Linear relations among double zeta values in positive characteristic

Dedicated to the memory of my father

Chieh-Yu Chang*

For each integer $n \geq 2$, we study linear relations among weight n double zeta values and the nth power of the Carlitz period over the rational function field $\mathbb{F}_q(\theta)$. We show that all the $\mathbb{F}_q(\theta)$-linear relations are induced from the $\mathbb{F}_q[t]$-linear relations among certain explicitly constructed special points in the nth tensor power of the Carlitz module. We then establish a principle of Siegel's lemma for computing and determining the $\mathbb{F}_q[t]$-linear relations mentioned above, and thus obtain an effective criterion for computing the dimension of weight n double zeta values space.

1. Introduction

1.1. Classical theory

Classical multiple zeta values (abbreviated as MZV's) are defined by the series: for $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $s_1 \geq 2$,

$$\zeta(\mathfrak{s}) := \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}} \in \mathbb{R}^{\times}.$$

Here $\mathrm{wt}(\mathfrak{s}) := \sum_{i=1}^{r} s_i$ is called the weight and r is called the depth of the presentation $\zeta_A(\mathfrak{s})$. MZV's have many connections with various research topics. For example, see [Z93, Z94, Cart02, André04, B12, Zh16].

Let $Z_0 := \mathbb{Q}$, $Z_1 := \{0\}$ and Z_n be the $\mathbb{Q}$-vector space spanned by the weight n MZV's for integers $n \geq 2$. Putting $Z := \sum_{n \geq 0} Z_n$. It is well known that $Z_m Z_n \subseteq Z_{m+n}$, and so Z has a $\mathbb{Q}$-algebra structure. The Goncharov's direct sum conjecture [Gon97] asserts that Z is a graded algebra (graded by weights). Therefore, understanding the $\mathbb{Q}$-algebraic relations among MZV's boils down to understanding the $\mathbb{Q}$-linear relations among the same weight MZV's. However, to date computing the dimension of Z_n for each n is out of reach. Note that Zagier's dimension conjecture predicts that $\{\dim_{\mathbb{Q}} Z_n\}$ satisfies the recursive relation: $d_0 = 1$, $d_1 = 0$ and

$$d_n = d_{n-2} + d_{n-3} \text{ for } n \geq 3,$$

and one knows by Goncharov and Terasoma that $\dim_{\mathbb{Q}} Z_n \leq d_n$ for each n (see [Te02, DG05]).

Now, we focus on depth two MZV's, which are called double zeta values. It is a natural question to ask how to compute the dimension of the $\mathbb{Q}$-vector space

$$\mathrm{DZ}_n := \mathrm{Span}_{\mathbb{Q}} \left\{ (2\pi\sqrt{-1})^n, \zeta(2, n-2), \zeta(3, n-3), \cdots, \zeta(n-1, 1) \right\}$$

for $n \geq 3$. This is still a very difficult problem in the classical theory as $\dim_{\mathbb{Q}} \mathrm{DZ}_n$ is only known for $n = 3, 4$. Zagier (cf. [Z94]) gave a conjectural formula for d_n in terms of the dimension of weight n cusp forms.

Conjecture 1.1.1. (Zagier) *For $n \geq 3$, we put*

$$s_n := \begin{cases} \frac{n}{2} - 1 - \dim_{\mathbb{C}} S_n(\mathrm{SL}_2(\mathbb{Z})) & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd,} \end{cases}$$

where $S_n(\mathrm{SL}_2(\mathbb{Z}))$ is space of weight n cusp forms for $\mathrm{SL}_2(\mathbb{Z})$. Then we have

$$\dim_{\mathbb{Q}} \mathrm{DZ}_n = s_n.$$

The best known result toward this conjecture is due to Gangl, Kaneko and Zagier [GKZ06], who showed that $\dim_{\mathbb{Q}} \mathrm{DZ}_n \leq s_n$ for each $n \geq 3$. One of their approaches is to introduce and study the double Eisenstein series to explain the relation between double zeta values and cusp forms for $\mathrm{SL}_2(\mathbb{Z})$. The main result of this paper is to establish an effective criterion for computing the analogue of $\dim_{\mathbb{Q}} \mathrm{DZ}_n$ in the positive characteristic function field setting.

1.2. The main result

Let $A := \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over the finite field $\mathbb{F}_q$ of q elements with characteristic p, and k be its quotient field. Denote by ∞ the infinite place of k. Let $k_\infty := \mathbb{F}_q((\frac{1}{\theta}))$ be the ∞-adic completion of k, and $\overline{k_\infty}$ be a fixed algebraic closure of k_∞. Denote by $\mathbb{C}_\infty$ the ∞-adic completion of $\overline{k_\infty}$. Finally, we let A_+ be the set of all monic polynomials in A. We then have the following comparisons:

$$A_+ \leftrightarrow \mathbb{N}, \ A \leftrightarrow \mathbb{Z}, \ k \leftrightarrow \mathbb{Q}, \ k_\infty \leftrightarrow \mathbb{R}, \ \mathbb{C}_\infty \leftrightarrow \mathbb{C}.$$

For any r-tuple of positive integers $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, Thakur [T04] introduced the multiple series

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \cdots a_r^{s_r}} \in k_\infty,$$

where the sum is taken over r-tuples $(a_1, \ldots, a_r) \in A_+^r$ satisfying the strict inequalities $\deg_\theta a_1 > \cdots > \deg_\theta a_r$. In analogy with the classical MZV's, r is called the depth and $\mathrm{wt}(\mathfrak{s}) := \mathrm{s}_1 + \cdots + \mathrm{s_r}$ is called the weight of the presentation $\zeta_A(\mathfrak{s})$. These special values $\zeta_A(\mathfrak{s})$ are called multizeta values (abbreviated as MZV's too) and each of them is non-vanishing by [T09a]. Moreover, these MZV's have a t-motivic interpretation in the sense that they occur as periods of certain mixed Carlitz-Tate t-motives by the work of Anderson and Thakur [AT09].

In [T10], Thakur showed that the product of two MZV's can be expressed as an $\mathbb{F}_p$-linear combination of some MZV's with the same weight, which is regarded as a kind of shuffle product relation (cf. (1.2.2)), and so the $\mathbb{F}_p$-vector space spanned by MZV's has a ring structure. Further, an analogue of Goncharov's direct sum conjecture was shown by the author [C14], that the k-algebra generated by all MZV's is a graded algebra (graded by weights). In other words, all k-linear relations among MZV's are generated by those k-linear relations among the same weight MZV's.

In the classical theory of MZV's, (regularized) double shuffle relations give rise to rich $\mathbb{Q}$-linear relations among the same weight MZV's (see [IKZ06]). Unlike the classical situation, there is no natural total order on A_+ and so far a nice analogue of the iterated integral expression for MZV's is not developed yet. To date there is no different expression for the product of two MZV's other than Thakur's relation mentioned above, and hence we do not have the analogue of double shuffle relations to produce k-linear relations among MZV's naturally. In his Ph.D. thesis, Todd [To15]

tried to produce k-linear relations among the same weight MZV's using power sums and used lattice reduction methods to give a conjecture on the dimensions in question.

Let $\tilde{\pi}$ be a fixed fundamental period of the Carlitz $\mathbb{F}_q[t]$-module $\mathbf{C}$, which plays the analogous role of $2\pi\sqrt{-1}$, and let $\mathbf{C}^{\otimes n}$ be the nth tensor power of the Carlitz module $\mathbf{C}$ for a positive integer n (see §§ 2.2 for the definitions). The main result of this paper gives a new point of view to completely determine the k-linear relations among weight n double zeta values together with $\tilde{\pi}^n$. It is stated as follows and its proof is given in Corollary 5.2.1 and Theorem 6.1.1.

Theorem 1.2.1. *Let $n \geq 2$ be a positive integer. Put*

$$\mathscr{V} := \left\{(s_1, s_2) \in \mathbb{N}^2; s_1 + s_2 = n \text{ and } (q-1)|s_2\right\}.$$

(1) *For each $\mathfrak{s} \in \mathscr{V}$, we explicitly construct a special point $\Xi_{\mathfrak{s}} \in \mathbf{C}^{\otimes n}(A)$ so that*

$$\dim_k \operatorname{Span}_k \left\{\tilde{\pi}^n, \zeta_A(1, n-1), \zeta_A(2, n-2), \cdots, \zeta_A(n-1, 1)\right\}$$
$$= n - \lfloor \tfrac{n-1}{q-1} \rfloor + \operatorname{rank}_{\mathbb{F}_q[t]} \operatorname{Span}_{\mathbb{F}_q[t]} \left\{\Xi_{\mathfrak{s}}\right\}_{\mathfrak{s} \in \mathscr{V}}.$$

(2) *We establish an effective algorithm for computing the rank*

$$\operatorname{rank}_{\mathbb{F}_q[t]} \operatorname{Span}_{\mathbb{F}_q[t]} \left\{\Xi_{\mathfrak{s}}\right\}_{\mathfrak{s} \in \mathscr{V}}.$$

In other words, we relate the k-linear relations among double zeta values to the $\mathbb{F}_q[t]$-linear relations among the special points $\{\Xi_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathscr{V}}$, and which can be effectively computed and determined.

We mention that although Todd [To15] provided some ways of producing k-linear relations among the same weight MZV's, it is not clear how to derive k-linear relations among weight n double zeta values together with $\tilde{\pi}^n$ from Todd's relation. When the given weight $n \geq 2$ is A-even, i.e., $(q-1)|n$ (as $q - 1$ is the cardinality of the unit group $A^{\times}$), one can use the following formula to produce linear relations. For two positive A-even integers r and s with $r + s = n$, one has

(1.2.2)
$$\zeta_A(r)\zeta_A(s) = \zeta_A(r, s) + \zeta_A(s, r) + \zeta_A(r + s)$$
$$+ \sum_{\substack{i+j=n \\ (q-1)|j}} \left[(-1)^{s-1}\binom{j-1}{s-1} + (-1)^{r-1}\binom{j-1}{r-1}\right] \zeta_A(i, j)$$

(see [T10] for the existence of such relations and [Chen15] for the explicit formula). By work of Carlitz [Ca35], we know that

$$\zeta_A(r)/\tilde{\pi}^r \in k, \ \zeta_A(s)/\tilde{\pi}^s \in k, \ \zeta_A(n)/\tilde{\pi}^n \in k,$$

and so (1.2.2) gives rise to a nontrivial linear relation among $\tilde{\pi}^n$ and weight n double zeta values. As all the coefficients of the double zeta values are in $\mathbb{F}_p$, Thakur call it an "$\mathbb{F}_p$-linear relation". Our effective algorithm based on Theorem 1.2.1 is able to find all the independent k-linear relations in question. As observed by Thakur [T09b], those $\mathbb{F}_p$-linear relations produced by (1.2.2) can not generate all the k-linear relations, and our computational data can capture the difference precisely. We refer the reader to the end of this paper.

We mention that there is a difference between Conjecture 1.1.1 and our results, and refer the reader to Remark 3.1.12 about the detailed comparison. We further mention that in [Chen16], Drinfeld double Eisenstein series are introduced. Since double zeta values occur as the constant terms of Drinfeld double Eisenstein series, one naturally expects that Drinfeld cusps forms [Go80, Ge88] can have connections with double zeta values if they can be shown to be a subspace of the space of Drinfeld double Eisenstein series (cf. [GKZ06]).

1.3. Methods of proof

Let notation and hypothesis be given as in Theorem 1.2.1. We outline the major steps in the proof of Theorem 1.2.1.

(I) *A necessary condition.* We show that all the k-linear relations among the set

$$\{\tilde{\pi}^n\} \cup \{\zeta_A(1, n-1), \ldots, \zeta_A(n-1, 1)\}$$

are those coming from the k-linear relations among $\{\tilde{\pi}^n\} \cup \{\zeta_A(\mathfrak{s})\}_{\mathfrak{s} \in \mathscr{V}}$, and so we are reduced to studying the k-linear relations among $\{\tilde{\pi}^n\} \cup \{\zeta_A(\mathfrak{s})\}_{\mathfrak{s} \in \mathscr{V}}$. This result is Corollary 3.1.10.

(II) *Logarithmic interpretion.* Let $r \geq 2$. For any $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $\zeta_A(s_2, \ldots, s_r)$ Eulerian, i.e.,

$$\zeta_A(s_2, \ldots, s_r)/\tilde{\pi}^{s_2 + \cdots + s_r} \in k,$$

we relate $\zeta_A(\mathfrak{s})$ to the last coordinate of the logarithm of $\mathbf{C}^{\otimes(s_1 + \cdots + s_r)}$ at an explicit integral point. This result is Theorem 4.1.1.

(III) *The identity.* Using the results in (I) and (II), we establish the equality in Theorem 1.2.1 (1) by appealing to Yu's transcendence theory [Yu91]

for the last coordinate of the logarithm of $\mathbf{C}^{\otimes n}$ at algebraic points. This result is Corollary 5.2.1.

(IV) *A Siegel's Lemma.* We establish a principle of Siegel's lemma for integral points in $\mathbf{C}^{\otimes n}$ to achieve Theorem 1.2.1 (2). This result is Theorem 6.1.1.

The four steps laid out above give an approach toward producing k-linear relations among the same weight MZV's of arbitrary depths. Actually, for weight $n \geq 2$ combining the ideas of (II), (III) and (IV) above one determine all the k-linear relations among the set

$$\{\zeta_A(n)\} \cup \{\zeta_A(\mathfrak{s}); \mathfrak{s} \in \mathscr{E}_n\},$$

where $\mathscr{E}_n$ is the set consisting of all $\mathfrak{s} \in \mathbb{N}^r$ with $r \geq 2$ and $\mathrm{wt}(\mathfrak{s}) = n$ satisfying that $\zeta_A(\mathfrak{s}')$ is Eulerian, where $\mathfrak{s}' = (s_2, \ldots, s_r)$ for $\mathfrak{s} = (s_1, \ldots, s_r)$. This result is Corollary 5.1.3 together with Theorem 6.1.1. In [CPY14] an effective criterion for Eulerian MZV's is established, and conjecturally one can describe the set $\mathscr{E}_n$ precisely (see [CPY14, § 6.2]). We note that assuming Todd's dimension conjecture [To15], for weight $n \geq 2$ the k-linear relations among the set $\{\zeta_A(n)\} \cup \{\zeta_A(\mathfrak{s}); \mathfrak{s} \in \mathscr{E}_n\}$ are not enough to generate all the k-linear relations among weight n MZV's.

