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Dynamic treatment regimes: the mathematics of unstable switched systems

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In this paper we begin the study of dynamic treatment regimes. We use switched systems that are neither stable for all switching patterns nor unstable for all switching patterns. We find conditions on the dynamics of the systems for which we can contain the orbits between two concentric balls. A patient with a chronic condition cannot be cured but on the other hand it is necessary to keep the condition from becoming more serious. In the treatment of chronic conditions such as high blood pressure, Type II diabetes, obesity, addictions and many others a "cure" is not really possible and so the goal of treatments is to maintain the patient at an acceptable level. It is well understood that a single treatment is seldom successful over a long time periods and therefore, treatments are adapted to the patient. Medicines are changed or modified and in many such treatments counseling is adapted to the patient along with pharmaceuticals. All of these cases are compatible with switched systems. While there is a significant amount of work being done on modeling the bodys response to treatment there has been hardly any effort to model multiple treatments. This paper is a step in that direction.

1. Introduction

It has always been standard medical practice when treating a difficult disease, such as cancer, to use multiple treatments, sometimes simultaneously but more often in sequence. The question that we pose in this paper, one of the main question of dynamic treatment scheduling, is "When to switch treatments?" Susan Murphy has pioneered the idea of decision rules for this problem in her papers, [11, 15]. Those papers are an excellent source of information about dynamic regimes or as they are sometimes referred adaptive treatment regimes. Murphy has provided a very detailed statistical analysis of such trials as has Marie Davidian and her colleagues, [16, 17]. In this paper we do not attempt to analyze the statistics but we will present a schematic to model such treatments using the theory of switched dynamical systems.

The theory of switched systems is well developed in the area of stability. Dayawansa and Martin, [2] gave the first necessary and sufficient condition for stability of switched systems and an excellent book by Daniel Lieberzon, [9], surveys the status of the theory up to the time of publishing. In the recent past the stability theory has been greatly advanced by the work of Ogura, [12–14]. The concept of instability has been avoided in the literature for the simple reason that in engineering the focus is almost always on the stability of systems. Du and Martin, [5], use the fact that there is a pair of stable linear systems that can be driven to infinity by switching. In that paper they show that to drive a system to infinity the switches should take place when the norm of the subsystem is locally maximum and relate this to dynamical clinical trials. There are a few results on controllability of switched linear systems which relate to instability. See, for example, the paper of Klamka and coauthors, [6].

Recently there has been significant work of the construction of mathematical models of the treatment of certain tumors. Dorothy Wallace, [4, 10], and colleagues have written a series of papers describing the reactions of tumors to various types of chemotherapy. These models are basically linear but nonlinearity is introduced when certain rates are bounded. Ledzewicz and Schttler, [7, 8], have developed very involved models of the growth of tumors. However, these models are based on partial differential equations and are difficult to use in the theory of switched systems.

In the treatment of chronic conditions such as high blood pressure, Type II diabetes, obesity, addictions and many others a “cure” is not really possible and so the goal of treatments is to maintain the patient at an acceptable level. It is well understood that a single regime of treatment is seldom successful over a long time periods and so treatments are adapted to the patient. Medicines are changed or simply modified and in many such treatments counseling is adapted to the patient along with pharmaceuticals. All of these cases are compatible with switched systems. There is a significant amount of work being done on modeling the bodys response to treatment. What has not been done is to model multiple treatments. This paper is a step in that direction.

In this paper we attack the problem of instability directly. We will work with a system of two dynamical systems. It would not be difficult to include any finite number of systems but it would complicate the statement of results. The basic system will be

$$\dot{x}(t) = \delta f(x(t)) + (1 - \delta)g(x(t)) \tag{1.1}$$

where $\delta \in \{0, 1\}$. We seek conditions on f and g so that the system is not asymptotically stable and conditions under which we can force

$$\|x(t)\| \geq \alpha$$

by choice of δ and where α is a predetermined constant. It will also be of great interest to determine when

$$\beta \leq \|x(t)\| \leq \alpha.$$

This condition is relevant in cancer treatment when it is desired to keep a tumor from growing but not necessarily to destroy it. This is sometimes the case in the treatment of childhood tumors. It is also the standard when treating a chronic disease. It is seldom possible to affect a cure and the goal is to keep the symptoms under control. In Type II diabetes the goal is to keep the A1C at a low level, say between 6 and 7. Inshort it means excursions above this level are undesirable but if the level can be brought back to the interval the patient will do well. The same is true for high blood pressure; the goal is not to ensure that the patient's blood pressure never gets outside the normal range but instead, to ensure that the bllod pressure stays at a moderate level. Recently HIV patients are considered to have a chronic disease and are treated accordingly.

In the next section we give the basic theorems for switched systems which are the basis for the theory. We give conditions on the models for which the trajectories can be contained in an annulus. In section three we examine a model which meets the conditions of the theorem. In the last section we conclude the paper. The proofs of the propositions are moved to an appendix for the benefit of the reader.

