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Master adjoint systems in mean-field-type games

Hamidou Tembine*

This article presents mean-field-type games with both atomic and non-atomic decision-makers. We establish sufficiency conditions for equilibria in state-and-mean-field-type feedback strategies using Bellman systems in the space of measures. We then derive master adjoint systems. The computation of the equilibrium system is illustrated in variance-aware games with heterogeneous decision-makers.

1. Introduction

The term 'mean-field' has been referred to a physics concept that attempts to describe the effect of an infinite number of particles on the motion of a single particle. Researchers began to apply the concept to social sciences in the early 1960s to study how an infinite number of factors affect individual decisions. However, the key ingredient in a game-theoretic context is the influence of the distribution of states and or control actions into the payoffs of the decision-makers. There is no need to have large population of decision-makers. A mean-field-type game is a game in which the payoffs and/or the state dynamics coefficient functions involve not only the state and actions profiles but also the distributions of state-action process (or its marginal distributions).

In a mean-field-type game (MFTG), the quantities-of-interest (such as instantaneous payoff, cost, performance functionals, state kernel/coefficients) depend not only on the type-state-actions of the decision-makers but also on the distribution of them. This allows us to consider higher order (non-linear) dependence in the distribution of state and or actions in the instantaneous quantities-of-interest. It also allows us to capture risk-awareness in the performance metric whenever one or more decision-makers are facing uncertainties.

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1.1. Literature review

One decision-maker Different methods have been developed to solve MFTG problems with one decision-maker. These methods include (Pontryagin) Stochastic Maximum Principle (SMP), (Bellman) Dynamic Programming Principle (DPP), Chaos expansion approach, Direct Method, Lagrangian or Calculus of Variations approach.

Maximum Principle In the context of one decision-maker, the authors [1, 19, 59] considered the system dynamics and cost functional of mean-field type. They established stochastic maximum principle of mean-field type of first and second order adjoint process and possibly non-convex-valued action set. These optimality equations differ from the ones obtained from the classical mean-field-free control problems in which the instantaneous cost and the coefficients of the state are independent of the probability measure of the state-action. Therefore the problem was referred to as a non-standard control problem or *mean-field-type control problem*. It can also be seen as a control of McKean-Vlasov dynamics with a cost functional of mean-field type. The SMP has been extended to include action mean-field type in [40], where authors has established the second-order adjoint processes. The work in [33] studies discrete-time mean-field-type control problems. In [49], the SMP is apply to solve jump-diffusion mean-field problems involving mean-variance terms, i.e., of mean-field-type. The well-posedness of mean-field-type forward-backward stochastic differential equations is studied in [18] motivated by the work in [20]. In [29] the SMP has been extended to the risk-sensitive mean-field-type control problem, which involve not only first and second moment terms, but also higher terms. In [45] a stochastic robust control of mean-field-type with state and control input dependent noise is studied. In [46], the well-posedness and the solvability of a indefinite linear-quadratic mean-field-type control problem in discrete time. In [26], mean-field-type games are characterized with time-inconsistent cost functionals and SMP is applied. Partial observation mean-field-type control problems are investigated in [38, 58] in the risk-neutral case. The risk-sensitive case under partial observation is analyzed in [28, 44], and applying the backward separation method in [39] and [43].

Dynamic Programming Principle (DPP) solutions for mean-field-type control problems have been discussed by using methods based on either Hamilton-Jacobi-Bellman and Fokker-Planck coupled equations or stochastic maximum principle in [17]. The works in [41, 42] proposed to solve mean-field-type control problems by applying dynamic programming and two examples are presented, i.e., a portfolio optimization and a systemic risk model. In [38] the SMP is used to solve mean-field-type control problems with partial observation and results are applied to financial engineering, and in [58] addresses a partially observed optimal control problem, whose cost functional is of mean-field type and the authors solve it by means of a maximum principle using Girsanov's theorem and convex variation. A class of mean-field-type control problems with common noise has been considered in [37]. Randomized dynamic programming principle is established in [15].

Chaos expansion approach Uncertainty quantification has been studied in [55] in the context of mean-field-type teams and using Kosambi-Karhunen-Loeve expansion (chaos expansion approach), which allows representing the stochastic process as a linear orthogonal functions combination.

Two or more decision-makers: Mean-Field-Type Games

DPP The work in [54] examines risk-sensitive mean-field-type games and sufficient optimality equations are established via infinite-dimensional dynamic programming principle. The basic foundations of discrete time and continuous mean-field-type games are formulated and analyzed in [27, 57]. A class of mean-field-type games with jump and regime switching are studied by using dynamic programming principle in [16].

Direct method In [53, 32, 8] the so-called direct method is used in order to compute semi-explicit solutions for mean-field-type games for the linear state and variance-aware case. The work is extended in [5, 10] to non-linear problems and beyond Brownian motions including multi-fractional Brownian motions, Gauss-Volterra processes and non-Gaussian features such as Rosenblatt processes. Moreover, other game theoretical solution concepts such as partial cooperation, co-opetition (competitive cooperation or cooperation competition), partial altruism, spite, bargaining etc., are studied by means of the co-opetitive mean-field-type games in [6].

SMP A zero-sum mean-field-type game has been introduced in [48] in which sufficiency conditions for existence of saddle point is provided. The work in [23] presents cooperative mean-field-type games. The work in [21] considers a two-player nonzero sum mean-field-type game with backward stochastic differential equations with jump and under partial information.

An open-loop mean-field-type Nash equilibrium is investigated under restrictive conditions on the data. The two-player game scenario has been studied for both non-zero-sum and zero-sum cases in [2, 25]. *Notice that in the context of games with at least two decision-makers, open-loop Nash equilibrium may differ from state (-and-mean-field-type) feedback Nash equilibrium.* Even if the open-loop solution obtained via the stochastic maximum principle has a state-feedback representation, that representation can be different from the one obtained from Hamilton-Jacobi-Bellman system. The difference between the two solution concepts appears even in simple linear-quadratic (mean-field-type) games [53, 32].

Applications of MFTG

- Data-driven:
  a *blockchain-based distributed power network* is considered in [30, 52]. Therein, the decision-makers are investors, producers, prosumers (producer-consumer), consumers, and verifiers who decide to engage a validation of consensus algorithms and protocols. The data of prosumers are divided into demand, supply and storage and added to the price dynamics. In addition, the insurance option is offered to the entities making the problem of mean-field type in the dynamics of the state as well. Mean-field type data were also collected for mobility patterns in *road traffic* and a new class mean-field-type filters for big data assimilation was designed in [35]. Deep learning and traffic video analytics are studied in [36] using mean-field-type filtering. A network effect of several congestion areas was shown to emerge in *multi-level building evacuation* [31] where a localized congestion measure is introduced. The model is modified to reflect some of the observed behaviors from data. The localized congestion measure is a counting process around the position of the individual and a risk term. Including the localized mean-field term and its evolution along the path causes a sort of dispersion of the flow: the agents will try to avoid high density areas in order to reduce their overall walking costs and queuing costs at the stairs and exits. Each agent individual state is represented by a center of a box that follows a simple dynamical system in an Euclidean space. Each agent will move to one of the closest exits that is safer and with less congested path. The authors in [47] examines how much does users' psychology matter in engineering mean-field-type games [13, 14].

- Model-driven:
  Network security have been studied by means of a mean-field-type game of public good. Each participant makes some effort in improving network security. However, when the global network security is high enough, free-riders can emerge. Given that different users have different investment strategies, how do we optimally invest into security resources such that the basic security level is provided as public good in the society? The work [50] provides a basic model of variance-aware mean-field-type game for investment into network security.
  A demand-supply management in smart grid is analyzed in [56] using constrained mean-field-type games. In [24], electricity price formation are investigated in the smart grids. In [3, 4] a mean-field-type-based pedestrian crowd model is presented. In [9], a class of control input constraints are considered and an application of water distribution system is presented. A class of variance-aware model-predictive control was developed based on the idea of mean-field-type game theory [7] and applied to energy storage of micro-grid. A mean-field-type model predictive control was implemented for the water distribution networks in [11].
  Another example of variance-awareness is an Air Conditioning thermostat which is a feedback system. At a given temperature, the air-conditioner turns on cooling mode and bring the temperature a comfort zone. Once in the comfort zone it tries to reduce the variance of the temperature with respect to the desired comfort temperature of the user and minimizes the error. In this context, there is only one decision-maker, the user who acts on the controller. The control action variable belongs to $\{1, 0, -1\}$ i.e., Heating, Do Nothing, or Cooling. Clearly, the control action has significant impact on the variance of the temperature [27].
  In [51], the performance of hub-based airline networks is analyzed using mean-field-type game theory. Three types of interactions between within the game: interaction between passengers, passengers-airlines, and interaction between airlines. The key mean-field terms are the airline traffic (or frequency of flights), number of people at the same slot per flight/airline.
  In [12], blockchain cryptographic tokens are examined by means of mean-field-type game theory with several classes of decision-makers. The x-chain system has the following classes of decision-makers:
  - blockchain and x-chain users
  - x-chain verifiers and validators

- investors and developers
- x-chain companies and tokens
- network security suppliers and last mile integration owners

where some of them are atomic and some others are non-atomic.
The authors have introduced the variance-aware utility function per decision-maker to capture the risk of cryptographic tokens associated with the uncertainties of technology adoption, network security, regulatory legislation, and market volatility. They established a relationship between the network characteristics, token price, number of token holders, and token supply. Both in-chain diversification and cross-chain diversification among tokens are examined by using a mean-variance approach. The authors showed that the number of tokens in circulation needs to be adjusted in order to capture risk-awareness and self-regulatory behavior in blockchain token economics.