The idea of proving (I) above is to construct a suitable system of Frobenius difference equations for each k-linear relation among the double zeta values and $\tilde{\pi}^n$, and then use ABP-criterion [ABP04, Thm. 3.1.1] in the study of certain Ext^1-modules. For the proof of (II) above, we need an explicit formula for the bottom row of each coefficient matrix of the logarithm of $\mathbf{C}^{\otimes n}$ due to Papanikolas [P14]. Combining with the period interpretation of MZV's given by Anderson-Thakur [AT09], one is able to relate $\zeta_A(\mathfrak{s})$ for those $\mathfrak{s} \in \mathscr{V}$ in Theorem 1.2.1 to the last coordinate of the logarithm of $\mathbf{C}^{\otimes n}$. The proof of (III) is to use the functional equation of the exponential function of $\mathbf{C}^{\otimes n}$ and apply Yu's theory [Yu91]. To achieve (IV), we translate the effectiveness question to a question of the type of Siegel's lemma for certain difference equations, and we prove it directly.

1.4. Outline of this paper

In order to let the present paper be self-contained, in §§ 2 we give some preliminaries about some major results in [AT90, Yu91, CPY14]. We then give proofs of (I)-(IV) above in §§ 3-6 respectively. The proof of Theorem 1.2.1 is given in Corollary 5.2.1 and Theorem 6.1.1. In §§ 6.2, we give an effective algorithm for implementing Theorem 1.2.1, and at the end of this paper we provide some data of this computation using Magma.

Acknowledgements

2. Preliminaries

2.1. Notation

We adopt the following notation.

$\mathbb{F}_q$	=	the finite field with q elements, for q a power of a prime number p.
θ, t	=	independent variables.
A	=	$\mathbb{F}_q[\theta]$, the polynomial ring in the variable θ over $\mathbb{F}_q$.
A_+	=	set of monic polynomials in A.
k	=	$\mathbb{F}_q(\theta)$, the fraction field of A.
k_∞	=	$\mathbb{F}_q((1/\theta))$, the completion of k with respect to the infinite place ∞.
$\overline{k_\infty}$	=	a fixed algebraic closure of k_∞.
$\overline{k}$	=	the algebraic closure of k in $\overline{k_\infty}$.
$\mathbb{C}_\infty$	=	the completion of $\overline{k_\infty}$ with respect to the canonical extension of ∞.
$\lvert \cdot \rvert_\infty$	=	a fixed absolute value for the completed field $\mathbb{C}_\infty$ so that $\lvert \theta \rvert_\infty = q$.
$\mathbb{C}_\infty[\![t]\!]$	=	ring of formal power series in t over $\mathbb{C}_\infty$.
$\mathbb{C}_\infty(\!(t)\!)$	=	field of Laurent series in t over $\mathbb{C}_\infty$.
$\mathbb{T}$	=	the ring of power series in $\mathbb{C}_\infty[\![t]\!]$ convergent on the closed unit disc.
$\tilde{\pi}$	=	a fixed fundamental period of the Carlitz module $\mathbf{C}$.
$\mathbb{G}_a$	=	the additive group scheme over A.

2.2. Anderson t-modules revisited

We let $\tau := (x \mapsto x^q)$ be the qth power endomorphism of $\mathbb{C}_\infty$, and define $\mathbb{C}_\infty\{\tau\}$ to be the twisted polynomial ring in the variable τ over $\mathbb{C}_\infty$ subject to the relation

$$\tau\alpha = \alpha^q\tau \text{ for } \alpha \in \mathbb{C}_\infty.$$

It follows that we have the matrix ring $\mathrm{Mat}_n\left(\mathbb{C}_\infty\{\tau\}\right)$ with entries in $\mathbb{C}_\infty\{\tau\}$ and any element in this ring can be expressed as

$$\varphi = \sum_{i \geq 0} a_i\tau^i$$

with $a_i \in \mathrm{Mat}_n(\mathbb{C}_\infty)$ and $a_i = 0$ for $i \gg 0$. We denote by $\partial\varphi := a_0$, the constant matrix of φ. For convenience, we still denote by τ the operator on $\mathbb{C}_\infty^n$ which raises each component to the qth power. We denote by $\mathbb{G}_a^n$ the n-dimensional additive group scheme over A and note that $\mathrm{Mat}_n(\mathbb{C}_\infty\{\tau\})$ can be identified with $\mathrm{End}_{\mathbb{F}_q}\left(\mathbb{G}_a^n(\mathbb{C}_\infty)\right)$, the ring of $\mathbb{F}_q$-linear endomorphisms of the algebraic group $\mathbb{G}_a^n(\mathbb{C}_\infty) = \mathbb{C}_\infty^n$. Via this identification $\partial\varphi$ is the tangent map of the morphism $\varphi : \mathbb{G}_a^n(\mathbb{C}_\infty) \to \mathbb{G}_a^n(\mathbb{C}_\infty)$ at the identity.

By an *n-dimensional t-module* we mean a pair $E = (\mathbb{G}_a^n, \phi)$, where the underlying space of E is $\mathbb{G}_a^n(\mathbb{C}_\infty)$, which is equipped with an $\mathbb{F}_q[t]$-module structure via the $\mathbb{F}_q$-linear ring homomorphism

$$\phi : \mathbb{F}_q[t] \to \mathrm{Mat}_n\left(\mathbb{C}_\infty\{\tau\}\right)$$

so that $(\partial\phi_t - \theta I_n)$ is a nilpotent matrix. For such a t-module, Anderson [A86] showed that there is a unique n-variable power series $\exp_E$ defined on the whole $\mathbb{C}_\infty^n$, called the *exponential* of the t-module E, for which:

- $\exp_E : \mathrm{Lie}\left(\mathbb{G}_a^n(\mathbb{C}_\infty)\right) = \mathbb{C}_\infty^n \to E(\mathbb{C}_\infty)$ is $\mathbb{F}_q$-linear.
- $\exp_E$ is of the form $I_n + \sum_{i=1}^\infty \alpha_i\tau^i$ with $\alpha_i \in \mathrm{Mat}_n(\mathbb{C}_\infty)$.
- $\exp_E$ satisfies the functional identity: for all $a \in \mathbb{F}_q[t]$,

$$\exp_E \circ \partial\phi_a = \phi_a \circ \exp_E.$$

One typical example of a nontrivial t-module is the *nth tensor power of the Carlitz module* denoted by $\mathbf{C}^{\otimes n} = (\mathbb{G}_a^n, [\cdot]_n)$ for $n \in \mathbb{N}$ (see [AT90]). Here $[\cdot]_n$ is the $\mathbb{F}_q$-linear ring homomorphism $[\cdot]_n : \mathbb{F}_q[t] \to \mathrm{Mat}_n(\mathbb{C}_\infty[\tau])$ determined by

$$[t]_n = \begin{pmatrix} \theta & 1 & \cdots & 0 \\ & \theta & \ddots & \vdots \\ & & \ddots & 1 \\ & & & \theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \tau.$$

When $n = 1$, $\mathbf{C} := \mathbf{C}^{\otimes 1}$ is called the Carlitz $\mathbb{F}_q[t]$-module.

We denote by $\exp_n := \exp_{\mathbf{C}^{\otimes n}}$ the exponential of $\mathbf{C}^{\otimes n}$. We define $\log_n :=$ $\log_{\mathbf{C}^{\otimes n}}$ to be the unique power series of n variables, called the *logarithm* of $\mathbf{C}^{\otimes n}$, for which:

- $\log_n$ is of the form $\log_n = I_n + \sum_{i=1}^{\infty} P_i \tau^i$ with $P_i \in \mathrm{Mat}_n(k)$.
- $\log_n$ satisfies the functional identity: for any $a \in \mathbb{F}_q[t]$,

$$\log_n \circ [a]_n = \partial[a]_n \circ \log_n.$$

As formal power series we note that $\exp_n$ and $\log_n$ are inverses of each other:

$$\exp_n \circ \log_n = \mathrm{identity} = \log_n \circ \exp_n.$$

In the case of $n = 1$, $\exp_{\mathbf{C}}$ and $\log_{\mathbf{C}}$ are called the Carlitz exponential and Carlitz logarithm respectively, and these two functions can be written down explicitly as follows. Putting $D_0 = 1$ and $D_i := \prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j})$ for $i \in \mathbb{N}$, then

$$\exp_{\mathbf{C}} = \sum_{i=0}^{\infty} \frac{1}{D_i} \tau^i.$$

We further put $L_0 := 1$ and $L_i := (\theta - \theta^q) \cdots (\theta - \theta^{q^i})$ for $i \in \mathbb{N}$, and then

$$\log_{\mathbf{C}} = \sum_{i=0}^{\infty} \frac{1}{L_i} \tau^i.$$

For the details, see [Go96, T04].

2.3. Review of the theories of Anderson-Thakur and Yu

2.3.1. Theory of Anderson-Thakur. In their seminal paper [AT90], Anderson and Thakur first showed that for each $n \in \mathbb{N}$, $\exp_n : \mathbb{C}_\infty^n \to \mathbf{C}^{\otimes n}(\mathbb{C}_\infty)$ is surjective and its kernel is of rank one over A in the sense that

$$\mathrm{Ker} \exp_n = \partial[\mathbb{F}_q[t]]_n \lambda_n,$$

where $\lambda_n \in \mathbb{C}_\infty^n$ is of the form

$$\lambda_n = \begin{pmatrix} * \\ \vdots \\ \tilde{\pi}^n \end{pmatrix}.$$

The $\tilde{\pi}$ above is a fundamental period of the Carlitz module $\mathbf{C}$ in the sense that $\mathrm{Ker}\,\exp_{\mathbf{C}} = A\tilde{\pi}$, and it is fixed throughout this paper.

For a non-negative integer n, we express n as

$$n = \sum_{i=0}^{\infty} n_i q^i \quad (0 \le n_i \le q-1,\, n_i = 0 \text{ for } i \gg 0).$$

Then the Carlitz factorial is defined by

$$\Gamma_{n+1} := \prod_{i=0}^{\infty} D_i^{n_i} \in A.$$

One of the major results in [AT90] is to relate $\zeta_A(n)$ to the last coordinate of the logarithm of $\mathbf{C}^{\otimes n}$. It is stated as follows.

Theorem 2.3.1. (Anderson-Thakur [AT90, Thm. 3.8.3]) *For each positive integer n, one explicitly constructs an integral point $\mathbf{v}_n \in \mathbf{C}^{\otimes n}(A)$ so that there exists a vector $Y_n \in \mathbb{C}_\infty^n$ of the form*

$$Y_n = \begin{pmatrix} * \\ \vdots \\ \Gamma_n \zeta_A(n) \end{pmatrix}$$

satisfying

$$\exp_n(Y_n) = \mathbf{v}_n.$$

For an integer m, we define m-fold Frobenius twisting by

$$\begin{array}{rcl} \mathbb{C}_\infty((t)) & \to & \mathbb{C}_\infty((t)), \\ f := \sum_i a_i t^i & \mapsto & f^{(m)} := \sum_i a_i^{q^m} t^i. \end{array}$$

We extend this to matrices with entries in $\mathbb{C}_\infty((t))$ by twisting entry-wise.

We put $G_0(y) := 1$ and define polynomials $G_n(y) \in \mathbb{F}_q[t, y]$ for $n \in \mathbb{N}$ by the product

$$G_n(y) = \prod_{i=1}^{n} \left(t^{q^n} - y^{q^i}\right).$$

For $n = 0, 1, 2, \ldots$, we define the sequence of Anderson-Thakur polynomials $H_n \in A[t]$ by the generating function identity

$$\left(1 - \sum_{i=0}^{\infty} \frac{G_i(\theta)}{D_i|_{\theta=t}} x^{q^i}\right)^{-1} = \sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}|_{\theta=t}} x^n.$$

We put

$$\Omega(t) := (-\theta)^{\frac{-q}{q-1}} \prod_{i=1}^{\infty}\left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{C}_\infty[\![t]\!],$$

where $(-\theta)^{\frac{1}{q-1}}$ is a suitable choice of $(q-1)$-st root of $-\theta$ so that $\frac{1}{\Omega(\theta)} = \tilde{\pi}$ (cf. [ABP04] and [AT09]). The function Ω satisfies the difference equation $\Omega^{(-1)} = (t - \theta)\Omega$. One important identity established in [AT90, AT09] is the following: for any positive integer n and non-negative integer i, we have

$$(2.3.2) \qquad (\Omega^n H_{n-1})^{(i)}|_{t=\theta} = \frac{\Gamma_n S_i(n)}{\tilde{\pi}^n},$$

where $S_i(n)$ is the partial sum

$$S_i(n) := \sum_{a \in A_{+,i}} \frac{1}{a^n} \in k.$$

Here $A_{+,i}$ denotes by the set of all monic polynomials in A with degree i. For any $(s_1, \ldots, s_r) \in \mathbb{N}^r$ we define the following series

$$(2.3.3) \qquad \mathscr{L}_{(s_1,\ldots,s_r)}(t) := \sum_{i_1 > \cdots > i_r \geq 0} (\Omega^{s_1} H_{s_1-1})^{(i_1)} \cdots (\Omega^{s_r} H_{s_r-1})^{(i_r)}.$$

Then by (2.3.2), specialization at $t = \theta$ of $\mathscr{L}_{(s_1,\ldots,s_r)}$ gives

$$\mathscr{L}_{(s_1,\ldots,s_r)}(\theta) = \sum_{i_1 > \cdots > i_r \geq 0} \Gamma_{s_1} S_{i_1}(s_1) \cdots \Gamma_{s_r} S_{i_r}(s_r) \tilde{\pi}^{s_1 + \cdots + s_r}$$

$$= \Gamma_{s_1} \cdots \Gamma_{s_r} \zeta_A(s_1, \ldots, s_r) / \tilde{\pi}^{s_1 + \cdots + s_r}.$$

We single out the following useful lemma, which is rooted in [C14, Lem. 5.3.5] (see also [CPY14, Prop. 2.3.3]).