2. The models

We will build a series of models that are based on switching systems that are unstable in the sense that they have no common Lyapunov function, [2]. We will work with systems in which each individual system is stable and will find conditions that will allow us to force a sequence of switches that keep the system away from the origin. We begin with systems that are Lyapunov stable. Recall that Lyapunov stability implies that given $\epsilon > 0$ there exists a $\delta > 0$ so that whenever $\|x(0)\| < \delta$ we have $\|x(t)\| < \epsilon$. In other words, if the initial data is small, the state remains small. For example, the system $\ddot{x} = -x$ is Lyapunov stable but is not asymptotically stable. Thus, to state that for a Lyapunov function $V(x(t))$ to have the property that $\dot{V}(x(t)) = 0$ is to simply imply that $x(t)$ is following the level sets of $V(x)$. Through out we will denote the complement of a set S by S' and the closure of set B by $\bar{B}$.

In this first part we are relying on a theorem in [9].

Theorem 2.1. *[9] Let $\dot{x} = f_p(x)$, $p \in P$ be a finite family of globally asymptotically stable systems, and let V_p, $p \in P$ be a family of corresponding radially unbounded Lyapunov functions. Let $\sigma(t) : [0, \infty) \to P$ be the switching signal. Suppose that there exists a family of positive definite continuous functions W_p, $p \in P$ such that for every pair of switching times (t_i, t_j),$i < j$ such that $\sigma(t_i) = \sigma(t_j) = p$ and $\sigma(t_k) \neq p$ for $t_i < t_k < t_j$, we have*

$$V_p(x(t_j)) - V_p(x(t_i)) \leq -W_p(x(t_i)).$$

Then, the switched system is globally asymptotically stable.

First, we consider a family of systems where each member is Lyapunov stable but not asymptotically stable. We will begin our analysis with a very simple proposition. The following proposition states that if the trajectory starts from a point outside of a level set at the initial time the system p becomes active, the trajectory remains in the complement of that level set at least until the next switching time.

Proposition 2.2. *[Trajectory outside a Level Set] Let $\dot{x} = f_p(x)$, $p \in P$ be a switched system and each member of the system is stable in the sense of Lyapunov with $\dot{V}_p(x) = 0$, where $V_p(x) : R^n \to R$ is the Lyapunov function associated with f_p. Let $\sigma(t) : [0, \infty) \to P$ be the switching signal. If $\sigma(t) = p$, for all t such that $t_{i_k} \leq t < t_{j_k}$, and $x(t_{i_k}) \in S'$, where $S = \{x : \ V_p(x) \leq m \text{ and } m > 0\}$, then, $x(t) \in S'$, for all t such that $t_{i_k} \leq t < t_{j_k}$.*

Proof. The proof is trivial and thus, omitted. □

The idea is to violate the conditions of Theorem 2.1 to understand how a trajectory can be kept away from the origin for such systems. The following proposition is such an example and it provides checkable conditions to keep the trajectory out of a ball of given radius. Proposition 2.2 is used in the proof of this proposition. Constraints are placed on the initial condition and

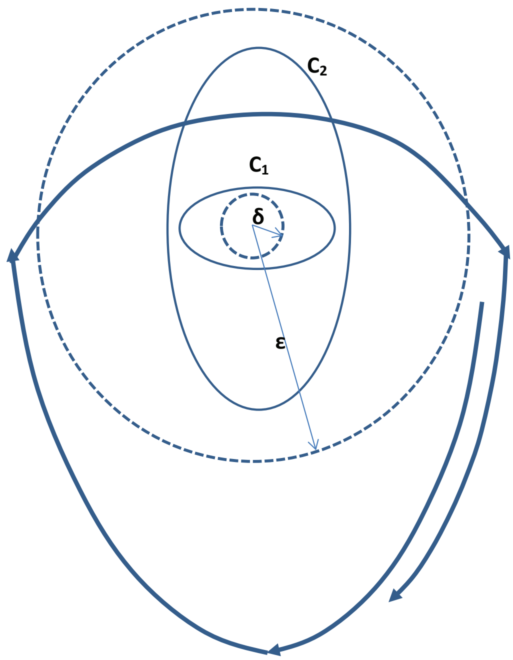

Figure 1: The concept of Proposition 2.3.

the class of admissible switching signal in the sense that certain relationships between Lyapunov functions must be satisfied at the switching times.

Proposition 2.3. *(Nonasymtotic Case) Let* $\dot{x} = f_p(x)$, $p \in P = \{1, 2\}$ *be a switched system and each member of the system is stable in the sense of Lyapunov with* $\dot{V}_p(x) = 0$, *where* $V_p(x) : R^n \to R$ *is the Lyapunov function associated with* f_p. *Let* $\sigma(t) : (0, \infty) \to P$ *be the switching signal. For any given* $\delta > 0$, *there exists an* $\epsilon > 0$ *such that* $x(0) \in B'_\epsilon(0)$ *implies* $x(t) \in B'_\delta(0)$, *for all* $t \geq 0$ *provided that for every pair of switching time* (t_i, t_j), $i < j$ *such that* $\sigma(t_i) = \sigma(t_j) = p$ *and* $\sigma(k) \neq p$ *for* $t_i < t_k < t_j$, *we have*

$$V_p(x(t_j)) > V_p(x(t_i)).$$

Proof. The proof is moved to the appendix. □

It would be rare for the conditions of Proposition 2.3 to be met in practice. The conditions on asymptotic stability are used to model a situation in which the origin represents death or at least complete failure of a treatment. Thus we are assuming that both treatments will fail eventually and the goal is to use multiple treatments to keep the system away for the origin. Keeping the trajectory on a surface is almost never feasible.