- Token-less Blockchain economy: we show that the optimal investment of decision-maker in the token-free blockchain shares the following interesting features: it increases with the total number of active participants to the token-free blockchain, increases with the productivity, and decreases with the probability of success of an attack of the blockchain by malicious nodes.
- Token-based Blockchain economy: The optimal number of tokens of a decision-maker has the following interesting features: It increases with the total number of active participants in the blockchain, decreases with the probability of success of an attack, decreases with the token price, increases with the productivity, increases with interest rate of the token price.
- Token adoption: The authors also provided sufficient conditions under which the token enables and improves the adoption of blockchain technology. Token diversification within the chain: We show how to diversity in-chain tokens under correlated prices. Token diversification cross-chains: The methodology extends to cross-chain tokens with different access costs.

1.2. Content of the present article

In this article we consider a distribution-dependent interaction with both atomic and non-atomic decision-makers. Two categories of mean-field quantities: individual mean-field and population mean-field, are introduced.

There is a finite number of atomic decision-makers who have a non-negligible effect in the population mean-field quantities. The atomic decision-makers can have different characters and goals, and no symmetry assumption is made. There are several classes of non-atomic decision-makers. The non-atomic decision-makers are distinguishable when they belong to different classes but remain indistinguishable within each class. Each class is composed of a large (sub-)population of non-atomic decision-makers. A non-atomic decision-maker has a negligible effect on the population mean-field but can still have a big impact on its own individual mean-field terms. The total number of classes is assumed to be finite for simplicity. In practice, a nonatomic decision-maker in a certain class has a negligible influence of the situation but the aggregative behavior of all non-atomic decision-makers in that class will have an impact on the quantity-of-interest.

In Section 2 we establish Bellman systems which provide sufficiency conditions for mean-field-type equilibria in state-and-mean-field-type feedback form. We then derive a class of master adjoint systems (MASS). The measure of the state does not need to be normalized and include birth-and-death terms (inflow and outflow). Section 3 presents nonatomic MFTG problems. Section 4 extends the results to atomic and a multi-class non-atomic MFTG problems. Section 5 concludes the article.

Notations

Throughout this article, I represents an arbitrary positive integer and $\mathcal{I} = \{1,\ldots,I\}$ the set of atomic decision-makers. There are $C \geq 1$ classes of non-atomic decision-makers denoted by $\mathcal{C} := \{I+1, I+2, \ldots, I+C\}$. We consider two time instants $t_0 < t_1$, and ν is a Radon measure over $\Theta = \mathbb{R}_+ \backslash \{0\}$.

Two type of mean-field terms are involved in each of the two categories of decision-makers. For atomic decision-makers, the two mean-field terms are

- μ_{do} is the measure of the state s_{do}, is called individual mean-field of the state of the atomic decision-makers.
- $\tilde{\mu}_i$ is the individual mean-field of the action a_i of the atomic decision-maker i.

For nonatomic decision-makers the two mean-field terms are given by

- m_c is the population mean-field of states in class c i.e., the probability distribution of states generated by all non-atomic decision-makers of class c,

- $\tilde{m}_c$ is the distribution of actions picked by all non-atomic decision-makers in class c in equilibrium. This is another population mean-field of actions of class c. The population mean-field of actions of non-atomic decision-makers is $\tilde{m} = (\tilde{m}_c)_{c\in\mathcal{C}}$.

A deviating nonatomic decision-maker has also two individual mean-field terms given by

- μ_c is the probability measure of s_c, is called individual mean-field of the state of a generic deviating nonatomic decision-makers,
- $\tilde{\mu}_c$ is the individual mean-field of the action a_c of a deviating nonatomic decision-maker in class c.

Let $\mathcal{M}_2(\mathbb{R}^d)$ be the set of Borel measures on $\mathbb{R}^d$ with finite second moments. We introduce the Gâteaux (variational) derivative. Let $f : \mathcal{M}_2(\mathbb{R}^d) \to \mathbb{R}$. We say that $f_\rho(x,\rho) := f_\rho[\rho](x)$ is a Gâteaux derivative of f with respect to a measure ρ if

$$\lim_{\epsilon\to 0_+} \tfrac{d}{d\epsilon}\left[f(\rho+\epsilon\hat{\rho})\right] = \int f_\rho[\rho](x)\hat{\rho}(dx). \tag{1}$$

If $\int \hat{\rho}(dx) = 0$ then by adding a constant to $f_\rho[\rho](x)$ does not change the value of the integral in (1). For any scalar $\lambda = \lambda\int \frac{1}{\kappa}\rho(dx)$ where $\kappa = \int_{\mathbb{R}^d}\rho(dx)$. Thus, λ is also a Gâteaux-derivative of the constant λ with respect to the probability measure $\frac{1}{\kappa}\rho(dx)$. However in our problem, the term $f_{x\rho}$ which is the gradient of $x \mapsto f_\rho[\rho](x)$ will be used in the Hamiltonian, and $f_{x\rho}$ does not have the constant ambiguity.

2. MFTG with finitely many atomic decision-makers

Consider $I \geq 2$ decision-makers with distribution-dependent quantities-of-interest interacting over a finite time horizon $(t_0, t_1),\ t_0 < t_1$. A strategy a_i of decision-maker $i \in \mathcal{I} = \{1,\dots,I\}$ is an A_i-valued measurable function, where A_i is a non-empty convex set. Let $a = (a_i)_{i\in\mathcal{I}}$ be the action profile of all decision-makers. The collective state s of all decision-makers, can be a common and/or collection of individual states, is in a finite dimensional Euclidean space $\mathcal{S} := \mathbb{R}^d,\ d \geq 1$. The running cost function of decision-maker $i \in \mathcal{I}$ is l_i and the terminal cost function is h_i.

$$\begin{aligned} l_i :\ & D \to \mathbb{R}, \\ h_i : \mathcal{S} \times \mathcal{M}_2(\mathcal{S})\ & \to \mathbb{R}. \end{aligned} \tag{2}$$

where $D := (t_0, t_1) \times \mathcal{S} \times \prod_i A_i \times \mathcal{M}_2(\mathcal{S}) \times \mathcal{M}_2(\prod_i A_i)$. The measure μ of the state is given by a Kolmogorov equation with coefficients

$$
\begin{array}{l}
b, \tilde{b}, \tilde{d} : D \to \mathbb{R}^d, \\
\sigma : D \to \mathbb{R}^{d\times k}, \\
\gamma : D \times \Theta \to \mathbb{R}^d,
\end{array}
\tag{3}
$$

where b represents the drift, $\tilde{b}$ inflow into the state, $\tilde{d}$ outflow from state, σ represents the diffusion and γ represents a jump coefficient with jump measure ν over Θ. Let μ be a Borel measure of the state on $\mathbb{R}^d$. The data of the mean-field-type game MFTG is given by

$$\mathcal{G} = \{[t_0, t_1], \mathcal{I}, (A_i, l_i, h_i)_{i\in\mathcal{I}}, b, \sigma, \gamma, \tilde{b}, \tilde{d}, \mu_0\}.$$

The game is played as follows. Given a initial state distribution μ_0 at time t_0 the game $\mathcal{G}$ proceeds as follows. At each instant $t \in (t_0, t_1)$, each decision-maker observes the state (perfect monitoring, perfect state observation), chooses a control action according to her strategy and observes/measures her cost. We restrict ourselves to the class of admissible strategies in state-and-mean-field-type feedback form. The risk-neutral mean-field-type Nash equilibrium problem is given by

$$
\left\{
\begin{array}{l}
\inf_{a_i} \int h_i(s, \mu(t_1))\mu(t_1, ds) + \int_{t_0}^{t_1} \int l_i(t, s, a, \mu, \tilde{\mu})\mu(t, ds)dt, \\
\text{subject to} \\
\mu_t = -\text{div}_s(b\mu) + \frac{1}{2} trace[(\sigma'\sigma\mu)_{ss}] \\
+ \int_\Theta [\mu(s-\gamma) - \mu(s) + \langle \mu_s, \gamma\rangle]\nu(d\theta) + (\tilde{b} - \tilde{d})\mu, \\
\mu(t_0, ds) = \mu_0(ds), \\
i \in \mathcal{I}.
\end{array}
\right.
\tag{4}
$$

Definition 1. A mean-field-type Nash equilibrium is a solution $(a_i)_{i\in\mathcal{I}}$ to the system (4).