Lemma 2.3.4. *For any $(s_1, \ldots, s_r) \in \mathbb{N}^r$ and any nonnegative integer N, we have*

$$\mathscr{L}_{(s_1,\ldots,s_r)}(\theta^{q^N}) = \left(\Gamma_{s_1} \cdots \Gamma_{s_r} \zeta_A(s_1, \ldots, s_r) / \tilde{\pi}^{s_1 + \cdots + s_r}\right)^{q^N}.$$

2.3.2. Yu's theory. In [Yu91], Yu proved the transcendence of $\zeta_A(n)$ for each $n \in \mathbb{N}$. The key ingredient in Yu's proof is to establish the following theorem.

Theorem 2.3.5. (Yu [Yu91, Thm. 2.3]) *Let n be a positive integer, and $Y = (y_1, \ldots, y_n)^{\mathrm{tr}} \in \mathbb{C}_\infty^n$ be a nonzero vector satisfying that $\exp_n(Y) \in \mathbf{C}^{\otimes n}(\overline{k})$. Then y_n is transcendental over k.*

Remark 2.3.6. Combining Theorems 2.3.1 and 2.3.5, one can show (see [Yu91, Thm. 3.2]):

- $\zeta_A(n)/\tilde{\pi}^n \in k$ if and only if $\mathbf{v}_n$ is an $\mathbb{F}_q[t]$-torsion point in $\mathbf{C}^{\otimes n}(A)$.
- $\mathbf{v}_n$ is an $\mathbb{F}_q[t]$-torsion point if and only if n is divisible by $q - 1$.

2.4. Review of the CPY criterion for Eulerian MZV's

In what follows, by a *Frobenius* module we mean a left $\overline{k}[t, \sigma]$-module that is free of finite rank over $\overline{k}[t]$, where $\overline{k}[t, \sigma] := \overline{k}[t][\sigma]$ is the twisted polynomial ring generated by σ over $\overline{k}[t]$ subject to the relation $\sigma f = f^{(-1)}\sigma$ for $f \in \overline{k}[t]$. Morphisms of Frobenius modules are defined to be left $\overline{k}[t, \sigma]$-module homomorphisms and we denote by $\mathscr{F}$ the category of Frobenius modules.

In what follows, an object M in $\mathscr{F}$ is said to be defined by a matrix $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ if M is free of rank r over $\overline{k}[t]$ and the σ-action on a given $\overline{k}[t]$-basis of M is represented by the matrix Φ. We denote by $\mathbf{1}$ the *trivial* object in $\mathscr{F}$, where the underlying space of $\mathbf{1}$ is $\overline{k}[t]$ and on which σ acts as $\sigma f = f^{(-1)}$ for $f \in \mathbf{1}$. We further denote by $C^{\otimes n}$ the nth tensor power of the Carlitz motive for $n \in \mathbb{N}$. The underlying space of $C^{\otimes n}$ is $\overline{k}[t]$, and on which σ acts by $\sigma f := (t - \theta)^n f^{(-1)}$ for $f \in C^{\otimes n}$.

By an *Anderson t-motive* we mean an object $M' \in \mathscr{F}$ that possesses the following properties.

- M' is also a free left $\overline{k}[\sigma]$-module of finite rank.
- $\sigma M' \subseteq (t - \theta)^n M'$ for all sufficiently large integers n.

Here we follow the terminology of Anderson t-motives in [P08] (cf. [A86, ABP04]).

For a fixed Anderson t-motive M' of rank d over $\overline{k}[\sigma]$, we are interested in $\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, M')$, the set of equivalence classes of Frobenius modules M fitting into a short exact sequence of Frobenius modules

$$0 \to M' \hookrightarrow M \twoheadrightarrow \mathbf{1} \to 0,$$

and we denote by $[M]$ the equivalence class of M in $\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, M')$. Since $\mathbb{F}_q[t]$ is contained inside the center of $\overline{k}[t, \sigma]$, left multiplication by any element

of $\mathbb{F}_q[t]$ on M' is a morphism and hence $\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, M')$ has a natural $\mathbb{F}_q[t]$-module structure coming from Baer sum and pushout of morphisms of M'. The following is a review of the $\mathbb{F}_q[t]$-module isomorphisms established by Anderson

$$\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, M') \cong M'/(\sigma - 1)M' \cong E'(\overline{k}),$$

where

$$(2.4.1) \qquad\qquad E' = (\mathbb{G}_a^d, \rho)$$

is the t-module over $\overline{k}$ associated to M' in the sense that the $\overline{k}$-valued points of E' is isomorphic to $M'/(\sigma-1)M'$ as $\mathbb{F}_q$-vector spaces and the $\mathbb{F}_q[t]$-module structure on E' via ρ is induced by the $\mathbb{F}_q[t]$-action on $M'/(\sigma - 1)M'$. For a detailed description of the isomorphisms above, see [CPY14, § 5.2]. We also refer the reader to [S97, PR03, HP04, Ta10, BP, CP12, HJ16] for related discussions.

For example, let n be a positive integer. Then the nth tensor power of Carlitz motive $C^{\otimes n}$ has a $\overline{k}[\sigma]$-basis $\{(t-\theta)^{n-1}, \ldots, (t-\theta), 1\}$ and every $f \in C^{\otimes n}/(\sigma-1)C^{\otimes n}$ has a unique representative polynomial (with degree $\leq n-1$) of the form $u_1(t-\theta)^{n-1} + \cdots + u_n \in \overline{k}[t]$. Then the maps above can be characterized as

$$(2.4.2) \qquad \begin{array}{ccccc} \mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, C^{\otimes n}) & \cong & C^{\otimes n}/(\sigma - 1)C^{\otimes n} & \cong & \mathbf{C}^{\otimes n}(\overline{k}) \\ [M_f] & \mapsto & f & \mapsto & (u_1, \ldots, u_n)^{\mathrm{tr}}, \end{array}$$

where $M_f \in \mathscr{F}$ is defined by the matrix

$$(2.4.3) \qquad \Phi_f := \begin{pmatrix} (t-\theta)^n & 0 \\ f^{(-1)}(t-\theta)^n & 1 \end{pmatrix} \in \mathrm{Mat}_2(\overline{k}[t]).$$

Remark 2.4.4. For $n \in \mathbb{N}$, we note that $H_{n-1}^{(-1)}(t-\theta)^n = H_{n-1} + (\sigma-1)H_{n-1}$ in $C^{\otimes n}$ and so

$$H_{n-1}^{(-1)}(t-\theta)^n \equiv H_{n-1} \in C^{\otimes n}/(\sigma - 1)C^{\otimes n}.$$

The special point $\mathbf{v}_n$ given in Theorem 2.3.1 is defined to be the image of $H_{n-1}^{(-1)}(t-\theta)^n$ under the isomorphism $C^{\otimes n}/(\sigma-1)C^{\otimes n} \cong \mathbf{C}^{\otimes n}(\overline{k})$. For the details, see [CPY14, p. 26].

Let Z be an MZV of weight w. Following [T04], we say that Z is *Eulerian* if the ratio $Z/\tilde{\pi}^w$ is in k. In [CPY14], an effective criterion for Eu-

lerian MZV's is established and we describe it as follows. Let r be a positive integer and fix an r-tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$. We define the matrix $\Phi_{\mathfrak{s}} \in \mathrm{Mat}_{r+1}(\overline{k}[t])$,

(2.4.5)

$$
\Phi_{\mathfrak{s}} := \begin{pmatrix}
(t-\theta)^{s_1+\cdots+s_r} & 0 & 0 & \cdots & 0 \\
H_{s_1-1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & 0 & \cdots & 0 \\
0 & H_{s_2-1}^{(-1)}(t-\theta)^{s_2+\cdots+s_r} & \ddots & & \vdots \\
\vdots & & \ddots & (t-\theta)^{s_r} & 0 \\
0 & & \cdots & 0 & H_{s_r-1}^{(-1)}(t-\theta)^{s_r} & 1
\end{pmatrix}.
$$

Define $\Phi_{\mathfrak{s}}' \in \mathrm{Mat}_r(\overline{k}[t])$ to be the square matrix of size r cut off from the upper left square of $\Phi_{\mathfrak{s}}$:

(2.4.6)

$$
\Phi_{\mathfrak{s}}' := \begin{pmatrix}
(t-\theta)^{s_1+\cdots+s_r} & & & \\
H_{s_1-1}^{(-1)}(t-\theta)^{s_1+\cdots+s_r} & (t-\theta)^{s_2+\cdots+s_r} & & \\
& \ddots & \ddots & \\
& & H_{s_{r-1}-1}^{(-1)}(t-\theta)^{s_{r-1}+s_r} & (t-\theta)^{s_r}
\end{pmatrix}.
$$

Define

(2.4.7)

$$
\Psi_{\mathfrak{s}} := \begin{pmatrix}
\Omega^{s_1+\cdots+s_r} & & & & \\
\Omega^{s_2+\cdots+s_r}\mathscr{L}_{s_1} & \Omega^{s_2+\cdots+s_r} & & & \\
\Omega^{s_3+\cdots+s_r}\mathscr{L}_{(s_1,s_2)} & \Omega^{s_3+\cdots+s_r}\mathscr{L}_{s_2} & \ddots & & \\
\vdots & \vdots & \ddots & \Omega^{s_{r-1}+s_r} & \\
\Omega^{s_r}\mathscr{L}_{(s_1,\ldots,s_{r-1})} & \Omega^{s_r}\mathscr{L}_{(s_2,\ldots,s_{r-1})} & & \Omega^{s_r}\mathscr{L}_{s_{r-1}} & \Omega^{s_r} \\
\mathscr{L}_{(s_1,\ldots,s_r)} & \mathscr{L}_{(s_2,\ldots,s_r)} & \cdots & \mathscr{L}_{(s_{r-1},s_r)} & \mathscr{L}_{s_r} & 1
\end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{T})
$$

and let

(2.4.8)

$$
\Psi_{\mathfrak{s}}' := \begin{pmatrix}
\Omega^{s_1+\cdots+s_r} & & & & \\
\Omega^{s_2+\cdots+s_r}\mathscr{L}_{s_1} & \Omega^{s_2+\cdots+s_r} & & & \\
\Omega^{s_3+\cdots+s_r}\mathscr{L}_{(s_1,s_2)} & \Omega^{s_3+\cdots+s_r}\mathscr{L}_{s_2} & \ddots & & \\
\vdots & \vdots & \ddots & \Omega^{s_{r-1}+s_r} & \\
\Omega^{s_r}\mathscr{L}_{(s_1,\ldots,s_{r-1})} & \Omega^{s_r}\mathscr{L}_{(s_2,\ldots,s_{r-1})} & & \Omega^{s_r}\mathscr{L}_{s_{r-1}} & \Omega^{s_r}
\end{pmatrix} \in \mathrm{GL}_r(\mathbb{T})
$$

be the square matrix cut off from the upper left square of $\Psi_{\mathfrak{s}}$. Then we have that

$$(2.4.9) \qquad \Psi_{\mathfrak{s}}^{\prime(-1)} = \Phi_{\mathfrak{s}}^{\prime}\Psi_{\mathfrak{s}}^{\prime} \text{ and } \Psi_{\mathfrak{s}}^{(-1)} = \Phi_{\mathfrak{s}}\Psi_{\mathfrak{s}}$$

(see [AT09, C14, CPY14]).

We let $M_{\mathfrak{s}}$ (resp. $M_{\mathfrak{s}}^{\prime}$) be the Frobenius module defined by the matrix $\Phi_{\mathfrak{s}}$ (resp. $\Phi_{\mathfrak{s}}^{\prime}$). Then $M_{\mathfrak{s}}$ represents a class in $\mathrm{Ext}_{\mathscr{F}}^{1}(\mathbf{1}, M_{\mathfrak{s}}^{\prime}) \cong M_{\mathfrak{s}}^{\prime}/(\sigma - 1)M_{\mathfrak{s}}^{\prime} \cong E_{\mathfrak{s}}^{\prime}(\overline{k})$ and we define $\mathbf{v}_{\mathfrak{s}} \in E_{\mathfrak{s}}^{\prime}(\overline{k})$ to be the image of $[M_{\mathfrak{s}}]$ under the composition of the isomorphisms above. Note that it is shown in [CPY14, Thm. 5.3.4] that actually $E_{\mathfrak{s}}^{\prime}$ is defined over A and $\mathbf{v}_{\mathfrak{s}}$ is an integral point in $E_{\mathfrak{s}}^{\prime}(A)$.

The criterion of Chang-Papanikolas-Yu for Eulerian MZV's is as follows.

Theorem 2.4.10. [CPY14, Thms. 5.3.5 and 6.1.1] *For each r-tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$, let $E_{\mathfrak{s}}^{\prime}$ and $\mathbf{v}_{\mathfrak{s}}$ be defined as above. Then we explicitly construct a polynomial $a_{\mathfrak{s}} \in \mathbb{F}_q[t]$ so that $\zeta_A(\mathfrak{s})$ is Eulerian if and only if $\mathbf{v}_{\mathfrak{s}}$ is an $a_{\mathfrak{s}}$-torsion point in $E_{\mathfrak{s}}^{\prime}(A)$.*

3. Step I: A necessary condition

3.1. The formulation

In this section, our main goal is to show the following necessary condition, which will be applied to compute the dimension of double zeta values.

Theorem 3.1.1. *Let $n \geq 2$ be an integer and for $i = 1, \ldots, m$, let $\mathfrak{s}_i = (s_{i1}, s_{i2}) \in \mathbb{N}^2$ be chosen with $s_{i1} + s_{i2} = n$. Suppose that we have*

$$(3.1.2) \qquad c_0\zeta_A(n) + \sum_{i=1}^{m} c_i\zeta_A(\mathfrak{s}_i) = 0 \text{ for some } c_0, c_1, \ldots, c_m \in k.$$

If the coefficient c_j is nonzero for some $1 \leq j \leq m$, then we have that $(q-1)|s_{j2}$. In other words, all k-linear relations among $\{\zeta_A(n), \zeta_A(\mathfrak{s}_1), \ldots, \zeta_A(\mathfrak{s}_m)\}$ are those coming from the k-linear relations among $\{\zeta_A(n)\} \cup \{\zeta_A(\mathfrak{s}_j); (q-1)|s_{j2}\}$.