We now turn to the case of a family of asymptotically stable systems and here we will use a theorem from [2]. This theorem essentially states that a switched system is globally asymptotically stable if and only if each of the subsystems has a common Lyapunov function. Theoretically this is a very powerful theorem but it is extremely hard to manufacture non quadratic Lyapunov functions. It is known from an example in the that paper that quadratic Lyapunov functions do not suffice. The following proposition provides sufficient conditions to keep the trajectory outside a ball of a given radius for a family of switched systems with asymptotically stable members. In this case the switching needs to be sufficiently fast to keep the trajectory outside of a ball, therefore, additional constraints are placed on the dwell times.

Proposition 2.4. *(Asymptotic Case) Let $\dot{x} = f_p(x)$, $p \in P = \{1, 2\}$ be a switched system and each member of the system is globally and asymptotically stable with $\dot{V}_p(x) \geq -k_p$, $k_p > 0$, where $V_p(x) : R^n \to R$ is the Lyapunov function associated with f_p. Let $\sigma(t) : [0, \infty) \to P$ be the switching signal and suppose the switching sequence starts with $\sigma = 2$, for all t, $0 \leq t < t_1$. For any given $\delta > 0$, there exists an $\epsilon > 0$ such that $x(0) \in B'_\epsilon(0)$ implies $x(t) \in B'_\delta(0)$, for all $t \geq 0$ provided the followings are true.*

1) *For every pair of switching time (t_i, t_j), $i < j$ such that $\sigma(t_i) = \sigma(t_j) = p$ and $\sigma(k) \neq p$ for $t_i < t_k < t_j$, we have*

$$V_p(x(t_j)) > V_p(x(t_i)).$$

2) *If $\sigma(t) = p$, $t_{i_p} \leq t < t_{i_p+1}$, then,*

$$t_{i_p+1} - t_{i_p} < (V_p(x(t_{i_p})) - m_p)/k_p,$$

where $m_1 = \max_{x \in \bar{B}_\delta} V_1(x)$ and m_2 is any positive number such that $C_1 = \{x : V_1(x) \leq m_1\} \subset C_2 = \{x : V_2(x) \leq m_2\}$.

Proof. The proof is moved to the appendix. □

While it is desirable to keep the trajectory away from the origin in the context where the origin corresponds to malignant equilibrium, which represents the patients' death, it is also important that the trajectory be bounded since the unbounded trajectory will cease to be physically meaningful. Proposition 2.5 states the conditions under which the trajectory is bounded.

Proposition 2.5. *(Bounded trajectory) Let $\dot{x} = f_p(x)$, $p \in P = \{1, 2\}$ be a switched system and each member of the system is globally and asymptotically stable (GAS) and $V_p(x) : R^n \to R$ is the Lyapunov function associated with f_p. Let $\sigma(t) : [0, \infty) \to P$ be the switching signal and let t_{p_i}, $i = 1, 2, \ldots$ be switching times at which f_p is activated for the i^{th} time. Then, there exists a $\delta > 0$ and a finite N such that $||x(t)|| < N$, for all t, $t \geq 0$, provided $||x(0)|| < \delta$ and the following sum converges for all $p \in P$.*

$$\sum_{i=1}^{\infty} \Delta V_{p_i} = M_p < \infty,$$

where $\Delta V_{p_i} = |V(x(t_{p_i})) - V(x(t_{p_{i-1}}))|$, and $\Delta V_{p_1} = V_p(x(t_{p_1}))$.

Proof. The proof is moved to the appendix. □

In some situation, it might be necessary to contain the trajectory within a ball of of a certain radius. The following proposition provides the sufficient conditions for containing the trajectory in a ball of given radius.

Proposition 2.6. *Let $\dot{x} = f_p(x)$, $p \in P = \{1, 2\}$ be a switched system and each member of the system is globally and asymptotically stable (GAS) and $V_p(x) : R^n \to R$ is the Lyapunov function associated with f_p. Let $\sigma(t) : [0, \infty) \to P$ be the switching signal and let t_{p_i}, $i = 1, 2, \ldots$ be switching times at which f_p is activated for the i^{th} time. Then, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $||x(0)|| < \delta$ implies that $||x(t)|| \leq \epsilon$ provided the following sum converges for all $p \in P$ and*

$$\sum_{i=1}^{\infty} \Delta V_{p_i} < M_p,$$

where ΔV_{p_i} are defined as in Proposition 2.5 and M_p is the largest number such that $S_p = \{x : V_p(x) \leq M_p\} \in B_\epsilon(0)$.

Proof. The proof is moved to the appendix. □

The following theorem generalizes the previous results. It provides the sufficient condition to confine the trajectory in an annulus consisting of two given radius ϵ_1 and ϵ_2. This theorem ties together all the previous results and the previous propositions are special cases of the following Theorem.

