The black hole stability problem

Andras Vasy

Contents

1. The setup

These lectures concern the stability of black holes, which are certain Lorentzian manifolds solving Einstein's equation – the first lecture will mostly discuss what these black holes are and what stability means. These notes follow the general structure of the actual lectures, with Sections 1–5 comprising the first lecture, while the rest covers the second, more technical, lecture as well as various extensions.

We adopt the convention that Lorentzian metrics on an n-dimensional manifold M have signature $(1, n-1)$. That is to say, they are symmetric non-degenerate bilinear forms on the tangent space T_pM at each $p \in M$ such that the largest dimensions of a subspace on which they are positive, resp. negative, definite are 1, resp. $n-1$. The non-degeneracy means that the map $T_pM \ni V \mapsto g(V, .) \in T_p^*M$ is invertible; this allows one to identify

These lecture notes are based on joint work with Peter Hintz, and in part with Dietrich Häfner. The author gratefully acknowledges support from the National Science Foundation under grant number DMS-1953987.

vectors and covectors, and in particular to put a non-degenerate pairing also on T_p^*M; this is the dual metric written as g^{-1}.

For instance, the Minkowski metric on $\mathbb{R}^4 = \mathbb{R}^{1+3}$, with coordinates z_0, z_1, z_2, z_3, is the Lorentzian metric

$$g = dz_0^2 - dz_1^2 - dz_2^2 - dz_3^2;$$

this is translation invariant. This corresponds to the pairing

$$g((x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3)) = x_0 y_0 - (x_1 y_1 + x_2 y_2 + x_3 y_3)$$

on the tangent space (with which the vector space $\mathbb{R}^4$ itself can be identified). Here z_0 is 'time', (z_1, z_2, z_3) 'space', but there are many other timelike and spacelike coordinate functions on it. The dual metric is

$$g^{-1} = \partial_{z_0}^2 - \partial_{z_1}^2 - \partial_{z_2}^2 - \partial_{z_3}^2.$$

In general, for a tangent vector V timelike means $g(V,V) > 0$, spacelike means $g(V,V) < 0$ and lightlike means $g(V,V) = 0$, so e.g. on Minkowski space $2\partial_{z_0} + \partial_{z_1}$ is timelike, $\partial_{z_0} + \partial_{z_1}$ is lightlike and $\partial_{z_0} + 2\partial_{z_1}$ is spacelike. For a covector α timelike means $g^{-1}(\alpha, \alpha) > 0$, spacelike means $g^{-1}(\alpha, \alpha) < 0$, lightlike or null, means $g^{-1}(\alpha, \alpha) = 0$; this can then be applied to any function f by considering its differential df. Thus, on Minkowski space, e.g. $2z_0 - z_1$ is timelike, $z_0 - z_1$ is lightlike and $z_0 - 2z_1$ is spacelike. In view of the isomorphism induced by g, the differentials of these are in fact the one-forms corresponding to the three previously listed tangent vectors. Finally for a hypersurface Σ the statement spacelike refers to non-zero tangent vectors to Σ: Σ is spacelike if all non-zero tangent vectors are spacelike. Equivalently (and more conveniently, as this immediately extends to the timelike and lightlike cases), one can use its conormal bundle: Σ is spacelike if all non-zero vectors in the conormal bundle are timelike, Σ is timelike if all non-zero covectors in the conormal bundle are spacelike, and Σ is lightlike if all covectors in the conormal bundle are lightlike.

In 4 dimensions Einstein's equation in vacuum is an equation for the metric tensor g of the form

$$\mathrm{Ric}(g) + \Lambda g = 0, \tag{1}$$

where Λ is the (given) cosmological constant, and $\mathrm{Ric}(g)$ is the Ricci curvature of the metric; the n dimensional generalization is $\mathrm{Ric}(g) + \frac{2\Lambda}{n-2} g = 0$. The slightly complicated numerology is due to the natural phrasing of the equation in terms of the Einstein tensor, which we discuss in Section 4. If there were matter present, there would be a non-trivial right hand side of the equation, given by a modification of the matter's stress-energy tensor; again the modification corresponds to the natural phrasing of the equation using the Einstein tensor.

In local coordinates, the Ricci curvature is a non-linear expression in up to second derivatives of g; thus, (1) is a partial differential equation. Only a few properties of Ric matter for our purposes; we point these out as they arise.

The type of PDE that Einstein's equation is most similar to, albeit with certain issues, is the class of (tensorial, non-linear) wave equations. The typical formulation of such a wave equation is that one specifies 'initial data' or 'initial conditions' (IC), at a spacelike hypersurface, such as $\Sigma = \{z_0 = C\}$, C constant, in Minkowski space. For linear wave equations $\Box u = f$ on spaces like $\mathbb{R}^{1+3}$, where $\Box = d^*d = D_{z_0}^2 - D_{z_1}^2 - D_{z_2}^2 - D_{z_3}^2$, the solution u for given data (which are the prescribed value of u and a transversal derivative at this hypersurface: $u|_\Sigma$ and $D_{z_0}u|_\Sigma$) exists globally (i.e. on $\mathbb{R}^{1+3}$) and is unique.

The analogue of the question how solutions of Einstein's equation behave in this much simpler setting is: if one has a solution u_0 of $\Box u = 0$, say $u_0 = 0$ with vanishing data, we ask how the solution u changes when we slightly perturb data (to be still close to 0). For instance, does u stay close to u_0 everywhere? Does it perhaps even tend to u_0 as $z_0 \to \infty$? This is the question of *stability* of solutions, with the latter being *asymptotic stability*. Since one cannot expect that the universe is given by some explicit solution of Einstein's equation, even if it is close to it, answering this question is of great importance.

Now, Ric is diffeomorphism invariant (i.e. is invariant under changes of local coordinates), so if Ψ is a diffeomorphism, then $\mathrm{Ric}\Psi^* g = \Psi^*\mathrm{Ric}g$. Hence if g solves Einstein's equation, then so does the pullback $\Psi^* g$. This means that if there is one solution, there are many (even with same initial conditions), since there are many diffeomorphisms fixing even a neighborhood of the initial hypersurface Σ. While this does not seem to conflict with the *existence* of solutions to the equation (since it says there are many solutions if there is one), in practice a typical way of solving non-linear PDE relies on a reduction to linear PDE by an iterative argument, while solving linear PDE is based on duality arguments, for which one would want an injective adjoint operator (with estimates!), so as the adjoint of the linearization has the same properties as the linearization (which was far from being injective), this means that it may not be easy to solve the equation at all!

To summarize, Einstein's equation is not quite a wave equation since it does not have unique solutions for given initial conditions, but it can be turned into one by imposing extra gauge conditions. Concretely, imposing that the local coordinates solve wave equations enabled Choquet-Bruhat [**16**] to show local well-posedness in the 1950s. A closely related version, is DeTurck's trick [**31**]; we discuss this in more detail in Section 4.

With the local theory well understood, it is natural to turn to questions of the global behavior. The first stability results were obtained for Minkowski space and de Sitter space (the latter will be discussed in detail soon), respectively, and are due to Christodoulou and Klainerman [**17**] in the 1990s, later simplified by Lindblad and Rodnianski [**68**] in the 2000s (and extended by Bieri and Zipser [**9**]), resp. Friedrich [**43**] in the 1980s. In the late 2010s Hintz and the author [**56**], gave a different proof of the

Minkowski result that provided a full asymptotic expansion (polyhomogeneous, with logarithmic terms) of the metric. As the timeline indicates, the de Sitter stability problem is better behaved than the Minkowski one, for reasons that will become clear later.

The first main result [**55**], joint with Peter Hintz, in this lecture is the *global non-linear asymptotic stability* of the Kerr-de Sitter family of metrics for the initial value problem for small angular momentum a ($\Lambda > 0$). The second main result [**50**] with Dietrich Häfner and Peter Hintz is the analogus *linearized* stability of the Kerr family for small a ($\Lambda = 0$). The Kerr and Kerr-de Sitter families are Lorentzian metrics in 4 dimensions depending on two parameters, called mass m and angular momentum a (as well as the cosmological constant Λ), whose geometric features we explore at first.

We will discuss both $\Lambda = 0$ and $\Lambda > 0$, though we focus on the latter as the results are more complete then. We remark that the observed accelerating expansion of the universe is consistent with a positive cosmological constant, which plays the role of a positive vacuum energy density; indeed, in theoretical physics $\Lambda > 0$ *is* what plays a dominant role. Roughly, in the author's admittedly biased opinion, $\Lambda > 0$ is the geometer's problem, as it has all the interesting black hole features without serious analytic complications, while $\Lambda = 0$ is the analyst's problem as most of the additional difficulties are ultimately of analytic nature.

We finally note that while physically $\Lambda > 0$ is small, on the scale of stability, i.e. 'time tends to ∞' behavior, there is no such thing as small Λ: on the relevant time scale the only relevant distinction is whether Λ is zero, or it is positive (or indeed negative, corresponding to anti-de Sitter like geometries that we do not discuss here).

2. Geometry

The simplest solution of Einstein's equation with $\Lambda = 0$ is Minkowski space, which is of course flat, i.e. the curvature tensor vanishes: it is the Lorentzian version of Euclidean space. Its counterpart in $\Lambda > 0$ is de Sitter space. This is a symmetric space, i.e. all points behave the same way; it is a Lorentzian version of hyperbolic space.

There is a simple description of de Sitter space is in terms of Minkowski space of one higher dimension: n-dimensional de Sitter space dS^n is the hyperboloid

$$z_0^2 - (z_1^2 + \ldots + z_n^2) = -1 \tag{2}$$

in $\mathbb{R}^{n+1}$ with the Minkowski metric $dz_0^2 - (dz_1^2 + \ldots + dz_n^2)$. Pulling back the metric to dS^n one obtains a signature $(1, n-1)$ Lorentzian manifold which solves Einstein's equation with cosmological constant $\frac{(n-1)(n-2)}{2}$. This is very much like how hyperbolic space, which is half of the two-sheeted hyperboloid

$$z_0^2 - (z_1^2 + \ldots + z_n^2) = 1,$$

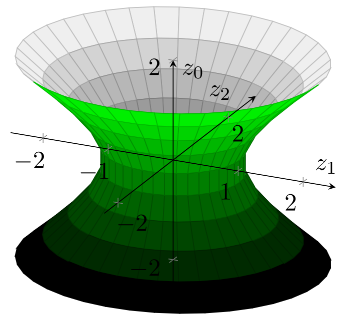

FIGURE 1. As a manifold, $dS^n = \mathbb{R}_{t^*} \times \mathbb{S}^{n-1}$, where t_* is given by an explicit expression in terms of z_0 (roughly $\log z_0$ for $z_0 \gg 2$); the metric is then $dt_*^2 - (\cosh^2 t_*)\, h$, h the round metric. Here $n = 2$.

in Minkowski space, also solves Einstein's equation, in Riemannian signature; the overall sign of the Lorentzian metric here means that the hyperbolic space metric is the *negative* of the pull back of the Minkowski metric to the hyperboloid.

Another family of explicit solutions to Einstein's equations with $\Lambda \geq 0$ (in $1+3$ dimensions here) is the Schwarzschild, resp. Schwarzschild-de Sitter (SdS) family of metrics depending on a parameter, called mass $m > 0$:

$$g = \mu(r)\, dt^2 - \mu(r)^{-1}\, dr^2 - r^2\, h,\ \ \mu(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \tag{3}$$

h the metric on the 2-sphere, $m > 0$ a parameter. Some salient features are:

- $\Lambda = 0$ gives the Schwarzschild metric, discovered about a month after Einstein's 1915 paper; $\Lambda > 0$ is the Schwarzschild-de Sitter metric.
- Depending on Λ, $m = 0$ gives the Minkowski/de Sitter metric in different coordinates.
- Thus, this family describes a black hole in Minkowski/de Sitter space in a certain sense.

Returning to the form (3) of the metric:

- $\mu(r) = 0$ has two positive solutions r_+, r_- if $m, \Lambda > 0$ and $9\Lambda m^2 < 1$ (SdS); if $\Lambda = 0$, $m > 0$, the only root is $r_- = 2m$ (Schwarzschild); if $m = 0$, $\Lambda > 0$, the only root is $r_- = \sqrt{3/\Lambda}$ (dS).
- In this form the metric makes sense where $\mu > 0$: $\mathbb{R}_t \times (r_-, \infty)_r \times \mathbb{S}^2_\omega$ ($\Lambda = 0$), resp. $\mathbb{R}_t \times (r_-, r_+)_r \times \mathbb{S}^2_\omega$ ($\Lambda > 0$).
- It is spherically symmetric,
- ∂_t is a Killing vector field, i.e. pullback by translation in t (fixing r, ω) preserves the metric.

It is not hard to see that $r = r_\pm$ are coordinate singularities.

A better coordinate than t is, with $c_\pm$ *smooth up to* $r_\pm$,

$$t_* = t - F(r),\ F'(r) = \pm(\mu(r)^{-1} + c_\pm(r)) \text{ near } r = r_\pm.$$

Notice that F' is singular at the roots of μ, so as μ has simple zeros under these assumptions, F introduces a logarithmic, in $|r - r_\pm|$, change in t. In (r_-, r_+) (or $(r_-, +\infty)$ if $\Lambda = 0$) the choice of F is not very important as long as F is smooth; the same applies to the smooth terms $c_\pm$ even near $r_\pm$. The effect of this coordinate change is 'dragging' the time variable: the sign of F' assures that $F > 0$ near $r_\pm$ and indeed tends to $+\infty$ as $r \to r_\pm$, so asymptotically t_* is much smaller than t is. A different way of thinking about this is that there is nothing special with the original presentation (3), and one should have written it in better coordinates in the first place, but these Boyer-Lindquist coordinates is how they first arose.

In the coordinates (t_*, r, ω), the metric makes sense (as a Lorentzian metric) on

$$\mathbb{R}_{t_*} \times (0, \infty)_r \times \mathbb{S}^2_\omega,$$

thus for $r \le r_-$ and $r \ge r_+$ as well. The submanifold $r = r_-$ is called the *event horizon*, $r = r_+$ the *cosmological horizon* (if $\Lambda > 0$); the geometry of the spacetime, which we discuss promptly, is very similar near these two respective horizons. Thus, locally near the event horizon the analysis and geometry is no worse than near the cosmological horizon, and at least in de Sitter space the latter is a truly artificial concept as we shall see, so we should *not* think of the horizons as 'worrisome' places (though of course they do have some global causal impacts).

The Schwarzschild/SdS metric fits into an even bigger family discovered by Kerr and Carter in the 1960s: the *Kerr/Kerr-de Sitter* family of metrics depending on 2 parameters, called mass m and angular momentum a; $a = 0$ gives the Schwarzschild/Schwarzschild-de Sitter metric.

The general Kerr(-de Sitter) metric has a somewhat intimidating coordinate expression, so we start by mentioning that the underlying manifold is *still* $\mathbb{R}_{t_*} \times (0, \infty)_r \times \mathbb{S}^2$, and ∂_{t_*} is a Killing vector field, i.e. translation in t_* preserves the metric. These metrics are axisymmetric around the axis of rotation; in the case $a = 0$ they are spherically symmetric (like the de Sitter metric). There are restrictions on a to preserve the geometric features; if $\Lambda = 0$, this is $|a| < m$; if $\Lambda > 0$ they are more complicated.

With these preliminary comments, given the angular momentum $a = |\mathbf{a}|$ of a black hole of mass m (spinning around the axis $\mathbf{a}/|\mathbf{a}| \in \mathbb{R}^3$ for $a \ne 0$), one can explicitly write down the Kerr–de Sitter metric with parameters $b = (m, \mathbf{a})$ in Boyer–Lindquist coordinates. Technically, these are valid away from the axis of rotation on the sphere $\mathbb{S}^2$, i.e. in spherical coordinates relative to this axis, for $(t, r, \phi, \theta) \in \mathbb{R} \times (r_-, r_+) \times \mathbb{S}^1_\phi \times (0, \pi)$, with $(r_-, r_+) = (r_{b,-}, r_{b,+})$ defined momentarily. Namely, the metric takes the form

$$g_b = -\rho_b^2\Big(\frac{dr^2}{\widetilde{\mu}_b} + \frac{d\theta^2}{\kappa_b}\Big) - \frac{\kappa_b \sin^2\theta}{(1+\lambda_b)^2\rho_b^2}(a\,dt - (r^2+a^2)\,d\phi)^2 + \frac{\widetilde{\mu}_b}{(1+\lambda_b)^2\rho_b^2}(dt - a\sin^2\theta\,d\phi)^2, \tag{4}$$

where

$$\widetilde{\mu}_b(r) = (r^2+a^2)\Big(1-\frac{\Lambda r^2}{3}\Big) - 2mr, \qquad \rho_b^2 = r^2 + a^2\cos^2\theta, \quad \lambda_b = \frac{\Lambda a^2}{3}, \quad \kappa_b = 1 + \lambda_b\cos^2\theta. \tag{5}$$

Then, much as for Schwarzschild-de Sitter space, $r_- = r_{b,-} < r_+ = r_{b,+}$ are positive roots of $\widetilde{\mu}_b$, with $\widetilde{\mu}_b'(r_{b,-}) > 0$, $\widetilde{\mu}_b'(r_{b,+}) < 0$. Notice that if $a \neq 0$ there is a third positive root as $\widetilde{\mu}_b(0) > 0$ in that case; $r_{b,\pm}$ are the largest two positve roots of this quartic polynomial, and they are perturbations of the SdS roots if a is small. The Kerr case is completely similar, with (r_-, r_+) replaced by $(r_-, +\infty)$, with r_- the larger positive root (there are two in this case), and $\Lambda = 0$.

To better understand the relationship between the various spaces we mentioned for $\Lambda > 0$, we start by discussing de Sitter space. It is then useful to conformally compactify $\mathrm{dS}^4 = \mathbb{R} \times \mathbb{S}^3$ by compactifying $\mathbb{R}$ to an interval. Here we concentrate on $z_0 \geq 1$ in the notation of (2); then $\tau = z_0^{-1}$, adding $\tau = 0$ as infinity, the metric is

$$\tau^{-2}\big((1+\tau^2)^{-1}d\tau^2 - (1+\tau^2)h\big),$$

h the standard metric on the sphere. In terms of the Kerr-de Sitter (modified Boyer-Lindquist) coordinates τ can be taken to be e^{-t_*}; see [**94**, Section 4]. Indeed, asymptotically as $t_* \to \infty$, i.e. $\tau \to 0$, the metric is roughly like

$$\tau^{-2}(d\tau^2 - h);$$

in the arguments below we use this simpler form for transparency as it does not affect the results in a significant way. Such a metric, being conformal to a non-degenerate smooth Lorentzian metric, is called *conformally compact*; cf. the Riemannian analogue, the Poincaré model of hyperbolic space, where there is extensive literature on this conformally compact generalization, see e.g. [**71**, **48**].

