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A General Fredholm Theory and Applications

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The theory described here results from an attempt to find a general abstract framework in which various theories, like Gromov-Witten Theory (GW), Floer Theory (FT), Contact Homology (CH) and more generally Symplectic Field Theory (SFT) can be understood from a general point of view. Let us describe the general landscape in a somewhat oversimplified form. The common feature (with the exception of GW which has less structure) is the fact that we have infinitely many different Fredholm problems defined on spaces with boundary with corners, where the boundary strata can be explained in terms of products (or more generally fibered products) of other problems (on the list). In oversimplified form, the solution sets are zeros of a section f of some bundle $\tau : Y \to X$, where the space has a boundary ∂X, and where moreover there exists a recipe (or even many recipes) to construct from two given solutions[1] x' and x'' of $f = 0$ a new solution, say the product, $x = x' \circ x''$. The recipes for constructing new solutions are defined even for non-solutions and ∂X is precisely the space of points which are products. Hence we have

$$\partial X = X \circ X.$$

Moreover, if we denote the restriction of f to ∂X by ∂f and define $f \circ f$ on ∂X as the set-valued section

$$f \circ f(x) = \{f(x') \circ f(x'') \mid x = x' \circ x''\},$$

then we say that f is compatible with the recipe $\circ$ provided

$$\partial f = f \circ f.$$

Assuming f to be compatible the $\circ$-structure generates a certain amount of algebra which can be captured on a rather rudimentary level. Then the more sophisticated algebra we see in the description of SFT can be

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[1]In general one requires them to satisfy a compatibility condition.

viewed as obtained by some kind of representation theory of the underlying "primitive" data. One can develop a general theory which covers a variety of problems. The only difference in application between seemingly different problems is that the underlying structure of the family of Fredholm problems, i.e., their interaction, is different.

The starting point for the investigation is the Symplectic Field Theory (SFT) as initiated by Eliashberg, Givental and the author in [**9**]. Wysocki, Zehnder and the author developed a powerful nonlinear Fredholm theory with operations (FTO) which can be used to describe SFT, [**15, 16, 17**]. The Fredholm theory takes place in a new kind of spaces called polyfolds. These spaces are needed since all phenomena of interest are coming from analytically difficult phenomena like bubbling-off, stretching the neck, breaking of trajectories and blowing-up[2]. In [**10**] this theory will be used to develop SFT and many ideas around it in full generality (and absolutely rigorously). In fact, the polyfold language allows to completely remove the analysis before carrying out topological and algebraic considerations contrasting the current situation where many arguments are relying on a combination of arguments across the board.

Let us discuss for a moment the issues which we have to address in developing a general framework allowing us to describe seemingly different problems. Gromov–Witten theory, Floer–Theory, Contact–Homology, or more generally Symplectic Field Theory are theories built on the study of certain compactified moduli spaces, or even infinite families of such spaces. These moduli spaces are measured and the data is encoded in convenient ways, quite often as a so-called generating function. Common features include:

1) The moduli spaces are solutions of elliptic PDE's quite often exhibiting compactness problems, at least as seen from a more classical analytical viewpoint.
2) Very often these moduli spaces, when they are not compact, admit nontrivial compactifications usually based on surviving analytic phenomena carrying names like "Bubbling-off", "Stretching the Neck", "Blow-up", "Breaking of Trajectories" hinting to borderline analytic behavior.
3) In problems like Floer-Theory, Contact-Homology or Symplectic Field Theory precisely the algebraic structures of interest are those created by the "violent analytic behavior" and its "taming" by finding a workable compactification. In fact the algebra is created by the fact that many different moduli spaces interact with each other in a complicated way.

[2]Perhaps folklore-wise known as the "Analytical Chamber of Horror".

We begin with the shortcomings of classical Fredholm theory. The classical Fredholm theory can be viewed as the study of Fredholm sections of some Banach bundle $Y \to X$. For definiteness we assume that Y is a Banach space bundle over the Banach manifold X. Let us denote the fiber over $x \in X$ by Y_x. If $f(x) = 0$ we can build the linearisation $f'(x) : T_xX \to Y_x$ and if $f'(x)$ is surjective we have a solution manifold near x in fact inheriting its manifold structure as a submanifold of the (big) ambient space. From a practical point of view the bubbling-off phenomena usual cannot be described within this classical framework. The key question is therefore if there is a generalized Fredholm theory in which interesting problems of the type described above can be handled. Keeping this in mind it is worthwhile to have a critical look at the classical case. We may raise the following question in the classical context. Is it not "unnecessary luxury" that the ambient space has a lot of "hard structure" whereas we only seem to use a little of it in order to obtain a smooth structure on the solution set $f^{-1}(0)$ (assuming transversality)? This question is very much justified, since in many cases, once the solution spaces are constructed, the ambient spaces are discarded and considered irrelevant. The hope is, of course, that analyzing the situation, we might be able to see what is the bare minimum of structure needed for a suitable generalization. More precisely we have to address the following question:

What (perhaps new) structures do we need on the ambient space and bundle (with a preferred section called 0*) to talk about transversality and an abstract perturbation theory for a section f so that at points of transversality the solution set $f^{-1}(0)$ carries in a natural way the structure of a smooth orbifold with boundary with corners? In addition we require the theory to be so general that in applications the compactified moduli spaces in Gromov-Witten theory, Floer theory, or SFT would be the solution sets of the generalized Fredholm operators.*

Analyzing the before-mentioned theories it becomes immediately clear that one has to address a certain number of very serious issues. For example, if one of the "violent analytical phenomena" occurs, any natural candidate for an ambient space seems to have locally varying dimensions. In other words, in particular, the spaces are locally not isomorphic to open sets in Banach or Frechet spaces. Hence, if we still think we should devise a manifold-type theory, the local models cannot be open sets in some Banach or Frechet space. They need to be more general. If we still want to talk about a linearisation of a problem, which, as every analyst knows, has its undeniable benefits, we should look for some class of local models which in some way admit tangent spaces. Moreover, there are some other unpleasant phenomena to deal with. For example in some constructions we have divide out by families of diffeomorphisms acting on the domain of maps. Analysts know that such actions (for example in any Banach space set-up) will always be only continuous, but never smooth. In addition, the applications like SFT will require the theory to have certain features of a theory of (infinite-dimensional) orbifolds with boundaries with corners.

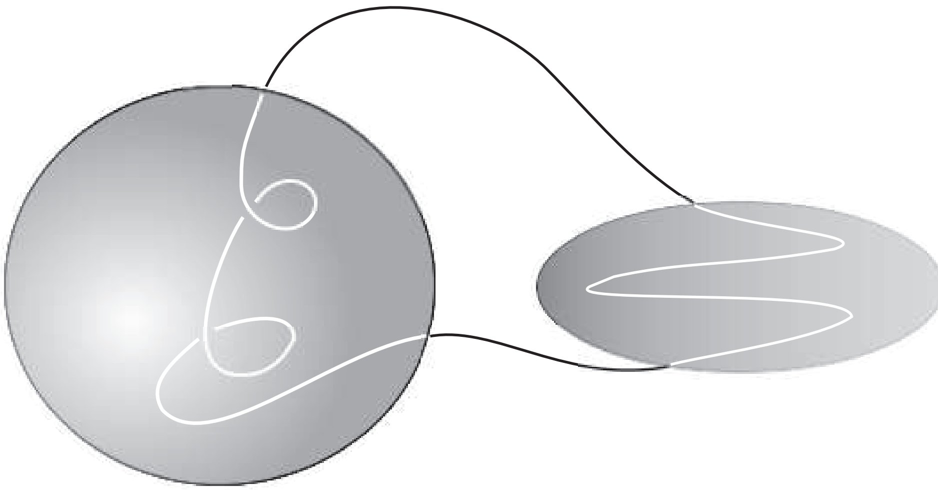

FIGURE 1. This figure shows a finite-dimensional M-polyfold, say X, homeomorphic to the space obtained from an open three-ball and an open two-ball connected by two curves having a one-dimensional S^1-like submanifold. This submanifold could arise as the zero of a transversal section of a strong (finite-dimensional) M-polyfold bundle Y over X, which has varying dimensions. Namely over the three-ball it is two-dimensional, over the two-disk one-dimensional and otherwise trivial. The polyfold theory guarantees natural smooth structures on such solution sets.

Summarizing, there is a whole basket of issues which call for a more general theory. If we have a look at our list of requirements it seems that the problem of finding an adequate theory is "over-determined". Surprisingly, however, there is such a general Fredholm theory, and even more surprisingly, it is not much more difficult than the classical one. This leads to our theory of M-polyfolds (or more generally polyfolds) and an adapted Fredholm theory. There are even finite-dimensional polyfolds and Figure 1 shows a finite-dimensional M-polyfold having a one-dimensional "submanifold". In this new theory we can formalize new structures which could not be formalized before. This gives a unified perspective on a variety of theories in symplectic geometry. It also seems that the theory should have applications in other fields as well, since the addressed analytical issues arise in geometric pde's of Riemannian geometry as well as the theory of nonlinear pde in general. In this note we will explain some parts of the theory. For the proofs and further discussions we refer the reader to [**15, 16, 17, 10**] and [**9**].

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The organization of this paper is as follows:

Contents

1. New Smoothness Concepts and Spaces

In a first step we introduce a new concept of smoothness for a Banach space and define the notion of a smooth map in this new context. One might, alternatively, view our first definition as a new interpretation for certain classes of interpolation spaces[3].

[3]Interpolation theory is an important part of functional analysis and the theory of function spaces. We refer to the comprehensive book by Triebel [**38**].

1.1. Concepts of Smoothness.

DEFINITION 1.1. Let E be a Banach space. A sc-smooth structure on E is given by a nested sequence of Banach spaces E_m, $m \in \mathbb{N}$, satisfying

1) For $m \leq n$ the space E_n is a linear subspace of E_m and $E_0 = E$.
2) The inclusion $E_n \to E_m$ for $m < n$ is a compact operator.
3) The vector space E_∞ defined by
$$E_\infty = \bigcap_{m \in \mathbb{N}} E_m$$
is dense in every E_m.

Of course, one can build a linear functional analytic theory on Banach spaces with sc-structures. There is a large body of such a theory usually as part of interpolation theory, where the focus is quite different from ours. The interpretation of a scale as a generalization of a smooth structure as being developed below seems to be new.

Here is an important example.

EXAMPLE 1.2. Let $0 < \delta_0 < \delta_1 < \ldots$ be a strictly increasing sequence of weights. We denote by E the Banach space consisting of maps $u : \mathbb{R} \times S^1 \to \mathbb{R}^N$ of Sobolev class H^3_{loc} so that for every multi-index α of order at most 3 the weighted partial derivative
$$(s,t) \to |D^\alpha u(s,t)| \cdot e^{\delta_0 |s|}$$
belongs to L^2. We define the sc-structure on E by taking E_m to consist of maps of regularity $(m+3, \delta_m)$, i.e., $m+3$ derivatives integrable with the weight associated to δ_m. That E_m defines a sc-structure on E follows from the compact Sobolev embedding theorem for bounded domains and the fact that the weights are strictly increasing.

If $U \subset E$ is an open subset we define a sc-smooth structure to be the nested sequence $U_m = U \cap E_m$. Given a sc-smooth structure on U we observe that U_m inherits a sc-smooth structure by defining $(U_m)_k = U_{m+k}$. We shall write U^m for the sc-space defined by
$$(U^m)_k = U_{m+k}.$$
Given two sc-spaces E and F there is a well-defined direct sum $E \oplus F$. We would like to note here that there are three important linear concepts. The first is that of a linear sc-operator $T : E \to F$, which by definition is a linear operator inducing bounded operators between the same levels. The next one is that of a linear sc-Fredholm operator (to be explained later), and the latter is that of a sc$^+$-operator $A : E \to F$. By definition this is a sc-operator inducing one from E to F^1, i.e., $E_m \to F_{m+1}$ for every level m.

If $U \subset E$ is open we define its tangent as $TU = U^1 \oplus E$. In particular

$$(TU)_m = U_{m+1} \oplus E_m.$$

A map $f : U \to V$, where U and V are open in sc-Banach spaces, is said to be sc^0 provided it induces a continuous map between every level. Next we define the notion of a sc-smooth map. We give two equivalent definitions.

DEFINITION 1.3. Let E and F be sc-Banach spaces and $U \subset E$ an open subset. The sc^0-map $f : U \to F$ is said to be sc^1 if the following holds:

- For every $x \in U_1$ there exists a bounded linear operator $Df(x) \in L(E_0, F_0)$ so that for $h \in E_1$
$$\lim_{h \to 0} \frac{1}{\| h \|_1} \| f(x+h) - f(x) - Df(x)h \|_0 = 0.$$
- The map $Tf : TU \to TF$ defined by
$$(Tf)(x, h) = (f(x), Df(x)h)$$
is of class sc^0.

Let us observe that for every $x \in U_1$ there can be at most one map $Df(x)$ with the properties described above. This map will be called the linearization of f at x. We call $Tf : TU \to TF$ the tangent map of the sc^1-map $f : U \to F$. Let us observe that Tf for every $m \in \mathbb{N}$ is given by

$$Tf : U_{1+m} \oplus E_m \to F_{m+1} \oplus F_m : (x, h) \to (f(x), Df(x)h).$$

There is an equivalent definition for being sc^1 which relates it to the notion of C^1-map between different levels.

DEFINITION 1.4. Let E and F be sc-smooth Banach spaces and $U \subset E$ an open subset. A sc^0-map $f : U \to F$ is said to be sc^1 provided the following holds:

1) For every $m \geq 1$ the induced map
$$f : U_m \to F_{m-1}$$
is of class C^1. In particular the derivative gives the continuous map
$$U_m \to L(E_m, F_{m-1}) : x \to Df(x).$$
2) For $x \in U_m$ and $m \geq 1$ the map $Df(x)$ induces a continuous linear operator $Df(x) : E_{m-1} \to F_{m-1}$ and the resulting map
$$U_m \times E_{m-1} \to F_{m-1} : (x, h) \to Df(x)h$$
is continuous.

As already emphasized

Proposition 1.5. *The two definitions for a sc^0-map $f : U \to F$ to be of class sc^1 are equivalent.*

For a proof see [**15**]. A sc^1-map $f : U \to V$ has a well-defined tangent map

$$Tf : TU \to TV$$

and inductively we can define the notion of being sc^k. An important result is the validity of the chain rule:

Theorem 1.6 (Chain Rule). *If $f : U \to V$ and $g : V \to W$ are sc^1 so is $g \circ f$ and $T(g \circ f) = (Tg) \circ (Tf)$.*

In view of the second characterization of being sc^1 it is not clear at all that a chain rule has to hold. In fact, as the proof reveals, it just works.

We give an example of a sc^1-map which will be important in the application of the theory to SFT.

Example 1.7. Recall the sc-space E of maps $\mathbb{R} \times S^1 \to \mathbb{R}^N$ from Example 1.2. We have an action of the group $\mathbb{R} \times S^1$ by sc-operators on E defined by

$$((c, \rho) * u)(s, t) = u(s + c, t + \rho),$$

where $c \in \mathbb{R}$ and $\rho \in \mathbb{R}/\mathbb{Z} = S^1$. The map

$$\mathbb{R} \times S^1 \times E_m \to E_m : ((c, \rho), u) \to (c, \rho) * u$$

is continuous for every m, so that it defines a sc^0-map

$$\Phi : (\mathbb{R} \times S^1) \times E \to E.$$

The important fact is now that the map Φ is sc$^\infty$. The proof is somewhat lengthy and a variation of the proof can be found in [**15**].

At this point we can develop a whole theory of manifolds built on the pseudogroup of sc-diffeomorphisms. The fact that the whole manifold theory and its constructions are functorial allows to build a parallel theory based on this pseudogroup. In other words we could define a second countable Hausdorff space X to have a sc-smooth structure provided it is equipped with an atlas so that the transition maps are sc-smooth. Note that X will inherit a filtration X_m. Also X has a tangent bundle $TX \to X^1$.

Here is an important example.

Example 1.8. Let M be a complete Riemannian manifold and $\Phi : M \to \mathbb{R}$ a Morse-function. Let us assume for simplicity that the critical points can be totally ordered by $a < b$ via $f(a) < f(b)$. For every a fix a sequence $\delta^a =$ of weights $\delta^a_0 = 0 < \delta^a_1 < \ldots$. Usually the limit should be finite and smaller than the spectral gap of the Hessian around a. Then denote for $a < b$ by $X(a, b)$ the quotient of H^2-maps (Sobolev class) connecting at $-\infty$ the point a with b at $+\infty$ by the obvious $\mathbb{R}$-action. The space has a sc-smooth structure where level-m-elements are represented by $(H^{2+m,\delta_m^a,\delta_m^b})$-maps, i.e., we have different exponential weights at $\pm\infty$.

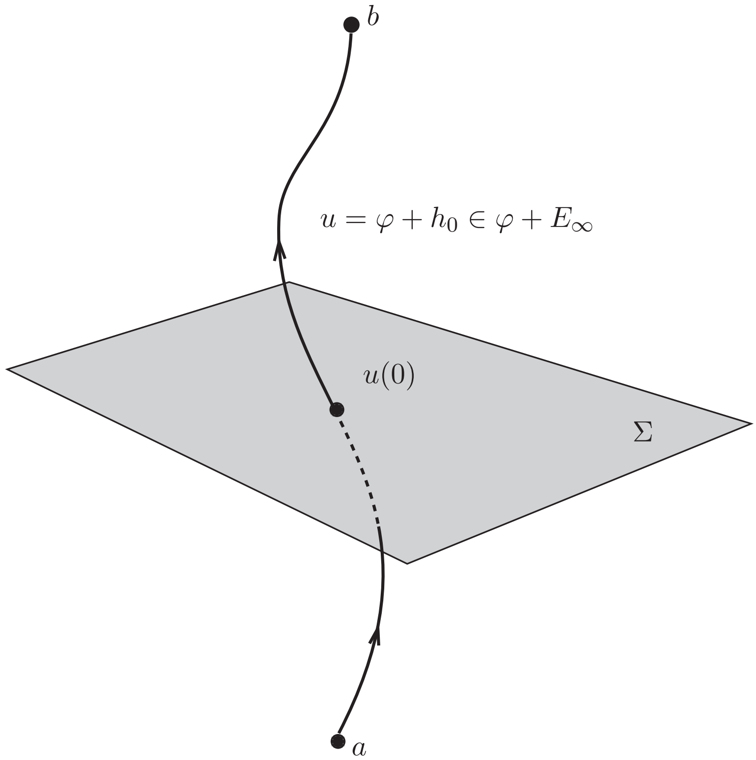

FIGURE 2.

The local model is the codimension one sc-subspace E of $H^2(\mathbb{R},\mathbb{R}^n)$ consisting of functions $h=(h_1,\dots,h_n)$ with $h_1(0)=0$, equipped with the sc-structure E_m given by $E_m = H^{m+2,\delta_m^a,\delta_m^b}\cap E$. We explain this with the special case $M=\mathbb{R}^N$ and also assume that the weight sequence is independent of the critical point. Let $a\neq b$ be two different points in $\mathbb{R}^N$. Pick a smooth map $\varphi:\mathbb{R}\to\mathbb{R}^N$ so that $\varphi(s)=a$ for $s\ll 0$ and $\varphi(s)=b$ for $s\gg 0$. Define $\hat{X}=\varphi+H^2(\mathbb{R},\mathbb{R}^N)$. Then we have by time-shift a natural $\mathbb{R}$-action on $\hat{X}$. Denote the quotient $\hat{X}/\mathbb{R}$ by $X(a,b)$ with its induced quotient topology. The space $H^2(\mathbb{R},\mathbb{R}^N)$ has a sc-structure where E_m consists of maps of regularity $(m+2,\delta_m)$. Given a smooth representative u for a class $[u]\in X(a,b)$, i.e., $u=\varphi+v$ with $v\in E_\infty$, we can define the inverse of a chart as follows. Since u connects two different points a and b there exists a time t_0 with $u'(t_0) \neq 0$. We may assume without loss of generality that $t_0 = 0$ by replacing u by $t_0 * u$. Let Σ be the hyperplane orthogonal to $u'(0)$. Then define a codimension one subspace of E to consist of all h with $h(0) \in \Sigma$. One can show that H has a one-dimensional sc-complement in E. For $h \in H$ in a H^2-neighborhood of 0 the map

$$h \to [u + h]$$

is a homeomorphism onto an open neighborhood of $[u] \in X(a, b)$. One can show that the collection of the inverses of all these maps defines an atlas of charts with sc-smooth transition maps. Hence $X(a, b)$ carries the structure of a sc-manifold. Going back to our original case of maps into M we can modify the construction for the $\mathbb{R}^N$-case using the exponential map for a suitable Riemannian metric on M and can construct a sc-manifold structure in the general case as well. See Figure 2 for an illustration.

1.2. M-Polyfolds. The local models for the new spaces, which are needed to construct ambient spaces for the moduli spaces (for example occurring in SFT), can now easily be constructed (at least those which have a "manifold flavor"). We call these new type of spaces polyfolds. We begin with M-polyfolds, where the M stands for "manifold flavor". Polyfolds, which are defined later, have an orbifold flavor. These spaces have locally varying dimensions. We also exhibit a (rather tame) finite-dimensional example.

1.2.1. *M-Polyfolds.* We will describe first M-polyfolds which carry a manifold flavor in contrast to polyfolds which are in general potentially complicated objects resembling something like "orbifolds with varying dimensions and with boundary with corners". Let us call a subset C of some finite-dimensional vector space A a partial cone if there is a linear isomorphism $T : A \to \mathbb{R}^n$ mapping C onto $[0, \infty)^k \times \mathbb{R}^{n-k}$.

Definition 1.9. Let V be a (relatively) open subset of some partial cone C, E a Banach space with a sc-smooth structure and $\pi_v : E \to E$ a family of sc-projections so that the induced map

$$V \oplus E \to E : (v, e) \to \pi_v(e)$$

is sc-smooth. Then we call the triple $\mathcal{S} = (\pi, E, V)$ a sc-smooth splicing.

Every splicing $\mathcal{S} = (\pi, E, V)$ is accompanied by a complementary splicing $\mathcal{S}^c = (Id - \pi, E, V)$. Observe that a splicing $\mathcal{S}$ decomposes the space $V \oplus E$ as a fibered sum over V; namely a point (v, x) can be decomposed as

$$(v, x) = (v, u_v + u_v^c),$$

where $\pi_v(u) = u$ and $\pi_v(u^c) = 0$. Also note that the sc-smoothness of $(v, e) \to \pi_v(e)$ is a rather weak condition. In fact, the dimension of the image of π_v is in general locally not constant.

