Communications in Information and Systems
Vol. 6, No. 4, pp. 273-298, 2006

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A POLYNOMIAL CRITERION FOR ADAPTIVE STABILIZABILITY OF DISCRETE-TIME NONLINEAR SYSTEMS

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Abstract. In this paper, we will investigate the maximum capability of adaptive feedback in stabilizing a basic class of discrete-time nonlinear systems with both multiple unknown parameters and bounded noises. We will present a complete proof of the polynomial criterion for feedback capability as stated in [12], by providing both the necessity and sufficiency analyzes of the stabizability condition, which is determined by the growth rates of the system nonlinear dynamics only.

Key words: Uncertain nonlinear systems, Adaptive control, Robust stability, Discrete-time systems, Feedback capability

1. Introduction. Although much progress on adaptive control has been made over the past three decades, (see e.g. [1], [2], [3], [4]) there are only a few results on global stabilizability in the literature for discrete-time uncertain nonlinear systems when the growth rate of the nonlinear dynamics is faster than linear. The difficulty involved with adaptive control of discrete-time nonlinear systems was clearly demonstrated by the negative conclusion drawn in [5], which states that it is impossible in general to stabilize a discrete-time nonlinear system with even only a scalar unknown parameter if the nonlinear growth rate is too high. In contrast, for a continuous-time counter-part, no matter how high the nonlinear growth rate is, it can always be stabilized by, say, a nonlinear damping controller with a higher order. This inspired us to study the capability and limitations of feedback for discrete-time uncertain nonlinear systems as started in [5].

The benchmark model considered by [5] is as follows:

$$y_{t+1} = \theta y_t^b + u_t + w_{t+1}, \quad t = 0, 1, \ldots \tag{1}$$

where, u_t, y_t and w_t are the system input, output and noise respectively, θ is an unknown parameter, and the exponent $b \geq 1$ is a known real number which is regarded as the nonlinear growth rate of the system.

For the system (1), under the assumption that both the unknown parameter θ and the noise $\{w_t\}$ are Gaussian distributed, it is proved in [5] that if the nonlinear growth rate $b \geq 4$, then however you design the feedback control, there always exists a set with positive probability, on which the closed-loop dynamics is unstable in a standard sense. On the other hand, if $b < 4$, then it was also shown in [5] that

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the standard least-square-based adaptive control scheme can ensure the closed-loop stability almost surely.

Later on, the negative conclusion of [5] is extended in [8] to systems with multiple unknown parameters and with Gaussian white noises:

$$y_{t+1} = \theta_1 y_t^{b_1} + \theta_2 y_t^{b_2} + \cdots + \theta_p y_t^{b_p} + u_t + w_{t+1} \tag{2}$$

by providing the following polynomial rule: (2) is not almost surely stabilizable by feedback if there is a point $x \in [1, b_1]$ such that $P(x) < 0$, where

$$P(x) = x^{p+1} - b_1 x^p + (b_1 - b_2) x^{p-1} + \cdots + b_p. \tag{3}$$

This negative result implies that the usual linear growth condition is indispensable in general for stabilizability of uncertain nonlinear systems (see [8] for related discussions). This polynomial rule was further extended in [10] to the case where the uncertain parameters are known *a priori* to lie in a bounded region and the systems are allowed to have a more general structure:

$$y_{t+1} = \theta^T f(y_t, y_{t-1}, \ldots, y_{t-p+1}) + u_t + w_{t+1} \tag{4}$$

It should be noticed that all the above-mentioned results need the assumption that the noise is Gaussian distributed. It would be interesting to ask what happens if the noise is only bounded (see, e.g. [13], p.229). Let us again take the basic model (1) as the starting point to answer this question. One may suspect that the boundedness assumption on the noise w_t would be helpful for designing feedback stabilizers, which would at least result in a less stringent requirement on the nonlinear growth rate b. In fact, [11] demonstrated that this is not the case, since it was showed that $b < 4$ is still necessary for the existence of a feedback stabilizer, even if the noise are assumed to be bounded and with a known upper bound. However, the boundedness assumption on the noise will indeed be helpful in designing the feedback stabilizers when $b < 4$.

In the multiple unknown parameter case with bounded noises, the necessary and sufficient condition for stabilizability by feedback turns out to be governed by a polynomial rule, which is identical to the necessity condition obtained in [8] for the Gaussian white noise case. The corresponding theorem was stated in [12], but only partial analysis was given there. This paper will give a complete analysis of the feedback capability criterion found and stated in [12], by providing the proofs for both the necessity and sufficiency of the polynomial rule. Finally, we remark that the analysis in the current deterministic framework is completely different from that in the stochastic case [8] where the sufficiency of the criterion is still remains open.

2. Main Results. Consider the following system

$$y_{t+1} = \theta^T f(y_t) + u_t + w_{t+1} \tag{5}$$

where y_t, u_t and w_t are the system output, input and noise sequences respectively, and $\theta = (\theta_1, \theta_2, \cdots, \theta_p)^T$ is a p-dimensional unknown vector, $f(y_t) = [f_1(y_t), f_2(y_t), \cdots, f_p(y_t)]^T$ is assumed to be known nonlinear vector function with its components satisfying the following growth condition [1],

$$|f_l(x)| = \Theta(|x|^{b_l}), \quad l = 1, ..., p$$

where the exponents $b_1, b_2, \cdots, b_p$ are real numbers which, without loss of generality, are assumed to be arranged in a decreasing order: $b_1 > b_2 > \cdots > b_p > 0$ with $b_1 > 1$.

The above condition implies that there exist some x' and $c_2 \geq c_1 > 0$ such that for any $l = 1, 2, \cdots, p$,

$$c_1 \leq \frac{|f_l(x)|}{|x|^{b_l}} \leq c_2, \qquad \forall x \geq x' \tag{6}$$

We assume that the unknown parameters and the noise satisfy the following two conditions:

A1) The unknown parameter θ_i lies in a certain interval $[\underline{\theta}_i, \overline{\theta}_i] \subset \mathbb{R}^1$ with $\overline{\theta}_i - \underline{\theta}_i > 0$, for any $i = 1, 2, \ldots, p$.

A2) The noise sequence is bounded with a known bound $w > 0$, i.e.,

$$\sup_{t \geq 1} |w_t| \leq w. \tag{7}$$

Before presenting our main result, we restate the definition of feedback law [10].

Definition 2.1. *A sequence $\{u_t\}$ is called a feedback control law if at any time $t \geq 0$, u_t is a (causal) function of all the observations up to the time t, $\{y_i, i \leq t\}$, i.e.,*

$$u_t = h_t(y_0, \cdots, y_t) \tag{8}$$

where $h_t(\cdot) : \mathbb{R}^{t+1} \to \mathbb{R}^1$ can be any (nonlinear) mapping.

Apparently, the feedback thus defined includes all possible feedback inputs that can be designed based on the online observations. Hence the impossibility results to be established later on will have a celebrated "universality".

Definition 2.2. *The system (5) under the assumptions A1)-A2) is said to be globally stabilizable by feedback, if there exists a feedback control law $\{u_t\}$ such that for any $y_0 \in \mathbb{R}^1$, any θ and $\{w_t\}$ satisfying A1)-A2), the outputs of the closed-loop*

[1] $|f_l(x)| = \Theta(|x|^{b_l})$ means $0 < \liminf_{x\to\infty} \frac{|f_l(x)|}{|x|^{b_l}} \leq \limsup_{x\to\infty} \frac{|f_l(x)|}{|x|^{b_l}} < \infty$.

system are bounded as follows:

$$\sup_{t\geq 0}|y_t|<\infty. \tag{9}$$

Now, our main result of this paper is as follows:

THEOREM 2.1. *Under the assumptions A1)-A2), the system (5) is globally stabilizable by feedback* **if and only if** *for any* $x\in(1,b_1)$*,*

$$P(x)>0 \tag{10}$$

where $P(x)$ *is a polynomial defined by*

$$P(x)=x^{p+1}-b_1x^p+(b_1-b_2)x^{p-1}+\cdots+(b_{p-1}-b_p)x+b_p.$$

REMARK 2.1. *For* $p=1$*, we have* $P(x)=x^2-b_1x+b_1$*. Then it is easy to see that the condition (10) is equivalent to* $b_1<4$*. This simple criterion was established first in the Gaussion noise case by [5], then in the bounded noise case by [11].*

To facilitate the analysis, we divide the proof into three sections. The first section contains a series of basic lemmas, the second one gives the proof of sufficiency, and the last one gives the proof of necessity.

3. Some Basic Lemmas. In this section, we will prove four basic lemmas which will be used in the next two sections.

Denote

$$z=\left(\frac{c_2^p}{c_1^p}\cdot p!\frac{p}{p-1}\right)^{\overline{\min_{1\leq k\leq p-1}(b_k-b_{k+1})}^{\,1}}.$$

where the constants c_1 and c_2 are defined as in (6).

LEMMA 3.1. *Let us consider the functions* $f_i(x)$ *as defined in (5) and (6). If* $a_i\in\mathbb{R}^1, i=1,2,\dots,p$ *are constants which satisfy* $|a_i|>z|a_{i+1}|, i=1,2,\cdots,p-1$ *with* $|a_p|\geq\max\{1,x'\}$*, and if* D *denotes the determinant of the matrix* $(d_{ij})_{p\times p}\triangleq(f_j(a_i))_{p\times p}$*, then*

$$\frac{1}{p}\prod_{s=1}^{p}|f_s(a_s)|<|D|<2\prod_{s=1}^{p}|f_s(a_s)|;$$

Furthermore, we have

$$\frac{c_1^p}{p}\prod_{s=1}^{p}|a_s|^{b_s}<|D|<2c_2^p\prod_{s=1}^{p}|a_s|^{b_s}.$$

Proof. Obviously, D is a summation of terms of the form $(-1)^r\prod_{s=1}^{p}f_{j_s}(a_s)$ where $r=0$ or 1, $(j_1,j_2,\dots,j_p)=\pi(1,2,\dots,p)$, and $\pi\underline{X}$ denotes a permutation of a vector $\underline{X}$.