A nice feature in the context of conformally compact metrics is that null-geodesics, i.e. geodesics with null (or light-like), i.e. $g(V,V) = 0$, tangent vectors V, with the notion of geodesics being similar to the Riemannian setting, are simply reparameterized by such a conformal factor, i.e. they are essentially the same as those of $d\tau^2 - h$ under our rough comparison. (We will recall the notion of geodesics from a Hamiltonian perspective in Section 8.) This is of course very far from being true for non-null geodesics, as is shown by the Poincaré ball model of hyperbolic space.

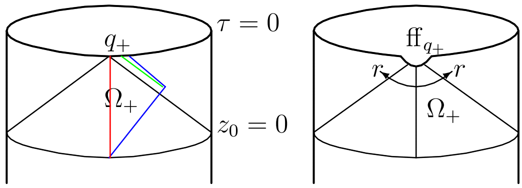

FIGURE 2. Left: the conformal compactification of de Sitter space dS^n, $n = 2$, with the backward light cone (null-geodesics) Ω_+ from q_+. The red line is the path of an observer (or particle) who tends to q_+. The blue line is that of another who leaves Ω_+... then no matter how desperately she/he/it tries, cannot get back to it. Even the green flashlight signal cannot make it back!!! Right: the blow up of de Sitter space at q_+. This desingularizes the tip of the light cone, and the interior of the light cone inside the front face ff_{q_+} can be identified with a ball, which itself is a conformal compactification of hyperbolic space $\mathbb{H}^{n-1}$. The radial variable r for the SdS presentation (3) of dS is that of the ball; $r = 1$ is the light cone.

The interior of the backward light cone from a point at $\tau = 0$ (future infinity) can be identified with $\mathbb{R}_{t_*} \times \mathbb{B}^3$; in the coordinates $(0, \infty)_r \times \mathbb{S}^2$ of the Schwarzschild-de Sitter like presentation of de Sitter space (3) (with $m = 0$), *singular* at $r = 0$ (with the usual spherical coordinate singularity), this is $r < 1$, often called the static (region of) de Sitter space; see Figure 2.

Notice that de Sitter space has the feature that if a forward timelike or lightlike curve leaves such a backward light cone, it can never return (while remaining timelike or lightlike). Thus, the lightcone, $r = 1$, acts as a *horizon*; it is called the *cosmological horizon*. Note that nothing drastic happens at the horizons though; the manifold and the metric continue smoothly across it! For Kerr-de Sitter space then we consider an analogue of this region, or rather that of its slight enlargement $r < 1 + \epsilon_0$; see Figure 3.

Kerr-de Sitter space has two such horizons, at $r = r_\pm$, with r_+ called the *cosmological horizon*, r_- the black hole *event horizon*. They are extremely similar: once one leaves, one cannot return along timelike or lightlike curves if one crosses them from the exterior region $r \in (r_-, r_+)$.

There is one more relevant null-feature of Kerr-de Sitter space: there are some trapped null-geodesics in the exterior region $r \in (r_-, r_+)$, i.e. null-geodesics that do not cross either horizon. (This does not happen in de Sitter space.) This is the photonsphere in Schwarzschild-de Sitter space, deformed in Kerr-de Sitter space. Unfortunately, given the limitations of these notes, these will not be discussed in much detail, but the basic difficulties they pose, as well as some indication how to resolve these, will be discussed in Section 8.

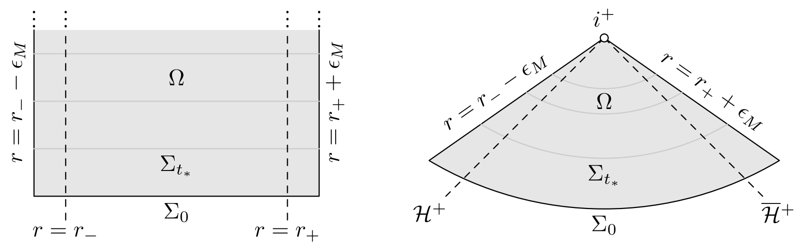

FIGURE 3. Setup for the initial value problem for perturbations of a Schwarzschild–de Sitter spacetime (M, g_{b_0}), showing the Cauchy surface Σ_0 of Ω and a few translates (level sets of the nonsingular time t_*) Σ_{t_*}; here $\epsilon_M > 0$ is small. *Left:* Product-type picture, illustrating the stationary nature of g_{b_0}. *Right:* Penrose diagram of the same setup. The event horizon is $\mathcal{H}^+ = \{r = r_-\}$, the cosmological horizon is $\overline{\mathcal{H}}^+ = \{r = r_+\}$, and the (idealized) future timelike infinity is i^+.

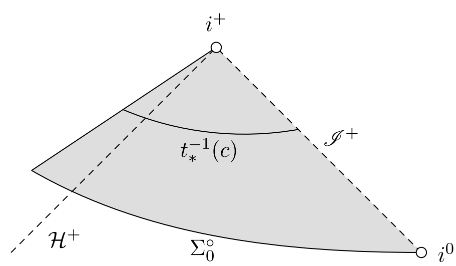

FIGURE 4. Part of the Penrose diagram of a Kerr spacetime: the event horizon $\mathcal{H}^+$, null infinity $\mathscr{I}^+$, timelike infinity i^+ and spacelike infinity i^0. We show the domain $\{\mathfrak{t} \geq 0\}$ inside of M° in gray, the Cauchy surface $\Sigma_0^\circ = \mathfrak{t}^{-1}(0)$, and a level set of t_*; $t_* = t - (r + 2m\log(r - 2m))$, r large.

The Kerr spacetime behaves completely analogously to Kerr-de Sitter near the event horizon. The key difference is the presence of the Minkowski infinity, i.e. as $r \to \infty$. For this purpose it is useful to have a time function $\mathfrak{t}$ that is equal to t_* near the event horizon (i.e. r close to r_-), and is equal to the standard t for r large. Then the underlying manifold is still $\mathbb{R}_{\mathfrak{t}} \times (0, \infty)_r \times \mathbb{S}^2$. See Figure 4.

3. Stability

We now return to the stability questions for Einstein's equation. Recall that one subtlety is the diffeomorphism invariance of the equation, causing non-uniqueness; this invariance is the only cause of non-uniqueness locally. On the flipside, one cannot specify the initial data completely arbitrarily:

they need to satisfy certain equations, called the constraint equations, implied by Einstein's equation.

In general, for a manifold M with Σ_0 a codimension 1 hypersurface, the initial data are a Riemannian metric h and a symmetric 2-cotensor k which satisfy the constraint equations, and one calls a Lorentzian metric g on M a solution of Einstein's equation with initial data (Σ_0, h, k) if the pull-back of g to Σ_0 is $-h$, and k is the second fundamental form of Σ_0 in M. Here h, k are symmetric 2-tensors on Σ_0, so in local coordinates pointwise they are 3-by-3 matrices.

As an example of this setup, a very roughly (and weakly!) stated version of stability of Minkowski space $\mathbb{R}^4_z$, $\Sigma_0 = \{0\} \times \mathbb{R}^3$, due to Christodoulou and Klainerman [**17**], is that given initial data (h, k) close to $(g_{\mathrm{Eucl}}, 0)$ in an appropriate sense (in particular decaying), there is a global solution of Einstein's equation on $\mathbb{R}^4$, and it tends to g_{Mink} as $|z| \to \infty$.

The Kerr-de Sitter stability is simplest phrased by considering a fixed background Schwarzschild-de Sitter metric, g_{b_0}, $b_0 = (m, \mathbf{0})$, where we use $\mathbf{a} \in \mathbb{R}^3$ as the angular momentum parameter instead of the scalar a. Let Σ_{t_*} denote the translate of Σ_0 by the ∂_{t_*} flow. Let

$$\Omega = \cup_{t_* \geq 0} \Sigma_{t_*}.$$

THEOREM 1 (Stability of the Kerr–de Sitter family for small a; informal version, Hintz-V., [**55**]). *Suppose (h, k) are smooth initial data on Σ_0, satisfying the constraint equations, which are close to the data (h_{b_0}, k_{b_0}) of a Schwarzschild–de Sitter spacetime in a high regularity norm. Then there exist a solution g of Einstein's equation in Ω attaining these initial data at Σ_0, and black hole parameters b which are close to b_0, so that*

$$g - g_b = \mathcal{O}(e^{-\alpha t_*})$$

for a constant $\alpha > 0$ independent of the initial data; that is, g decays exponentially fast to the Kerr–de Sitter metric g_b. Moreover, g and b are quantitatively controlled by (h, k).

By comparison, the strongest $\Lambda = 0$ nonlinear black hole result at the time of the lecture was the very recent work of Klainerman and Szeftel [**63**] under polarized axial symmetry assumptions. This is a restricted version of axial symmetry, indicating that we still have some distance to go to understand this problem. However, as these notes were prepared, an initial version of a much more general stability result by Dafermos, Holzegel, Rodnianski and Taylor [**21**] was posted, corresponding to the full problem under the constraint that the asymptotic limit is Schwarzschild spacetime, rather than another member of the Kerr family, illustrating the rapid progress in the field.

What Theorem 1 states is that the metric 'settles down to' a Kerr-de Sitter metric at an exponential rate. Note that even if we perturb a Schwarzschild-de Sitter metric, in general we get a Kerr-de Sitter limit,

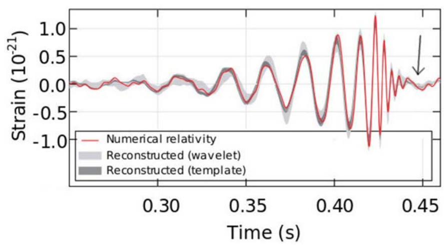

FIGURE 5. Courtesy Caltech/MIT/LIGO Laboratory 2016

since after all the slowly rotating Kerr-de Sitter metrics (and thus their initial data) are perturbations of Schwarzschild-de Sitter metrics (and their initial data), and they themselves solve Einstein's equation! This 'settling down' means that gravitational waves are being emitted; far away observers (hopefully us!) can see this 'tail'. LIGO exactly aimed (successfully!) at capturing these waves, using numerical computations as a template to see what one would expect from the merger of binary black holes.

In $\Lambda = 0$ in full generality (no additional symmetry assumptions) only linearized results are available. For the following statement recall that at the linearized level pullbacks by diffeomorphisms correspond to Lie derivatives along vector fields.

THEOREM 2 (Linearized stability of the Kerr family for small a; informal version, Häfner-Hintz-V., [**50**]). *Let $b = (m, a)$ be close to $b_0 = (m_0, 0)$; let $\alpha \in (0, 1)$. Suppose $\dot{h}, \dot{k} \in C^\infty(\Sigma_0^\circ; S^2 T^*\Sigma_0^\circ)$ satisfy the linearized constraint equations, and decay according to $|\dot{h}(r, \omega)| \le Cr^{-1-\alpha}$, $|\dot{k}(r, \omega)| \le Cr^{-2-\alpha}$, together with their derivatives along $r\partial_r$ and ∂_ω (spherical derivatives) up to order 8. Let $\dot{g}$ denote a solution of the linearized Einstein vacuum equations on Ω which attains the initial data $\dot{h}$, $\dot{k}$ at Σ_0°. Then there exist linearized black hole parameters $\dot{b} = (\dot{m}, \dot{a}) \in \mathbb{R} \times \mathbb{R}^3$ and a vector field V on Ω, lying in a 6-dimensional space, consisting of generators of spatial translations and Lorentz boosts, such that*

$$\dot{g} = \dot{g}_b(\dot{b}) + \mathcal{L}_V g_b + \dot{g}',$$

where for bounded r the tail $\dot{g}'$ satisfies the bound $|\dot{g}'| \le C_\eta \mathfrak{t}^{-1-\alpha+\eta}$ for all $\eta > 0$.

There are a number of closely related linearized $\Lambda = 0$ black hole results: linearized Schwarzschild, plus Teukolsky in the slowly rotating case: Dafermos, Holzegel and Rodnianski [**20**, **19**], as well as the linearized stability result of Andersson, Bäckdahl, Blue and Ma [**3**] also in the slowly rotating case, with also a more restricted general result, under a strong asymptotic assumption.

There has been extensive research in the area, including works by (in addition to the authors already mentioned) Wald, Kay, Finster, Kamran, Smoller, Yau, Tataru, Tohaneanu, Marzuola, Metcalfe, Sterbenz, Donninger, Schlag, Soffer, Sá Barreto, Wunsch, Zworski, Wang, Bony, Dyatlov, Luk, Alexakis, Ionescu, Shlapentokh-Rothman [**61**, **105**, **89**, **88**, **70**, **91**, **32**, **25**, **24**, **18**, **4**, **10**, **83**, **107**, **35**, **36**, **11**, **41**, **42**, **69**, **2**, **1**, **62**]...

4. The proof

We now return to the nonlinear setting, making our first steps towards proving Theorem 1. Recall that for the purposes of analysis, we need an additional gauge condition. There are two separate parts to these: the actual gauge and the implementation.

Following *DeTurck's trick* [**31**], as presented by Graham and Lee [**47**] (to solve Einstein's equation in Riemannian signature in their case, for asymptotically hyperbolic spaces, i.e. in a setting of the aforementioned [**71**, **48**]), one fixes a background metric $\mathfrak{g}_0$, and requires that the identity map $(M, g) \to (M, \mathfrak{g}_0)$ be a wave map (solve a wave equation). This is *implemented* by working with the equation (called a gauge fixed, or reduced, Einstein's equation)

$$\mathrm{Ric}(g) + \Lambda g - \Phi(g, \mathfrak{g}_0) = 0, \tag{6}$$

where

$$\Phi(g, \mathfrak{g}_0) = \delta_g^* \Upsilon(g), \ \ \Upsilon(g) = g\mathfrak{g}_0^{-1}\delta_g G_g \mathfrak{g}_0.$$

Here δ_g^* is the symmetric gradient mapping one-forms to symmetric 2-cotensors, δ_g its adjoint (negative divergence), G_g is the trace-reversal (in 4-dimensions; in general it is still given by this formula) operator

$$G_g r = r - \frac{1}{2}(\mathrm{tr}_g\, r)g,$$

and $\Upsilon(g)$ is the gauge one-form, whose vanishing is equivalent to the wave map condition. Note that $G_g\mathrm{Ric}(g)$ is the Einstein tensor, and the natural way of writing Einstein's equation in vacuum is $G_g\mathrm{Ric}(g) - \Lambda g = 0$; when matter is present, the right hand side is replaced by the stress energy tensor (appropriately normalized). One reason $G_g\mathrm{Ric}(g)$ plays a special role is that it is always divergence free by the second Bianchi identity:

$$\delta_g G_g \mathrm{Ric}(g) = 0, \tag{7}$$

which is true for any metric g.

We reiterate that there are two separate considerations: *first*, what is the gauge, i.e. the additional conditions one imposes, here the vanishing of $\Upsilon(g)$, and *second*, how is it imposed, here by adding the term Φ, which in particular involves the symmetric gradient in addition to Υ.

The analytic point is that (6) *is* a (quasilinear) wave-type equation, so the problems with diffeomorphism invariance have been eliminated, thus at least one has local existence and uniqueness near the initial surface Σ_0! Of course, the downside is that this is a different equation than what we wanted to solve, so we need to see that solving this actually gives a solution of Einstein's equation, at least if appropriate preliminary steps were made.

So let us discuss how DeTurck's gauge works. To see that for given initial data solving the gauged Einstein's equation (6) actually gives a solution of the original, ungauged, problem, one constructs Cauchy data for the gauged problem for g which give rise to the required initial data and moreover solve $\Upsilon(g) = 0$ at Σ_0 (Υ is a first order differential operator, so this is determined by Cauchy data).

Solving the gauged Einstein equation with these data (which can be done locally since this is a wave equation), the constraint equations show that the normal derivative of $\Upsilon(g)$ at Σ_0 also vanishes, and then applying $\delta_g G_g$ to the gauged Einstein's equation, in view of the second Bianchi identity, (7), gives

$$\Box_g^{\mathrm{CP}} \Upsilon(g) = 0,\ \Box_g^{\mathrm{CP}} = 2\delta_g G_g \delta_g^*.$$

Here $\Box_g^{\mathrm{CP}}$ is a (one-form) wave operator, so by the vanishing of the Cauchy data for $\Upsilon(g)$ we see that $\Upsilon(g)$ vanishes identically.

While any choice of $\mathfrak{g}_0$ works for this local theory, for the global solvability $\mathfrak{g}_0$ makes a difference; it is natural to choose $\mathfrak{g}_0 = g_{b_0}$.

The analytic framework we use is based on:

- non-elliptic linear global analysis with coefficients of finite Sobolev regularity,
- with a simple Nash-Moser iteration to deal with the loss of derivatives corresponding to both non-ellipticity and trapping;

this will be explained in more detail in Sections 7–8. But the most important feature for us for our geometric purposes is that this gives global solvability for quasilinear wave equations like the gauged Einstein's equation provided

- certain dynamical assumptions are satisfied (only trapping is normally hyperbolic trapping, with an appropriate subprincipal symbol condition) and
- there are no exponentially growing modes (with a precise condition on non-decaying ones), i.e. non-trivial solutions of the linearized equation at g_{b_0} of the form $e^{-i\sigma t_*}$ times a function of the spatial variables r, ω only, with $\operatorname{Im}\sigma > 0$.

Unfortunately, in the DeTurck gauge, while the dynamical assumptions are satisfied, there *are* growing modes, although only a finite dimensional space of these. The key to proving the theorem (given the analytic background, discussed in the second lecture, starting with Section 6) is to overcome this issue.

Typically when solving a non-linear equation any growing modes of the linearization destroy stability; even non-decaying ones typically do.

For instance, for the ODE $u' = u^2$ with initial condition at 0, $u \equiv 0$ is a solution, which is stable on $[0, T]$ for any T, but for any positive initial condition ϕ the solution $u = \phi/(1 - t\phi)$ blows up in finite time, so there cannot be any stability on $[0, \infty)$. Here the linearized operator is $v \mapsto v'$, which has the non-decaying mode $v \equiv 1$ (i.e. $\sigma = 0$).

The Kerr-de Sitter *family* automatically gives rise to non-decaying modes with $\sigma = 0$ (corresponding to infinitesimally varing the Kerr-de Sitter parameters), but as these correspond to non-linear solutions, one may expect these not to be a problem with some work. However, in the DeTurck gauge there are even growing modes, which are definitely problematic!

The reason this problem can be overcome is that the PDE is not fixed: one can modify $\Phi(g, \mathfrak{g}_0)$ as long as it gives a wave-type equation which asymptotically behaves like a Kerr-de Sitter wave equation.