In Definition 1, A mean-field-type Nash equilibrium is a strategy profile in which no decision-maker has incentive to change her strategy unilaterally. This means that no atomic decision-maker can lower her expected cost by changing her choice alone.

We introduce the equilibrium payoff (if any) as $\hat{V}_i : (t_0, t_1) \times \mathcal{M}_2(\mathcal{S}) \to \mathbb{R}$

given by

$$(5)\quad \begin{cases} \hat{V}_i(t,\mu) := \\ \inf_{a_i} \int h_i(s,\hat{\mu}(t_1))\hat{\mu}(t_1,ds) + \int_t^{t_1}\int l_i(t,s,a,\hat{\mu},\tilde{\mu})\hat{\mu}(t,ds)dt, \\ \text{subject to} \\ \hat{\mu}_t = -\text{div}_s(b\hat{\mu}) + \frac{1}{2}trace[(\sigma'\sigma\hat{\mu})_{ss}] \\ +\int_\Theta [\hat{\mu}(s-\gamma) - \hat{\mu}(s) + \langle \hat{\mu}_s, \gamma\rangle]\nu(d\theta) + (\tilde{b}-\tilde{d})\hat{\mu}, \\ \hat{\mu}(t,ds) = \mu(ds), \\ i \in \mathcal{I}, \end{cases}$$

starting from $t \in (t_0, t_1)$ with the measure μ.

A sufficiency condition for mean-field-type Nash equilibrium is obtained by the HJB system given below.

$$(6)\quad \begin{cases} \hat{V}_{i,t}(t,\mu) + H_i(t,\mu,\hat{V}_\mu,\hat{V}_{s\mu},\hat{V}_{ss\mu}) = 0, & \text{on } (t_0,t_1)\times\mathcal{M}_2(\mathcal{S}) \\ \hat{V}_i(t_1,\mu) = \int h_i(s,\mu)\mu(ds), & \text{on } \{t_1\}\times\mathcal{M}_2(\mathcal{S}) \\ i\in\mathcal{I}, \end{cases}$$

where the Hamiltonian H_i is given by

$$H_i(t,\mu,\hat{V}_\mu,\hat{V}_{s\mu},\hat{V}_{ss\mu}) = \inf_{a_i\in A_i}\int \Big\{ l_i + \langle b, \hat{V}_{i,s\mu}\rangle + \frac{1}{2}trace[\sigma'\sigma\hat{V}_{i,ss\mu}] \\ + \int_\Theta [\hat{V}_{i,\mu}(t_-, s+\gamma) - \hat{V}_{i,\mu} - \langle \gamma, \hat{V}_{i,s\mu}\rangle]\nu(d\theta) + \hat{V}_{i,\mu}(t,s,\mu)(\tilde{b}-\tilde{d}) \Big\}\mu(ds).$$

The system (6) is a system of partial-integro differential equations (PIDEs) in the space of measures. It provides not only a mean-field-type Nash equilibrium $(a_i^*)_i$ solution to (4) but also the associated equilibrium cost $\hat{V}_i(t_0,\mu_0)$ for decision-maker i. One of the main challenges of the Nash MFTG problem is to understand (6). In particular, the existence of solution to the infinite dimensional HJB $(\hat{V}_i)_{i\in\mathcal{I}}$ is an open issue in the general setting. To date, only very special cases of (4) are known to have a solution [57, 27, 32, 5]. We provide below an example where the HJB system (6) admits a solution even with non-convex and non-Lipschitz Hamiltonian. Another approach for finding equilibrium strategies is to consider a master adjoint system (MASS). The MASS associated with the equilibrium problem (4) is given by

$$(7)\quad \begin{cases} \hat{U}_{i,t}(t,s,\mu) + \partial_\mu[H_i(t,\mu,\hat{U},\hat{U}_s,\hat{U}_{ss})](s) = 0, & \text{on } (t_0,t_1)\times\mathcal{S}\times\mathcal{M}_2(\mathcal{S}) \\ \hat{U}_i(t_1,s,\mu) = h_i(s,\mu) + \int h_{i,\mu}(s,s',\mu)\mu(ds'), & \text{on } \{t_1\}\times\mathcal{S}\times\mathcal{M}_2(\mathcal{S}), \\ i\in\mathcal{I}, \end{cases}$$

where $\partial_\mu[f(p,\mu)](s) := \frac{\delta f}{\delta\mu}(p,s,\mu) := f_\mu(p,s,\mu)$, is the Gâteaux differentiation of f and the Hamiltonian is given by

$$\begin{array}{l} H_i(t,\mu,\hat{U},\hat{U}_s,\hat{U}_{ss}) = \inf_{a_i\in A_i}\int\Big\{l_i + \langle b,\hat{U}_{i,s}\rangle + \frac{1}{2}trace[\sigma'\sigma\hat{U}_{i,ss}] \\ +\int_\Theta[\hat{U}_i(t_-,s+\gamma,\mu) - \hat{U}_i - \langle\gamma,\hat{U}_{i,s}\rangle]\nu(d\theta) + \hat{U}_i(t,s,\mu)(\tilde{b}-\tilde{d})\Big\}\,\mu(ds). \end{array}$$

Again, one of the challenges in (7) is the well-posedness of MASS $\hat{U}$. We provide an example where MASS admits a solution even with non-convex and non-Lipschitz Hamiltonian.

Example 2 (Distributed variance reduction)**.** Consider the distributed variance reduction problem with finitely many atomic agents over an Euclidean space with the following data

$$\begin{cases} l_i = \frac{1}{2}r_i\|a_i(t,s,\mu) - \int a_i(t,y,\mu)\mu(t,dy)\|^2 + \frac{1}{2}\bar{r}_i\|\int a_i(t,y,\mu)\mu(t,dy)\|^2 \\ +\frac{1}{2}q_i\|s-\int y\mu(t,dy)\|^2 + \frac{1}{2}\bar{q}_i\|\int y\mu(t,dy)\|^2, \\ b = \sum_{j\in\mathcal{I}} a_j(t,s,\mu), \\ h_i = q_i\int\|s-\int y\mu(t,dy)\|^2\mu(ds) + \bar{q}_i\|\int y\mu(t,dy)\|^2, \\ \sigma,\gamma \ \ \text{constant}, \\ \tilde{b} = \tilde{d}, \\ i\in\mathcal{I}, \end{cases}$$

where $\|.\| = \|.\|_2$ is the 2-norm, $q_i,\bar{q}_i,r_i,\bar{r}_i > 0$. The MFTG equilibrium strategy is given by

$$a_i^*(t,s,\mu) = -\frac{1}{r_i}(\hat{V}_{i,s\mu} - \int\hat{V}_{i,y\mu}\mu(dy)) - \frac{1}{\bar{r}_i}\int\hat{V}_{i,y\mu}\mu(dy).$$

The MFTG equilibrium cost system is given by the following HJB system.

$$\begin{array}{l} \hat{V}_{i,t} + \frac{1}{2}q_i\int\|s\|^2\mu(ds) + \frac{1}{2}(\bar{q}_i - q_i)\|\int y\mu(dy)\|^2 \\ -\frac{1}{2r_i}\int\|\hat{V}_{i,s\mu}\|^2\mu(ds) - (\frac{1}{2\bar{r}_i} - \frac{1}{2r_i})\|\int\hat{V}_{i,s\mu}\mu(ds)\|^2 \\ -\int\langle\sum_{j\neq i}\frac{1}{r_j}\hat{V}_{j,s\mu},\hat{V}_{i,s\mu}\rangle\mu(ds) \\ -\langle\sum_{j\neq i}(\frac{1}{\bar{r}_j} - \frac{1}{r_j})\int\hat{V}_{j,s\mu}\mu(ds),\Big(\int\hat{V}_{i,y\mu}\mu(dy)\Big)\rangle \\ +\frac{1}{2}\int trace[\sigma'\sigma\hat{V}_{i,ss\mu}]\mu(ds) \\ +\int\int_\Theta[\hat{V}_{i,\mu}(t_-,s+\gamma) - \hat{V}_{i,\mu} - \langle\gamma,\hat{V}_{i,s\mu}\rangle]\nu(d\theta)\mu(ds) \ = 0, \\ \hat{V}_i(t_1,\mu) = q_i\int\|s-\int y\mu(dy)\|^2\mu(ds) + \bar{q}_i\|\int y\mu(dy)\|^2. \end{array}$$

