COMMUNICATIONS IN INFORMATION AND SYSTEMS
Vol. 1, No. 1, pp. 33–50, January 2001

003

INTERNETS IN THE SKY: THE CAPACITY OF THREE DIMENSIONAL WIRELESS NETWORKS*

PIYUSH GUPTA† AND P. R. KUMAR‡

Abstract. Consider n nodes located in a sphere of volume V cubic meters, each capable of transmitting at a rate of W bits/sec. Under a protocol based model for successful receptions, the entire network can carry only $\Theta\left(WV^{\frac{1}{3}}n^{\frac{2}{3}}\right)$ bit-meters/sec, where 1 bit carried a distance of 1 meter is counted as 1 bit-meter. This is the best possible even assuming the node locations, traffic patterns, and the range/power/timing of each transmission, are all optimally chosen.

If the node locations and their destinations are randomly chosen, and all transmissions employ the same power/range, then each node only obtains a throughput of $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec, if the network is optimally operated.

Similar results hold under an alternate physical model where a minimum signal-to-interference ratio is specified for successful receptions.

The proofs of these results require determination of the VC-dimensions of certain geometric sets, which may be of independent interest.
Keywords: Wireless networks, ad hoc networks, multi-hop radio networks, throughput, capacity, transport capacity, Internet-in-the-sky.

1. Introduction. In [1], the capacity of multi-hop wireless networks was analyzed when nodes are located in a disk on the plane. It was shown that when n nodes are randomly and uniformly distributed in a disk of area A m^2, with each node capable of transmitting at W bits/sec and using a fixed range, the throughput obtained by each node for a randomly chosen destination is $\Theta(\frac{W}{\sqrt{n\log n}})$ bits/sec under a non-interference protocol. It was shown that even when node locations, origin-destination pair assignments, and transmission ranges are optimally chosen, the bit-distance product that can be transported in the network is $\Theta(W\sqrt{An})$ bit-meters/sec.

In this paper we obtain the traffic-carrying capacity of three dimensional wireless networks. Such wireless networks arise when the network consists of both terrestrial and satellite-based or aircraft-based communication links, or in building networks where nodes are located on different floors.

Consider n nodes located in a sphere of volume V cubic meters, with each node capable of transmitting at W bits/sec. We show that in the random case where the n nodes are randomly located in the sphere and each node's destination is randomly chosen, the throughput obtained by each node is $\Theta(\frac{W}{(n\log^2 n)^{\frac{1}{3}}})$ bits/sec under a Protocol

*Invited paper; received May 25, 2000; accepted for publication August 28, 2000. This material is based upon work partially supported by the Office of Naval Research under Contract No. N00014-99-1-0696, the Air Force Office of Scientific Research under Contract No. AF-DC-5-36128, by EPRI/DOD Complex Interactive Networks Initiative under contract W08333-04 and the U.S. Army Research Office under Army OSP 35352-6086. Any opinions, findings, and conclusions are those of the authors and do not necessarily reflect the views of the above agencies. Please address all correspondence to the second author at University of Illinois, Coordinated Science Laboratory, 1308 West Main Street, Urbana, IL 61801. Web http://black.csl.uiuc.edu/~prkumar

†Bell Laboratories, Lucent Technologies, 600 Mountain Avenue, Murray Hill, NJ 07974, USA, pgupta@research.bell-labs.com

‡ Dept. of Electrical and Computer Engineering, and Coordinated Science Laboratory, University of Illinois 1308 West Main Street Urbana, IL 61801, USA, prkumar@decision.csl.uiuc.edu

Model of non-interference, if the network is optimally operated. In the best case where the node locations, OD-pair assignments and traffic patterns are optimally chosen, and the network is optimally operated, i.e., the transmission ranges, routes, and schedules of all transmissions are optimal, the entire network can transport $\Theta(WV^{\frac{1}{3}}n^{\frac{2}{3}})$ bit-meters/sec. Thus, even under optimal conditions, the throughput still decreases as $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ bits/sec for each node for a destination nonvanishingly far away.

As in [1], we also consider an alternate Physical Model of non-interference where a required signal-to-interference ratio is specified for successful receptions. Under this model, the lower bounds on the capacity are the same as those above, while the upper bounds on throughput are $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ for the random case, and $\Theta\left(\frac{W}{n^{\frac{1}{\alpha}}}\right)$ for the best case, where α is the signal power path loss exponent.

In both the random and best cases, the capacity of a wireless network is higher when the nodes are located in a sphere, than when they are located in a disk (or on the surface of a sphere). Nevertheless, the throughput obtained by each node still diminishes to zero as the number of nodes in the network is increased. Thus, the implications discussed in [1] continue to hold for 3-D wireless networks. In particular, wireless networks connecting fewer number of users, or allowing connections mostly with nearby neighbors, may be more likely to find acceptance.

While proving the above results, we also determine the VC-dimensions of the following geometric sets: The set of all spheres in $\Re^k$, the set of all discs on the surface of a sphere in $\Re^k$, and the collection of the sets of lines intersecting spheres in $\Re^k$. These results may be of independent interest.

The rest of the paper is organized as follows. In Section 2 we describe the model for Arbitrary 3-D Wireless Networks. In Section 3 we obtain upper bounds on the transport capacity of such networks, which are of the form $cWn^{\frac{2}{3}}$ bit-meters/sec and $c'Wn^{\frac{\alpha-1}{\alpha}}$ bit-meters/sec, under the Protocol and Physical Models, respectively. In Section 4 we show that a transport capacity of $c''Wn^{\frac{2}{3}}$ bit-meters/sec is also feasible for Arbitrary 3-D Networks. In Section 5 we discuss the model for Random 3-D Wireless Networks. In Section 6 we show that $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec and $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ bits/sec are upper bounds on the throughput obtainable by each node in Random 3-D Networks, under the Protocol and Physical Models, respectively. In Section 7 we construct a scheme which provides a throughput of $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec with high probability for Random 3-D Networks.

2. Arbitrary 3-D Networks. In Arbitrary 3-D Networks, n nodes are arbitrarily located in a sphere S of volume V cubic meters. Each node can have traffic to send to an arbitrary destination. Each node can transmit over any subset of M independent channels with capacities $W_1, W_2, \ldots, W_M$ bits/sec, where $\sum_{m=1}^{M} W_m = W$. Each node can use an arbitrary transmission range for each such transmission.

Let X_i, $1 \leq i \leq n$, denote the location of node i (hereafter, we will also use X_i to denote node i itself). Suppose $\{(X_k, X_{R(k)}) : k \in \mathcal{T}\}$ is the set of all active transmitter-receiver pairs at some instant over a certain channel. Then we consider the following two models for successful reception of a transmission over one hop.