In spite of this gauge freedom, we actually cannot arrange a gauge in which there are no non-decaying modes, even beyond the trivial Kerr-de Sitter family induced ones. However, we can arrange for a partial success: we can modify Φ by changing δ_g^* by a 0th order term:

$$\widetilde{\delta}^*\omega = \delta_{g_0}^*\omega + \gamma_1\, dt_* \otimes_s \omega - \gamma_2 g_0 \operatorname{tr}_{g_0}(dt_* \otimes_s \omega),$$

$$\Phi(g, \mathfrak{g}_0) = \widetilde{\delta}^*\Upsilon(g).$$

For suitable choices of $\gamma_1, \gamma_2 \gg 0$, this preserves the dynamical requirements, and while the gauged Einstein's equation does still have growing modes, it has a new feature:

$$\widetilde{\Box}_g^{\mathrm{CP}} = 2\delta_g G_g \widetilde{\delta}^*,\ \ g = g_{b_0}$$

has no non-decaying modes! It should not be a surprise that such a change is useful: there is no reason to expect that the DeTurck gauge is well-behaved in any way except in a high differential order sense, relevant for the local theory. Note that what we are thus changing at this point is not the gauge, but how it is implemented!

We call this property SCP, or *stable constraint propagation*; by a general feature of our analysis, this property is preserved under perturbations of the metric around which we linearize. Such a change to the gauge term, called *'constraint damping'*, has been successfully used in the numerical relativity literature by Pretorius [**80**] and others, following the work of Gundlach et al [**49**] (who in turn put a new twist on the approach of Brodbeck et al [**13**]; these authors already used the SCP terminology), to damp numerical errors in $\Upsilon(g) = 0$; here we show rigorously why such choices work well. Notice that damping for the gauge term is crucial, for one might well expect the gauge errors to grow exponentially otherwise (as indeed would typically happen in the Kerr-de Sitter setting), which would cause the breakdown of the numerical computation by itself.

SCP is useful because it means that, for $g = g_{b_0}$, any non-decaying mode h of the linearized gauge fixed Einstein equation is a solution of

$$D_g(\mathrm{Ric}(g) + \Lambda g)h = 0.$$

Indeed this follows by applying $\delta_g G_g$ to the gauge fixed Einstein's equation, using Bianchi's second identity, giving that $\widetilde{\Box}_g^{\mathrm{CP}}(D_g\Upsilon)h$ and thus $(D_g\Upsilon)h$ vanish. Thus, properties of a gauge dependent equation are reduced to those of one independent of the gauge!

Growing modes are disastrous for non-linear equations, such as Einstein's, so we also need a statement that the above modes are actually pure gauge modes, i.e. given by linearized diffeomorphisms, so of the form $\delta_g^*\omega$ for a one-form ω, corresponding to infinitesimal diffeomorphisms. We call this, together with a precise treatment of the zero modes, UEMS, ungauged Einstein mode stability.

UEMS is actually well-established in the physics literature in a form that is close to what one needs for a precise theorem; this goes back to Regge and Wheeler [**82**], Vishveshwara [**104**], Zerilli [**108**], Whiting [**106**] and others; the invariant form we use is due to Ishibashi, Kodama and Seto [**65, 64, 60**].

Now, if we assume away the Kerr-de Sitter-family zero modes (we call such a setting UEMS*, which holds for de Sitter space), we could easily have a framework to show global stability: namely consider

$$\Phi(g, \mathfrak{g}_0; \theta) = \widetilde{\delta}^*(\Upsilon(g) - \theta),$$

where θ is an unknown, lying in a finite dimensional space Θ of gauge terms of the form $D_{g_{b_0}}\Upsilon(\delta^*_{g_{b_0}}(\chi\omega))$, where $\chi \equiv 1$ for $t_* \gg 1$, $\chi \equiv 0$ near $t_* = 0$, and such that $\delta^*_{g_{b_0}}\omega$ is a non-decaying resonance of the gauged Einstein operator.

As $D_{g_{b_0}}\Upsilon(\delta^*_{g_{b_0}}(\omega)) = 0$ by SCP, $D_{g_{b_0}}\Upsilon(\delta^*_{g_{b_0}}(\chi\omega))$ is compactly supported, away from Σ_0, i.e. elements of Θ are also such.

Then we could solve

$$\mathrm{Ric}(g) + \Lambda g - \Phi(g, \mathfrak{g}_0; \theta) = 0$$

for g and θ, with $g - g_{b_0}$ in a decaying function space. So crucially θ is also treated as an unknown. This can be seen by solving the linearized equation without θ in a space which is decaying apart from finitely many non-decaying resonant modes, and then subtracting away cut off versions of these resonant terms and checking the equation they satisfy. Notice that we have now *changed the gauge*: we are solving Einstein's equation in the gauge $\Upsilon(g) \in \Theta$, so $\Upsilon(g)$ is not fixed, rather is in a finite dimensional space. The full Kerr-de Sitter version is not much harder, and both of these are discussed in Section 6.

5. Outlook

There are some very interesting questions that the works discussed leave open.

First of all, it is natural to ask whether the Kerr-de Sitter family stable even if a is not small. The biggest issue here is checking UEMS, which is harder due to the lack of symmetry. There is extensive work on the analogue of this for $\Lambda = 0$, and in particular for the scalar wave equation mode stability has been shown in the full subextremal parameter range $|a| < m$. These rely on Whiting's transformation [**106**] for which there is no known analogue if $\Lambda > 0$.

Another natural question is that of cosmic censorship. This concerns the behavior of solutions farther in the black hole ($r < r_-$). Namely, one might want to say that in a suitable sense the solution generically does not extend beyond the next root r_C of μ, called the Cauchy horizon. The reason is that the solution there would not be determined by the initial data, and this breakdown of determinism is a physical concern. Recent work of Dafermos and Luk [**23, 22**] in the $\Lambda = 0$ setting gives a conditional result (conditional on the stability of Kerr spacetimes in a particular quantitative sense). It is expected that this work should have an unconditional analogue, and the modification of the Dafermos-Luk argument should not be too hard.

We can also ask whether one can show an expansion of the solution in terms of decaying modes. This would mathematically justify the ringdown that physicists are studying. This should again not be overly difficult, at least including the first few decaying modes: it might become more difficult once one wants to impose sufficient decay that there is an infinite number of modes in play due to the trapping.

Last, but certainly not least: the biggest question is whether we can extend the non-linear stability results to the case $\Lambda = 0$. There are good reasons for believing this, and there are many partial results in this directions, already mentioned, with this being an extremely active area of research, so it is reasonable to expect significant progress in the very near future.

6. The gauge fixing

We now return to the claim that under UEMS* (i.e. ignoring the Kerr-de Sitter-family zero modes, which e.g. would be the case for de Sitter space) we can solve

$$\mathrm{Ric}(g) + \Lambda g - \Phi(g, \mathfrak{g}_0; \theta) = 0,$$

for g and θ, with $g - g_{b_0}$ in a decaying function space. Here $\Phi(g, \mathfrak{g}_0; \theta) = \widetilde{\delta}^*(\Upsilon(g) - \theta)$.

This can be seen by solving the linearized equation without θ in a function space which is decaying apart from finitely many non-decaying resonant modes, and then subtracting away cut off versions of these resonant terms and checking the equation they satisfy. Concretely we can proceed as follows.

The linearization of the gauged Einstein equation at $(g_{b_0}, 0)$, in (g, θ) (with the linearized change of g denoted by r, that in θ is still denoted by θ since the equation is linear in θ) is

$$(D_{g_{b_0}}\mathrm{Ric} + \Lambda)r - \widetilde{\delta}^*(D_{g_{b_0}}\Upsilon)r + \widetilde{\delta}^*\theta = 0. \tag{8}$$

We claim that this *can* be solved in a decaying function space.

Indeed, consider the equation with the θ term dropped:

$$(D_{g_{b_0}}\mathrm{Ric} + \Lambda)r - \widetilde{\delta}^*(D_{g_{b_0}}\Upsilon)r = 0.$$

The arguments of the Section 7–8 show that this equation can be solved with solution $\tilde{r}$ with

$$\tilde{r} = \sum_j r_j + r'$$

r' in a decaying function space, r_j finitely many non-decaying terms, given by the resonances, which satisfy the linearized gauged Einstein equation (but of course not the initial conditions).

By UEMS*, we have that $r_j = \delta^*_{g_{b_0}} \omega_j$, so as

$$(D_{g_{b_0}} \mathrm{Ric} + \Lambda)\delta^*_{g_{b_0}} \omega = 0$$

for any one-form ω, due to g_{b_0} solving Einstein's equation and the diffeomorphism invariance of Ric, the tensor

$$r = \tilde{r} - \sum_j \delta^*_{g_{b_0}}(\chi\omega_j)$$

satisfies

$$(D_{g_{b_0}} \mathrm{Ric} + \Lambda)r - \widetilde{\delta}^*(D_{g_{b_0}}\Upsilon)r = \sum_j \widetilde{\delta}^*(D_{g_{b_0}}\Upsilon)\delta^*_{g_{b_0}}(\chi\omega_j),$$

which is exactly of the form given above in (8)!

Analytically, the point is that the operator

$$L_{b_0} = (D_{g_{b_0}} \mathrm{Ric} + \Lambda) - \widetilde{\delta}^*(D_{g_{b_0}}\Upsilon)$$

is not surjective between appropriate decaying function spaces, though the range is closed with a finite dimensional complement. So we need to add a finite dimensional complementary subspace W so that

$$L_{b_0} r = f$$

for given f is replaced by

$$L_{b_0} r = f + h,$$

$h \in W$ undetermined, for this equation to become solvable in these function spaces.

For us, $W = \widetilde{\delta}^*\Theta$, and it is important that this lies in the range of $\widetilde{\delta}^*$ because this assures (much like without the θ term) that the solution of the gauged Einstein equation actually gives a solution of the ungauged one!

An important point is that the analytic framework is stable under perturbations since complementary spaces are such, so if one has a metric g which is close to g_{b_0} in the appropriate sense then for the gauged Einstein's equation, linearized at g,

$$L_g = (D_g \mathrm{Ric} + \Lambda) - \widetilde{\delta}^*(D_g\Upsilon),$$

$L_g r = f + h$ is also solvable with h in the same space W. In particular, this holds for Kerr-de Sitter metrics with small a (and their perturbations!).

This assures that the non-linear equation is also solvable for perturbations of the initial data, near g_{b_0}, in the same decaying function spaces, which then gives (under UEMS*) the non-linear stability result.

We interpret as saying that solving the equation finds the gauge, $\Upsilon(g) = \theta$, in which the solution is stable as well as the actual solution of Einstein's equation.

Now, UEMS* does not hold for the Kerr-de Sitter family (exactly because it is a family, so the existence of infinitesimal deformations assures a 0-resonance) but it does hold for de Sitter space, giving a new proof of its stability. Note that this differs from the 'de Sitter space as a symmetric space stability' established by Friedrich [**43**], since analogously to the Kerr-de Sitter case, in terms of the conformal compactification of de Sitter space, it is based on a slight enlargement of the backward light cone from a point on the boundary, cf. Figure 2. In some ways this is thus a weaker result, since it is not the 'de Sitter space as a symmetric space stability', i.e. it is not global in the sense of the symmetric space sense, in other ways it is stronger as the hypotheses are also localized to such an enlarged backward lightcone (so are in a small region), which is very natural by causality considerations.

However, it is not hard to actually deal with the full Kerr-de Sitter family by modifying our equation by adding another term to it which corresponds to the family and somewhat enlarging the space Θ. The result is that for an appropriate finite dimensional space $\overline{\Theta}$ the nonlinear equation

$$(\mathrm{Ric}(g) + \Lambda g) - \widetilde{\delta}^*(\Upsilon(g) - \Upsilon(g_{b_0,b}) - \theta) = 0$$

with prescribed initial condition is solvable for g, θ, b with $\theta \in \overline{\Theta}$, b near b_0, and $g - g_b$ exponentially decaying; here $g_{b_0,b} = (1-\chi)g_{b_0} + \chi g_b$ is the asymptotic Kerr-de Sitter metric with parameter b. Thus, *both b and θ are found along with g in the nonlinear iteration!* This proves the nonlinear stability of the KdS family with small a. We refer to [**55**] for details.

7. Analytic aspects

7.1. Overview. Recall that the analytic framework we use:

- non-elliptic linear global analysis with coefficients of finite Sobolev regularity,
- with a simple Nash-Moser iteration to deal with the loss of derivatives corresponding to both non-ellipticity and trapping,

gives global solvability for quasilinear wave equations like the gauged Einstein's equation provided

- certain dynamical assumptions are satisfied (only trapping is normally hyperbolic trapping, with an appropriate subprincipal symbol condition) and
- there are no exponentially growing modes (with a precise condition on non-decaying ones), i.e. non-trivial solutions of the linearized equation at g_{b_0} of the form $e^{-i\sigma t_*}$ times a function of the spatial variables r, ω only, with $\mathrm{Im}\,\sigma > 0$.

This analytic framework encompasses a much broader class of linear and non-linear problems from general relativity, quantum field theory (QFT), dynamical systems and inverse problems, such as boundary rigidity. The non-linear aspects of the analysis in these problems (when it arises) can be reduced to a precise understanding of underlying linear problems, via linearization and an iteration such as Picard, Newton or Nash-Moser, or 'pseudolinearization'. (This latter is useful for uniqueness and stability estimates, as in the boundary rigidity problem, see [**86, 87**].)

In all of these problems one solves the linear (and non-linear) problems *globally* on a certain underlying 'physical space'; the expression 'underlying' refers to the operators actually acting on a function space on this 'physical space'. Here 'globally' still leaves us freedom in deciding what region of perhaps an even bigger physical space we care about, but *once we decide this*, we need to work globally in the region.

7.2. Non-linear analysis. The non-linear aspects usually simply mean that the linear analysis needs to be 'done right', so we discuss these very briefly only. Three simpler examples, relative to Einstein's equation, in the geometric context of de Sitter and Kerr-de Sitter spaces are semilinear wave equations, where Picard iteration is possible [**53**], quasilinear wave equations on de Sitter space (no trapping) where a Newton iteration is possible [**51**] and quasilinear wave equations on Kerr-de Sitter space using a Nash-Moser iteration to deal with the loss of the derivatives at the trapped set [**54**].

In order to illustrate the differences, consider a general nonlinear PDE $P(u) = f$. Using the linearization $(DP)|_{u_0}$ around some u_0, we can rewrite this by adding and subtracting appropriate terms:

$$(DP)|_{u_0}(u - u_0) = -\big(P(u) - P(u_0) - (DP)|_{u_0}(u - u_0)\big) - (P(u_0) - f).$$

Assuming $(DP)|_{u_0}$ is invertible, we obtain

$$\begin{aligned} u - u_0 = &- (DP)|_{u_0}^{-1}\big(P(u) - P(u_0) - (DP)|_{u_0}(u - u_0)\big) \\ &- (DP)|_{u_0}^{-1}\big(P(u_0) - f\big). \end{aligned}$$

Since the first term on the right hand side is quadratic in $u - u_0$, it might be expected to be negligible for sufficiently small $u - u_0$, as measured in a norm. Thus, one might expect to be able to run a Picard iteration, i.e. a fixed point argument, at least if $P(u_0) - f$ is also suitably small. This would take the form of starting with some u_1, and then defining u_{k+1} by

$$\begin{aligned} u_{k+1} - u_0 = &- (DP)|_{u_0}^{-1}(P(u_k) - P(u_0) - (DP)|_{u_0}(u_k - u_0)) \\ &- (DP)|_{u_0}^{-1}(P(u_0) - f), \end{aligned}$$

showing that the map

$$v \mapsto u_0 - (DP)|_{u_0}^{-1}(P(v) - P(u_0) - (DP)|_{u_0}(v - u_0)) - (DP)|_{u_0}^{-1}(P(u_0) - f) \tag{9}$$

is a contraction mapping, at least for v close to u_0, so that the sequence $\{u_k\}_{k=1}^{\infty}$ converges. Now, when P is an mth order operator, typically the error of the linearization at u_0,

$$v \mapsto P(v) - P(u_0) - (DP)|_{u_0}(v - u_0)$$

is also an mth order nonlinear operator, and so for instance (pretending we are on some compact manifold without boundary so the only issue is regularity) it is bounded between the Sobolev spaces H^s and H^{s-m}. Thus, (9) is not going to be bounded, let alone continuous, unless $(DP)|_{u_0}^{-1}$ gains back these derivatives, i.e. unless $(DP)|_{u_0}$ satisfies elliptic estimates. On the other hand, for elliptic equations the method indeed works for perturbations of a nonlinear solutions, for instance by working in sufficiently high order Sobolev spaces to ensure that they form an algebra, and indeed the multiplicative properties are preserved for the relevant first m derivatives. As emphasized already, we actually work globally, so if the manifold is non-compact, one typically needs the appropriate weighted Sobolev spaces; below we discuss the latter in the context of linear estimates.

If P is not elliptic, the Picard method may still work, but P needs to be semilinear to the appropriate extent, namely the error of the linearization should be an order $m-k$ nonlinear differential operator if $(DP)|_{u_0}^{-1}$ satisfies estimates with a loss of k derivatives relative to elliptic estimates, i.e. is continuous from H^{s-m+k} to H^s (again, we are pretending that we are working on a compact manifold without boundary). A typical example is then a semilinear wave equation, so $P(u) = Lu + Q(u)$, where L is a linear wave operator, and Q is a first order differential operator. Then the error of the linearization is given by the error of the linearization of Q (often one linearizes around 0 in this case), and is thus a first order nonlinear differential operator. On the other hand, from the local theory perspective (i.e. pretending that one is working on a compact manifold without boundary), discussed later in this section, L^{-1} maps H^{s-1} to H^s. This exactly compensates the one derivative loss of Q (assuming the Sobolev order is sufficiently high for algebra considerations), and indeed one again obtains a contraction mapping.

An improved version of the Picard iteration is Newton's iteration in which one linearizes at the current point in the iteration. Thus, rather than linearizing at some fixed u_0, if one is at u_k in step k, one uses the linearization at u_k in order to obtain a better approximation u_{k+1}. Since typically for an mth order P the error of this linearization is still an order m nonlinear operator, this does not help with non-elliptic problems directly, though might give faster convergence rates. However, if one considers quasilinear wave equations, there is a crucial advantage as we discuss next.