We observe that the HJB system exhibits a non-linearity in the measure μ via the terms $\|\int\hat{V}_{i,s\mu}\mu(ds)\|^2$ and $\langle\int\hat{V}_{j,s\mu}\mu(ds),\Big(\int\hat{V}_{i,y\mu}\mu(dy)\Big)\rangle$. The

Hamiltonian of i non-convex in μ and non-Lipschitz in μ. The system $(\hat{V}_i)_{i\in\mathcal{I}}$ is semi-explicitly solvable and it is given by

$$\begin{aligned}&\hat{V}_i(t,\mu)\\&=\alpha_i(t)\int\|s-\int y\mu(dy)\|^2\mu(ds)+\bar{\alpha}_i(t)\|\int y\mu(dy)\|^2+\delta_i(t)\int\mu(dy),\end{aligned}$$

where $(\alpha_i,\bar{\alpha}_i,\delta_i)_{i\in\mathcal{I}}$ solve coupled ordinary differential systems ([32]).

3. MFTG with non-atomic decision-makers

We consider a class of MFTG with infinitely many non-atomic decision-makers. Let $a_i = a$ denotes the action of a generic non-atomic decision-maker and $s_i = s$ denotes the state of a generic decision-maker. The best response problem of a representative decision-maker is given by

$$\tag{8}\left\{\begin{array}{l}\inf_a\int h(s,\mu(t_1),m(t_1))\mu(t_1,ds)+\int_{t_0}^{t_1}\int l(t,s,a,\mu,\tilde{\mu},m,\tilde{m})\mu(t,ds)dt,\\ \text{subject to}\\ \mu_t=-div_s(b\mu)+\frac{1}{2}trace[(\sigma'\sigma\mu)_{ss}]\\ +\int_\Theta[\mu(s-\gamma)-\mu(s)+\langle\mu_s,\gamma\rangle]\nu(d\theta)+(\tilde{b}-\tilde{d})\mu,\\ \mu(t_0,ds)=m_0(ds),\\ m_t=-div_s(b_*m)+\frac{1}{2}trace[(\sigma_*'\sigma_*m)_{ss}]\\ +\int_\Theta[m(s-\gamma_*)-m(s)+\langle m_s,\gamma_*\rangle]\nu(d\theta)+(\tilde{b}_*-\tilde{d}_*)m,\\ m(t_0,ds)=m_0(ds),\end{array}\right.$$

where μ is a measure of the individual state, referred to as a *individual mean-field* term. A single decision-maker has a big influence on its own-state measure μ. The term m is a measure of *other* decision-makers' states referred to as a *population mean-field* term. A generic decision-maker has a negligible influence on m. The terms $\tilde{b}_*$ (resp. $\tilde{d}_*$) denote $\tilde{b}$ (resp. $\tilde{d}$) evaluated at the equilibrium strategy a^*, m. $\phi_*=\phi_{|a^*},\ \phi\in\{b,\sigma,\gamma\}$. The consistency condition is $\mu^*=\mu^{a^*,m_0}=m$.

The unnormalized global HJB system associated with (8) with infinite number of agents yields

$$\tag{9}\left\{\begin{array}{l}\hat{V}_t+H(t,\mu,m,\hat{V}_\mu,\hat{V}_{s\mu},\hat{V}_{ss\mu},\hat{V}_m)=0,\\ \hat{V}(t_1,\mu,m)=\int h(s,\mu,m)\mu(ds),\end{array}\right.$$

(10)

$$\begin{cases} H = \inf_a \int \Big\{ l + \langle b, \hat{V}_{s\mu}\rangle + \frac{1}{2} trace[(\sigma'\sigma)\hat{V}_{ss\mu}] \\ + \int_\Theta [\hat{V}_\mu(t, s+\gamma, \mu, m) - \hat{V}_\mu(t, s, \mu, m) - \langle \hat{V}_{s\mu}, \gamma\rangle]\nu(d\theta) + (\tilde{b} - \tilde{d}).\hat{V}_\mu \Big\} \mu(ds) \\ + \int \langle b_*, \hat{V}_{ym}\rangle m(dy) + \int \frac{1}{2} trace[(\sigma_*'\sigma_*)\hat{V}_{yym}] m(dy) \\ + \int\int_\Theta [\hat{V}_m(t, y+\gamma_*, \mu, m) - \hat{V}_m(t, y, \mu, m) - \langle \hat{V}_{ym}, \gamma_*\rangle]\nu(d\theta) m(dy) \\ + \int (\tilde{b}_* - \tilde{d}_*)(y).\hat{V}_m(t_1, y, \mu, m) m(dy) \end{cases}$$

If there exists a classical smooth solution to (12) then $(\hat{V}(t_0, m_0, m_0), a^*)$ provides a mean-field-type Nash equilibrium cost and strategy where $a^* \in \arg\min \hat{H}$ with $\hat{H}$ being the integrand Hamiltonian given by

$$\hat{H} = l + \langle b, \hat{V}_{s\mu}\rangle + \frac{1}{2} trace[(\sigma'\sigma)\hat{V}_{ss\mu}] + \int_\Theta [\hat{V}_\mu(t, s+\gamma, \mu, m) - \hat{V}_\mu(t, s, \mu, m) - \langle \hat{V}_{s\mu}, \gamma\rangle]\nu(d\theta) + (\tilde{b} - \tilde{d}).\hat{V}_\mu. \tag{11}$$

By consistency the Hamiltonian above is reduced to the diagonal case with (μ, m) replaced by (μ, μ). The system of $\hat{W}(t_1, m) = \hat{V}(t_1, m, m)$ becomes

$$\begin{cases} \hat{W}_t + H(t, m, m, \hat{V}_\mu, \hat{V}_{s\mu}, \hat{V}_{ss\mu}, \hat{V}_m) = 0, \\ \hat{W}(t_1, m) = \int h(s, m, m)\mu(ds). \end{cases} \tag{12}$$

Next, we describe a class of MFTG with infinite number of agents with separable Hamiltonians.

Example 3 (Separable Hamiltonians)**.** The best-response problem of a representative agent is the following: Given Borel probability measures $m(t, .) \in \mathcal{P}(\mathbb{R}^d)$, for $t \in [t_0, t_1]$ find a, μ such that

$$\begin{cases} \inf_a \int \tilde{h}(s - \int y\mu(t_1, dy), m(t_1))\mu(t_1, ds) + \bar{h}(\int y\mu(t_1, dy), m(t_1)) \\ + \int_{t_0}^{t_1} \{\int \frac{1}{2} a^2 \mu(t, ds) + \bar{q} \| \int a(t, y)\mu(t, dy)\|^2\} dt, \\ + \int_{t_0}^{t_1} \int \{\tilde{f}(s - \int y\mu(t, dy), m)\mu(t, ds) + \bar{f}(\int y\mu(t, dy), m)\} dt, \\ \text{subject to} \\ \mu_t = -div_s(a\mu) + \frac{1}{2} trace[(\sigma'\sigma\mu)_{ss}], \\ \mu(t_0, ds) = m_0(ds), \end{cases} \tag{13}$$

where the initial Borel probability measure $\mu(t_0, ds) = \mu_0(ds) = m_0(ds)$ over $\mathbb{R}^d$ is assumed to have a finite second moment, i.e., $m_0 \in \mathcal{P}_2(\mathbb{R}^d)$. The functions $\tilde{h}, \bar{h}, \tilde{f}, \bar{f}$ are real-valued functions. The coefficient $\bar{q}$ is chosen such that $\bar{q} > -\frac{1}{2}$ and σ is a constant matrix of appropriate dimension such that $\sigma'\sigma \succ \epsilon\mathbb{I}$. Problem (13) contains important classes of variance-aware MFTG problems such as distributed variance-reduction problems.