Theorem 2.7. *Let* $\dot{x} = f_p(x), p \in P = \{1, 2\}$ *be a switched system and each member of the system is globally and asymptotically stable (GAS) and* $V_p(x) : R^n \to R$ *is the Lyapunov function associated with* f_p. *Let* $\sigma(t) : [0, \infty) \to P$ *be the switching signal, which is continuous from the right. Given* $0 < \epsilon_1 < \epsilon_2$, *let the set* $K_p = \{x : V_p(x) \leq N_p\}$ *be the the smallest level set of* V_p *that contains* $B_{\epsilon_1}(0)$, *and the set* $L_p = \{x : V_p(x) \leq M_p\}$ *be the the largest level set of* V_p *that is contained in* $B_{\epsilon_2}(0)$. *If* $N_p < M_p$, *then, the followings are sufficient conditions for confining the trajectory in* $B'_{\epsilon_1}(0) \bigcap B_{\epsilon_2}(0)$.

1) $M_p \geq V_p(x(0)) + \sum_{i=0}^{n-1} \int_{t_{2i+p-1}}^{t_{2i+p}} L_{f_p} V_p(x) dt + \int_{t_{2i+p}}^{t_{2i+p+1}} L_{f_q} V_p(x) dt.$
2) $N_p \leq V_p(x(0)) + \sum_{i=0}^{n-1} \int_{t_{2i+p}}^{t_{2i+p+1}} L_{f_p} V_p(x) dt + \int_{t_{2i+p+1}}^{t_{2i+p+2}} L_{f_q} V_p(x) dt.$
3) *The inequalities also hold when* $n \to \infty$,
   for all $p \in P$, $p \neq q$ *and for all* $n \geq 1$.
   When $n = 0$ *the first and second conditions becomes* $M_p \geq V(x(0))$ *and* $N_p \leq V(x(0)) + \int_0^{t_1} L_{f_p} V_p(x) dt$, *respectively.*

Here $t_0, t_1, t_2 \ldots$ *are consecutive switching times with* $t_0 = 0$, $L_{f_p} V_p$ *denotes the Lie-derivative of* V_p *with respect to the vector field* f_p.

Proof. First define $W_1 = \bigcap_{p \in P} K_p$, $W_2 = \bigcup_{p \in P} L_p$. See Figure 2. Also denote $T_p = \{t : \sigma(t) = p\}$. Since σ is continuous from the right $\{T_p : p \in P\}$ form a partition for $[0, \infty)$. For notational simplification it will be assumed without loss of generality $\sigma(t) = 1$, for all $t \in [0, t_1)$. Notice that $V_p(x(t_{2n+p-1}))$ with $n \geq 0$ represents V_p value at the beginning of the interval for which f_p is active. See Figure 3 and observe,

$$V_p(x(t_{2n+p-1})) = V_p(x(0)) + \sum_{i=0}^{n-1} \int_{t_{2i+p-1}}^{t_{2i+p}} L_{f_p} V_p(x) dt + \int_{t_{2i+p}}^{t_{2i+p+1}} L_{f_q} V_p(x) dt.$$

Since f_p is GAS, $V_p(x(t)) \leq V_p(x(t_{2n+p-1}))$, for all $t \in [t_{2n+p-1}, t_{2n+p})$. If conditions $(1), (3)$ are true, $V_p(x(t) \leq M_p$, for all $t \in T_p$, for all $p \in P$. Thus, $x(t) \in W_2 = \bigcup_{p \in P} L_p$, for all $t \in [0, \infty)$. On the other hand, $V_p(x(t_{2n+p}))$ with $n \geq 0$ represents V_p value at the end of the interval for which f_p is active. Also observe

$$V_p(x(t_{2n+p})) = V_p(x(0)) + \sum_{i=0}^{n-1} \int_{t_{2i+p}}^{t_{2i+p+1}} L_{f_p} V_p(x) dt + \int_{t_{2i+p+1}}^{t_{2i+p+2}} L_{f_q} V_p(x) dt.$$

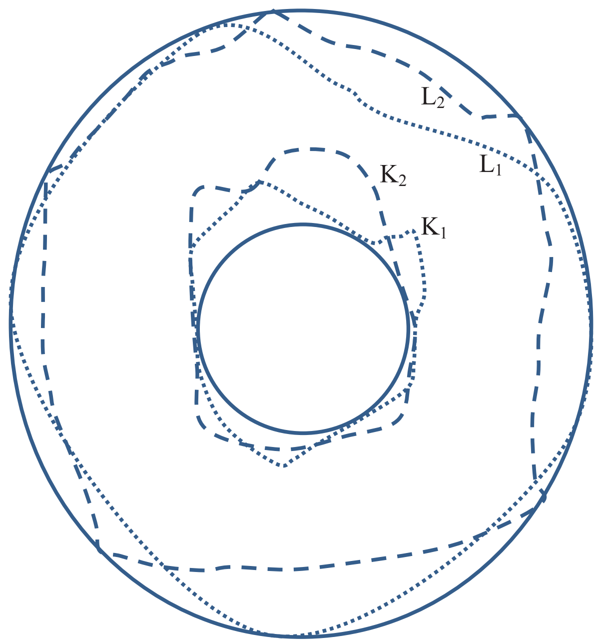

Figure 2: The trajectory needs to be confined within $(K_1 \bigcap K_2)' \bigcap (L_1 \bigcup L_2)$.