Let us first consider the following terms $(-1)^r \prod_{s=1}^{p} a_s^{b_{j_s}}$. Taking logarithm on its absolute value, we have $\log|\prod_{s=1}^{p} a_s^{b_{j_s}}| = \sum_{s=1}^{p} b_{j_s} \log|a_s|$. Since

$$\begin{aligned} & b_1 > b_2 > \cdots > b_p > 0 \\ & \log|a_1| > \log|a_2| > \cdots > \log|a_p| \geq 0, \end{aligned}$$

by the inequality in [7, p.341], we have

$$\sum_{s=1}^{p} b_{j_s} \log|a_s| \leq \sum_{s=1}^{p} b_s \log|a_s|,$$

which means $|a_1^{b_1} a_2^{b_2} a_3^{b_3} \cdots a_p^{b_p}|$ is the maximum term. For any other terms $\prod_{s=1}^{p} |a_s|^{b_{j_s}}$ with $(b_{j_1}, \ldots, b_{j_p}) \neq (b_1, \ldots, b_p)$, let $m = \min\{s : j_s \neq s, 1 \leq s \leq p-1\}$, i.e., $j_s = s$ for $s < m$, and $j_m = n > m$, then by the above argument, $\left|\prod_{s=1}^{p} a_s^{b_{j_s}}\right|$ is not larger than

$$\left|a_1^{b_1} \cdots a_{m-1}^{b_{m-1}} a_m^{b_n} a_{m+1}^{b_m} a_{m+2}^{b_{m+1}} \cdots a_n^{b_{n-1}} a_{n+1}^{b_{n+1}} \cdots a_p^{b_p}\right|.$$

Now, let us denote $\delta_m := b_m - b_{m+1}$, $m = 1, \ldots, p-1$ and $\delta := \min_{1 \leq m \leq p-1} \delta_m$. We then have the following uniform bound:

$$\begin{aligned} & \frac{\left|a_1^{b_1} \cdots a_{m-1}^{b_{m-1}} a_m^{b_n} a_{m+1}^{b_m} a_{m+2}^{b_{m+1}} \cdots a_n^{b_{n-1}} a_{n+1}^{b_{n+1}} \cdots a_p^{b_p}\right|}{\left|\prod_{s=1}^{p} a_s^{b_s}\right|} \\ & \leq \left|a_m^{b_n - b_m} a_{m+1}^{b_m - b_{m+1}} \cdots a_n^{b_{n-1} - b_n}\right| \\ & = \left|a_m^{(b_{m+1} - b_m) + \cdots + (b_n - b_{n-1})} a_{m+1}^{b_m - b_{m+1}} \cdots a_n^{b_{n-1} - b_n}\right| \\ & \leq \frac{1}{z^{\delta_m} z^{2\delta_{m+1}} \cdots z^{(n-m)\delta_{n-1}}} \\ & \leq \frac{1}{z^{(n-m)(n-m+1)\delta/2}} \leq \frac{1}{z^\delta} = \frac{p-1}{pp!} \cdot \frac{c_1^p}{c_2^p} \end{aligned}$$

So we have

$$\begin{aligned} |D| & \geq \prod_{s=1}^{p} |f_s(a_s)| - \sum_{(j_1, \ldots, j_p) \neq (1, \ldots, p)} \left|\prod_{s=1}^{p} f_{j_s}(a_s)\right| \\ & \geq \prod_{s=1}^{p} |f_s(a_s)| \left(1 - \frac{c_2^p}{c_1^p} \cdot \sum_{(j_1, \ldots, j_p) \neq (1, \ldots, p)} \frac{|\prod_{s=1}^{p} a_s^{b_{j_s}}|}{|\prod_{s=1}^{p} a_s^{b_s}|}\right) \\ & > \prod_{s=1}^{p} |f_s(a_s)| \left(1 - (p! - 1)\frac{p-1}{pp!}\right) \qquad (11) \\ & > \frac{1}{p} \prod_{s=1}^{p} |f_s(a_s)| > \frac{c_1^p}{p} \prod_{s=1}^{p} |a_s|^{b_s} \end{aligned}$$

and similarly,

$$|D| < \prod_{s=1}^{p} |f_s(a_s)| \left(1 + (p! - 1)\frac{p-1}{pp!}\right)$$
$$< 2\prod_{s=1}^{p} |f_s(a_s)| < 2c_2^p \prod_{s=1}^{p} |a_s|^{b_s}.$$

□

Similar to the proof above, we further have

LEMMA 3.2. *Let $D_{k,l}$ be the kl-th cofactor of D. If the conditions of Lemma 3.1 hold, then*

$$\frac{1}{p}\prod_{s=1}^{l-1} |f_s(a_s)| \prod_{s=l}^{p-1} |f_{s+1}(a_s)| < \sum_{k=1}^{p} |D_{k,l}| < 2\prod_{s=1}^{l-1} |f_s(a_s)| \prod_{s=l}^{p-1} |f_{s+1}(a_s)|, \tag{12}$$

furthermore,

$$\frac{c_1^p}{p}\prod_{s=1}^{l-1} |a_s|^{b_s} \prod_{s=l}^{p-1} |a_s|^{b_{s+1}} < \sum_{k=1}^{p} |D_{k,l}| < 2c_2^p \prod_{s=1}^{l-1} |a_s|^{b_s} \prod_{s=l}^{p-1} |a_s|^{b_{s+1}}. \tag{13}$$

The following two lemmas are only involved in the proof of necessity.

LEMMA 3.3. *Let c and Δ be two constants satisfying $\left|\sum_{i=2}^{s+1}\prod_{j=i}^{p+i-s}\lambda_j\right|\Delta + c \geq 0$ for some $1 \leq s \leq p$, where $\lambda_l, l = 1, 2, \cdots, p+1$ are the $p+1$ roots of the polynomial $P(x)$ defined as in Theorem 2.1 with $\lambda_1 \in (1, b_1)$. Also, let $\{a_i, i = 0, ..., p+1\}$ be real numbers which satisfy the inequality $a_{p+1} \geq b_1(a_p - a_{p-1}) + b_2(a_{p-1} - a_{p-2}) + \cdots + b_p(a_1 - a_0) + c$. If $a_k - \lambda_1 a_{k-1} \geq 0, k = 1, 2, \ldots, p$, and $a_s = \lambda_1 a_{s-1} + \Delta$, then $a_{p+1} - \lambda_1 a_p \geq 0$.*

Proof: According to the relationship of roots and coefficients, we have

$$\begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_{p+1} = b_1 \\ \lambda_1\lambda_2 + \cdots + \lambda_p\lambda_{p+1} = b_1 - b_2 \\ \vdots \\ \lambda_1\lambda_2\cdots\lambda_{p+1} = (-1)^{p-1}b_p \end{cases} \tag{14}$$

which implies that

$$(-1)^{p-1}(\lambda_2\lambda_3\cdots\lambda_{p+1}) = \frac{(-1)^{2(p-1)}b_p}{\lambda_1} > 0. \tag{15}$$

Furthermore, by the last equation but one in (14),

$$\begin{aligned} &(-1)^{p-2}(b_{p-1} - b_p) \\ =\ &\lambda_1\lambda_2\cdots\lambda_p + \cdots + \lambda_2\lambda_3\cdots\lambda_{p+1} \\ =\ &\lambda_1(\lambda_2\lambda_3\cdots\lambda_p + \cdots + \lambda_3\lambda_4\cdots\lambda_{p+1}) + \lambda_2\lambda_3\cdots\lambda_{p+1} \end{aligned}$$

hence by multiplying $(-1)^{p-2}$ and dividing λ_1 on both sides, we have by (85) that

$$\begin{aligned}&(-1)^{p-2}(\lambda_2\lambda_3\cdots\lambda_p+\cdots+\lambda_3\lambda_4\cdots\lambda_{p+1})\\=&\frac{1}{\lambda_1}[(-1)^{p-1}\lambda_2\lambda_3\cdots\lambda_{p+1}+(-1)^{2(p-2)}(b_{p-1}-b_p)]>0.\end{aligned}$$

By a similar argument, we can prove in succession that

$$\left\{\begin{array}{l}(-1)^{p-1}(\lambda_2\lambda_3\cdots\lambda_{p+1})>0\\ \vdots\\ (-1)^1(\lambda_2\lambda_3+\cdots+\lambda_p\lambda_{p+1})>0\\ (-1)^0(\lambda_2+\lambda_3+\cdots+\lambda_{p+1})>0\end{array}\right. \tag{16}$$

Finally, by using (16) we have

$$\begin{aligned}a_{p+1}-\lambda_1a_p&\geq b_1(a_p-a_{p-1})+b_2(a_{p-1}-a_{p-2})+\cdots+b_p(a_1-a_0)+c-\lambda_1a_p\\&=(\lambda_2+\lambda_3+\cdots+\lambda_{p+1})(a_p-\lambda_1a_{p-1})\\&+(-1)^1(\lambda_2\lambda_3+\cdots+\lambda_p\lambda_{p+1})(a_{p-1}-\lambda_1a_{p-2})\\&+\cdots+(-1)^{p-1}\lambda_2\lambda_3\cdots\lambda_{p+1}(a_1-\lambda_1a_0)+c\\&\geq((-1)^{p-s}\lambda_2\cdots\lambda_{p+2-s}+\cdots+\lambda_{s+1}\cdots\lambda_{p+1})\Delta+c\\&\geq 0,\end{aligned} \tag{17}$$

where the "="follows from (14), and the last two inequalities follow from the assumptions of the lemma. □

Let $A=(a_{ij})_{s\times r}$ be a real matrix with dimension $s\times r$, and $E_{s+j}\subset\mathbb{R}^1$ be some intervals, $j=1,2,\cdots,r$, and denote $\overline{E}=\max_{1\leq j\leq r}\{|E_{s+j}|\}$ with $|\cdot|$ being the length of the interval concerned. We now consider the projection properties of the following polyhedron:

$$E=\{(\alpha^T,\beta^T)^T:\alpha=A\beta,\ \beta\in\prod_{j=1}^{r}E_{s+j}\}, \tag{18}$$

where α and β are s and r dimensional vectors respectively. Also, denote the projection of E on its i-th component as $E_i, i=1,2,\cdots,s$. The following lemma shows how the lengths of the projected components will vary due to the change of the first component.

