Communications in Information and Systems
Volume 21, Number 4, 475–479, 2021

For Tyrone Duncan

Alain Bensoussan and Sanjoy K. Mitter

Tyrone Duncan has made seminal contributions in the field of Filtering, Stochastic Control and the interface of probability theory and geometry. In his pioneering thesis, he presented a fundamental theory of nonlinear filtering and was instrumental in introducing the modern theory of martingales as developed by Doob, Meyer, Kunita, Watanabe and others in the study of nonlinear filtering. As he said to one of us, he was fortunate to learn the subject from Watanabe in the course on Martingale theory which Watanabe gave at Stanford when Tyrone was a graduate student. The celebrated stochastic partial differential equation describing the evolution of the unnormalized conditional density of the state given the observations was introduced in his thesis (independently by Mortensen and Zakai). As an outgrowth of the ideas presented in his thesis, he was the first to give a causal representation for the Radon–Nikodym derivative of the probability measure representing the Gaussian signal plus (white) noise with respect to the probability measure representing noise only – the celebrated likelihood ratio formula obtained via the Girsanov Theorem. In important work, he also gave the formula for mutual information between a Gauss–Markov Diffusion (the state) and an observation where the state is observed in additive white noise.

One of us is a direct beneficiary of Tyrone's research in this field. Tyrone's other research which belongs to the same period is his joint work with Pravin Varaiya on existence theorems for Optimal Stochastic Control, one of the first general theorems on existence questions.

Motivated by problems in Quantum Field Theory, Irving Segal, in the fifties, undertook a fundamental investigation of the Absolute Continuity of Gaussian measures obtained as a translation of white noise measure in a canonical Hilbert space setting. These ideas were further developed by Leonard Gross in the framework of Abstract Wiener spaces. Tyrone was one of the first to appreciate the importance of this work to Stochastic Systems Theory. One of us distinctly remembers his lecture on this subject given at Imperial College in the mid-seventies.

The study of Stochastic Differential Equations on manifolds was initiated by K. Ito and continued by R. Gangooli, in his MIT thesis directed by

H.P. McKean. Amongst system theorists, Tyrone was again the first to recognize the importance of this work and is credited as the first one to define stochastic integrals in Riemannian Manifolds and to the study of Estimation and Stochastic Control on Riemannian Manifolds. He presented explicit solutions to estimation and stochastic control problems defined on manifolds, some of the few explicit solutions known for Estimation and Stochastic Control.

Browninan motion and the heat equation play important roles in geometry. What is less well known is that the first probabilistic proof of the Atiyah–Singer Index theorem is due to Tyrone. This was published in *Partial Differential Equations and Geometry*, Proceedings of the Park City Conference, Marcel Dekker (1979), pp. 57–76. H.P. McKean and I. Singer were present at the conference. It is to be noted that Bismut, in his 1984 paper on the probabilistic proof of the Atiyah–Singer Index theorem, references Gilkey in the same proceedings but fails to reference Tyrone. This work represents a striking example of Tyrone's original thinking. His work on the representation of Affine Lie algebras using Browninan motion is another example of his unique capability of making connections between seemingly diverse intellectual fields,

> A (Jacobi) theta function has two expressions, an infinite product and an infinite sum. Tyrone showed that the infinite product is the Jacobian of a change of variables in an infinite dimensional Hilbert space and it follows that the sum arises as an infinite series. The infinite product arises as a change of variables by conditioning the Wiener measure in the group at the final time to a special orbit for the compact simple Lie group. The infinite sum arises by considering the orbits in the compact Lie group intersecting a maximal torus. Jacobi's Triple Product result considered the simplest (nontrivial) compact Lie group, SU(2), the special unitary group in two complex dimensions, whose Lie algebra has a basis of one positive root, its negative and one element in the center. These three provide the structure of the triple product. Macdonald extended Jacobi's Triple Product result to an arbitrary compact simple Lie group using affine Lie algebras that are an important example of Kac–Moody Lie algebras. Frenkel used semigroups and Wiener measure to provide another proof of this Macdonald identity. Again, this identity has terms from the root space and the center, Cartan subalgebra. Tyrone used some stochastic analysis as a change of variables in Wiener space (or the canonical normal distribution of I. Segal) to show that the product form of the theta function arose from a change of variables determinant for a suitably pinned Brownian motion. The infinite sum arises from the intersection of the orbits with a maximal torus. The infinite product for a product of simple Lie groups can be used to verify directly some difficult number theoretic results of Ramanujan. In his paper, he also provides a stochastic interpretation of the radial part of the Laplacian for a compact simple Lie group describing it from the Radon–Nikodym derivative for the addition of a drift term to Brownian motion in the group.