Since f_p is GAS, and $x(t)$ and $V_p(x(t))$ are continuous in t, $V_p(x(t)) \geq V_p(x(t_{2n+p}))$, for all $t \in [t_{2n+p}, t_{2n+p-1})$. If conditions (2), (3) are true, $N_p \leq V_p(x(t)$, for all $t \in T_p$, for all $p \in P$. Thus, $x(t) \in W_1' = (\bigcap_{p\in P} K_p)'$, for all $t \in [0, \infty)$. Therefore if all three conditions are satisfied, the trajectory is confined in $W_1' \bigcap W_2$ which is a subset of $B'_{\epsilon_1}(0) \bigcap B_{\epsilon_2}(0)$, $\forall t \geq 0$ and the theorem is proved. □

Corollary 2.8. *Let*

$$V_{p,n} = V_p(x(0)) + \sum_{i=0}^{n-1} \int_{t_{2i+p-1}}^{t_{2i+p}} L_{f_p} V_p(x) dt + \int_{t_{2i+p}}^{t_{2i+p+1}} L_{f_q} V_p(x) dt.$$

Then the switched system in Theorem 2.6 is asymptotically stable if and only if $V_{p,n}$, *is a monotone nonincreasing sequence and* $\lim_{n\to\infty} V_{p,n} = 0$, *for all* $p \in P$.

Proof. If $V_{p,n}$ is monotone nonincreasing, the implication of weak Lyapunov stability is obvious from the proof of Theorem 2.6. It is also obvious that it is

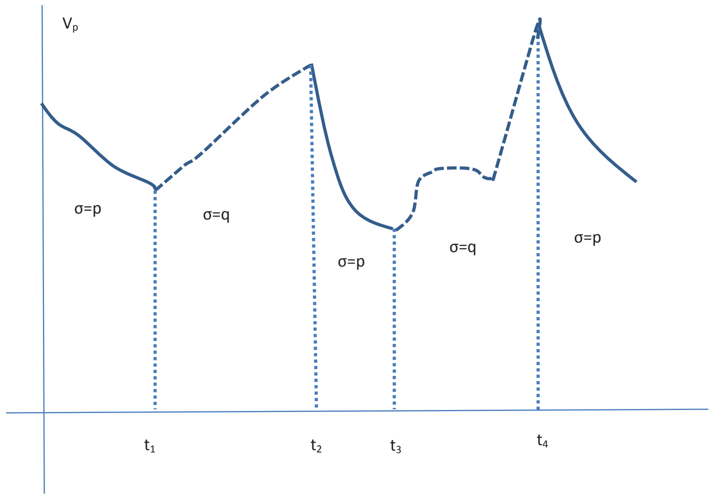

Figure 3: Variation of $V_p(x(t))$ when p is active and when p is inactive.

a necessary condition for the weak Lyapunov stability. However, by positive definiteness of $V_p(x)$, $\lim_{t\to\infty} ||x(t)|| = 0$ if and only if $\lim_{t\to 0} V_p x(t) = 0$, for all $p \in P$. Therefore, $\lim_{t\to\infty} ||x(t)|| = 0$ if and only if $\lim_{n\to\infty} V_{p,n} = 0$, for all $p \in P$. Thus, the system is asymptotically stable if and only if $V_{p,n}$ is nonincreasing and $\lim_{n\to\infty} V_{p,n} = 0$, $\forall p \in P$. □

Remark on Corollary 2.8. If a switched system satisfies the hypothesis of Theorem 2.1, it also satisfies the hypothesis of Corollary 2.8. To satisfy the hypothesis of Theorem 2.1, the difference $V_{p,n+1} - V_{p,n}$ must be bounded by a negative definite and continuous function $-W_p(x)$. This implies $V_{p,n}$ is nonincreasing for all n and $\inf_n V_{p,n} = 0$. Thus, the monotone sequence converges to its infimum and $\lim_{n\to\infty} V_{p,n} = 0$. However, switched systems with the switching sequence such that
$V_{p,n+1} - V_{p,n} \leq 0$, for all $n \geq 0$, and $\lim_{n\to\infty} V_{p,n} = 0$ do not satisfy the hypothesis of Theorem 2.1, but satisfy that of Corollary 2.8 and hence it is asymptotically stable.

Corollary 2.9. *The system has unbounded trajectory if and only if* $\lim_{n\to\infty} V_{p,n} = \infty$, *for some* $p \in P$.

Suppose $\lim_{n\to\infty} V_{p,n} = \infty$, *and the trajectory is bounded, i.e,* $\sup_{t\geq 0} ||x(t)|| \leq R$ *for some finite R. Then,* $\lim_{t\to\infty} V_p(x(t)) = \infty$ *and* $\{x : ||x|| \leq R\}$ *is a compact set. By continuity of* V_p*, the preimage of a noncompact set cannot be a compact set. Thus, the trajectory must be unbounded. The other direction is obvious from radial unboundedness condition of* $V_p(x)$*.*

3. Example

In this section we will give an example of a simple linear switching system. Both subsystems are stable with given Lyapunov functions. We show that it is possible to keep the trajectory with in an annulus for a prescribed period of time. It follows that if we can do it for a small number of switches it can be done indefinitely. It is possible to construct highly nonlinear examples but we feel that the example we present illustrates the concept of the theorem. In a future paper we will work with systems that are based on models of physiological systems.