LEMMA 3.4. *Let $E_1'\subset E_1$ be an interval such that $|E_1'|=\frac{1}{4}|E_1|$. Denote $E'=E_1'\times E_2\times\cdots\times E_{s+r}\cap E$, and let E_l' be the projection of E' on its l-th component, $l=1,2,\cdots,s+r$. If $\left|\frac{a_{1,1}}{a_{1,j}}\right|\geq\frac{4r\overline{E}}{|E_{s+1}|}, j=2,3,\cdots,r$, then*

$$\begin{aligned}&|E_i'|\geq\frac{1}{4}|E_i|,\quad i=2,\cdots,s+1\\&E_j'=E_j,\quad j=s+2,s+3,\cdots,s+r.\end{aligned}$$

Proof. Without loss of generality, we suppose that $a_{1j} \geq 0$ for $j = 1, 2, \cdots, r$, since otherwise, similar proof techniques also apply. Denote A_i as the i-th row of A, and denote the projected intervals as $E_l \triangleq [\underline{e}_l, \overline{e}_l], E'_l \triangleq [\underline{e}'_l, \overline{e}'_l], l = 1, 2, \cdots, s+r$. By the definition of E_i,

$$|E_i| = \sum_{j=1}^{r} |a_{ij}||E_{s+j}|, \quad i = 1, 2, \cdots, s. \tag{19}$$

Now, we introduce the projection of E' on β as

$$E_\beta = \{\beta : A_1\beta \in [\underline{e}'_1, \overline{e}'_1], \beta \in \prod_{j=1}^{r} E_{s+j}\}.$$

Obviously, E'_{s+j} are the existence intervals of the $(s+j)$-th component of E_β, $j = 1, 2, \cdots, r$.

We proceed to show that $|E'_i| \geq \frac{1}{4}|E_i|, i = 1, 2, \cdots, s+1$. Since $|E'_1| = \frac{1}{4}|E_1|$, either $\underline{e}'_1$ or $\overline{e}'_1$ must belong to $[\underline{e}_1 + \frac{1}{4}|E_1|, \overline{e}_1 - \frac{1}{4}|E_1|]$, say,

$$\underline{e}'_1 = \underline{e}_1 + k|E_1| \quad \text{with} \quad \frac{1}{4} \leq k \leq \frac{3}{4}.$$

Denote $\underline{\beta} = (\underline{e}_{s+1}, \underline{e}_{s+2}, \cdots, \underline{e}_{s+r})^T$, since all $a_{1j} \geq 0$, $A_1\underline{\beta} = \underline{e}_1$. Then the point $\beta^c = (e^c_{s+1}, e^c_{s+2}, \cdots, e^c_{s+r})^T$ defined by

$$e^c_{s+j} = \underline{e}_{s+j} + k|E_{s+j}| \quad j = 1, 2, \ \cdots, r \tag{20}$$

satisfies

$$\begin{aligned} A_1\beta^c - \underline{e}_1 &= A_1\beta^c - A_1\underline{\beta} \\ &= A_1(\underline{e}_{s+1} + k|E_{s+1}|, \underline{e}_{s+2} + k|E_{s+2}|, \cdots, \underline{e}_{s+r} + k|E_{s+r}|)^T \\ &\quad - A_1(\underline{e}_{s+1}, \underline{e}_{s+2}, \cdots, \underline{e}_{s+r})^T \\ &= k\sum_{j=1}^{r} a_{1j}|E_{s+j}| = k|E_1|, \end{aligned} \tag{21}$$

which means $A_1\beta^c = \underline{e}'_1$, and hence $\beta^c \in E_\beta$.

Let us consider the points

$$\beta^i = (e^c_{s+1} + \frac{1}{4}|E_{s+1}|, e^c_{s+2} + \operatorname{sgn}(\frac{a_{i1}}{a_{i2}})\frac{1}{4}|E_{s+2}|, \cdots, e^c_{s+r} + \operatorname{sgn}(\frac{a_{i1}}{a_{ir}})\frac{1}{4}|E_{s+r}|)^T$$

for $i = 2, 3, \cdots, s$, where $\operatorname{sgn}(x)$ is the sign function. Note that each component of β^i belongs to E_{s+j} by (20).

According to the assumption of the lemma,

$$\sum_{j=2}^{r} a_{1j}|E_{s+j}| \leq \sum_{j=2}^{r} \frac{|E_{s+1}|}{4r\overline{E}} a_{11}|E_{s+j}| \leq \frac{1}{4}a_{11}|E_{s+1}|, \tag{22}$$

hence by (22) and similar to (21) we have $0 \le A_1\beta^i - A_1\beta^c \le \frac{1}{4}|E_1|$, namely, $\underline{e}_1' \le A_1\beta^i \le \overline{e}_1'$, and we have $\beta^i \in E_\beta$. This in conjunction with $\beta^c \in E_\beta$, gives $|E_{s+1}'| \ge \frac{1}{4}|E_{s+1}|$. Furthermore,

$$|A_i\beta^i - A_i\beta^c| = \frac{1}{4}\sum_{j=1}^{r}|a_{ij}||E_{s+j}| = \frac{1}{4}|E_i|,$$

which implies $|E_i'| \ge \frac{1}{4}|E_i|, i = 2, 3, \cdots, s$.

Hence, it remains to prove that $E_{s+j}' = E_{s+j}, j = 2, s, \cdots, r$. Since there exists a point $\beta' = (e_{s+1}', e_{s+2}', \cdots, e_{s+r}')^T$ such that $A_1\beta' = \overline{e}_1'$, we consider

$$\underline{\beta} = (e_{s+1}', \underline{e}_{s+2}, \cdots, \underline{e}_{s+r})^T,$$

where $\underline{e}_{s+2}, \cdots, \underline{e}_{s+r}$ are defined as above. By (22), we have

$$\begin{aligned}\overline{e}_1' - A_1\underline{\beta} &= A_1(e_{s+1}', e_{s+2}', \cdots, e_{s+r}')^T - A_1(e_{s+1}', \underline{e}_{s+2}, \cdots, \underline{e}_{s+r})^T \\ &\le \sum_{j=2}^{r} a_{1j}|E_{s+j}| \le \frac{1}{4}a_{11}|E_{s+1}| \le \frac{1}{4}|E_1|,\end{aligned}$$

which means that $\underline{\beta} \in E_\beta$. By a similar argument together with the fact that $A_1\beta^c = \underline{e}_1'$, we have $A_1\overline{\beta} - \underline{e}_1' \le \frac{1}{4}|E_1|$ where $\overline{\beta} = (e_{s+1}^c, \overline{e}_{s+2}, \cdots, \overline{e}_{s+r})^T$, and so $\overline{\beta} \in E_\beta$. Finally, comparing the two points $\overline{\beta}$ and $\underline{\beta}$ in E_β, we know that $E_{s+j}' = E_{s+j}, j = 2, s, \cdots, r$. Hence the conclusion of the lemma is true. □

4. The Proof of Sufficiency. For any $t \ge 1$, let

$$i_1(t) := \operatorname*{argmax}_{0 \le i \le t-1} |y_i|, \tag{23}$$

$$i_j(t) := \operatorname*{argmax}_{\substack{0 \le i \le t-1 \\ z|y_i| < |y_{i_{j-1}(t)}|}} |y_i|, \qquad 2 \le j \le p \tag{24}$$

and

$$|y_{i_p(t)}| \ge \max\{1, x'\}. \tag{25}$$

Let $u_0 = u_1 = \cdots = u_{p-2} = 0$. Starting with $t = p$, if $i_j(p), 1 \le j \le p$ as defined in (23)-(25) can not be found, then let $u_{t-1} = 0$, $t = p, p+1, \ldots$ until $i_j(t), 1 \le j \le p$ can be found. If $i_j(t)$ can never be found for any t, then it is easy to show that $\sup_{t \ge 0}|y_t| < \infty$. We can prove this by contradiction. In fact, if $\sup_{t \ge 0}|y_t| = \infty$, then it is easy to find $k_i, i = 1, 2, \ldots, p$ such that $|y_{k_1}| \ge \max\{1, x'\}$ and $z|y_{k_{i-1}}| < |y_{k_i}|, i = 2, \ldots, p$. Obviously, for $t = k_p + 1$, $i_j(t)$ in (23)-(25) are well defined. Moreover, it is obvious that $i_j(t)$ are well defined for all $t > k_p + 1$.

So we only need to consider the case where starting from some t_0, $i_j(t)$ in (23)-(25) are all well defined. Then for any $t \geq t_0$, we have from the system equation that

$$(d_{kj}(t))_{p\times p} \cdot \theta = \varepsilon(t) \tag{26}$$

where $(d_{kj}(t))_{p\times p} \triangleq (f_i(y_{i_k(t)}))_{p\times p}$, $\varepsilon(t) \triangleq (\varepsilon_1(t), \cdots, \varepsilon_p(t))^T$ with $\varepsilon_k(t) \triangleq y_{i_k(t)+1} - u_{i_k(t)} - w_{i_k(t)+1}$ and θ is the unknown parameter vector.

Let $D(t)$ be the determinant of the matrix $(d_{kj}(t))_{p\times p}$, and $D_l(t)$ be the determinant of the matrix that is obtained by replacing the l-th column in $(d_{kj}(t))_{p\times p}$ by the R.H.S of (26).

By (23)-(25) and Lemma 3.1, we have

$$|D(t)| > \frac{c_1^p}{p} \prod_{s=1}^{p} |y_{i_s(t)}|^{b_s} > 0. \tag{27}$$

Hence by the Cramer principle, $\theta_l = \dfrac{D_l(t)}{D(t)}$. At the time t, let the parameter estimate be $\hat{\theta}_l(t) \triangleq \dfrac{\hat{D}_l(t)}{D(t)}$, where $\hat{D}_l(t)$ is defined in the same way as $D_l(t)$ but with $w_{i_k(t)+1} = 0$, $k = 1, \cdots, p$.