Proof. Note that each double zeta value $\zeta_A(\mathfrak{s}_i)$ is associated to the system of Frobenius difference equations

$$(3.1.3) \qquad \Psi_i^{(-1)} = \Phi_i\Psi_i,$$

where

$$\Phi_i := \begin{pmatrix} (t-\theta)^n & 0 & 0 \\ H_{s_{i1}-1}^{(-1)}(t-\theta)^n & (t-\theta)^{s_{i2}} & 0 \\ 0 & H_{s_{i2}-1}^{(-1)}(t-\theta)^{s_{i2}} & 1 \end{pmatrix}$$

and

$$\Psi_i := \begin{pmatrix} \Omega^n & 0 & 0 \\ \Omega^{s_{i2}} \mathscr{L}_{21}^{[i]} & \Omega^{s_{i2}} & 0 \\ \mathscr{L}_{31}^{[i]} & \mathscr{L}_{32}^{[i]} & 1 \end{pmatrix}.$$

Here for each $1 \le i \le m$,

$$\mathscr{L}_{21}^{[i]} := \mathscr{L}_{s_{i1}} = \sum_{\ell=0}^{\infty} (\Omega^{s_{i1}} H_{s_{i1}-1})^{(\ell)} \in \mathbb{T}$$

and

$$\mathscr{L}_{31}^{[i]} := \mathscr{L}_{(s_{i1},s_{i2})} := \sum_{\ell_1 > \ell_2 \ge 0} (\Omega^{s_{i2}} H_{s_{i2}-1})^{(\ell_2)} (\Omega^{s_{i1}} H_{s_{i1}-1})^{(\ell_1)} \in \mathbb{T},$$

which satisfy for any integer $N \in \mathbb{Z}_{\ge 0}$,
(3.1.4)
$$\mathscr{L}_{21}^{[i]}(\theta^{q^N}) = (\Gamma_{s_{i1}}\zeta_A(s_{i1})/\tilde{\pi}^{s_{i1}})^{q^N} \quad \text{and} \quad \mathscr{L}_{31}^{[i]}(\theta^{q^N}) = (\Gamma_{s_{i1}}\Gamma_{s_{i2}}\zeta_A(s_{i1},s_{i2})/\tilde{\pi}^n)^{q^N}$$

(see Lemma 2.3.4).

Moreover, $\zeta_A(n)$ is associated to the system of Frobenius difference equations

$$(3.1.5) \qquad \begin{pmatrix} \Omega^n \\ \mathscr{L}_n \end{pmatrix}^{(-1)} = \begin{pmatrix} (t-\theta)^n & 0 \\ H_{n-1}^{(-1)}(t-\theta)^n & 1 \end{pmatrix} \begin{pmatrix} \Omega^n \\ \mathscr{L}_n \end{pmatrix},$$

where

$$\mathscr{L}_n := \sum_{i=0}^{\infty} (\Omega^n H_{n-1})^{(i)} \in \mathbb{T}$$

has the property that for $N \in \mathbb{Z}_{\ge 0}$,

$$\mathscr{L}_n(\theta^{q^N}) = (\Gamma_n \zeta_A(n)/\tilde{\pi}^n)^{q^N} \text{ (see Lemma 2.3.4).}$$

To prove this theorem, without loss of generality we assume that each coefficient $c_i \ne 0$ for $i = 1, \ldots, m$. Multiplying the equation (3.1.2) by a

suitable element in $\mathbb{F}_q[\theta]$, without loss of generality we may assume that $a_i := \frac{c_i}{\Gamma_{s_{i1}}\Gamma_{s_{i2}}}|_{\theta=t}$ and $a_0 := \frac{c_0}{\Gamma_n}|_{\theta=t}$ are in $\mathbb{F}_q[t]$ for $i = 1, \ldots, m$. Note that $a_1 \neq 0, \ldots, a_m \neq 0$. So we have that

$$a_0(\theta)\Gamma_n\zeta_A(n) + \sum_{i=1}^{m} a_i(\theta)\Gamma_{s_{i1}}\Gamma_{s_{i2}}\zeta(\mathfrak{s}_i) = 0.$$

Define

$$\Phi := \begin{pmatrix} (t-\theta)^n & & & & \\ H_{s_{11}-1}^{(-1)}(t-\theta)^n & (t-\theta)^{s_{12}} & & & \\ \vdots & & \ddots & & \\ H_{s_{m1}-1}^{(-1)}(t-\theta)^n & & & (t-\theta)^{s_{m2}} & \\ a_0 H_{n-1}^{(-1)}(t-\theta)^n & a_1 H_{s_{12}-1}^{(-1)}(t-\theta)^{s_{12}} & \cdots & a_m H_{s_{m2}-1}^{(-1)}(t-\theta)^{s_{m2}} & 1 \end{pmatrix}$$

$$\in \mathrm{Mat}_{m+2}(\overline{k}[t])$$

and

$$\psi := \begin{pmatrix} \Omega^n \\ \Omega^{s_{12}}\mathscr{L}_{21}^{[1]} \\ \vdots \\ \Omega^{s_{m2}}\mathscr{L}_{21}^{[m]} \\ a_0\mathscr{L}_n + \sum_{i=1}^{m} a_i\mathscr{L}_{31}^{[i]} \end{pmatrix} \in \mathrm{Mat}_{(m+2)\times 1}(\mathbb{T}).$$

Using (3.1.3) and (3.1.5) one has $\psi^{(-1)} = \Phi\psi$.

By hypothesis we have

$$\left(a_0\mathscr{L}_n + \sum_{i=1}^{m} a_i\mathscr{L}_{31}^{[i]}\right)(\theta) = \frac{c_0\zeta_A(n) + \sum_{i=1}^{m} c_i\zeta_A(\mathfrak{s}_i)}{\tilde{\pi}^n} = 0.$$

It follows from the ABP-criterion [ABP04, Thm. 3.1.1] that there exists $\mathbf{f} = (f_0, f_1, \ldots, f_m, f_{m+1}) \in \mathrm{Mat}_{1\times(m+2)}(\overline{k}[t])$ so that

$$\mathbf{f}\psi = 0 \text{ and } \mathbf{f}(\theta) = (0, \ldots, 0, 1).$$

Put $\tilde{\mathbf{f}} := \frac{1}{f_{m+1}}\mathbf{f}$ and note that $\tilde{\mathbf{f}}\psi = 0$. We take the (-1)-fold Frobenius twist of the equation $\tilde{\mathbf{f}}\psi = 0$ and then subtract it from itself, and then we have that

$$(\tilde{\mathbf{f}} - \tilde{\mathbf{f}}^{(-1)}\Phi)\psi = 0.$$

Note that the last coordinate of the vector $\tilde{\mathbf{f}} - \tilde{\mathbf{f}}^{(-1)}\Phi$ is zero. Define

$$(R_0, R_1, \ldots, R_m, 0) := \tilde{\mathbf{f}} - \tilde{\mathbf{f}}^{(-1)}\Phi$$

and note that
(3.1.6)
$$R_0 := \tilde{f}_0 - \tilde{f}_0^{(-1)}(t - \theta)^n - \sum_{i=1}^m \tilde{f}_i^{(-1)} H_{s_{i1}-1}^{(-1)}(t - \theta)^n - a_0 H_{n-1}^{(-1)}(t - \theta)^n$$
$$R_1 := \tilde{f}_1 - \tilde{f}_1^{(-1)}(t - \theta)^{s_{12}} - a_1 H_{s_{12}}^{(-1)}(t - \theta)^{s_{12}}$$
$$\vdots$$
$$R_m := \tilde{f}_m - \tilde{f}_m^{(-1)}(t - \theta)^{s_{m2}} - a_m H_{s_{m2}-1}^{(-1)}(t - \theta)^{s_{m2}},$$

where $(\tilde{f}_1, \cdots, \tilde{f}_m, 0) := \tilde{\mathbf{f}}$. We claim that $R_0 = R_1 = \cdots = R_m = 0$.

Assume this claim first. Put

$$\gamma := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \tilde{f}_0 & \tilde{f}_1 & \ldots & \tilde{f}_m & 1 \end{pmatrix}$$

and note that the claim above implies the following difference equations

$$\gamma^{(-1)}\Phi = \begin{pmatrix} \Phi' & \\ & 1 \end{pmatrix}\gamma,$$

where Φ' is the square matrix of size $m+1$ cut off from the upper left square of Φ.

By [CPY14, Prop. 2.2.1] the rational functions $\tilde{f}_0, \ldots, \tilde{f}_m$ have a (nonzero) common denominator $b \in \mathbb{F}_q[t]$ so that $b\tilde{f}_i \in \overline{k}[t]$ for $i = 0, 1, \ldots, m$. Since $b^{(-1)} = b$, multiplication by b on the both sides of (3.1.6) shows that if we put

$$\delta := \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ b\tilde{f}_0 & b\tilde{f}_1 & \ldots & b\tilde{f}_m & 1 \end{pmatrix}$$

and $\nu := (ba_0 H_{n-1}^{(-1)}(t - \theta)^n, ba_1 H_{s_{12}-1}^{(-1)}(t - \theta)^{s_{12}}, \ldots, ba_m H_{s_{m2}-1}^{(-1)}(t - \theta)^{s_{m2}}) \in$

$\operatorname{Mat}_{(m+1)\times 1}(\overline{k}[t])$, then we have

(3.1.7) $$\delta^{(-1)}\begin{pmatrix} \Phi' & \\ \nu & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & \\ & 1 \end{pmatrix}\delta.$$

Let M' (resp. M) be the Frobenius module defined by Φ' (resp. Φ) and note that $[M] \in \operatorname{Ext}^1_{\mathscr{F}}(\mathbf{1}, M')$. One checks directly that M' is an Anderson t-motive (cf. the proofs of [P08, Prop. 6.1.3] and [CY07, Lem. A.1]). So the action of b on M denoted by $b * M \in \mathscr{F}$, is defined by the matrix

$$b * \Phi := \begin{pmatrix} \Phi' & \\ \nu & 1 \end{pmatrix}$$

(see [CPY14, §§ 2.4]). So the system of difference equations (3.1.7) implies that $b * M$ represents the trivial class in $\operatorname{Ext}^1_{\mathscr{F}}(\mathbf{1}, M')$. Furthermore, if we let $\{x_0, x_1, \ldots, x_m\}$ be a $\overline{k}[t]$-basis of M' on which the action of σ is represented by Φ', then we have the isomorphism of $\mathbb{F}_q[t]$-modules (see [CPY14, Thm. 5.2.1]):

$$\operatorname{Ext}^1_{\mathscr{F}}(\mathbf{1}, M') \cong M'/(\sigma-1)M'$$
$$[b * M] \mapsto \widetilde{b * M} := ba_0 H^{(-1)}_{n-1}(t-\theta)^n x_0$$
$$+ \sum_{i=1}^m ba_i H^{(-1)}_{s_{i2}-1}(t-\theta)^{s_{i2}} x_i + (\sigma-1)M'.$$

Note that M' fits into the short exact sequence of Frobenius modules

$$0 \to C^{\otimes n} \to M' \twoheadrightarrow \oplus_{i=1}^m C^{\otimes s_{i2}} \to 0,$$

where the projection map $\pi : M' \twoheadrightarrow \oplus_{i=1}^m C^{\otimes s_{i2}}$ is given by $\pi := (\sum_{i=0}^m g_i x_i \mapsto (g_1, \ldots, g_m))$. However, since the $\mathbb{F}_q[t]$-linear map $(\sigma - 1) : \oplus_{i=1}^m C^{\otimes s_{i2}} \to \oplus_{i=1}^m C^{\otimes s_{i2}}$ is injective, the snake lemma shows that we have the following short exact sequence of $\mathbb{F}_q[t]$-modules:

$$0 \to C^{\otimes n}/(\sigma-1)C^{\otimes n} \to M'/(\sigma-1)M'$$
$$\twoheadrightarrow \oplus_{i=1}^m \left(C^{\otimes s_{i2}}/(\sigma-1)C^{\otimes s_{i2}}\right) \to 0.$$
$$\widetilde{b * M} \mapsto \left(ba_i H^{(-1)}_{s_{i2}-1}(t-\theta)^{s_{i2}}\right)_i$$

For any $s \in \mathbb{N}$ we recall the identification $C^{\otimes s}/(\sigma-1)C^{\otimes s} \cong \mathbf{C}^{\otimes s}(\overline{k})$, which is an $\mathbb{F}_q[t]$-module isomorphism and under which $H^{(-1)}_{s-1}(t-\theta)^s$ is

mapped to the special point $\mathbf{v}_s \in \mathbf{C}^{\otimes s}(A)$ (by Remark 2.4.4), itself associated to $\zeta_A(s)$. It follows that $(ba_i H_{s_{i2}-1}^{(-1)}(t-\theta)^{s_{i2}})_i$ is mapped to $([ba_i]_{s_{i2}}\mathbf{v}_{s_{i2}})_i \in \oplus_{i=1}^m \mathbf{C}^{\otimes s_{i2}}(A)$. Since $b * M$ represents the trivial class in $\mathrm{Ext}_{\mathscr{F}}^1(\mathbf{1}, M')$, $\widehat{b * M}$ is also trivial in $M'/(\sigma-1)M'$ and hence $[ba_i]\mathbf{v}_{s_{i2}} = \mathbf{0} \in \mathbf{C}^{\otimes s_{i2}}(A)$ for each $1 \le i \le m$. That is, each $\mathbf{v}_{s_{i2}}$ is ba_i-torsion as ba_i is nonzero. It follows by Remark 2.3.6 that s_{i2} must be divisible by $q-1$ for each $1 \le i \le m$.

To finish the proof, it suffices to prove the claim above. Since $s_{i1}+s_{i2} = n$ for each $1 \le i \le m$, without loss of generality we may assume that $s_{12} > s_{22} > \cdots > s_{m2}$. From the equation $(\tilde{\mathbf{f}} - \tilde{\mathbf{f}}^{(-1)}\Phi)\psi = 0$ we have that

$$(3.1.8) \qquad R_0 \Omega^n + R_1 \Omega^{s_{12}}\mathscr{L}_{21}^{[1]} + \cdots + R_m \Omega^{s_{m2}}\mathscr{L}_{21}^{[m]} = 0.$$

Dividing the equation above by $\Omega^{s_{m2}}$ one has

$$(3.1.9) \qquad R_0 \Omega^{s_{m1}} + R_1 \Omega^{s_{12}-s_{m2}}\mathscr{L}_{21}^{[1]} + \cdots + R_m \mathscr{L}_{21}^{[m]} = 0.$$

Note that Ω has simple zero at each $t = \theta^{q^N}$ for each $N \in \mathbb{N}$. Since each R_i is a rational function having only finitely many poles, we can pick a sufficiently large integer N so that R_i is regular at $t = \theta^{q^N}$ for $i = 1, \ldots, m$. Specializing both sides of the equation (3.1.9) at $t = \theta^{q^N}$ and using the formula (3.1.4) show that

$$R_m(\theta^{q^N})\,(\Gamma_{s_{m2}}\zeta_A(s_{m2})/\tilde{\pi}^{s_{m2}})^{q^N} = 0.$$

Since each MZV is non-vanishing by [T09a] and the equality above is valid for $N \gg 0$, R_m has to be zero as it has only finitely many zeros.