Tyrone spent the years 1979 and 80 at Harvard. One of us remembers with nostalgia the Friday afternoon excursions to the Ha'Penny Pub (which alas does not exist anymore) with Chris Byrnes, Peter Caines and Tyrone where some of the most exciting intellectual discussions on Systems Theory took place. This was also a time of great ferment and excitement in the field. One of the topics of discussion was the role of Algebraic Geometry and Topology in understanding the structure of systems. Here, for the first time, one of us learnt about the Bott Periodicity Theorem and what it had to say about the invariants of linear systems. Tyrone's own research at this time is reflected in his joint work with Chris Byrnes on topological invariants of systems.

As mentioned previously, the contributions of Tyrone Duncan to stochastic control are fundamental and bear some characteristics which are unique in the literature. He has the ability to work in a vast domain and explore it in all its aspects. He progresses from simple to more complex situations, develops the mathematical theory, considers the numerical aspects, and looks at very concrete applications. This may appear to be the natural way to conduct research in Applied Mathematics, but it is not so common, and certainly not carried out as systematically as in the works of Tyrone Duncan.

This is best illustrated by considering the field of Adaptive Control, on which he decided to work twenty years ago, and in which he is now a well established leader. With Bozenna Pasik-Duncan and some other colleagues and students he has covered all aspects of this domain. Adaptive Control is the problem of making decisions when the system has unknown parameters. The basic challenge is to obtain a design which eventually converges to the exact values of the parameters and the corresponding optimal control. Tyrone's articles are among the main references in the field. Following the systematic approach described above, he has considered all possible situations and all possible problems of interest: proven consistency, obtained rates of convergence both for finite dimensional systems, infinite dimensional systems, differential delay systems, fully and partially observable systems, linear and nonlinear systems, boundary control, discrete time and continuous time, numerical approximations and computational aspects. From the very beginning he has looked at models motivated by applications: portfolio and consumption models, manufacturing models, risk-sensitive models.

More recently (in 2000) Tyrone started working on stochastic models with fractional Brownian motion as inputs. One knows that the traditional Brownian motion models have serious limitations in modeling extreme phenomenon which occur in the crash of financial markets or in communication networks. This motivates the use of fractional Brownian motion in modeling such phenomena. Unfortunately, these processes lack the nice properties of Wiener processes. They are not Markovian, they are not semimartingales. That creates a serious hurdle which has, over a long period, prevented progress in their use for applications.

With his energy and renowned talent Tyrone is systematically covering this new domain in all aspects, with remarkable success. He has developed basic techniques like stochastic calculus and Ito's formula and then studied large classes of stochastic differential equations with fractional Brownian motion inputs. This is probably the most comprehensive set of results concerning stochastic dynamical systems driven by Fractional Brownian motion. At the same time he has considered many engineering problems, whose solutions are well known for the case of stochastic differential equations with Brownian motion inputs but totally open for fractional Brownian motion inputs.

Tyrone has used these models in problems of identification, estimation and control with remarkable success. In addition, he has studied many applied models (Finance, ATM traffic, queues) where modeling with fractional Brownian motion inputs is particularly relevant. This domain will particularly benefit from the pioneering work of Tyrone and will undoubtedly undergo substantial development.

In view of Tyrone's versatility, it would appear strange that he did not join the community of scholars in Mean Field Games. He has done this recently in his own way, namely rejuvenating the famous method of completion of squares, which was introduced in the early days of Control Theory, to provide a direct approach for the optimal solution, without relying on the theory of necessary conditions (Pontryagin Maximum Principle) or the theory of sufficient conditions (Bellman Dynamic Programming). Remarkably, Tyrone has shown that the method solves some nonlinear control and game problems. Even more, he was able to use this method for controlling fractional Brownian motion, Gauss–Volterra processes, Rosenblatt processes, and jump processes. Some non-quadratic payoff functions can also be handled with this approach, in particular exponential quadratic (risk sensitive) control and game problems.

There is a gold mine in the old methodologies of control theory. They can work for new problems they were not invented for. What a beautiful message from an experienced scientist to the young generation of researchers.

This short account of Tyrone's scientific work demonstrates the uniqueness and versatility of his contributions. As a person, he is known for his gentleness and kindness. He is a gentleman and a scholar, a faithful friend and an esteemed colleague. With Bozenna and their charming daughter Dominique, they form a model family, a wonderful team that will continue to make great contributions to Science and Engineering.

Alain Bensoussan
Lars Magnus Ericsson Chair
University of Texas at Dallas
Dallas
USA
Chair Professor of Risk and Decision Analysis
City University Hong Kong
Hong Kong
China
E-mail address: alain.bensoussan@utdallas.edu; abensous@cityu.edu.hk

Sanjoy K. Mitter
Department of Electrical Engineering and Computer Science
The Laboratory for Information and Decision Systems
Cambridge, MA 02139
USA
E-mail address: mitter@mit.edu