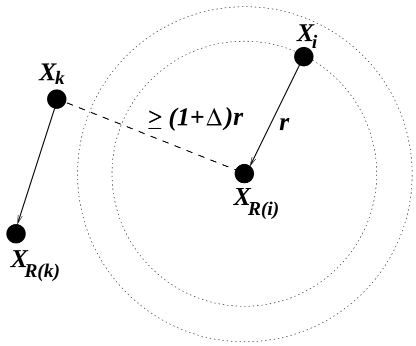

FIG. 2.1. *The Protocol Model.*

The Protocol Model. The transmission from node X_i, $i \in \mathcal{T}$, is successfully received by its intended receiver $X_{R(i)}$ if

$$|X_k - X_{R(i)}| \geq (1+\Delta)|X_i - X_{R(i)}|, \tag{2.1}$$

for every $k \in \mathcal{T} \setminus i$ (Figure 2.1).

The quantity $\Delta > 0$ models situations where a guard zone is specified by the protocol to prevent a neighboring node from transmitting on the same channel at the same time. It also allows for imprecision in the achieved range of transmissions.

The second model that is more related to physical layer considerations is the following:

The Physical Model. Let P_k, $k \in \mathcal{T}$, be the power level at which node X_k transmits. Then the transmission from node X_i, $i \in \mathcal{T}$, is successfully received by its intended receiver $X_{R(i)}$ if

$$\frac{\frac{P_i}{|X_i - X_{R(i)}|^\alpha}}{N + \sum_{\substack{k \in \mathcal{T} \\ k \neq i}} \frac{P_k}{|X_k - X_{R(i)}|^\alpha}} \geq \beta. \tag{2.2}$$

This models a situation where a minimum signal to interference ratio (SIR) of β is necessary for successful receptions, the ambient noise power level is N, and signal power decays with distance r as $\frac{1}{r^\alpha}$. For 3-D wireless networks we will suppose that $\alpha > 3$. The reason is that if $\alpha \leq 3$, and nodes are uniform in space, then the interference level everywhere is unbounded as the number of nodes in the network increases.

Variants of the Protocol Model. The capacity results obtained in this paper also hold for the following two variants of the Protocol Model:

1. Node $X_{R(i)}, i \in \mathcal{T}$, can successfully receive the transmission from node X_i if it does not lie within $(1+\Delta)$ times the range of any other concurrent transmitter, i.e., if, for every $k \in \mathcal{T} \setminus i$,

$$|X_k - X_{R(i)}| \geq (1+\Delta)|X_k - X_{R(k)}|. \tag{2.3}$$

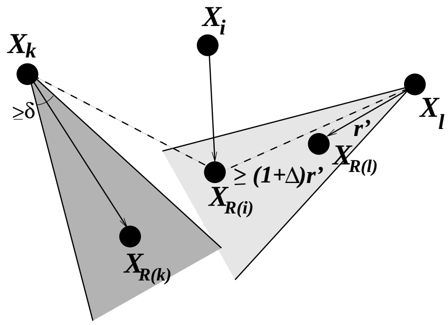

FIG. 2.2. *Variant of the Protocol Model that allows directed transmission of nonvanishing angle of beam spread.*

This models a situation where each transmitter adjusts the power level of its transmitted signal such that the receiver gets the signal at a prespecified power level.

2. Each node can do separate beamforming for each of its transmission. Because of dispersion, however, the angle of beam spread is lower bounded by δ (Figure 2.2). The transmission from node $X_i, i \in \mathcal{T}$, is successfully received by $X_{R(i)}$ if, for each $k \in \mathcal{T} \setminus i$, either $X_{R(i)}$ lies outside the beam spread of the transmission of X_k or (2.3) holds.

The Protocol Model and its variants above satisfy a special property.

LEMMA 2.1. *In all three versions of the Protocol Model, there exists a $\Delta' > 0$, dependent only on Δ and (possibly) δ, such that spheres of radius $\frac{\Delta'}{2}$ times hop length centered at the receivers over the same channel at the same time are essentially disjoint.*

Proof. First consider the original Protocol Model (Figure 2.1). From the triangle inequality and (2.1), the following holds for each $i, k \in \mathcal{T}$

$$\begin{aligned} |X_{R(i)} - X_{R(k)}| &\geq |X_{R(i)} - X_k| - |X_{R(k)} - X_k| \\ &\geq (1+\Delta)|X_i - X_{R(i)}| - |X_{R(k)} - X_k|. \end{aligned}$$

Similarly,

$$|X_{R(k)} - X_{R(i)}| \geq (1+\Delta)|X_k - X_{R(k)}| - |X_{R(i)} - X_i|.$$

Adding the two inequalities, we obtain

$$|X_{R(k)} - X_{R(i)}| \geq \frac{\Delta}{2} \left(|X_k - X_{R(k)}| + |X_i - X_{R(i)}|\right).$$

Thus spheres of radius $\frac{\Delta}{2}$ times the lengths of hops centered at the receivers over the same channel in the same slot are essentially disjoint.

Next consider the first variant of the Protocol Model. Again from the triangle inequality, the following holds for each $i, k \in \mathcal{T}$

$$\begin{aligned} |X_{R(i)} - X_{R(k)}| &\geq |X_{R(i)} - X_k| - |X_{R(k)} - X_k| \\ &\geq (1+\Delta)|X_k - X_{R(k)}| - |X_{R(k)} - X_k| \\ &= \Delta\,|X_k - X_{R(k)}|. \end{aligned}$$

Similarly,

$$|X_{R(k)} - X_{R(i)}| \geq \Delta\,|X_i - X_{R(i)}|.$$

Hence, here too, spheres of radius $\frac{\Delta}{2}$ times the lengths of hops centered at the concurrent receivers are essentially disjoint.

Finally, consider the second variant of the Protocol Model that allows directed transmissions. For every $k \in \mathcal{T} \setminus i$, either (2.3) holds, in which case the triangle inequality gives

$$|X_{R(i)} - X_{R(k)}| \geq \Delta |X_k - X_{R(k)}|,$$

or $X_{R(i)}$ lies outside the beam spread of X_k, in which case

$$|X_{R(i)} - X_{R(k)}| \geq \sin\frac{\delta}{2} |X_k - X_{R(k)}|.$$

Thus

$$|X_{R(i)} - X_{R(k)}| \geq \min\left\{\Delta,\, \sin\frac{\delta}{2}\right\} |X_k - X_{R(k)}|.$$

Similarly,

$$|X_{R(k)} - X_{R(i)}| \geq \min\left\{\Delta,\, \sin\frac{\delta}{2}\right\} |X_i - X_{R(i)}|.$$

Hence we have that spheres of radius $\frac{1}{2}\min\left\{\Delta,\, \sin\frac{\delta}{2}\right\}$ times hop length centered at the concurrent receivers are essentially disjoint. □

To give a unified treatment for the three variants of the Protocol Model in the following, we will assume that $\Delta' = \Delta$. This may require redefining Δ in the second variant of the Protocol Model.