We turn back to the equation (3.1.8) and repeat the arguments above. We then eventually have $R_m = \cdots = R_2 = 0$ and so obtain

$$R_0 \Omega^n + R_1 \Omega^{s_{12}}\mathscr{L}_{21}^{[1]} = 0.$$

Again, by dividing $\Omega^{s_{12}}$, the arguments above show that $R_1 = 0$ and hence $R_0 = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 3.1.10. *Let $n \ge 2$ be an integer and for $i = 1, \ldots, m$, let $\mathfrak{s}_i = (s_{i1}, s_{i2}) \in \mathbb{N}^2$ be chosen with $s_{i1}+s_{i2} = n$. Suppose that we have*

$$c_0 \tilde{\pi}^n + \sum_{i=1}^m c_i \zeta_A(\mathfrak{s}_i) = 0 \text{ for some } c_0, c_1, \ldots, c_m \in k.$$

If the coefficient c_j is nonzero for some $1 \leq j \leq m$, then we have that $(q-1)|s_{j2}$. In other words, all k-linear relations among $\{\tilde{\pi}^n, \zeta_A(\mathfrak{s}_1), \ldots, \zeta_A(\mathfrak{s}_m)\}$ are those coming from the k-linear relations among $\{\tilde{\pi}^n\} \cup \{\zeta_A(\mathfrak{s}_j); (q-1)|s_{j2}\}$.

Proof. If n is A-even, then $\zeta_A(n)/\tilde{\pi}^n \in k$ by Carlitz [Ca35]. So the result follows from Theorem 3.1.1. If n is A-*odd* (i.e., n is not divisible by $q-1$), then $\tilde{\pi}^n \notin k_\infty$ and hence $\tilde{\pi}^n$ is k-linearly independent from all MZV's as each MZV is in k_∞. Therefore, we can put $c_0 = 0$ in Theorem 3.1.1 and the desired result follows. $\square$

Remark 3.1.11. The result above shows that the set $\{\tilde{\pi}^n\} \cup \{\zeta_A(\mathfrak{s}_j); (q-1) \nmid s_{j2}\}$ is linearly independent over k. It verifies the parity conjecture of Thakur [T09b] in this depth two setting.

Remark 3.1.12. For an even integer $n > 2$, Gangl, Kaneko and Zagier [GKZ06, Thms. 1 and 2] showed that

$$\mathscr{O} := (2\pi\sqrt{-1})^n \cup \{\zeta(3, n-3), \zeta(5, n-5), \ldots, \zeta(n-3, 3)\}$$

is a set of generators for the vector space DZ_n. Our point of view is that the set $\mathscr{O}$ is generated by $(2\pi\sqrt{-1})^n$ and weight n double zeta values $\zeta(odd, odd)$, but excluding $\zeta(n-1, 1)$ (ζ is not defined at 1 and so the odd 1 is special). Note that $\frac{n}{2} - 1$ is the cardinality of $\mathscr{O}$ and hence Conjecture 1.1.1 in the case of even $n > 2$ is equivalent to that there are $\dim_{\mathbb{C}} S_n(SL_2(\mathbb{Z}))$ independent $\mathbb{Q}$-linear relations among the set $\mathscr{O}$.

When the given weight $n \geq 2$ is A-even, we consider the set

$$\mathscr{O}_A := \{\tilde{\pi}^n\} \cup \{\zeta_A(s_1, s_2) | s_1 + s_2 = n \text{ and } (q-1) \nmid s_2\},$$

which is the analogue of the set $\mathscr{O}$ since each $\zeta_A(s_1, s_2) \in \mathscr{O}_A$ has the property that s_1 and s_2 are both A-odd. So it is not analogous to Conjecture 1.1.1 as $\mathscr{O}_A$ is k-linearly independent by Corollary 3.1.10.

4. Step II: Logarithmic interpretation

4.1. The formulation

In this section, our goal is to establish the following result.

Theorem 4.1.1. *Let $r \geq 2$ be an integer and let $\mathfrak{s} := (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $n := \sum_{i=1}^r s_i$. Put $\mathfrak{s}' := (s_2, \ldots, s_r)$ and suppose that $\zeta_A(\mathfrak{s}')$ is Eulerian,*

i.e., $\zeta_A(s_2, \ldots, s_r)/\tilde{\pi}^{s_2 + \cdots + s_r} \in k$. Let $\alpha_{\mathfrak{s}} := a_{\mathfrak{s}'} \in \mathbb{F}_q[t]$ be given in Theorem [2.4.10]. Then we explicitly construct an integral point $\Xi_{\mathfrak{s}} \in \mathbf{C}^{\otimes n}(A)$ so that there exists a vector $Y_{\mathfrak{s}} \in \mathbb{C}_\infty^n$ of the form

$$Y_{\mathfrak{s}} = \begin{pmatrix} * \\ \vdots \\ \alpha_{\mathfrak{s}}(\theta)\Gamma_{s_1}\cdots\Gamma_{s_r}\zeta_A(s_1, \ldots, s_r) \end{pmatrix}$$

satisfying

$$\exp_n(Y_{\mathfrak{s}}) = \Xi_{\mathfrak{s}}.$$

We divide the proof into the following steps.

4.2. Difference equations arising from algebraic points of $\mathbf{C}^{\otimes n}$

For each positive integer n, we denote by $\log_n$ the logarithm of $\mathbf{C}^{\otimes n}$. We first recall the convergence domain $\mathbb{D}_n$ of $\log_n$ (see [AT90, Prop. 2.4.3]):

$$\mathbb{D}_n := \left\{ \mathbf{z} := (z_1, \ldots, z_n)^{\mathrm{tr}} \in \mathbf{C}^{\otimes n}(\mathbb{C}_\infty); \; |z_i|_\infty < |\theta|_\infty^{i-n+\frac{nq}{q-1}} \text{ for } i = 1, \ldots, n. \right\}$$

For a fixed nonzero point $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{tr}} \in \mathbf{C}^{\otimes n}(\overline{k}) \cap \mathbb{D}_n$, we define the following polynomial associated to $\mathbf{u}$:

$$(4.2.1) \qquad f := f_{\mathbf{u}} := u_1(t - \theta)^{n-1} + \cdots + u_{n-1}(t - \theta) + u_n \in \overline{k}[t].$$

We let M_f be the Frobenius module defined by the matrix Φ_f in (2.4.3) and note that $[M_f] \in \mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, C^{\otimes n})$.

Now we define the series

$$(4.2.2) \qquad \mathscr{L}_f := f + \sum_{i=1}^\infty \frac{f^{(i)}}{(t - \theta^q)^n \ldots (t - \theta^{q^i})^n} \in \overline{k}[\![t]\!],$$

and note that this kind of series was introduced by Papanikolas [P08] and later on studied in [CY07, C14, M14, CPY14].

Proposition 4.2.3. *Let notation and assumptions be as above. Then there exists $\Psi_f \in \mathrm{GL}_2(\mathbb{T})$ so that $\Psi_f^{(-1)} = \Phi_f \Psi_f$. So M_f is a t-motive in the sense of [P08].*

Proof. Note that by the definition of $\mathscr{L}_f$ we have

$$\mathscr{L}_f^{(-1)} = f^{(-1)} + \sum_{i=0}^{\infty} \frac{f^{(i)}}{(t-\theta)^n \cdots (t-\theta^{q^i})^n} = f^{(-1)} + \frac{\mathscr{L}_f}{(t-\theta)^n},$$

and hence

(4.2.4) $$(\Omega^n \mathscr{L}_f)^{(-1)} = f^{(-1)}(t-\theta)^n \Omega^n + \Omega^n \mathscr{L}_f.$$

It follows that if we put

(4.2.5) $$\Psi_f := \begin{pmatrix} \Omega^n & 0 \\ \Omega^n \mathscr{L}_f & 1 \end{pmatrix},$$

then we have

$$\Psi_f^{(-1)} = \Phi_f \Psi_f.$$

The hypotheses of **u** imply that $\Omega^n \mathscr{L}_f$ satisfies the condition of [C14, Lemma 5.3.1], whence $\Omega^n \mathscr{L}_f$ is an entire function and therefore M_f is a t-motive (cf. [P08, Prop. 6.1.3] and [CY07, §§ 3.2]). □

4.3. Some lemmas

Lemma 4.3.1. *Let* **u** *be a nonzero point in* $\mathbf{C}^{\otimes n}(\overline{k}) \cap \mathbb{D}_n$, *and let* y *be the last coordinate of* $\log_n(\mathbf{u})$. *Let* $f \in \overline{k}[t]$ *be the polynomial associated to* **u** *given in (4.2.1) and* Ψ_f *be defined in (4.2.5). Then we have*

$$\Psi_f(\theta) = \begin{pmatrix} 1/\tilde{\pi}^n & 0 \\ y/\tilde{\pi}^n & 1 \end{pmatrix}.$$

To prove Lemma 4.3.1, we need the following formula due to Papanikolas.

Proposition 4.3.2. *(Papanikolas [P14]) We write* $\log_n = \sum_{i=0}^{\infty} P_i \tau^i$, *where* $P_0 = I_n$ *and* $P_i \in \mathrm{Mat}_n(k)$. *For each positive integer* i, *the bottom row vector of* P_i *is given by*

$$\left(\frac{(-1)^{n-1}(\theta^{q^i} - \theta)^{n-1}}{L_i^n}, \ldots, \frac{(-1)^{n-\ell}(\theta^{q^{i-\ell}} - \theta)^{n-\ell}}{L_i^n}, \ldots, \frac{1}{L_i^n} \right).$$

Proof of Lemma 4.3.1. As we know that $\Omega^n(\theta) = 1/\tilde{\pi}^n$, by (4.2.5) it suffices to show that

$$\mathscr{L}_f(\theta) = y.$$

We interpret $\mathscr{L}_f$ as

$$
\mathscr{L}_f = \left(u_1(t-\theta)^{n-1} + \cdots + u_{n-1}(t-\theta) + u_n\right)
$$
$$
+ \sum_{i=1}^{\infty} \frac{(t-\theta^{q^i})^{n-1}u_1^{q^i} + \cdots + (t-\theta^{q^i})u_{n-1}^{q^i} + u_n^{q^i}}{(t-\theta^q)^n \ldots (t-\theta^{q^i})^n}.
$$

By specializing at $t = \theta$ we have

$$
\mathscr{L}_f(\theta) = u_n + \sum_{i=1}^{\infty} \frac{(-1)^{n-1}(\theta^{q^i}-\theta)^{n-1}u_1^{q^i} + \cdots + (-1)(\theta^{q^i}-\theta)u_{n-1}^{q^i} + u_n^{q^i}}{L_i^n}.
$$

Hence by Proposition 4.3.2 we obtain that $y = \mathscr{L}_f(\theta)$.

Lemma 4.3.3. *Let n be a positive integer and let $\Xi \in \mathbf{C}^{\otimes n}(\overline{k})$ be an algebraic point. Then there exists a positive integer m and an algebraic point $\mathbf{u} \in \mathbf{C}^{\otimes n}(\overline{k})$ (depending on m) satisfying*

- $[t^m]_n(\mathbf{u}) = \Xi$ *(i.e., $\mathbf{u}$ is a t^m-division point of Ξ),*
- $\log_n(\mathbf{u})$ *converges.*

Proof. We recall that $\exp_n$ is an entire function on $\mathbb{C}_\infty^n$ and is of the form $\exp_n = I_n + \sum_{i=1}^{\infty} Q_i \tau^i$. By the inverse function theorem, there exists open subsets $U, V \subset \mathbb{C}_\infty^n$ so that $\exp_n : U \to V$ is one to one and its inverse is also continuous. Note that as formal power series $\log_n$ is inverse to $\exp_n$ and so $\log_n$ is defined on V. Since $\exp_n$ is surjective, there exists a vector $X \in \mathbb{C}_\infty^n$ for which $\exp_n(X) = \Xi$.

For each positive integer m, $\partial[t^m]_n$ is an upper triangular matrix with θ^m down to the diagonals, and the other nonzero entries off the diagonals have degrees in θ strictly less than m. It follows that if we define $\| \partial[t^m]_n^{-1} \|$ to be the maximum of the absolute values of entries of $\partial[t^m]_n^{-1}$, then we have

$$
\| \partial[t^m]_n^{-1} \| < |\theta|_\infty^{-m}.
$$

Hence, we can pick a sufficiently large integer m for which

$$
\partial[t^m]_n^{-1}(X) \in U.
$$

Put $\mathbf{u} := \exp_n\left(\partial[t^m]_n^{-1}(X)\right) \in V$, at which $\log_n$ converges. By the functional equation of $\exp_n$, we observe that

$$
[t^m]_n(\mathbf{u}) = [t^m]_n\left(\exp_n\left(\partial[t^m]_n^{-1}(X)\right)\right) = \exp_n(X) = \Xi. \qquad \square
$$

4.4. Proof of Theorem 4.1.1

Put $n := s_1 + \cdots + s_r$ and $\mathfrak{s}' := (s_2, \ldots, s_r) \in \mathbb{N}^{r-1}$. We let $(E'_{\mathfrak{s}}, \mathbf{v}_{\mathfrak{s}}, a_{\mathfrak{s}})$ (resp. $(E'_{\mathfrak{s}'}, \mathbf{v}_{\mathfrak{s}'}, a_{\mathfrak{s}'})$) be given in Theorem 2.4.10 corresponding to $\mathfrak{s}$ (resp. $\mathfrak{s}'$). Note that in this setting we have the following exact sequence of t-modules defined over A:

$$0 \longrightarrow \mathbf{C}^{\otimes n}(\bar{k}) \longrightarrow E'_{\mathfrak{s}}(\bar{k}) \overset{\pi}{\longrightarrow\!\!\!\!\rightarrow} E'_{\mathfrak{s}'}(\bar{k}) \longrightarrow 0,$$

and note that the projection π maps $\mathbf{v}_{\mathfrak{s}}$ to $\mathbf{v}_{\mathfrak{s}'}$ (see the proof [CPY14, Thm. 6.1.1]).