The map $\Phi : V \oplus E \to E : (v, e) \to \pi_v(e)$ is sc-smooth. Taking its tangent map we can define

$$\Pi_{(v,\delta v)} : TE \to TE : (e, \delta e) \to (\Phi(v, e), D\Phi(v, e)(\delta v, \delta e))$$

which has the property that the induced map

$$TV \oplus TE \to TE : (a, b) \to \Pi_a(b)$$

is sc-smooth since it is modulo the identification $TV \oplus TE = T(V \oplus E)$ the tangent map of Φ. One easily verifies that (Π, TV, TE) defines a sc-smooth splicing. We call it the tangent of the splicing $\mathcal{S}$ and denote it by $T\mathcal{S}$.

Let $\mathcal{S} = (\pi, E, V)$ be a sc-smooth splicing. Then the associated splicing core is the subset $K = K^{\mathcal{S}}$ of $V \oplus E$ consisting of all pairs (v, e) with $\pi_v(e) = e$. Observe that we have a natural map

$$K^{T\mathcal{S}} \to K^{\mathcal{S}} : (v, \delta v, e, \delta e) \to (v, e).$$

Clearly the fiber over any point is a sc-Banach space in a natural way. We define the tangent of $K^{\mathcal{S}}$, denoted by $TK^{\mathcal{S}}$, by

$$TK^{\mathcal{S}} := K^{T\mathcal{S}}.$$

In fact it is useful to keep track of the underlying splicings and the above should be read

$$T(K^{\mathcal{S}}, \mathcal{S}) = (K^{T\mathcal{S}}, T\mathcal{S}).$$

Now we can define our new local models for spaces.

Definition 1.10. A local M-polyfold model consists of a pair $(O, \mathcal{S})$ where O is an open subset of the splicing core $K^{\mathcal{S}}$ associated to the sc-smooth splicing $\mathcal{S}$. The tangent $T(O, \mathcal{S})$ of the local M-polyfold model $(O, \mathcal{S})$ is defined by

$$T(O, \mathcal{S}) = (K^{T\mathcal{S}}|O^1, T\mathcal{S}),$$

where $K^{T\mathcal{S}}|O^1$ denotes the collection of all points in $K^{T\mathcal{S}}$ which project under the canonical projection

$$K^{T\mathcal{S}} \to (K^{\mathcal{S}})^1$$

onto O^1.

The above discussion gives us a natural projection

$$T(O, \mathcal{S}) \to (O, \mathcal{S})^1 : (v, \delta v, e, \delta e) \to (v, e).$$

In the following we shall write O instead of $(O, \mathcal{S})$, but observe that $\mathcal{S}$ is part of the structure. Note that for an open subset O of a splicing core we have an induced filtration. Hence we may talk about sc^0-maps. We continue by introducing the notion of a sc^1-map between open sets of splicing cores.

Definition 1.11. Let O and O' be open subsets of splicing cores. Assume that $f : O \to O'$ is a sc^0-map. We say that f is sc^1 provided the map

$$(v, e) \to f(v, \pi_v(e))$$

which is defined on some open subset of $C \oplus E$ and takes image in O' (which we view lying in the obvious Banach space with sc-smooth structure) is sc^1.

A sc^1-map induces in a canonical way a tangent map

$$Tf : TO \to TO'.$$

To see this start with the map

$$\hat{f} : (v, e) \to f(v, \pi_v(e)) = (f_1(v, \pi_v(e)), f_2(v, \pi_v(e))) = (\hat{f}_1(v, e), \hat{f}_2(v, e)),$$

which is defined on an open subset $\hat{O}$ of a sc-smooth Banach space and which takes its image in a sc-smooth Banach space G. By assumption it is sc^1. On easily verifies that the map Tf defined by

$$Tf(v, \delta v, e, \delta e) = (T\hat{f}_1(v, e, \delta v, \delta e), T\hat{f}_2(v, e, \delta v, \delta e))$$

maps $K^{T\mathcal{S}}|O$ into $K^{T\mathcal{S}'}|O'$. This is by definition the induced tangent map. We have

Theorem 1.12 (Chain Rule for sc^1-maps). *Let O, O', O'' be open sets in splicing cores and $f : O \to O'$ and $g : O' \to O''$ be sc^1. Then $g \circ f$ is sc^1 and $T(g \circ f) = Tg \circ Tf$. Moreover Tf and Tg are sc^0.*

This is a consequence of the sc-chain rule, the definition and the fact that our reordering of the terms in our definition of the tangent map is consistent. Hence given a sc^1-map $f : O \to O'$ between open sets of splicing cores we obtain an induced tangent map $Tf : TO \to TO'$. Inductively we can define the notion of being sc^k.

Here one should point out the following. If we recall that the constructions of Differential Geometry are functorial with respect to the input being: a) the notion of a smooth map between two open subsets of Euclidean spaces and b) the chain rule and the functoriality of the tangent functor, then we can easily imagine if we replace open sets in Euclidean spaces by open sets in splicing cores and smooth maps by sc-smooth maps that many constructions of Differential Geometry carry over and that many more constructions become possible. We should however note that in finite dimensions the existence of a smooth partition of unity is automatic, whereas in general (with the exception of sc-Hilbert spaces) it has to be required[4]. The situation is, however, somewhat better in the sc-situation than in the Banach space case in view of the following criterion for sc-smoothness of a real-valued function, which is proved in [**15**].

[4]The existence of a smooth partitions of unity in a Banach space context is a subtle problem and we refer the reader to [**1**].

Proposition 1.13. *Let E be a Banach space with a sc-smooth structure and let $U \subset E$ be an open subset. Assume that $f : U \to \mathbb{R}$ is sc-continuous and that the induced maps*

$$f_m := f|_{U_m} : U_m \to \mathbb{R}, \quad m \geq 0,$$

are of class C^{m+1}. Then f is of class sc^∞.

This should allow to define sc-smooth partitions on spaces which classically do not admit smooth partitions of unity. It would be interesting to see some worked-out examples for interesting (i.e., relevant for applications) sc-structures.

Armed with the philosophical point of view that most constructions of Differential Geometry should carry over if we replace open sets of Banach spaces by open sets of splicing cores, we can introduce the notion of a M-polyfold. This is in the new context the object corresponding to the classical notion of a manifold.

Definition 1.14. Let X be a second countable Hausdorff space. A M-polyfold chart is a triple $(U, \varphi, \mathcal{S})$, where U is an open subset of X and $\varphi : U \to K^{\mathcal{S}}$ a homeomorphism onto an open subset of a splicing core. We say two charts are compatible if the transition map between open subsets of splicing cores is sc-smooth in the sense defined above. A maximal atlas of sc-smoothly compatible M-polyfold charts is called a M-polyfold structure on X.

Let us observe that a M-polyfold is necessarily metrizable. If X is a M-polyfold so is X^1 and moreover X^n for any $n \geq 1$. Here $X^n = (X^{n-1})^1$. Given a M-polyfold X we can construct its tangent TX in a natural way. The projection

$$TX \to X^1$$

is a sc-smooth map. One may view TX as a bundle over X^1. As it will turn out we need to introduce the notion of a strong bundle in order to develop a Fredholm theory. The tangent bundle will in general not be a strong bundle.

For the convenience of the reader let us give some examples illustrating the new notions. We begin with an example for a splicing.

Example 1.15. Let $E = L^2(\mathbb{R})$ be equipped with the sc-structure defined by $E_m = H^{m,\delta_m}$, i.e., maps of Sobolev class H^m_{loc} with derivatives up to order m weighted by $e^{\delta_m |s|}$ belonging to L^2. Here $\delta_0 = 0 < \delta_1 < \dots$ is a strictly increasing sequence of weights. Pick a smooth compactly supported map $\gamma : \mathbb{R} \to [0, 1]$ with

$$\int_{\mathbb{R}} \gamma(t)^2 dt = 1.$$

Next we put $V = \mathbb{R}$ and define for $t \in V$ a family of sc-projections π_t by $\pi_t = 0$ for $t \leq 0$ and for $t > 0$

$$\pi_t(u) = \langle u, \gamma_t \rangle \cdot \gamma_t,$$

where

$$\gamma_t(s) = \gamma(s + e^{\frac{1}{t}}).$$

One can show that $\mathcal{S} := (\pi, E, V)$ is a sc-smooth splicing. The splicing core $K^{\mathcal{S}}$ is homoeomorphic to

$$X = ((-\infty, 0] \times \{0\}) \bigcup ((0, \infty) \times \mathbb{R}).$$

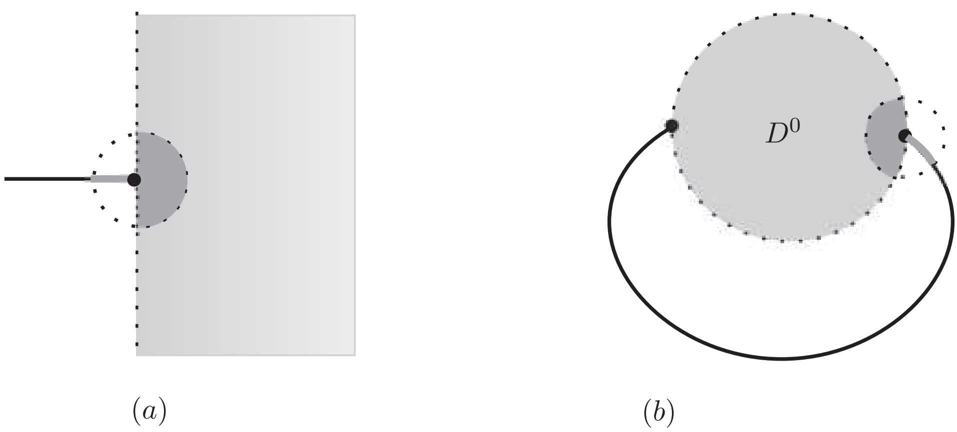

FIGURE 3. The splicing core in the example is shown in a). Picture b) shows a topological space which in view of a) admits the structure of a M-polyfold.

From this it follows that X can be equipped with a M-polyfold structure. The reader will easily modify this example to construct splicings where the local dimensions vary between 1 and any given natural number N. As a consequence one can show that the subspace X of $\mathbb{R}^3$ defined below admits a M-polyfold structure:

$$\begin{aligned} X &= \{(x,0,0) \mid x \in (-\infty, -1] \cup [1,2]\} \bigcup \{(x,y,0) \mid x^2 + y^2 < 1\} \\ &\bigcup \{(x,y,z) \mid |x-3|^2 + y^2 + z^2 < 1\}. \end{aligned}$$

Then X^m is independent of m. Moreover TX would be over the open unit disk a plane bundle and over the line a line bundle, etc. There are, in fact, much more complicated examples with the dimensions even allowed to locally vary between finite and infinite.

An interesting feature is that sc-smooth maps recognize corners. To prove this requires some efforts and we refer the reader for a proof in [**15**]. As a consequence a M-polyfold has its corner structure as an invariant. Take a M-polyfold chart $U \to O$; then O is in an open subset of a splicing core $K = \{(v, e) \in V \oplus E \mid \pi_v(e) = e\}$. Here V is open in a partial cone, say $[0, \infty)^k \times \mathbb{R}^{n-k}$. We can assign to a point $x \in X$ the number $d(x)$ of vanishing first k-coordinates. It turns out that this does not depend on the choice of local coordinates and every point x has an open neighborhood so that $d|U(x) \leq d(x)$.

Definition 1.16. For a M-polyfold X the map $d : X \to \mathbb{N}$ is called the degeneration map.

This map will be important in our Fredholm theory with operations. We need another definition.

Definition 1.17. Given a M-polyfold X we call the closure of a connected component F of $X(1) = \{x \in X \mid d(x) = 1\}$ a face.

It is an easily established fact that around every point $x_0 \in X$ there exists an open neighborhood $U = U(x_0)$ so that every $x \in U$ belongs to exactly $d(x)$ many faces of U. Globally it is always true that $x \in X$ belongs to at most $d(x)$ many faces and strict inequality is possible. For example a two-dimensional closed domain with one corner point, homeomorphic to the closed disk, is not face-structured.

Definition 1.18. We call a M-polyfold face-structured if every point x belongs to $d(x)$ many faces.

Face-structure M-polyfolds and polyfolds will be important in SFT, or more generally in a Fredholm theory with operations, since they have an interesting algebraic structure. For known facts about finite-dimensional manifolds with boundary and corners see for example [**27**] and [**29**].

We continue with our Morse-Theory example in a separate subsection.

1.2.2. *Example of a M-polyfold in Morse-theory.* Consider again our Morse function $\Phi : M \to \mathbb{R}$. For any finite sequence $(a_0, \ldots, a_k)$ of critical points with $a_i < a_{i+1}$ and $k \geq 1$ define $X(a_0, \ldots, a_k) = X(a_0, a_1) \times \ldots \times X(a_{k-1}, a_k)$ and let $\overline{X}$ be the disjoint union of all these $X(a_0, \ldots, a_k)$. One can equip X with a natural second countable Hausdorff topology inducing on all parts the already defined topology and in addition the closure of $X(a, b)$ contains all $X(a_0, \ldots, a_k)$ where $a_0 = a$ and $a_k = b$. Moreover there is a natural M-polyfold structure on $\overline{X}$ so that $\overline{X}$ is faced structured and $d(x) = k - 1$ provided $x \in X(a_0, \ldots, a_k)$ for some sequence $(a_0, \ldots, a_k)$ of ordered critical points. It requires some work to write down the relevant splicing cores and the charts. The main point is, of course, the understanding of the space $\overline{X}$ near a broken trajectory. For this we have to introduce a particular splicing which we discuss now. Similar versions will be crucial for the constructions of polyfolds in SFT. We will only discuss a model situation and refer the reader to [**15**] for full details.

Let us assume we are given three mutually different points a, b and c in $\mathbb{R}^N$. We have seen that $X(a, c)$ is a sc-manifold in a natural way given a sequence of weights (δ_i) starting with $\delta_0 = 0$. Assuming that a and c are critical points for a Morse function Φ the gradient lines connecting a with c (modulo parametrization) would lie in $X(a, c)$. The space of these gradient lines is in general not compact and a gradient line might split into a broken gradient line first going from a to another critical point b and then to c. In order to compactify the space of gradient lines (which we might view as the solution space of a nonlinear elliptic problem) we have to add suitable broken ones. If we want to develop a Fredholm theory for which the compactified space is the solution space, the ambient space needs to contain broken trajectories. A natural choice as a set in our case is obviously

$$\bar{X}(a, c) = (X(a, b) \times X(b, c)) \bigcup X(a, c).$$

The spaces $X(a, b), X(b, c)$ and $X(a, c)$ have natural paracompact second countable topologies. One can show that there is a natural second countable paracompact topology on $\bar{X}(a, c)$ inducing on $X(a, c)$ and $X(a, b) \times X(b, c)$ the given topology so that $X(a, c)$ is dense in $\bar{X}(a, c)$. The topological space $\bar{X}(a, c)$ is not obviously homeomorphic to any open subset of a Banach space. However, it is a nontrivial fact that it is homeomorphic to an open subset of some splicing core. Moreover, all these local homeomorphisms can be picked in such a way that the transition maps are sc-smooth.

The construction of the relevant splicing is closely related to some gluing construction. We begin with "nonlinear gluing" which quite often in literature is referred to as pre-gluing. It associates to a curve connecting a with b, and one connecting b with c, and a gluing parameter $r \in (0, 1)$ a curve connecting a with c. Then we define a (linear) gluing and anti-gluing for vector fields along the underlying given curves. Finally we will show how these constructions are related to each other and how we can construct an associated splicing. It will be a punch-line that the splicing idea can be viewed as a generalization of some constructions arising around the gluing procedure.

Let $\beta : \mathbb{R} \to [0, 1]$ be a smooth cut-off function so that

$$\begin{gathered} \beta(s) = 1 \quad \text{for } s \leq -1 \\ \beta'(s) < 0 \quad \text{for } s \in (-1, 1) \\ \beta(s) + \beta(-s) = 1 \quad \text{for all } s \in \mathbb{R}. \end{gathered}$$

Let us assume we are given $u, v : \mathbb{R} \to \mathbb{R}^N$ with $u(-\infty) = a$, $u(+\infty) = b$, $v(-\infty) = b$ and $v(+\infty) = c$. These two maps are representatives of classes $[u] \in X(a,b)$ and $[v] \in X(b,c)$. For a real number $R \geq 0$ we define the glued map $\oplus_R(u,v)$ by

$$\oplus_R(u,v)(s) = \beta\left(s - \frac{R}{2}\right)u(s) + \left(1 - \beta\left(s - \frac{R}{2}\right)\right)v(s-R).$$

Of interest for us will be the class $[\oplus_R(u,v)] \in X(a,c)$. We will also define a gluing for $R = \infty$ by

$$\oplus_\infty(u,v) = (u,v).$$

The number $R \in [0,\infty) \cup \{\infty\} =: [0,\infty]$ we will call the gluing length. At this point we have defined a gluing for any gluing length in $[0,\infty]$. On the level of equivalence classes we would like to view $([u],[v])$ as on the boundary of the family $\{[\oplus_R(u,v)] \mid R \in [0,\infty)\}$. For this we have to identify the family, say, with $[0,1)$ so that $([u],[v])$ corresponds to 1. One has to be precise here, since there are many ways of identifying $[0,1]$ with $[0,\infty]$. Also a precise choice is required for the definition of the M-polyfold structure (sc-smoothness of the transition maps!). Let us call a number $r \in [0,1]$ a gluing parameter. In order to make a consistent construction which leads to sc-smooth transition maps only the following piece of data, namely a gluing profile, is needed.

Definition 1.19. A gluing profile φ is a diffeomorphism

$$\varphi : (0,1] \to [0,\infty).$$

A gluing profile is obviously a rule of how a gluing parameter is converted into a gluing length. Clearly $r = 0$ is associated to $R = \infty$. We refer the reader to [**18**], where different gluing profiles are studied in the context of Deligne-Mumford Theory of stable Riemann surfaces. Usually, in any application a gluing profile has to satisfy certain growth conditions. A very useful gluing profile is the "exponential profile"

$$\varphi(r) = e^{\frac{1}{r}} - e,$$

and we will use it in the following. Using the letters r and R one should always have in mind that $R = \varphi(r)$.

Let us recall that the inverses of the charts for $X(a,b)$ were of the form

$$h \to [u+h],$$

where $h(0) \in \Sigma$ and Σ is a hypersurface in $\mathbb{R}^N$. One should interpret $u+h$ as $\exp_u(h)$, where h is a vector field along the underlying curve u. Similarly

$$k \to [v+k].$$

Using the gluing construction we can define

$$(r,h,k) \to [\oplus_R(u+h, v+k)].$$

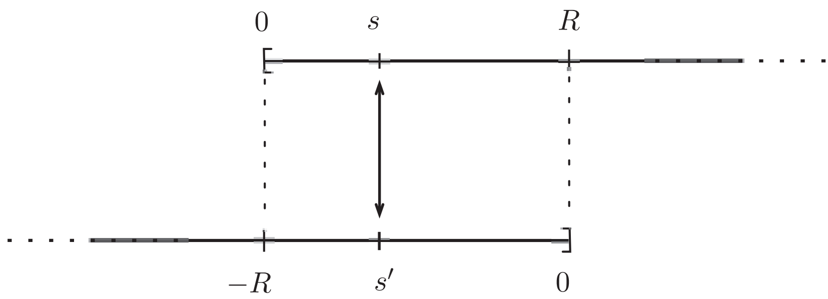

FIGURE 4. Identification $-R+s=s'$.

Any curve $[w]$ near to the broken one $([u],[v])$ can be written in such a way. Of course there is a large ambiguity. The class $[w]$ can be written in many such ways (taking different choices of u and v) and in the following we introduce the concept of anti-gluing to get rid of this ambiguity.

We begin by introducing the "linear gluing" on the level of vector fields. We use the same formula as before:

$$\oplus_R(h,k)(s) = \beta\left(s-\frac{R}{2}\right)h(s) + \left(1-\beta\left(s-\frac{R}{2}\right)\right)k(s-R).$$

It is, of course, important to find a proper interpretation for $\oplus_R(h,k)$, particularly since we want to generalize our model situation to the Morse-theory situation for a manifold M. The right interpretation for $\oplus_R(h,k)$ is that of a vector field along $\oplus_R(u,v)$. Then the obvious relationship

$$\oplus_R(u+h,v+k) = \oplus_R(u,v) + \oplus_R(h,k)$$

can be rewritten as

$$\oplus_R(\exp_u(h),\exp_v(k)) = \exp_{\oplus_R(u,v)}(\oplus_R(h,k)).$$

One can define gluing in the general case in such a way that this formula is true for certain Riemannian metrics. After having defined the "linear gluing" on the level of vector fields along the underlying curves we introduce now the "linear anti-gluing" $\ominus_R(h,k)$. Again h and k are vector fields along u and v as described before. Then we define

$$\ominus_R(h,k)(s) = -\left(1-\beta\left(s-\frac{R}{2}\right)\right)h(s) + \beta\left(s-\frac{R}{2}\right)k(s-R).$$

Observe that the map

$$(h,k)\to(\oplus_R(h,k),\ominus_R(h,k))$$

is an isomorphism (for fixed R). The right interpretation for $\ominus_R(h,k)$ is that of a map

$$\mathbb{R} \to T_b(\mathbb{R}^N).$$

With the gluing constructions at hand we can define the splicing. Denote by E the sc-space consisting of pairs (h,k) of vector fields along u and v, respectively, which satisfy $h(0) \in \Sigma$ and $k(0) \in \Sigma'$.

Then E has for every $r \in [0,1)$ two distinguished sc-complementary subspaces, namely $\ker(\ominus_R)$ and $\ker(\oplus_R)$:

$$E = \ker(\ominus_R) \oplus_{sc} \ker(\oplus_R).$$

In case $r = 0$ we have $\ker(\ominus_\infty) = E$ and $\ker(\oplus_\infty) = \{0\}$. We define $V = [0,1)$ which is an open subset of the cone $[0,\infty) \subset \mathbb{R}$ and denote by π_r the projection onto $\ker(\ominus_R)$ along $\ker(\oplus_R)$. It is a nontrivial result that

THEOREM 1.20. *The triple (π, E, V) is a sc-smooth splicing.*

Then, which is again a nontrivial result, we have

THEOREM 1.21. *Let (u,v) be a smooth pair of paths connecting a via b to c as described before. Then the map*

$$(r,h,k) \to [\oplus_R(\exp_u(h), \exp_v(k))]$$

defined for (r,h,k) sufficiently close to $(0,0,0)$ in the splicing core $K^{\mathcal{S}}$ is a homeomorphism onto an open subset of $\bar{X}(a,c)$. Hence its inverse is a chart with image being an open subset of the splicing core $K^{\mathcal{S}}$. Here $\mathcal{S}$ is the splicing core from Theorem 1.20*. Moreover all these charts together with the charts of the sc-manifold $X(a,c)$ previously constructed are sc-smoothly compatible.*

The previously discussed construction can be brought into a manifold set-up and we can define a M-polyfold structure on the space of broken curves connecting a critical point a with b. We refer the reader to [**15**] for the precise construction.