Proposition 4. *If there exists $\hat{V}(t,\mu,m)$ satisfying*

$$(14)\quad \begin{cases} \hat{V}_t(t,\mu,m) - \frac{1}{2}\int \|\hat{V}_{s\mu}\|^2\mu(ds) + \frac{1}{2}[1-\frac{1}{(2\bar{q}+1)}]\|\int \hat{V}_{y\mu}\mu(dy)\|^2 \\ +\int \tilde{f}(s-\int y\mu(dy),m)\mu(ds) + \bar{f}(\int y\mu(dy),m) \\ +\int \frac{1}{2}trace[(\sigma'\sigma)\hat{V}_{ss\mu}]\mu(ds) \\ -\int \|\hat{V}_{y\mu}\|^2 m(dy) + [1-\frac{1}{(2\bar{q}+1)}]\langle \int \hat{V}_{s\mu}\mu(ds), \int \hat{V}_{ym}m(dy)\rangle \\ +\int \frac{1}{2}trace[(\sigma'\sigma)\hat{V}_{yym}]m(dy), \\ \hat{V}(t_1,\mu,m) = \int \tilde{h}(s-\int y\mu(dy),m)\mu(ds) + \bar{h}(\int y\mu(dy),m), \end{cases}$$

then $\hat{V}(t_0,\mu_0,m_0)$ the best-response cost associated with Problem (13) and the best-response strategy is given by

$$a^*(t,s,\mu) = -\hat{V}_{s\mu}(t,s,\mu,m) + (1-\frac{1}{2\bar{q}+1})\int \hat{V}_{y\mu}(t,y,\mu,m)\mu(dy).$$

Proof. We use (12) with the augmented state (μ,m) to solve Problem (13). The HJB system of the problem is given by the following system:

$$(15)\quad \begin{cases} \hat{V}_t(t,\mu,m) + \inf_a \int \left\{ l + \langle b, \hat{V}_{s\mu}\rangle \right\}\mu(ds) \\ +\int \frac{1}{2}trace[(\sigma'\sigma)\hat{V}_{ss\mu}]\mu(ds) \\ +\int \langle a^*, \hat{V}_{ym}\rangle m(dy) + \int \frac{1}{2}trace[(\sigma'\sigma)\hat{V}_{yym}]m(dy), \\ \hat{V}(t_1,\mu,m) = \int \tilde{h}(s-\int y\mu(dy),m)\mu(ds) + \bar{h}(\int y\mu(dy),m). \end{cases}$$

The Hamiltonian term is

$$(16)\quad \begin{array}{l} \inf_a \int \left\{ l + \langle b, \hat{V}_{s\mu}\rangle \right\}\mu(ds) \\ = \inf_a \{\int \frac{1}{2}\|a(t,s) - \int a(t,y)\mu(dy)\|^2\mu(ds) \\ +\int \langle a(t,s) - \int a(t,z)\mu(dz), \hat{V}_{s\mu} - \int \hat{V}_{y\mu}\mu(dy)\rangle\mu(ds) \\ +(\bar{q}+\frac{1}{2})\|\int a(t,y)\mu(dy)\|^2 + \langle \int a(t,s)\mu(ds), \int \hat{V}_{y\mu}\mu(dy)\rangle \} \\ +\int \tilde{f}(s-\int y\mu(dy),m)\mu(ds) + \bar{f}(\int y\mu(dy),m) \end{array}$$

By Legendre-Fenchel duality, one has

$$(17)\quad \begin{array}{l} \inf_a \int \left\{ l + \langle b, \hat{V}_{s\mu}\rangle \right\}\mu(ds) \\ = -\frac{1}{2}\int \|\hat{V}_{s\mu})\|^2\mu(ds) - \frac{1}{2}[\frac{1}{(2\bar{q}+1)} - 1]\|\int \hat{V}_{y\mu}\mu(dy)\|^2 \\ +\int \tilde{f}(s-\int y\mu(dy),m)\mu(ds) + \bar{f}(\int y\mu(dy),m) \end{array}$$

which is non-convex in μ and the best-response strategy is given by

$$a^*(t,s,\mu) = -\hat{V}_{s\mu}(t,s,\mu,m) + (1-\frac{1}{2\bar{q}+1})\int \hat{V}_{y\mu}(t,y,\mu,m)\mu(dy).$$

Hence, the Bellman system becomes

$$(18)\quad \begin{cases} \hat{V}_t(t,\mu,m) - \frac{1}{2}\int \|\hat{V}_{s\mu}\|^2\mu(ds) - \frac{1}{2}[\frac{1}{(2\bar{q}+1)} - 1]\|\int \hat{V}_{y\mu}\mu(dy)\|^2 \\ +\int \tilde{f}(s-\int y\mu(dy),m)\mu(ds) + \bar{f}(\int y\mu(dy),m) \\ +\int \frac{1}{2}trace[(\sigma'\sigma)\hat{V}_{ss\mu}]\mu(ds) \\ +\int \langle a^*,\hat{V}_{ym}\rangle m(dy) + \int \frac{1}{2}trace[(\sigma'\sigma)\hat{V}_{yym}]m(dy), \\ \hat{V}(t_1,\mu,m) = \int \tilde{h}(s-\int y\mu(dy),m)\mu(ds) + \bar{h}(\int y\mu(dy),m). \end{cases}$$

This completes the proof. □

Proposition 4 provides the global HJB system to be solved. Note that the Hamiltonian in Proposition 4 is non-convex and non-Lipschitz. We study the well-posedness of (14). The next result provides a complexity reduction for MFTG Problem (13).

Proposition 5. *If there exist two functions $\tilde{V}(t,\tilde{\mu},m), \bar{u}(t_1,\bar{s},m)$ classical solutions to the following system*

$$(19)\quad \begin{cases} \tilde{V}_t - \int \frac{1}{2}\|\tilde{V}_{\tilde{s}\tilde{\mu}}\|^2\tilde{\mu}(d\tilde{s}) \\ +\int \tilde{f}(\tilde{s},m)\tilde{\mu}(d\tilde{s}) + \int \frac{1}{2}trace[(\sigma'\sigma)\tilde{V}_{\tilde{s}\tilde{s}\tilde{\mu}}]\tilde{\mu}(d\tilde{s}) \\ -\int \langle \tilde{V}_{y\tilde{\mu}},\tilde{V}_{ym}\rangle m(dy) + \int \frac{1}{2}trace[(\sigma'\sigma)\tilde{V}_{yym}]m(dy) = 0, \\ \tilde{V}(t_1,\tilde{\mu},m) = \int \tilde{h}(\tilde{s},m)\tilde{\mu}(d\tilde{s}), \\ \bar{u}_t - \frac{1}{2(1+2\bar{q})}\|\bar{u}_{\bar{s}}\|^2 + \bar{f}(\bar{s},m) \\ -\frac{1}{(1+2\bar{q})}\int \langle \bar{u}_{\bar{s}},\bar{u}_{ym}m(dy)\rangle + \int \frac{1}{2}trace[(\sigma'\sigma)\bar{u}_{yym}]m(dy) = 0, \\ \bar{u}(t_1,\bar{s},m) = \bar{h}(\bar{s},m), \end{cases}$$

then $\tilde{V}(t_0,\tilde{\mu}_0,\mu_0) + \bar{u}(t_0,\int y\mu_0(dy),\mu_0)$ is an equilibrium cost and $a = [-\tilde{V}_{\tilde{s}\tilde{\mu}} + \int \tilde{V}_{\tilde{s}\tilde{\mu}}\tilde{\mu}(d\tilde{s})] - \frac{\bar{u}_{\bar{s}}}{1+2\bar{q}}$ is an equilibrium strategy for the MFTG problem (13).

Proof. We propose a guess functional in the following form: $\tilde{V}(t,\tilde{\mu},m) + \bar{u}(t,\int y\mu(dy),m)$. Given the structure of the separated functions $(\tilde{f},\bar{f})$ and $(\tilde{h},\bar{h})$ in terms of the control action of the generic decision-maker $(\tilde{a},\bar{a})$, we transform Problem (13) into a combination of two simpler sub-problems.