3. Arbitrary 3-D Networks: An Upper Bound on Transport Capacity. Given a set of successful transmissions, we say that the network transports one *bit-meter* when one bit has been transported over a distance of one meter towards its destination. The *transport capacity* of a network is defined as the supremum of the bit-distance product that can be transported by the entire network per second.

Suppose an Arbitrary Network transports a total of $\lambda n T$ bits in T seconds. Suppose the average distance between the source and the destination of a bit is $\bar{L}$ meters. In other words, the network achieves a transport capacity of $\lambda n \bar{L}$ bit-meters/sec. Then, the following holds:

THEOREM 3.1. *i) In the Protocol Model, the transport capacity $\lambda n \bar{L}$ of any Arbitrary 3-D Wireless Network is bounded by*

$$\lambda n \bar{L} \leq 2\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \frac{1}{\Delta} W n^{\frac{2}{3}} \quad \textit{bits-meters/sec.}$$

(ii) In the Physical Model,

$$\lambda n \bar{L} \leq \left(\frac{2\beta + 2}{\beta}\right)^{\frac{1}{\alpha}} \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} W n^{\frac{\alpha-1}{\alpha}} \quad \textit{bit-meters/sec.}$$

(iii) If the ratio $\frac{P_{max}}{P_{min}}$ between the maximum and minimum powers that transmitters can employ is strictly bounded above by β, then

$$\lambda n \bar{L} \leq 2 \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} \frac{1}{(\frac{\beta P_{min}}{P_{max}})^{\frac{1}{\alpha}} - 1} W n^{\frac{2}{3}} \text{ bit-meters/sec.}$$

Proof. Let $h(b)$ be the number of hops taken by bit b for $1 \leq b \leq \lambda n T$, and r_b^h be the distance traversed by b in hop h. Then

$$\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} r_b^h \geq \lambda n T \bar{L}. \tag{3.1}$$

For simplicity in exposition, suppose that transmissions in the network are slotted into synchronized slots of length τ secs. Then, in any slot s, at most $n/2$ nodes can transmit over any channel m. Hence, we have

$$\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} 1(\text{The } h\text{-th hop of bit } b \text{ is over channel } m \text{ in slot } s) \leq \frac{W_m \tau n}{2}.$$

Summing over the channels and the slots, and noting that there can be no more than $\frac{T}{\tau}$ slots in T secs, we get

$$H := \sum_{b=1}^{\lambda n T} h(b) \leq \frac{WTn}{2}. \tag{3.2}$$

Consider now the Protocol Model. Let $\{(X_k, X_{R(k)}) : k \in \mathcal{T}_m(s)\}$ denote the set of all active transmitter-receiver pairs in slot s over channel m. From Lemma 2.1, we have that spheres of radius $\frac{\Delta}{2}$ times the lengths of hops centered at the receivers $\{X_{R(k)} : k \in \mathcal{T}_m(s)\}$ are essentially disjoint. Taking edge effects into account and noting that a range greater than the diameter is unnecessary, we deduce that at least a quarter of such a sphere is within sphere S. Since at most $W_m \tau$ bits can be carried in slot s from a transmitter to a receiver over the channel m, we have

$$\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} 1(\text{The } h\text{-th hop of bit } b \text{ is over channel } m \text{ in slot } s) \frac{1}{4} \frac{4\pi}{3} \left(\frac{\Delta r_b^h}{2}\right)^3 \leq W_m \tau V. \tag{3.3}$$

Summing over the channels and the slots, we get

$$\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} \frac{\pi \Delta^3}{24} (r_b^h)^3 \leq WTV,$$

which can be rewritten as

$$\sum_{b=1}^{\lambda n T} \sum_{h=1}^{h(b)} \frac{1}{H} (r_b^h)^3 \leq \frac{24WTV}{\pi \Delta^3 H}. \tag{3.4}$$

Now x^3 is a convex function over $x \geq 0$. Hence

$$(3.5)\qquad (\sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)}\frac{1}{H}r_b^h)^3 \leq \sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)}\frac{1}{H}(r_b^h)^3.$$

Combining (3.4) and (3.5) yields,

$$(3.6)\qquad \sum_{b=1}^{\lambda nT}\sum_{h=1}^{h(b)} r_b^h \leq \left(\frac{24WTVH^2}{\pi\Delta^3}\right)^{\frac{1}{3}}.$$

Now substituting (3.1) in (3.6) gives,

$$(3.7)\qquad \lambda nT\bar{L} \leq \left(\frac{24WTVH^2}{\pi\Delta^3}\right)^{\frac{1}{3}}.$$

Substituting (3.2) in (3.7) yields the result.

Proofs of (ii) and (iii) proceed along similar lines as in Theorem 2.1 [1], with the only difference arising due to the diameter $2\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$ of sphere S. □

4. Arbitrary 3-D Networks: A Constructive Lower Bound on Transport Capacity. We will now show that the $O(n^{\frac{2}{3}})$ order of the upper bound on the transport capacity in the previous section is tight, by exhibiting a scenario where it is achieved.

THEOREM 4.1. *Nodes can be placed in sphere S, and traffic patterns assigned, such that the network can achieve $\frac{WV^{\frac{1}{3}}}{1+2\Delta}n^{\frac{2}{3}}$ bit-meters/sec under the Protocol Model, and $\frac{WV^{\frac{1}{3}}}{\left(16\beta\left(21+2^{\frac{\alpha}{2}}+\frac{4\alpha-2}{\alpha-3}\right)\right)^{\frac{1}{\alpha}}}n^{\frac{2}{3}}$ bit-meters/sec under the Physical Model.*

Proof. First consider the Protocol Model. Let $r = \frac{V^{\frac{1}{3}}}{1+2\Delta}\frac{2}{n^{\frac{1}{3}}}$. With the center of sphere S taken as the origin, place transmitters at locations $(j(1+2\Delta)r \pm \Delta r, k(1+2\Delta)r, l(1+2\Delta)r)$, $(j(1+2\Delta)r, k(1+2\Delta)r \pm \Delta r, l(1+2\Delta)r)$ and $(j(1+2\Delta)r, k(1+2\Delta)r, l(1+2\Delta)r \pm \Delta r)$ where $|j+k+l|$ is odd. Also place receivers at $(j(1+2\Delta)r \pm \Delta r, k(1+2\Delta)r, l(1+2\Delta)r)$, $(j(1+2\Delta)r, k(1+2\Delta)r \pm \Delta r, l(1+2\Delta)r)$ and $(j(1+2\Delta)r, k(1+2\Delta)r, l(1+2\Delta)r \pm \Delta r)$, where $|j+k+l|$ is even. Each transmitter can transmit to its nearest receiver, which is at a distance r away, without interference from any other transmitter. Furthermore, the above allows for $\frac{n}{2}$ transmitter–receiver pairs to be placed within S. Under this placement of nodes, there are a total of $\frac{n}{2}$ concurrent transmissions, each of range r, and each at W bits/sec. This achieves the transport capacity indicated.