1.2.3. *Local Strong sc-Bundles.* Let E and F be two Banach spaces with sc-smooth structures. We define their $\triangleleft$-product $E \triangleleft F$ which consists of $E \oplus F$ with the double filtration

$$(E \triangleleft F)_{m,k} = E_m \oplus F_k$$

defined for $0 \leq k \leq m+1$. Observe that the product is not symmetric. Given an open subset U of E we define in the obvious way $U \triangleleft F$. We have a canonical map

$$U \triangleleft F \to U.$$

We refer to the above as a local strong sc-bundle. We define the $\triangleleft$-tangent space by

$$T_\triangleleft(U \triangleleft F) = (TU) \triangleleft (TF).$$

Again we have a double filtration with (m,k) and $k \leq m+1$ by

$$T_{\triangleleft}(U \triangleleft F)_{m,k} = U_{m+1} \oplus E_m \oplus F_{k+1} \oplus F_k.$$

Given $U \triangleleft F \to U$ we can build the associated derived sc-spaces

$$U \oplus F \quad \text{and} \quad U \oplus F^1.$$

DEFINITION 1.22. Let $U \triangleleft F \to U$ and $V \triangleleft G \to V$ be two local strong sc-bundles. A $\text{sc}^0_{\triangleleft}$-map is a map

$$f : U \triangleleft F \to V \triangleleft G$$

of the form

$$f(u,h) = (a(u), \ell(u,h))$$

inducing sc^0-maps between the associated derived sc-spaces, i.e., we have induced maps

$$f : U \oplus F \to V \oplus G$$

and

$$f : U \oplus F^1 \to V \oplus G^1$$

which are sc^0.

We can define a $\text{sc}^1_{\triangleleft}$-notion as follows:

DEFINITION 1.23. We say that the $\text{sc}^0_{\triangleleft}$-map $f : U \triangleleft F \to V \triangleleft G$ is $\text{sc}^1_{\triangleleft}$ provided it induces sc^1 maps between the associated derived sc-spaces.

If f is $\text{sc}^1_{\triangleleft}$ we obtain an induced $\text{sc}^0_{\triangleleft}$-map

$$T_{\triangleleft} f : T_{\triangleleft}(U \triangleleft F) \to T_{\triangleleft}(V \triangleleft G)$$

defined by

$$(T_{\triangleleft} f)(u,h,v,b) = (a(u), Da(u)h, \ell(u,v), D\ell(u,v)(h,b)).$$

The following is easily obtained:

THEOREM 1.24 (Chain Rule for $\text{sc}^1_{\triangleleft}$-maps). *Let $f : U \triangleleft F \to R \triangleleft G$ and $g : V \triangleleft G \to W \triangleleft H$ be $sc^1_{\triangleleft}$, where $U \subset E$ and $V \subset R$ are open with the induced sc-structures, and $g(U \triangleleft F) \subset V \triangleleft G$. Then $g \circ f$ and $sc^1_{\triangleleft}$ and*

$$T_{\triangleleft}(g \circ f) = (T_{\triangleleft} g) \circ (T_{\triangleleft} f).$$

Moreover $T_{\triangleleft}(g \circ f)$ is $sc^0_{\triangleleft}$.

Inductively we can define the notion of being $\text{sc}^k_{\triangleleft}$. If $\Phi : U \triangleleft E \to V \triangleleft F$ is $\text{sc}^k_{\triangleleft}$ and has the form

$$\Phi(x,h) = (a(x), \phi(x,h))$$

and is linear in h we call it a $\text{sc}^k_{\triangleleft}$-vector bundle map. We call Φ a $\text{sc}_{\triangleleft}$ vector bundle isomorphism if it is $\text{sc}_{\triangleleft}$-smooth and the same holds for the inverse.

Given a local strong sc-bundle there are two important classes of sc-smooth sections. Let $U \triangleleft F \to U$ be the bundle. A sc-smooth section f is a map of the form

$$u \to (u, \bar{f}(u))$$

so that the principal part $\bar{f} : U \to F$ is sc-smooth. A sc-section f is called a sc$^+$-smooth section provided the principal part induces a sc-smooth map $\bar{f} : U \to F^1$. The tangent $T_\triangleleft f$ of a sc$^+$-section f is a sc$^+$-section of $TU \triangleleft TF \to TU$. Let us denote the space of sc-smooth sections by $\Gamma(U \triangleleft F)$ and that of sc$^+$-sections by $\Gamma^+(U \triangleleft F)$. Assume that $\Phi : U \triangleleft F \to V \triangleleft G$ is a sc$_\triangleleft$-smooth vector bundle isomorphism. Then the pull-back maps induce isomorphisms

$$\Gamma(V \triangleleft G) \to \Gamma(U \triangleleft G)$$

and

$$\Gamma^+(V \triangleleft G) \to \Gamma^+(U \triangleleft G).$$

The same, of course, holds for the push-forward. As a consequence of the previous discussion we can construct a theory of strong sc-bundles over sc-manifolds. More precisely, let $b : Y \to X$ be a continuous surjective map between second countable Hausdorff spaces so that the preimage of a point $x \in X$ has the structure of Banach space. Then we can equip Y with charts preserving the algebraic structure in the fiber so that the transition maps are sc$_\triangleleft$-smooth vector bundle isomorphisms. The space X then has the underlying structure of a smooth sc-manifold. Note that X inherits a filtration X_m, whereas Y has a double-filtration $Y_{m,k}$ with $0 \leq k \leq m+1$. We denote by $\Gamma(b)$ the vector space of sc-smooth sections of b and by $\Gamma^+(b)$ the vector space of $(+)$-sections.

Let us explain the philosophy behind this concept again with our usual Morse-theory situation. We will stay in the sc-manifolds setting. Let $a < b$ be critical points and consider for u connecting a with b and belonging to H^2, sections of class H^1 along u^*TM. Clearly the map

$$u \to \dot{u} - \Phi'(u)$$

maps u of class H^2 to a H^1-section along u. If we insist on notational grounds that sections preserve the filtration index, then H^2-maps into M lie on the same level as the H^1-sections along them. The following observation is crucial. Note that it makes sense to talk about H^2-sections along an underlying H^2-curve as well, i.e., it makes sense to talk about sections of a somewhat higher regularity along a base curve then a priori seems to be needed for the Fredholm theory. Denote by $Y(a, b)$ the space of equivalence classes defined by the $\mathbb{R}$-action. We have a canonical map

$$Y(a, b) \to X(a, b) : [(u, h)] \to [u].$$

The fiber over a point $[u]$ has a natural Banach space structure. One can equip $Y(a, b)$ with a second countable Hausdorff topology so that the projection map is continuous. Moreover, it possesses the structure of a strong sc$_\triangleleft$-bundle, where the subspace $Y_{m,k}$ consists of all $(k+1, \delta_k^a, \delta_k^b)$-sections along a $(m+2, \delta_m^a, \delta_m^b)$-map. The map

$$f : [u] \to [\dot{u} - \Phi'(u)]$$

defines a sc-smooth section. This section will turn out to be a Fredholm operator in our generalized sense. Clearly f maps X_m to $Y_{m,m}$. A sc$^+$-section s maps X_m to $Y_{m,m+1}$. It will turn out that as a consequence of the compactness property for sc-structures a perturbation of f by s still will be a Fredholm section. However, not for the bundle b, but for the bundle with shifted index b^1:

$$b^1 : (Y^1)_{m,k} := Y_{m+1,k+1} \to (X^1)_m := X_{m+1}.$$

1.2.4. *M-Polyfold Bundles.* The next step consists in introducing M-polyfold bundles. For this we need a particular notion of splicing. Of course, not only the base should be spliced but also the fiber.

Definition 1.25. A spliced sc-fibered Banach scale is a triple $\mathcal{S}_\triangleleft = (\Pi, E \triangleleft H, V)$ where $\Pi = (\pi, \sigma)$ and $\pi_v : E \to E$ and $\sigma_v : H \to H$ are splicing families parameterized by V.

Note that the above data gives splicings $\mathcal{S}_0 = (\pi, E, V)$ and $\mathcal{S}_1 = (\sigma, H, V)$. Then we build the fibered $\triangleleft$-product

$$K^{\mathcal{S}_\triangleleft} := K^{\mathcal{S}_0} \triangleleft_V K^{\mathcal{S}_1}$$

with the double filtration by $[m, k]$ (we write $[m, k]$ to indicate that $0 \leq k \leq m+1$) given by

$$\left(K^{\mathcal{S}_0} \triangleleft_V K^{\mathcal{S}_1}\right)_{m,k} = \{(v, e, h) \in V \oplus E_m \oplus H_k \mid \pi_v(e) = e,\ \rho_v(h) = h\}.$$

The natural projection

$$(V \oplus E) \triangleleft H \to V \oplus E$$

induces a natural projection

$$K^{\mathcal{S}_\triangleleft} \to K^{\mathcal{S}_0}.$$

As before we can define a tangent $T\mathcal{S}_\triangleleft$ of the splicing $\mathcal{S}_\triangleleft$ and associated $K^{T\mathcal{S}_\triangleleft}$ so that $T\,\mathrm{pr}_1$ induces (using the definition)

$$T\,\mathrm{pr}_1 : TK^{\mathcal{S}_\triangleleft} \to TK^{\mathcal{S}_0}.$$

We are interested in pairs $(K^{\mathcal{S}_\triangleleft}|O, \mathcal{S}_\triangleleft)$, where $\mathcal{S}_\triangleleft$ is a spliced sc-fibered Banach scale $(\pi, E \triangleleft H, V)$ and

$$K^{\mathcal{S}_\triangleleft}|O$$

stands for the preimage under the canonical projection

$$K^{\mathcal{S}_\triangleleft} \to K^{\mathcal{S}}$$

of the open subset of O of $K^{\mathcal{S}_0}$. A sc$_\triangleleft$-smooth morphism

$$\Phi : (K^{\mathcal{S}_\triangleleft}|O, \mathcal{S}_\triangleleft) \to (K^{\mathcal{S}'_\triangleleft}|O', \mathcal{S}'_\triangleleft)$$

is a map

$$K^{\mathcal{S}_\triangleleft}|O \to K^{\mathcal{S}'_\triangleleft}|O'$$

of the form

$$(a, b) \to (\phi(a), \Phi(a, b))$$

so that its composition with the projections is a $\text{sc}_\triangleleft$-smooth map. To be more precise, if $a = (r, e, h)$ with $r \in V$, $h \in H$ and $e \in E$, then

$$(r, e, b) \to (\phi(r, \pi_r(e)), \Phi(r, \pi_r(e), \sigma_r(b)))$$

is $\text{sc}_\triangleleft$-smooth. Similarly we can define sc-sections and sc^+-sections $\Gamma(\mathcal{S}_\triangleleft, K^{\mathcal{S}_\triangleleft}|O)$ and $\Gamma^+(\mathcal{S}_\triangleleft, K^{\mathcal{S}_\triangleleft}|O)$. In future, if irrelevant, we might suppress the $\mathcal{S}_\triangleleft$ in the notation and write for example $\Gamma(K^{\mathcal{S}_\triangleleft}|O)$. We also consider $\text{sc}_\triangleleft$-smooth vector bundle morphisms which are those which are linear in the fiber.

In a way similar to how we introduced M-polyfolds and strong sc-vector bundles, we can define M-polyfold bundles $b : Y \to X$. Let us also remark that one can develop a good notion of connection for b. These strong sc-connections have special properties reflecting the fact that we have a grading of the fiber, say $Y_{m,k} \to X_m$ with $0 \leq k \leq m+1$ and the compact inclusion from level $m + 1$ to m. As a consequence covariant derivatives of a section f of b with respect to different connections in this class differ by a sc^+-operator. In particular the difference is a linear sc-compact operator. This will be important for the finer aspects of the Fredholm theory as needed for the "operation theory" in [**17**], f.e. orientation questions.

Coming back to our Morse-theory example we can define a M-polyfold bundle $\overline{Y}$ over the M-polyfold $\overline{X}$. Its elements are sequences $([h_1], \ldots, [h_k])$ of equivalence classes of H^1-sections h_i along underlying curves u_i, where $([u_1], \ldots, [u_k]) \in \overline{X}$. With the notion of sc-Fredholm section, which we are going to introduce in the next section, it will turn out the map

$$f : \overline{X} \to \overline{Y} : ([u_1], \ldots, [u_k]) \to ([\dot{u}_1 - \Phi'(u_1)], \ldots, [\dot{u}_k - \Phi'(u_k)])$$

will be a sc-smooth Fredholm section. Moreover, for every section $s \in \Gamma^+(\overline{Y})$ the section $f + s$ is always sc-Fredholm for the bundle $\overline{Y}^1 \to \overline{X}^1$. (Here all indices are lifted by one.) Further, if f is on every connected component of $\overline{X}$ proper, the same will be true for $f + s$ if s is small enough and has its support in a suitable open neighborhood of $f^{-1}(0)$.

1.3. Polyfold Groupoids and Polyfolds. In dealing with Gromov-Witten theory or more generally with SFT we need an orbifold version of the notion of M-polyfold. Orbifolds always arise if we consider objects modulo some equivalence. Objects with self-symmetries occur as singular points, i.e., true orbifold points. There are different ways of defining orbifolds and similarly (with some modifications) different ways of defining their polyfold-generalization. The approach via groupoids seems particularly useful, even from an analysis viewpoint. We refer the reader to the excellent article by Moerdijk about a groupoid approach to orbifolds, [**32**], as well as the book [**33**].

1.3.1. *Polyfold groupoids.* Recall that a groupoid $\mathfrak{G}$ is a small category where every morphism is invertible. We shall write G for the objects and $\mathbf{G}$ for the morphism set. Given a groupoid we have a certain number of obvious structure maps. There are the source and target maps

$$s, t : \mathbf{G} \to G$$

which associate to a morphism its source or target. Since every morphism is invertible we have the inversion map

$$i : \mathbf{G} \to \mathbf{G} : \phi \to \phi^{-1}.$$

In addition we have the unit map

$$u : G \to \mathbf{G} : x \to 1_x.$$

Finally we can build the fibered product $\mathbf{G}_s{\times}_t\mathbf{G}$ consisting of all pairs of morphisms (ϕ, ψ) with $s(\phi) = t(\psi)$. Then we can define the multiplication map

$$m : \mathbf{G}_s{\times}_t\mathbf{G} \to \mathbf{G} : (\phi, \psi) \to \phi \circ \psi.$$

Now we are almost in the position to introduce the notion of a polyfold groupoid. As a final preparation we need the notion of a fred-submersion between two M-polyfolds X and Y. If $\mathcal{T} = (\pi, E, V)$ and $\mathcal{S} = (\rho, F, V)$ are splicings with common parameter set V we can build the Whitney sum $\mathcal{T} \oplus \mathcal{S}$ by defining

$$\mathcal{T} \oplus \mathcal{S} = (\tau, E \oplus F, V),$$

where

$$\tau_v(e, f) = (\pi_v(e), \rho_v(f)).$$

A particular situation arises if $\mathcal{S} = (Id, \mathbb{R}^N, V)$. In that case we will simply write $\mathcal{T} \oplus \mathbb{R}^N$ instead of $\mathcal{T} \oplus \mathcal{S}$.

Definition 1.26. A sc-smooth map $f : X \to Y$ between the M-polyfolds X and Y is said to be a fred-submersion, if at every point $x_0 \in X$ resp. $f(x_0) \in Y$ there exists a chart $(U, \varphi, \mathcal{T} \oplus \mathbb{R}^N)$ resp. $(W, \psi, \mathcal{T})$ satisfying $f(U) \subset W$ and

$$\psi \circ f \circ \varphi^{-1}(v, e', e'') = (v, e').$$

Note the following easy consequence of the definition of a fred-submersion:

Proposition 1.27. *If $f : X \to Y$ is a fred-submersion between the M-polyfolds X and Y, then for every smooth $y \in Y$ the preimage f^{-1} carries in a natural way the structure of a smooth finite-dimensional manifold.*

Given three M-polyfolds X, X' and Y and sc-smooth maps $s : X \to Y$ and $t : X' \to Y$, we can build as a set the fibered product $X_s{\times}_t X'$ by defining

$$X_s{\times}_t X' = \{(x, x') \mid s(x) = t(x')\}.$$

In certain situations this set carries in a natural way the structure of a M-polyfold. We have

PROPOSITION 1.28. *If at least one of the maps s or t is a fred-submersion the fibered product $X_s{\times}_t X'$ carries in a natural way the structure of a M-polyfold. Further, if s is a fred-submersion the projection*

$$X_s{\times}_t X' \to X'$$

is also a fred-submersion. If t is a fred-submersion the same is true for the projection

$$X_s{\times}_t X' \to X.$$

Now we can give the definition of a polyfold groupoid.

DEFINITION 1.29. A polyfold groupoid is a groupoid $\mathfrak{X}$, together with a M-polyfold structure for the set of objects X and the set of morphisms $\mathbf{X}$ so that the source and target maps s and t are surjective fred-submersions and all structure maps are sc-smooth. We assume the induced topologies on X and $\mathbf{X}$ to be second countable and paracompact.

Note that $\mathbf{X}_s{\times}_t\mathbf{X}$ is a M-polyfold since s and t are fred-submersions, so that it makes sense to talk about the sc-smoothness of the multiplication map m.

Clearly, the notion of a polyfold groupoid is a straightforward modification of that of a Lie groupoid, where we have replaced the notion of a finite-dimensional manifold by that of a M-polyfold and the notion of submersion is modified by that of a fred-submersion; see [**32, 33**]. For the convenience of the reader let us recall the definition:

DEFINITION 1.30. A Lie groupoid is a small category $\mathfrak{X}$, where the set of objects X and morphisms $\mathbf{X}$ is equipped with a smooth manifold structure[5], so that the source and target maps are surjective submersions and all structure maps are smooth.

The orbit space $|\mathfrak{X}|$ of a polyfold groupoid consists of the quotient space $X/\sim$, where two points are identified if they are related by a morphism. Observe that $|\mathfrak{X}|$ inherits a filtration from X. The maps between two polyfold groupoids are the sc-smooth functors $F : \mathfrak{X} \to \mathfrak{Y}$. That means F induces sc-smooth maps $X \to Y$ and $\mathbf{X} \to \mathbf{Y}$. A sc-smooth functor F induces a sc^0-map between the orbit-spaces.

[5]For our purposes we can assume that the manifolds are second countable. However, for certain applications (not relevant for us) one should allow non-Hausdorff manifolds, see [**33**].

For the construction of polyfolds it will be important to introduce the notion of a generalized map. The discussion is similar to that in the Lie groupoid situation. For this let us first introduce the notion of an equivalence.

Definition 1.31. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be M-polyfold groupoids. A sc-smooth functor $F:\mathfrak{X}\rightarrow\mathfrak{Y}$ is called an equivalence provided the following holds:

1) The map $t\pi_1:\mathbf{Y}_s{\times}_F X\rightarrow Y$ is a surjective fred-submersion.
2) The square

$$\begin{array}{ccc}\mathbf{Y} & \xrightarrow{F} & \mathbf{X}\\ \big\downarrow{\scriptstyle (s,t)} & & \big\downarrow{\scriptstyle (s,t)}\\ Y\times Y & \xrightarrow{F\times F} & X\times X\end{array}$$

is a fibered product.

An equivalence is usually not invertible. At this point we have a category where the objects are M-polyfold groupoids with the sc-smooth functors being the morphisms between them. Further we have a distinguished family of special morphisms, namely the equivalences. There is now a very particular, purely category-theoretic procedure for inverting a distinguished class of arrows in a category, while at the same time keeping the objects and only minimally changing (given the fact that we must invert a certain number of given arrows) the morphisms. The general procedure is described in [**13**]. Here we will describe the procedure for our special situation. We need a certain amount of preparation.

Definition 1.32. Assume that $F,G:\mathfrak{A}\rightarrow\mathfrak{B}$ are sc-smooth functors. They are called equivalent if there exists a sc-smooth map

$$\tau:A\rightarrow\mathbf{B}$$

associating to every object $x\in A$ a morphism

$$\tau(x):F(x)\rightarrow G(x)$$

so that for every $h:x\rightarrow x'$ we obtain the commutative diagram

$$\begin{array}{ccc}F(x) & \xrightarrow{\tau(x)} & G(x)\\ \big\downarrow{\scriptstyle F(h)} & & \big\downarrow{\scriptstyle G(h)}\\ F(x') & \xrightarrow{\tau(x')} & G(x').\end{array}$$

The map τ is called a natural transformation.

In order to define generalized maps start with a diagram

$$\mathfrak{X}\xleftarrow{F}\mathfrak{A}\xrightarrow{\phi}\mathfrak{Y},$$

where F is an equivalence and ϕ a sc-smooth functor. Let us call $\mathfrak{X}$ the domain and $\mathfrak{Y}$ the codomain (of the diagram). Consider a second such diagram

$$\mathfrak{X} \xleftarrow{F'} \mathfrak{B} \xrightarrow{\phi'} \mathfrak{Y},$$

with identical domain and codomain. We call it a refinement of the first if there exists a sc-smooth functor $H : \mathfrak{B} \rightarrow \mathfrak{A}$ so that $F \circ H$ and F' are naturally equivalent as well as $\phi \circ H$ and ϕ. Finally we say that two diagrams of maps, say

$$\mathfrak{X} \xleftarrow{F} \mathfrak{A} \xrightarrow{\phi} \mathfrak{Y} \text{ and } \mathfrak{X} \xleftarrow{F'} \mathfrak{A}' \xrightarrow{\phi'} \mathfrak{Y}$$

are equivalent if they have a common refinement. It takes a certain amount of work to show that this indeed defines an equivalence relation. We associate to a smooth functor $\phi : \mathfrak{X} \rightarrow \mathfrak{Y}$ the equivalence class of

$$\mathfrak{X} \xleftarrow{Id} \mathfrak{X} \xrightarrow{\phi} \mathfrak{Y}.$$

Let us denote this equivalence class by $[\phi]$. Similarly we denote for an equivalence $F : \mathfrak{X} \rightarrow \mathfrak{Y}$ the class of the diagram

$$\mathfrak{Y} \xleftarrow{F} \mathfrak{X} \xrightarrow{Id} \mathfrak{X}$$

by $[F]^{-1}$. Then one verifies immediately that this is the inverse of $[F]$.