Given a population mean-field m over the time horizon $[t_0,t_1]$ we would like to find $\tilde{a},\tilde{\mu},\bar{a}$ such that $\int \tilde{a}(t,y)\tilde{\mu}(dy) = 0$, $a = \tilde{a}+\bar{a}$, with $\tilde{a}(t,s) = a(t,s) - \int a(t,y)\mu(t,dy)$, and $\bar{a}(t) = \int a(t,s)\mu(t,ds)$ is an orthogonal decomposition. Let us consider the following two sub-problems:

$$(20)\quad \begin{cases} \inf_{\tilde{a}} \int \tilde{h}(\tilde{s}, m(t_1))\tilde{\mu}(t_1, d\tilde{s})) + \int_{t_0}^{t_1} \int [\frac{1}{2}\|\tilde{a}\|^2 + \tilde{f}(\tilde{s}, m)]\tilde{\mu}(t, d\tilde{s})\}dt, \\ \text{subject to} \\ \tilde{\mu}_t = -div_{\tilde{s}}(\tilde{a}\tilde{\mu}) + \frac{1}{2}trace[(\sigma'\sigma\tilde{\mu})_{\tilde{s}\tilde{s}}] \\ \tilde{\mu}(t_0, \tilde{s}) = \mu_0(\tilde{s} + \int y\mu_0(dy)), \\ \int \tilde{a}(t, y)\tilde{\mu}(t, dy) = 0, \end{cases}$$

$$(21)\quad \begin{cases} \inf_{\bar{a}} \bar{h}(\bar{s}(t_1), m(t_1)) + \int_{t_0}^{t_1} \{\frac{1+2\bar{q}}{2}\bar{a}^2 + \bar{f}(\bar{s}, m)\}dt, \\ \text{subject to} \\ \frac{d}{dt}\bar{s} = \bar{a}, \quad \bar{s}(t_0) = \int y\mu_0(dy). \end{cases}$$

Let $\tilde{\mu}(t, z) = \mu(t, z + \int y\mu(t, dy))$. Then,

$$(22)\quad \begin{aligned} &\int z\tilde{\mu}(t, dz) = \int z\mu(t, dz + \int y\mu(t, dy)) \\ &= \int w\mu(t, dw) - [\int y\mu(t, dy)] = 0 \end{aligned}$$

The two sub-problems (20) and (21) are coupled via m. The reduced systems of the sub-problems (20) and (21) are given by

$$(23)\quad \begin{cases} \tilde{v}_t(t, \tilde{s}) - \frac{1}{2}\|\tilde{v}_{\tilde{s}}\|^2 + \tilde{f}(\tilde{s}, \tilde{m}) + \frac{1}{2}trace[(\sigma'\sigma)\tilde{v}_{\tilde{s}\tilde{s}}] = 0, \\ \tilde{v}(t_1, \tilde{s}) = \tilde{h}(\tilde{s}, \tilde{m}(t_1)) \\ \tilde{m}_t = div_{\tilde{s}}((\tilde{v}_{\tilde{s}} - \int \tilde{v}_{\tilde{s}}\tilde{\mu}(d\tilde{s}))\tilde{m}) + \frac{1}{2}trace[(\sigma'\sigma\tilde{m})_{\tilde{s}\tilde{s}}] \\ \tilde{m}(t_0, \tilde{s}) = m_0(\tilde{s} + \int ym_0(dy)), \\ \tilde{a} = -\tilde{v}_{\tilde{s}} + \int \tilde{v}_{\tilde{s}}\tilde{\mu}(d\tilde{s}), \\ \int \tilde{v}_{\tilde{s}}(t, y)\tilde{\mu}(dy) = 0, \\ \bar{v}_t(t, \bar{s}) - \frac{1}{2(1+2\bar{q})}.\|\bar{v}_{\bar{s}}\|^2 + \bar{f}(\bar{s}, m) = 0, \\ \hat{v}(t_1, \bar{s}) = \bar{h}(\bar{s}, m), \\ \bar{a} = -\frac{\bar{v}_{\bar{s}}}{1+2\bar{q}}. \end{cases}$$

We observe that $\hat{V}(t, \mu, m) = \int \tilde{v}(t, s - \int y\mu(dy), m)\mu(ds) + \bar{v}(t, \int y\mu(dy), m)$ and $a = [-\tilde{v}_{\tilde{s}} + \int \tilde{v}_{\tilde{s}}\tilde{\mu}(d\tilde{s})] - \frac{\bar{v}_{\bar{s}}}{1+2\bar{q}}$.

The mean-field m solves

$$(24)\quad \begin{cases} m_t = div_s((\tilde{v}_{\tilde{s}} - \int \tilde{v}_{\tilde{s}}\tilde{\mu}(d\tilde{s}) + \frac{\bar{v}_{\bar{s}}}{1+2\bar{q}})m) + \frac{1}{2}trace[(\sigma'\sigma m)_{ss}] \\ m(t_0, ds) = m_0(ds). \end{cases}$$

We then use the reduced system solution to the two coupled sub-problems to construct a solution to the following system of two HJB equations on the space of measures.

(25)

$$\begin{cases} \tilde{V}_t - \int \frac{1}{2}\|\tilde{V}_{\tilde{s}\tilde{\mu}} - \int \tilde{V}_{\tilde{y}\tilde{\mu}}\tilde{\mu}(d\tilde{y})\|^2 \tilde{\mu}(d\tilde{s}) \\ + \int \tilde{f}(\tilde{s}, m)\tilde{\mu}(d\tilde{s}) + \int \frac{1}{2} trace[(\sigma'\sigma)\tilde{V}_{\tilde{s}\tilde{s}\tilde{\mu}}]\tilde{\mu}(d\tilde{s}) \\ - \int \langle \tilde{V}_{y\tilde{\mu}} - \int \tilde{V}_{\tilde{s}\tilde{\mu}}\tilde{\mu}(d\tilde{s}), \tilde{V}_{ym}\rangle m(dy) + \int \frac{1}{2} trace[(\sigma'\sigma)\tilde{V}_{yym}]m(dy) = 0, \\ \tilde{V}(t_1, \tilde{\mu}, m) = \int \tilde{h}(\tilde{s}, m)\tilde{\mu}(d\tilde{s}), \\ \bar{u}_t - \frac{1}{2(1+2\bar{q})}\|\bar{u}_{\bar{s}}\|^2 + \bar{f}(\bar{s}, m) \\ -\frac{1}{(1+2\bar{q})}\int \langle \bar{u}_{\bar{s}}, \bar{u}_{ym} m(dy)\rangle + \int \frac{1}{2} trace[(\sigma'\sigma)\bar{u}_{yym}]m(dy) = 0, \\ \bar{u}(t_1, \bar{s}, m) = \bar{h}(\bar{s}, m) \end{cases}$$

where $\hat{V}(t,\mu,m) = \tilde{V}(t,\tilde{\mu},m) + \bar{u}(t, \int y\mu(dy), m)$, and $a = [-\tilde{V}_{\tilde{s}\tilde{\mu}} + \int \tilde{V}_{\tilde{s}\tilde{\mu}}\tilde{\mu}(d\tilde{s})] - \frac{\bar{u}_{\bar{s}}}{1+2\bar{q}}$.

This completes the proof. □

It can also be solved using a system of two MASS given by

$$\begin{cases} \tilde{U}_t - \frac{1}{2}\|\tilde{U}_{\tilde{s}}\|^2 + \tilde{f}(\tilde{s}, \tilde{m}) + \frac{1}{2} trace[(\sigma'\sigma)\tilde{U}_{\tilde{s}\tilde{s}}] \\ - \int \langle \tilde{U}_{\tilde{s}}, \tilde{U}_{\tilde{m}}\rangle \tilde{m}(dy) + \int \frac{1}{2} trace[(\sigma'\sigma)\tilde{U}_{yy\tilde{m}}]\tilde{m}(dy) = 0, \\ \tilde{U}(t_1, \tilde{s}, \tilde{m}) = \tilde{h}(\tilde{s}, \tilde{m}), \\ \bar{U}_t - \frac{1}{2(1+2\bar{q})}\|\bar{U}_{\bar{s}}\|^2 + \bar{f}(\bar{s}, m) \\ -\frac{1}{(1+2\bar{q})}\int \langle \bar{U}_{\bar{s}}, \bar{U}_{ym} m(dy)\rangle + \int \frac{1}{2} trace[(\sigma'\sigma)\bar{U}_{yym}]m(dy) = 0, \\ \bar{U}(t_1, \bar{s}, m) = \bar{h}(\bar{s}, m) \end{cases} \tag{26}$$

It follows that the best-response cost is given by $\hat{V}(t,\mu,m) = \int \tilde{U}(t, s - \int y\mu(dy), m)\mu(ds) + \bar{U}(t, \int y\mu(dy), m)$, and the equilibrium cost is $\hat{V}(t_0, m_0, m_0)$. The mean-field-type Nash equilibrium strategy is given by $a = [-\tilde{U}_{\tilde{s}} + \int \tilde{U}_{\tilde{y}}\tilde{\mu}(d\tilde{y})] - \frac{\bar{U}_{\bar{s}}}{1+2\bar{q}}$.