For the Physical Model, a calculation of the SIR under the above placement shows that it is lower bounded at all receivers by $\frac{(1+2\Delta)^\alpha}{16\left(21+2^{\frac{\alpha}{2}}+\frac{4\alpha-2}{\alpha-3}\right)}$. Choosing Δ to make this lower bound equal to β yields the result. □

5. Random 3-D Networks. Above we have determined the best case behavior where nodes can be optimally placed and traffic patterns optimally designed. We now address a scenario when the network itself is random.

In a random scenario, there are n nodes uniformly and independently distributed in a sphere S of volume 1 cubic meter. Each node sends data at $\lambda(n)$ bits/sec to a randomly chosen destination node. This destination node is picked as follows. A uniformly and independently distributed point in S is chosen, and the node nearest to this location is chosen as the destination node. Thus, the average separation between source-destination pairs is on the order of 1 meter.

In this random setting, we assume that all transmissions employ the same nominal range r or power level P. As in Arbitrary 3-D Networks, we consider two models for successful reception of a transmission.

The Protocol Model. All nodes employ a common *range* r for all their transmissions. Let $\{(X_k, X_{R(k)}) : k \in \mathcal{T}\}$ be the set of all active transmitter-receiver pairs at some time instant over a certain channel. Then transmission from $X_i, i \in \mathcal{T}$, is successfully received by $X_{R(i)}$ if:
(i) The distance between X_i and $X_{R(i)}$ is no more than r, i.e.,

$$|X_i - X_{R(i)}| \leq r. \tag{5.1}$$

(ii) For every other node X_k, $k \in \mathcal{T}$,

$$|X_k - X_{R(i)}| \geq (1+\Delta)r. \tag{5.2}$$

As for Arbitrary Networks, the capacity results obtained in the following also hold under two variants of the model above. These variants are similar to those given in Section 2 except that all nodes now employ a common range r, which may however depend on n, the number of nodes in the network.

The Physical Model. All nodes transmit at a common power level P. A transmission from a node X_i, $i \in \mathcal{T}$, is successfully received by node $X_{R(i)}$ if

$$\frac{\frac{P}{|X_i - X_{R(i)}|^\alpha}}{N + \sum_{\substack{k\in\mathcal{T}\\ k\neq i}} \frac{P}{|X_k - X_{R(i)}|^\alpha}} \geq \beta. \tag{5.3}$$

We say that the *throughput capacity* of Random 3-D Wireless Networks is of order $\Theta(f(n))$ bits/sec if there are deterministic constants $c > 0$ and $c' < +\infty$ such that

$$\lim_{n\to\infty} \text{Prob}(\lambda(n) = cf(n) \text{ is feasible}) = 1,$$
$$\lim_{n\to\infty} \text{Prob}(\lambda(n) = c'f(n) \text{ is feasible}) = 0.$$

We next obtain an $O\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ upper bound on the throughput capacity of random 3-D wireless networks. In Section 7 we will construct a scheme which achieves the same order of throughput capacity.

6. Random 3-D Networks: An Upper Bound on Throughput Capacity. An essential requirement for any positive throughput level to be feasible is that there exists a path between each node and its chosen destination. In particular, every node must have at least one node in its range with which it can communicate. We therefore first obtain a necessary condition on the transmission range $r(n)$ such that every node in the network has at least one node in its range with probability approaching one as the number of nodes $n \to +\infty$.

We recall two results from [2].

LEMMA 6.1. *(i) For any* $p \in [0,1]$

$$(1-p) \ \leq \ e^{-p}.$$

(ii) For any given $\theta \geq 1$, *there exists* $p_0 \in [0,1]$, *such that*

$$e^{-\theta p} \ \leq \ (1-p), \quad \textit{for all} \ \ 0 \leq p \leq p_0.$$

If $\theta > 1$, *then* $p_0 > 0$.

LEMMA 6.2. *If* $A(n) = \frac{\log n + \kappa}{n}$, *then, for any fixed* $\theta < 1$ *and for all sufficiently large* n

$$n(1-A(n))^{n-1} \geq \theta e^{-\kappa}.$$

Given the n nodes in S, denote by $\mathcal{G}(n, r(n))$ the graph which results from connecting by an edge nodes separated by a distance of at most $r(n)$. Let $P^{(1)}(n, r(n))$ denote the probability that a graph $\mathcal{G}(n, r(n))$ has at least one isolated node. Then the following holds.

LEMMA 6.3. *If the range* $r(n)$ *of each transmission is such that* $\frac{4\pi}{3} r^3(n) = \frac{\log n + \kappa_n}{n}$ *with* $\lim_{n\to\infty} \kappa_n = \kappa < +\infty$, *then*

$$\liminf_{n\to\infty} P^{(1)}(n, r(n)) \geq e^{-\kappa}\left(1 - e^{-\kappa}\right).$$

Proof. In the following, we will neglect edge effects. A proof similar to the one in [2] can be given to show that the edge effects do not alter the result.

Consider first the case where $\frac{4\pi}{3} r^3(n) = \frac{\log n + \kappa}{n}$ for a fixed κ. The probability that $\mathcal{G}(n, r(n))$ has at least one isolated node, satisfies

$$\begin{aligned} P^{(1)}(n, r(n)) \geq\ & nP(\{i \text{ is isolated in } \mathcal{G}(n, r(n))\}) - \\ & n(n-1)P(\{i \text{ and } j \text{ are isolated in } \mathcal{G}(n, r(n))\}). \end{aligned} \tag{6.1}$$

Let $V(r) := \frac{4\pi}{3} r^3$ be the volume of a sphere of radius r. Neglecting edge effects

$$P(\{\text{Node } i \text{ is isolated in } \mathcal{G}(n, r(n))\}) \sim (1 - V(r(n)))^{n-1}, \tag{6.2}$$

$$\begin{aligned} P(\{\text{Nodes } i \text{ and } j \text{ are isolated in } \mathcal{G}(n, r(n))\}) < (V(2r(n)) - V(r(n))) \cdot \\ (1 - \frac{3}{2} V(r(n)))^{n-2} + \ (1 - V(2r(n)))(1 - 2V(r(n)))^{n-2}, \end{aligned} \tag{6.3}$$

where the first term on the RHS above takes into account the case where the distance between i and j is between $r(n)$ and $2r(n)$. Substituting (6.2) and (6.3) in (6.1), and using the definition of $V(r)$, we get

$$\begin{aligned} P^{(1)}(n, r(n)) \geq n(1 - \frac{4\pi}{3} r^3(n))^{n-1} - n(n-1)\Big(\frac{28\pi}{3} r^3(n)(1 - \frac{3}{2}\frac{4\pi}{3} r^3(n))^{n-2} \\ + (1 - 2\frac{4\pi}{3} r^3(n))^{n-2}\Big). \end{aligned}$$