1.3.2. *Polyfolds.* Now we are in the position to define polyfolds. These type of spaces suffice to carry out all the functional analytic constructions in Gromov-Witten, Floer-Theory, and SFT.

DEFINITION 1.33. Consider a polyfold groupoid $\mathfrak{X}$.

1) We say that $\mathfrak{X}$ is étale provided the source and target maps are local sc-diffeomorphisms.
2) We say that $\mathfrak{X}$ is proper if for every $x \in X$ there exists an open neighborhood $U(x)$ so that the map

$$t : s^{-1}(\overline{U(x)}) \rightarrow X$$

is proper.
3) A polyfold groupoid which is étale and proper is called an ep-polyfold groupoid.

Note that we could interchange the role of s and t in the definition of proper defining the same property.

DEFINITION 1.34. Let Z be a second countable paracompact topological space. An ep-polyfold structure on Z is given by a pair $(\mathfrak{X}, \beta)$, where $\mathfrak{X}$ is a ep-polyfold groupoid and $\beta : |\mathfrak{X}| \rightarrow Z$ a homeomorphism. We say that two polyfold structures $(\mathfrak{X}, \beta)$ and $(\mathfrak{X}', \beta')$ are equivalent if there exist equivalences

$$F : \mathfrak{X}'' \rightarrow \mathfrak{X} \text{ and } \mathfrak{X}'' \rightarrow \mathfrak{X}'$$

so that

$$\beta \circ |F| = \beta' \circ |F'|.$$

Let us observe that $(\mathfrak{X}'', \beta \circ F)$ is also a polyfold structure equivalent to the two other ones. Finally we can introduce the notion of a polyfold:

Definition 1.35. A second countable paracompact topological space Z equipped with an equivalence class of polyfold structures is called a polyfold.

For a polyfold Z we can define a degeneracy function $d : Z \to \mathbb{N}$ by

$$d(z) = d_X(x)$$

where $(\mathfrak{X}, \beta)$ is a defining polyfold structure and $\beta(x) = z$. Here d_X is the degeneracy map for X. This is well-defined, i.e., independent of the representative of the polyfold structure. We leave it to the reader to verify that it makes sense to talk about a face-structured polyfold.

1.3.3. *Polyfold bundles.* We have already defined strong bundles over M-polyfolds. In a first step we introduce strong bundles over a polyfold groupoid. Let us start with a given polyfold groupoid $\mathfrak{X}$ and a strong M-polyfold bundle $\tau : E \to X$ over X. Using the fact that the source map $s : \mathbf{X} \to X$ is a fred-submersion we can build the pull-back bundle via s over $\mathbf{X}$. This pull-back bundle is, of course,

$$\mathbf{X}_s{\times}_\tau E \to \mathbf{X}.$$

This is a strong bundle over $\mathbf{X}$. Assume we are given a strong bundle map $\mu : \mathbf{X}_s{\times}_\tau E \to E$ covering t. To be more precise μ gives the following commutative diagram

$$\begin{array}{ccc} \mathbf{X}_s{\times}_\tau E & \xrightarrow{\mu} & E \\ \Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \tau} \\ \mathbf{X} & \xrightarrow{t} & X. \end{array}$$

We require now μ to be compatible with the morphisms in $\mathbf{X}$. More precisely we require with the abbreviation $g \cdot e = \mu(g, e)$ that

- The identity $1_x \cdot e = e$ holds.
- Moreover $(g \circ h) \cdot e = g \cdot (h \cdot e)$.

The following definition is useful:

Definition 1.36. Let $\mathfrak{X}$ be a polyfold groupoid. A strong linear $\mathfrak{X}$-space is given by a pair (E, μ), where $\tau : E \to X$ is a strong bundle over X and $\mu : \mathbf{X}_s{\times}_\tau E \to E$ a strong vector bundle isomorphism satisfying the properties above.

Given a strong linear $\mathfrak{X}$-space (E, μ) we can build a category $\mathfrak{E}$, with objects E and morphism set $\mathbf{E}$ defined by

$$\mathbf{E} := \mathbf{X}_s{\times}_\tau E.$$

We define the source map s by $s(g,e) = e$ and the target map by $t(g,e) = g \cdot e$. This defines a polyfold groupoid $\mathfrak{E}$. Note, however, that we have more structure since $E \to X$ for example is a strong bundle. The projection map $\tau : E \to X$ extends to a sc-smooth functor

$$\tau : \mathfrak{E} \to \mathfrak{X}.$$

We might view the latter diagram as the strong linear $\mathfrak{X}$-space, keeping the extra structure of $\mathfrak{E}$ in mind.

Next we introduce polyfold bundles. Assume we start with a surjective continuous map $\tau : L \to Z$ between second countable paracompact spaces. We assume in addition that the fibers are equipped with Banach space structures. Consider a strong bundle $\pi : \mathfrak{E} \to \mathfrak{X}$ over the ep-polyfold groupoid $\mathfrak{X}$ and assume we are given homeomorphisms

$$\beta : |\mathfrak{E}| \to L \text{ and } \beta_0 : |\mathfrak{X}| \to Z,$$

so that

$$\tau \circ \beta = \beta_0 \circ |\pi|.$$

We can define equivalence classes of such objects $(\pi : \mathfrak{E} \to \mathfrak{X}, \beta, \beta_0)$ as before by refinements.

Definition 1.37. Assume that we are given two second countable paracompact spaces together with a surjective continuous map $\tau : L \to Z$. A strong polyfold bundle structure for τ is given by an equivalence class of triples $(\pi : \mathfrak{E} \to \mathfrak{X}, \beta, \beta_0)$ as described above.

A sc-smooth section f of $\tau : L \to Z$ would we be represented by a sc-smooth section (functor) F of some $\pi : \mathfrak{E} \to \mathfrak{X}$, where the latter occurs in a triple defining the strong polybundle structure. We would like to point out that these abstract bundles together with the Fredholm theory developed in the following section are sufficient as a functional analytic framework for Gromov-Witten theory. Our Fredholm theory will allow for an abstract multi-valued perturbation theory so that the solutions spaces in Gromov-Witten theory are smooth compact branched manifolds. McDuff has developed a convenient frame-work, [**31**], in which integration, differential forms etc. make sense so that many formulas in Gromov-Witten theory can be in fact obtained through integration of suitable quantities over the moduli spaces. So far the smoothness theory (in the symplectic case) for the moduli spaces was not developed to the extent necessary.

1.4. Comments. Let us finish the section with some remarks about possible variations of our previous definitions. For example one could remove the condition that the embedding $E_n \to E_m$ is compact for $n > m$. In that case the sequence $E_m = E$ would be allowed. However, in order to obtain the chain rule the notion of a sc-smooth map has to be modified as well. In fact, one has to replace the requirement that

$$U_{m+1} \oplus E_m \to F_m : (x, h) \to Df(x)h$$

is continuous by the requirement that

$$U_{m+1} \to L(E_m, F_m) : x \to Df(x)$$

is continuous. In the case of the constant sequence $E_m = E$ we recover the standard notion of Frechet differentiability in a Banach space. As it turns out this modified concept and the associated theory parallel to the previous discussion is applicable neither to Gromov-Witten theory, nor to Floer Theory and SFT. Observe that our notion of smoothness and the above modified notion only coincide in the finite-dimensional case, but are otherwise strikingly different. Hence the notion of differentiability which works in our applications of interest generalizes the finite-dimensional notion, but does not contain Frechet-differentiability in infinite-dimensional Banach spaces (which is the commonly used generalization) as a special case.

One can also generalize the notion of splicing allowing for an infinite-dimensional set of splicing parameters, but the author is not currently aware of any good application.

The notion of ep-polyfold can be generalized by not requiring the étale condition. In this case one could study objects with a compact Lie group as a symmetry group (isotropy group). This would be a necessary generalization to deal with Yang-Mills type problems.

2. Generalized Fredholm Theory

In this section we describe the Fredholm theory. The article [**3**] gives an overview of the classical Fredholm theory. The constructions and ideas described in [**3**] can be carried out within our generalized Fredholm context. In addition many more constructions and concepts are possible and lead to a theory applicable to a much wider range of problems. Moreover, the "Fredholm Theory with Operations" which we describe in some simple cases later, gives a broad range of new structures on Fredholm problems. In principle one could have formulated such an abstract theory also within the classical Fredholm theory. Unfortunately, these structures have only been observed as consequences of the lack of compactness and clever compactifications of the moduli spaces, i.e., in situations where the classical theory is not applicable.

We forget sc-structures for the moment and have a look at the classical Fredholm-situation. Assume that $f : U \to F$ is a smooth map, which is defined on the open neighborhood $U \subset E$ of 0 and takes values in F. Moreover $f(0) = 0$. Here E and F are Banach spaces. We assume that $f'(0)$ is a linear Fredholm operator. Then there exist topological splittings of the domain $E = K \oplus X$ and the target $F = C \oplus Y$ with the property that

$$f'(0) : X \to Y$$

is a linear isomorphism. Define an isomorphism

$$\sigma : F = C \oplus Y \to C \oplus X : (c, y) \to (c, f'(0)^{-1}(y)).$$

Then the composition

$$K \oplus X \to C \oplus X : (k, x) \to \sigma \circ f'(0)(k + x)$$

has the form $(k, x) \to (0, x)$. If we consider now $\sigma \circ f(k, x)$ and project in the target onto X along C we obtain a map of the form

$$(k, x) \to x - B(k, x)$$

where for k small $x \to B(k, x)$ is a contraction. In other words, by appropriately taking a suitable coordinate representation of f, a suitable splitting of the domain and a projection onto a finite codimension subspace this new map looks like a parameterized contractive perturbation of the identity. If $f(x_0) = y_0 \neq 0$ and we want to understand the behavior near x_0, then we can look at $x \to g(x) := f(x + x_0) - f(x_0)$ which brings us back into the first case. It is not difficult to verify that the contraction "normal form" together with a smoothness requirement is equivalent to saying that the original f is Fredholm. This equivalent formulation fits well into our sc-framework, as can be seen in the following.

2.1. Contraction Germs. Now we introduce the relevant result for our situation. If we talk about a sc-germ $f : \mathfrak{O}(E, 0) \to (F, 0)$ we mean that for every level m we have an open neighborhood around 0 (for the topology on the m-level) on which f is defined and maps it into m-level. Below $\mathbb{N}$ denotes the non-negative integers.

Definition 2.1. Let $f : \mathfrak{O}(V \oplus E, 0) \to (E, 0)$ be a sc^0-germ with $f(0, 0) = 0$, where V is an open subset of a partial cone in some finite-dimensional vector space and E is a sc-Banach space. We call f a sc-contraction germ if f can be written in the form

$$f(v, u) = u - B(v, u)$$

so that the following holds. For every level $m \in \mathbb{N}$, and a suitable $\Theta_m \in (0, 1)$ we have an estimate

$$\| B(v, u) - B(v, u') \leq \Theta_m \cdot \| u - u' \|_m$$

provided v, u, u' are close enough to 0 (the notion of close depending on m and Θ_m).

Banach's fixed point theorem applied to every level gives a sc^0-germ $\delta : \mathfrak{O}(V, 0) \to \mathfrak{O}(E, 0)$ so that its graph $gr(\delta)$ satisfies

$$f \circ gr(\delta) = 0.$$

The main result is the following "Germ-Implicit Function Theorem":

THEOREM 2.2 (Germ-Implicit Function Theorem). *Let $f : \mathfrak{O}(V \oplus E, 0) \to \mathfrak{O}(E, 0)$ be sc-smooth and a sc^0-contraction germ. Then the solutions germ δ of $f \circ gr(\delta) = 0$ is sc-smooth.*

To be more precise, the conclusion is that for every m and k and $|v|$ small enough the map $v \to \delta(v)$ goes into the m-level and is C^k. In particular δ is C^∞ at the point 0. As it turns out, describing more globally the solution set of a problem of the form $f = 0$, all these local solution germs fit smoothly together to give the solution set a smooth structure. Via this observation the above theorem will be one of the key building blocks for all versions of the implicit function theorem, as well as transversality theory.

2.2. Fillers. We would like to use the above discussion to generalize the idea of a Fredholm section in our polyfold set-up. The study above only centers at a particular local situation which takes place in open subsets of Banach spaces (with a sc-structure). Clearly we need to explain how splicings come in if we want to develop a theory in polyfolds. Of course, one might expect that locally varying dimensions of the ambient spaces lead to a cumbersome definition of what a Fredholm operator in such a context would really mean. Surprisingly the idea of a "Filler", which is a rather simple object, makes it a non-issue.

We start with the local set-up. Assume that we have a strong local sc-vector bundle

$$U \triangleleft F \to U,$$

where U is open in the sc-Banach space E. Since we will only be interested in the neighborhoods of smooth points we may without loss of generality assume that $0 \in U$. Also, being only interested in a neighborhood of 0 we will write $\mathfrak{O}(U \triangleleft F, 0)$ to emphasize that we consider the germ of a bundle. Given a germ of a section $[f, 0]$ we denote by $\bar{f}$ the principal part of f. Then

$$\bar{f} : \mathfrak{O}(U, 0) \to \mathfrak{O}(F, \bar{f}(0)).$$

Linearize $\bar{f}$ at 0, say $\bar{f}'(0) : E \to F$. We say that $[f, 0]$ is linearized Fredholm if $\bar{f}'(0)$ is sc-Fredholm. Observe, that the linearization of a section f of some abstract vector bundle at some point q with $f(q) \neq 0$ is not an intrinsic object, whereas it is at a zero. It depends on the choice of local coordinates.

In our case, however, due to the notion of strong bundle the property of being linearized Fredholm is in fact intrinsic in the following sense. If $\Psi : \mathfrak{O}(U \triangleleft F, 0) \to \mathfrak{O}(V \triangleleft G, 0)$ is a germ of a sc-vector bundle isomorphism, then the push-forward germ $\Psi_*([f, 0])$ is linearized Fredholm if and only this is true for $[f, q]$. This is a deeper consequence of the property of a strong(!) sc-vector bundle. The reader might verify, that the linearizations taken of two different local coordinate representations (using strong bundle coordinates), say L_1 and L_2, are related by

$$L_1 = AL_2B + K,$$

where A and B are sc-isomorphisms and K is a sc^+-operator. In particular K is level-wise a compact perturbation. Hence, given a section of a strong sc-vector bundle $b : Y \to X$, saying that for a $q \in X^\infty$, i.e., a smooth q, the germ $[f, q]$ is linearized Fredholm has an intrinsic meaning (of course the actual linearization depends on the choice of local coordinates). Up to this point we have discussed (germs of) sections of a strong bundle over a sc-manifold.

Let us assume next that we have a germ of sc-smooth section $[f, 0]$ of a local M-polyfold bundle. In other words for (v, e) in the splicing core $K^{\mathcal{S}_0}$ near 0 the map f takes the form

$$f(v, e) = ((v, e), \bar{f}(v, e))$$

with image in $K^{\mathcal{S}_0} \triangleleft_V K^{\mathcal{S}_1}$. Let us abbreviate $K_i = K^{\mathcal{S}_i}$. If we fix v the space $K_{0,v} = \{e \mid \pi_v(e) = e\}$ is a sc-Banach space and similarly $K_{1,v}$. The principal part $\bar{f}$ has the property that $e \to \bar{f}(v, e)$ maps $K_{0,v}$ to $K_{1,v}$. We say that $[f, 0]$ is linearized Fredholm provided the derivative of map $e \to \bar{f}(0, e)$ at 0 is sc-Fredholm. It turns out that this is again an intrinsic definition invariant under changes of coordinates.

If we have a splicing $\mathcal{S} = (\pi, E, V)$ we also have the complementary splicing $\mathcal{S}^c = (I - \pi, E, V)$. Clearly $V \oplus E$ can be viewed as the fiber sum over V

$$V \oplus E = K^{\mathcal{S}} \oplus_V K^{\mathcal{S}^c}.$$

In particular a point (v, e) in $V \oplus E$ can be written as

$$(v, e) = (v, u_v + u_v^c),$$

where $v \in V$, $u_v \in K_v^{\mathcal{S}}$ and $u_v^c \in K_v^{\mathcal{S}^c}$. A section germ $[f, 0]$ for $\mathfrak{O}(K^{\mathcal{S}_0} \triangleleft_V K^{\mathcal{S}_1}, 0)$ is called fillable if there exists a section germ $[\hat{f}, 0]$ of $\mathfrak{O}((V \oplus E) \triangleleft F, 0)$ having the form

$$\hat{f}(v, u_v + u_v^c) = ((v, u_v + u_v^c), \bar{f}(v, u_v) + \bar{f}^c(v, u_v, u_v^c)),$$

where $\bar{f}^c$ is defined on an open neighborhood of $(0, 0)$ in $V \oplus E$ mapping its points to K_1^c in such a way that $u_v^c \to \bar{f}^c(v, u_v, u_v^c)$ is a linear sc-isomorphism $K_{0,v}^c \to K_{1,v}^c$.

What is the significance of a filler? We would like to study the section f, meaning that we are interested in the solution set of $f = 0$. Here, for $(v, u_v) \in K_v^{\mathcal{S}_0}$ we have $(v, \bar{f}(v, u_v)) \in K_v^{\mathcal{S}_1}$. Consider now the filled section $\hat{f}$. If $\hat{f}(v, u) = 0$ we conclude that

$$\bar{f}^c(v, u_v, u_v^c) = 0.$$

By the properties of the filler this means that $u_v^c = 0$. Hence we conclude that $(v, u) = (v, u_v) \in K_v^{\mathcal{S}_0}$ and

$$\bar{f}(v, u) = 0.$$

In other words, the modification by a filler does not change the solution set. Hence, locally the study of $\bar{f}$, which is defined on a perhaps very bad space with varying dimensions, is equivalent to the study of the section $\hat{f}$ on open sets of Banach spaces with sc-structure. The nice fact is that in applications, there are usually obvious choices for fillers. For example in Morse-theory every critical point b has an associated filler

$$h \rightarrow \dot{h} - \Phi''(b)h,$$

where $h : \mathbb{R} \rightarrow T_bM$ belongs to a suitable sc-Hilbert space of functions and $\Phi''(b)$ is the Hessian. In Gromov-Witten theory the situation is slightly more complicated. The fillers are associated to the images of the nodal points which can be any point on the symplectic manifold W. Then the filler associated to $w \in W$ is

$$h \rightarrow h_s + J(w)h_t,$$

i.e., the linear Cauchy-Riemann operator acting on maps

$$h : \mathbb{R} \times S^1 \rightarrow T_wW,$$

where h takes antipodal values at $\pm\infty$:

$$h(-\infty) + h(+\infty) = 0$$

and belongs to a certain Sobolev class. In SFT the asymptotic periodic orbits behave like one-dimensional Morse-Bott manifolds of critical points and the fillers are one-dimensional families of (linear) perturbed Cauchy-Riemann type problems associated to periodic orbits. We refer the reader to a comprehensive discussion of fillers in [**15**] including a complete discussion of the Morse-theory case. The fillers for SFT are constructed in full detail in [**16**].

Let us conclude, that the bottom line of the "filler discussion" is that we can reduce the local study of sections of strong M-polyfold bundles to the study of sections of strong local bundles. In this context the previous results on "contraction germs" allows us to derive suitable implicit function theorems.

2.3. Fredholm Operators. Let $b : Y \rightarrow X$ be a smooth M-polyfold bundle and f a section. We will define the notion of a Fredholm section. We give a less condensed form (than is possible) since it is more instructive. First of all f is sc-smooth and regularizing, i.e., if $f(x) \in Y_{m,m+1}$ then $x \in X_{m+1}$. Of course, if f is regularizing and $f(x) = 0$ we can conclude that $x \in X_\infty$. Let us also note the following important fact for the perturbation theory using sc^+-sections, $s \in \Gamma^+(b)$. If $f(x) + s(x) \in Y_{m,m+1}$ for $s \in \Gamma^+(b)$, then $x \in X_m$ and $s(x) \in Y_{m,m+1}$. The latter is true by definition of a sc^+-section. Consequently $f(x) \in Y_{m,m+1}$ implying that $x \in X_{m+1}$. In other words $f + s$ is also regularizing.

Secondly, for every smooth $q \in X$, i.e., $q \in X_\infty$, there is a local strong M-polyfold bundle trivialization mapping the germ $[f, q]$ to a fillable $[f_1, 0]$. Further there exists a filled section $[\hat{f}, 0]$ which is a section germ of $\mathfrak{O}((V \oplus E) \triangleleft F, 0)$. Thirdly there is a sc-Banach space W, a finite dimensional vector space R, another finite-dimensional vector space Q and an open subset B of some partial cone in Q and a local strong $\text{sc}_\triangleleft$-vector bundle isomorphism

$$\Psi : \mathfrak{O}((V \oplus E) \triangleleft F, 0) \to \mathfrak{O}((B \oplus W) \triangleleft (R \oplus W), 0)$$

so that the push-forward $[g, 0]$ of $[\hat{f}, 0]$ has the following property. If $P : R \oplus W \to W$ is the projection and $\bar{g}$ the principal part, then

$$(b, w) \to P\bar{g}(b, w) - P\bar{g}(0, 0)$$

is a sc^0-contraction germ. In other words:

"A section of a M-polyfold bundle is Fredholm provided it is regularizing and in suitable local coordinates it admits a filler, so that the filled section gives under another coordinate change and suitable splittings and projection a contraction germ."

Let us denote by Fred(b) the Fredholm sections of b. The definition of a Fredholm section is very general. It looks not very practical at first sight, but at least as far as applications are concerned, it indeed is and the methods for showing that the nonlinear elliptic pde's in GW, FT, CH and SFT are Fredholm in our generalized sense are almost identical. Let us elaborate somewhat on this point. The advantage of the classical implicit function theorem is, of course, the fact that we can conclude something about the local properties of the solution set by knowing something about the linearization at a single point. In our applications it is, however, not applicable. Nevertheless, in practice our problems will in general only have a finite-dimensional set of bad parameters which will not enter smoothly (but they enter in a sc-smooth way). With respect to the remaining variables we will have smoothness and linearizations (in the classical way on every level). A certain uniformity of behavior of these linearizations with respect to the bad parameters allows to show the "contraction normal form". Hence, in applications, the analysis needed is concerned with the standard implicit function theorem applied to continuous (finite-dimensional) families of maps and the normal form is a consequence of certain uniformity of the estimates. This is reminiscent of the uniformity of estimates in the gluing constructions occurring in FT or GW, see f.e. [**30**]. We also refer the reader to [**15**], where the Fredholm property is shown in the context of Morse-theory, validating our previous remarks.