To simplify the notation, we drop the subscript and put $\alpha := \alpha_{\mathfrak{s}} := a_{\mathfrak{s}'}$. Since by hypothesis $\zeta_A(\mathfrak{s}')$ is Eulerian, Theorem 2.4.10 shows that $\mathbf{v}_{\mathfrak{s}'}$ is an α-torsion point in $E'_{\mathfrak{s}'}$. It follows that $\Xi_{\mathfrak{s}} := \rho_\alpha(\mathbf{v}_{\mathfrak{s}}) \in \mathrm{Ker}\,\pi$, and so we identify $\Xi_{\mathfrak{s}}$ as a point in $\mathbf{C}^{\otimes n}(A)$. Recall that ρ is the map defining the $\mathbb{F}_q[t]$-module structure on $E'_{\mathfrak{s}}$ (see (2.4.1)).

We write $\Xi_{\mathfrak{s}} = (v_1, \ldots, v_n)^{\mathrm{tr}} \in \mathbf{C}^{\otimes n}(A)$ and let g be the polynomial in $A[t]$ associated to $\Xi_{\mathfrak{s}}$ in (4.2.1). By Lemma 4.3.3 there exists a positive integer m and $\mathbf{u}_{\mathfrak{s}} \in \mathbf{C}^{\otimes n}(\bar{k})$ (depending on m) so that $[t^m]_n(\mathbf{u}_{\mathfrak{s}}) = \Xi_{\mathfrak{s}}$ and $\log_n(\mathbf{u}_{\mathfrak{s}})$ converges. We write $\mathbf{u}_{\mathfrak{s}} = (u_1, \ldots, u_n)^{\mathrm{tr}} \in \mathbf{C}^{\otimes n}(\bar{k})$ and let f be the polynomial in $\bar{k}[t]$ associated to $\mathbf{u}_{\mathfrak{s}}$ in (4.2.1).

We recall the isomorphism $\Delta_n : \mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, C^{\otimes n}) \cong \mathbf{C}^{\otimes n}(\bar{k})$ and note that Δ_n maps $[M_f]$ (resp. $[M_g]$) to $\mathbf{u}_{\mathfrak{s}}$ (resp. $\Xi_{\mathfrak{s}}$) by (2.4.2), where M_f (resp. M_g) is the Frobenius module defined by the matrix

$$\Phi_f := \begin{pmatrix} (t-\theta)^n & 0 \\ f^{(-1)}(t-\theta)^n & 1 \end{pmatrix} \text{ (resp. } \Phi_g := \begin{pmatrix} (t-\theta)^n & 0 \\ g^{(-1)}(t-\theta)^n & 1 \end{pmatrix} \text{).}$$

Note that

$$\Delta_n\left([t^m * M_f]\right) = [t^m]_n\left(\Delta_n([M_f])\right) = [t^m]_n\left(\mathbf{u}_{\mathfrak{s}}\right) = \Xi_{\mathfrak{s}},$$

where $t^m * M_f$ is the Frobenius module defined by

$$t^m * \Phi_f := \begin{pmatrix} (t-\theta)^n & 0 \\ t^m f^{(-1)}(t-\theta)^n & 1 \end{pmatrix}$$

(see [CPY14, §§ 2.4]). It follows that Φ_g and $t^m * \Phi_f$ define the same class of Frobenius modules in $\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, C^{\otimes n})$ since the classes of their defining Frobenius modules are mapped to same point $\Xi_{\mathfrak{s}}$ under Δ_n. In other words, there

exists a polynomial $h \in \overline{k}[t]$ satisfying the following Frobenius difference equations

$$\begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} (t-\theta)^n & 0 \\ g^{(-1)}(t-\theta)^n & 1 \end{pmatrix} = \begin{pmatrix} (t-\theta)^n & 0 \\ t^m f^{(-1)}(t-\theta)^n & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h & 1 \end{pmatrix},$$

from which we derive the following identity

$$
\begin{aligned}
\text{(4.4.1)} \quad & \begin{pmatrix} I_r & \\ h^{(-1)}, 0, \ldots, 0 & 1 \end{pmatrix} \begin{pmatrix} \Phi' & \\ g^{(-1)}(t-\theta)^n, 0, \ldots, 0 & 1 \end{pmatrix} \\
& = \begin{pmatrix} \Phi' & \\ t^m f^{(-1)}(t-\theta)^n, 0, \ldots, 0 & 1 \end{pmatrix} \begin{pmatrix} I_r & \\ h, 0, \ldots, 0 & 1 \end{pmatrix},
\end{aligned}
$$

where $\Phi' := \Phi'_\mathfrak{s}$ is given in (2.4.6).

We define the following two matrices

$$\widetilde{\Phi}_g := \begin{pmatrix} \Phi' & \\ g^{(-1)}(t-\theta)^n, 0, \ldots, 0 & 1 \end{pmatrix} \text{ and }$$

$$\widetilde{\Phi}_{t^m f} := \begin{pmatrix} \Phi' & \\ t^m f^{(-1)}(t-\theta)^n, 0, \ldots, 0 & 1 \end{pmatrix}.$$

Let $\Psi' := \Psi'_\mathfrak{s} \in \mathrm{GL}_r(\mathbb{T})$ (resp. $\Psi := \Psi_\mathfrak{s} \in \mathrm{GL}_{r+1}(\mathbb{T})$) be given in (2.4.8) (resp. (2.4.7)) and note that Ψ is of the form

$$\Psi = \begin{pmatrix} \Psi' & \\ \mu & 1 \end{pmatrix},$$

where

$$\mu = \left(\mathscr{L}_{(s_1,\ldots,s_r)}, \ldots, \mathscr{L}_{s_r} \right).$$

We further put

$$\widetilde{\Psi}_{t^m f} := \begin{pmatrix} \Psi' & \\ t^m \Omega^n \mathscr{L}_f, 0, \ldots, 0 & 1 \end{pmatrix}$$

and note that using (2.4.9) we have

$$\widetilde{\Psi}_{t^m f}^{(-1)} = \widetilde{\Phi}_{t^m f} \widetilde{\Psi}_{t^m f}.$$

Claim. There exists a matrix ν of the form

$$\nu = \begin{pmatrix} I_r & \\ \nu_1, \ldots, \nu_r & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\overline{k}[t])$$

so that

$$\nu^{(-1)} \left(\alpha * \Phi \right) = \widetilde{\Phi}_{t^m f} \cdot \nu,$$

where

$$\Phi := \Phi_{\mathfrak{s}} := \begin{pmatrix} \Phi' & \\ \varphi & 1 \end{pmatrix} \text{ given in } (2.4.5),$$

and

$$\alpha * \Phi := \begin{pmatrix} \Phi' & \\ \alpha\varphi & 1 \end{pmatrix}.$$

We first assume the claim above to finish the proof. Define

$$\alpha * \Psi := \begin{pmatrix} \Psi' & \\ \alpha\mu & 1 \end{pmatrix} \in \mathrm{GL}_{m+1}(\mathbb{T}).$$

Since $\alpha \in \mathbb{F}_q[t]$, using (2.4.9) we have

$$(\alpha * \Psi)^{(-1)} = \alpha * \Phi \cdot \alpha * \Psi.$$

Note that the claim above implies

$$(\nu \cdot \alpha * \Psi)^{(-1)} = \nu^{(-1)} \cdot \alpha * \Phi \cdot \alpha * \Psi = \widetilde{\Phi}_{t^m f} \left(\nu \cdot \alpha * \Psi \right).$$

In other words, $\nu \cdot \alpha * \Psi$ is also a *fundamental matrix* for $\widetilde{\Phi}_{t^m f}$ in the sense of [P08, §§ 4.1.6] and hence by [P08, § 4.1.6] that there exists a matrix $\gamma \in \mathrm{GL}_{r+1}(\mathbb{F}_q(t))$ of the form

$$\gamma = \begin{pmatrix} I_r & \\ \gamma_1, \ldots, \gamma_r & 1 \end{pmatrix}$$

so that

$$\nu \cdot \alpha * \Psi = \widetilde{\Psi}_{t^m f} \cdot \gamma.$$

By comparing with the $(r+1, 1)$-entries of both sides of the equation above we obtain the following identity
(4.4.2)
$$\nu_1 \Omega^n + \nu_2 \Omega^{s_2 + \cdots + s_r} \mathscr{L}_{s_1} + \cdots + \nu_r \Omega^{s_r} \mathscr{L}_{(s_1, \ldots, s_{r-1})} + \alpha \mathscr{L}_{(s_1, \ldots, s_r)} = t^m \Omega^n \mathscr{L}_f + \gamma_1.$$

By Lemma 2.3.4 we have that for each $N \in \mathbb{Z}_{\geq 0}$

$$\mathscr{L}_{(s_1, \ldots, s_r)}(\theta^{q^N}) = \mathscr{L}_{(s_1, \ldots, s_r)}(\theta)^{q^N} = (\Gamma_{s_1} \cdots \Gamma_{s_r} \zeta_A(s_1, \ldots, s_r) / \tilde{\pi}^n)^{q^N},$$

and by Lemma 4.3.1 and [CPY14, Prop. 2.3.3] we have that for each $N \in \mathbb{Z}_{\geq 0}$,

$$(\Omega^n \mathscr{L}_f)(\theta^{q^N}) = (\Omega^n \mathscr{L}_f)(\theta)^{q^N} = (y/\tilde{\pi}^n)^{q^N},$$

where y is the last coordinate of $\log_n(\mathbf{u}_{\mathfrak{s}})$ as f is the polynomial associated to $\mathbf{u}_{\mathfrak{s}}$. We mention that γ_1 must be in $\mathbb{F}_q[t]$ since all other terms of (4.4.2) are in the Tate algebra $\mathbb{T}$. Note that Ω has a simple zero at $t = \theta^{q^N}$ for each $N \in \mathbb{N}$ and hence by Lemma 2.3.4

$$\Omega^{s_2 + \cdots + s_r} \mathscr{L}_{(s_1)}(\theta^{q^N}) = \cdots = \Omega^{s_r} \mathscr{L}_{(s_1,\ldots,s_{r-1})}(\theta^{q^N}) = 0.$$

It follows that specializing (4.4.2) at $t = \theta^{q^N}$ for a positive integer N and taking the q^N-th root we obtain the following identity

$$\alpha(\theta)\Gamma_{s_1} \cdots \Gamma_{s_r} \zeta_A(\mathfrak{s}) = \theta^m y + \gamma_1(\theta)\tilde{\pi}^n.$$

Now we put

$$Y_{\mathfrak{s}} := \partial[t^m]_n \cdot \log_n(\mathbf{u}_{\mathfrak{s}}) + \partial[\gamma_1]_n \lambda_n$$

and note that the last coordinate of $Y_{\mathfrak{s}}$ is $\theta^m y + \gamma_1(\theta)\tilde{\pi}^n = \alpha(\theta)\Gamma_{s_1} \cdots \Gamma_{s_r} \zeta_A(\mathfrak{s})$. Since $\partial[\gamma_1]_n \lambda_n \in \Lambda_n$, by the functional equation of $\exp_n$ we see that $\exp_n(Y_{\mathfrak{s}}) = [t^m]_n(\mathbf{u}_{\mathfrak{s}}) = \Xi_{\mathfrak{s}}$, whence completing the proof of Theorem 4.1.1.

The rest task is to prove the claim above. We recall that M' is the Frobenius module defined by the matrix Φ' with respect to a $\overline{k}[t]$-basis $\{m_1, \ldots, m_r\}$ of M', and we have the following isomorphism as $\mathbb{F}_q[t]$-modules

$$\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, M') \cong E'_{\mathfrak{s}}(\overline{k})$$

and note that $[M]$ is mapped to $\mathbf{v}_{\mathfrak{s}}$. Hence $[\alpha * M]$ is mapped to

$$\Xi_{\mathfrak{s}} := \rho_\alpha(\mathbf{v}_{\mathfrak{s}}) \in \mathbf{C}^{\otimes n}(A) \hookrightarrow E'_{\mathfrak{s}}(A).$$

We note that M' is a free left $\overline{k}[\sigma]$-module with a natural $\overline{k}[\sigma]$-basis:
(4.4.3)
$$\left\{(t-\theta)^{s_1 + \cdots + s_r - 1} m_1, \cdots, (t-\theta)m_1, m_1, \ldots, (t-\theta)^{s_r - 1} m_r, \ldots, (t-\theta)m_r, m_r\right\}$$

(see [CPY14, Proof of Thm. 5.2.1]). We recall that $g \in A[t]$ is the polynomial associated to $\Xi_{\mathfrak{s}}$ and hence via the isomorphism $\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, M') \cong E'_{\mathfrak{s}}(\overline{k})$, the class of the Frobenius module defined by the matrix $\widetilde{\Phi}_g$ is mapped to

$\Xi_{\mathfrak{s}} \in \mathbf{C}^{\otimes n}(A) \hookrightarrow E'_{\mathfrak{s}}(A)$ via the basis (4.4.3) (see [CPY14, (5.2.2)]). In other words, the two matrices $\widetilde{\Phi}_g$ and

$$\alpha * \Phi = \begin{pmatrix} \Phi' & \mathbf{0} \\ 0, \ldots, \alpha H_{s_r-1}^{(-1)}(t-\theta)^{s_r} & 1 \end{pmatrix}$$

define the same class of Frobenius modules in $\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, M')$. Therefore there exists a matrix

$$\delta = \begin{pmatrix} I_r & \\ \delta_1, \ldots, \delta_r & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\overline{k}[t])$$

so that

(4.4.4) $$\delta^{(-1)} \cdot \alpha * \Phi = \widetilde{\Phi}_g \cdot \delta.$$

Put

$$\eta := \begin{pmatrix} I_r & \mathbf{0} \\ h, 0, \ldots, 0 & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\overline{k}[t])$$

and note that by (4.4.1) we have $\eta^{(-1)} \cdot \widetilde{\Phi}_g = \widetilde{\Phi}_{t^m f} \cdot \eta$. Putting $\nu := \eta\delta \in \mathrm{GL}_{r+1}(\overline{k}[t])$ and using (4.4.4) and (4.4.1) we have the desired identity

$$\nu^{(-1)} \cdot \alpha * \Phi = \eta^{(-1)}\delta^{(-1)} \cdot \alpha * \Phi = \eta^{(-1)} \cdot \widetilde{\Phi}_g \cdot \delta = \widetilde{\Phi}_{t^m f} \cdot \eta\delta = \widetilde{\Phi}_{t^m f} \cdot \nu.$$

From the explicit forms of δ and η we see that ν has the desired form, whence proving the claim above.