These quasilinear wave equations take the form $\Box_{g(u)}u = q(u, du) + f$, where we explicitly note the dependence on the derivatives of u. This is thus linear in the highest (second) order derivatives of u. Now, if one linearizes $\Box_{g(u)}u$ in u, the error of the linearization is still a second order differential operator. But one can simply take the Newton iteration given by

$$u_{k+1} = \Box_{g(u_k)}^{-1}(q(u_k, du_k) + f).$$

Notice that though $\Box_{g(u_k)}^{-1}$ loses one derivative relative to elliptic estimates, it is only applied to first order derivatives of u_k, hence the number of derivatives is preserved under the iteration, and indeed as shown by Hintz [**51**], under appropriate hypotheses this provides a convergent scheme. A crucial feature of the Newton iteration is that it relies on $\Box_{g(u)}^{-1}$ where u is not simply some background function rather the current stage of the iteration, hence has limited regularity of the coefficients. At the level of a multiplication operator (multiplication by u here), this corresponds to an estimate

$$\|uv\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{H^s},$$

which is valid for $s > n/2$, and indeed represents the (topological) algebra property of Sobolev spaces.

For more general quasilinear wave equations, including Einstein's equation on Kerr-de Sitter space, there are additional issues, such as the solution operator losing additional derivatives relative to elliptic estimates, due to the trapped set of the geodesic flow. In order to deal with this, we use a Nash-Moser iterative scheme, which is a combination of a Newton iteration with smoothing. Here for simplicity we use X. Saint Raymond's version [**84**]: one solves

$$\phi(u; f) = 0,\ \phi(u; f) = \Box_{g(u,du)}u - q(u, du) - f,$$

by using the solution operator $\psi(u; f)$ for the linearization $\phi'(u; f)$ of ϕ in u:

$$\psi(u; f)\phi'(u; f)w = w,$$

and letting $u_0 = 0$,

$$u_{k+1} = u_k - S_{\theta_k}\psi(u_k; f)\phi(u_k; f),$$

where $\theta_k \to \infty$, S_{θ_k} is a smoothing operator that smooths less and less as $k \to \infty$. Again, as the Newton iteration, this needs the linear theory for H^s-coefficients. Further, one needs *tame* estimates. At the level of a multiplication operator, this corresponds to

$$\|uv\|_{H^s} \leq C(\|u\|_{H^{s_0}}\|v\|_{H^s} + \|u\|_{H^s}\|v\|_{H^{s_0}}),$$

which is valid for $s \geq s_0 > n/2$. Here s is a 'high' (regularity), s_0 a 'low' norm; what one does not want is the product of high norms, i.e. one wants an estimate with a linear bound in high norms. Again, we actually work globally on non-compact spaces, thus in fact the tame estimates are on weighted Sobolev spaces. This approach was introduced in [**54**], and then used in the proof of the stability of Kerr-de Sitter space [**55**].

7.3. Fredholm theory. To reiterate, the linear problems are solved globally and this is used to solve the non-linear problems so as well, rather than using the finite time non-linear solvability and attempting to control it uniformly as time goes to infinity.

Thus, one decides on an underlying 'physical space' (often a complete manifold for quantum field theory, possibly a region bounded by horizons, or an enlargement of this, in space-time in general relativity, a domain for the inverse problems) M, and one considers an operator P on M.

More precisely, one needs to specify some function spaces (usually with considerable freedom) X, Y, and consider the continuous map

$$P : X \to Y.$$

In spite of the considerable freedom, it is *crucial* to be able to fix these spaces. Note also that while many choices may be equivalent, other choices may result in very different operators (cf. boundary conditions)!

- Solving equations amounts to a surjectivity statement for P
- Inverse problems/rigidity amount to an injectivity statement for P.

Since function spaces are infinite dimensional, we also need estimates: these are (semi-)Fredholm estimates.

The almost-injectivity (in that the nullspace is finite dimensional) estimate is

$$\|u\|_X \leq C(\|Pu\|_Y + \|u\|_{Z_1})$$

and the almost surjectity (in that the range has finite codimension) estimate is

$$\|v\|_{Y^*} \leq C(\|P^* v\|_{X^*} + \|u\|_{Z_2}),$$

where the inclusion maps $X \to Z_1$ and $Y^* \to Z_2$ are compact. Compactness of these maps typically comes from the Z_j being weaker in the sense of *derivatives*, and if M is non-compact, or has a degenerate structure, then from in addition the Z_j being weaker in the sense of *decay*. (Actual invertibility, as opposed to almost invertibility, amounts to being able to drop the relatively compact terms.)

While far from widespread for evolution equations, global analysis is standard for elliptic PDE, like Laplace's equation: after all, one cannot solve an elliptic PDE by solving it locally, namely the restriction of the actual global solution restricted to a region is not given by solving the equation in that region by imposing some arbitrary boundary condition! In this sense the behavior of evolution equations is quite unusual: if one imposes initial conditions at 'time' $t = t_0$, one can solve them for $t \in [t_0, t_1]$ without solving them for t outside this interval. Wave equations are even more extreme as they also allow spatial localization due to the finite speed of propagation. However, while these localizations allow results like the global hyperbolicity statements, which give up any kind of uniform control outside compact sets, they are much less suited for stability considerations which are by necessity global.

The simplest example for Fredholm theory is elliptic operators P on compact manifolds without boundary M; there is a similar theory for M with smooth boundary with boundary conditions that we do not discuss here.

Recall that for $P \in \mathrm{Diff}^m(M)$, $m \in \mathbb{N}$, the *principal symbol* $\sigma_m(P)$ captures the leading terms. In local coordinates, if

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha,$$

then

$$p(x,\xi) = \sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

Ellipticity is the statement that p does not vanish (or more generally, in the non-scalar setting, is invertible) if $\xi \neq 0$. Then

- $X = H^s = H^s(M)$, $Y = H^{s-m}(M)$, $s \in \mathbb{R}$,
- so $X^* = H^{-s}(M)$, $Y^* = H^{-s+m}(M)$,
- $Z_1 = H^{-N}(M)$, $Z_2 = H^{-N}(M)$, N large.

The Fredholm property follows from the elliptic estimate

$$\|\phi\|_{H^r} \leq C(\|L\phi\|_{H^{r-m}} + \|\phi\|_{H^{-N}}),$$

with $L = P$, $r = s$, resp. $L = P^*$, $r = -s + m$. Note that the choice of s is irrelevant here (this is a result of elliptic regularity).

The non-elliptic problems we consider are problems in which the elliptic estimate is replaced by estimates of the form

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}), \tag{10}$$

i.e. with a loss of one derivative relative to the elliptic setting, and

$$\|v\|_{H^{s'}} \leq C(\|P^*v\|_{H^{s'-m+1}} + \|v\|_{H^{-N'}}), \tag{11}$$

with $s' = -s + m - 1$ being the case of interest. These are often proved by *propagation* estimates using microlocal analysis.

Such estimates imply that $P : X \to Y$ is Fredholm if

$$X = \{u \in H^s : \ Pu \in H^{s-m+1}\}, \ Y = H^{s-m+1}.$$

Such a space X is automatically a Hilbert space. More interestingly, it is easy to see that C^∞ is still dense in X, so X is a reasonable space. Ultimately, the density comes down to commuting a regularizer with P with no more than 1 differential order loss, which in turn corresponds to commutators of operators being one order lower than the product (just like the commutator of vector fields is a vector field and not just a second order differential operator; see also Section 8), so in general this space X is on the borderline of being well-behaved: if we replaced $Pu \in H^{s-m+1}$ by $Pu \in H^{s-m+1+\delta}$, $\delta > 0$, it would typically not be such.

7.4. Non-elliptic problems on non-compact or degenerate underlying spaces. Our non-elliptic problems are usually more complicated, except in the case of dymanical systems, since there *is* an infinity in the underlying manifold (i.e. in the way it is usually considered it is not compact, which is not to say that we cannot compactify it, and indeed we will), which means that the Sobolev spaces will only have a compact inclusion if the error term is in a weaker *weighted* space; this is how resonances enter. In (the simplest) dynamical systems, such as Anosov flows on compact manifolds M, e.g. the unit cotangent bundle of an underlying manifold of negative curvature, however, this setup is already satisfactory, and the work of Faure and Sjöstrand [38], Dyatlov and Zworski [37] introduced a new approach and insights by these considerations.

In these cases with an infinity for the underlying manifold M (i.e. M is not compact) there is typically a 2-step process of obtaining estimates on weighted Sobolev spaces. This is already present in the elliptic setting where it has an extensive literature using various approaches, see e.g. [76, 71]. For the sake of definiteness, we consider settings with decay measured relative to e^{-t_*}, such as de Sitter and Kerr-de Sitter spaces; thus from the compactification perspective $\tau = e^{-t_*}$ is our boundary defining function. Thus, we work with spaces $H^{s,\ell} = e^{-\ell t_*} H^s$, and the estimates to prove are

$$\|u\|_{H^{s,\ell}} \leq C(\|Pu\|_{H^{s-m+1,\ell}} + \|u\|_{H^{-N,-N}}). \tag{12}$$

In Step 1 one proves an estimate

$$\|u\|_{H^{s,\ell}} \leq C(\|Pu\|_{H^{s-m+1,\ell}} + \|u\|_{H^{-N,\ell}}). \tag{13}$$

Thus, the error term is lower order in the differential sense but not in the decay sense, and hence the inclusion from $H^{s,\ell}$ into $H^{-N,\ell}$ is *not* compact. Again, this step is often proved using microlocal analysis. Then, *in the simplest settings* in Step 2 one proves an estimate for a model operator at infinity

$$\|u\|_{H^{s',\ell}} \leq C\|P_0 u\|_{H^{s'-m+1,\ell}}. \tag{14}$$

Taking $s' \geq -N$, one applies this to u cut off to large t_*, i.e. to χu, where $\chi \equiv 1$ for t_* large and vanishes outside a collar neigborhood of infinity (so e.g. vanishes at the initial Cauchy hypersurface). Combining (13) and (14) (with the latter applied to χu), and writing $u = \chi u + (1-\chi)u$ in the last term on the right hand side of (13), we get

$$\|u\|_{H^{s,\ell}} \leq C(\|Pu\|_{H^{s-m+1,\ell}} + \|P_0(\chi u)\|_{s'-m+1,\ell} + \|(1-\chi)u\|_{H^{-N,\ell}}).$$

Commuting χ through P_0 gives another term that is supported in bounded t_* (with $m-1$ extra derivatives falling on u), like $(1-\chi)u$, and thus both of these can be estimated in a weighted space with *any* weight, so with ℓ' arbitrary,

$$\|u\|_{H^{s,\ell}} \leq C(\|Pu\|_{H^{s-m+1,\ell}} + \|\chi P_0 u\|_{s'-m+1,\ell} + \|u\|_{H^{s',\ell'}}).$$

Finally writing $\chi P_0 = \chi P + \chi(P - P_0)$, the first term can be absorbed into Pu, while $P - P_0$ has decaying coefficients, thus maps into a more decaying space, one gets (12).

For nonlinear stability problems, P is the linearization of the nonlinear problem at a current step in the iteration, and P_0 is the linearization at a metric *to which* the current iterate is asymptotic. Thus, for KdS stability, this would be a KdS space.

In order to have the P_0 estimate (14), one conjugates it by the Fourier transform to obtain a family $\hat{P}_0(\sigma)$ where σ is the (complex!) Fourier dual of $-t_*$; this is where the stationarity is used. That is, one takes

$$(\mathcal{F}u)(\sigma,.) = \int e^{i\sigma t_*} u(t_*,.)\, dt_*,$$

where the . refers to the remaining 'spatial' variables, r, ω, so

$$\hat{P}(\sigma)(\mathcal{F}u)(\sigma,.) = (\mathcal{F}Pu)(\sigma,.).$$

Now, the Fourier transform is an isomorphism between a space like $e^{-\ell t_*}H^s$ and a family of spaces, defined on the line $\operatorname{Im}\sigma = -\ell$, with a norm that depends on σ. More precisely, the norms for various σ are all equivalent, but this equivalence is not uniform as $|\sigma| \to \infty$ (with fixed imaginary part). As an indication of the form this takes, for s a positive integer, the square of the H^s norm of u is a sum of the squared L^2 norms of up to s derivatives of u with respect to t_* and the 'spatial variables' r, ω; the norm on the Fourier transformed family is the analogous expression with u replaced by $\mathcal{F}u(\sigma,.)$, and each occurrence of ∂_{t_*} is replaced by σ. This is a 'large parameter' or, after rescaling, a 'semiclassical' Sobolev space. One then automatically has a(n analytic) Fredholm theory for $\hat{P}_0(\sigma)$, corresponding to the Step 1 estimate. Thus, the question is invertibility, i.e. whether $\hat{P}_0(\sigma)$ has a non-trivial nullspace (index 0 follows from large σ considerations); this is *how* the resonances (points of non-invertibility of $\hat{P}_0(\sigma)$) play a role. The net result is that as long as $-\ell$ is not the imaginary part of a resonance σ, one has the desired estimate (14), and so the Fredholm theory for P for all but a discrete set of weights (at least as long as the Step 1 theory allows this: there can be issues with trapping!).

For the stability problems, we want *forward* solutions, which is to say we want to work on distributions supported in the future of our Cauchy hypersurface.

On these spaces we get invertibility for sufficiently negative ℓ (which corresponds to a growing space!), i.e. $\operatorname{Im}\sigma$ sufficiently positive, due to the absence of resonances in an upper half space. A contour deformation integral for the inverse Fourier transform from the line $\operatorname{Im}\sigma = -\ell$ to a line $\operatorname{Im}\sigma = -\ell'$ allows one to conclude invertibility in more decaying spaces *up to having to allow finitely many terms corresponding to the more growing resonances* in the solution.

7.5. A bit of scattering theory. For Kerr spacetime (i.e. if $\Lambda = 0$), the fixed σ theory already has a similar aspect in 'Step 2', i.e. it already is on a non-compact/degenerate (if compactified) manifold, due to the Minkowski end. It turns out that an analogous, but more subtle, theory also works in this case. This has been analyzed in the papers [**100, 102**]. Here we very briefly comment on the general structure of the non-zero σ behavior; $\sigma = 0$ is an additional degeneracy, studied in [**102**]. Thus, the rest of this secion is rather terse, and the reader should feel welcome to skip it.

In the Kerr setting ($\Lambda = 0$) for the Fourier transformed model operator, $\hat{P}_0(\sigma)$, $\sigma \neq 0$, there are various possibilites for setting up a Fredholm theory.

Since the interesting aspect is at Minkowski infinity, which after the Fourier transform acts as infinity in Euclidean space, this amounts to setting up Fredholm theory for essentially a perturbation of the spectral family of the Euclidean Laplacian, $\Delta_{\mathbb{R}^n} - \sigma^2$, with $n = 3$. More precisely, if one used the standard time function t for the Fourier transform, one would end up with the aforementioned perturbation of the spectral family of the Euclidean Laplacian. Here we use $t_* = t-(r+2m\log(r-2m))$, which in the Minkowski case is $t_* = t - r$, to perform the Fourier transform; as is clear from the Fourier transform formula, this amounts to a conjugation of the spectral family relative to the t Fourier transform. An advantage of the t_*-Fourier transform, or equivalently this conjugation, is that it directly gives rise to function spaces which encode the precise asymptotics at future null infinity $\mathscr{I}^+$; notice that t_* is finite in the interior of $\mathscr{I}^+$, cf. Figure 4.

The analysis of the spectral family of asymptotically Euclidean Laplacians at the spectrum is typically called *scattering theory*. The presence of the spectrum directly indicates the lack of invertibility of members of the family with target space L^2, but of course this does not mean that invertibility between other spaces is impossible, just that more sophisticated function spaces must be used. In the mathematics literature typically the approach is more round-about: one inverts the spectral family off spectrum (to get the resolvent family), and then studies function spaces in which this inverse has limits in a suitable sense at the spectrum. This is called the *limiting absorption principle*, which has a long history; microlocal tools to its description (though not quite a Fredholm framework) were introduced by Melrose [**77**], though partial microlocalization was present in many earlier works, especially in the quantum N-body scattering, see [**28**], and references therein. A key point is that though one is working on weighted spaces, i.e. in terms of our general description above it would seem that one needs a 2-step approach, *in fact microlocal tools are also available to analyze the decay behavior.* A direct Fredholm approach was introduced in [**98**, Section 5.4.8]; [**100**] introduced an alternative, relying on different function spaces, discussed below.

Now, both the unconjugated and the conjugated spectral family can be placed in a Fredholm framework in different ways; one way is using variable order Sobolev spaces, which are discussed in the next section, *except that here it is the decay order that is variable.* For the unconjugated spectral family when $\operatorname{Im}\sigma > 0$ one has a much simpler constant order theory: just as $\Delta_{\mathbb{R}^n} - \sigma^2$ is, the actual operator is also fully elliptic. (These spaces are then *standard* weighted Sobolev spaces on $\mathbb{R}^n_z$, like $(1+|z|^2)^{-\ell/2}H^s(\mathbb{R}^n)$.) But for the conjugated operator even the case $\operatorname{Im}\sigma > 0$ is non-trivial, essentially because one is capturing the precise exponential decay of the solutions: the conjugating factor exactly encodes this! On the other hand, for real, non-zero σ, the analysis of the unconjugated and conjugated spectral families is very similar, as the conjugation is by a purely oscillatory factor then (so the two are equivalent in a much stronger sense than if $\operatorname{Im}\sigma > 0$).

Both can be handled by variable decay order Sobolev spaces, see e.g. [**98**, Section 5.4.8] for a treatment of the spectral family, and [**101**] for its $\sigma = 0$ limit, which requires an additional step. The advantage of this approach is that microlocal analysis, discussed in Section 8, is applicable even to the handling of the decay aspects, i.e. there is no 'Step 2' for the operator $\hat{P}_0(\sigma)$ itself: even though the manifold is non-compact, Step 1 can take care of decay as well.

An alternative is to work in stronger Sobolev spaces [**100, 102**] in which the derivatives are not just with respect to the translation structure of $\mathbb{R}^n_z$ (i.e. ∂_{z_j}), as in the standard Sobolev spaces, but also the dilation structure (i.e. $r\partial_r$ and angular derivatives ∂_ω; note that these are r times stronger than the translation based derivatives, which are roughly like ∂_r and $\frac{1}{r}\partial_\omega$). These are then the b-Sobolev spaces which are discussed in the Kerr-de Sitter context in Section 8, and in the Euclidean/Minkowski context in Section 9. The advantage is that, being stronger spaces, these give a more precise description of the asymptotics of both the inverse of the spectral family applied to 'nice' inputs, such a Schwartz functions, and of solutions of the wave equation; the disadvantage is that they still need a '2-step' Fredholm theory. Here we simply refer to [**100, 102**] for a thorough introduction in this context. However, we do mention that in fact these two kinds of Sobolev spaces, namely the standard one corresponding to the translation structure, and the b-Sobolev spaces reflecting the dilation structure, can be combined into what are called 'scattering-b' Sobolev spaces in [**101, 100**]. These in fact are an analogue of second microlocalization at a Lagrangian manifold in standard microlocal analysis, introduced by Bony [**12**], see also [**85**] and [**93**] in a somewhat simpler semiclassical setting. This combination barely adds overhead relative to the basic b-Sobolev space Fredholm treatment, and indeed is more transparent in certain ways.

