{R}^3 - B_{m/2}(0), \mathcal{U}_t(x)^4\delta_{ij})$, where

$$\mathcal{U}_t(x) = \begin{cases} e^{-t} + \frac{m}{2r}e^t, & \text{for } r \geq \frac{m}{2}e^{2t} \\ \sqrt{\frac{2m}{r}}, & \text{for } r < \frac{m}{2}e^{2t}. \end{cases} \tag{91}$$

We note that on this metric the outermost horizon (and also the outermost minimal area enclosure of the original horizon) is the coordinate sphere given by $r = \frac{m}{2}e^{2t}$, so by equation 13 we define this horizon to be $\Sigma(t)$.

Next we recall from equation 12 that $g_t = u_t(x)^4 g_0$. Hence, $u_t(x) = \mathcal{U}_t(x)/\mathcal{U}_0(x)$. Furthermore, by equation 15 we must have $v_t(x) = \frac{d}{dt}u_t(x)$, so

$$v_t(x) = \frac{1}{\mathcal{U}_0(x)} \begin{cases} -e^{-t} + \frac{m}{2r}e^t, & \text{for } r \geq \frac{m}{2}e^{2t} \\ 0, & \text{for } r < \frac{m}{2}e^{2t}. \end{cases} \tag{92}$$

By equation 94, $v_t(x)$ is harmonic on (M^3, g_0) outside $\Sigma(t)$ since $a + b/r$ is harmonic in $(\mathbf{R}^3, \delta_{ij})$. Then since $v_t(x)$ goes to $-e^{-t}$ at infinity, is continuous, and equals zero inside $\Sigma(t)$, it follows that equation 14 is satisfied. Hence, (M^3, g_t) is a solution to the first order conformal flow of metrics defined by equations 12, 13, 14, and 15.

This example is a good example to keep in mind when considering the main theorems of section 2. For example, we notice that by definition 2 the total mass $m(t)$ of (M^3, g_t) equals m and hence is nonincreasing as claimed in section 2.3, and the area $A(t)$ of the horizon $\Sigma(t)$ in (M^3, g_t) is constant as claimed in theorem 3. Also, we see that the diameter of $\Sigma(t)$ is growing exponentially and contains any given bounded set in a finite amount of time, which turns out to be a general characteristic of the flow and is proven in [**6**].

Finally, we note that for all $t \geq 0$ in this example, (M^3, g_t) is isometric to a Schwarzschild metric of total mass m outside their respective horizons. Hence, even though the metric is shrinking pointwise, it is not changing at all outside its horizon, *after a reparametrization of the metric.* It is in this sense that theorem 4 states that no matter what the initial metric is, it eventually converges to a Schwarzschild metric outside its horizon.

Appendix E. The Harmonic Conformal Class of a Metric

In this appendix we define a new equivalence class and partial ordering of conformal metrics following [**6**]. As we will see, the existence of this equivalence class of metrics provides a natural motivation for the techniques used in section 2 to prove the Riemannian Penrose conjecture.

Let

$$g_2 = u(x)^{\frac{4}{n-2}} g_1 \tag{93}$$

where g_2 and g_1 are metrics on an n-dimensional manifold M^n, $n \geq 3$. Then we get the surprisingly simple identity that

$$\Delta_{g_1}(u\phi) = u^{\frac{n+2}{n-2}} \Delta_{g_2}(\phi) + \phi \Delta_{g_1}(u) \tag{94}$$

for any smooth function ϕ.

This motivates us to define the following relation.

DEFINITION 19. *Define*

$$g_2 \sim g_1$$

if and only if equation 93 is satisfied with $\Delta_{g_1}(u) = 0$ *and* $u(x) > 0$.

Then from equation 94 we get the following lemma.

LEMMA 6. *The relation* $\sim$ *is reflexive, symmetric, and transitive, and hence is an equivalence relation.*

Thus, we can define the following equivalence class of metrics.

DEFINITION 20. *Define*

$$[g]_H = \{\bar{g} \mid \bar{g} \sim g\}$$

to be the **harmonic conformal class** *of the metric* g.

Of course, this definition is most interesting when (M^n, g) has nonconstant positive harmonic functions, which happens for example when (M^n, g) has a boundary.

Also, we can modify the relation $\sim$ to get another relation $\succeq$.

DEFINITION 21. *Define*

$$g_2 \succeq g_1$$

if and only if equation 93 is satisfied with $-\Delta_{g_1}(u) \geq 0$ *and* $u(x) > 0$.

Then from equation 94 we get the following lemma.

LEMMA 7. *The relation $\succeq$ is reflexive and transitive, and hence is a partial ordering.*

Since $\succeq$ is defined in terms of superharmonic functions, we will call it the superharmonic partial ordering of metrics on M^n. Then it is natural to define the following set of metrics.