Let $\tilde{\theta}_l(t) = \theta - \hat{\theta}_l(t)$. Let $D_{k,l}(t)$ be the kl-th cofactor of $D(t)$, i.e. by taking out the k-th row and the l-th column of $D(t)$. Hence, the estimation error is

$$\tilde{\theta}_l(t) = \sum_{k=1}^{p} (-1) w_{i_k(t)+1} \frac{D_{k,l}(t)}{D(t)}. \tag{28}$$

By (23)-(25), (27), (28) and Lemma 3.2, we have

$$\begin{aligned} |\tilde{\theta}_l(t) f_l(y_t)| &\leq c_2 |\tilde{\theta}_l(t) y_t^{b_l}| \leq c_2 \frac{\sum_{k=1}^{p} |D_{k,l}(t)|}{|D(t)|} w |y_t|^{b_l} \\ &< \frac{2c_2^{p+1} \cdot pw}{c_1^p} \Big| \frac{y_t}{y_{i_l(t)}} \Big|^{b_l} \prod_{s=l}^{p-1} \Big| \frac{y_{i_s(t)}}{y_{i_{s+1}(t)}} \Big|^{b_{s+1}}. \end{aligned} \tag{29}$$

Now we define

$$u_t = -\sum_{l=1}^{p} \hat{\theta}_l(t) \cdot f_l(y_t) \quad \text{ for any } t \geq t_0, \tag{30}$$

so the closed-loop dynamics is

$$y_{t+1} = \sum_{l=1}^{p} \tilde{\theta}_l(t) \cdot f_l(y_t) + w_{t+1}. \tag{31}$$

We use a contradiction argument to prove that $\sup_{t\geq 0} |y_t| < \infty$. Suppose there exist some $y_0 \in \mathbb{R}^1$, $\{\theta_l, l = 1, 2, \cdots, p\}$ and a sequence of $\{w_t\}$, such that for the control defined in (30), $\sup_{t\geq 0}|y_t| = \infty$. From this sequence $\{|y_t|,\, t \geq t_0\}$, we can pick out a monotonously increasing subsequence $\{|y_{t_k}|,\, k \geq 1\}$ with

$$|y_{t_1}| > \frac{3c_2^{p+1}\cdot p^2 w z^{b_1}}{c_1^p}, \tag{32}$$

$$t_{k+1} = \inf\{t > t_k : |y_t| > z|y_{t_k}|\}. \tag{33}$$

For any $k \geq p+1$, let $m = t_{k+1}-1$, and it is easy to check that

$$|y_m| \leq z|y_{t_k}| \tag{34}$$

$$|y_{t_{k-1}}| \leq |y_{i_1(m)}| \leq z|y_{t_k}| \tag{35}$$

$$|y_{t_{k-j}}| \leq |y_{i_j(m)}| \quad \text{for any } j = 1, 2, \ldots, p. \tag{36}$$

In fact, (34) is obvious, and (35) follows by $t_{k-1} \leq t_k - 1 \leq t_{k+1} - 2 = m-1$, and (36) can be proved by induction: By (35),

$$z|y_{t_{k-2}}| < |y_{t_{k-1}}| \leq |y_{i_1(m)}| \Rightarrow |y_{t_{k-2}}| \leq |y_{i_2(m)}|,$$

and this can be continued for $j = 3, 4, \ldots, p$.

Hence by (29), (34)-(36), for any $k \geq p+1$, we have

$$\begin{aligned}
|y_{t_{k+1}}| &\leq \sum_{l=1}^{p} |\tilde{\theta}_l(m) f_l(y_m)|^{b_l} + w \\
&\leq \frac{2c_2^{p+1}\cdot pw}{c_1^p} \sum_{l=1}^{p} \left|\frac{y_m}{y_{i_l(m)}}\right|^{b_l} \prod_{s=l}^{p-1} \left|\frac{y_{i_s(m)}}{y_{i_{s+1}(m)}}\right|^{b_{s+1}} + w \\
&\leq \frac{2c_2^{p+1}\cdot pwz^{b_1}}{c_1^p} \sum_{l=1}^{p} \left|\frac{y_{t_k}}{y_{t_{k-l}}}\right|^{b_l} \prod_{s=l}^{p-1} \left|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\right|^{b_{s+1}} + w \\
&\leq \frac{3c_2^{p+1}\cdot p^2 wz^{b_1}}{c_1^p} \left|\frac{y_{t_k}}{y_{t_{k-1}}}\right|^{b_1} \left|\frac{y_{t_{k-1}}}{y_{t_{k-2}}}\right|^{b_2} \cdots \left|\frac{y_{t_{k-p+1}}}{y_{t_{k-p}}}\right|^{b_p},
\end{aligned} \tag{37}$$

where the last inequality follows from the fact $y_{t_i} \geq y_{t_{i-1}}$ and the monotonicity of the terms

$$\begin{aligned}
&\left|\frac{y_{t_k}}{y_{t_{k-l-1}}}\right|^{b_{l+1}} \prod_{s=l+1}^{p-1} \left|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\right|^{b_s+1} \\
&= \left|\frac{y_{t_{k-l}}}{y_{t_k}}\right|^{b_l - b_{l+1}} \left|\frac{y_{t_k}}{y_{t_{k-l}}}\right|^{b_l} \prod_{s=l}^{p-1} \left|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\right|^{b_{s+1}} \\
&< \left|\frac{y_{t_k}}{y_{t_{k-l}}}\right|^{b_l} \prod_{s=l}^{p-1} \left|\frac{y_{t_{k-s}}}{y_{t_{k-s-1}}}\right|^{b_{s+1}}, \quad \forall\, l = 1, \ldots, p-1.
\end{aligned}$$

Let $a_k = \ln|y_{t_k}| - \ln 3c_2^{p+1}\cdot p^2 wz^{b_1} + \ln c_1^p > 0$. By (32)-(33), $\{a_k\}$ is monotonically increasing with $a_1 > 0$ and by (37),

$$a_{k+1} \leq b_1(a_k - a_{k-1}) + b_2(a_{k-1} - a_{k-2}) + \cdots + b_p(a_{k-p+1} - a_{k-p}). \tag{38}$$

Let $x_k = \dfrac{a_k}{a_{k-1}}$. Obviously, $x_k > 1$, and by (38), we have

$$x_{k+1} \le b_1 - (b_1 - b_2)\frac{1}{x_k} - \cdots - (b_{p-1} - b_p)\frac{1}{\prod_{s=0}^{p-2} x_{k-s}} - b_p \frac{1}{\prod_{s=0}^{p-1} x_{k-s}}. \tag{39}$$

Therefore, it follows that for $k \ge p+1$, $x_k \le b_1$.

Hence, $\overline{x} := \overline{\lim_{k\to\infty}}\, x_k \in [1, b_1]$. By (39) we have

$$\overline{\lim_{k\to\infty}}\, x_{k+1} \le b_1 - (b_1 - b_2)\frac{1}{\overline{\lim\limits_{k\to\infty}}\, x_k} - (b_2 - b_3)\frac{1}{\overline{\lim\limits_{k\to\infty}}\, x_k x_{k-1}} - \cdots - b_p \frac{1}{\overline{\lim\limits_{k\to\infty}} \prod_{s=0}^{p-1} x_{k-s}},$$

which means

$$\overline{x} \le b_1 - (b_1 - b_2)\frac{1}{\overline{x}} - (b_2 - b_3)\frac{1}{\overline{x}^2} - \cdots - (b_{p-1} - b_p)\frac{1}{\overline{x}^{p-1}} - b_p \frac{1}{\overline{x}^p}.$$

So $P(\overline{x}) \le 0$, which contradicts to (10). Hence the sufficiency is proven. □

5. The Proof of Necessity. The proof of the necessity part is quite involved, and we therefore divide the total proof into several subsections.

5.1. Construction of the Feasible Uncertainty Domain.. We know that the information about the system is increasing with the time t, so the uncertainty of the unknown parameter vector $(\theta_1, \theta_2, \cdots, \theta_p)$ should be reduced with the time. In this section, a proposition about the feasible domain of $(\theta_1, \theta_2, \cdots, \theta_p)$ will be established, which is instrumental to the proof of the necessity part.

For this, we need to introduce some notation first, which will be used throughout the sequel.

$$\begin{aligned}
&f_t(\cdot) = 1,\ b_t = 0, \quad \text{for} \quad t \le 0 \text{ or } t \ge p+1,\\
&\delta = \max_{2\le l\le p}\{|\underline{\theta}_l|, |\overline{\theta}_l|\},\\
&m = \min\{2w, \delta, |\Theta_l(0)|, l = 1, 2, \cdots, p\},\\
&M = \max\{2w, \delta, |\Theta_l(0)|, l = 1, 2, \cdots, p\}.
\end{aligned}$$

Now, we consider the following cuboid in R^p,

$$\Theta(0) := \Theta_1(0) \times \Theta_2(0) \times \cdots \times \Theta_p(0),$$

where $\Theta_l(0) := [\underline{\theta}_l, \overline{\theta}_l], l = 1, 2, \cdots, p$. Let $\Theta_1'(0) \subset \Theta_1(0)$ be some interval with the length $|\Theta_1'(0)| = \dfrac{1}{4}|\Theta_1(0)|$, and we denote

$$\Theta'(0) := \Theta_1'(0) \times \Theta_2(0) \times \cdots \times \Theta_p(0).$$

With any initial value y_0, and for any $t \geq 1$ and any given $\{y_t, u_{t-1}\}$, we can recursively define

$$\Theta(t) := \{(\theta_1, \theta_2, \cdots, \theta_p) \in \Theta'(t-1) : y_t = \sum_{l=1}^{p} \theta_l f_l(y_{t-1}) + u_{t-1} + w_t, \tag{40}$$
$$\text{for some } |w_t| \leq w\}.$$

Let $\Theta_1'(t) \subset \Theta_1(t)$ be some interval with the length $|\Theta_1'(t)| = \frac{1}{4}|\Theta_1(t)|$ (the construction details of $\Theta_1'(t)$ will be discussed later on in Proposition 5.2), and denote

$$\Theta'(t) := \Theta_1'(t) \times \Theta_2(t) \times \cdots \times \Theta_p(t) \cap \Theta(t).$$

It is obviously that $\Theta(t) \subset \Theta'(t-1) \subset \Theta(t-1)$. Furthermore, denote

$$S(0) := \Theta(0),$$
$$S(t) := \Big\{(\theta_1, \cdots, \theta_{t\wedge p}) \in R^{t\wedge p}, (\theta_{t+1}, \cdots, \theta_p) \in \prod_{j=t+1}^{p} \Theta_j(0) : \tag{41}$$
$$y_i = \sum_{l=1}^{p} \theta_l f_l(y_{i-1}) + u_{i-1} + w_i, \text{ for some} \, |w_i| \leq w, i = 1 \vee (t-p+1), \cdots, t\Big\},$$

which obviously is a closed set at any time $t \geq 0$.