8. Microlocal analysis

8.1. Overview. We still need to prove the estimates with a gain in differential order (13). The tool we use is *microlocal analysis*, originating from the work of Lax [**67**], Kohn and Nirenberg [**66**] and Hörmander [**58**] among others. Microlocal analysis is local (in a sense to be made precise momentarily) in phase space, T^*M, which is locally $\mathbb{R}^n_z \times \mathbb{R}^n_\zeta$, with ζ the momentum variables. A key point is that this is both *perturbation stable*, and (for nonlinear non-elliptitc problems, such as general relativity) *works in a limited regularity setting*, with work on this going back to Beals and Reed [**8**] in the 1980s, extended to the setting needed for general relativity by Hintz [**51**]. A basic general reference in the spirit of the forthcoming discussion is [**72**]; a more targeted reference is the author's lecture notes from a summer school in Grenoble [**98**]. A more concise version than these thorough references, with a different presentation than below, is given in the author's ICM lecture notes, [**96**]. We also refer to [**57, 90**] for different treatments of some of the material.

Let us discuss the 'location' of microlocalization more precisely. Recall that the Fourier transform (or Fourier series) turns differentiability into decay. Microlocal analysis keeps track of both the original position, and its Fourier dual (momentum), but this is still an accurate correspondence. Thus, in view of the differential order gain we are after, we are interested in what happens as $|\zeta| \to \infty$, referred to as 'fiber infinity' (infinity in the fibers of the cotangent bundle). This is often encoded by using dilations in the fibers (i.e. in ζ) (or a compactification), so the phase space can be considered as $T^*M \setminus o$ modulo dilations in the fibers $(z, \zeta) \mapsto (z, \lambda\zeta)$, $\lambda > 0$, i.e. $S^*M = (T^*M \setminus o)/\mathbb{R}^+$. Thus, one thinks of a dilation orbit as really representing an ideal point at infinity in the fibers of $T^*M \setminus o$; this can be made concrete by actually compactifying the fibers of the cotangent bundle much like we compactify Minkowski space in Section 9, and a bit like how we already compactified de Sitter space, but we mostly refrain from this for the sake of a more transparent presentation.

Before turning to the actual estimates, let us discuss a perhaps more hands on use of microlocalization. Namely, using microlocal analysis one can say where in *phase space*, i.e. at which point z and which codirection ζ, is a distribution u in a Sobolev space H^s or is C^∞; the complement of the latter (i.e. points near which the distribution is not C^∞) is the wave front set $\mathrm{WF}(u)$. For instance, for the distribution

$$(z_1 + i0)^{-1} = \lim_{\epsilon \to 0+} (z_1 + i\epsilon)^{-1}$$

(distributional limit),

$$\mathrm{WF}((z_1 + i0)^{-1}) = \{(0, z', \zeta_1, 0) : \zeta_1 > 0\},$$

which says that the distribution is only singular at $z_1 = 0$, and there only in the conormal direction (as $\zeta' = 0$), and moreover only at one half of the conormal bundle ($\zeta_1 > 0$, with $\zeta_1 < 0$ excluded). Notice that in the covectors ζ we are considering directions corresponding to the quotient $S^*\mathbb{R}^n = (T^*\mathbb{R}^n \setminus o)/\mathbb{R}^+$, so in fact for a fixed z', the ray (half-line) $\{\zeta_1, 0) : \zeta_1 > 0\}$ is a single point in the quotient space.

8.2. Quantization, pseudodifferential operators and ellipticity. The power of microlocal analysis for our purposes derives from the ability to prove estimates *locally in* S^*M, i.e. *microlocally*. This microlocalization is carried out by *pseudodifferential operators*, which we will discuss more explicitly after (19), but for now all that matters is that they can in particular be associated to functions b on S^*M so that the associated $B = \mathrm{Op}(b) \in \Psi^0(M)$ (called a *quantization* of b) is localizing to $\operatorname{supp} b$, and is non-degenerate, namely *elliptic* on $\{b > 0\}$; b is the *principal symbol* of B, extending the notion for differential operators. Here 'localization to $\operatorname{supp} b$' means that if there is another function a on S^*M and $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$, then, with $A = \mathrm{Op}(a)$, $A(Bu)$ is 'trivial', i.e. is in the Sobolev space H^s for every s (locally), so localization to $\operatorname{supp} b$ followed by localization to the disjoint set $\operatorname{supp} a$ gives a trivial result. So one can think of there being an object (which is simply a useful figment of the imagination for our purposes here) associated to u on S^*M, and $\mathrm{Op}(b)$ acting on u is like multiplying this object by b, then $\mathrm{Op}(a)$ acting on the result is like multiplying the resulting object by a, so the triviality of $A(Bu) = (AB)u$ arises from $ab = 0$ as AB multiplies the original object by ab.

We also need operators of order other than 0; for these the analogous impressionistic idea is to replace the function b on S^*M, which is thus equivalently a homogeneous degree 0 function on $T^*M \setminus o$ (homogeneous with respect to the dilation action), by a homogeneous degree m function on $T^*M \setminus o$. The corresponding quantization, $B = \mathrm{Op}(b) \in \Psi^m(M)$ is called a pseudodifferential operator of order m. If m is a non-negative integer, a particular example of such homogeneous functions is a homogeneous degree m polynomial in the fibers of T^*M, depending smoothly on the base variable: these are exactly the principal symbols of differential operators: $\sum_{|\alpha|=m} a_\alpha(z)\zeta^\alpha$ quantizes as $\sum_{|\alpha|=m} a_\alpha(z) D_z^\alpha$, plus lower order terms. As the 'plus lower order terms' indicates, for a full description we really should be considering more than just homogeneous degree m functions, and at the very least we should allow homogeneous degree m' terms added, for $m' < m$, to the homogeneous degree m ones as 'lower order' as $|\zeta| \to \infty$. We briefly comment on the actual framework after (19) below, but for our present purposes this simplified discussion is sufficient.

For a wave operator $P = \Box_g$, the principal symbol is given by the dual metric, i.e. the inverse $G = (g^{ij})$ of g, which we think of as an 'energy function' on phase space: $p(z,\zeta) = \sum g^{ij}(z)\zeta_i\zeta_j$.

One says that P is elliptic at $\alpha \in S^*M$ if the homogeneous function p is non-zero at α, i.e. in our relativistic setting at non-null covectors. Near such points we have elliptic estimates:

$$\|B_1 u\|_{H^s} \leq C(\|B_3 P u\|_{H^{s-m}} + \|u\|_{H^{-N}}),$$

$B_j = \mathrm{Op}(b_j) \in \Psi^0$, provided $b_3 \neq 0$ on $\operatorname{supp} b_1$ and $p \neq 0$ on $\operatorname{supp} b_1$. Here the idea is that $B_1 u$ is localized to $\operatorname{supp} b_1$, on which $b_3 p$ is non-zero, so one can write $b_1 = q b_3 p$ for an appropriate q (which has order $-m$ to cancel the order m of p), which in turn implies that $B_1 u = \mathrm{Op}(q) B_3 P u + R u$, where R is order -1. This directly gives a version of the elliptic estimate with the H^{-N} norm replaced by the H^{s-1} norm, and an iterated and improved version of this argument is the elliptic estimate as stated.

8.3. Propagation estimates. The basic non-elliptic estimates for P with *real principal symbol* relate to the Hamiltonian formulation of classical mechanics, which encodes geodesics as follows.

- The lifted geodesics are integral curves of the *Hamilton vector field* given by the symplectic structure on T^*M; locally

$$H_p = \sum_{j=1}^n \frac{\partial p}{\partial \zeta_j}\frac{\partial}{\partial z_j} - \frac{\partial p}{\partial z_j}\frac{\partial}{\partial \zeta_j},$$

i.e. $H_p = (\partial_\zeta p, -\partial_z p)$; these are also called *bicharacteristics*.
- Each point in T^*M gives rise to a unique lifted geodesic (the integral curve through the point).

The 'quantum' or analytical, version, due to Hörmander [**59**], then propagates H^s-estimates in the *characteristic set*

$$\mathrm{Char}(P) = \{p = 0\}$$

along bicharacteristics. Here in general (this is automatic for the lifted geodesics) one needs that H_p is non-radial, i.e. is not a multiple of the dilation vector field, $\sum \zeta_j \partial_{\zeta_j}$, in the region one is proving the estimates (so say on $\operatorname{supp} b_3$ in (20) below). The meaning of this condition becomes more transparent by considering the quotient space, S^*M. By possibly multiplying by a positive homogeneous factor, one may assume that H_p is homogeneous of degree 0 (which is automatic if p is homogeneous of degree 1). Thus, it induces a map on homogeneous degree zero functions. Considering a representative of S^*M given by a transversal S to the dilation action (say, the unit cosphere bundle with respect to a Riemannian metric), this amounts to considering the pushforward of H_p to this transversal via the projection map; this is actually a smooth vector field. Then the non-radial nature of H_p amounts to the statement that the induced vector field on the transversal S does not vanish in the relevant region.

Hörmander's propagation estimate is an estimate of the form

$$\|B_1 u\|_{H^s} \leq C(\|B_2 u\|_{H^s} + \|B_3 P u\|_{H^{s-m+1}} + \|u\|_{H^{-N}}), \tag{15}$$

$B_j \in \Psi^0$, provided $\operatorname{supp} b_1 \subset \{b_3 \neq 0\}$, and all bicharacteristics from points in $\operatorname{supp} b_1 \cap \mathrm{Char}(P)$ reach $\{b_2 \neq 0\}$ of while remaining in $\{b_3 \neq 0\}$. This is usually proved via a positive commutator estimate, which is a microlocal version of an energy estimate.

The basic idea of such an estimate is to take $A = A^* \in \Psi^{m'}(M)$, and compute

$$\langle iPu, Au\rangle - \langle iAu, Pu\rangle = \langle (i[A,P] + i(P-P^*)A)u, u\rangle, \tag{16}$$

It is important to keep in mind that while a product like AP is a pseudodifferential operator whose order is the sum of the two individual orders, $AP, PA \in \Psi^{m+m'}$, the commutator $[A,P] = AP - PA$ is actually 1 order lower, i.e. is in $\Psi^{m+m'-1}$. This is closely related to the analogous property of differential operators, and in particular to the statement that the commutator of two vector fields is a vector field, rather than a second order differential operator. In addition, as the principal symbol of P is real, $P - P^*$ is also one order lower than P or P^* respectively, so $P - P^* \in \Psi^{m-1}$.

Now, the principal symbol of $i[A,P] + i(P-P^*)A$ in $\Psi^{m+m'-1}(M)$ is $-H_p a - 2\tilde p a$ if $\tilde p$ is the principal symbol of $\frac{1}{2i}(P - P^*) \in \Psi^{m-1}(M)$. One arranges, in a manner that we comment on momentarily, that

$$-H_p a - 2\tilde{p} a = b_1^2 + e, \tag{17}$$

where e has support in the region where the a priori assumptions are imposed (such as $\operatorname{supp} b_2$ above). Taking B_1, E with principal symbols b_1, e, one has $i[A,P] + i(P - P^*)A = B_1^* B_1 + E + F$, with F lower order, so substituting into (16), one controls $B_1 u$ in H^s, $s = \frac{m+m'-1}{2}$ (for a particular desired s this is what dictates the choice of m'), in terms of Eu as well as A^*Pu measured in the appropriate Sobolev spaces. Since Eu is controlled by $B_2 u$ in view of support properties of e and b_2, this proves the theorem after a regularization argument, combined with a Cauchy-Schwartz inequality for the A^*Pu term. There are actually subtleties with these technical steps, but in these expository notes we simply refer to [**98**] for more details.

Now assuming that $\tilde{p} = 0$, an assumption which can be removed by making a slightly more careful choice below, (17) simply means that the function a is decreasing along the H_p-flow, at least modulo the error term e, which is supported in the a priori controlled region $\operatorname{supp} b_2$. Again, there are some technical complications, but for the sake of the following argument let us assume that $m = 1$, $m' = 0$; then H_p and a are homogeneous of degree 0, and by the non-radial nature of H_p, H_p does not vanish when considered as a vector field on the transversal S to the dilation orbits. Then such a function a can be constructed particularly simply by straightening out the H_p flow on S, so introducing coordinates $(q_1, \ldots, q_{2n-1}) = (q', q_{2n-1})$ so that that $H_p = \partial_{q_{2n-1}}$, and taking a of the form $\chi(q_{2n-1})\chi_2(q')$, χ is a bump function, decreasing outside the region $\operatorname{supp} b_2$. This function a can be regarded as a homogeneous degree zero function on $T^*M \setminus o$, satisfying (17). While we assumed $m' = 0$, this is simply remedied by including an additional positive, homogeneous degree m', factor in a. Its derivative causes an additional term in $H_p a$, but choosing χ so that χ' is greater, in absolute value, than a large constant times χ outside $\operatorname{supp} b_2$, one can absorb this into the χ' term. In fact, the way of dealing with a non-zero $\tilde{p}$ and the regularization is completely similar: a suitable choice of χ, with $|\chi'|$ large relative to χ, allows these to be absorbed into the χ' term. In practice one often uses a somewhat different construction, roughly corresponding to the geometry of Figure 6, to have a more stable estimate; these choices go back to the work of Melrose and Sjöstrand on the propagation of waves in domains with boundary [**73, 74**].

8.4. Making the initial step in a propagation problem. Having understood elliptic estimates, valid away from the characteristic set of P, as well as the propagation estimates in the characteristic set, in view of the goal (13) the key question is how one *starts* the propagation estimate, i.e. how one controls the $B_2 u$ term in (15).

For wave equations, one option is Cauchy hypersurfaces (in the base manifold M); this gives the usual finite time formulation of wave propagation. Namely, consider $\Box u = f$, where f is supported in the causal future of a

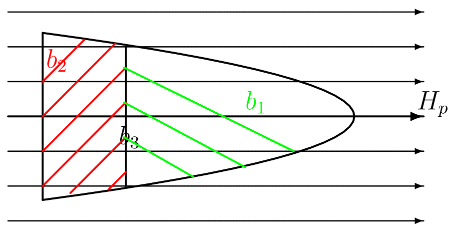

FIGURE 6. The stucture of propagation estimates. Control of u in the region $\operatorname{supp} b_2$ (upward stripes) as well as of Pu in the whole region $\operatorname{supp} b_3$ (including both the upward and the downward striped regions) gives control of u in the region $\operatorname{supp} b_1$ (downward stripes).

Cauchy hypersurface, and demand that u be also so supported. Then with b_2 localized to the past of the Cauchy hypersurface, B_2u is trivial (since $u = 0$ there), and thus can be dropped (absorbed into the error term $\|u\|_{H^{-N}}$).

Another possibility is to have a structured bicharacteristic flow: we need that there are submanifolds L of S^*M (and thus corresponding conic submanifolds Λ of $T^*M \setminus o$, with $L = \Lambda/\mathbb{R}^+$) which act as sources/sinks in the normal direction: it turns out that on *high regularity spaces* (which we specify below), one can get an estimate in which the B_2 term can be dropped. This plays a key role in scattering theory, where it was introduced by Melrose in the 1990s [**77**], though has a long history in a non-microlocal way, and around 2009 Faure and Sjöstrand also introduced this to dynamical systems [**38**].

Having such an initial estimate near a (normal) source or a sink L_1, one can propagate them along the flow, i.e. along bicharacteristics that tend to L_1 in the relevant direction (so e.g. tend to L_1 backward for a source, then one propagates the estimates forward along the bicharacteristics, away from L_1).

One would like to then propagate the estimates into a source/sink manifold L_2 at the other limit of the bicharacteristics, and fortunately one can do so, using *low regularity spaces*, specified below. In this case the B_2 term is not dropped, but b_2 is supported in the region where we already have control on u thanks to the propagation from L_1!

Thus, if we *assume* that the bicharacteristic flow in the characteristic set is globally well behaved, namely there are source/sink manifolds L_1 and L_2, and all bicharacteristics outside these (in the characteristic set) tend to L_1, resp. L_2 in the appropriate directions (so if L_1 is a source, L_2 a sink, then in the forward direction for L_2, backward for L_1), then one obtains complete control on the characteristic set; this is global assumption is called a *non-trapping* assumption. Due to the high, resp. low regularity requirements at the two ends of the flow, often one needs *variable order* (s), or *anisotropic*, spaces, which goes back to Unterberger [**92**] and Duistermaat [**33**], see also

[**6**, Appendix A] where they are used in our current context. Crucially, these also give estimates for the adjoint on dual spaces.

The high, resp. low, regularity mentioned above are measured relative to a threshold, s_Λ, which depends on the order m of P, the principal symbol of the skew-adjoint part $\frac{P-P^*}{2i}$ of P (which, we recall, is order $m-1$ if the principal symbol of P is real), at Λ, as well as the Hamilton vector field H_p acting on 'weight functions', i.e. homogeneous functions of order 1 (or -1). If say ρ_0 is *positive homogeneous of degree* -1 near Λ, then $H_p\rho_0$ is homogeneous of degree $m-2$ (as H_p is homogeneous of degree $m-1$), so $\rho_0^{m-2}H_p\rho_0$ is a homogeneous degree 0 function. Now, what is actually required in the source/sink argument is not just that L is a source or a sink in the normal direction in S^*M, but that Λ 'at fiber infinity' is such, i.e. if L is a source, then ρ_0 be increasing along the flow (corresponding to the bicharacteristics coming from fiber infinity, thus increasing the homogeneous degree -1 function ρ_0), while if L is a sink, then ρ_0 be decreasing along the flow (corresponding to the bicharacteristics tending towards fiber infinity). Thus $\beta_0 = \mp\rho_0^{m-2}H_p\rho_0 > 0$, where the top sign corresponds to the normal sink and the bottom to the normal source. The renormalized principal symbol of $\frac{P-P^*}{2i}$ (renormalized relative to ρ_0 and H_p) is the homogeneous degree 0 function $\pm\beta_0^{-1}\rho_0^{m-1}$ times the actual principal symbol of $\frac{P-P^*}{2i}$. In many applications this is actually constant (otherwise one needs to take appropriate sup's and inf's), and then the threshold is

$$s_\Lambda = \frac{m-1}{2} \mp \beta_0^{-1}\rho_0^{m-1}\sigma_{m-1}\Big(\frac{P-P^*}{2i}\Big). \tag{18}$$

The reason for this threshold is transparent from the positive commutator argument, and concretely from the necessity of assuring (17). Due to the normal source/sink structure of L, at L the positivity of the b_1^2 term is arranged by the positive homogeneous degree m' weight function multiplying the function one constructs on S^*M, like $\chi(q_{2n-1})\chi_2(q')$ in the discussion following (17). Hence, if $\tilde{p}$ vanishes, one needs to have m' to have an appropriate sign (which then in turn makes the threshold into $\frac{m-1}{2}$ since $s = \frac{m+m'-1}{2}$), matching the nature of the dynamics. For general $\tilde{p}$, there is simply a shift corresponding to (17), giving rise to s_Λ as in (18).