Using Lemmas 6.1 and 6.2, for $\frac{4\pi}{3}r^3(n) = \frac{\log n + \kappa}{n}$, and any fixed $\theta < 1$ and $\epsilon > 0$, we have

$$P^{(1)}(n, r(n)) \geq \theta e^{-\kappa} - n(n-1)\Big(\frac{28\pi}{3}r^3(n)e^{-(n-2)\frac{3}{2}\frac{4\pi}{3}r^3(n)} + e^{-(n-2)2\frac{4\pi}{3}r^3(n)}\Big)$$
$$\geq \theta e^{-\kappa} - (1+\epsilon)e^{-2\kappa} \text{ for all } n > N(\epsilon, \theta, \kappa).$$

Now consider the case where the range $r(n)$ satisfies $\frac{4\pi}{3}r^3(n) = \frac{\log n + \kappa_n}{n}$, with $\lim_{n\to\infty} \kappa_n = \kappa < +\infty$. We have to replace κ by κ_n in the above. Note that for any $\epsilon > 0$, $\kappa_n \leq \kappa + \epsilon$ for all $n \geq N'(\epsilon)$. Also, the probability of an isolated node is monotone decreasing in κ. Hence

$$P^{(1)}(n, r(n)) \geq \theta e^{-(\kappa+\epsilon)} - (1+\epsilon)e^{-2(\kappa-\epsilon)}$$

for $n \geq \max\{N(\epsilon, \theta, \kappa+\epsilon), N'(\epsilon)\}$. Taking limits,

$$\liminf_{n\to\infty} P^{(1)}(n, r(n)) \geq \theta e^{-(\kappa+\epsilon)} - (1+\epsilon)e^{-2(\kappa-\epsilon)}.$$

Since this holds for all $\epsilon > 0$ and $\theta < 1$, and since $P^{(1)}(n, r(n)) \leq P_d(n, r(n))$, the result follows. □

The following corollary is immediate.

COROLLARY 6.1. *The asymptotic probability that graph $\mathcal{G}(n, r(n))$ has an isolated node is strictly positive if $\frac{4\pi}{3}r^3(n) = \frac{\log n + \kappa_n}{n}$ and $\limsup_n \kappa_n < +\infty$.*

As in (3.3), under the non-interference protocol requirement, spheres of radius $\frac{\Delta}{2}r(n)$ around concurrent receivers on a channel are disjoint. Hence there can be at most $R(n) = \frac{1}{(1/4)(4/3\pi(\Delta r(n)/2)^3)}$ concurrent receptions on any channel. Thus the total transmission rate in the network at any time can be at most $WR(n)$ bits/sec. On the other hand, if $\bar{L}$ denotes the mean length of the random line from a source to its destination, then the total traffic in the network is at least $n\lambda(n)\frac{\bar{L}-o(1)}{r(n)}$ bits/sec, since $\frac{\bar{L}-o(1)}{r(n)}$ is a lower bound on the mean number of hops traversed by a packet. For stability, we therefore need

$$\frac{(\bar{L} - o(1))n\lambda(n)}{r(n)} \leq \frac{24W}{\pi\Delta^3 r^3(n)}.$$

Thus,

$$\lambda(n) \leq \frac{24W}{\pi\Delta^3(\bar{L} - o(1))nr^2(n)}.$$

From Corollary 6.1, $r(n) > (\frac{3\log n}{4\pi n})^{\frac{1}{3}}$ is necessary to ensure that every node has at least one node in its range. Hence we obtain the following:

THEOREM 6.1. *For Random 3-D Wireless Networks under the Protocol Model, there is a deterministic constant $c' < +\infty$, not depending on n, Δ or W, such that*

$$\lim_{n\to\infty} Prob\Big(\lambda(n) = \frac{c'W}{\Delta^3(n\log^2 n)^{\frac{1}{3}}} \textit{ bits/sec is feasible}\Big) = 0.$$

For the Physical Model, the following upper bound follows from Theorem 3.1 and an argument similar to that used in Theorem 5.2 of [1].

THEOREM 6.2. *For Random 3-D Networks under the Physical Model, there is a deterministic sequence $\epsilon(n) \to 0$, not depending on N, α, β or W, such that*

$$\lim_{n\to\infty} Prob\left(\lambda(n) \;=\; 2\left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \frac{W}{\bar{L}(\beta^{\frac{1}{\alpha}}-1)} \frac{1+\epsilon(n)}{n^{\frac{1}{3}}} \;\; \textit{bits/sec is feasible}\right) = 0.$$

7. Random 3-D Networks: A Constructive Lower Bound on Throughput Capacity. In this section we describe a constructive scheduling and routing scheme similar to the one in [1], which shows that in a random 3-D wireless network each node can obtain a throughput of $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec with high probability. For this purpose, we need to compute the *VC-dimensions* of certain geometric sets.

Let $\mathcal{F}$ be a collection of sets. A set A is said to be *shattered* by $\mathcal{F}$ if for every $B \subseteq A$ there is a set $F \in \mathcal{F}$ such that $A \cap F = B$. The *VC-dimension* of $\mathcal{F}$, denoted by VC-dim($\mathcal{F}$), is defined as the largest integer m such that there exists a set A of cardinality m that is shattered by $\mathcal{F}$ [3, 4].

For a collection of sets with finite VC-dimension, the following uniform convergence in the weak law of large numbers holds:

THEOREM 7.1. **The Vapnik–Chervonenkis Theorem.** *If $\mathcal{F}$ is a collection of sets of finite VC-dimension, VC-dim($\mathcal{F}$), and $\{X_j\}$ is an i.i.d. sequence with common probability distribution P, then for every $\epsilon, \delta > 0$,*

$$Prob(\sup_{F\in\mathcal{F}} |\frac{1}{N}\sum_{j=1}^{N} I(X_j \in F) - P(F)| \le \epsilon) > 1-\delta,$$

whenever

$$N > \max\{\frac{8\,\textit{VC-dim}(\mathcal{F})}{\epsilon}\log\frac{16e}{\epsilon}, \frac{4}{\epsilon}\log\frac{2}{\delta}\}.$$

Above $I(\cdot)$ is the indicator function.

We next determine the VC-dimensions of certain collections of sets associated with our constructive scheme.

THEOREM 7.2. *Let $\mathcal{S}(k)$ be the set of all spheres in $\Re^k$. Then VC-dim($\mathcal{S}(k)$) = $k+1$.*

We prove the result through a sequence of lemmas. We first obtain a close upper bound on VC-dim($\mathcal{S}(k)$) by mapping the problem into one for which the solution is already known.