It is worth pointing out the difference between the two parameter sets $\Theta(t)$ and $S(t)$ defined above: For any $\theta \in \Theta(t)$, we know by definition that θ is a feasible parameter for all system equations up to time t, which will be convenient in the contradiction proof of the necessity part. However, if $\theta \in S(t)$, we can only guarantee that θ is a feasible parameter for the latest p system equations when $t \geq p$, and the advantage of this property is that such θ can be conveniently and explicitly expressed. The striking fact is that these two parameter sets can be made identical successively by carefully selecting the output values $\{y_t\}$ for any given input sequence $\{u_t\}$, and which is the content of the following proposition.

PROPOSITION 5.1. *For any time $t \geq 0$, let the following two conditions hold:*

B1) *For any $1 \leq k \leq t$, and any $(\theta_1, \theta_2, \cdots, \theta_p) \in \Theta'(k)$, the output satisfies $|y_k| \geq d|y_{k-1}|$ with*

$$d = \max\{z, (\frac{256Mp^3c_2^{2p}}{mc_1^{2p}})^{\max_{1\leq i\leq p}\{\frac{1}{b_i - b_{i+1}}\}}\}, \tag{42}$$

where $z > 1$ is defined in Lemma 3.1.

B2) $\Theta(t) = S(t)$.

Then for any given u_t, there is an output value y_{t+1} such that the $\Theta(t+1)$ and the $S(t+1)$ respectively defined as in (40) and (41) are identical, when y_0 is large enough.

Proof. The proof is involved and is placed in Appendix A.

5.2. Analysis of the Growth Rate of Output.. According to the above section, we know that B1 is a key condition in Proposition 5.1. The following proposition shows that this condition can also be guaranteed successively.

PROPOSITION 5.2. *Let the polynomial $P(x)$ defined in Theorem 2.1 have a root $\lambda_1 \in (1, b_1)$. For any time $t \geq 0$, any given u_{t+1} and any $w_{t+2} \in [-w, w]$, if $\Theta(t+1) = S(t+1)$, then we can find some $\Theta'(t+1) \subset \Theta(t+1)$ such that for any $(\theta_1, \theta_2, \cdots, \theta_p) \in \Theta'(t+1)$,*

$$|y_{t+2}| \geq c_0^{1-\lambda_1}|y_{t+1}|^{\lambda_1} \geq d|y_{t+1}|$$

when y_0 is large enough, provided that the following assumption holds:
C1) *For any $0 \leq k \leq t$, and $(\theta_1, \theta_2, \cdots, \theta_p) \in \Theta'(k)$, the output satisfies*

$$|y_{k+1}| \geq c_0^{1-\lambda_1}|y_k|^{\lambda_1}, \tag{43}$$

where $c_0 = \dfrac{m}{16p}(\dfrac{c_1}{c_2})^p(c_1 \wedge 1)$.

Proof. First of all, it is easy to see that Condition C1 implies Condition B1 at time $t+1$, if we take the initial condition to satisfy

$$c_0^{1-\lambda_1}|y_0|^{\lambda_1-1} \geq d. \tag{44}$$

Let us further assume that $|y_0|$ satisfies the following conditions throughout the sequel,

$$|f_l(y)| \geq p|f_{l+1}(y)| \quad \text{for} \quad \forall y \geq y_0, \quad l = 1, 2, \cdots, p, \tag{45}$$

$$|y_0| \geq \max\{1, x'\} \tag{46}$$

where x' is defined in (6).

Now, at time $t+1$, for any $(\theta_1, \theta_2, \cdots, \theta_p) \in \Theta(t+1)$, the new system equation is

$$y_{t+2} = \theta_1 f_1(y_{t+1}) + \theta_2 f_2(y_{t+1}) + \cdots + \theta_p f_p(y_{t+1}) + u_{t+1} + w_{t+2}.$$

To estimate the growth rate of $|y_{t+2}|$, we first note that the first term on the right-hand-side of the above equation will be dominating for large y_{t+1}. This inspires us to take θ_i^c as the center points of $\Theta_i(t+1)$, $i = 2, \cdots, t+1$, and to deduce the following from the above equation:

$$\begin{aligned} |y_{t+2}| &\geq |\theta_1 f_1(y_{t+1}) + u_{t+1} + \sum_{i=2}^{p}\theta_i^c f_i(y_{t+1})| - \left(\sum_{i=2}^{p}|\theta_i - \theta_i^c||f_i(y_{t+1})| + w\right) \\ &\geq |\theta_1 + \hat{\theta}_1||f_1(y_{t+1})| - \left(\frac{1}{2}\sum_{i=2}^{p}|\Theta_i(t+1)||f_i(y_{t+1})| + w\right) \end{aligned} \tag{47}$$

where $\hat{\theta}_1 \triangleq \dfrac{u_{t+1} + \sum_{l=2}^{p}\theta_i^c f_i(y_{t+1})}{f_i(y_{t+1})}$.

We will estimate the above inequality term by term. To estimate the first term, notice that no matter what is the choice of u_{t+1}, there always exists some interval

$$\Theta_1'(t+1) \subset \Theta_1(t+1)$$

with the length

$$|\Theta_1'(t+1)| \geq \frac{1}{4}|\Theta_1(t+1)|$$

such that for any $\theta_1 \in \Theta_1'(t+1)$, we will have

$$|\theta_1 + \hat{\theta}_1| \geq \frac{1}{4}|\Theta_1(t+1)| \geq \frac{m}{8p}R(1, t+1), \tag{48}$$

where the first inequality is illustrated by Fig.1, and the second inequality follows from Lemma 6.1. This gives an estimation for the first term in (47).

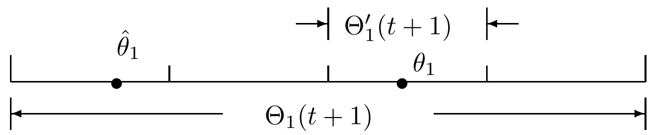

FIG. 1. *The choice of* Θ_{t+1}'

To estimate the second term, we first note that

$$\frac{R(i,t)}{R(i+1,t)} = \frac{|f_{i+1}(y_{t-i})|}{|f_i(y_{t-i})|}, \tag{49}$$

then by (42)

$$\begin{aligned}\frac{R(i,t+1)|f_i(y_{t+1})|}{R(i+1,t+1)|f_{i+1}(y_{t+1})|} &= \frac{|f_{i+1}(y_{t+1-i})|}{|f_i(y_{t+1-i})|} \cdot \frac{|f_i(y_{t+1})|}{|f_{i+1}(y_{t+1})|} \geq \frac{c_1^2}{c_2^2}\left|\frac{y_{t+1}}{y_{t+1-i}}\right|^{b_i - b_{i+1}} \\ &\geq \frac{32p^3M}{m}, \quad i = 1, 2, \cdots, p, \end{aligned} \tag{50}$$

which implies that $R(i, t+1)|f_i(y_{t+1})|$ is a non-increasing function of i, hence we have

$$\sum_{i=2}^{p+1} R(i, t+1)|f_i(y_{t+1})| \leq p \cdot R(2, t+1)|f_2(y_{t+1})|. \tag{51}$$

Furthermore, (50) implies

$$2p^2M \cdot R(2, t+1)|f_2(y_{t+1})| \leq \frac{m}{16p}R(1, t+1)|f_1(y_{t+1})|. \tag{52}$$

Now, by Lemma 6.1, (51) and (52), we have

$$\begin{aligned}\frac{1}{2}\sum_{i=2}^{p} |\Theta_i(t+1)||f_i(y_{t+1})| + w &\leq 2pM\sum_{i=2}^{p+1} R(i, t+1)|f_i(y_{t+1})| \\ &\leq 2p^2M \cdot R(2, t+1)|f_2(y_{t+1})| \leq \frac{m}{16p}R(1, t+1)|f_1(y_{t+1})|. \end{aligned} \tag{53}$$

Now, let $\Theta'(t+1) := \Theta'_1(t+1) \times \Theta_2(t+1) \times \cdots \times \Theta_p(t+1) \cap \Theta(t+1)$. Then, for any $(\theta_1, \theta_2, \cdots, \theta_p) \in \Theta'(t+1)$, and any $w_{p+1} \in [-w, w]$, by (47) (48) and (53)

$$\begin{aligned} |y_{t+2}| &\geq \frac{m}{8p} R(1, t+1)|f_1(y_{t+1})| - \frac{m}{16p} R(1, t+1)|f_1(y_{t+1})| \\ &\geq \frac{m}{16p} R(1, t+1)|f_1(y_{t+1})| \\ &\geq c_0 \Big|\frac{y_{t+1}}{y_t}\Big|^{b_1} \Big|\frac{y_t}{y_{t-1}}\Big|^{b_2} \cdots \Big|\frac{y_1}{y_0}\Big|^{b_{t+1}} \cdot |y_0|^{b_{t+2}}, \end{aligned} \tag{54}$$

where we have used the convention $f_t(\cdot) = 1, b_t = 0$ for $t > p$ as mentioned at the beginning of Section 5.1.

Now, taking logarithm on both sides of (47) gives

$$\ln|y_{t+2}| \geq b_1(\ln|y_{t+1}| - \ln|y_t|) + \cdots + b_{t+1}(\ln|y_1| - \ln|y_0|) + b_{t+2}\ln|y_0| + \ln c_0.$$

In order to apply Lemma 3.3, we take $a_i = \ln|y_{t-p+1+i}| - \ln c_0$, for $i = (p-t-1) \vee 0, \cdots, p+1$ and $a_i = 0$ for $i < (p-t-1) \vee 0$, and rewrite the above inequality into the following form

$$a_{p+1} \geq b_1(a_p - a_{p-1}) + b_2(a_{p-1} - a_{p-2}) + \cdots + b_p(a_1 - a_0) + b_{t+2}\ln c_0.$$

By C1 and taking logarithm on both sides of (43), we have $a_i - \lambda_1 a_{i-1} \geq 0$ for $i = 1, \cdots, p$. Note that $b_{t+2} = 0$ for $t+2 \geq p+1$, hence by Lemma 3.3 it is obvious that the desired conclusion holds. Hence, we need only to consider the case where $t < p-1$. To apply Lemma 3.3 again, we define $s = p-t-1$, and then by definition we have $a_s = \ln|y_0| - \ln c_0$ and $a_i = 0, i < s$. Furthermore, we define $\Delta = a_s - \lambda_1 a_{s-1} = \ln|y_0| - \ln c_0$. It is obvious that for sufficiently large $|y_0|$ we will have

$$\Big|\sum_{i=2}^{s+1}\prod_{j=i}^{p+i-s}\lambda_j\Big|\Delta + b_{t+2}\ln c_0 \geq 0, \tag{55}$$

since the first term is positive by (16).