5. Step III: The identity

5.1. The dimension formula

In this section, we will give a proof of Theorem 1.2.1 (1). Combining Theorems 4.1.1 and 2.3.5, we first prove the following.

Theorem 5.1.1. *Let $n \geq 2$ be an integer and let $\mathbf{v}_n \in \mathbf{C}^{\otimes n}(A)$ be the special point given in Theorem 2.3.1. For $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ with $r \geq 2$, we denote by $\mathfrak{s}' := (s_2, \ldots, s_r)$. Put*

$$\mathscr{E}_n := \left\{ \mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r; r \geq 2, \mathrm{wt}(\mathfrak{s}) = n \text{ and } \zeta_A(\mathfrak{s}') \text{ is Eulerian} \right\}.$$

For each $\mathfrak{s} \in \mathscr{E}_n$, let $\Xi_{\mathfrak{s}} \in \mathbf{C}^{\otimes n}(A)$ be given in Theorem 4.1.1. Then we have

$$\dim_k \mathrm{Span}_k \left\{ \widetilde{\pi}^n, \zeta_A(n), \zeta_A(\mathfrak{s}); \mathfrak{s} \in \mathscr{E}_n \right\} = 1 + \mathrm{rank}_{\mathbb{F}_q[t]} \mathrm{Span}_{\mathbb{F}_q[t]} \left\{ \mathbf{v}_n \right\} \cup_{\mathfrak{s} \in \mathscr{E}_n} \left\{ \Xi_{\mathfrak{s}} \right\}.$$

Proof. To prove the theorem, it suffices to prove the following equivalent statements:

$$\{\tilde{\pi}^n, \zeta_A(n), \zeta_A(\mathfrak{s}); \mathfrak{s} \in \mathscr{E}_n\} \text{ are linearly dependent over } k$$
$$\Leftrightarrow \quad \{\mathbf{v}_n\} \cup_{\mathfrak{s} \in \mathscr{E}_n} \{\Xi_\mathfrak{s}\} \text{ are linearly dependent over } \mathbb{F}_q[t].$$

Proof of ($\Rightarrow$). Suppose that there exist polynomials (not all zero) $\{\eta\} \cup \{\eta_\mathfrak{s}\}_{\mathfrak{s} \in \mathscr{E}_n} \subseteq \mathbb{F}_q[t]$ for which

$$[\eta]_n(\mathbf{v}_n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} [\eta_\mathfrak{s}]_n (\Xi_\mathfrak{s}) = \mathbf{0}.$$

By Theorem 2.3.1, there exists a vector Y_n of the form

$$Y_n = \begin{pmatrix} * \\ \vdots \\ \Gamma_n \zeta_A(n) \end{pmatrix} \in \mathbb{C}_\infty^n$$

for which $\exp_n(Y_n) = \mathbf{v}_n$. For each $\mathfrak{s} \in \mathscr{E}_n$, by Theorem 4.1.1 there exists vectors $Y_\mathfrak{s} \in \mathbb{C}_\infty^n$ satisfying the property in Theorem 4.1.1. We define the vector

$$Y := \partial[\eta]_n Y_n + \sum_{\mathfrak{s} \in \mathscr{E}_n} \partial[\eta_\mathfrak{s}]_n Y_\mathfrak{s}$$

and note that

$$\exp_n(Y) = [\eta]_n (\exp_n(Y_n)) + \sum_{\mathfrak{s} \in \mathscr{E}_n} [\eta_\mathfrak{s}]_n (\exp_n(Y_\mathfrak{s}))$$
$$= [\eta]_n(\mathbf{v}_n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} [\eta_\mathfrak{s}]_n (\Xi_\mathfrak{s}) = \mathbf{0},$$

whence

$$Y \in \operatorname{Ker} \exp_n = \partial[\mathbb{F}_q[t]]_n \lambda_n.$$

Note that the last coordinate of λ_n is $\tilde{\pi}^n$ and that for each $a \in \mathbb{F}_q[t]$, $\partial[a]_n$ is an upper triangular matrix with $a(\theta)$ down the diagonals. Taking the last coordinates from both sides of the equality above gives the desired result.

Proof of ($\Leftarrow$). We suppose that there exist polynomials $\{\delta_0, \delta, \delta_\mathfrak{s}; \mathfrak{s} \in \mathscr{E}_n\} \in A$ (not all zero) so that

$$\delta_0 \tilde{\pi}^n + \delta \zeta_A(n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} \delta_\mathfrak{s} \zeta_A(\mathfrak{s}) = 0,$$

which can be also written as

$$\delta_0 \tilde{\pi}^n + \frac{\delta}{\Gamma_n} \Gamma_n \zeta_A(n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} \frac{\delta_\mathfrak{s}}{\alpha_\mathfrak{s}(\theta)\Gamma_\mathfrak{s}} \alpha_\mathfrak{s}(\theta)\Gamma_\mathfrak{s}\zeta_A(\mathfrak{s}) = 0,$$

where $\Gamma_\mathfrak{s} := \Gamma_{s_1} \ldots \Gamma_{s_r}$ for $\mathfrak{s} = (s_1, \ldots, s_r)$ and $\alpha_\mathfrak{s} \in \mathbb{F}_q[t]$ is given in Theorem 4.1.1. Multiplying a common denominator of the coefficients of the equation above shows that there exist polynomials $\{\eta_0, \eta, \zeta_\mathfrak{s}; \mathfrak{s} \in \mathscr{E}_n\} \subseteq \mathbb{F}_q[t]$ (not all zero) so that

$$\eta_0(\theta)\tilde{\pi}^n + \eta(\theta)\Gamma_n\zeta_A(n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} \eta_\mathfrak{s}(\theta)\alpha_\mathfrak{s}\Gamma_\mathfrak{s}\zeta_A(\mathfrak{s}) = 0.$$

For each $\mathfrak{s} \in \mathscr{E}_n$, let $Y_\mathfrak{s} \in \mathbb{C}_\infty^n$ be given in Theorem 4.1.1 and note that its last coordinate is $\alpha_\mathfrak{s}(\theta)\Gamma_\mathfrak{s}\zeta_A(\mathfrak{s})$. So the last coordinate of

$$Y := \partial[\eta_0]_n \lambda_n + \partial[\eta]_n Y_n + \sum_{\mathfrak{s} \in \mathscr{E}_n} \partial[\eta_\mathfrak{s}]_n Y_\mathfrak{s}$$

is zero by the equation above. Since

$$\exp_n(Y) = \exp_n \left(\partial[\eta_0]_n \lambda_n + \partial[\eta]_n Y_n + \sum_{\mathfrak{s} \in \mathscr{E}_n} \partial[\eta_\mathfrak{s}]_n Y_\mathfrak{s} \right)$$

$$= [\eta]_n (\mathbf{v}_n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} [\eta_\mathfrak{s}]_n (\Xi_\mathfrak{s}) \in \mathbf{C}^{\otimes n}(A),$$

Theorem 2.3.5 implies that Y has to be zero, and hence

$$[\eta]_n (\mathbf{v}_n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} [\eta_\mathfrak{s}]_n (\Xi_\mathfrak{s}) = \mathbf{0}. \qquad \square$$

Remark 5.1.2. For a positive integer n, we recall that $\zeta_A(n)/\tilde{\pi}^n \in k$ for A-even n by [Ca35] and in which case $\mathbf{v}_n$ is an $\mathbb{F}_q[t]$-torsion point by Remark 2.3.6. When n is A-odd, we have $\tilde{\pi}^n \notin k_\infty$ and so $\tilde{\pi}^n$ is k-linearly independent from MZV's, whence from the proof above we derive that

$$[\eta]_n(\mathbf{v}_n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} [\eta_\mathfrak{s}]_n(\Xi_\mathfrak{s}) = \mathbf{0}$$

if and only if

$$\eta(\theta)\Gamma_n\zeta_A(n) + \sum_{\mathfrak{s} \in \mathscr{E}_n} \eta_\mathfrak{s}(\theta)\Gamma_\mathfrak{s}\zeta_A(\mathfrak{s}) = 0$$

for $\{\eta\} \cup_{\mathfrak{s} \in \mathscr{E}_n} \{\eta_{\mathfrak{s}}\} \subseteq \mathbb{F}_q[t]$.

By this remark, we immediately obtain the following two consequences.

Corollary 5.1.3. *For an integer $n \geq 2$, we continue with the notation in Theorem 5.1.1. Then we have*

$$\dim_k \operatorname{Span}_k \{\zeta_A(n), \zeta_A(\mathfrak{s}); \mathfrak{s} \in \mathscr{E}_n\}$$

$$= \begin{cases} 1 + \operatorname{rank}_{\mathbb{F}_q[t]} \operatorname{Span}_{\mathbb{F}_q[t]} \{\Xi_{\mathfrak{s}}; \mathfrak{s} \in \mathscr{E}_n\} & \text{if } n \text{ is } A\text{-even,} \\ \operatorname{rank}_{\mathbb{F}_q[t]} \operatorname{Span}_{\mathbb{F}_q[t]} \{\mathbf{v}_n, \Xi_{\mathfrak{s}}; \mathfrak{s} \in \mathscr{E}_n\} & \text{if } n \text{ is } A\text{-odd.} \end{cases}$$

Corollary 5.1.4. *Let $n \geq 2$ be a positive integer. Put*

$$\mathscr{V} := \left\{(s_1, s_2) \in \mathbb{N}^2; s_1 + s_2 = n \text{ and } (q-1)|s_2\right\}.$$

For each $\mathfrak{s} \in \mathscr{V}$, let $\Xi_{\mathfrak{s}}$ be given in Theorem 4.1.1. Then we have

$$\dim_k \operatorname{Span}_k \{\tilde{\pi}^n, \zeta_A(\mathfrak{s}); \mathfrak{s} \in \mathscr{V}\} = 1 + \operatorname{rank}_{\mathbb{F}_q[t]} \operatorname{Span}_{\mathbb{F}_q[t]} \{\Xi_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathscr{V}}.$$

5.2. Proof of Theorem 1.2.1 (1)

Here we give a proof for part (1) of Theorem 1.2.1, which is addressed as the following result.

Corollary 5.2.1. *Let $n \geq 2$ be an integer. Put*

$$\mathscr{S} := \{\tilde{\pi}^n, \zeta_A(1, n-1), \zeta_A(2, n-2), \ldots, \zeta_A(n-1, 1)\},$$

and

$$\mathscr{V} := \left\{(s_1, s_2) \in \mathbb{N}^2; s_1 + s_2 = n \text{ and } (q-1)|s_2\right\}.$$

For each $\mathfrak{s} \in \mathscr{V}$, let $\Xi_{\mathfrak{s}}$ be given in Theorem 4.1.1. Then we have

$$\dim_k \operatorname{Span}_k \mathscr{S} = n - \lfloor \frac{n-1}{q-1} \rfloor + \operatorname{rank}_{\mathbb{F}_q[t]} \operatorname{Span}_{\mathbb{F}_q[t]} \{\Xi_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathscr{V}}.$$

Proof. Note that we have the following equalities

$$\begin{aligned} \dim_k \operatorname{Span}_k \mathscr{S} &= |\mathscr{S} \setminus \{\tilde{\pi}^n, \zeta_A(\mathfrak{s}); \mathfrak{s} \in \mathscr{V}\}| + \dim_k \operatorname{Span}_k \{\tilde{\pi}^n, \zeta_A(\mathfrak{s}); \mathfrak{s} \in \mathscr{V}\} \\ &= \left(n - 1 - \lfloor \frac{n-1}{q-1} \rfloor\right) + \left(1 + \operatorname{rank}_{\mathbb{F}_q[t]} \operatorname{Span}_{\mathbb{F}_q[t]} \{\Xi_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathscr{V}}\right) \\ &= n - \lfloor \frac{n-1}{q-1} \rfloor + \operatorname{rank}_{\mathbb{F}_q[t]} \operatorname{Span}_{\mathbb{F}_q[t]} \{\Xi_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathscr{V}}, \end{aligned}$$

where the first equality comes from Corollary 3.1.10, and the second equality comes from Corollary 5.1.4. □

Remark 5.2.2. Recently, some algebraic independence results of certain MZV's are obtained by Mishiba [M14], but the coordinates of those MZV's are restricted to be A-odd with other hypotheses. Concerning this issue, we refer the reader to [M14].

6. Step IV: A Siegel's lemma

6.1. The main result

The primary result in this section is the following theorem, which implies Theorem 1.2.1 (2) and so allows us to compute the exact quantity in Theorem 1.2.1 (1).

Theorem 6.1.1. *Let n be a positive integer and $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbf{C}^{\otimes n}(A)$. Then we have an effective algorithm to compute the following rank*

$$r_m := \mathrm{rank}_{\mathbb{F}_q[t]} \mathrm{Span}_{\mathbb{F}_q[t]} \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}.$$

Proof. We let $\mathbf{v}_i := (v_{i1}, \ldots, v_{in})^{\mathrm{tr}} \in \mathbf{C}^{\otimes n}(A)$, and let $f_i := v_{i1}(t - \theta)^{n-1} + \cdots + v_{in} \in A[t]$ be its associated polynomial. Note that the class $[M_i] \in \mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, C^{\otimes n}) \cong \mathbf{C}^{\otimes n}(\overline{k})$ is mapped to $\mathbf{v}_i$, where $M_i \in \mathscr{F}$ is defined by the matrix

$$\begin{pmatrix} (t - \theta)^n & 0 \\ f_i^{(-1)}(t - \theta)^n & 1 \end{pmatrix}.$$

Fix m polynomials $a_1, \ldots, a_m \in \mathbb{F}_q[t]$ and put $F := \sum_{i=1}^m a_i f_i$. Then the class $[M_F] \in \mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, C^{\otimes n})$ is mapped to the integral point $\sum_{i=1}^m [a_i]_n \mathbf{v}_i$ (see [CPY14, §§ 2.4]), where $M_F \in \mathscr{F}$ is defined by the matrix

$$\begin{pmatrix} (t - \theta)^n & 0 \\ F^{(-1)}(t - \theta)^n & 1 \end{pmatrix}.$$

It follows that $\sum_{i=1}^m [a_i]_n \mathbf{v}_i = \mathbf{0}$ if and only if M_F presents the trivial class in $\mathrm{Ext}^1_{\mathscr{F}}(\mathbf{1}, \mathbf{C}^{\otimes n})$. Therefore, we have the following equivalence:

- $\sum_{i=1}^m [a_i]_n \mathbf{v}_i = \mathbf{0} \in \mathbf{C}^{\otimes n}(A)$.
- there exists a polynomial $\delta \in \overline{k}[t]$ for which

$$\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} (t - \theta)^n & 0 \\ F^{(-1)}(t - \theta)^n & 1 \end{pmatrix} = \begin{pmatrix} (t - \theta)^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix},$$

which is equivalent to

(6.1.2) $$\delta^{(-1)}(t - \theta)^n + F^{(-1)}(t - \theta)^n = \delta.$$

Step I. Assume that there exist $a_1, \ldots, a_m \in \mathbb{F}_q[t]$ for which $\sum_{i=1}^m [a_i]_n \mathbf{v}_i = \mathbf{0}$, i.e., the equation (6.1.2) holds. Then the δ in (6.1.2) must be in $A[t]$.