LEMMA 7.1. *VC-dim($\mathcal{S}(k)$) $\le k+2$.*

Proof. Let S be a sphere in $\Re^k$ of radius r and centered at $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$. Then

$$\begin{aligned} S &= \{x \in \Re^k : |x-\bar{x}| \le r\} \\ &= \{x \in \Re^k : \sum_{i=1}^{k} x_i^2 - 2\sum_{i=1}^{k}\bar{x}_i x_i \le r^2 - \sum_{i=1}^{k}\bar{x}_i^2 =: \bar{r}\}. \end{aligned} \tag{7.1}$$

We now map the sphere S in $\Re^k$ to the affine halfspace H in $\Re^{k+1}$ given by $\{(x_1, x_2, \ldots, x_k, x_{k+1}) \in \Re^{k+1} : x_{k+1} - 2\sum_{i=1}^{k}\bar{x}_i x_i \le \bar{r}\}$. Note that $x \in \Re^k$ belongs to

S if and only if $(x, |x|^2) \in \Re^{k+1}$ belongs to H. Hence, if a set of p points, say $\{x^{(1)}, x^{(2)}, \dots, x^{(p)}\} \subset \Re^k$, is shattered by $\mathcal{S}(k)$, then the set of points $\{(x^{(1)}, |x^{(1)}|^2), (x^{(2)}, |x^{(2)}|^2), \dots, (x^{(p)}, |x^{(p)}|^2)\} \subset \Re^{k+1}$ is shattered by $\mathcal{H}(k+1)$, the collection of all affine halfspaces in $\Re^{k+1}$. It is known that the VC-dimension of $\mathcal{H}(l)$ is $l+1$ [3]. Hence $p \le k+2$, which proves the result. $\square$

The above argument can also be used to show finiteness of VC-dimensions of more general collections of sets. For instance, using the same reasoning we can deduce that the collection of all closed sets in $\Re^k$ enclosed by order-l polynomials, has a finite VC-dimension (in fact, upper bounded by $(k+1)^l$). More generally, we have the following result, which is not only of independent interest, but is also specifically useful in what follows.

THEOREM 7.3. *Let $\mathcal{B}$ be some subset of $\Re^k$. The VC-dimension of the set of closed sets in $\Re^k$ given by $\{\{x \in \mathcal{B} : \sum_{i=1}^{m} a_i f_i(x) \le b\} : (a_1, a_2, \dots, a_m, b) \in \Re^{m+1}\}$, where the mapping $x \in \mathcal{B} \mapsto (f_1(x), f_2(x), \dots, f_m(x)) \in \Re^m$ is one-to-one, is at most $m+1$.*

We next obtain a lower bound on the VC-dimension of $\mathcal{S}(k)$.

LEMMA 7.2. *VC-dim$(\mathcal{S}(k)) \ge k+1$.*

Proof. As mentioned above, VC-dim$(\mathcal{H}(k)) = k+1$, i.e., there exists a set of $k+1$ points, $\{x^{(1)}, x^{(2)}, \dots, x^{(k+1)}\}$, which is shattered by the set of all affine halfspaces in $\Re^k$. Clearly, these points are in "general" positions: for each $2 \le l \le k$, any subset of $l+1$ points does not lie in an $(l-1)$-dimensional affine subspace of $\Re^k$. Hence there exits a unique sphere, say S_0, whose boundary includes these $k+1$ points. Let the center of S_0 be $x^{(0)}$ and the radius be r_0. Now choose a new coordinate system in which the origin is $x^{(0)}$. For simplicity in notation, continue to denote the $k+1$ points by $x^{(1)}, x^{(2)}, \dots, x^{(k+1)}$. Now consider an affine halfspace H_A which contains $A \subseteq \{x^{(1)}, x^{(2)}, \dots, x^{(k+1)}\}$, but not $\{x^{(1)}, x^{(2)}, \dots, x^{(k+1)}\} \setminus A$. Specifically, suppose H_A is given by $\{x \in \Re^k : xa^T \le b\}$. Now define the sphere $S_A \subset \Re^k$ by $S_A := \{x : xx^T + 2xa^T + aa^T \le r_0^2 + 2b + aa^T\}$. Then, since $xx^T = r_0^2$ for $x \in S_0$, $x \in H_A \cap S_0$ if and only if $x \in S_A \cap S_0$. Hence, $\{x^{(1)}, x^{(2)}, \dots, x^{(k+1)}\}$ is shattered by the set of all spheres in $\Re^k$, which proves the result. $\square$

Note that the intersection of an affine halfspace in $\Re^k$ with a spherical shell S^{k-1} (i.e., the boundary of a sphere in $\Re^k$) is a disk. Conversely, given a disk D on S^{k-1}, there exists an affine halfspace in $\Re^k$ such that its intersection with S^{k-1} is D. Hence, the above argument also leads to the following:

THEOREM 7.4. *The VC-dimension of the set of all disks on S^{k-1} is $k+1$.*

In order to complete the proof of Theorem 7.2, we next close the gap between the upper and lower bounds given in Lemmas 7.1 and 7.2.

LEMMA 7.3. *The VC-dimension of the set of all affine halfspaces in $\Re^k$ of the type $\{x \in \Re^k : x(1, a_2, \dots, a_k)^T \le b\}$, denoted by $\mathcal{H}_{\frac{1}{2}}(k)$, is k.*

Proof. By restricting to the subspace $\{x_1 = 0\}$, it is clear that the VC-dimension of $\mathcal{H}_{\frac{1}{2}}(k)$ is at least k. To prove that it is exactly k, we observe that $\{x^{(1)}, x^{(2)}, \dots, x^{(p)}\}$ is shattered by $\mathcal{H}_{\frac{1}{2}}(k)$ if and only if the set of p halfspaces in the normal space $\{\{(a_1, a_2, \dots, a_k, -b) \in \Re^{k+1} : (x^{(i)}, 1)(a_1, a_2, \dots, a_k, -b)^T \le 0\}, 1 \le i \le p\}$

partition the k-dimensional affine subspace $\{a_1 = 1\}$ into 2^p distinct cells. Now it is well known (see for example [5]) that p hyperplanes in general position partition $\Re^d$ into $C_d(p)$ cells, where

$$C_d(p) = \sum_{i=0}^{d} \binom{p}{i}. \tag{7.2}$$

For $d = k$ and $p = k + 1$, (7.2) gives that $C_d(p) = 2^{k+1} - 1 < 2^p$. Hence the VC-dimension of $H_{\frac{1}{2}}(k)$ can be at most k, which completes the proof of the result. □

Next let $\mathcal{S}(k)$ be the set of all spheres in $\Re^k$. Also, let $\mathcal{L}(S)$ denote the set of all lines in $\Re^k$ which intersect a sphere $S \in \mathcal{S}(k)$. Then the following holds.