Hence, by Lemma 3.3, we have $a_{p+1} - \lambda_1 a_p \geq 0$, which means

$$|y_{t+2}| \geq c_0^{1-\lambda_1}|y_{t+1}|^{\lambda_1} \geq d|y_{t+1}|.$$

□

REMARK 5.1. *The condition C1 always holds for $k = t = 0$, if the initial condition y_0 is large enough. This can be seen by taking $t = -1$ in the equation (54).*

5.3. The Proof of Necessity.. We use the contradiction method. Suppose that there exists an $x_0 \in (1, b_1)$ such that $P(x_0) \leq 0$, we proceed to show that for any feedback sequence $\{u_t\}$, there must exist at least one parameter vector $(\theta_1, \theta_2, \cdots, \theta_p)$

and a bounded sequence $\{w_t\}$ with the prescribed upper bound w, such that the corresponding dynamical system is not globally stable.

Since $P(x_0) \leq 0$ and $P(1) > 0$, there must exist a point $\lambda_1 \in (1, b_1)$ such that $P(\lambda_1) = 0$.

Taking the initial value $|y_0|$ large enough to satisfy the requirements in Propositions 5.1 and 5.2, and in Remark 5.1, we will first show inductively that there exists a sequence of domains $\{\Theta(t), \Theta'(t)\ , t \geq 0\}$ such that for any time $t \geq 0$, the conditions B2 and C1 hold.

At time $t = 0$, this assertion holds trivially, since $\Theta(0)$ satisfies B2 by the definition of $S(0)$, and since by Remark 5.1, there exists some $\Theta'(0)$ satisfying C1 and hence B1.

Suppose that B2 and C1 hold for some $t \geq 0$. Since C1 implies B1, by Proposition 5.1 we can construct $\Theta(t+1)$ defined as in (40) such that $\Theta(t+1) = S(t+1)$. Now according to Proposition 5.2, we can find some $\Theta'(t+1)$ satisfying C1 at time $t+1$. So B2 and C1 also hold at time $t+1$, and then hold for all the time by induction.

Finally, by the theorem for nested closed sets, we know that $\Theta(\infty) := \lim_{t\to\infty} \Theta(t) \neq \varnothing$. This means that there exists at least one parameter vector $(\theta_1, \theta_2, \cdots, \theta_p) \in \Theta(\infty)$ such that the corresponding output sequence diverges to infinity exponentially fast. This completes the proof of the necessity part of Theorem 2.1 . □

6. Appendix A. The proof of Proposition 5.1 is divided into three lemmas, which are given below.

LEMMA 6.1. *Under the conditions of Proposition 5.1, if* y_0 *is large enough, then (i)* $\Theta(t)$ *satisfies*

$$\begin{cases} \dfrac{m}{2p} R(i,t) \leq |\Theta_i(t)| \leq 4Mp \cdot R(i,t), & i = 1, 2, \cdots, t \wedge p \\ \Theta_j(t) = \Theta_j(0), & j = t+1, t+2, \cdots, p \end{cases}, \tag{56}$$

where

$$R(i,t) \triangleq \begin{cases} \dfrac{\prod_{s=i}^{t} |f_{s+1}(y_{t-s})|}{\prod_{s=i}^{t} |f_s(y_{t-s})|}, & i \leq t \\ 1, \quad i > t \end{cases}.$$

(ii) $\Theta'(t)$ *satisfies*

$$\begin{cases} |\Theta_i'(t)| \geq \dfrac{1}{4} |\Theta_i(t)|, & i = 2, \cdots, (t+1) \wedge p \\ \Theta_j'(t) = \Theta_j(t), & j = t+2, t+3, \cdots, p \end{cases}.$$

Proof. We first prove the Assertion (i). It holds trivially at time $t = 0$. For any time $t \geq 1$, let us denote $D(t) \triangleq |(d_{ij})_{(p\wedge t)\times(p\wedge t)}| \triangleq |\{f_j(y_{((t-p)\vee 0)+i-1})\}_{(p\wedge t)\times(p\wedge t)}|$ and let $D(t)^{j,i}$ be the determinant of the matrix with the i-th column in $D(t)$ being replaced by $(f_j(y_{0\vee t-p}),\cdots,f_j(y_{t-1}))^T$, and let $D(t)_{k,l}$ be the kl-th cofactor of the matrix in $D(t)$.

We remark that the upper and lower bounds to the $|D(t)^{j,i}|$ can be easily derived, since by the fact that the absolute value of the determinant of a matrix does not change if the $i-th$ column is moved to the last column, we know that similar to Lemma 3.1, we have for $i=1,\cdots t, j=t+1,\cdots,p$ if $t<p$,

$$\frac{1}{t}\le\frac{|D(t)^{j,i}|}{\prod_{s=1}^{i-1}|f_s(y_{t-s})|\cdot\prod_{s=i}^{t-1}|f_{s+1}(y_{t-s})||f_j(y_0)|}\le 2. \tag{57}$$

Now, by the definition of $S(t)$, we have

$$\begin{cases} y_{(t-p\vee 0)+1}=\theta_1 f_1(y_{t-p\vee 0})+\theta_2 f_2(y_{t-p\vee 0})+\cdots \\ \qquad\qquad +\theta_p f_p(y_{t-p\vee 0})+u_{t-p\vee 0}+w_{(t-p\vee 0)+1} \\ \vdots \\ y_t=\theta_1 f_1(y_{t-1})+\theta_2 f_2(y_{t-1})+\cdots+\theta_p f_p(y_{t-1})+u_{t-1}+w_t \\ \theta_j\in\Theta_j(0),\quad j=t+1,t+2,\cdots,p \\ w_k\in[-w,w], k=(t-p)\vee 0+1,\cdots,t. \end{cases}. \tag{58}$$

Let$(\theta_1^c(t-1),\theta_2^c(t-1),\cdots,\theta_{t\wedge p}^c(t-1))$ be the solution of the following equations

$$y_k=\sum_{l=1}^p\theta_l^c(t-1)f_l(y_{k-1})+u_{k-1},\quad k=(t-p)\vee 0+1,\cdots,t. \tag{59}$$

where $\theta_j^c(t-1)$ is the center point of $\Theta_j(0)$ for $j=t+1,\cdots,p$.

Then, subtracting the above equation from (58) and by using the Cramer rule, it is not difficult to see that $S(t)$ can be equivalently defined by

$$\begin{cases} \theta_i=\theta_i^c(t-1)-\sum_{j=t+1}^p(\theta_j-\theta_j^c(t-1))\dfrac{D(t)^{j,i}}{D(t)} \\ \qquad\qquad -\sum_{k=1}^{t\wedge p}w_{t-p\vee 0+k}\dfrac{D(t)_{k,i}}{D(t)},\quad 1\le i\le t\wedge p \\ \theta_j\in\Theta_j(0),\quad j=t+1,t+2,\cdots,p \\ w_k\in[-w,w],\quad k=(t-p)\vee 0+1,\cdots,t. \end{cases}. \tag{60}$$

Hence, by the Assumption B2 we see that $\Theta_j(t)=S_j(t)=\Theta_j(0),\ j=t+1,\cdots,p$. Then, by (60), (57), Lemma 3.1 and Lemma 3.2, the length of the interval in which θ_i belongs to can be bounded by

$$\begin{aligned} &\sum_{j=t+1}^p\frac{|D(t)^{j,i}|}{|D(t)|}|\Theta_j(0)|+2w\sum_{k=1}^{t\wedge p}\frac{|D(t)_{k,i}|}{|D(t)|} \\ &\le 2pM\frac{\prod_{s=i}^{t-1}|f_{s+1}(y_{t-s})|}{\prod_{s=i}^{t}|f_s(y_{t-s})|}(\sum_{j=t+1}^p|f_j(y_0)|+1) \\ &\le 4pM\frac{\prod_{s=i}^{t}|f_{s+1}(y_{t-s})|}{\prod_{s=i}^{t}|f_s(y_{t-s})|}=4pM\cdot R(i,t),\quad 1\le i\le t\wedge p \end{aligned} \tag{61}$$

where for the last but one inequality we have used the following fact which follows from (45),

$$|f_{t+1}(y_0)| \geq p|f_{t+2}(y_0)| \geq \sum_{l=t+2}^{p} |f_l(y_0)| + 1. \tag{62}$$

Similarly, we can obtain the lower bound to $|\Theta_i(t)|$, $1 \leq i \leq t \wedge p$ as in (56), hence (i) is true.

To prove (ii), we proceed to apply Lemma 3.4. Let us take $s = t \wedge p, r = p, \alpha = (\theta_1, \theta_2, \cdots, \theta_{t\wedge p})^T, \beta = (\theta_{t+1}, \cdots, \theta_p, w_{t-p+1} \cdots, w_t)^T$,

$$A_i = \left(-\frac{D(t)^{t+1,i}}{D(t)}, \cdots, -\frac{D(t)^{p,i}}{D(t)}, -\frac{D(t)_{1,i}}{D(t)} \cdots - \frac{D(t)_{t\wedge p,i}}{D(t)}\right),$$

where A_i is the i-th row of A. Set $E_j = \Theta_j(0)$, for $j = t+1, \cdots, p$ and $E_{p+1} = \cdots = E_{p+t} = [-w, w]$. Then by the definitions of E in (18) and $\Theta(t)$, it is evident that $E = \Theta(t)$, and so $E_1 = \Theta_1(t)$. Now let us take $E_1' = \Theta_1'(t)$, obviously, $E' = \Theta'(t)$. We divide our further discussion into two cases.