Example. Consider the following switched system that consists of

$$A_1 = \begin{bmatrix} -3 & 10 \\ -1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & \sqrt{2}/2 \\ -5\sqrt{2} & -2 \end{bmatrix}.$$

Then, it can be verified both A_1 and A_2 are asymptotically stable and $V_1(x) = x_1^2/2 + 5x_2^2$, and $V_2 = 5x_1^2 + x_2^2/2$ are their respective Lyapunov functions. Suppose $x_1(0) = 16$, $x_2(0) = 10$, we need to confine the trajectory to the annulus of inner radius 5 and outer radius 25. The switching sequence $t_1 = 0.3s,\ t_2 = 0.6s,\ t_3 = 1.1s,\ t_4 = 1.4s,\ t_5 = 1.83s,\ t_6 = 2.13s,\ t_7 = 2.56s$ is an example of the switching regime, which keeps the trajectory within the target annulus at least for $2.56s$. Without switching, suppose $\sigma(t) = 1$ $\forall t \geq 0$; then the the trajectory will enter $B_{\epsilon_1}(0)$ at $0.7s$. If $\sigma(t) = 2$, $\forall t \geq 0$, then it will enter $B_{\epsilon_1}(0)$ in $8s$. The simulation results are depicted in Figures 4, 5, 6.

4. Conclusion

In this paper we presented a case for the use of switched systems for the modeling of dynamic treatment regimes. We prove a series of results that extends the theory of switched systems to those that are neither stable nor totally unstable. We give conditions for which the trajectories of a switched system can be kept in an annulus. This is important for the modeling of the

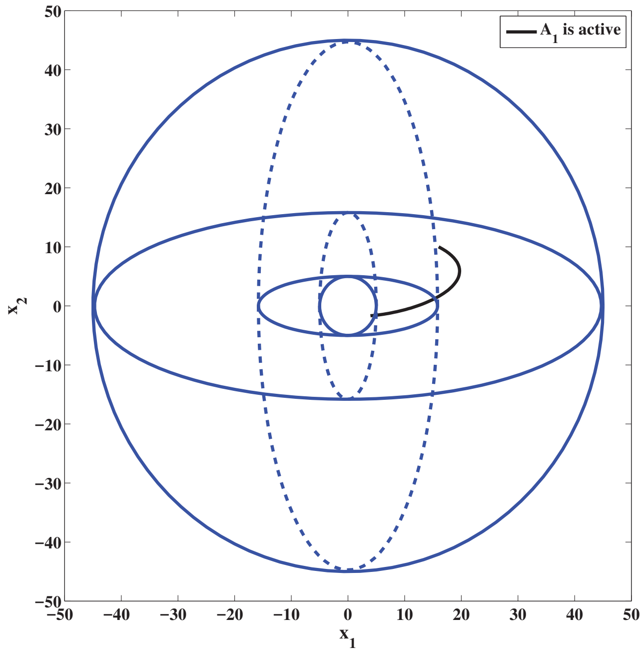

Figure 4: With $\sigma(t) = 1$, $t \geq 0$, the trajectory enters B_{ϵ_1} in $0.7s$.

treatment of chronic diseases such as high blood pressure, type II diabetes and chronic depression as well as for the treatment of certain malignancies.

5. Appendix

Proof of Proposition 2.3. First it will be shown that the claim is true for the switching sequence that starts with $\sigma(t) = 2$, for $0 \leq t < t_1$. Given a $\delta > 0$, consider $B_\delta(0)$ and denote its closure with $\bar{B}_\delta$. Since $\bar{B}_\delta$ is compact and $V_1(x)$ is continuous, $V_1(x)$ has a maximum, M on $\bar{B}_\delta$. Let $C_1 = \{x : V_1(x) \leq m_1\}$ with $m_1 > M$. Then, C_1 contains $B_\delta(0)$. Let C_2 be any set of the form $C_2 = \{x : V_2(x) \leq m_2\}$, which contains C_1. Choose ϵ_1 such that $B_{\epsilon_1}(0)$ contains C_2. Suppose $x(0) \in B'_{\epsilon_1}(0)$. Then, $x(0) \in C'_2$, which implies it is in C'_1. Therefore, by Proposition 2.2, $x(t) \in C'_1$, for all t, $0 \leq t < t_1$. Now, when the signal is switched at t_1 such that $\sigma(t) = 1$, for $t_1 \leq t < t_2$, by continuity, $x(t_1)$ is in the complement of C_1. Again by Proposition 2.2, $x(t) \in C'_1$, for all $t_1 \leq t < t_2$. Next the signal is switched again at t_2 such that $\sigma(t) = 2$, $t_2 \leq t < t_3$. Now that the system $p = 2$ is being repeated,