THEOREM 7.5. *VC-dim*$(\{\mathcal{L}(S) : S \in \mathcal{S}(k)\}) \leq \frac{k^2+3k+4}{2}$.

Proof. Let S be a sphere in $\Re^k$ of radius r and centered at $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k)$. Given a line L in $\Re^k$, there exist $\hat{x}, d \in \Re^k$ such that

$$\begin{aligned} |d| &= 1, \\ \hat{x} d^T &= 0, \\ L &= \{\hat{x} + l d : l \in \Re\}. \end{aligned} \tag{7.3}$$

Now $L \in \mathcal{L}(S)$ if and only if the distance between L and $\bar{x}$ is at most r, i.e.,

$$\begin{aligned} & \min_{l \in \Re} |\bar{x} - (\hat{x} + l d)| \leq r, \\ \Leftrightarrow \quad & \min_{l \in \Re} |(\bar{x} - \hat{x}) - l d|^2 \leq r^2, \\ \Leftrightarrow \quad & |\bar{x} - \hat{x}|^2 - \left(\bar{x} d^T\right)^2 \leq r^2. \end{aligned} \tag{7.4}$$

Thus, $\mathcal{L}(S) = \{(\hat{x}, d) : \sum_{i=1}^{k} \bar{x}_i^2 f_i(\hat{x}, d) + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \bar{x}_i \bar{x}_j g_{ij}(\hat{x}, d) + \sum_{i=1}^{k} \bar{x}_i h_i(\hat{x}, d) + l(\hat{x}, d) \leq r^2\}$, where

$$\begin{aligned} f_i(\hat{x}, d) &= 1 - d_i^2, \\ g_{ij}(\hat{x}, d) &= -2 d_i d_j, \\ h_i(\hat{x}, d) &= -2 \hat{x}_i, \\ l(\hat{x}, d) &= |\hat{x}|^2. \end{aligned}$$

Clearly, distinct lines lead to distinct (f_i, g_{ij}, h_i, l) vectors. The result now follows from Theorem 7.3. □

We are now ready to show that a constructive scheme similar to the one in [1] achieves the same order of throughput capacity as upper bounded by Theorem 6.1.

THEOREM 7.6. *(i) For Random 3-D Wireless Networks under the Protocol Model, there is a deterministic constant $c > 0$ not depending on n, Δ or W, such that*

$$\lambda(n) = \frac{c}{(1+\Delta)^3} \frac{W}{(n \log^2 n)^{\frac{1}{3}}} \quad \textit{bits/sec}$$

is feasible with probability approaching 1 as $n \to \infty$.

(ii) For Random 3-D Wireless Networks under the Physical Model, there is a deterministic constant $c' > 0$ not depending on n, α, β or W, such that

$$\lambda(n) = \frac{c'}{\left(2\left(18\beta\left(3+\frac{1}{\alpha-1}+\frac{1}{\alpha-2}+\frac{1}{\alpha-3}\right)\right)^{\frac{1}{\alpha}}-1\right)^3}\frac{W}{(n\log^2 n)^{\frac{1}{3}}} \quad \textit{bits/sec}$$

is feasible with probability approaching 1 as $n \to \infty$.

Proof. The proof proceeds along lines similar to the one given in [1] for the 2-D case. In the following we give the main steps involved in the proof without repeating the detailed arguments of [1], except where they do not exactly carry over to the 3-D case.

First consider the Protocol Model.

- Construct a Voronoi tessellation $\mathcal{V}_n$ of sphere S (similar to the one in Section IV.A of [1], but now in 3-D) such that each cell V contains a ball of radius $\rho(n)$ and is contained in a ball of radius $2\rho(n)$, where $\rho(n)$ is such that $\frac{4\pi}{3}\rho^3(n) = \frac{100\log n}{n}$.
- Choose the range of each node to be $r(n) = 8\rho(n)$.
- This range allows any two nodes in neighboring cells to directly communicate (Lemma 4.2 in [1]).
- As in Lemma 4.3 [1], every cell in $\mathcal{V}_n$ has at most a constant number $c_1 = O((1+\Delta)^3)$ of interfering neighbors.
- Using a result from graph theory on vertex coloring, the above allows us to construct a transmission schedule such that each cell gets to transmit at least once in every (c_1+1) consecutive slots (Lemma 4.4(i) [1]).
- Let $\{Y_i\}_{i=1}^n$ be independently and uniformly distributed (i.i.d.) points in S chosen independently of $\{X_i\}_{i=1}^n$. The destination node $X_{\text{dest}(i)}$ for the traffic generated at source node X_i is chosen as the node closest to Y_i.
- Let L_i be the line joining X_i and Y_i. Then, $\{L_i\}_{i=1}^n$ are i.i.d.
- Packets originating at source node X_i are routed to their destination node $X_{\text{dest}(i)}$ to follow L_i. That is, the packets from X_i are relayed from one cell to another in the order in which the cells intersect L_i. On reaching the cell containing Y_i, the packets are sent on to their final destination $X_{\text{dest}(i)}$, which is within one hop of Y_i with high probability.
- By Theorem 7.2, the VC-dimension of the set of all spheres in $\Re^3$ is 4. Hence, as in Lemma 4.8 [1], every cell in the Voronoi tessellation $\mathcal{V}_n$ contains at least one node with probability approaching one as $n \to +\infty$.
- We next obtain a uniform bound on the amount of traffic that needs to be carried by each cell V of $\mathcal{V}_n$. For this, we first bound the expected number

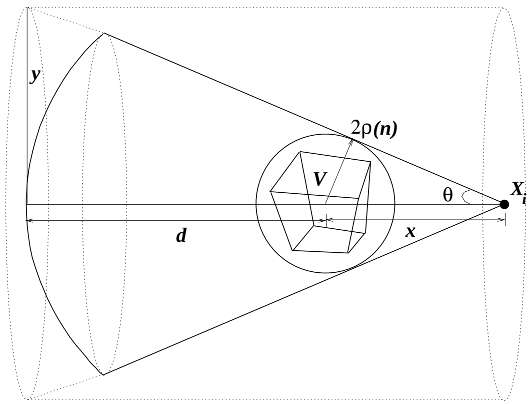

FIG. 7.1. *Computing the probability that a line L_i intersects cell V. Note that $d + x \leq 2r_0$.*

of lines intersecting cell V as follows

$$
\begin{aligned}
&E[\text{Number of lines in } \{L_i\}_{i=1}^{n} \text{ intersecting cell } V] \\
&= n \text{ Prob(Line } L_i \text{ intersects } V) \\
&\leq n\Big(\text{Prob}(d(X_i, V) \leq c_0\rho(n)) + \text{ Prob}(L_i \text{ intersects } V \;\Big|\; d(X_i, V) > c_0\rho(n))\Big) \\
&\leq n\Bigg(\frac{4\pi}{3}((c_0+2)\rho(n))^3 + \\
(7.5)\quad &\int_{(c_0+1)\rho(n)}^{2r_0} \text{Prob}(L_i \text{ intersects } V \;\Big|\; |X_i - c(V)| = x)\; 4\pi x^2 dx\Bigg),
\end{aligned}
$$

where we have used the facts that V contains a ball of radius $\rho(n)$ and is contained in a ball of radius $2\rho(n)$, c_0 is a constant to be specified later, $c(V)$ is the center of the ball containing the cell V, and $r_0 = (3/(4\pi))^{\frac{1}{3}}$ is the radius of the unit-volume sphere S. For the conditional probability in the second term, we upper bound the volume of the cone at X_i and touching the ball of radius $2\rho(n)$ containing V, by that of a cylinder as shown in Figure 7.1.