Case I) $t \geq p$: By Lemma 3.1, Assumption B1, (42) and (45),

$$\begin{aligned}
\left|\frac{a_{1,j}}{a_{1,1}}\right| &= \frac{|D(t)_{j,1}|}{|D(t)_{1,1}|} \leq \frac{2p\prod_{s=1}^{p-j}|f_{s+1}(y_{t-s})|\prod_{s=p-j+2}^{p}|f_s(y_{t-s})|}{\prod_{s=1}^{p-1}|f_{s+1}(y_{t-s})|} \\
&= \frac{2p\prod_{s=p-j+2}^{p}|f_s(y_{t-s})|}{\prod_{s=p-j+1}^{p-1}|f_{s+1}(y_{t-s})|} \qquad (63)\\
&\leq \frac{2p|f_p(y_{t-p})|}{|f_{p-j+2}(y_{t-p+j-1})|} \leq \frac{2pc_2}{c_1}\left|\frac{y_{t-p}}{y_{t-p+j-1}}\right|^{b_p} \\
&\leq \frac{2pc_2}{c_1}\left(\frac{1}{d}\right)^{b_p} \leq \frac{m}{4Mp}, \quad j = 2, \cdots, p. \qquad (64)
\end{aligned}$$

Case II) $t \leq p-1$: By Lemma 3.1 and (63), taking sufficiently large $|y_0|$,

$$\left|\frac{a_{1,j}}{a_{1,1}}\right| = \frac{|D(t)^{j+t,1}|}{|D(t)^{t+1,1}|} \leq \frac{2p|f_{j+t}(y_0)|}{|f_{t+1}(y_0)|} \leq \frac{m}{4Mp}, \quad j = 2, \cdots, p-t. \tag{65}$$

$$\left|\frac{a_{1,j}}{a_{1,1}}\right| \leq \frac{|D(t)_{1,1}|}{|D(t)^{t+1,1}|} \leq \frac{2p}{|f_{t+1}(y_0)|} \leq \frac{m}{4Mp}, \quad j = p-t+1, \cdots, p. \tag{66}$$

Therefore, Lemma 3.4 can be applied, and this completes the proof. □

LEMMA 6.2. *Let $\theta_l^c(0)$ be the center points of $\Theta_l(0)$, $l = t+2, \cdots, p$. Under the conditions of Proposition 5.1, if y_0 is large enough, we can find some point $(\theta_1^c(t), \theta_2^c(t), \cdots, \theta_p^c(t)) \in \Theta'(t)$ satisfying*

$$y_k = \sum_{l=1}^{p} \theta_l^c(t) f_l(y_{k-1}) + u_{k-1}, \quad k = (t-p+2) \vee 1, \cdots, t. \tag{67}$$

and

$$(68)\quad \begin{cases} \theta_j^c(t) = \theta_j^c(0), \quad j = t+2, t+3\cdots, p \\ \min\{|\underline{\theta}_i'(t) - \theta_i^c(t)|, |\overline{\theta}_i'(t) - \theta_i^c(t)|\} \geq \dfrac{m}{32p} R(i,t), \quad i = 1, 2, \cdots, (t+1)\wedge p \end{cases}$$

where $\Theta_i'(t) = [\underline{\theta}_i'(t), \overline{\theta}_i'(t)]$ *and* $R(i,t)$ *is defined in Lemma 6.1.*

Proof. Let us introduce the definition:

$$\Gamma(t) := \{(\theta_1, \cdots, \theta_p) \in S(t) : y_k = \sum_{l=1}^{p} \theta_l f_l(y_{k-1}) + u_{k-1} \text{ for } k = 1 \vee (t-p+2), \cdots, t$$
$$\theta_j = \theta_j^c(0),\ j = t+2, \cdots, p\}.$$

Obviously $\Gamma(t) \subset S(t)$ is a line segment, which is formed by taking $w_{1\vee(t-p+2)} = \cdots = w_t = 0$ and $\theta_j = \theta_j^c(0) = \theta_j^c(t-1)$, $j = t+2, \cdots, p$, in (59) and (60). Hence, it is not difficult to see that any $\theta \in \Gamma(t)$ can be equivalently defined by

$$(69)\quad \begin{cases} \theta_i = \theta_i^c(t-1) - (\theta_{t+1} - \theta_{t+1}^c(t-1)) \dfrac{D(t)^{t+1,i}}{D(t)} - w_{t-p+1} \dfrac{D(t)_{1,i}}{D(t)}, \quad 1 \leq i \leq t \wedge p \\ \theta_{t+1} \in \Theta_{t+1}(0); \quad \theta_j = \theta_j^c(0),\ j = t+2, \cdots, p \end{cases}$$

where $\theta_i^c(t-1)$ is defined in (59) and $w_k := 0$ for $k \leq 0$.

Now, define $\Gamma'(t) := \Gamma(t) \cap \Theta'(t)$. To estimate $|\Gamma_1'(t)|$, we first estimate $\Delta := |S_1(t) - \Gamma_1(t)|$. We prove this lemma by considering two cases separately.

Case I) $t \leq p-1$: Comparing (69) with $S(t)$ in (60), and similar to (61), we have by Lemma 6.1

$$\begin{aligned} \Delta &\leq \left(\sum_{j=t+2}^{p} |\Theta_j(0)| \left| \frac{D(t)^{j,1}}{D(t)} \right| + 2w \sum_{k=1}^{t} \left| \frac{D(t)_{k,1}}{D(t)} \right| \right) \\ &\leq 4pM \frac{\prod_{s=1}^{t-1} |f_{s+1}(y_{t-s})| \cdot |f_{t+2}(y_0)|}{\prod_{s=1}^{t} |f_s(y_{t-s})|} \\ &\leq \frac{m}{16p} R(1,t) \leq \frac{1}{8} |\Theta_1(t)|, \end{aligned} \quad (70)$$

where the third inequality holds by taking $|y_0|$ large enough so that

$$4pM|f_{t+2}(y_0)| \leq \frac{m}{16p} |f_{t+1}(y_0)|.$$

Now, notice that $\Theta(t) = S(t)$, by (70) and the definition of Δ, we know that

$$\begin{aligned} |\Gamma_1'(t)| &= |\Gamma_1(t) \cap \Theta_1'(t)| = |(\Theta_1(t) - \Delta) \cap \Theta_1'(t)| \\ &= |\Theta_1'(t)| - |\Delta \cap \Theta_1'(t)| \geq |\Theta_1'(t)| - \Delta \\ &\geq \frac{1}{4} |\Theta_1(t)| - \frac{1}{8} |\Theta_1(t)| = \frac{1}{8} |\Theta_1(t)|. \end{aligned} \quad (71)$$

Let $\theta_i^c(t)$ be the center point of $\Gamma_i'(t)$, $i = 1, 2, \cdots, t+1$ and $\theta_j^c(t) \triangleq \theta_j^c(0)$, $j = t+2, \cdots, p$. Obviously, $(\theta_1^c(t), \theta_2^c(t), \cdots, \theta_p^c(t)) \in \Gamma'(t) \subset \Theta'(t)$ by the definition of $\Gamma'(t)$, and hence (67) holds.

Now, by (69), we know that for any $(\theta_1, \cdots, \theta_p) \in \Gamma'(t)$, its first $t+1 (\leq p)$ components can be rewritten as

$$\theta_1 \in \Gamma_1'(t), \quad \theta_i = \frac{D(t)^{t+1,i}}{D(t)^{t+1,1}}\theta_1 + e_i(t), \quad i = 2, \cdots, t+1, \tag{72}$$

where $e_i(t)$, $i = 1, \cdots, t+1$, are some constants.

Now, let us denote $\Gamma_i'(t) = [\underline{\gamma}_i(t), \overline{\gamma}_i(t)]$ for $i = 1, \cdots, t+1$. Since $\theta_i^c(t)$ is the center point of $\Gamma_i'(t)$, by (72), (71), (57), (49) and Lemma 6.1, we have

$$\begin{aligned}
|\theta_i^c(t) - \underline{\gamma}_i(t)| &= |\overline{\gamma}_i(t) - \theta_i^c(t)| = |\overline{\gamma}_1(t) - \theta_1^c(t)| \Big| \frac{D(t)^{t+1,i}}{D(t)^{t+1,1}} \Big| \geq \frac{1}{16} |\Theta_1(t)| \Big| \frac{D(t)^{t+1,i}}{D(t)^{t+1,1}} \Big| \\
&\geq \frac{m}{32p} R(1,t) \cdot \prod_{s=1}^{i-1} \frac{|f_s(y_{t-s})|}{|f_{s+1}(y_{t-s})|} \\
&= \frac{m}{32p} R(1,t) \cdot \prod_{s=1}^{i-1} \frac{R(s+1,t)}{R(s,t)} = \frac{m}{32p} R(i,t),
\end{aligned}$$

which in conjunction with $\Gamma'(t) \subset \Theta'(t)$ gives (68).

Case II) $t \geq p$: To estimate Δ, we first notice that by (42) and by using the same argument as that used from (63) to (64), we have

$$\frac{2pM \prod_{s=p-k+2}^{p} |f_s(y_{t-s})|}{\prod_{s=p-k+1}^{p-1} |f_{s+1}(y_{t-s})|} \leq \frac{2pMc_2}{c_1} \left(\frac{1}{d}\right)^{b_p} \leq \frac{1}{p} \cdot \frac{m}{16p}, \quad k = 2, \cdots, p. \tag{73}$$

Now, comparing the definition of (69) with (60), we know that the only difference is the absence of noises at times $t-p+2, \cdots, t$ in the equations in $\Gamma(t)$. Hence by (73), Lemma 3.1, we have

$$\begin{aligned}
\Delta &\leq \sum_{k=2}^{p} \Big| \frac{D(t)_{k,1}}{D(t)} \Big| w \leq 2pM \sum_{k=2}^{p} \frac{\prod_{s=1}^{p-k} |f_{s+1}(y_{t-s})| \prod_{s=p-k+2}^{p} |f_s(y_{t-s})|}{\prod_{s=1}^{p} |f_s(y_{t-s})|} \\
&\leq \frac{m}{16p} \frac{\prod_{s=1}^{p-1} |f_{s+1}(y_{t-s})|}{\prod_{s=1}^{p} |f_s(y_{t-s})|} = \frac{m}{16p} R(1,t) \leq \frac{1}{8} |\Theta_1(t)|,
\end{aligned}$$

which is the same as (70). This completes the proof of Lemma 6.2 since the rest of the proof is completely similar to Case I. □

LEMMA 6.3. *The conclusion of Proposition 5.1 holds.*

Proof. Note that for any given u_t, the set $\Theta(t+1)$ defined as in (40) depends on the value of y_{t+1}, and our desired parameter set $\Theta(t+1)$ will be constructed to be consistent with the following output value:

$$y_{t+1} = \sum_{l=1}^{p} \theta_l^c(t) f_l(y_t) + u_t, \tag{74}$$

where $(\theta_1^c(t), \theta_2^c(t), \cdots, \theta_p^c(t))$ is defined in Lemma 6.2.