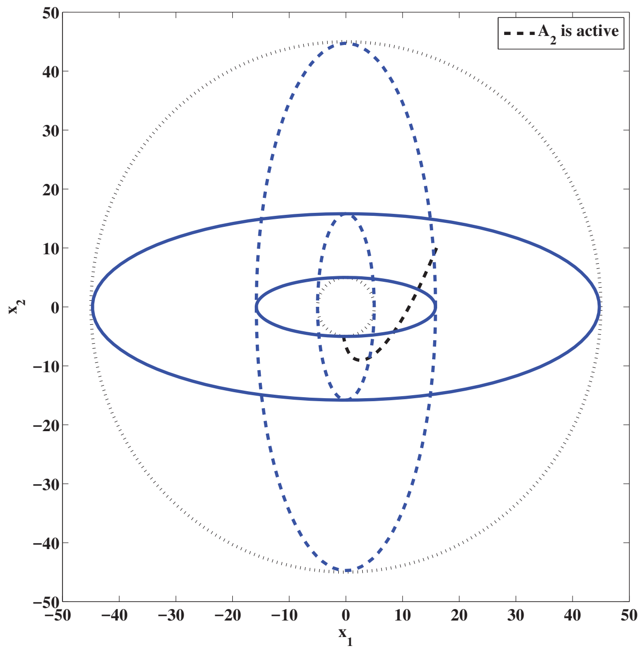

Figure 5: With $\sigma(t) = 2$, $t \geq 0$, the trajectory enters B_{ϵ_1} in $0.8s$.

by the hypothesis of the proposition, $V_2(x(t_2)) > V_2(0)$. Since the set $\{x : V_2(x) \leq V_2(0)\}$ contains $B_{\epsilon_1}(0)$, $x(t) \in B'_{\epsilon_1}(0)$, $\forall t_2 \leq t < t_3$. Now the process repeats itself with further switching without the trajectory ever entering C_1. Thus, it never enters $B_\delta(0)$ and $||x(t)|| \geq \delta$, for all t, $t \geq 0$. Therefore the claim is true for the case where the switching sequence starts with $\sigma = 2$. For the switching sequence that starts with $\sigma = 1$, C_1, C_2, ϵ_2 will be defined so that $C_2 \subset C_1 \subset B_{\epsilon_2}(0)$. Now define $\epsilon = \max\{\epsilon_1, \epsilon_2\}$. If $x(0) \in B'_\epsilon(0)$, the trajectory never enters $B_\delta(0)$ for all $t \geq 0$. To generalize this proof to an arbitary finite P, ϵ is chosen as the maximum over all permutation of P. The concept of the proof is depicted in Figure 1. $\square$

Proof of Proposition 2.4. Choose ϵ such that $C_2 \subset B_\epsilon(0)$. Suppose $x(0) \in B'_\epsilon(0)$, then, $x(0) \in C'_2$ and thus $V_2(x(0)) > m_2$. Let t_{m_2} be the time where

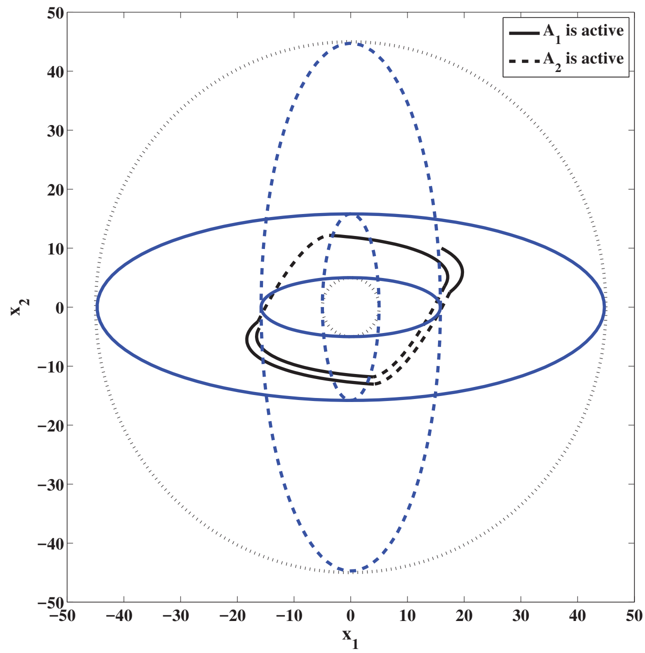

Figure 6: With $\sigma(t)$ defined in Example 1, the trajectory is confined within the annulus at least up to $2.56s$.

$V_2(x(t_{m_2})) = m_2$ with $\sigma(t) = 2$, on $[0, t_{m_2}]$. Then,

$$V_2(x(t_{m_2})) - V_2(x(0)) = \int_0^{t_{m_2}} \dot{V}_2(x)dt \tag{5.1}$$
$$\geq -k_2 t_{m_2}$$
$$t_{m_2} \geq \frac{V_2(x(t_{m_2})) - V_2(x(0))}{-k_2}$$
$$= \frac{V_2(x(0)) - m_2}{k_2}.$$

Thus, if $V_2(x(t_{m_2})) = m_2$, then $t_{m_2} \geq (V_2(x(0)) - m_2)/k_2$. For any $t < (V_2(x(0)) - m_2)/k_2$, $V_2(x(t)) \neq m_2$. By the hypothesis of the proposition, $t_1 < (V_2(x(0)) - m_2)/k_2 \leq t_{m_2}$ and thus $V_2(x(t_1)) \neq m_2$. But by monontonicity of $V_2(x(t))$ in t, $t_1 < t_{m_2}$ implies $V_2(x(t_1)) \geq V_2(x(t_{m_2})) = m_2$. Since the equality is impossible, $V_2(x(t_1)) > m_2$. Therefore at the switching time t_1, $x(t_1) \in C_2'$ and $x(t_1) \in C_1'$. Thus, the trajectory remains in C_1' on $[0, t_1]$.