There are quite a number of consequences of this definition. For example if f is a Fredholm section of $b : Y \to X$, and s is a sc^+-section then $f + s$ is a Fredholm section of $b^1 : Y^1 \to X^1$. Hence, Proposition 2.3. *There is a well-defined map*

$$Fred(b) \times \Gamma^+(b) \to Fred(b^1) : (f, s) \to f + s.$$

An important consequence of the germ-implicit function theorem and the definition of Fredholm section is the following implicit function theorem:

Theorem 2.4. *Assume that $b : Y \to X$ is a smooth M-polyfold bundle and f a Fredholm section. Suppose further that $\partial X = \emptyset$. If $f(q) = 0$ and the linearization $f'(q) : T_qX \to Y_q$ is onto then the solution set of $f(x) = 0$ is near q a smooth manifold (in a natural way).*

2.4. Comments. There are also results concerning the case where X has a boundary with corners, i.e., $\partial X \neq \emptyset$. We refer the reader for more results to [**15**].

One can show that if f is Fredholm there are many small perturbations by sc^+-sections s so that for every solution of $f(x) + s(x) = 0$ the linearisation is onto, i.e., $(f + s)^{-1}(0)$ is a smooth manifold. Moreover, if f is proper then $f + s$ will be proper if s is small enough in a suitable sense. In addition generic perturbations put the solution set into general position to ∂X, see [**15**].

In the case of polyfold bundles where the section is represented by a smooth section functor F of $\mathfrak{E} \to \mathfrak{X}$, the compatibility with the morphisms asks for a perturbation theory respecting the morphisms. As is well-known this required compatibility obstructs transversality in general, if we are allowed only single-valued perturbation. This changes if multi-valued perturbations are allowed. In this case we obtain for generic multi-valued sc^+-perturbation as a solution set a smooth branched weighted manifold with boundary with corners (in general position to the boundary) as defined by D. McDuff, see [**31**]. For details of the Fredholm theory in this context we refer the reader to [**15**].

3. Operations

In this section we describe the important theory of operations, which allows to capture the "algebra" underlying the structure of having infinitely many interacting Fredholm operators. We will describe only a somewhat simplified version in order to expose the ideas. We refer the reader to [**17**] for extensions which are in fact necessary for the applications we have in mind. We begin with a very useful rudimentary algebraic structure.

3.1. Degeneration Structures. A set with relators is a pair (S, R) with a set S and a subset R of $S \times S \times S$. We write $(A, B; C)$ for an element in R and call A the left-source, B the right-source and C the target. Given (S, R) consider $(k + 1)$-tuples $z_k = (A_0, \ldots, A_k)$ with

$A_i \in S$. A 1-step degeneration is a diagram

$$z_k \leftarrow z_{k-1}$$

where z_k is obtained from z_{k-1} by replacing an occurring element C by two elements (A, B) if $(A, B; C) \in R$. A "Short Degeneration Sequence" has the form $z_0 \to z_1 \to z_2$.

Definition 3.1. A degeneration structure $\mathfrak{S} = (S, R)$ consists of a set with relators (S, R) so that the following properties hold:

1) (Degeneration Finiteness) Given $Z \in S$ the number of degeneration sequences starting at (Z) is finite.
2) (Associativity) The set of short degeneration sequences having prescribed target and source is either empty or consists of precisely two elements. In the latter case these have the form

$$(Z) \to (A, B) \to (A, I, E) \text{ and } (Z) \to (A^*, E) \to (A, I, E).$$

3) (Minimality) If $(A, B; C)$ and $(A', B'; C)$ belong to R and either $A = A'$ and A is not decomposable or if $B = B'$ and B is not decomposable then $(A, B) = (A', B')$.

Here A is called decomposable if there exist $X, Y \in S$ with $(X, Y; A) \in R$. Observe that for a degeneration structure $(A, B; C) \in R$ implies $A \neq C$ and $B \neq C$. This is a consequence of the finiteness axiom. It is important to note that

Proposition 3.2. *If there exists a degeneration sequence*

$$z_n \leftarrow \cdots \leftarrow z_0 = (Z),$$

then there exist exactly $n!$ *degeneration sequences connecting* z_0 *with* z_n.

Degeneration structures will be used to organize large families of interacting Fredholm problems. Here is an example of how they might occur. If f is a Fredholm section of the polyfold bundle $Y \to X$ we might consider the set S of connected components of X, i.e., $S = \pi_0(X)$. All symplectic problems we have mentioned have the following structure as Fredholm problems. First of all the underlying strong bundle $\tau : Y \to X$ is defined over a face-structured polyfold. The compactified moduli space is defined by $f = 0$ for a suitable Fredholm section f of τ. In general X has infinitely components and the moduli space in every component is compact, whereas the overall solution space is not compact. Now here is a rough description of a common feature of FT, CH, SFT and in fact Morse Theory viewed in an appropriate way. If we are given a point z which solves the Fredholm problem $f(z) = 0$ and z belongs to a face of Z then z can be viewed as the "product" $z' \circ z''$ of two solutions z' and z'' in different components of X. (For example, a boundary face of the moduli space of gradient lines from a to c consist of the broken trajectories (z', z'') factoring over the same intermediate critical point b. In other words z' is a perhaps broken gradient line connecting a with an intermediate critical point b and z' connects b with c. Hence we may view z as the product $z' \circ z''$.) Coming back to the abstract situation, if one of them, say z', is again a boundary point, the point z' is again a product and so on. Here, of course, product refers to some kind of composition law how to build out of two solutions a new one. In general the way to write z as a product is not unique and the non-uniqueness depends on the degeneracy $d(z)$. (For example, a two-broken gradient line can be viewed in precisely two different ways as product. Moreover, we can view it as triple product and, of course, this example already shows an associativity property which should be required in general.) Moreover, in general, given two solutions there might in fact be finitely many different recipes to produce new solutions, i.e., the product is in fact multi-valued. This "product structure" satisfies some basic axioms which are common to all examples and we in fact do not need to know more in order to develop our general theory. We will call the rule or method of producing out of two solutions a new one an operation. This will be discussed in more detail later. The operation will make sense not only for solutions but for elements in the ambient space X, or the bundle, as well. In short, given a point a in the component A and a point b in the component B we can produce a point z in some component Z provided the components A, B and Z are related, i.e.,

$$(A, B; Z) \ \text{ is a relator.}$$

Further z will belong to a face of Z which we will denote by $[A, B; Z]$. We will make this more precise in the next subsection. Before that let us describe a little bit more the landscape of degeneration structure and related concepts. There are in addition to degeneration structures notions like degeneration modules. Degenerations modules compare to degenerations structures as modules compare to rings. Degeneration modules occur when organizing homotopies of inter-depending families of Fredholm operators. For example, organizing holomorphic curves in symplectic cobordisms between contact manifolds we have a positive and negative boundary component. Degeneration structures help to organize the Fredholm problems associated to the boundary components and a degeneration module (in fact a bi-module over the other two structures) organizes the Fredholm operators associated to the cobordisms. We refer the reader to [**17**] for more details, and [**9**] for some inspiration where these concepts might show up (look out for bi-modules in the usual sense).

Let us mention one algebraic aspect of degeneration structures. Assume that Λ is a ring. Consider the group $C(S, \Lambda)$ of maps $S \to \Lambda$. Define the convolution $\alpha * \beta$ by

$$(\alpha * \beta)(C) = \sum_{(A,B;C) \in R} \alpha(A)\beta(B).$$

Then the properties of a degeneration structure imply that this is well-defined and the convolution is associative. We will explain this later on our Morse-theory example. Applying a similar procedure to degeneration modules we will obtain a bi-module (in the usual sense) over the previously constructed rings (for the right and left Fredholm problem). See the introduction to Section 5 for some suggestive formulas. Now we are in the position to define operations.

3.2. Operations on M-Polyfolds. Let $\pi : Y \to X$ be a M-polyfold bundle over a face-structured M-polyfold and (S, R) a degeneration structure. Let us begin with a process which we might call indexing. Denote the set of connected components of X by S and assume we are given a degeneration structure (S, R) where the set of relators is in 1-1 correspondence with the set of faces of X. More precisely, for every connected component $Z \in S$ the faces of Z are in 1-1 correspondence with the subset R_Z of R consisting of all relators of the form $(A, B; Z)$.

Definition 3.3. An operation[6] for π consists of the following data:

1) A degeneration structure (S, R), where $S = \pi_0(X)$ together with an indexing of X.
2) A map $\circ_{(A,B;C)} : [A] \times [B] \to [A, B; C]$, increasing the degeneracy by 1, which is a sc-diffeomorphism so that the following properties hold:
   2.1) With $(A, B, C) \leftarrow (A, E) \leftarrow (D)$ and $(A, B, C) \leftarrow (F, C) \leftarrow (D)$ dual degeneration sequences we have for $a \in [A], b \in [B]$ and $c \in [C]$ that
   $$a \circ_D \circ (b \circ_E c) = (a \circ_F b) \circ_D c.$$
   2.2) If $x = a \circ_E b = a' \circ_E b'$ with $a \in [A]$, $a' \in [A']$ and $d(a) = d(a')$ then $A = A'$.

We also assume that $\circ$ extends to linear isomorphisms on the corresponding fibers.

The definition just given is too special for most applications we have in mind[7]. Nevertheless it is the most instructive one. We outline necessary modifications later on.

As a consequence of the axioms of an operation every element $x \in X$ with $d(x) \geq 1$ is decomposable. Every element has a prime decomposition in the sense that for $x \in [Z]$ with $d(x) \geq 1$ there is a uniquely

[6]For our purposes in this paper the definition is a scaled back version of the one in [**17**].

[7]For example: for SFT we need that the operation is compatible with morphisms, has to be multi-valued, and rather than being as in item 2) sc-diffeomorphisms, we would have only covering maps.

determined sequence $(A_0, \ldots, A_{d(x)})$ so that there are uniquely determined $a_i \in [A_i]$ and so that following any degeneration sequence

$$(A_0, \ldots, A_{d(x)}) \leftarrow \cdots \leftarrow (Z)$$

the associated $\circ$-maps map x onto $(a_0, \ldots, a_{d(x)})$. We define the spectrum of x, denoted by $\sigma(x)$ as the generalized relator

$$\sigma(x) = (A_0, \ldots, A_{d(x)}; Z).$$

We call a Fredholm section f of $\pi : Y \to X$ compatible with the operation provided for $(A, B; C)$ and $a \in [A]$, $b \in [B]$ we have

$$f(a \circ_C b) = f(a) \circ_C f(b).$$

If f is a section of $\pi : Y \to X$ define a perhaps multi-valued section $f \circ f$ on ∂X by

$$(f \circ f)(z) = \{f(a) \circ_C f(b) \mid a \circ_C b = z\}.$$

Also define ∂f to be the restriction of f to ∂X. Clearly the compatibility with the operations means that

$$\partial f = f \circ f.$$

We call this the "Master Equation". If f is multi-valued[8], compatibility would be defined by the same equation. If Q is any subset of X we can define

$$\partial Q = Q \cap (\partial X),$$

where we recall that $\partial X = \{x \in X \mid d(x) \geq 1\}$, and

$$Q \circ Q = \{a \circ_C b \mid (A, B; C) \in R,\ a \in A \cap Q,\ b \in B \cap Q\}.$$

Then we can define compatibility of Q with the operation by

$$\partial Q = Q \circ Q.$$

The Master Equation turns up everywhere in the theory. We know already what is means that f is compatible with $\circ$. If $K = f^{-1}(0)$ and f is compatible with $\circ$ we have $\partial K = K \circ K$. Not surprisingly, the perturbation theory for a $\circ$-compatible f has to be consistent with $\circ$ by requiring that the perturbation $s \in \Gamma^+(\pi)$ satisfies

$$\partial s = s \circ s.$$

It turns out that there is a very good abstract $\circ$-compatible perturbation theory. However, one should point out that depending on the circumstances the transversality-theory can be quite complicated and elaborate. This is in particular the case if the indexing involves so-called diagonal relators, i.e., relators of the form $(A, A; Z)$. In that case we have a situation where we can take two copies of the same object and can create a new one. This in general implies an additional symmetry

[8]Multi-valued sections have to be considered in the polyfold case due to serious transversality issues.

in the problem which causes some problems in transversality questions, which in most cases can only be resolved by multi-valued perturbations. The resulting moduli spaces then will be branched manifolds, see [**31**], rather than manifolds, and counting of solutions can usually be done only over the rational numbers[9]. Similarly, in the polyfold context, we need to have the perturbations not only compatible with the operation, but also with the morphisms. In general, transversality issues can only be resolved by multi-valued perturbations.

Let us observe that there are many possible generalizations of the above definition. For example we might have some group action on the set S which happens for example in SFT where π_2 acts on the connected components. The (special) definition we have given for an operation does not take into account that we might have different recipes which allow us to associate to a pair of points a bunch of other points in different components. Moreover, it might occur that the map $(a, b) \to a \circ_C b$ is not a diffeomorphism between $A \times B$ and a face F, but only a finite-to-one covering map. Then, of course, we would like to incorporate symmetries which is best dealt with the groupoid set-up we discussed previously. Clearly, it is not necessary to take $S = \pi_0(X)$. In fact, for a given problem there might be better criteria for indexing the space than just connectivity components. For example one would like to bundle several connected components and denote this subspace by a letter A and the collection of all these subspaces will be the set S. We refer the reader to [**17**] for a quite comprehensive picture.

Let us informally illustrate some of the ideas in the case of Morse-theory. In the case of a Morse-function $\Phi : M \to \mathbb{R}$ we can take S to consist of all pairs (a, b) of critical points with $a < b$. The relators consist of all triple $((a, b), (b, c); (a, c))$. The operation is defined by associating to (a, b) the component $[(a, b)]$ which is the union of all $X(a_0, \ldots, a_k)$ with $a_0 = a$ and $b = a_k$. If $(a, b) \in S$ then $R_{(a,b)}$ consist of all admissible symbols $((a, c), (c, b); (a, b))$ and

$$[(a, c), (c, b); (a, b)] = [(a, c)] \times [(c, b)].$$

The map $\circ_{((a,c),(c,b);(a,b))}$ associates to elements $x \in [(a, c)]$ and $y \in [(c, b)]$ the broken trajectory obtained from x and y. The spectrum of a broken trajectory x connecting $a_0, a_1, \ldots, a_n$ is

$$\sigma(x) = ((a_0, a_1), (a_1, a_2), \ldots, (a_{n-1}, a_n); (a_0, a_n)).$$

Of course, we could also take a much finer indexing by, in fact, taking connected components for the indexing, which can be described by elements in the relative first homotopy $\pi_1(M; a, b)$.

Coming back to the abstract Fredholm theory it is useful to introduce an auxiliary concept. In the following, given an indexing by a degeneration structure (S, R), the symbol $[A]$ denotes the closed and open

[9]This then in general requires that the Fredholm problems are orientable.

subspace of X associated to $A \in S$. Observe that we allow ourselves to be a little bit more general and do not require A to be a connected component. Assume we are given a Fredholm operator with operations $\mathfrak{f} = (\pi, f, \circ)$. For technical reasons we will restrict to M-polyfolds built on sc-Hilbert spaces. An auxiliary norm N is a continuous map defined on $Y_{0,1}$ with image in $[0, \infty)$ introducing on every fiber $Y_{0,1;x}$ a complete norm and having some additional properties. First of all

$$N(h \circ_A k) = \max\{N(h), N(k)\}.$$

Further, it is not difficult to see that there is a well-defined concept of "mixed convergence" for sequences in $Y_{0,1}$, which in local coordinates would mean that the data in the base is converging on the 0-level, whereas the data in the fiber is converging weakly on level 1 and strongly on level 0. We write $y_k \overset{\rightharpoonup}{\longrightarrow} y$ for mixed convergence. We require that N has the property

$$N(y) \leq \liminf_{k \to \infty} N(y_k).$$

One can show that such auxiliary norms exist. There are different useful concepts of properness for a Fredholm section of $\pi : Y \to X$.

1) We say f is component-proper if on every connected component the induced operator is proper.
2) If f is compatible with an operation $\circ$, we say that it is $\circ$-proper if for every $A \in S$ the operator induced on $[A]$ is proper.

Of course 2) implies 1). Recall that a subset K of X is said to be $\circ$-compatible if $\partial K = K \circ K$. Here $\partial K = (\partial X) \cap K$ and $K \circ K$ is defined by

$$K \circ K = \{a \circ_C b \mid a \in K \cap [A], b \in K \cap [K], (A, B; C) \in R\}.$$

For example if f is compatible with an operation $\circ$ then $f^{-1}(0)$ is $\circ$-compatible. If f is component-wise proper so is $f^{-1}(0)$. An open $\circ$-neighborhood U of a $\circ$-invariant subset K of X is an open set containing K which is $\circ$-invariant. A $\circ^+$-section is a sc$^+$-section compatible with $\circ$. We denote the whole collection by $\Gamma^+(\pi, \circ)$,

An important compactness result is the following[10]

THEOREM 3.4. *Assume that $\mathfrak{f}$ is a Fredholm operator with operations which is component-wise proper. Let an auxiliary norm N be given. Then there exists an open $\circ$-neighborhood U of $f^{-1}(0)$ so that for every $s \in \Gamma^+(\pi, \circ)$ with support in U and satisfying $N(s(x)) \leq 1$ for all x the section $f + s$ is $\circ$-compatible and component-wise proper.*

The next theorem gives an abstract transversality result:

[10]One needs for the following results that the Fredholm section has the contraction germ property around every point on level 0 and not only around smooth points, see [**17**]. In applications the proof of this type of Fredholm property by exhibiting the contraction germ form on the 0-level at non-smooth points is usually identical to the proofs at smooth points on an arbitrary level.

THEOREM 3.5. *Let $\mathfrak{f}$ be a Fredholm operator with operation and N an auxiliary norm. Assume that $\overline{\mathcal{O}}$ is the set of $\circ^+$-sections satisfying $N(s(x)) \leq 1$ for all x and having support in the open $\circ$-neighborhood U of $f^{-1}(0)$. Then the following holds. There exists a section $s \in \mathcal{O}$ so that $f + s$ is transversal with the zero-section and in general position with respect to the boundary strata.*

At this point we would like to count solutions on components where the Fredholm index is 0. We can do this over $\mathbb{Z}_2$, or if we can orient the determinants of the linearized Fredholm sections in a coherent way, we can work with more general coefficients. As we pointed out earlier there is a natural class of strong sc-connections. Taking the covariant derivative of a section with respect to two such connections the difference will be a sc$^+$-operator, in particular compact. As a consequence there is a convex set of possible linearisations differing by compact operators. We will not address these issues here, but point out that for SFT, orientations over $\mathbb{Z}$ have to be taken, since due to intrinsic "orbifold difficulties" counting has to be done over the rational numbers.

In our example for the Morse-function Φ let us do the counting over $\mathbb{Z}_2$. Given (a, b) with Morse-index difference 1, i.e., $m(b) - m(a) = 1$, denote by $Q(a, b)$ the number of solutions of $(f + s)(x) = 0$ in $X(a, b)$, otherwise define the map to be 0. Observe that the Fredholm index $i(a, b)$ is given by

$$i(a, b) = m(b) - m(a) - 1.$$

Counting gives a map

$$Q : S \to \mathbb{Z}_2.$$

For the convolution product $*$ one can verify that $Q * Q = 0$. We can partition S into even and odd elements by saying that (a, b) is odd if the difference of the Morse-indices is odd and even otherwise[11]. If we consider the space of all functions from S to $\mathbb{Z}_2$, say $C(S, \mathbb{Z}_2)$, we therefore obtain a decomposition

$$C(S, \mathbb{Z}_2) = C_0(S, \mathbb{Z}_2) \oplus C_1(S, \mathbb{Z}_2).$$

Using the convolution product we can define a commutator compatible with the grading. (For two odd elements the commutator has a $(+)$-sign, for all the other cases it has a $(-)$-sign). Then

$$[Q, Q] = 2Q * Q = 0.$$

At this point we have produced the data (S, Q). The next step is the representation of this data. There are in fact different possibilities. For example we can define

$$D_Q : C(S, \mathbb{Z}_2) \to C(S, \mathbb{Z}_2)$$

[11] On the level of Fredholm indices a component is even (odd) if the Fredholm index is odd (even) since we divided out by the $\mathbb{R}$-action.

by

$$D_Q(\lambda) = [Q, \lambda].$$

Then $D_Q^2 = 0$ and we obtain a Homology group. If we do it for S^2 with the height function we see that S consists of one point, say $*$, which is even. The counting function Q is 0. If we take a different Morse-function on S^2 having for example four critical points we obtain a more complicated homology. This kind of homology can always be defined for any abstract situation and has some invariance properties with respect to small perturbations. It seems that this homology in our Morse-theory example can be used to define an isotopy invariant for Morse-functions with prescribed type of critical points. If we know more about the data we can do some "representation theorem". For example, in our Morse-theory case denote by V the vector space of maps $Cr(\Phi) \to \mathbb{Z}_2$. We can use Q to define a linear operator

$$Q : V \to V,$$

by

$$(Qh)(a) = \sum_{(a,b)\in S} Q(a,b)h(b).$$

Then $Q^2 = 0$ and the homology of (V, Q) is the usual homology with $\mathbb{Z}_2$-coefficients, which would be invariant under arbitrary (generic) perturbations of Φ if M is compact.

The following fact should be pointed out. The data (S, Q) is a homologically invariant way of counting solutions for a Fredholm problem with operation. This kind of counting or versions applying more sophisticated topological methods to the whole moduli space always can be done in the abstract framework. What we have illustrated above is the instance of a representation theory for this "counting data". Depending on additional structure of the counting data we might be able to use it to construct new algebraic objects. For example in SFT super-Weyl-differential algebras, see [**9**]. Note, however, that when [**9**] was written, the technical tools for separating out the analytical, topological and algebraic aspects as cleanly as it is possible now, had not been developed.

3.3. Comments. There is an interesting algebra of formal differential equations in the background which can be used to describe homotopies and other relevant concepts. We will not explain this here and refer the reader to [**17**]. The reader might also have a look at [**9**], where a certain number of mysterious differential equations arise in the study of cobordisms and homotopies. These formulas have a completely rigorous definition within some new algebra of formal differential equations.