DEFINITION 22. *Define*

$$[g]_S = \{\bar{g} \mid \bar{g} \succeq g\}.$$

This set of metrics has the property that if $\bar{g} \in [g]_S$, then $[\bar{g}]_S \subset [g]_S$

Also, the scalar curvature transforms nicely under a conformal change of the metric. In fact, assuming equation 93 again,

$$R(g_2) = u(x)^{-(\frac{n+2}{n-2})}\left(-c_n\Delta_{g_1} + R(g_1)\right)u(x) \tag{95}$$

where $c_n = \frac{4(n-1)}{n-2}$. This gives us the following lemma.

LEMMA 8. *The sign of the scalar curvature is preserved pointwise by $\sim$. That is, if $g_2 \sim g_1$, then $sgn(R(g_2)(x)) = sgn(R(g_1)(x))$ for all $x \in M^n$.*

Also, if $g_2 \succeq g_1$, and g_1 has non-negative scalar curvature, then g_2 has non-negative scalar curvature.

Hence, the harmonic conformal equivalence relation $\sim$ and the superharmonic partial ordering $\succeq$ are useful for studying questions about scalar curvature. In particular, these notions are useful for studying the Riemannian Penrose inequality which concerns asymptotically flat 3-manifolds (M^3, g) with non-negative scalar curvature. Given such a manifold, define $m(g)$ to be the total mass of (M^3, g) and $A(g)$ to be the area of the outermost horizon (which could have multiple components) of (M^3, g). Define $P(g) = \frac{m(g)}{\sqrt{A(g)}}$ to be the Penrose quotient of (M^3, g). Then an interesting question is to ask which metric in $[g]_S$ minimizes $P(g)$.

Section 2 can be viewed as an answer to the above question. We defined a conformal flow of metrics (starting with g_0) for which the Penrose quotient was non-increasing, and in fact this conformal flow stays inside $[g_0]_S$. Furthermore, $g_{t_2} \in [g_{t_1}]_S$ for all $t_2 \geq t_1 \geq 0$. Also, no matter which metric we start with, the metric converges to a Schwarzschild metric outside its horizon [**6**]. Hence, the minimum value of $P(g)$ in $[g]_S$ is achieved in the limit by metrics converging to a Schwarzschild metric (outside their respective horizons).

In the case that the g is harmonically flat at infinity, a Schwarzschild metric (outside the horizon) is contained in $[g]_S$. More generally, given any asymptotically flat manifold (M^3, g), we can use $\mathbf{R}^3 \backslash B_r(0)$ as a coordinate chart for the asymptotically flat end of (M^3, g) which we are interested in, where the metric g_{ij} approaches δ_{ij} at infinity in this coordinate chart. Then we can consider the conformal metric

$$g_C = \left(1 + \frac{C}{|x|}\right)^4 g \tag{96}$$

in this end. In the limit as C goes to infinity, the horizon will approach the coordinate sphere of radius C. Then outside this horizon in the limit as C goes to infinity, the function $(1 + \frac{C}{|x|})$ will be close to a superharmonic function on (M^3, g) and the metric g_C will approach a Schwarzschild metric (since the metric g is approaching the standard metric on $\mathbf{R}^3$). Hence, the Penrose quotient of g_C will approach $(16\pi)^{-1/2}$, which is the Penrose quotient of a Schwarzschild metric.

As a final note, we prove that the first order o.d.e. for $\{g_t\}$ defined in equations 12,13, 14, and 15 is naturally defined in the sense that the rate of change of g_t is a function only of g_t and not of g_0 or t. To see this, given any solution $g_t = u_t(x)^4 g_0$ to equations 12,13, 14, and 15, choose any $s > 0$ and define $\bar{u}_t(x) = u_t(x)/u_s(x)$ so that

$$g_t = \bar{u}_t(x)^4 g_s \tag{97}$$

and $\bar{u}_s(x) \equiv 1$. Then define $\bar{v}_t(x)$ such that

$$\begin{cases} \Delta_{g_s} \bar{v}_t(x) & \equiv & 0 & \text{outside } \Sigma(t) \\ \bar{v}_t(x) & = & 0 & \text{on } \Sigma(t) \\ \lim_{x\to\infty} \bar{v}_t(x) & = & -e^{-(t-s)} & \end{cases} \tag{98}$$

and $\bar{v}_t(x) \equiv 0$ inside $\Sigma(t)$. Then what we want to show is

$$\bar{u}_t(x) = 1 + \int_s^t \bar{v}_r(x)dr \tag{99}$$

To prove the above equation, we observe that from equations 94, 98, and 14 it follows that

$$v_t(x) = \bar{v}_t(x)\, u_s(x) \tag{100}$$

since $\lim_{x\to\infty} u_s(x) = e^{-s}$. Hence, since

$$u_t(x) = u_s(x) + \int_s^t v_r(x)dr \tag{101}$$

by equation 15, dividing through by $u_s(x)$ yields equation 99 as desired. Thus, we see that the rate of change of $g_t(x)$ at $t = s$ is a function of $\bar{v}_s(x)$ which in turn is just a function of $g_s(x)$ and the horizon $\Sigma(s)$. Hence, to understand properties of the flow we need only analyze the behavior of the flow for t close to zero, since any metric in the flow may be chosen to be the base metric. This point is used several times in section 2.

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