In order to illustrate this, consider the operator family (depending on σ)

$$P = D_x x D_x - \sigma D_x + D_y^2$$

near $x=0$, where y is in a compact manifold Y and D_y^2 stands for a positive Laplacian on Y (with respect to some Riemannian metric on Y). These operators are also called Keldysh-type, and the reflect the structure of the Fourier transformed, in t_*, black hole wave operators near the horizon, which would be given by $x=0$. Then the principal symbol of P is

$$p = x\xi^2 + |\eta|^2,$$

where ξ is dual to x, η is dual to y, and

$$H_p = 2x\xi\partial_x + 2\eta\cdot\partial_y - \xi^2\partial_\xi.$$

The source/sink manifold is the conormal bundle $\{x=0,\ \eta=0\}$ of $\{x=0\}$ (take away the zero section, where $\xi=0$), so one can take

$$\rho_0 = |\xi|^{-1},$$

and

$$\beta_0 = (\operatorname{sign}\xi)\xi^2\partial_\xi|\xi|^{-1} = 1,$$

with $\xi>0$ giving a normal source, $\xi<0$ a normal sink. On the other hand, (with respect to the standard inner product)

$$\frac{P-P^*}{2i} = -\operatorname{Im}\sigma D_x,$$

so its principal symbol of $-\operatorname{Im}\sigma\xi$, so its renormalized principal symbol is $\operatorname{Im}\sigma$. The threshold is then

$$s_\Lambda = \frac{1}{2} - \operatorname{Im}\sigma.$$

The actual estimates then are:

- If $s\geq s_0>s_\Lambda$, then

$$\|B_1u\|_{H^s} \leq C(\|B_3Pu\|_{H^{s-m+1}} + \|u\|_{H^{s_0}}),$$

  $B_j\in\Psi^0$, B_1 elliptic on L, provided $\operatorname{supp} b_1\subset\{b_3\neq 0\}$, and all bicharacteristics from points in $\operatorname{supp} b_1\cap\operatorname{Char}(P)$ tend to L while remaining in $\{b_3\neq 0\}$.
- If $s<s_\Lambda$ then

$$\|B_1u\|_{H^s} \leq C(\|B_2u\|_{H^s} + \|B_3Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

  $B_j\in\Psi^0$, B_1 elliptic on L, provided $\operatorname{supp} b_1\subset\{b_3\neq 0\}$, and all bicharacteristics from points in $(\operatorname{supp} b_1\cap\operatorname{Char}(P))\setminus L$ reach the elliptic set $\{b_2\neq 0\}$ of B_2 while remaining in $\{b_3\neq 0\}$.

Replacing P by P^* changes s_Λ as it reverses the sign of the principal symbol of $\frac{P-P^*}{2i}$. Thus, for P^* the analogous quantity is

$$s^*_\Lambda = \frac{m-1}{2} \pm \beta_0^{-1}\rho_0^{m-1}\sigma_{m-1}\Big(\frac{P-P^*}{2i}\Big) = m-1-s_\Lambda.$$

In view of (10)–(11), this naturally leads to estimates on the required dual spaces.

As a consequence, if there are radial sets L_1, L_2 such that all bicharacteristics in $\operatorname{Char}(P)\setminus(L_1\cup L_2)$ escape to L_1 in one of the directions along the bicharacteristics and to L_2 in the other, one has the required Fredholm estimate provided one can arrange the Sobolev spaces so that

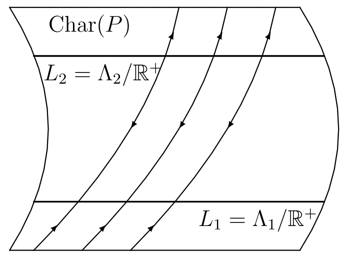

Figure 7. A globally well-structured (non-trapping) flow on the characteristic set shown as a torus (within S^*M). The source manifold is L_2, the sink manifold is L_1, and all bicharacteristics (in the characteristic set) outside these tend to the appropriate one of these in the forward/backward direction along the Hamilton flow.

- at L_1 the Sobolev order is above the threshold s_{Λ_1} for P,
- at L_2 the Sobolev order is above the threshold $s^*_{\Lambda_2}$ for P^*;

cf. Figure 7.

Typically this requires *variable order* Sobolev spaces, i.e. the order s is a function on S^*M, in which case we also need

- the Sobolev order is monotone decreasing from L_1 to L_2,

for the propagation estimates to be valid in that case.

Namely, under these assumptions on the dynamics, we conclude that

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m+1}} + \|u\|_{H^{-N}}),$$

$$\|v\|_{H^{s'}} \leq C(\|P^*v\|_{H^{s'-m+1}} + \|v\|_{H^{-N'}}),$$

with $s' = -s + m - 1$, i.e. (10)–(11) hold.

- A frequent place these normal sources and sinks arise are *radial sets*, i.e. points in T^*M where H_p is tangent to the fiber dilation orbits; propagation provides no information here as in S^*M (or the preferred transversal S of the dilation orbits) the induced vector field vanishes.
- In non-degenerate settings, i.e. when H_p is non-zero, the biggest possible dimension of a radial set is that of M, in which case it is a conic Lagrangian submanifold of T^*M.
- In this case, the radial sets necessarily act as source or sink within $\mathrm{Char}(P)$; in the source case H_p flows to the zero section within Λ, in the sink case from the zero section: these are the red shift (flow to the zero section, i.e. source, means frequencies are becoming 'redder' under the flow) and blue shift (flow away from the zero section, i.e. to fiber infinity, means frequencies are becoming 'bluer' under the flow).

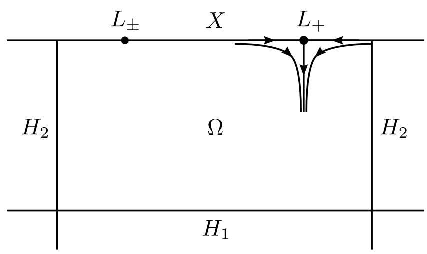

FIGURE 8. The saddle point structure at the horizons at the ideal boundary at infinity. This picture is very partial since it only shows the position space and the projection of bicharacteristics rather than the full phase space in which the bicharacteristics actually are.

- More generally there may be a non-trivial flow within the sources and sinks; this is the case for dynamical systems as well as rotating black holes, where these are at the conormal bundles of the horizons.

8.5. The extension to non-compact or degenerate underlying spaces. The just described theory, introduced in [**94**], works for instance for the Fourier transformed family $\hat{P}_0(\sigma)$ of the model Kerr-de Sitter wave operator, with the model P_0 itself arising in (14). For the operator P itself it is best to think of M as a compact manifold with boundary via compactifying it to $\overline{M}$; the new boundary is an ideal boundary added to infinity with defining function $e^{-t_*} = \tau$, see Figure 8. The microlocal analysis is then on the phase space of this compactified space, which is ${}^bS^*\overline{M}$; here 'b' encodes the compactification used; this was introduced by Melrose for the study of wave propagation [**75**], and he gave a detailed and extended treatment in [**76**].

In order to demystify this phase space and the corresponding algebra, recall that Kerr-de Sitter space is of the form $\mathbb{R}_{t_*} \times (0,\infty)_r \times \mathbb{S}^2$, and we actually put 'final Cauchy hypersurfaces' at $r = r_- - \epsilon$ and $r = r_+ + \epsilon$, where r_- is the event horizon and r_+ is the cosmological horizon, as well as an 'initial Cauchy hypersurface' at $t_* = T_0$ so in fact we are working on

$$M = [T_0, \infty)_{t_*} \times [r_- - \epsilon, r_+ + \epsilon]_r \times \mathbb{S}^2.$$

The only noncompactness thus is in t_*. However, we do have a uniform structure as $t_* \to \infty$; after all the metric is t_*-translation invariant! Thus, we want to have an operator algebra based on the vector fields reflecting the structure of the metric in a uniform manner, thus based on ∂_{t_*} as well as the 'spatial derivatives' ∂_r and vector fields on the sphere. The most direct way the uniformity could be introduced is by making the coefficients independent of t_*, i.e. making them lie in a 'spatial' function space, such as

$$H^s([r_- - \epsilon, r_+ + \epsilon]_r \times \mathbb{S}^2).$$

But of course for the actual stability problem we do have an actual evolution, so we need to allow t_* dependence that should be arbitrary when t_* is in bounded sets, such as locally in t_* being in

$$H^s([T_0, T_1] \times [r_- - \epsilon, r_+ + \epsilon]_r \times \mathbb{S}^2),$$

but well controlled as $t_* \to \infty$. There are various reasonable choices for the latter, such as the Sobolev space

$$H^s([T_0, \infty) \times [r_- - \epsilon, r_+ + \epsilon]_r \times \mathbb{S}^2),$$

which corresponds to the above stated derivatives, up to s of them (when s is a positive integer), leaving the function in L^2 (with respect to the standard measure), or the analogous C^0_∞-based version, i.e. (for the infinite regularity case) all derivatives (of any order) with respect to the above stated vector fields are bounded continuous functions. The latter is exactly the (localized to this region) version of Hörmander's uniform algebra [**57**]: differential operators of the form $\sum_{|\alpha|\leq m} a_\alpha(z) D_z^\alpha$ on $\mathbb{R}^n$ ($n = 4$ here) with $D^\beta a_\alpha$ bounded for all $\beta \in \mathbb{N}^n$. The pseudodifferential version is then analogous, with symbols on the phase space $T^*\mathbb{R}^n$ satisfying uniform estimates

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \leq C_{\alpha\beta}(1 + |\zeta|)^{m-|\alpha|}. \tag{19}$$

This is a good place to recall how these pseudodifferential operators are defined. One writes, for e.g. Schwartz functions u on $\mathbb{R}^n$,

$$\begin{aligned}(\mathrm{Op}(a)u)(z) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} a(z,\zeta)\, u(z')\, d\zeta\, dz' \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iz\cdot\zeta} a(z,\zeta)(\mathcal{F}u)(\zeta)\, d\zeta,\end{aligned}$$

where the last integral expression is actually absolutely convergent, since $\mathcal{F}u$ itself is Schwartz, while the middle expression is interpreted as an oscillatory integral, which is to say that it is essentially defined by integrating by parts sufficiently many times in z', using that

$$e^{i(z-z')\cdot\zeta} = (1 + |\zeta|^2)^{-1}(1 + D_{z'}^2)e^{i(z-z')\cdot\zeta},$$

to gain sufficient decay of the integrand in ζ. (It is technically better to phrase this slightly differently; see [**57, 72, 98**].) This is actually called the left quantization of a; if in the intermediate formula we had $a(z', \zeta)$, it would be called the right quantization: either works equally well for our purposes.

One consequence of this setup is that these pseudodifferential operators act on the standard Sobolev spaces H^s, which for non-negative integers s possess s derivatives (with respect to ∂_{t_*} and the spatial derivatives) in the standard L^2 space. They also act on polynomially weighted (in terms of t_*) spaces. However, it turns out that with a little care in their definition one can also assure that they act on exponentially weighted spaces $e^{-\ell t_*} H^s$ of decay order ℓ. Writing $z = (t_*, r\omega) \in \mathbb{R} \times \mathbb{R}^3$, staying in a compact subset of $\mathbb{R}^3$, and similarly in z', this modification is essentially to add a localization factor $\psi(t_* - t_*')$, where ψ has compact support and is identically 1 near

0. (For the sake of completeness, one needs to add one more 'trivial' term to define the pseudodifferential operator algebra; this is the analogue of adding smooth Schwartz kernels on compact manifolds. See [**98**, Section 6] for details.) For reasons that become clear soon, the collection of the slightly more carefully constructed pseudodifferential operators from symbols (19) is called *b-pseudodifferential operators* Ψ_b^m (or 'conormal b', Ψ_{bc}^m, to emphasize the uniform regularity, as opposed to the full expansion discussed below), and $e^{-\ell t_*}H^s$ is called the *b-Sobolev space* is $H_b^{s,\ell}$.

One might want something stronger than uniform estimates, for instance an asymptotic expansion at infinity. The matching expansion in this structure is in terms of powers of $\tau = e^{-t_*}$, with coefficients that are smooth in the 'spatial variables' (r, ω). But having such an expansion,

$$a(t_*, r, \omega) \sim \sum_{j=0}^{\infty} e^{-jt_*} a_j(r, \omega)$$

exactly amounts to the statement that a is a smooth function of $\tau = e^{-t_*}, r, \omega$. So one can naturally work on the *compactification* given by (taking $T_0 > 0$ for the simplicity of discussion)

$$\overline{M} = [0, -\log T_0]_\tau \times [r_- - \epsilon, r_+ + \epsilon]_r \times \mathbb{S}^2.$$

Notice that $(0, -\log T_0]_\tau \times [r_- - \epsilon, r_+ + \epsilon]_r \times \mathbb{S}^2$ is still M, simply in different coordinates! The phase space is then

$${}^bT^*\overline{M} = [0, -\log T_0]_\tau \times [r_- - \epsilon, r_+ + \epsilon]_r \times \mathbb{S}^2 \times (\mathbb{R}^4)^*$$

with the coordinates on $(\mathbb{R}^4)^*$ given by the dual variable ζ. While $(\mathbb{R}^4)^*$ is non-compact, this is exactly the same non-compactness that we are familiar with from basic microlocal analysis, and one can simply quotient out by dilations to obtain the b-cosphere bundle

$${}^bS^*\overline{M} = ({}^bT^*\overline{M} \setminus o)/\mathbb{R}^+.$$

One can again quantize these symbols, such as homogeneous degree m smooth functions a on ${}^bT^*\overline{M} \setminus o$ to obtain pseudodifferential operators $\mathrm{Op}(a) = A \in \Psi_b^m(\overline{M})$; a is then the principal symbol of A. Thus, a is a smooth function on the compact space ${}^bS^*\overline{M}$, and the same sort of analysis as in the basic microlocal analysis setting applies.

One again says that P is elliptic at $\alpha \in {}^bS^*\overline{M}$ if the homogeneous function p is non-zero at α, i.e. in our relativistic setting, where P is a wave operator, at non-null covectors. Near such points we have elliptic estimates, with arbitrary s, ℓ, N:

$$\|B_1 u\|_{H_b^{s,\ell}} \leq C(\|B_3 P u\|_{H_b^{s-m,\ell}} + \|u\|_{H_b^{-N,\ell}}),$$

$B_j = \mathrm{Op}(b_j) \in \Psi_b^0$, provided $b_3 \neq 0$ on $\operatorname{supp} b_1$ and $p \neq 0$ on $\operatorname{supp} b_1$. Roughly, the proof again relies on the fact that $B_1 u$ is localized to $\operatorname{supp} b_1$, on which $b_3 p$ is non-zero, so one can write $b_1 = q b_3 p$ for an appropriate q (which has order $-m$ to cancel the order m of p), which in turn implies that

$B_1 u = \mathrm{Op}(q) B_3 P u + R u$, where R is order $(-1, 0)$. This directly gives a version of the elliptic estimate with the $H_b^{-N,\ell}$ norm replaced by the $H_b^{s-1,\ell}$ norm, and an iterated and improved version of this argument is the elliptic estimate as stated.

Now, in the characteristic set of P, i.e. at non-elliptic points, one can again propagate estimates along integral curves of H_p (assuming p is real-valued). A key point is that if $P \in \Psi_b^m(\overline{M})$ then H_p is a homogeneous degree $m-1$ vector field on ${}^bT^*\overline{M} \setminus o$ *which is tangent to* $\partial\overline{M}$. Much as in the compact M setting, we can also induce a vector field on ${}^bS^*\overline{M}$, which is still tangent to the boundary $\partial\overline{M}$, and now we have a compact manifold, albeit with boundary, to work on. This gives a dynamical system on ${}^bS^*\overline{M}$, and away from points at which H_p is radial (i.e. a multiple of the dilation vector field), i.e. the induced vector field on ${}^bS^*M$ has a critical point, one can propagate estimates along the H_p integral curves.

Concretely, one again has estimates of the form

$$\|B_1 u\|_{H^{s,\ell}} \leq C(\|B_2 u\|_{H^{s,\ell}} + \|B_3 P u\|_{H^{s-m+1,\ell}} + \|u\|_{H^{-N,\ell}}), \tag{20}$$

$B_j \in \Psi^0$, provided $\operatorname{supp} b_1 \subset \{b_3 \neq 0\}$, and all bicharacteristics from points in $\operatorname{supp} b_1 \cap \mathrm{Char}(P)$ reach $\{b_2 \neq 0\}$ of while remaining in $\{b_3 \neq 0\}$, and the proof again proceeds by a positive commutator estimate completely analogous to the local/compact manifold discussion.

In order to obtain estimates leading towards Fredholm theory the most immediate obstacle is how to start these estimates. For wave equations Cauchy hypersurfaces are one possibility, and indeed for the Kerr-de Sitter stability problem they play a key role. Another possibility would be to have a source/sink structure, which indeed plays a role in Minkowski-like spaces, with the source/sink at the conormal bundle of the light cone at the boundary of the compactification, and which is discussed in Section 9.

Concretely, in Kerr-de Sitter space, $t_* = T_0$ is the initial Cauchy hypersurface, and $r = r_+ + \epsilon$ and $r = r_- - \epsilon$ are final Cauchy hypersurfaces; they are all space-like. These are *artificial boundaries*: we choose them. While they are important for the complete framework, they do behave as for finite time problems. But once they are chosen, we work globally.