Counting solutions in general calls for a theory of orientations. An orientation for Fredholm problems is by definition an orientation for the determinant bundle associated to the point-wise linearized Fredholm operator. For a general Fredholm problem there is no orientation. However, if the linearized operators are compactly homotopic to complex linear Fredholm operators such a determinant bundle is orientable. This underlying reason is for example valid in the case of Gromov-Witten Theory. In SFT the orientation question is very subtle and can be viewed as an extension of the study of coherent orientations in Floer Theory, see [**11**]. A basic outline for the general orientation questions in SFT has been given in [**9**], and worked out in some variation in [**6**]. In our general situation where we deal with Fredholm Theory with Operations a coherent orientation will refer to a choice of orientation of the linearized Fredholm section (linearized using a special class of connections which is intrinsic to the theory of strong bundles) so that on every component the orientation changes continuously, i.e., by prolongation along paths, and, most importantly, for a given relator $(A, B; Z)$ there is a well-defined relationship between the orientations $\mathfrak{o}_A$, $\mathfrak{o}_B$ and $\mathfrak{o}_Z$. Clearly using the operation $\circ$ one can come up with a standardized procedure to define an orientation for Z if we have given ones for A and B. This results in an orientation $\mathfrak{o}_A \circ_Z \mathfrak{o}_A$ which is obtained from orientations for A and B following some specific constructions (i.e., conventions). Assuming that Z is already oriented the requirement for a coherent orientation is the relationship

$$\mathfrak{o}_A \circ_Z \mathfrak{o}_B = (-1)^{p(A)} \mathfrak{o}_Z.$$

Here $p(A)$ is some parity associated to A and related to the Fredholm index on A. As there are important sign conventions in Homology theory we also have crucial sign conventions here. The above formula contains such a convention which is (keeping in mind that counting is a homological process) compatible with the sign conventions of Homology.

Let us also mention the following. Rather than counting the solutions in the components where the Fredholm index is 0 we could look at the full solution space $K = f^{-1}(0)$ which of course satisfies

$$\partial K = K \circ K.$$

In general, i.e., if studying $\circ$-proper Fredholm sections of a strong polyfold bundle, we need multi-valued $\circ$-compatible sc^+-perturbations. In that case we will obtain a branched finite-dimensional manifold M, see [**31**]. Many differential geometric concepts make sense in this context. Of course we have as a consequence of the $\circ$-compatibility of the perturbation that $\partial M = M \circ M$. It would be useful to develop algebraic topology concepts for spaces with an operation. It is clear that one can develop a theory of differential forms satisfying $\partial \omega = \omega \circ \omega$, but other concepts should also carry over.

In the interesting paper [**2**] by Barraud and Cornea it is shown how the equation $\partial A = A \circ A$ can be exploited algebraically in Floer theory without bubbling, and even in the case of Morse theory leading to results going far beyond the discussion of the usual Morse complex. A crucial input in their discussion is some kind of representation of the moduli spaces in loop spaces. The moduli spaces have boundaries and they introduce a spectral sequence which in effect allows them to systematically forget the boundaries allowing them to define a reduced homology class which is a new invariant representing higher dimensional moduli spaces. Their results depend on a representation theory of the moduli spaces in loop spaces which is quite natural in their context. It would be interesting to study the question of representations of other moduli spaces. Floer theory with bubbling seems already to be a challenging start.

Finally let us describe in some detail what a general theory of operations should be. Of course, one would like a notion as general as possible, subject to a constraint. Namely one wants easy axioms and within such a general theory one would like to have an abstract perturbation and transversality theory. Currently we have a theory which goes beyond what we described here, and covers FT, CH and SFT, but still doesn't achieve what we describe now (but is close). Assume we are given a polyfold Z with boundary ∂Z. One would like to explain ∂Z in terms of (fibered) products of its components. So given a pair of points (a, b) for suitable components A and B one can produce a point z in a suitable face of ∂Z, i.e., one has a relator $(a, b; z)$. If we now vary a and b the target z should change smoothly in dependence of a and b. Let us view this, and refer to it, as a "smoothly changing recipe". Now we could imagine that we have a whole family of smoothly changing recipes. Then for two points (a, b) there might indeed be several relators $(a, b; z_j)$. On the other hand, sometimes different recipes applied to different points might imply the same result. In other words one would like to have a theory of families of interacting smoothly changing recipes described by a simple set of axioms, so that the concept allows to develop an abstract transversality and perturbation theory. Of course, everything should be so general that the concrete theories of interest fit into this scheme, but in addition, the level of generality is right in the sense that its description and necessary constructions are relatively easy.

4. Gromov-Witten Theory

We begin with Gromov-Witten theory. The scale-analysis which has to be carried out to construct the ambient spaces of SFT is not much more difficult than that needed for Gromov-Witten theory. Besides that it is interesting to have a polyfold set-up in that case as well.

An important input in constructing the polyfold set-up for GW or SFT is the Deligne-Mumford theory of stable Riemann surfaces, however in a modified form. Our description of Deligne-Mumford theory and its modifications geared towards applications in SFT is taken from [**18**].

4.1. Deligne Mumford Type Spaces. For the analysis of SFT it is important to understand certain variants of the Deligne-Mumford theory of stable Riemann surfaces. In fact there are a certain number of issues which will be important to understand and which are not classical and nonstandard and deal with the fact that the (smooth) SFT-constructions need differentiable structures which are (with exceptions) not compatible with the standard DM-Theory. The DM-background as used in SFT is being developed in much detail in [**18**]. We give some minimal background here.

We consider tuples (S, j, M, D), where (S, j) is a closed Riemann surface, $M \subset S$ a finite subset of un-numbered points, and D a finite collection of un-ordered pairs of points $\{x, y\}$, where $x \neq y$. We call (S, j, M, D) a noded Riemann surface with (un-ordered) marked points. We say it is connected, provided the topological space obtained by identifying x with y for every nodal pair is a connected topological space. Moreover, we assume that $\{x, y\} \cap \{x', y'\} \neq \emptyset$ implies $\{x, y\} = \{x', y'\}$. In addition the union $|D|$ of all these two-point sets is disjoint from the points in M. We will refer to M as the marked points and D the set of nodal pairs. We say that (S, j, M, D) is equivalent to (S', j', M', D') provided there exists a biholomorphic map $\phi : (S, j) \to (S', j')$ such that $\phi(M) = M'$ and $\phi(D) = D'$, where

$$\phi(D) := \{\{\phi(x), \phi(y)\} \mid \{x, y\} \in D\}.$$

We call (S, j, M, D) stable provided its automorphism group is finite. Denote by $\overline{\mathcal{N}}$ the collection of all equivalence classes of stable noded Riemann surfaces with marked points. In SFT one will need usually somewhat more complicated objects. For example one needs some of the points in M to be ordered, some un-ordered, and some carrying a distinguished oriented real line in their tangent space. However, stripping away all additional data we end up with the objects just introduced. Given an equivalence class $\alpha = [S, j, M, D] \in \overline{\mathcal{N}}$, we can define its type as follows. We associate to α the isomorphism class of a decorated graph by declaring the components of S to be the vertices together with a number giving the genus and a second number giving the number of points from M lying on the component. In addition we draw an edge for every $\{x, y\}$ connecting the components on which the points x, y reside, see Figure 5. Note that x, y can lie on the same component. Let us define the arithmetic genus g_a of α by

$$g_a = 1 + \sharp D + \sum_C [g(C) - 1]$$

where the sum is taken over all connected components of S. Assume that α is stable, i.e., for every vertex the union of twice $g(C)$ plus the number of marked points plus the number of nodal points should be at least 3.

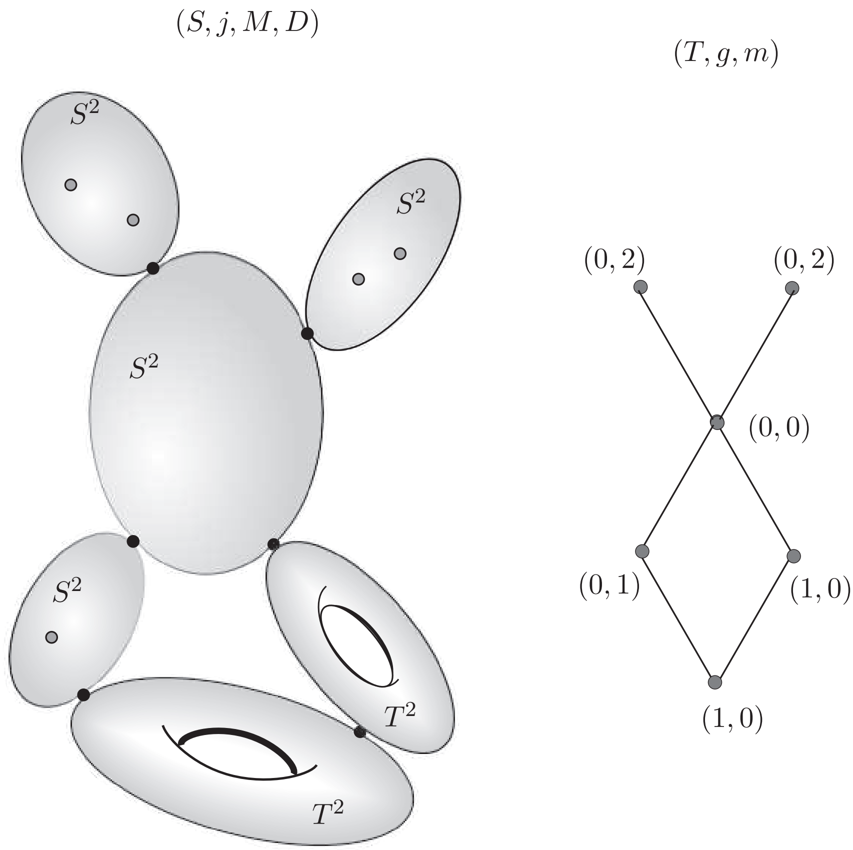

FIGURE 5. A noded Riemann surface (S, j, M, D) and its associated graph (T, g, m).

Denote by $\Omega^1(\alpha)$ the space of smooth maps associating to $x \in S$ a complex anti-linear map $\phi(x) : (T_xS, j) \to (T_xS, j)$. Let $\Gamma(\alpha)$ be the space of vector fields on S which vanish at the points in D and M. Then the Cauchy-Riemann operator $\bar{\partial}$ defines a Fredholm map

$$\Gamma(\alpha) \to \Omega^{0,1}$$

which is injective and has index $3 - 3g_a - \sharp M - \sharp D$. We denote by $H^1(\alpha)$ the complex vector space of dimension $3g_a + \sharp M + \sharp D - 3$ defined by

$$H^1(\alpha) = \Omega^{0,1}/\bar{\partial}\Gamma(\alpha).$$

If we denote by τ the type of α and by $\overline{\mathcal{N}}_\tau$ the subset of $\overline{\mathcal{N}}$ of type τ then one can identify $H^1(\alpha)$ with the (orbi-) tangent space of $\overline{\mathcal{N}}_\tau$ at

$[\alpha]$. Note that the automorphism group G of α acts on $H^1(\alpha)$ and that isomorphisms $\phi : \alpha \to \alpha'$ induce isomorphisms $H^1(\alpha) \to H^1(\alpha')$. Given a smooth family of complex structures $v \to j(v)$ on S, say with $v \in V \subset E$, where E is some finite-dimensional vector space with $j(0) = j$, we can take its derivative $Dj(v)$ at v and observe that it induces a linear map

$$E \to H^1(\alpha_v)$$

where $\alpha_v = (S, j(v), M, D)$, called the Kodaira differential and denoted by $[Dj(v)]$.

It is convenient to take $E = H^1(\alpha)$ which is a complex vector space with a natural action of the automorphism group G of α on it. Let $V \subset H^1(\alpha)$ be a G-invariant open neighborhood of 0. We call a family $v \to j(v)$ with $j(0) = j$ effective if at every $v \in V$ the Kodaira differential is a real linear isomorphism. We call it complex if at every v the Kodaira differential is complex linear. We call it symmetric provided for every $v \in V$ and $g \in G$ the map

$$g : (S, j(v), M, D) \to (S, j(g * v), M, D)$$

is an isomorphism. A family $v \to j(v)$ is called good if it is effective, symmetric and smooth. One can show that for every α there exists a good complex family so that the maps

$$v \to [S, j(v), M, D]$$

define a smooth orbifold structure for $\overline{\mathcal{N}}_\tau$ with holomorphic transition maps. Next we would like to address the problem of defining smooth orbifold structure on $\overline{\mathcal{N}}$. This is more delicate if one keeps in mind an important goal: Such smooth structures should be compatible with a "to be constructed" theory of smooth structures for the moduli spaces (or their perturbations) of stable finite energy surfaces, a notion we will introduce later. For the moment it suffices to know that these are the solutions of our Fredholm problems arising in SFT.

Start with α of type τ. Then one can find a good complex family $v \to j(v)$ with $j(v) = j$ near the points in D and M. A small disk structure consists of a family of disks D_x, for every nodal point x, with smooth boundaries so that their union is invariant under the G-action. Assume that $j(v) = j$ on these disks. Fix for every x a biholomorphic map $\bar{h}_x : (D, 0) \to (D_x, x)$ and complex anti-linear maps $\varphi_{(x,y)} : T_y S \to T_x S$ so that $\varphi^{-1}_{(x,y)} = \varphi_{(y,x)}$ and the following compatibility holds

$$T\bar{h}_y(0)^{-1} \circ \varphi_{(y,x)} \circ T\bar{h}_x(0) : \mathbb{C} \to \mathbb{C}$$

is complex conjugation. Next take for every nodal pair $\{x, y\}$ a copy of the complex plane $\mathbb{C}_{(x,y)}$ and let

$$N = \bigoplus_{\{x,y\} \in D} \mathbb{C}_{(x,y)}.$$

Then there is a natural unitary action of G on N which will be compatible with the following construction where small elements in N occur as gluing parameters. Pick $\{x, y\}$ and define $h_x(s,t) = \bar{h}_x(e^{-2\pi(s+it)})$ and $h_y(s',t') = \bar{h}_y(e^{2\pi(s'+it')})$, where $t, t' \in S^1$ and $s \leq 0$ and $s' \geq 0$. Note here that we pick for one of the points positive holomorphic polar coordinates and for the other negative ones. However, the following construction does not depend on this choice. For $R \geq 0$ define D_x^R and D_y^R by

$$D_x^R = \{z \in D_x \mid z = h_x(s,t),\ s \in [0,R],\ t \in S^1\}$$
$$D_y^R = \{z \in D_y \mid z = h_y(s',t'),\ s' \in [-R,0],\ t \in S^1\}.$$

Here we arrive at the important point where we have to discuss the gluing (also in this context sometimes called plumbing). Observe that we have to make a choice how a gluing parameter $a = a_{\{x,y\}} \in \mathbb{C}_{\{x,y\}}$ is being converted into a gluing length R and a gluing angle ϑ. There is not too much choice with the angle, but there are plenty of choices for R. Observe that this discussion parallels the one in the Morse-theory example. Let us recall the notion of a gluing profile:

DEFINITION 4.1. A gluing profile φ is a diffeomorphism

$$\varphi : (0,1] \to [0,\infty).$$

Fix a gluing profile φ and define if $0 < |a_{\{x,y\}}| < 1$

$$R = \varphi(|a_{\{x,y\}}|) \text{ and } a_{\{x,y\}} = |a_{\{x,y\}}|e^{-2\pi i\vartheta}.$$

Now call points $z \in D_x^R$ and $z' \in D_y^R$ equivalent provided

$$s - s' = R \text{ and } t - t' = \vartheta \text{ (mod 1)}.$$

Doing this for all gluing parameters a we obtain a new family of Riemann surfaces $\alpha_{(v,a)} = (S_a, j(v,a), M_a, D_a)$. Moreover every $g \in G$ defines in a natural way an isomorphism

$$g : \alpha_{(v,a)} \to \alpha_{g*(v,a)}.$$

We have

THEOREM 4.2. *The space $\overline{\mathcal{N}}$ possesses a natural paracompact topology which is locally compact. The following holds:*

- *Fixing a gluing profile φ and taking for every α of type τ a good family the above construction defines via*

$$(v,a) \to [\alpha_{(v,a)}],$$

  if restricted to a small enough G-invariant neighborhood $(0,0)$ a family of C^0-uniformizers for $\overline{\mathcal{N}}$.
- *Taking the gluing profile $\varphi(x) = -\frac{1}{2\pi}\ln(|x|)$ and starting with good complex families the associated uniformizer defines a holomorphic orbifold structure.*

- *Starting with the gluing profile* $\varphi(x) = e^{\frac{1}{x}} - e$ *and any good family the construction gives a smooth orbifold structure together with a natural homotopy class of almost complex structures.*

This result is proved in [**18**]. The logarithmic gluing profile gives in fact the classical Deligne-Mumford structure. Unfortunately it is not compatible with a smoothness theory for SFT or even with Gromov-Witten theory in the smooth non-integrable case. The exponential gluing profile, however, works very well. The identity map from the exponential smooth structure to the DM-structure is, however, smooth. The above theorem, and in particular some estimates which one obtains during the proofs, are important for the SFT-theory.

Let us give a somewhat different description of the previous results, which were formulated in the classical V-manifold or orbifold language, by describing them via the groupoid approach (see [**33**] for the relevant Lie groupoid theory). The space $\overline{\mathcal{N}}$ constructed above will be viewed as the orbit space of a Lie groupoid. To motivate the "groupoid approach" consider the huge category (not a set) where the objects are tuples $\alpha := (S, j, M, D)$ of stable Riemann surfaces. Let us define a morphism $\alpha \to \alpha'$ to be a biholomorphic map

$$\phi : (S, j) \to (S', j')$$

satisfying $\phi(D) = D$ and $\phi(M) = M'$. Clearly, every morphism is invertible. Calling two objects equivalent if there is a morphism between them, we can build the quotient of the category, which is a set and, in fact, the orbit space we are interested in. Since the quotient space is a set there is an enormous redundancy in the original category and we have to cut it down to a suitable full subcategory which still has the same quotient space, but is a set. In order to do so in a sensible way observe that it is possible to say in our category that two objects which are not isomorphic are close (up to isomorphism), see the discussion of the DM-theory in [**5**]. This then allows us to put a topology on the orbit space. Having this all in place we notice that we can describe all equivalence classes by a suitable set of parameterized models. Pick an element $\alpha = (S, j, M, D)$ and fix a good family $v \to j(v)$, together with a small disk structure and define

$$(a, v) \to \alpha_{(a,v)} = (S_a, j(a, v), M_a, D_a).$$

If we restrict this map to a sufficiently small open neighborhood of $(0, 0) \in N \times E$, say W, it would induce a unifomizer as constructed before. Consider the set

$$\Delta := \{(a, v, \alpha_{(a,v)}) \mid (a, v) \in W\}.$$

Then Δ possesses a natural smooth manifold structure by requiring that the map

$$\Delta \to W : ((a, v), \alpha_{(a,v)}) \to (a, v)$$

is smooth. Construct in a similar way Δ'. Assume that we have an isomorphism

$$\phi : \alpha_{(a,v)} \to \alpha'_{(a',v')}.$$

Then one can show that ϕ lies in a uniquely determined family of isomorphisms $\phi_{(b,w)}$ between $\alpha_{(b,w)}$ and $\alpha'_{u(b,w)}$. Here u stands for a uniquely determined smooth local diffeomorphism

$$(b,w) \to (b',w') = u(b,w)$$

with $u(a,v) = (a',v')$. We view

$$(u(b,w), \phi_{(b,w)}, (b,w)) : ((b,w), \alpha_{(b,w)}) \to (u(b,w), \alpha'_{u(b,w)})$$

as a morphism. Note that the set of morphisms

$$\Delta \to \Delta'$$

has the structure of a smooth manifold of the same dimension as Δ. We can now find a countable family of Δ's, say Δ_i, $i \in I$, so that with X defined by the disjoint union

$$X = \coprod_{i \in I} \Delta_i$$

the map $X \to \overline{\mathcal{N}}$ which assigns to an element $(a, v, \alpha_{(a,v)})$ the isomorphism class of $\alpha_{(a,v)}$ is a surjection. The previous discussion already identified the morphisms and we know that the collection of all morphisms $\mathbf{X}$ has a manifold structure as well. It follows immediately from the construction that the associated category $\mathfrak{X}$ is an étale Lie groupoid[12] whose orbit space is $\overline{\mathcal{N}}$. Consider $(\mathfrak{X}, \beta)$, where

$$\beta : |\mathfrak{X}| \to \overline{\mathcal{N}}$$

is the obvious homeomorphism. Then we can say that the pair defines an orbifold structure on $\overline{\mathcal{N}}$. As in the polyfold case we can define equivalence of two such pairs by refinement. This gives an alternative way of defining an orbifold structure. The method will immediately generalize to the case where we also have a map on S. This applies then to Gromov-Witten theory.

4.2. The Splicings for Gromov-Witten Theory. The main ingredient, as in the Morse-Theory case, is a gluing and anti-gluing procedure. The situation in GW differs somewhat from the latter case, since the isolated critical points in Morse-Theory are replaced by constant loops, i.e., the symplectic manifold. In other words we are dealing in fact with a Morse-Bott situation. This requires some modifications.

We begin with the discussion of the procedure how maps u on a nodal Riemann surface S can be used to describe neighboring curves on (un-noded or glued) Riemann surfaces S_a by the so-called gluing construction $\oplus_a(u) : S_a \to W$. This construction, which can be carried

[12]We could modify it and make it even proper.

out in local coordinates and implanted into a manifold setting by a chart, will be referred to as "nonlinear gluing". The gluing also exists on the level of vector fields and leads to vector fields along glued maps. As in the Morse theory case there is a certain ambiguity in gluing in the sense that gluing of different maps might lead to the same result. Here is the place where anti-gluing comes in. It is defined on the level of vector fields and precisely resolves this ambiguity. Moreover, the combination of gluing and anti-gluing, called total gluing, leads to the abstract concept of splicing which is crucial for our theory. In other words the whole philosophy is as discussed in the Morse theory case. Note, however, that there will be some modifications. This is mainly due to the fact that we deal with a Morse-Bott, rather than a Morse-situation.