Existence of solution

Proposition 6. *Let*

$$\begin{cases} \tilde{U}_t - \frac{1}{2}\|\tilde{U}_{\tilde{s}}\|^2 + \tilde{f}(\tilde{s}, \tilde{m}) + \frac{1}{2} trace[(\sigma'\sigma)\tilde{U}_{\tilde{s}\tilde{s}}] \\ - \int \langle \tilde{U}_{\tilde{s}}, \tilde{U}_{\tilde{m}}\rangle \tilde{m}(dy) + \int \frac{1}{2} trace[(\sigma'\sigma)\tilde{U}_{yy\tilde{m}}]\tilde{m}(dy) = 0, \\ \tilde{U}(t_1, \tilde{s}, \tilde{m}) = \tilde{h}(\tilde{s}, \tilde{m}), \end{cases} \tag{27}$$

Assume that $\bar{q} > -\frac{1}{2}$, $\sigma = \mathbb{I}$, the initial distribution μ_0 has a finite second moment, and $\tilde{m} \mapsto \tilde{H}(\tilde{m})$ and $\tilde{m} \mapsto \tilde{F}(\tilde{m})$ are strictly convex in the space of measure and $\tilde{F}_{\tilde{m}}(\tilde{m}) = \tilde{f}(\tilde{s}, \tilde{m})$, $\tilde{H}_{\tilde{m}}(\tilde{m}) = \tilde{h}(\tilde{s}, \tilde{m})$ then there is a solution to (27).

The proof follows from the representation formula of the original problem as a variational problem. A detailed proof can be found in [34]. It can be shown that the convexity assumption can be relaxed to a class of semi-convexity under suitable conditions [22].

4. MFTG with atomic and a multi-class non-atomic decision-makers

We now consider $I \geq 2$ number many atomic decision-makers and several classes of nonatomic decision-makers interacting within the time horizon $[t_0, t_1], \quad t_0 < t_1$. The set of atomic decision-makers is denoted by $\mathcal{I} = \{1, 2, \ldots, I\}$. Decision-maker $i \in \mathcal{I}$ has a control action $a_i \in A_i = \mathbb{R}^d$, $d \geq 1$. There are $C \geq 1$ classes of non-atomic decision-makers denoted by $\mathcal{C} := \{I+1, I+2, \ldots, I+C\}$. A representative decision-maker in class $c \in \mathcal{C}$ has a control action $a_c \in A_c = \mathbb{R}^{d_c}$.

State and cost functionals of atomic decision-makers

The data of the state and cost functionals of atomic decision-makers are given by

D_{do}

$:= [t_0, t_1] \times \mathcal{S}_{do} \times \prod_{i \in \mathcal{I}} A_i \times \mathcal{P}(\mathcal{S}_{do}) \times \prod_{i \in \mathcal{I}} \mathcal{P}(A_i) \times \prod_{c \in \mathcal{C}} [\mathcal{P}(\mathcal{S}_c) \times \mathcal{P}(A_c)]$,

Drift coefficient: $b_{do}: \; D_{do} \to \mathcal{S}_{do}$,

Diffusion coefficient: matrix $\sigma_{do}: \; D_{do} \to \mathbb{R}^{d \times d}$,

Jump rate coefficient: $\gamma_{do}: D_{do} \times \Theta \to \mathbb{R}^d$,

Birth rate: $\tilde{b}_{do}: D_{do} \to \mathcal{S}_{do}$,

Death rate: $\tilde{d}_{do}: D_{do} \to \mathcal{S}_{do}$,

where $\mathcal{P}(\mathcal{S}_{do})$ denotes the set of Borel measures on $\mathcal{S}_{do}$. Let

$$l_i: \ D_{do} \to \mathbb{R},$$
$$h_i: \mathcal{S}_{do} \times \mathcal{P}(\mathcal{S}_{do}) \times \prod_{c\in\mathcal{C}} \mathcal{P}(\mathcal{S}_c) \to \mathbb{R},$$

where l_i is the running cost and h_i is the terminal cost at time t_1. The performance functional of the atomic decision-maker i is given by

$$\begin{aligned}&\hat{L}_i((a_j)_{j\in\mathcal{I}})\\ &= \int h_i(s_{do}, \mu_{do}(t_1, .), m(t_1)).\mu_{do}(t_1, ds_{do})\\ &+ \int_{t_0}^{t_1} \int l_i(t, s_{do}, (a_j)_{j\in\mathcal{I}}, \mu_{do}, (\tilde{\mu}_j)_{j\in\mathcal{I}}, m, \tilde{m})\mu_{do}(t, ds_{do})dt\end{aligned}$$

The performance functional $\hat{L}_i$ of i is completely determined by the conditional measure $\mu_{do} \otimes m$.

State and cost functionals of non-atomic decision-makers

The data of the state and cost functionals of non-atomic decision-makers are given by

$$\begin{aligned}&D_c := [t_0, t_1] \times \mathcal{S}_{do} \times \mathcal{S}_c \times \prod_{i\in\mathcal{I}} A_i \times \mathcal{A}_c \times \mathcal{P}(\mathcal{S}_{do}) \times \mathcal{P}(\mathcal{S}_c)\\ &\times \prod_{i\in\mathcal{I}} \mathcal{P}(A_i) \times \mathcal{P}(A_c) \times \prod_{c'\in\mathcal{C}} [\mathcal{P}(\mathcal{S}_{c'}) \times \mathcal{P}(A_{c'})],\\ &b_c: \ D_c \to \mathcal{S}_c,\\ &\sigma_c: \ D_c \to \mathbb{R}^{d_c\times k_c},\\ &\gamma_c: D_c \times \Theta \to \mathbb{R}^{d_c},\\ &\nu_c \text{ is a Radon measure over } \Theta,\\ &\text{Birth rate: } \ \tilde{b}_c: D_c \to \mathcal{S}_c,\\ &\text{Death rate: } \ \tilde{d}_c: D_c \to \mathcal{S}_c,\end{aligned}$$

The performance functional $\hat{L}_c$ is given by

(28)
$$\begin{aligned}&\hat{L}_c((a_i)_{i\in\mathcal{I}}, a_c, m, \tilde{m}) := \int h_c(s_{do}, x_c, \mu_{do,c}(t_1, .), m(t_1)) \ \mu_{do,c}(t_1, ds_{do}dx_c)\\ &+ \int_{t_0}^{t_1} \int l_c(t, s_{do}, x_c, (a_i)_{i\in\mathcal{I}}, a_c, \mu_{do,c}, (\tilde{\mu}_i)_{i\in\mathcal{I}}, \tilde{\mu}_c, m, \tilde{m})\mu_{do,c}(t, ds_{do}dx_c)dt\end{aligned}$$

The expected performance functional $\hat{L}_c$ of c is completely determined by the measure $\mu_{do,c} \otimes m$.

Let

$$
\begin{aligned}
(29)\quad & H_i = \inf_{a_i} \int \hat{V}_{i,\mu_{do}}(\tilde{b}_{do} - \tilde{d}_{do})\mu_{do}(ds_{do}) \\
& + \sum_c \int \hat{V}_{i,m_c\mu_{do}}(\tilde{b}_{*,c} - \tilde{d}_{*,c})\mu_{do,c}(ds_c ds_{do}) \\
& + \int \mu_{do}(ds_{do}) \Big\{ l_i + \hat{V}_{i,s_{do}\mu_{do}} b_{do} + \tfrac{1}{2}\langle \sigma_{do}, \sigma_{do}\hat{V}_{i,s_{do}s_{do}\mu_{do}}\rangle \\
& + \int_\Theta \Big\{ \hat{V}_{i,\mu_{do}}(t, s_{do} + \gamma_{do}(t_-, ., \theta), m) \\
& -\hat{V}_{i,\mu_{do}}(t, s_{do}, m) - \langle \hat{V}_{i,s_{do}\mu_{do}}, \gamma_{do}(t, ., \theta)\rangle \Big\} \nu_{do}(d\theta) \\
& + \int [\textstyle\sum_c \langle \bar{b}_c, \hat{V}_{i,y_c m\mu_{do}}\rangle + \tfrac{1}{2}\sum_c \langle \bar{\sigma}_c, \bar{\sigma}_c \hat{V}_{i,y_c y_c m\mu_{do}}\rangle] m(dy) \\
& + \int \int_\Theta \Big\{ \hat{V}_{i,m\mu_{do}}(t, s_{do}, \mu_{do}, y + \bar{\gamma}(t_-, ., \theta), m) \\
& -\hat{V}_{i,m\mu_{do}}(t, s_{do}, \mu_{do}, y, m) \\
& - \textstyle\sum_{c\in\mathcal{C}} \langle \hat{V}_{i,y_c m\mu_{do}}, \bar{\gamma}_c(t, ., \theta)\rangle \Big\} \nu_{\mathcal{C}}(d\theta) m(dy) \ \}
\end{aligned}
$$