While for Kerr-de Sitter space we use Cauchy surfaces, there is a feature that is very similar to sources and saddles discussed for compact M: namely there are invariant manifolds L of the Hamilton flow over $\partial\overline{M}$, which are however *saddle points* in the normal directions to L; one of the stable/unstable manifolds, $\mathcal{L}_{\partial\overline{M}}$, is completely within ${}^bS^*_{\partial\overline{M}}\overline{M}$, corresponding to the flow in/flow out of the sources and sinks of $\hat{P}(\sigma)$, and the other, $\mathcal{L}_M$, is normal to it, corresponding to the conormal bundle of the horizons. One can again propagate estimates through these saddle points, and again one has a threshold quantity. In this case the role the threshold quantity plays is that when the regularity (Sobolev order) is above the threshold, one can propagate the estimates from M to $\partial\overline{M}$, while if the regularity is below the threshold, one can propagate the estimates from $\partial\overline{M}$ to M. A different feature from the compact M setting is that now the Sobolev spaces also have a decay order, and this decay order plays a role in determining the threshold. It is again simple to see how this comes about: at L only the weights can give positivity in the positive commutator argument, but now there are two: the homogeneous degree m' weight, $s = \frac{m+m'-1}{2}$, corresponding to the differential order, and $e^{2kt_*} = \tau^{-2k}$ corresponding to the weight of the Sobolev space. These are balanced by the relative sizes of the Hamilton derivatives, namely $\beta_0 = \mp\rho_0^{m-2}H_p\rho_0 > 0$, resp. $-\beta_0\tilde\beta = \mp\rho_0^{m-1}\tau^{-1}H_p\tau > 0$, with the top sign corresponding to a sink, the bottom to a source, within ${}^bS^*_{\partial\overline{M}}\overline{M}$, with the latter defining the relative constant $\tilde\beta$. Note that the overall sign difference in these two corresponds to the locally opposite asymptotic behavior of the defining function τ of $\partial\overline{M}$ and the defining function ρ_0 of fiber infinity under the Hamilton flow: as one increases, the other decreases. So for instance, as one moves to the boundary, i.e. $\tau \to 0$, ρ_0 increases, i.e. the frequency shifts to the red, and vice versa. Then the threshold is, assuming these quantities are constants,

$$s_\Lambda = \frac{m-1}{2} + \tilde\beta k \mp \beta_0^{-1}\rho_0^{m-1}\sigma_{m-1}\Big(\frac{P-P^*}{2i}\Big).$$

Concretely,

- If $s \geq s_0 > s_\Lambda$, then

  $$\|B_1u\|_{H_b^{s,\ell}} \leq C(\|B_2u\|_{H_b^{s,\ell}} + \|B_3Pu\|_{H_b^{s-m+1,\ell}} + \|u\|_{H_b^{s_0,\ell}}),$$

  $B_1 \in \Psi_b^0$ elliptic on L, provided $\operatorname{supp} b_1 \subset \{b_3 \neq 0\}$, and all bicharacteristics from points in $\operatorname{supp} b_1 \cap \operatorname{Char}(P)$ tend to L or $\{b_2 \neq 0\}$ while remaining in $\{b_3 \neq 0\}$, *and* all bicharacteristics from points in a neighborhood of L in $\mathcal{L}_M$ tend to the elliptic set $\{b_2 \neq 0\}$ of B_2 while remaining in $\{b_3 \neq 0\}$.
- If $s < s_\Lambda$ then

  $$\|B_1u\|_{H_b^{s,\ell}} \leq C(\|B_2u\|_{H_b^{s,\ell}} + \|B_3Pu\|_{H_b^{s-m+1,\ell}} + \|u\|_{H_b^{-N,\ell}}),$$

  $B_1 \in \Psi_b^0$ elliptic on L, provided $\operatorname{supp} b_1 \subset \{b_3 \neq 0\}$, and all bicharacteristics from points in $(\operatorname{supp} b_1 \cap \operatorname{Char}(P)) \setminus L$ reach the elliptic set $\{b_2 \neq 0\}$ of B_2 while remaining in $\{b_3 \neq 0\}$, *and* all bicharacteristics from points in a neighborhood of L in $\mathcal{L}_{\partial\overline{M}}$ tend to the elliptic set $\{b_2 \neq 0\}$ of B_2 while remaining in $\{b_3 \neq 0\}$.

The additional issue is the trapping, corresponding to bicharacteristics that even in a generalized sense (going through the saddle points) do not propagate to the initial/final 'Cauchy hypersurfaces'. In de Sitter space, in an analogous setting as Kerr-de Sitter space (the static patch), there is no trapping, so the our setup is directly applicable. In Kerr-de Sitter space, on the other hand the trapped set is a submanifold Γ of ${}^bS^*\overline{M}$, and it has a stable/unstable manifolds which are much like those at the saddle points we had discussed: one, $\Gamma_{\partial\overline{M}}$ in $\partial\overline{M}$, and one, Γ_M, normal to ∂M. The key difference to the saddle point is that this is neutral from the perspective of fiber infinity: after all, ∂_{t_*}, which is elliptic there, commutes with the model operator P_0, and its principal symbol can be taken as ρ_0^{-1}. The defining function τ of $\partial\overline{M}$ still gives a definite contribution, but this means that unlike at the saddle points, at the trapped set there is a limit to the amount of decay we can propagate through Γ as we propagate towards $\partial\overline{M}$ using positive commutator (microlocal energy) estimates; for the actual (formally self-adjoint) wave operator this amounts to $\ell < 0$, i.e. one has forward-in-time estimates on exponentially growing spaces, and backward in time estimates on the dual exponentially decaying spaces (these are the ones required for solvability by duality arguments). The key for resolving this, since one needs decay for black hole stability considerations, is to allow for a loss of derivatives.

There is a long history of such trapped estimates going back to Wunsch and Zworski [**107**], Nonnenmacher and Zworski [**79**] and Dyatlov [**35**]; the Kerr-de Sitter stability problem relies on a modification of Dyatlov's approach espoused in [**36**] (which since then has been further extended by Hintz in [**52**]). Very roughly, the idea is to perform a 2-step positive commutator estimate, with each step losing a derivative, which altogether results in the loss of two derivatives, i.e. an additional derivative relative to the usual wave propagation/microlocal energy estimates. (Some of the previous works have less of a loss, but Dyatlov's approach is particularly amenable to quasilinear equations.) At the end of the day, this is what necessitates the Nash-Moser iteration.

In summary, one propagates estimates from H_1 through the radial saddle points L at the horizons (and the normally hyperbolic trapping) to H_2. When all the parts are combined, this gives rise to the estimates

$$\|u\|_{H_b^{s,\ell}} \leq C(\|Pu\|_{H_b^{s-1,\ell}} + \|u\|_{H_b^{-N,\ell}}),$$

which is Step 1 of our general analytic setup, (13). In particular, as the inclusion of $H_b^{s,\ell}$ into $H_b^{-N,\ell}$ is not compact, there is still need for Step 2.

In Step 2, one inverts the actual Kerr-de Sitter operator P_0 on the Fourier transform (in $-t_*$) side, which relies on the compact manifold version of the estimates, in a uniform in the Fourier-dual variable σ version; these are 'large parameter' or, via a rescaling, semiclassical, estimates. We refer to [**94**] for a quick introduction to these, and [**109**] for a full dertailed treatment. This Step 2 inversion can be done on spaces $H_b^{s,\ell}$ whose decay order ℓ is not the negative of the imaginary part of a pole of the family $\hat{P}_0(\sigma)^{-1}$, as indicated in Section 7, completing the Fredholm setup.

9. The Feynman propagator

For the sake of perspective, let us consider an even simpler problem: the linear wave equation on a Lorentzian spacetime (M, g): $\Box_g u = f$ (f

given). Typically the Cauchy problem is considered, i.e. initial conditions are specified at an (embedded) spacelike hypersurface H. Then

- there is a unique local solution (near H), and
- if (M, g) is globally hyperbolic, i.e. each maximally extended time-like curve intersects H exactly once, or equivalently there is a global time function t, there is a unique global solution.

The Cauchy problem is equivalent to a forcing (or inhomogeneous) problem with trivial Cauchy data which can be restated as: solve $\Box_g u = f$ where f is supported in $t \geq t_0$, by finding u which is supported in $t \geq t_0$, together with its analogue where $\geq$ is replaced by $\leq$.

The solution operator $\Box_{g,R}^{-1} : f \mapsto u$ is the forward, or retarded, solution operator. If one replaces $\geq$ by $\leq$, one obtains the backward, or advanced, solution operator, $\Box_{g,A}^{-1}$.

While this is 'standard', essentially going back to the 18th century, this does not mean that these are the only inverses of $\Box_g$. Indeed, a natural question is: What are the natural inverses of $\Box_g$? Are the inverses beyond the advanced/retarded ones?

The work of Gell-Redman, Haber, Wrochna and the author (in various combinations) [**44, 103**] shows that in reasonable, but quite general geometric, settings there are two more natural inverses, the Feynman and anti-Feynman propagators; these were introduced by Feynman in the Minkowski setting for use in quantum field theory. One application of the Feynman propagator is to show the essential self-adjointness of the wave operator: the spectral theory inverse for $\Box_g - \lambda$ when $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is the Feynman or the anti-Feynman propagator depending on the sign of $\operatorname{Im} \lambda$ [**99**].

The basic idea is to encode propagators (inverses) via the choice of function spaces (the inverse depends on the choice!) on which $\Box_g$ is Fredholm. In terms of the source/sink structure we discussed in the previous section this amounts to deciding where the regularity is high vs. low.

In a 'parametrix' sense (modulo smoothing errors) this problem was analyzed by Duistermaat and Hörmander [**34**]: there is a *distinguished parametrix* for each choice of a direction in each connected component of the characteristic set. (These authors themselves were already motivated by quantum field theory considerations, as explained in the paper.) But the 'smoothing errors' inherent in the paramterix description are weak in non-compact manifold settings: in particular they lead to non-compact errors, and thus do not lead the Fredholm estimates. Thus, the point of the recent work is to set up Fredholm problems.

In fact, our discussion is not really specific for the wave equation, rather it is a general non-elliptic phenomenon. But back in the setting of second order PDE, another place where Feynman and anti-Feynman propagators arise is ultrahyperbolic PDE such as $\sum_{j=1}^k D_{x_j}^2 - \sum_{j=k+1}^n D_{x_j}^2$, $k, n-k \geq 2$ on $\mathbb{R}^n$. These are in fact *very much* like the wave equation *except* for the Cauchy problem — but our approach of constructing inverses works just as well! Indeed, it turns out that, due to the characteristic set in these cases being connected, there are two reasonable inverses, namely the Feynman and anti-Feynman propagators. Thus, the difficulty of solving ultrahyperbolic equations stems from unreasonably expecting an analogue of the Cauchy problems; the global Fredholm approach, much like for elliptic problems, remains perfectly valid, in this case producing the Feynman and anti-Feynman inverses.

There has been much work in mathematical quantum field theory on Feynman propagators, where the connection to microlocal analysis that already arose in the work of Duistermaat and Hörmander [**34**] was strengthened in the 1990s by Radzikowski [**81**]. The closest works in terms of general (non-algebraic) outlook have been due to Bär, Dereziński, Gérard, Häfner, Siemssen, Strohmaier and Wrochna [**29**, **30**, **46**, **45**, **5**]. Some others in the field are Brunetti, Dappiaggi, Fredenhagen, Köhler, Moretti, Pinamonti [**14**, **15**, **26**, **78**, **27**]...

Here we only consider an example of this setup, namely Minkowski space and its perturbations. In the setup we discuss the Minkowski metric, but the setup is perturbation stable within the same kind of problems: the metric being a perturbation of Minkowski metric in the b-geometry sense of Melrose, see [**44**]. So let $M = \mathbb{R}^n$ with the Minkowski metric and $\Box$ be the wave operator.

Let ρ be a homogeneous degree -1 positive function, e.g. the reciprocal of the Euclidean distance r from the origin, but any such function will do as all choices give equivalent results. This function ρ is an analogue of the Kerr-de Sitter τ above. Indeed, even in terms of the b-structure discussed in the previous section, $\rho = r^{-1}$ plays the analogue of $\tau = e^{-t_*}$, so if one wishes to use Hörmander's uniform algebra to construct the pseudodifferential operators, one should use $\log r$ as the replacement for t_*. Correspondingly, the natural compactification is $\overline{M}$ arises by gluing a sphere to M as the new boundary, using ρ as a boundary defining function:

$$\overline{M} = M \sqcup ([0,\infty)_\rho \times \mathbb{S}^{n-1})/\sim,$$

where $\sqcup$ is the disjoint union, and $\sim$ is the relation identifying a point $(\rho,\omega) \in (0,\infty)_\rho \times \mathbb{S}^{n-1}$ with $\rho^{-1}\omega \in \mathbb{R}^n \setminus \{0\}$.

The operator $P = \rho^{-2}\Box$ is then an element of $\mathrm{Diff}_b^2(\overline{M})$. It turns out that the Hamilton flow of its principal symbol p has a source/sink structure, with the source/sink being the (spherical) conormal bundle of the light cone at ∂M, which has two halves both in the future and in the past, corresponding to the two connected components of the characteristic set Σ_+ and Σ_-. If one writes the operator as $D_{x_0}^2 - (D_{x_1}^2 + \ldots + D_{x_{n-1}}^2)$ and the dual variable as ζ, these two components of the characteristic set are given by $\zeta_0 > 0$ and $\zeta_0 < 0$ (within the characteristic set). Writing the future, resp. past, radial set in Σ_+ as L_{++}, resp. L_{+-}, L_{++} is a sink and L_{+-} is a source. On the other hand, the future radial set L_{-+} in Σ_- is a source and the past radial

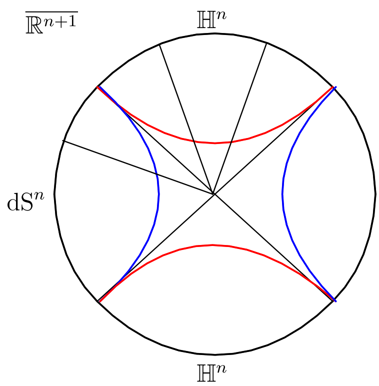

FIGURE 9. Minkowski space compactified to a ball; the sphere at infinity corresponds to 'endpoints' of the dilation orbits, of which some are shown as straight lines from the center. Also shown are the hyperboloids: hyperbolic space and de Sitter space. One can consider the latter equivalently as regions in the sphere at infinity, via the dilation action, or indeed in any smooth transversal to the dilation orbits.

set L_{--} in Σ_- is a sink. The global dynamics is that in Σ_+, the bicharacteristics go from the past component (source) of the radial set L_{+-} to the future one (sink) L_{++}; in the other component Σ_- they go from the future component (source) of the radial set L_{-+} to the past one (sink) L_{--}. This then has completely analogous microlocal structure to the compact manifold source/sink based theory. Note that the acceptable perturbations of the operator (or the metric) are in this b-setting, as discussed in [**44**]; a somewhat more stringent setting, which however gives more precise information on the behavior of waves at the light cone, was introduced earlier by Baskin, Wunsch and the author [**6, 7**].

Reasonable choices of Fredholm problems:

- Make the Sobolev spaces high regularity at the past radial sets and low at the future radial sets: this is the *forward, or retarded, propagator.*
- Make the Sobolev spaces low regularity at the past radial sets and high at the future radial sets: this is the *backward, or advanced, propagator.*
- Make the Sobolev spaces high regularity at the sources L_{+-} and L_{-+} and low regularity at the sinks, or vice versa. These are the Feynman propagators, and they propagate estimates for P in the direction of the Hamilton flow in the first case (Feynman), and against the Hamilton flow in the second (anti-Feynman).
- Note that the adjoint of these inverses always propagates estimates in the *opposite* direction on the relevant dual function spaces!
- All these choices require *variable order* Sobolev spaces.

So from the perspective of Hamiltonian dynamics, the most natural inverses are in fact the Feynman and anti-Feynman inverses as they always propagate estimates in the same direction relative to the Hamilton vector field in both components of the characteristic set!

Let us now consider the model operator theory at infinity which is still required to complete the Fredholm setup. Since the analogue of the Kerr-de Sitter t_* is $\log r$, the analogue of the Fourier transform in $-t_*$ is the Fourier transform in $-\log r$, i.e. the Mellin transform in r. Now, the conjugate of $\rho^{-2}\Box$ by the Mellin transform along the dilation orbits in $\mathbb{R}^n \setminus \{0\}$ gives a family of operators $\hat{P}(\sigma)$, σ the Mellin dual parameter, on the standard sphere, $\mathbb{S}^{n-1}$, which is simply a smooth transversal to the dilation orbits (so its metric structure is completely irrelevant).

The induced operators $\hat{P}(\sigma)$ are elliptic 2nd order differential operators on $\mathbb{S}^{n-1}$ inside the light cone, i.e. correspond to a Riemannian metric, but they correspond to a Lorentzian metric outside the light cone. In fact, these two metrics are essentially the hyperbolic space and de Sitter space metrics, respectively, considered in a 'projective compactification', rather than a 'conformal compactification', sense. This is *exactly* the same way as a $n-1$-dimensional de Sitter space is glued to a hyperbolic space for the Mellin-transformed normal operator family for the compactification of n-dimensional de Sitter space in the analogue of our treatment of Kerr-de Sitter space. We refer to [**95, 97**] for more details on these induced problems and the geometric structures, which indeed had arisen in earlier geometric work of Fefferman and Graham [**39, 40**].

The conormal bundle of the light cone (at $\mathbb{S}^{n-1}$) consists of radial points for these operators $\hat{P}(\sigma)$, much as it happened at the boundary $\partial\overline{M}$ of the compactified Minkowski space ('light cone at infinity'). Again, the characteristic set (which lies over the de Sitter region and the light cone) has two components, and there are four components of the radial set: a future and a past component within each component of the characteristic set; one needs to use function spaces which match those on $\overline{M}$ itself, thus e.g. for the Feynman propagator they are higher than threshold order at the sources and lower than threshold order at the sinks.

This combination yields a Fredholm setup for P *provided* one works in weighted Sobolev spaces $H_b^{s,\ell}$, satisfying the threshold requirements, and ℓ not being the negative of the imaginary part of a pole of $\hat{P}(\sigma)^{-1}$ on the corresponding function spaces. In fact, for an appropriate range of ℓ the Fredholm operator is in fact invertible; this thus constructs the Feynman and anti-Feynman propagators as actual operator theoretic inverses! We refer to [**44**] for more details about this approach.

References

[1] S. Alexakis, A. D. Ionescu, and S. Klainerman. Uniqueness of smooth stationary black holes in vacuum: small perturbations of the Kerr spaces. *Comm. Math. Phys.*, 299(1):89–127, 2010. MR 2672799

[2] S. Alexakis, A. D. Ionescu, and S. Klainerman. Rigidity of stationary black holes with small angular momentum on the horizon. *Duke Math. J.*, 163(14):2603–2615, 2014. MR 3273578

[3] Lars Andersson, Thomas Bäckdahl, Pieter Blue, and Siyuan Ma. Stability for linearized gravity on the Kerr spacetime. *Preprint, arXiv:1903.03859*, 2019.