We begin with the basic gluing $\oplus$. Fix a smooth cut-off function

$$\beta:\mathbb{R}\rightarrow[0,1]$$

so that

$$\beta(s)+\beta(-s)=1 \text{ for all } s\in\mathbb{R}$$
$$\beta'(s)<0 \text{ for all } s\in(-1,1).$$

Also recall our gluing profile $\varphi:(0,1]\rightarrow[0,\infty)$ defined by

$$\varphi(r)=e^{\frac{1}{r}}-e.$$

Let us first assume that we are given two abstract disk-like Riemann surfaces D_x and D_y with smooth boundaries and interior points x and y. We view $\{x,y\}$ as a nodal pair. Hence $(D_x\cup D_y,\{x,y\})$ is a noded Riemann surface. As explained in the background chapter about DM-theory, we can glue the two disks given a gluing parameter. We will however keep a little bit more information. Let us also assume that we have given holomorphic polar coordinates centered around x and y. To be more precise we are given biholomorphic maps $\bar{h}_x:(D_x,x)\rightarrow(D,0)$ and $\bar{h}_y:(D_y,y)\rightarrow(D,0)$. Then define

$$h_x:\mathbb{R}^+\times S^1\rightarrow D_x:(s,t)\rightarrow\bar{h}_x(e^{-(2\pi(s+it))})$$

and

$$h_y:\mathbb{R}^-\times S^1\rightarrow D_y:(s',t')\rightarrow\bar{h}_y(e^{2\pi(s+it)}).$$

If $a\in\mathbb{C}$ is a complex number with $|a|\leq\frac{1}{2}$ we define $Z_0=(D_x\cup D_y,\{x,y\})$, which is our original noded disk, and if $0<|a|\leq\frac{1}{2}$ we define the cylinder Z_a by identifying the points $z=h_x(s,t)\in D_x$ for $(s,t)\in[0,R]\times S^1$ with $z'=h_y(s',t')\in D_y$, where $s-s'=R$ and $t-t'=\vartheta$ and (R,ϑ) is associated to a. This cylinder has two preferred sets of coordinates

$$[0,R]\times S^1\rightarrow Z_a:(s,t)\rightarrow[h_x(s,t)]$$

and

$$[-R,0]\times S^1\to Z_a:(s',t')\to [h_y(s',t')].$$

Here [.] means the passing to equivalence classes. We will abbreviate the first by $[s,t]$ and the second by $[s',t']'$. Then

$$[s,t]=[s-R,t-\vartheta]'.$$

Here comes an important observation. We can define another Riemann surface as follows. Namely glue D_x and D_y with the same identification to obtain Σ_a which is a simply connected Riemann surface having two distinguished points x and y and a distinct annular subregion, namely Z_a. This, of course, only holds for $a\neq 0$. If $a=0$ we put $\Sigma_0=\emptyset$.

The purpose of gluing is to associate to a map u defined on $D_x\cup D_y$, with matching condition at the nodal pair $\{x,y\}$, a map $\oplus_a(u)$ on Z_a. We assume that the target manifold is $\mathbb{R}^{2n}$. Consider continuous maps $u:D_x\cup D_y\to\mathbb{R}^{2n}$ with matching condition $u(x)=u(y)$. We will define a map $\oplus_a(u):Z_a\to\mathbb{R}^{2n}$ which we might view as the "nonlinear gluing" of two maps from a noded Riemann surface with image in the manifold $\mathbb{R}^{2n}$.

Consider the Banach space consisting of u with matching condition at $\{x,y\}$ so that $u(x)=u(y)=c$ with

$$u^\pm - c\in H^{3,\delta_0}(\mathbb{R}^\pm\times S^1,\mathbb{R}^{2n}).$$

Here (u^+,u^-) is defined by

$$u^+=u\circ h_x(s,t)\ \text{and}\ \ u^-=u\circ h_y(s',t').$$

We call c the (common) asymptotic constant of u^+ and u^-. Here $\delta_0\in(0,2\pi)$. We equip E with the sc-structure where level m corresponds to regularity $(m+3,\delta_m)$ and (δ_m) is a strictly increasing sequence strictly bounded by 2π. We define $\oplus_a(u)$ as follows. We put $\oplus_0(u)=u$. If $0<|a|\leq\frac{1}{2}$ then

$$\oplus_a(u):Z_a\to\mathbb{R}^{2n}$$

given by

$$\begin{aligned}&\oplus_a(u^+,u^-)([s,t])\\&=\beta\left(s-\frac{R}{2}\right)u^+(s,t)\\&\quad+\left(1-\beta\left(s-\frac{R}{2}\right)\right)u^-(s-R,t-\vartheta).\end{aligned}$$

The interpretation of the procedure so far should be that of a "nonlinear gluing".

Assume next that h is a vector field along u, i.e., $h(z)\in T_{u(z)}\mathbb{R}^{2n}=\mathbb{R}^{2n}$. Again we have a matching condition $h(x)=h(y)$. Moreover, we define the gluing $\oplus_a(h)$ by the same formula and consider elements h of the same regularity as that of u. Observe that $u+h$ can be interpreted as

$$z \to (\exp_u(h))(z) = \exp_{u(z)}(h(z))$$

for the standard metric. The interpretation of $\oplus_a(h)$ has to be that of a vector field along $\oplus_a(u)$. Clearly,

$$\exp_{\oplus_a(u)}(\oplus_a(h)) = \oplus_a(\exp_u(h)).$$

Next we introduce the anti-gluing which is only defined for vector fields. We define the anti-gluing with respect to a reference map $u : D_x \cup D_y \to \mathbb{R}^{2n}$ as follows. Let c be the asymptotic constant of u, without loss of generality say $c = 0$. Then

$$\ominus_a(h) : \Sigma_a \to T_0\mathbb{R}^{2n}$$

is given for $a \neq 0$ by

$$\begin{aligned}
&\ominus_a (h)([s,t]) \\
&= -\left(1-\beta\left(s-\frac{R}{2}\right)\right)(h^+(s,t) - av_R(h)) \\
&\quad + \beta\left(s-\frac{R}{2}\right)(h^-(s-R,t-\vartheta) - av_R(h))
\end{aligned}$$

where

$$av_R(h) = \frac{1}{2}\left(\int_{S^1}\left(h^+\left(\frac{R}{2},t\right) + h^-\left(-\frac{R}{2},t\right)\right)dt\right).$$

Moreover $\ominus_0(h) = 0$. Observe that

$$\ominus_a(h)(x) = -\ominus_a (h)(y).$$

If we view the above construction as a description of curves near a noded curve with $u(x) = u(y) = 0$, then one should interpret the image of $\ominus_a(h)$ as $T_0\mathbb{R}^{2n} = \mathbb{R}^{2n}$.

The crucial observation of the total gluing construction is that the map

$$h \to \boxplus_a(h) = (\oplus_a(h), \ominus_a(h))$$

for fixed $a \neq 0$ is a bijective linear sc-operator

$$E \to H^3(Z_a, \mathbb{R}^{2n}) \oplus H^{3,\delta_0}_c(\Sigma_a, \mathbb{R}^{2n})$$

where the target spaces are equipped with the sc-structure where the first component is of class $m+3$ and the second $(m+3, \delta_m)$. The subscript c stands for antipodal asymptotic constant. Given a map $w : Z_a \to \mathbb{R}^{2n}$ near $\oplus_a(u)$ we find a unique vector field η along $\oplus_a(u)$ with

$$\exp_{\oplus_a(u)}(\eta) = w.$$

Then we find a unique h with

$$\oplus_a(h) = \eta \text{ and } \ominus_a (h) = 0.$$

We can define for $|a| \leq \frac{1}{2}$ a sc-decomposition of E by

$$E = \ker(\oplus_a) \oplus_{sc} \ker(\ominus_a)$$

and denote by $\pi_a : E \to E$ the projection onto $\ker(\ominus_a)$ along $\ker(\oplus_a)$. We will show

THEOREM 4.3. *The triple $\mathcal{S} = (\pi, E, B_{\frac{1}{2}})$ is a sc-smooth splicing.*

Observe that by construction the map

$$h \to \oplus_a(\exp_u(h))$$

where $h \in K_a^{\mathcal{S}}$ defines a bijection onto the maps $w : Z_a \to \mathbb{R}^{2n}$ of class H^3. This will be important in the construction of polyfold charts.

4.3. The Polyfold Structure for Gromov-Witten Theory. Consider a compact[13] symplectic manifold (W, ω). We will consider tuple (S, j, M, D, u) where (S, j, M, D) is a not necessarily stable noded Riemann surface with an ordered set of marked points M. We impose the following stability condition. For every connected component C of S at least one of the following holds:

1) $2g(C) + \sharp(C \cap (M \cup |D|)) \geq 3$ or
2) $\int_C u^*\omega > 0$.

Let us describe the quality of the function u in more detail. We say that u is of class $H^{m,\varepsilon}$ for some $m \geq 3$ and $\varepsilon > 0$ if u is of class H^m_{loc} on $S \setminus |D|$ and if for every nodal point $x \in \{x, y\} \in D$ there exists a smooth chart φ around $u(x)$ mapping $u(x)$ to 0 and positive holomorphic polar coordinates centered around x, say $\sigma : [0, \infty) \times S^1 \to S$ so that

$$v(s,t) = \varphi \circ u \circ \sigma(s,t)$$

belongs to $H^{m,\varepsilon}([R_0, \infty) \times S^1, \mathbb{R}^{2n})$ for some sufficiently large R_0. The definition of being of class $H^{m,\varepsilon}$ does not depend on the choices involved. Let us remind the reader that $H^{m,\varepsilon}([R_0, \infty) \times S^1, \mathbb{R}^{2n})$ consists of all functions so that the distributional partial derivatives up to order m weighted by $e^{\varepsilon|s|}$ belong to L^2. We call two such tuples equivalent, say

$$(S, j, M, D, u) \equiv (S', j', M', D', u')$$

provided there exists a biholomorphic map

$$\phi : (S, j, M, D) \to (S', j', M', D')$$

so that

$$u' \circ \phi = u.$$

We have

LEMMA 4.4. *If α is stable its automorphism group G is finite. In particular the set of isomorphisms between two stable elements α and α' is finite.*

[13]Compactness is not crucial.

Now fix any sequence $0 < \delta_0 < \delta_1 < \ldots < 2\pi$ of increasing weights and denote by X the collection of equivalence classes where u is of class $(3, \delta_0)$. We can define a filtration by nested subsets of X by declaring X_m to consist of all elements of class $(m+3, \delta_m)$. Let us begin with the topological side of things.

THEOREM 4.5. *The space X carries a natural paracompact second countable topology so that the un-noded curves are dense. Moreover the spaces X_m carry natural topologies as well, so that every point in X_m has a closed neighborhood which embeds as a precompact set into X_{m-1} provided $m \geq 1$.*

Having defined the topology one can define a polyfold structure on X. More precisely

THEOREM 4.6. *The second countable paracompact space X defined above admits a natural polyfold structure.*

As we will see the construction is quite similar to the groupoid construction in the Deligne-Mumford case. Natural here means, that given the filtration $X_m \subset X$ by having fixed the regularity requirement that level m corresponds to maps of quality $(m+3, \delta_m)$, and using the exponential gluing profile there is a standardized way of defining the polyfold structure.

Consider an element $\alpha = (S, j, M, D, u)$ of class $(3, \delta_0)$. By the stability condition the map u induces on every unstable component C a non-constant map u with $\int_C u^*\omega > 0$. By the Sobolev embedding theorem the map u is of class C^1. We can add a finite number of (un-ordered) points Ξ to S in order that $\alpha^* := (S, j, M \cup \Xi, D)$ is a stable Riemann surface. We view $M^* = M \cup \Xi$ as an un-ordered set of marked points. Assume we are given a good family $v \to j(v)$, where $V \subset E$ is an open invariant neighborhood of 0 in some complex space E with an action of the automorphism group G^* of α^*. We find an open neighborhood O in N so that

$$(a, v) \to (S_a, j(a, v), M_a^*, D_a)$$

induces a smooth uniformizer for $(a, v) \in O \times V$. Let us assume now that the set of points Ξ has the property that Ξ is invariant under G and that given points z_1, z_2 belonging to different orbits the values $u(z_1)$ and $u(z_2)$ are different and $Tu(z)$ is injective for any such point in Ξ. We also assume that the image $u(\Xi)$ does not intersect $u(|D| \cup M)$.

Having fixed Ξ with these properties pick $z_1, \ldots, z_k \in \Xi$ so that their orbits are disjoint and the union of the orbits is Ξ. For every orbit z_i pick a complement C_i of the image of $Tu(z_i)$. Let us define $w_i = u(z_i)$ for $i = 1, \ldots, k$ and let us denote by $w_{k+1}, \ldots, w_{k+\ell}$ an enumeration of the points in $u(|D|)$. We fix diffeomorphic charts around $w_1, \ldots, w_{k+\ell}$, by

$$\varphi_i : (\mathbf{R}(w_i), w_i) \to (\mathbb{R}^{2n}, 0).$$

Here the closures of the domains are mutually disjoint. Let us denote by $\mathbf{R}_r(w_i)$ the preimage of the r-ball. We pick a small disk structure for (S, j, M^*, D) so that the images of the D_x are contained in $\bigcup_{i=1}^{k+\ell} \mathbf{R}_1(w_i)$. Now we can implant the nonlinear gluing construction so that it is applicable to maps u' which map the D_x into $\bigcup_{i=1}^{k+\ell} \mathbf{R}_4(w_i)$. Next we consider the space of H^{3,δ_0}-sections of u^*TW. By the Sobolev embedding theorem these sections belong to $C^1_{loc}(S \setminus |D|)$. We are interested in the subspace of those sections over the points in Ξ which belong to Gz_i belong to C_i. Note that C_i is a complement for the image of $Tu(z)$, if $z \in Gz_i$. Let us denote this space of sections by Z_α. Consider now the exponential map for a Riemannian metric which on $\mathbf{R}(w_i)$ is the pull-back of the standard metric by φ_i. Then we consider for $(a, v) \in O \times V$ and η sufficiently small in C^0 and belonging to Z_α the map

$$(a, v, \eta) \to \Phi(a, v, \eta) := (S_a, j(a, v), M_a, D_a, \oplus_a(\exp_u(\eta))).$$

Recall that for every nodal point we can implant using the charts the (local) splicing construction giving a splicing (π, Z_α, U), where U is an open neighborhood of $(0, 0) \in O \times V$. Note that π only depends on the first component of (a, v). One could show that the map

$$(a, v, \eta) \to [\Phi(a, v, \eta)],$$

where we pass to equivalence classes, restricted to a sufficiently small open neighborhood of 0 in the splicing core, can be viewed as a uniformizer. Alternatively, as in the Deligne-Mumford case we define

$$\Delta = \{(a, v, \eta, \Phi(a, v, \eta)) \mid (a, v, \eta) \in P\},$$

where P is a sufficiently small open neighborhood of 0 in the splicing core. Then we can take Δ and Δ' and show that isomorphisms occur in families parameterized by the data in the domain, i.e., (a, v, η). After this, as in the Deligne-Mumford case, we can define a polyfold structure for X. We can also define a strong bundle $Y \to X$. The elements of Y are equivalence classes of tuples (S, j, M, D, u, h), where $h : (T_zS, j) \to (T_{u(z)}W, J)$ is complex anti-linear of Sobolev regularity H^2_{loc} with a particular behavior near the nodes. The equivalence classes are defined similarly to the equivalence classes defining X. This defines the strong bundle

$$\pi : Y \to X.$$

Then we can define a section f by

$$f([S, j, M, D, u]) = [S, j, M, D, u, \bar{\partial}_{J,j}(u)].$$

This is, of course, the Cauchy-Riemann operator in our context extended in the obvious way to nodal surfaces. Recall that $Y \to X$ has the structure of a polyfold bundle. In particular the polyfold structure is represented by a strong polyfold bundle over some ep-polyfold groupoid. As it turns out, for suitable representatives the section f is the map induced between orbit spaces coming from a sc-smooth section functor. Using this we show in [**16**]

THEOREM 4.7. *The section f of $\pi : Y \to X$ is a sc-smooth Fredholm section.*

As a consequence abstract perturbation theory using sc^+- multi-sections is applicable. For such a generic perturbation by a multi-section the solution space is a locally compact, smooth-branched, weighted manifold, which restricted to every connected component of X, is compact. We can define Gromov-Witten invariants by integrating suitable quantities over these moduli spaces. One can study the question of natural homotopy classes of smooth almost complex structures for these moduli spaces and many more questions. In our set-up quite a number of constructions in the GW-theory of smooth symplectic manifolds become easier than with other technology.

5. Symplectic Field Theory

In this section we explain how SFT fits into the picture of Fredholm theory with operations.

One of the ideas behind SFT is that it provides tools for computing Gromov-Witten invariants. It can be viewed as a theory of relative Gromov-Witten invariants and allows computations by cutting a compact symplectic manifold along suitable (real) hypersurfaces and to obtain invariants for the parts taking values in some algebraic object associated to the hypersurface. It turns out that these hypersurfaces need some geometrical properties in order to make this program possible. Then, of course, later one wants to cut along hypersurfaces in the hypersurfaces and so on. One also would like to allow hypersurfaces with singularities, for example chopping up the symplectic manifold via a triangulation. It seems, for example, that our polyfold language (at least in some suitable generalization) is able to deal with the analytic intricacies arising by degenerating almost complex structures along a triangulation.

In the above program it turns out that it is important to understand contact manifolds. Let (M, ξ) be a compact contact manifold of dimension $2n-1$ (without boundary). We assume that the contact form is co-oriented. Hence there is a distinguished class of contact forms inducing ξ. Any two of them are related by multiplication with a positive function. Having fixed such a λ denote its Reeb vector field by X. Recall that it is defined by

$$\lambda(X) \equiv 1 \quad \text{and} \quad d\lambda(X, .) \equiv 0.$$

We like to construct invariants for (M, ξ). We aim at associating to (M, ξ) an object $O(M, \xi)$ in some category $\mathfrak{O}$. The construction of $O(M, \xi)$ will be accomplished in the following way. For (M, ξ) we pick a generic contact form λ inducing ξ and a complex multiplication $J : \xi \to \xi$ compatible with $d\lambda$. Then we associate to (λ, J) a Fredholm problem with operations say $\mathfrak{f}_{(\lambda,J)}$. Usually these problems are not generic for any choice of (geometric) data. Then an abstract perturbation by sc$^+$-sections (in fact they have to be multi-sections) will make the data generic and we can produce via counting of solutions the data for an associated Homology independent of the (abstract) perturbation. Having made different choices (λ, J) and (λ', J') and taking a generic homotopy there is a new associated Fredholm problem F which has a "module structure" as left module over $\mathfrak{f}_{(J,\lambda)}$ and as right-module over the other. In addition it has some more properties which allows to show that taking different perturbations leads to the same results if one counts correctly. We will not discuss this further, but refer the reader to [**17**]. We just mention some suggestive formulas which occur everywhere in the theory and catch what we mean that the Fredholm problem F has a module structure over f and f' as left- and right module, respectively:

$$F(x \circ_B^L a) = f(x) \circ_B^L F(a) \text{ and } F(b \circ_A^R z) = F(b) \circ_A^R f'(z).$$

Of course there is a certain set of axioms "regulating" the interplay between the "algebra" of the left- and right- degeneration structures and the way they operate on the middle problem.

5.1. Background Material. We explain the necessary background material.

5.1.1. *Finite Energy Maps.* In order to describe the moduli spaces we are interested in, we fix a non-degenerate contact form λ inducing the contact structure ξ and respecting the co-orientation. We denote the associated Reeb vector field by X. A contact form is said to be non-degenerate provided all its periodic orbits, i.e., those of its Reeb vector field, are non-degenerate. Recall that this means that the eigenvalues of the linearized Poincaré section maps contain no root of unity. Non-degeneracy is generic, see [**23, 25**]. Next we pick a compatible complex multiplication $J : \xi \to \xi$ on the contact planes ξ. Compatibility here means that for $h, k \in \xi_m$, $m \in M$, the map

$$(h, k) \to d\lambda(h, J(m)k)$$

is a positive definite inner product on ξ_m. We extend J to a $\mathbb{R}$-invariant almost complex structure $\tilde{J}$ on $\mathbb{R} \times M$ by defining

$$\tilde{J}(a, u)(s, h) = (-\lambda(h), J(u)\pi(h) + sX(u)).$$

Here $\pi : TM \to \xi$ is the projection along the Reeb vector field.

We are interested in equivalence classes of solutions of the nonlinear Cauchy-Riemann associated to $\tilde{J}$. To be more precise consider tuples[14] $\alpha := (S, j, \Gamma, \tilde{u})$ where (S, j) is a closed Riemann surface, $\Gamma \subset S$ a finite

[14] Of course one also could allow marked points.

ordered subset, and $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M$ a proper map satisfying the partial differential equation

$$T\tilde{u} \circ j = \tilde{J} \circ T\tilde{u}$$

and the energy condition

$$\int_{S\setminus\Gamma} u^* d\lambda < \infty.$$

If $\tilde{u}$ solves the differential equation so does $\tilde{u}_c$ defined by $\tilde{u} = (a + c, u)$, where c is a real number. We call two such tuples equivalent, say $\alpha \equiv \alpha'$ provided there exists a biholomorphic map $\phi : (S, j) \to (S', j')$ mapping Γ onto Γ' preserving the ordering so that for a suitable c

$$\tilde{u}' \circ \phi = \tilde{u}_c.$$

If S is disconnected we only will allow a constant c and not a locally constant function c in the definition of equivalence. As it will turn out the invariants of contact manifolds are obtained by counting equivalence classes of objects (as above) where the underlying Riemann surface is connected. However, the fact that such counting leads to invariants independent of the choices involved also requires counting of certain classes of non-connected Riemann surfaces. We would like to point out that even for defining the invariants it is nevertheless important that non-connected surfaces are being considered. The deeper reason for this is the following fact. In general it is not possible (for intrinsic reasons) to pick geometric data in such a way that the occurring Fredholm operators are transversal to the zero-section. Nevertheless abstract (multi-valued) perturbations can be picked in such a way that transversality can be achieved. In order that these counts only depend on the connected elements of the moduli space the perturbations have to be picked in such a way that they respect the non-connectedness. Observe that due to the fact that all Fredholm problems are strongly interrelated there are infinitely many conditions to be satisfied by the perturbation to achieve a perturbed problem featuring the same structure. In fact, due to the relevant compactification which has a certain level structure, there are connected objects in the compactification which contain disconnected levels. The perturbation restricted to such disconnected levels is essentially the same than the perturbation on the curve in this specific level viewed as part of a different moduli space.