$$
\begin{aligned}
(30)\quad & H_c = \inf_{a_c} \\
& \int \mu_{doc}(ds_{do} ds_c) \Big\{ l_c + \hat{V}_{c,s_{do}\mu_{do,c}} b_{do} + \tfrac{1}{2}\langle \sigma_{do}, \sigma_{do}\hat{V}_{c,s_{do}s_{do}\mu_{do,c}}\rangle \\
& + \hat{V}_{c,s_c\mu_{do,c}} b_c + \tfrac{1}{2}\langle \sigma_c, \sigma_c \hat{V}_{c,s_c s_c \mu_{do,c}}\rangle \\
& + \int_\Theta \Big\{ \hat{V}_{c,\mu_{do,c}}(t, s_{do}(t_-) + \gamma_{do}(t_-, \theta_{do}), s_c(t_-) + \gamma_c(t_-, ., \theta_c), .) \\
& - \hat{V}_{c,\mu_{do,c}}(t, s_{do}(t_-), s_c(t_-), .) \\
& - \langle \hat{V}_{c,s_{do}\mu_{do,c}}, \gamma_{do}(t, ., \theta_{do})\rangle \\
& - \langle \hat{V}_{c,s_c\mu_{do,c}}, \gamma_c(t, ., \theta_c) \ \} \nu_{do}(d\theta_{do})\nu_c(d\theta_c) \\
& \} \\
& + \int [\textstyle\sum_{c'} \langle \bar{b}_{c'}, \hat{V}_{c,y_{c'} m\mu_{do,c}}\rangle + \tfrac{1}{2}\sum_{c'} \langle \bar{\sigma}_{c'}, \bar{\sigma}_{c'} \hat{V}_{c,y_{c'} y_{c'} m\mu_{do,c}}\rangle] m(dy) \\
& + \int \ \int_\Theta \Big\{ \hat{V}_{c,m\mu_{do,c}}(t, s_{do}, s_c, \mu_{do,c}, y(t_-) + \bar{\gamma}(t_-, ., \theta), m) \\
& - \hat{V}_{c,m\mu_{do,c}}(t, s_{do}, s_c, \mu_{do,c}, y(t_-), m) \\
& - \textstyle\sum_{c\in\mathcal{C}} \langle \hat{V}_{c,y_{c'} m\mu_{do,c}}, \bar{\gamma}_{c'}(t, ., \theta)\rangle \} \nu_{\mathcal{C}}(d\theta) \Big\} m(dy) \\
& + \int \hat{V}_{c,\mu_{do}}(\tilde{b}_{do} - \tilde{d}_{do})\mu(ds_{do}) + \int \hat{V}_{c,\mu_{doc}}(\tilde{b}_c - \tilde{d}_c)\mu_{doc}(ds_{do} ds_c) \\
& + \textstyle\sum_{c'} \int \hat{V}_{i,m_{c'}\mu_{doc}}(\tilde{b}_{*,c'} - \tilde{d}_{*,c'}) m(dy)\mu_{do}(ds_{do}),
\end{aligned}
$$

The next Proposition provides a sufficiency condition for mean-field-type Nash equilibria.

Proposition 7. *If there exists a classical smooth solution* $\hat{V}_i : [t_0, t_1] \times \mathcal{P}(\mathcal{S}_{do}) \times \prod_c \mathcal{P}(\mathcal{S}_c) \to$ *and* $\hat{V}_c : [t_0, t_1] \times \mathcal{P}(\mathcal{S}_{do}) \times \mathcal{P}(\mathcal{S}_c) \times \prod_{c'} \mathcal{P}(\mathcal{S}_{c'}) \to$ *such* *that*

(31)
$$\begin{cases} \hat{V}_{i,t}(t,\mu_{do}\otimes m)+H_i=0, \quad on\ (t_0,t_1)\times\mathcal{P}(\mathcal{S}_{do})\times\prod_c\mathcal{P}(\mathcal{S}_c)\\ \hat{V}_i(t_1,\mu_{do}\otimes m)=\int h_i(s_{do},\mu_{do}\otimes m)\mu_{do}(ds_{do}),\\ \hat{V}_{c,t}(t,\mu_{do,c}\otimes m)+H_c=0, \quad on\ (t_0,t_1)\times\mathcal{P}(\mathcal{S}_{do}\times\mathcal{S}_c)\times\prod_{c'}\mathcal{P}(\mathcal{S}_{c'}),\\ \hat{V}_c(t_1,\mu_{do,c}\otimes m)=\int h_c(s_{do},x_c,\mu_{do,c},m)\mu_{do,c}(ds_{do}dx_c),\\ \mu_c(t,.)=m_c(t,.),\\ \tilde{\mu}_c^*=\tilde{m}_c,\\ i\in\mathcal{I},\ c\in\mathcal{C} \end{cases}$$

then

- $\hat{V}_c(t_0,\mu_{do,c}(t_0,.)\otimes m(t_0,.))$ *is an equilibrium cost of a generic non-atomic decision-maker in class c,*
- $\hat{V}_i(t_0,\mu_{do,c}(t_0,.)\otimes m(t_0,.))$ *is an equilibrium cost of the atomic decision-maker* $i\in\mathcal{I}$ *and*
- *the strategy* a_c^* *minimizing the expression in* H_c *is an optimal strategy.*
- *The optimal strategy* a_i^* *minimizes in* H_i*.*

Generically, a_c^* is a function of $(t,s_{do},x_c,m,\mu_{do,c})$ which is in a state-and-mean-field feedback form, and a_i^* is a function of (t,s_{do},m,μ_{do}) which is in a state-and-mean-field-type feedback form.

Proof. Let $\hat{\phi}_i$ be a test function. We use the integration formula to obtain the following equality:

(32)
$$\begin{aligned}&\hat{\phi}_i(t_1,\mu_{do}\otimes m(t_1))-\hat{\phi}_i(t_0,\mu_{do}\otimes m(t_0,.))\\ &=\int_{t_0}^{t_1}[\hat{\phi}_{i,t}+\int O_{do,m}[\hat{\phi}_{i,\mu_{do}}]\mu_{do}(t,ds_{do})]dt,\end{aligned}$$

where $O_{do,m}$ is the infinitesimal generator associated with $\mu_{do}\otimes m$. We aim to establish a verification method. In order to do so we compute the difference between the expected cost functional $\hat{L}_i$ and the candidate guess functional $\hat{V}_i(t_0,\mu_{do}\otimes(t_0,.))$

(33)
$$\begin{aligned}&\hat{L}_i-\hat{V}_i(t_0,\mu_{do}\otimes m(t_0,.))\\ &=\int h_i\mu_{do}(t_1,ds_{do})-\hat{V}_i(t_1,\mu_{do}\otimes m(t_1,.))\\ &+\int_{t_0}^{t_1}\hat{V}_{i,t}+\int\{l_i+O_{do,m}[\hat{V}_{i,\mu_{do}}]\}\mu_{do}(t,ds_{do}),\end{aligned}$$

We do a similar computation with $\hat{V}_c$. Then, the announced result follows by taking the minimization in a_i (respectively in a_c) and by matching the functionals with $\hat{V}$. This completes the proof. □

A solution (if any) to the infinite dimensional PIDE (31) provides the equilibrium values and the equilibrium strategies of all the decision-makers.

5. Conclusion

In this article we have presented a class of mean-field-type games with both atomic and multi-class non-atomic decision-makers. We proposed a Hamilton-Jacobi- Bellman PIDE approach and provided sufficiency conditions for mean-field-type Nash equilibria in state-and-mean-field feedback strategies. As a corollary a class of master adjoint system is derived. We have provided an example which the resulting PIDE system has a non-convex and non-Lipschitz Hamiltonian but is nevertheless semi-explicitly solvable.

Number of questions remain unanswered and we leave them for future works. The first concern is about the existence of solutions to the Bellman system in the general case. The second concern is the solvability of the system of the PIDEs in the general case. The third plan is the solvability of hierarchical designs such as

- All the atomic decision-makers move first. Then follows the mass of non-atomic decision-makers
- All the non-atomic decision-makers move first. Then, the atomic decision-makers respond
- Finding the best design by means of any permutation of cluster sequential ordering of decision-makers

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Hamidou Tembine
Learning and Game Theory Lab
75116, Paris
France
E-mail address: tembine@landglab.com

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