We divide the proof into two cases.

Case I): $t \le p-1$. Since Lemma 6.1 gives $\Theta_j'(t) = \Theta_j(0)$, $j = t+2, \cdots, p$, the only difference between $S(t+1)$ and $\Theta(t+1)$ is the constraint on $\theta_i \in \Theta_i'(t)$, $i = 1, 2, \cdots, t+1$ in $\Theta(t+1)$. Obviously, $\Theta(t+1) \subset S(t+1)$, so if we can show that $S(t+1) \subset \Theta(t+1)$, then the two sets will be equal as desired.

Now, by the definition of the point $(\theta_1^c(t), \theta_2^c(t), \cdots, \theta_p^c(t))$ and (74), we have

$$y_k = \sum_{l=1}^{p} \theta_l^c(t) f_l(y_{k-1}) + u_{k-1}, \quad k = 1, 2, \cdots, t+1, \tag{75}$$

hence, similar to (61), it can be shown that $|S_i(t+1)|, i = 1, 2, \cdots, t+1$ is bounded by $4pM \cdot R(i, t+1)$. To further estimate this bound, we first notice that by (42) and (45),

$$\begin{aligned} \frac{R(i,t)}{R(i,t+1)} &= \left(\prod_{s=i}^{t} \Big| \frac{f_{s+1}(y_{t-s})}{f_s(y_{t-s})} \Big| \Big| \frac{f_s(y_{t+1-s})}{f_{s+1}(y_{t+1-s})} \Big| \right) \cdot \Big| \frac{f_{t+1}(y_0)}{f_{t+2}(y_0)} \Big| \\ &\ge \prod_{s=i}^{t} \left(\frac{c_1^2}{c_2^2} \Big| \frac{y_{t+1-s}}{y_{t-s}} \Big|^{b_s - b_{s+1}} \right) \ge \frac{c_1^{2p}}{c_2^{2p}} \Big| \frac{y_1}{y_0} \Big|^{b_t - b_{t+1}} \ge \frac{64p^2 M}{m}. \end{aligned} \tag{76}$$

Hence by the above fact and (68),

$$\frac{1}{2}|S_i(t+1)| \le 2pM \cdot R(i, t+1) \le \frac{m}{32p} R(i,t) \tag{77}$$

$$\le \min\{|\underline{\theta}_i'(t) - \theta_i^c(t)|, |\overline{\theta}_i'(t) - \theta_i^c(t)|\}. \tag{78}$$

Since $S(t+1)$ can be equivalently defined as in (60) with t replaced by $t+1$ by (75), it is easy to see that $\theta_i^c(t)$ is the center point of the feasible interval for θ_i in $S_i(t+1)$. Then we have $S_i(t+1) \subset \Theta_i'(t)$, $i = 1, 2, \cdots, t+1$ by (78), which implies $S(t+1) \subset \Theta(t+1)$, and hence the two sets are equal.

Case II): $t \ge p$. In this case, the number of equations in the definition of $\Theta(t+1)$ is more than that in $S(t+1)$. Since $\Theta(t)$=$S(t)$ by our assumption, we know that there is only one equation, ie., the equation at time $t-p+1$ which constricts on $\Theta(t+1)$ but not $S(t+1)$. Now, let us introduce the following set,

$$\begin{aligned} \Phi(t-p+1) := \{(\theta_1, \cdots, \theta_p) \in \prod_{l=1}^{p-1} \Theta_l'(t) \times R : y_{t-p+1} = \sum_{l=1}^{p} \theta_l f_l(y_{t-p}) + u_{t-p} + w_{t-p+1}, \\ \text{for some } |w_{t-p+1}| \le w\}. \end{aligned}$$

Obviously, $\Theta(t+1) = S(t+1) \cap \prod_{l=1}^{p} \Theta_l'(t) \cap \Phi(t-p+1)$, so if we can show $S(t+1) \subset \prod_{l=1}^{p} \Theta_l'(t)$ and $S(t+1) \subset \Phi(t-p+1)$, we have $S(t+1) = \Theta(t+1)$ as desired.

First, we prove $S(t+1) \subset \prod_{l=1}^{p} \Theta'_l(t)$. We notice that by using a similar argument as for (76), we can further get the following by (42)

$$\frac{R(l,t)}{R(l,t+1)} \geq \frac{128p^2M}{m},$$

and similar to (77), we have $\frac{1}{2}|S_l(t+1)| \leq \frac{m}{64p}R(l,t)$ for $l = 1, 2, \cdots, p$.

Moreover, as noted in Case I, $\theta_l^c(t)$ is the center point of $S_l(t+1)$, $l = 1, \cdots, p$, hence we have from the above inequality and Lemma 6.2 that

$$\prod_{l=1}^{p} S_l(t+1) \subset \prod_{l=1}^{p} \left[\theta_l^c(t) - \frac{m}{64p}R(l,t), \quad \theta_l^c(t) + \frac{m}{64p}R(l,t)\right] \tag{79}$$

$$\subset \prod_{l=1}^{p} |\Theta'_l(t)|. \tag{80}$$

Next, we proceed to prove $S(t+1) \subset \Phi(t-p+1)$. Without loss of generality, we assume that $f_p(y_{t-p}) > 0$ for convenience. We need two more definitions:

$$\begin{aligned} &\overline{\Phi}(t-p+1) \\ &= \{(\theta_1, \cdots, \theta_p) \in \prod_{l=1}^{p-1} \Theta'_l(t) \times R : y_{t-p+1} = \sum_{l=1}^{p} \theta_l f_l(y_{t-p}) + u_{t-p} - w\}, \end{aligned} \tag{81}$$

$$\begin{aligned} &\underline{\Phi}(t-p+1) \\ &= \{(\theta_1, \cdots, \theta_p) \in \prod_{l=1}^{p-1} \Theta'_l(t) \times R : y_{t-p+1} = \sum_{l=1}^{p} \theta_l f_l(y_{t-p}) + u_{t-p} + w\}. \end{aligned} \tag{82}$$

Notice that $\Theta'(t) \subset \Phi(t-p+1)$, by the assumption $f_p(y_{t-p}) > 0$ we have

$$\Theta'_p(t) \subset \Phi_p(t-p+1) = [\inf \underline{\Phi}_p(t-p+1), \sup \overline{\Phi}_p(t-p+1)],$$

which implies

$$\sup \overline{\Phi}_p(t-p+1) \geq \overline{\theta}'_p(t) \qquad \text{and} \qquad \inf \underline{\Phi}_p(t-p+1) \leq \underline{\theta}'_p(t). \tag{83}$$

Now, we estimate the upper bound of $|\overline{\Phi}_p(t-p+1)|$ and $|\underline{\Phi}_p(t-p+1)|$. First, we need to estimate the following inequality by (42) and (49) for $l = 2, 3, \cdots, p$,

$$\frac{R(l,t)|f_l(y_{t-p})|}{R(l-1,t)|f_{l-1}(y_{t-p})|} \geq \frac{c_1^2}{c_2^2}\left|\frac{y_{t-l+1}}{y_{t-p}}\right|^{b_{l-1}-b_l} \geq \frac{256p^3M}{m}, \tag{84}$$

which implies that $R(l,t)|f_l(y_{t-p})|$ is a increasing function of l. Now, by (84) and Lemma 6.1, $|\overline{\Phi}_p(t-p+1)|$ and $|\underline{\Phi}_p(t-p+1)|$ are bounded by

$$\begin{aligned} &\sum_{l=1}^{p-1} |\Theta'_l(t)| \frac{|f_l(y_{t-p})|}{|f_p(y_{t-p})|} \leq 4pM \sum_{l=1}^{p-1} R(l,t) \frac{|f_l(y_{t-p})|}{|f_p(y_{t-p})|} \\ &\leq 4p^2M \cdot R(p-1,t) \frac{|f_{p-1}(y_{t-p})|}{|f_p(y_{t-p})|} \leq \frac{m}{64p} R(p,t). \end{aligned}$$

By the above inequality, (68) and (83) we have

$$\begin{aligned}\inf \overline{\Phi}_p(t-p+1) &\geq \sup \overline{\Phi}_p(t-p+1) - \frac{m}{64p}R(p,t) \\ &\geq \overline{\theta}'_p(t) - \frac{m}{64p}R(p,t) \geq \theta^c_p(t) + \frac{m}{64p}R(p,t),\end{aligned}$$

and similarly,

$$\sup \underline{\Phi}_p(t-p+1) \leq \theta^c_p(t) - \frac{m}{64p}R(p,t).$$

Then by (79), the two inequalities above gives,

$$\begin{aligned}S_p(t+1) &\subset [\theta^c_p(t) - \frac{m}{64p}R(p,t),\ \theta^c_p(t) + \frac{m}{64p}R(p,t)] \\ &\subset [\sup \underline{\Phi}_p(t-p+1),\ \inf \overline{\Phi}_p(t-p+1)]. \qquad (85)\end{aligned}$$

Now, notice that for any $(\theta^*_1, \cdots, \theta^*_p) \in S(t+1)$, we have $\theta^*_l \in \Theta'_l(t)$, $l = 1, \cdots, p-1$ by (80). So, by solving θ_p in terms of θ^*_l , $l = 1, \cdots, p-1$ respectively for the equations in (81) and (82), we have by (85)

$$\frac{y_{t-p+1} - \sum_{l=1}^{p-1} \theta^*_l f_l(y_{t-p}) - u_{t-p} - w}{f_p(y_{t-p})} \leq \theta^*_p \leq \frac{y_{t-p+1} - \sum_{l=1}^{p-1} \theta^*_l f_l(y_{t-p}) - u_{t-p} + w}{f_p(y_{t-p})},$$

namely,

$$|y_{t-p+1} - \sum_{l=1}^{p} \theta^*_l f_l(y_{t-p}) - u_{t-p}| \leq w.$$

Consequently, by the definition of $\Phi(t-p+1)$, we have $(\theta^*_1, \cdots, \theta^*_p) \in \Phi(t-p+1)$. Hence $S(t+1) \subset \Phi(t-p+1)$, and the proof of Lemma 6.3 is completed. □

Acknowledgement. The authors are grateful to Prof. Y.Tang for a correction in the original proof of Lemma 3.1.

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