$$
\begin{aligned}
\text{Prob}(L_i \text{ intersects } V \;\Big|\; |X_i - c(V)| = x) &\leq \pi y^2 (d+x) \\
&\leq \pi\Big((d+x)\tan\theta\Big)^2 (d+x) \\
(7.6)\quad &\leq \pi\left(2r_0 \cdot 2\frac{2\rho(n)}{x}\right)^2 2r_0,
\end{aligned}
$$

for sufficiently large c_0. Substituting (7.6) in (7.5), we get

$$
\begin{aligned}
&E[\text{Number of lines in } \{L_i\}_{i=1}^{n} \text{ intersecting a cell } V] \\
&\leq n\left(\frac{4\pi}{3}((c_0+2)\rho(n))^3 + \pi(4\rho(n))^2 \cdot 4\pi(2r_0)^4\right) \\
&\leq n c_2 \rho^2(n) \\
(7.7)\quad &\leq c_3(n \log^2 n)^{\frac{1}{3}},
\end{aligned}
$$

for some constants c_2 and c_3, and sufficiently large n.
Next, from Theorem 7.5, the Vapnik-Chervonenkis Theorem is applicable to the collection of sets of lines intersecting spheres in $\Re^3$, $\{\mathcal{L}(S) : S \in \mathcal{S}(3)\}$. Thus, for some $\delta(n) \to 0$,

$$\text{Prob}\left(\sup_{V\in\mathcal{V}_n} (\text{Traffic needing to be carried by cell } V) \leq c_4\lambda(n)(n\log^2 n)^{\frac{1}{3}}\right) \geq 1-\delta(n).$$

- Since each cell can transmit at $W/(c_1+1)$ bits/sec, while it needs to transmit at most $c_4\lambda(n)(n\log^2 n)^{\frac{1}{3}}$ bits/sec with high probability, $\lambda(n)$ can be accommodated in the network with probability approaching one as $n \to +\infty$, if

$$c_4\lambda(n)\left(n\log^2 n\right)^{\frac{1}{3}} \leq \frac{W}{1+c_1},$$

i.e,

$$\lambda(n) \leq \frac{W}{(1+c_1)c_4(n\log^2 n)^{\frac{1}{3}}}.$$

Next consider the Physical Model. An argument similar to the one used in Lemma 4.4(ii) [1] shows that if the transmitters in the above constructive scheme use power level P, the SIR at each receiver is lower bounded by

$$\frac{\frac{P}{r^\alpha(n)}}{N+\sum_{k=1}^{+\infty}((k+2)^3-(k-1)^3)\frac{P}{k^\alpha(1+\frac{\Delta}{2})^\alpha r^\alpha(n)}} = \frac{1}{\frac{Nr^\alpha(n)}{P}+\frac{9}{(1+\frac{\Delta}{2})^\alpha}\sum_{k=1}^{+\infty}\frac{k^2+k+1}{k^\alpha}}.$$

For $\alpha > 3$, the sum in the denominator is smaller than $(3+\frac{1}{\alpha-1}+\frac{1}{\alpha-2}+\frac{1}{\alpha-3})$. Choosing $P = 2\beta r^\alpha(n)\max\{N,1\}$ and $\Delta = 2\left(\left(18\beta\left(3+\frac{1}{\alpha-1}+\frac{1}{\alpha-2}+\frac{1}{\alpha-3}\right)\right)^{\frac{1}{\alpha}}-1\right)$, ensures that the lower bound on the SIR at each receiver is at least β. The result follows. □

8. Conclusions. We have obtained the capacity of three dimensional wireless networks. We have shown that under a Protocol Model of non-interference, in a random 3-D network of n nodes randomly located in a sphere, with each node capable of transmitting at W bits/sec and using a common range, the throughput that each node can obtain for a randomly chosen destination is $\Theta\left(\frac{W}{(n\log^2 n)^{\frac{1}{3}}}\right)$ bits/sec. Even under optimal choices for node locations, traffic patterns, and origin-destination pairs, and optimal operation by choosing transmission schedules, ranges and routes, each node cannot obtain a throughput of more than $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ bits/sec for a destination on the order of 1 meter away. Under a Physical Model of non-interference, the lower bounds are the same as those above for the Protocol Model, while the upper bounds on throughput are $\Theta\left(\frac{W}{n^{\frac{1}{3}}}\right)$ for Random 3-D Networks and $\Theta\left(\frac{W}{n^{\frac{1}{\alpha}}}\right)$ for Arbitrary 3-D Networks.

In both the random and best case scenarios, 3-D wireless networks have higher capacity than 2-D networks. However, the throughput obtained by each node still diminishes to zero as the number of nodes in the network is increased. Hence, wireless networks connecting fewer number of users, or allowing connections mostly with nearby neighbors, may be more likely to find acceptance.

REFERENCES

[1] P. GUPTA AND P. R. KUMAR, *The capacity of wireless networks,* IEEE Transactions on Information Theory, Vol. IT-46:2(2000), pp. 388–404.

[2] P. GUPTA AND P. R. KUMAR, *Critical power for asymptotic connectivity in wireless networks,* in Stochastic Analysis, Control, Optimization and Applications: A Volume in Honor of W. H. Fleming. Edited by W. M. McEneany, G. Yin, and Q. Zhang, Eds. Birkhäuser, Boston, 1998.

[3] V. N. VAPNIK AND A. CHERVONENKIS, *On the uniform convergence of relative frequencies of events to their probabilities,* Theory of Prob. and Its Appl., 16:2(1971), pp. 264–280.

[4] V. N. VAPNIK, *Estimation of Dependences Based on Empirical Data.* Springer-Verlag, New York, 1982.

[5] R. GIBBS AND R. ENTRINGER, *Partitions of E^d by hyperplanes,* in Proceedings of the Twelveth Southeastern Conference on Combinatorics, Graph Theory, and Computing, Vol. I, Baton Rouge, LA, 1981.