5.1.2. *Behavior At Punctures.* From now on we assume that all periodic orbits for the Reeb vector field X are non-degenerate. A periodic orbit for us will be a pair (P, k), where P is a submanifold of M diffeomorphic to S^1 and tangent to the Reeb vector field X. Moreover k is a positive integer called the covering number. The following is known and follows from standard results for Hamiltonian systems:

Proposition 5.1. *Given a contact form λ there is a Baire set $\Theta \subset C^\infty(M,(0,\infty))$ so that for every $f \in \Theta$ the contact form $f\lambda$ is non-degenerate.*

Consider a non-degenerate contact form λ on the closed M^{2n-1}. A periodic orbit for the associated Reeb vector field is a pair (P,k) where $P \subset M$ is diffeomorphic to a circle and tangent to X and $k \geq 1$ is an integer called the covering number. We will write P instead of $(P,1)$. The non-degeneracy means that the linearized Poincare return maps and their iterates do not have 1 in the spectrum. The k-fold iterated linearized Poincare map of P is, of course, the linearized Poincare map for (P,k) if we take the same local section. Call an orbit P troublesome if $(-1,0)$ contains an odd number of eigenvalues. We call an orbit (P,k) even if the determinant of $Id-A$ where A is the linearized Poincare map is positive. Otherwise we call it odd. Now denote by $\mathcal{P}$ the collection of all period orbits (P,k) so that k is odd if P is troublesome. We denote the collection of all periodic orbits by $\mathcal{P}_{all}$. Let us denote by γ the elements of $\mathcal{P}$.

In order to study the behavior of a finite energy map near a puncture $\gamma \in \Gamma$ fix holomorphic polar coordinates $\sigma : \mathbb{R}^+ \times S^1 \to \dot{S}$ around γ. Namely take a disk-like closed neighborhood $\mathcal{D}$ of γ containing no other puncture and having a smooth boundary. Let D be the closed unit disk in $\mathbb{C}$. Take a biholomorphic map $h : D \to \mathcal{D}$ mapping 0 to γ and define σ by

$$\sigma(s,t) = h(e^{-2\pi(s+it)}).$$

With $\tilde{u}$ being the finite energy map define $\tilde{v} = \tilde{u} \circ \sigma$. Then $\tilde{v}$ satisfies

$$\begin{gathered}\tilde{v} : \mathbb{R} \times S^1 \to \mathbb{R} \times M \\ \tilde{v}_s + \tilde{J}\tilde{v}_t = 0 \\ E(\tilde{v}) < \infty.\end{gathered}$$

Write $\tilde{v} = (b,v)$.

The following result is well-known, see [**14**], [**21, 22**]:

Theorem 5.2. *Let λ be a non-degenerate contact form on the closed manifold M and assume that J is an admissible complex multiplication. With $\tilde{v}$ as described above the limit*

$$T := \lim_{s\to\infty} \int_{S^1} v(s)^*\lambda$$

exists. If $T = 0$ the puncture γ for $\tilde{u}$ is smoothly removable. If $T \neq 0$ the number $|T|$ is the period of a periodic orbit of the Reeb vector field X associated to λ. Further there exists a constant d and a $|T|$-periodic orbit x so that

$$\begin{gathered}b(s,t) - Ts - d \to 0 \text{ as } s \to \infty \ \in C^\infty(S^1,\mathbb{R}) \\ v(s,t) \to x(Tt) \ \in C^\infty(S^1,M) \text{ as } s \to \infty.\end{gathered}$$

Even more can be said about the convergence near the puncture and we give more details soon. According to the cases $T < 0, T = 0, T > 0$ we will distinguish negative, removable and positive punctures.

In order to study the behavior near a puncture in more detail we need special coordinates. In the lemma below we denote by λ_0 the standard contact form

$$\lambda_0 = d\vartheta + \sum_{i=1}^{n-1} x_i dy_i$$

on $S^1 \times \mathbb{R}^{2(n-1)}$ with coordinates $(\vartheta, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1})$.

Lemma 5.3. *Let (M, λ) be a $(2n-1)$-dimensional manifold equipped with a contact form, and let $x(t)$ be a T-periodic solution of the corresponding Reeb vector field $\dot{x} = X_\lambda(x)$ on M. Let τ be the minimal period such that $T = k\tau$ for some positive integer k. Then there is an open neighborhood $U \subset S^1 \times \mathbb{R}^{2(n-1)}$ of $S^1 \times \{0\}$ and an open neighborhood $V \subset M$ of $P = \{x(t) \mid t \in \mathbb{R}\}$ and a diffeomorphism $\varphi\colon U \to V$ mapping $S^1 \times \{0\}$ onto P such that*

$$\varphi^* \lambda = f \cdot \lambda_0, \tag{5.1}$$

with a positive smooth function $f\colon U \to \mathbb{R}$ satisfying

$$f \equiv \tau \quad \text{and} \quad df \equiv 0 \tag{5.2}$$

on $S^1 \times \{0\}$.

For large s we can write $\tilde{v}$ in local coordinates around the limiting periodic orbit as follows (assuming that $T \neq 0$).

$$\tilde{v}(s,t) = (b(s,t), \vartheta(s,t), z(s,t))$$

where $z = (x, y)$. Here $s \geq s_0$ and $t \in \mathbb{R}$. The function $\vartheta(s,t)$ satisfies $\vartheta(s, t+1) = k + \vartheta(s,t)$. The functions b and z are 1-periodic in t.

The main result concerning the asymptotic behavior is the following, [**21, 22**].

Theorem 5.4. *There exists a constant $d > 0$ and constants b_0, ϑ_0 so that for every multi-index α there is a constant C_α so that*

$$\begin{aligned} |\partial^\alpha[b(s,t) - b_0 - Ts]| &\leq C_\alpha e^{-ds} \\ |\partial^\alpha[\vartheta(s,t) - \vartheta_0 - kt]| &\leq C_\alpha e^{-ds} \\ |z(s,t)| &\leq C_\alpha e^{-ds}. \end{aligned} \tag{5.3}$$

This result is crucial for the functional analytic set-up, since it tells us which function spaces to take. The constant d is related to the spectral properties of some self-adjoint operator A associated to the limiting periodic orbit. In fact d should be smaller than the distance of the smallest positive or largest negative eigenvalue to 0.

5.2. The Moduli Spaces. Now we can introduce the moduli spaces we are interested in.

5.2.1. *Height-k-Curves.* In this subsection we introduce the relevant compactifications for the moduli spaces of pseudoholomorphic curves in symplectized contact manifolds. There are obvious extensions to symplectic cobordisms which are important. We refer the reader to [**5, 9**] for complete detail.

The following type of level-k-curves are important and were introduced as objects occurring in the compactification of moduli spaces. Let us begin with level-1-curves. Consider tuples $(S, j, \tilde{u}, \Gamma, M, D)$ where (S, j) is a closed Riemann surface and D a set of nodal pairs. The sets Γ and M are mutually disjoint from the nodal points. Here M is an ordered set of marked points and Γ is an ordered set of so-called punctures. Further $\tilde{u} : S \setminus \Gamma \to \mathbb{R} \times M$ is a $\tilde{J}$-holomorphic proper map and $\tilde{u}(x) = \tilde{u}(y)$ for every nodal pair.

We call two such tuples equivalent if there exists a biholomorphic map $\phi : (S, j, \Gamma, M, D) \to (S', j', \Gamma', M', D')$ and a constant c so that $\tilde{u}' \circ \phi = \tilde{u}_c$.

After having introduced level-1-curves we define the somewhat more complicated level-k-curves. We are given the following data. A finite sequence of closed Riemann surface (S_ℓ, j_ℓ) with $\ell = 1, \ldots, k$. For every S_ℓ a set of nodal pairs D_ℓ on S_ℓ, for every $\ell = 1, \ldots, k-1$ a set $\hat{D}_\ell$ of ordered pairs $(\hat{x}, \hat{y})$ where $\hat{x}$ is a positive decorated[15] puncture on S_ℓ and $\hat{y}$ a negative decorated puncture on $S_{\ell+1}$. Moreover an ordered set of punctures Γ which is contained in $S_1 \cup S_k$ and an ordered set of marked points M which lie on the union of the S_ℓ. The set of Γ-points on S_k are positive and those on S_1 negative punctures. Moreover we are given maps $\tilde{u}_\ell$ on the S_ℓ. All this data satisfies the following conditions. At a nodal pair $\{x, y\} \in D_\ell$ we have $\tilde{u}_\ell(x) = \tilde{u}_\ell(y)$. At the decorated nodes $(\hat{x}, \hat{y}) \in \hat{D}_\ell$ we require that $\hat{x}$ has a positive asymptotic limit which is the negative asymptotic limit at $\hat{y}$ and in addition the points on the periodic orbit associated to the asymptotic markers coincide. The equivalence of two such objects is defined via biholomorphic maps between the corresponding levels preserving all data[16] and establishing a correspondence between the maps $\tilde{u}$ where we are allowed a different $\mathbb{R}$-shift on every level. Here is an important definition:

DEFINITION 5.5. We say that a level-k-curve is stable provide for every component C of a level either $2g + r \geq 0$, where g is the genus

---

[15] A decorated puncture consists of a point $z \in S$ together with an oriented real half line in $T_z S$. The decoration allows to take a special class of holomorphic polar coordinates compatible with the asymptotic direction. In particular, the results about asymptotic convergence imply that there is in an obvious way a special point on the asymptotic periodic orbit associated to the asymptotic marker.

[16] Here we allow the common rotation of the asymptotic markers of a pair $(\hat{x}, \hat{y})$ beforehand.

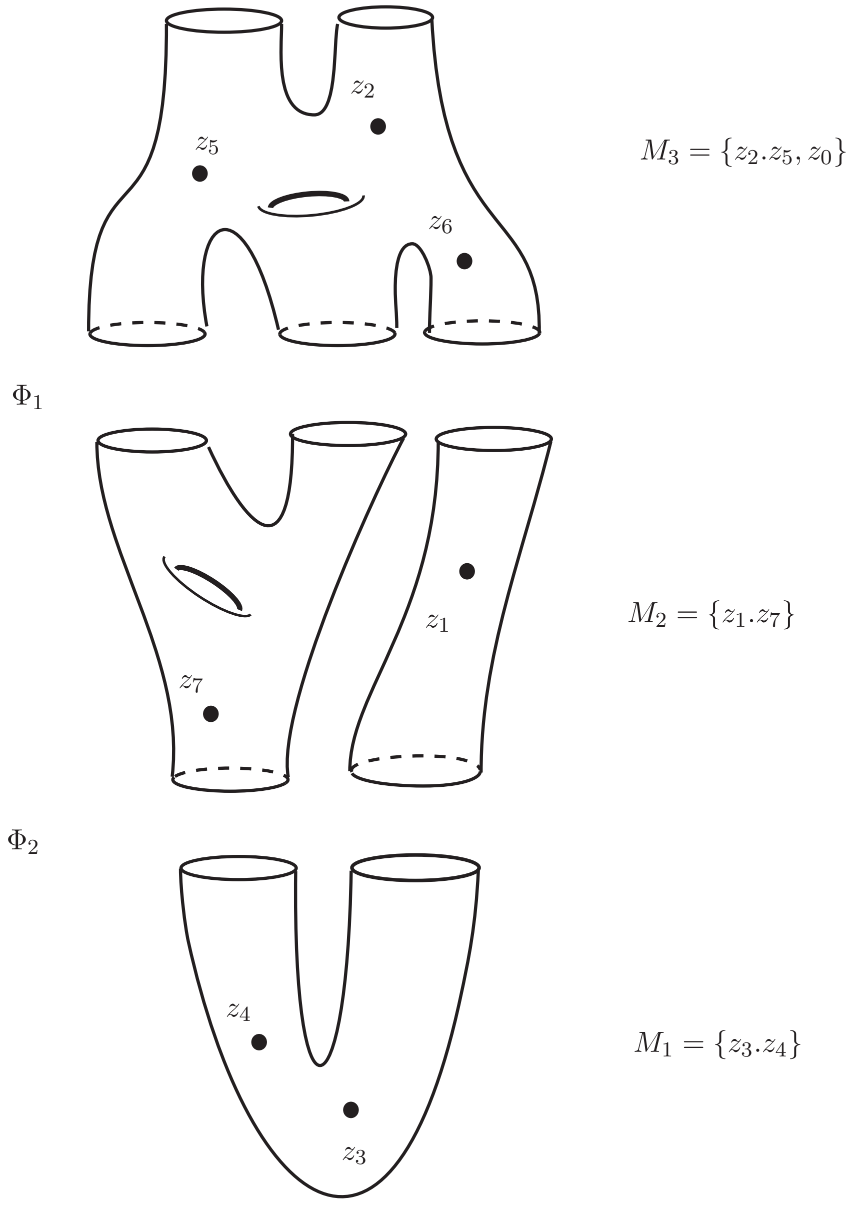

FIGURE 6. Holomorphic building of height three with an ordered set of marked points.

of C and r the number of special points on C. If that is not the case we require that the difference of the sum of the periods of the positive punctures minus the sum of the periods of the negative punctures is positive.

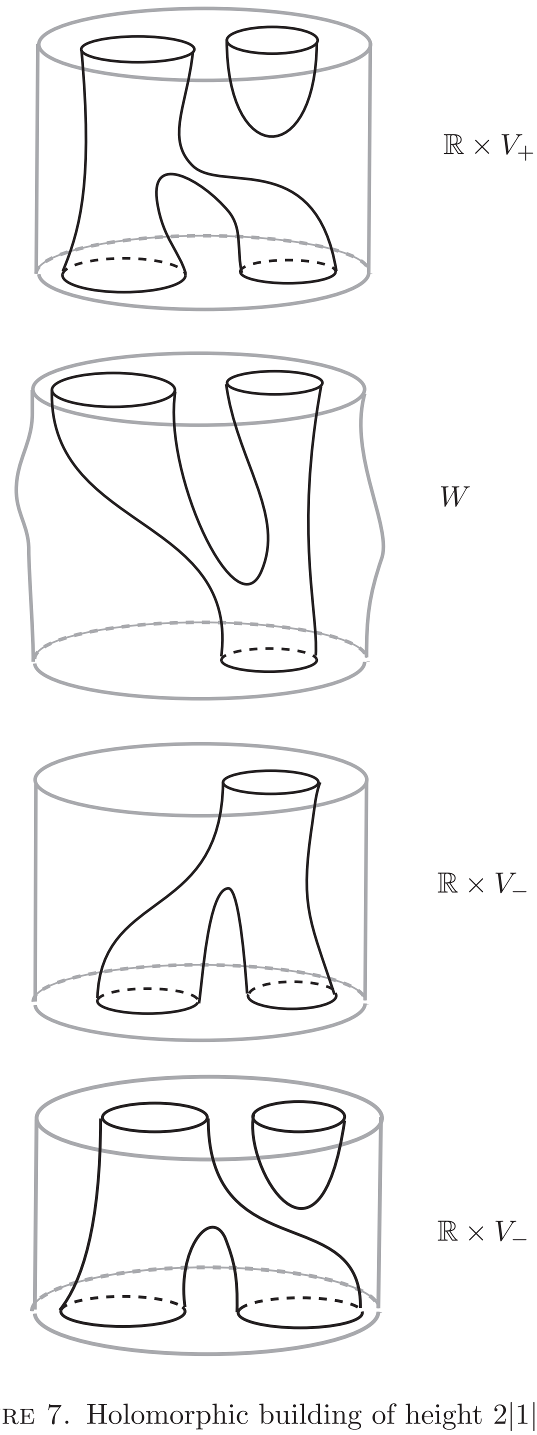

FIGURE 7. Holomorphic building of height 2|1|1.

Figure 6 shows a level-3-curve in $\mathbb{R} \times V$. We can also study $\tilde{J}$-holomorphic curves in a symplectic cobordism with contact-type boundary. Here one adds to the convex boundary V^+ the half-cylinder $[0,\infty) \times V^+$ and to the concave one $(-\infty, 0] \times V^-$. In these necks the analysis is quite similar to the one described for $\mathbb{R} \times M$. The analysis in the symplectic cobordism is like Gromov's analysis, see [**12, 26**]. The necessary compactification then consists of so-called $(k^-|1|k^+)$-curves. An example is depicted in Figure 7. It is almost apparent why this problem should have something like a module structure over the $\mathbb{R} \times V^{\pm}$-problems.

5.2.2. *The polyfold Fredholm set-up.* This compactification of level-1-curves via higher level curves can be viewed as the zero-set of a polyfold Fredholm problem with operations. In order to construct a polyfold set-up, we essentially take the same objects but do not require the maps to be $\tilde{J}$-holomorphic. The level-structure will incorporate the differentiability of the maps $\tilde{u}$ in some Sobolev class and exponential decay properties in their convergence to periodic orbits near the punctures and some similar properties near the nodes. We will not describe this in more detail here, but refer the reader to [**10, 15, 16, 17**]. Let us nevertheless describe the indexing of the operation.

5.2.3. *The operation.* Assuming that we have put the SFT problem into a polyfold set-up with bundle $Y \to X$ and a Fredholm section f, which, of course, is the nonlinear Cauchy-Riemann operator, we describe in this subsection the operation.[17]

For every periodic orbit γ introduce two symbols p_γ and q_γ. These symbols have a grading via a suitably normalized Conley-Zehnder index (or a mod 2 reduction thereof). This index plays the role of a Morse-index. We introduce a calculus of these symbols by allowing two "even" or one "even" and one "odd" symbol to commute. Two "odd" symbols anti-commute. Then we introduce an additional symbol $\hbar$ which is even. We add the relation

$$[p_\gamma, q_\gamma] = \kappa_\gamma \hbar,$$

where κ_γ is the covering number of γ. The commutator is, of course, a super-commutator. We allow now finite formal products of powers of these symbols of the form (and in the order as written)

$$\tau := \hbar^{g-1} q_{\gamma_1}^{k_1} \dots q_{\gamma_\ell}^{k_\ell} p_{\gamma_{\ell+1}}^{k_{\ell+1}} \dots p_{\gamma_{\ell+m}}^{k_{\ell+m}}.$$

The indexing then assigns to τ the component (not necessarily connected) of elements in the polyfold which have genus g, asymptotic negative limits (with multiplicities) $\gamma_1, \dots, \gamma_{k_\ell}$ and asymptotic positive limits $\gamma_\ell, \dots, \gamma_{\ell+m}$. We say this symbol sequence has standard form.

[17] There are many variations on what follows below. Our conventions are somewhat different from those in [**9**].

We say two symbols in standard form are equivalent if by permutation within the $q-$ and p-part using the commutation rules they can be brought into the same form. Denote for such a symbol sequence σ by $[\sigma]$ the equivalence class. Given $[\sigma]$ and $[\tau]$ define $[\sigma][\tau]$ by $[\sigma\tau]$. The latter symbol is not in standard form, but using all the rules is a formal finite sum

$$[\sigma][\tau] = \sum \lambda_{[\beta]}[\beta],$$

where the β are in standard form and the occurring classes are different. Further $\lambda_{[\beta]}$ are integers $\neq 0$. Then define a degeneration structure (S, R) as follows. The set S consists of all classes $[\sigma]$ in standard form using only symbols associated to "non-troublesome" periodic orbits, and the relators are triple $([\sigma], [\tau]; [\beta])$, where $[\beta]$ occurs with a nontrivial coefficient in the formal sum above.

The operation then assigns to $[\beta]$ level-k-curves of arithmetic genus g ($g-1$ is the exponent of $\hbar$) so that the top punctures are $(+)$-asymptotic to the γ occurring in p_γ and the bottom punctures are $(-)$-asymptotic to the γ occurring in a q-symbol. Also troublesome orbits will play a role in the theory. They occur in certain situations which one might call "geometric wall-crossing".

The homological data then produced from the counting of solutions in the moduli spaces has a very rich representation theory and we refer the reader to [**9**].

5.3. Comments. There are in general quite a number of different ways to turn a moduli problem into a Fredholm problem with operation. For example, in SFT one might fix asymptotic markers to the punctures and tag the periodic orbits with generically chosen points, or one might not make such choices at all. In certain situations this might lead to moduli spaces which in the first case have M-polyfold descriptions which are easier, or, in the second case, only polyfold descriptions. In some sense the first description is a covering of the second. Depending on the situation one might prefer one description over the other. In [**16**] we will describe a certain number of different approaches to the same problem.

6. Outlook and Thoughts

In the cases of Floer-Theory, Gromov-Witten Theory, and SFT there are many benefits of the polyfold theory. It gives a clean and easy language to describe and handle these problems. On the abstract level it offers the benefits of the usual "Fredholm package", i.e., transversality and perturbation theory. Of course, to bring the problems into such a framework is usually quite technical. (One shouldn't forget that the problems are for good reason considered very hard problems, i.e., one should not expect a free ride.) It is worthwhile to note that the known procedures of bringing the concrete problems into the polyfold set-up indicate standardized features. For example, it is quite feasible that the understanding of a wider range of applications would allow us to formulate a certain number of useful results in a “Scale-Analysis” which would simplify the transition from a concrete problem to a polyfold description.

It seems to be plausible that the theory described here, or suitable generalizations, should be applicable to quite a number of nonlinear problems. In fact a quick look at current research activities shows that problems with a lack of compactness are very prominent. There are, of course, problems like Yang-Mills or Seiberg-Witten-Floer Homology, which perhaps (under presumably mild generalizations) could be put into such a framework. It is likely that the analytical set-up for proving the Atiyah-Floer conjecture, which requires ultimately a homotopy from a Yang-Mills to a symplectic Lagrangian intersection problem, should be possible within our new framework.

Other problems of interest might be Ginzburg-Landau type problems, or elliptic problems with limiting Sobolev exponents. Here one should try to derive a good compactification of the problem. Even if in a physical context only particular solutions might be of interest it might still be the right point of view to consider a compactified solution space which carries invariants that cannot be destroyed and then to show later that for topological reasons there have to be solutions of physical interest as well[18]. It seems that currently this type of idea has been implemented quite successfully in an interesting array of problems with a geometric background, where the geometry quite often “dictates” a suitable compactification, but to a lesser degree in other problems.

Another interesting direction could be concerned with bubbling-off in a context of geometric evolution problems. It would be interesting to know if one can describe such phenomena in a polyfold context. Is there, for example, a theory of evolution equations in polyfolds and, importantly, are there good applications which would show the benefit of such an approach?

Then there are quite a number of “immediate spin-off ideas”. For example, as mentioned before one could try to develop some algebraic topology framework for spaces with operations, i.e., spaces with boundary with corners, where the faces are explained as products of their components or more generally as fibered products. In short, spaces with

$$\partial M = M \circ M.$$

Then, as our limited experience already shows, there should be some kind of representation theory of the rudimentary algebraic topology data leading to interesting algebraic objects. For example Floer-Theory,

[18]One should just recall how useful and fruitful the notion of a weak solution for a pde has been.

Contact Homology and more generally SFT show that the basic algebraic structures associated to a Fredholm problem with operations can have representations as differential algebras, (super-) Poisson algebras and (super-) Weyl algebras. In other words, the area looks interesting enough to give these issues some further thought.

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