Now, let t_{m_1} be such that $V_1(t_{m_1}) = m_1$ with the switching signal $\sigma(t) = 2$ on $[0, t_1)$ and $\sigma(t) = 1$ on $[t_1, t_{m_1}]$. Then similar arguments can be made as in the case of t_{m_2} to claim if the switching time t_2 is less than t_{m_1}, $x(t)$ remains in C_1'. Specifically, by the arguments similar to statements in 5.1, if $V_1(x(t_{m_1})) = m_1$, $t_{m_1} \geq (V_1(x(t_1)) - m_1)/k_1 + t_1$. Thus for any $t < (V_1(x(t_1)) - m_1)/k_1 + t_1$, $V_1(x(t)) \neq m_1$. By the hypothesis of the proposition, $t_2 < (V_1(x(t_1)) - m_1)/k_1 + t_1 \leq t_{m_1}$ and thus $V_1(x(t_2)) \neq m_1$. But by monontonicity of $V_1(x(t))$ in t, $t_2 < t_{m_2}$ implies $V_1(x(t_2)) \geq V_1(x(t_{m_1})) = m_1$. Since the equality is impossible, $V_2(x(t_2)) > m_1$. Therefore at the switching time t_2, $x(t_2) \in C_1'$ and the trajectory remains in C_1' on $[0, t_2]$. Now at $t = t_2$, $p = 2$ is repeated. By hypothesis (1) of the proposition, $V_2(x(t_2)) > V_2(x(0)) > m_2$. Therefore, $x(t_2) \in C_2'$. From this point, the process repeats itself with further switching without the trajectory ever entering C_1. Thus, $x(t) \in B_\delta'(0)$, for all t, $t \geq 0$. □

Proof of Proposition 2.5. If either one of the system is repeated only finitely many times, at least one dwell time needs to be infinite. In that case, clearly the trajectory will approach the origin as $t \to \infty$ since each f_p is GAS. Thus we will only consider the case where both f_p are repeated infinitely many times. Let T_{p_i} be the dwell time of f_p when it is repeated i^{th} time. Thus

$$\sigma(t) = p,$$

if and only if

$$t \in T_p = \bigcup_{i=1}^{\infty} [t_{p_i}, t_{p_i} + T_{p_i}).$$

Thus, the positive time axis is the following union of partitions.

$$[0, \infty) = \bigcup_{p \in P} T_p.$$

Let $S = \bigcap_{p \in P} \{x : V_p(x) \leq M_p\}$. It is sufficient to consider the case $M_p > 0$, $\forall p \in P$. Then by positive definiteness and continuity of V_p, the set S contains an open ball $B_\delta(0)$ with $\delta > 0$. Suppose $||x(0)|| < \delta$ and observe that $\forall p \in P$,

$$t \in [t_{p_i}, t_{p_i} + T_{p_i}) \implies x(t) \in \{x : V_p(x) \leq V_p(x(t_{p_i}))\}.$$

Thus, $\forall t \in T_p$,

$$x(t) \in \bigcup_{i=0}^{\infty} \{x : V_p(x) \leq V(x(t_{p_i}))\} \quad \subset \quad \{x : V_p(x) \leq \sum_{i=1}^{\infty} \Delta V_{p_i}\}.$$

If $\sum_{i=1}^{\infty} \Delta V_{p_i} \leq M_p < \infty$, $x(t) \in \{x : V_p(x(t) \leq M_p\}$, $\forall t \in T_p$. Now suppose $\sup_{T_p} ||x|| = \infty$, for some $p \in P$. Then this implies $||x(t)|| \to \infty$ on some subset of T_p. But, since f_p is GAS, by radial unbounded condition $V_p(x(t))$ must be unbounded. Hence this is a contradiction. Therefore, $\sup_{T_p} ||x(t)|| = N_p$ is finite for all p, $p \in P$. Denote $N = \max_{p\in P} N_p$, then $||x|| < N < \infty$ for all t, $t \in [0, \infty)$. □

Proof of Proposition 2.6. Let $S = \bigcap_{p\in P} S_p$. Clearly, $M_p > 0$, for all p, $p \in P$ and hence, by positive definiteness and continuity of V_p, the set S contains an open ball $B_\delta(0)$ with $\delta > 0$. Assume $||x(0)|| < \delta$ and define the partitions T_p of the positive time axis as in the proof of Proposition 2.5. Then, it is clear from Proposition 2.5 that if $t \in T_p$, $x(t) \in S_p$. Therefore, $x(t) \in \bigcup_{p\in P} S_p$ on $[0, \infty)$. Since $\bigcup_{p\in P} S_p \subset B_\epsilon(0)$, $x(t) \in B_\epsilon(0)$, for all t, $t \geq 0$. □

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