[4] Lars Andersson and Pieter Blue. Uniform energy bound and asymptotics for the Maxwell field on a slowly rotating Kerr black hole exterior. *J. Hyperbolic Differ. Equ.*, 12(4):689–743, 2015. MR 3450059

[5] Christian Bär and Alexander Strohmaier. An index theorem for Lorentzian manifolds with compact spacelike Cauchy boundary. *Amer. J. Math.*, 141(5):1421–1455, 2019. MR 4011805

[6] Dean Baskin, András Vasy, and Jared Wunsch. Asymptotics of radiation fields in asymptotically Minkowski space. *Amer. J. Math.*, 137(5):1293–1364, 2015. MR 3405869

[7] Dean Baskin, András Vasy, and Jared Wunsch. Asymptotics of scalar waves on long-range asymptotically Minkowski spaces. *Adv. Math.*, 328:160–216, 2018. MR 3771127

[8] Michael Beals and Michael Reed. Microlocal regularity theorems for nonsmooth pseudodifferential operators and applications to nonlinear problems. *Trans. Amer. Math. Soc.*, 285(1):159–184, 1984. MR 0748836

[9] Lydia Bieri and Nina Zipser. *Extensions of the stability theorem of the Minkowski space in general relativity*, volume 45 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2009. MR 2531716

[10] P. Blue and A. Soffer. Phase space analysis on some black hole manifolds. *J. Funct. Anal.*, 256(1):1–90, 2009. MR 2475417

[11] Jean-François Bony and Dietrich Häfner. Decay and non-decay of the local energy for the wave equation on the de Sitter-Schwarzschild metric. *Comm. Math. Phys.*, 282(3):697–719, 2008. MR 2426141

[12] Jean-Michel Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, 14(2):209–246, 1981. MR 0631751

[13] Othmar Brodbeck, Simonetta Frittelli, Peter Hübner, and Oscar A. Reula. Einstein's equations with asymptotically stable constraint propagation. *J. Math. Phys.*, 40(2):909–923, 1999. MR 1674255

[14] R. Brunetti, K. Fredenhagen, and M. Köhler. The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes. *Comm. Math. Phys.*, 180(3):633–652, 1996. MR 1408521

[15] Romeo Brunetti and Klaus Fredenhagen. Microlocal analysis and interacting quantum field theories: renormalization on physical backgrounds. *Comm. Math. Phys.*, 208(3):623–661, 2000. MR 1736329

[16] Y. Choquet-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta mathematica*, 88(1):141–225, 1952. MR 0053338

[17] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. MR 1316662

[18] Mihalis Dafermos, Gustav Holzegel, and Igor Rodnianski. A scattering theory construction of dynamical vacuum black holes. *Preprint, arxiv:1306.5364*, 2013.

[19] Mihalis Dafermos, Gustav Holzegel, and Igor Rodnianski. Boundedness and decay for the Teukolsky equation on Kerr spacetimes I: The case $|a| \ll M$. *Ann. PDE*, 5(1):Paper No. 2, 118, 2019. MR 3919495
[20] Mihalis Dafermos, Gustav Holzegel, and Igor Rodnianski. The linear stability of the Schwarzschild solution to gravitational perturbations. *Acta Math.*, 222(1):1–214, 2019. MR 3941803
[21] Mihalis Dafermos, Gustav Holzegel, Igor Rodnianski, and Martin Taylor. The non-linear stability of the Schwarzschild family of black holes. *Preprint, arXiv:2104.08222*, 2021.
[22] Mihalis Dafermos and Jonathan Luk. Stability of the Kerr Cauchy horizon. *In preparation.*
[23] Mihalis Dafermos and Jonathan Luk. The interior of dynamical vacuum black holes i: The C^0-stability of the Kerr Cauchy horizon. *Preprint, arXiv:1710.01722*, 2017.
[24] Mihalis Dafermos and Igor Rodnianski. A proof of Price's law for the collapse of a self-gravitating scalar field. *Invent. Math.*, 162(2):381–457, 2005. MR 2199010
[25] Mihalis Dafermos and Igor Rodnianski. The red-shift effect and radiation decay on black hole spacetimes. *Comm. Pure Appl. Math*, 62:859–919, 2009. MR 2527808
[26] Claudio Dappiaggi, Valter Moretti, and Nicola Pinamonti. Rigorous steps towards holography in asymptotically flat spacetimes. *Rev. Math. Phys.*, 18(4):349–415, 2006. MR 2245366
[27] Claudio Dappiaggi, Valter Moretti, and Nicola Pinamonti. Cosmological horizons and reconstruction of quantum field theories. *Comm. Math. Phys.*, 285(3):1129–1163, 2009. MR 2470919
[28] J. Dereziński and C. Gérard. *Scattering theory of classical and quantum N-particle systems.* Springer, 1997. MR 1459161
[29] Jan Dereziński and Daniel Siemssen. Feynman propagators on static spacetimes. *Rev. Math. Phys.*, 30(3):1850006, 23, 2018. MR 3770965
[30] Jan Dereziński and Daniel Siemssen. An evolution equation approach to the Klein-Gordon operator on curved spacetime. *Pure Appl. Anal.*, 1(2):215–261, 2019. MR 3949374
[31] Dennis M. DeTurck. Existence of metrics with prescribed Ricci curvature: local theory. *Invent. Math.*, 65(1):179–207, 1981/82. MR 0636886
[32] Roland Donninger, Wilhelm Schlag, and Avy Soffer. A proof of Price's law on Schwarzschild black hole manifolds for all angular momenta. *Adv. Math.*, 226(1):484–540, 2011. MR 2735767
[33] J. J. Duistermaat. On Carleman estimates for pseudo-differential operators. *Invent. Math.*, 17:31–43, 1972. MR 0322596
[34] J. J. Duistermaat and L. Hörmander. Fourier integral operators. II. *Acta Math.*, 128(3-4):183–269, 1972. MR 0388464
[35] Semyon Dyatlov. Resonance projectors and asymptotics for r-normally hyperbolic trapped sets. *J. Amer. Math. Soc.*, 28(2):311–381, 2015. MR 3300697
[36] Semyon Dyatlov. Spectral gaps for normally hyperbolic trapping. *Ann. Inst. Fourier (Grenoble)*, 66(1):55–82, 2016. MR 3477870
[37] Semyon Dyatlov and Maciej Zworski. Dynamical zeta functions for Anosov flows via microlocal analysis. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(3):543–577, 2016. MR 3503826
[38] Frédéric Faure and Johannes Sjöstrand. Upper bound on the density of Ruelle resonances for Anosov flows. *Comm. Math. Phys.*, 308(2):325–364, 2011. MR 2851145
[39] Charles Fefferman and C. Robin Graham. Conformal invariants. *Astérisque*, (Numero Hors Serie):95–116, 1985. The mathematical heritage of Élie Cartan (Lyon, 1984). MR 0837196

[40] Charles Fefferman and C. Robin Graham. *The ambient metric*, volume 178 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012. MR 2858236

[41] F. Finster, N. Kamran, J. Smoller, and S.-T. Yau. Decay of solutions of the wave equation in the Kerr geometry. *Comm. Math. Phys.*, 264(2):465–503, 2006. MR 2215614

[42] Felix Finster, Niky Kamran, Joel Smoller, and Shing-Tung Yau. Linear waves in the Kerr geometry: a mathematical voyage to black hole physics. *Bull. Amer. Math. Soc. (N.S.)*, 46(4):635–659, 2009. MR 2525736

[43] Helmut Friedrich. On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. *Comm. Math. Phys.*, 107(4):587–609, 1986. MR 0868737

[44] Jesse Gell-Redman, Nick Haber, and András Vasy. The Feynman propagator on perturbations of Minkowski space. *Comm. Math. Phys.*, 342(1):333–384, 2016. MR 3455153

[45] C. Gérard and M. Wrochna. The massive Feynman propagator on asymptotically Minkowski spacetimes. *Amer. J. Math.*, 141:1501–1546, 2019. MR 4030522

[46] Christian Gérard and Michał Wrochna. Hadamard property of the *in* and *out* states for Klein-Gordon fields on asymptotically static spacetimes. *Ann. Henri Poincaré*, 18(8):2715–2756, 2017. MR 3671549

[47] C. Robin Graham and John M. Lee. Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.*, 87(2):186–225, 1991. MR 1112625

[48] Colin Guillarmou. Meromorphic properties of the resolvent on asymptotically hyperbolic manifolds. *Duke Math. J.*, 129(1):1–37, 2005. MR 2153454

[49] Carsten Gundlach, Gioel Calabrese, Ian Hinder, and José M. Martín-García. Constraint damping in the Z4 formulation and harmonic gauge. *Classical and Quantum Gravity*, 22(17):3767, 2005. MR 2168553

[50] Dietrich Häfner, Peter Hintz, and András Vasy. Linear stability of slowly rotating Kerr black holes. *Invent. Math.*, 223(3):1227–1406, 2021. MR 4213773

[51] Peter Hintz. Global analysis of quasilinear wave equations on asymptotically de Sitter spaces. *Ann. Inst. Fourier (Grenoble)*, 66(4):1285–1408, 2016. MR 3494174

[52] Peter Hintz. Normally hyperbolic trapping on asymptotically stationary spacetimes. *Probability and Mathematical Physics*, 2:71–126, 2021. MR 4404817

[53] Peter Hintz and András Vasy. Semilinear wave equations on asymptotically de Sitter, Kerr–de Sitter and Minkowski spacetimes. *Anal. PDE*, 8(8):1807–1890, 2015. MR 3441208

[54] Peter Hintz and András Vasy. Global analysis of quasilinear wave equations on asymptotically Kerr–de Sitter spaces. *Int. Math. Res. Not. IMRN*, (17):5355–5426, 2016. MR 3556440

[55] Peter Hintz and András Vasy. The global non-linear stability of the Kerr–de Sitter family of black holes. *Acta Math.*, 220(1):1–206, 2018. MR 3816427

[56] Peter Hintz and András Vasy. Stability of Minkowski space and polyhomogeneity of the metric. *Ann. PDE*, 6(1):Paper No. 2, 146, 2020. MR 4105742

[57] L. Hörmander. *The analysis of linear partial differential operators,* vol. 1-4. Springer-Verlag, 1983. MR 0705278

[58] Lars Hörmander. Pseudo-differential operators. *Comm. Pure Appl. Math.*, 18:501–517, 1965. MR 0180740

[59] Lars Hörmander. On the existence and the regularity of solutions of linear pseudo-differential equations. *Enseignement Math. (2)*, 17:99–163, 1971. MR 0331124

[60] Akihiro Ishibashi and Hideo Kodama. Stability of higher-dimensional Schwarzschild black holes. *Progr. Theoret. Phys.*, 110(5):901–919, 2003. MR 2029760

[61] Bernard S. Kay and Robert M. Wald. Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere. *Classical Quantum Gravity*, 4(4):893–898, 1987. MR 0895907

[62] Sergiu Klainerman, Igor Rodnianski, and Jeremie Szeftel. The bounded L^2 curvature conjecture. *Invent. Math.*, 202(1):91–216, 2015. MR 3402797

[63] Sergiu Klainerman and Jérémie Szeftel. *Global nonlinear stability of Schwarzschild spacetime under polarized perturbations*, volume 210 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2020. MR 4298717

[64] Hideo Kodama and Akihiro Ishibashi. A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions. *Progr. Theoret. Phys.*, 110(4):701–722, 2003. MR 2033676

[65] Hideo Kodama, Akihiro Ishibashi, and Osamu Seto. Brane world cosmology: gauge-invariant formalism for perturbation. *Phys. Rev. D (3)*, 62(6):064022, 19, 2000. MR 1791026

[66] J. J. Kohn and L. Nirenberg. An algebra of pseudo-differential operators. *Comm. Pure Appl. Math.*, 18:269–305, 1965. MR 0176362

[67] Peter D. Lax. Asymptotic solutions of oscillatory initial value problems. *Duke Math. J.*, 24:627–646, 1957. MR 0097628

[68] Hans Lindblad and Igor Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)*, 171(3):1401–1477, 2010. MR 2680391

[69] Jonathan Luk. The null condition and global existence for nonlinear wave equations on slowly rotating Kerr spacetimes. *J. Eur. Math. Soc. (JEMS)*, 15(5):1629–1700, 2013. MR 3082240

[70] Jeremy Marzuola, Jason Metcalfe, Daniel Tataru, and Mihai Tohaneanu. Strichartz estimates on Schwarzschild black hole backgrounds. *Comm. Math. Phys.*, 293(1):37–83, 2010. MR 2563798

[71] Rafe R. Mazzeo and Richard B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75(2):260–310, 1987. MR 0916753

[72] R. B. Melrose. Lecture notes for '18.157: Introduction to microlocal analysis'. Available at http://math.mit.edu/~rbm/18.157-F09/18.157-F09.html, 2009.

[73] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31(5):593–617, 1978. MR 0492794

[74] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math.*, 35(2):129–168, 1982. MR 0644020

[75] Richard B. Melrose. Transformation of boundary problems. *Acta Math.*, 147(3-4):149–236, 1981. MR 0639039

[76] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1993. MR 1348401

[77] Richard B. Melrose. Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces. In *Spectral and scattering theory (Sanda, 1992)*, volume 161 of *Lecture Notes in Pure and Appl. Math.*, pages 85–130. Dekker, New York, 1994. MR 1291640

[78] Valter Moretti. Quantum out-states holographically induced by asymptotic flatness: invariance under spacetime symmetries, energy positivity and Hadamard property. *Comm. Math. Phys.*, 279(1):31–75, 2008. MR 2377628

[79] Stéphane Nonnenmacher and Maciej Zworski. Decay of correlations for normally hyperbolic trapping. *Invent. Math.*, 200(2):345–438, 2015. MR 3338007

[80] Frans Pretorius. Evolution of binary black-hole spacetimes. *Phys. Rev. Lett.*, 95:121101, 2005. MR 2169088

[81] Marek J. Radzikowski. Micro-local approach to the Hadamard condition in quantum field theory on curved space-time. *Comm. Math. Phys.*, 179(3):529–553, 1996. MR 1400751

[82] Tullio Regge and John A. Wheeler. Stability of a Schwarzschild Singularity. *Phys. Rev.*, 108:1063–1069, Nov 1957. MR 0091832

[83] Antônio Sá Barreto and Maciej Zworski. Distribution of resonances for spherical black holes. *Math. Res. Lett.*, 4(1):103–121, 1997. MR 1432814

[84] Xavier Saint Raymond. A simple Nash-Moser implicit function theorem. *Enseign. Math. (2)*, 35(3-4):217–226, 1989. MR 1039945

[85] Johannes Sjöstrand and Maciej Zworski. Fractal upper bounds on the density of semiclassical resonances. *Duke Math. J.*, 137(3):381–459, 2007. MR 2309150

[86] Plamen Stefanov, Gunther Uhlmann, and Andras Vasy. Boundary rigidity with partial data. *J. Amer. Math. Soc.*, 29(2):299–332, 2016. MR 3454376

[87] Plamen Stefanov, Gunther Uhlmann, and Andras Vasy. Local and global boundary rigidity and the geodesic X-ray transform in the normal gauge. *Ann. of Math. (2)* 194:1–95, 2021. MR 4276284

[88] Daniel Tataru. Local decay of waves on asymptotically flat stationary space-times. *Amer. J. Math.*, 135(2):361–401, 2013. MR 3038715

[89] Daniel Tataru and Mihai Tohaneanu. A local energy estimate on Kerr black hole backgrounds. *Int. Math. Res. Not. IMRN*, (2):248–292, 2011. MR 2764864

[90] Michael E. Taylor. *Partial differential equations II. Qualitative studies of linear equations*, volume 116 of *Applied Mathematical Sciences*. Springer, New York, second edition, 2011. MR 2743652

[91] Mihai Tohaneanu. Strichartz estimates on Kerr black hole backgrounds. *Trans. Amer. Math. Soc.*, 364(2):689–702, 2012. MR 2846348

[92] André Unterberger. Résolution d'équations aux dérivées partielles dans des espaces de distributions d'ordre de régularité variable. *Ann. Inst. Fourier (Grenoble)*, 21(2):85–128, 1971. MR 0599594

[93] A. Vasy and J. Wunsch. Semiclassical second microlocal propagation of regularity and integrable systems. *J. d'Analyse Mathematique*, 108:119–157, 2009. MR 2544756

[94] András Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces (with an appendix by Semyon Dyatlov). *Invent. Math.*, 194(2):381–513, 2013. MR 3117526

[95] András Vasy. Resolvents, Poisson operators and scattering matrices on asymptotically hyperbolic and de Sitter spaces. *J. Spectr. Theory*, 4(4):643–673, 2014. MR 3299810

[96] András Vasy. Some recent advances in microlocal analysis. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III*, pages 915–939. Kyung Moon Sa, Seoul, 2014. MR 3729058

[97] András Vasy. Analytic continuation and high energy estimates for the resolvent of the Laplacian on forms on asymptotically hyperbolic spaces. *Adv. Math.*, 306:1019–1045, 2017. MR 3581325

[98] András Vasy. A minicourse on microlocal analysis for wave propagation. In *Asymptotic analysis in general relativity*, volume 443 of *London Math. Soc. Lecture Note Ser.*, pages 219–374. Cambridge Univ. Press, Cambridge, 2018. MR 3792086

[99] András Vasy. Essential self-adjointness of the wave operator and the limiting absorption principle on Lorentzian scattering spaces. *J. Spectr. Theory*, 10(2):439–461, 2020. MR 4107521

[100] Andras Vasy. Limiting absorption principle on Riemannian scattering (asymptotically conic) spaces, a Lagrangian approach. *Comm. Partial Differential Equations* 46:780–822, 2021. MR 4265461

[101] Andras Vasy. Resolvent near zero energy on Riemannian scattering (asymptotically conic) spaces. *Pure Appl. Anal.* 3:1–74, 2021. MR 4265357

[102] Andras Vasy. Resolvent near zero energy on Riemannian scattering (asymptotically conic) spaces, a Lagrangian approach. *Comm. Partial Differential Equations* 46:823–863, 2021. MR 4265462

[103] András Vasy and Michał Wrochna. Quantum Fields from Global Propagators on Asymptotically Minkowski and Extended de Sitter Spacetimes. *Ann. Henri Poincaré*, 19(5):1529–1586, 2018. MR 3784921

[104] C. V. Vishveshwara. Stability of the Schwarzschild Metric. *Phys. Rev. D*, 1:2870–2879, 1970.

[105] Robert M. Wald. Note on the stability of the Schwarzschild metric. *J. Math. Phys.*, 20(6):1056–1058, 1979. MR 0534342

[106] Bernard F. Whiting. Mode stability of the Kerr black hole. *J. Math. Phys.*, 30(6):1301–1305, 1989. MR 0995773

[107] Jared Wunsch and Maciej Zworski. Resolvent estimates for normally hyperbolic trapped sets. *Ann. Henri Poincaré*, 12(7):1349–1385, 2011. MR 2846671

[108] Frank J. Zerilli. Effective Potential for Even-Parity Regge–Wheeler Gravitational Perturbation Equations. *Phys. Rev. Lett.*, 24:737–738, Mar 1970.

[109] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012. MR 2952218

Department of Mathematics, Stanford University, Stanford, CA 94305-2125, U.S.A.

Email address: andras@math.stanford.edu