Proof of Step I. Note that the equation (6.1.2) is equivalent to $\delta(t - \theta^q)^n + F(t - \theta^q)^n = \delta^{(1)}$. Then the result follows from H.-J. Chen's formulation in the proof of [KL15, Thm. 2 (a)].

For a polynomial $h = \sum_i u_i t^i \in A[t]$, we define its sup-norm by $\| h \| := \max_i \{|u_i|_\infty\}$. Note that for $h_1, h_2 \in A[t]$ we have:

- $\| h_1 h_2 \| = \| h_1 \| \cdot \| h_2 \|$.
- $\| h_1 + h_2 \| \leq \max \{\| h_1 \|, \| h_2 \|\}$.

For each $\mathbf{v}_i$, we define $\| \mathbf{v}_i \| := \max_{1 \leq j \leq n} \{|v_{ij}|_\infty\}$, and note that

$$\| f_i \| \leq \| \mathbf{v}_i \| \cdot |\theta|_\infty^{n-1}.$$

Put

$$D := \max_{1 \leq i \leq m} \{\| \mathbf{v}_i \|\} \cdot |\theta|_\infty^{n-1}$$

and so

$$\| F \| \leq D.$$

Step II. Let hypotheses be given as in Step I. Define

$$\ell := \max \left\{ \log_{|\theta|_\infty} D + 1, \frac{nq}{q-1} + 1 \right\}.$$

Then $\deg_\theta \delta < \ell$ when we regard δ as a polynomial in the variable θ over $\mathbb{F}_q[t]$.

Proof of Step II. Suppose on the contrary that $\deg_\theta \delta \geq \ell$, i.e., $\| \delta \| \geq |\theta|_\infty^\ell$. Note that by the definition of ℓ we have

(6.1.3) $$\| F(t - \theta^q)^n \| < |\theta|_\infty^\ell \cdot |\theta|_\infty^{qn} \leq \| \delta(t - \theta^q)^n \|.$$

Therefore, (6.1.3) and the equality

$$\delta(t - \theta^q)^n + F(t - \theta^q)^n = \delta^{(1)}$$

imply that

$$\| \delta(t - \theta^q)^n \| = \| \delta^{(1)} \|.$$

In other words, we have that

$$\deg_\theta \delta + nq = q \deg_\theta \delta,$$

whence

$$\deg_\theta \delta = \frac{nq}{q-1} < \ell,$$

a contradiction.

Step III: End of proof.   Now we write

(6.1.4) $\delta := c_1\theta^{\ell-1} + \cdots + c_\ell \in \mathbb{F}_q[t][\theta]$ and $F = d_1\theta^{\ell-1} + \cdots + d_\ell \in \mathbb{F}_q[t][\theta]$

and note that the coefficients $d_1,\ldots,d_\ell$ are $\mathbb{F}_q[t]$-linear combinations of $a_1,\ldots,a_m$. We recall that solving $a_1,\ldots,a_m$ in the equation

$$\sum_{i=1}^{m}[a_i]_n(\mathbf{v}_i) = \mathbf{0}$$

is equivalent to solving for δ and $a_1,\ldots,a_m$ satisfying

$$\delta(t-\theta^q)^n + F(t-\theta^q)^n = \delta^{(1)}.$$

However, putting the forms of δ and F (6.1.4) into the equation above and comparing the coefficients of each θ^i for $i = 0,\ldots,\ell-1$ we obtain a system of linear equations in $c_1,\ldots,c_\ell,a_1,\ldots,a_m$ over $\mathbb{F}_q[t]$. Using Gauss elimination we can solve for solutions $c_1,\ldots,c_\ell,a_1,\ldots,a_m$ effectively, and particularly obtain the rank of the solutions $a_1,\ldots,a_m$, whence establishing the desired result. $\square$

Remark 6.1.5. We mention that in [De91, De92] Denis studied the question of Siegel's lemma type for t-modules. For integral points $\mathbf{v}_1,\ldots,\mathbf{v}_m \in \mathbf{C}^{\otimes n}(A)$, Denis showed that there exists a constant c (depending on n and $\mathbf{v}_1,\ldots,\mathbf{v}_m$) so that the degrees of the coefficients of any $\mathbb{F}_q[t]$-linear relations among $\mathbf{v}_1,\ldots,\mathbf{v}_m$ can be bounded by c. However, the value of c is not explicit in Denis' results, and our approach is entirely different from his.

6.2. The algorithm

For weight $n \geq 2$, we provide the following algorithm to compute the dimension

$$d_n := \dim_k \mathrm{Span}_k \left\{ \tilde{\pi}^n, \zeta_A(1, n-1), \zeta_A(2, n-2), \ldots, \zeta_A(n-1, 1) \right\}.$$

Let $\mathscr{V}$ be given in Theorem 1.2.1. The algorithm is basically divided into two parts (most of the first part was given in [CPY14, § 6.1.1]).

(I) *Computing the integral points $\Xi_{\mathfrak{s}}$*: Fix an $\mathfrak{s} = (s_1, s_2) \in \mathscr{V}$.

I-1 Compute the Anderson-Thakur polynomials H_{s_1-1}, H_{s_2-1}.

I-2 Let $M'_{\mathfrak{s}}$ be the Frobenius module defined by $\Phi'_{\mathfrak{s}}$ as in (2.4.6) with $\overline{k}[t]$-basis $\{m_1, m_2\}$. Put $d = (s_1 + s_2) + s_2$ and let $\{\nu_1, \ldots, \nu_d\}$ be the $\overline{k}[\sigma]$-basis of $M'_{\mathfrak{s}}$ given by

$$(t-\theta)^{s_1+s_2-1}m_1, \ldots, (t-\theta)m_1, m_1, (t-\theta)^{s_2-1}m_2, \ldots, (t-\theta)m_2, m_2.$$

Identify $M'_{\mathfrak{s}}/(\sigma-1)M'_{\mathfrak{s}}$ with $\mathrm{Mat}_{d\times 1}(\overline{k})$ via $\nu_1, \ldots, \nu_d$ in [CPY14, (5.2.2)].

I-3 Write down the t-action on $M'_{\mathfrak{s}}/(\sigma-1)M'_{\mathfrak{s}}$, and so give a t-module structure on $\mathrm{Mat}_{d\times 1}(\overline{k})$, which we denote by $(E'_{\mathfrak{s}}, \rho)$.

I-4 Consider $H_{s_2-1}^{(-1)}(t-\theta)^{s_2}m_2 \in M'_{\mathfrak{s}}/(\sigma-1)M'_{\mathfrak{s}}$, which corresponds to an integral point $\mathbf{v}_{\mathfrak{s}} = (a_1, \ldots, a_d)^{\mathrm{tr}} \in E'(A)$ from the decomposition $H_{s_2-1}^{(-1)}(t-\theta)^{s_2}m_r \equiv \sum_{i=1}^{d} a_i\nu_i \pmod{\sigma - 1}$. See [CPY14, §§ 5.2].

I-5 Decompose $s_2 = p^\ell n_1 (q^h - 1)$ where $p \nmid n_1$ and h is the greatest integer such that $(q^h - 1) \mid s_2$. Define the polynomial $\alpha_{\mathfrak{s}} := (t^{q^h} - t)^{p^\ell} \in \mathbb{F}_q[t]$, and then compute $\Xi_{\mathfrak{s}} := \rho_{\alpha_{\mathfrak{s}}}(\mathbf{v}_{\mathfrak{s}})$, which is identified in $\mathbf{C}^{\otimes n}(A) \hookrightarrow E'_{\mathfrak{s}}(A)$.

(II) *Computing the dimension d_n*:

II-1 For each $\mathfrak{s} \in \mathscr{V}$, define $\| \Xi_{\mathfrak{s}} \|$ to the maximum of the absolute values of components of $\Xi_{\mathfrak{s}}$ and put $D := \max \{\| \Xi_{\mathfrak{s}} \|; \mathfrak{s} \in \mathscr{V}\} \cdot |\theta|_\infty^{n-1}$. Compute

$$\ell := \max \left\{ \log_{|\theta|_\infty} D + 1, \frac{nq}{q-1} + 1 \right\}.$$

II-2 Let $f_{\mathfrak{s}} \in A[t]$ be the polynomial associated to $\Xi_{\mathfrak{s}}$ given in (4.2.1). Let $\{a_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathscr{V}} \subseteq \mathbb{F}_q[t]$ be *parameters*, and put $F := \sum_{\mathfrak{s}\in\mathscr{V}} a_{\mathfrak{s}} f_{\mathfrak{s}}$.

II-3 Let $c_1, \ldots, c_\ell \in \mathbb{F}_q[t]$ be *parameters* and put $\delta := c_1\theta^{\ell-1} + \cdots + c_\ell \in \mathbb{F}_q[t][\theta]$. Write $F = d_1\theta^{\ell-1} + \cdots + d_\ell \in \mathbb{F}_q[t][\theta]$ and note that the coefficients $d_1, \ldots, d_\ell$ are $\mathbb{F}_q[t]$-linear combinations of $\{a_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathscr{V}}$.

II-4 Comparing the coefficients of θ^i from the equation $\delta(t-\theta^q)^n + F(t-\theta^q)^n = \delta^{(1)}$, we obtain a system of linear equations in $c_1, \ldots, c_\ell$ and $\{a_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathscr{V}}$ with coefficients in $\mathbb{F}_q[t]$. Using Gaussian elimination we solve for solutions $c_1, \ldots, c_\ell$ and $\{a_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathscr{V}}$, and particularly solve for rank $\tilde{r}_n$ of the solutions $[a_{\mathfrak{s}}]_{\mathfrak{s}\in\mathscr{V}}$, which is the number of independent $\mathbb{F}_q[t]$-linear relations among $\{\Xi_{\mathfrak{s}}\}_{\mathfrak{s}\in\mathscr{V}}$.

II-5 Compute $r_n := \lfloor \frac{n-1}{q-1} \rfloor - \tilde{r}_n$, which is the rank of the $\mathbb{F}_q[t]$-module

$$\mathrm{Span}_{\mathbb{F}_q[t]} \left\{ \Xi_{\mathfrak{s}} \right\}_{\mathfrak{s} \in \mathscr{V}}.$$

Compute $d_n := n - \lfloor \frac{n-1}{q-1} \rfloor + r_n$, which is the exact dimension we want by Theorem 1.2.1.

6.3. Computational data

In this section, we list some data of implementing the algorithm above using Magma. We thank Yi-Hsuan Lin for providing the code. In what follows, "Weight'' means the weight n, and "Dimension" means d_n above, and "$\mathbb{F}_p$-linear" means the number of independent linear relations arising from (1.2.2). When the weight n is A-odd (i.e., $(q-1) \nmid n$) the author does not know whether (1.2.2) can produce a linear relation as $\zeta_A(r)\zeta_A(s)$ is a "monomial" for one at least of r, s being A-odd. So we let the position of A-odd weight be blank. "Zeta-like" means the number of weight n double zeta values $\zeta_A(\mathfrak{s})$ for which $\zeta_A(\mathfrak{s})/\zeta_A(n) \in k$. We list the computation data below only for $q = 2$ and $q = 3$, although we have run the program for other q up to 11 with weight up to 150.

For $q = 2$, we have:

Weight	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Dimension	1	2	2	3	3	3	3	4	4	4	4	5	5	5
$\mathbb{F}_p$-linear	0	1	1	1	2	2	2	3	3	3	4	4	4	5
Zeta-like	1	1	1	0	0	2	1	0	0	0	0	0	0	2

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
5	6	6	6	6	7	7	7	7	8	8	8	8	8	8
5	5	6	6	6	7	7	7	8	8	8	9	9	9	10
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
8	8	9	9	9	9	10	10	10	10	11	11	11	11	11
10	10	11	11	11	12	12	12	13	13	13	14	14	14	15
2	0	0	0	0	0	0	0	0	0	0	0	0	0	0

46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
11	11	11	12	12	12	12	12	12	12	13	13	13	13
15	15	16	16	16	17	17	17	18	18	18	19	19	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

61	62	63	64	65	66	67	68	69	70	71	72	73	74	75
13	13	13	13	14	14	14	14	15	15	15	15	16	16	16
20	20	21	21	21	22	22	22	23	23	23	24	24	24	25
0	0	2	0	0	0	0	0	0	0	0	0	0	0	0

76	77	78	79	80	81	82	83	84	85	86	87	88	89	90
16	16	16	16	16	17	17	17	17	17	17	17	17	18	18
25	25	26	26	26	27	27	27	28	28	28	29	29	29	30
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

91	92	93	94	95	96	97	98	99	100	101	102	103	104	105
18	18	18	18	18	18	19	19	19	19	19	19	19	19	20
30	30	31	31	31	32	32	32	33	33	33	34	34	34	35
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
20	20	20	20	20	20	20	21	21	21	21	21	21	21	21
35	35	36	36	36	37	37	37	38	38	38	39	39	39	40
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

For $q = 3$, we have:

Weight	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Dimension	3	4	5	5	7	7	8	9	10	10	12	12	13	13
$\mathbb{F}_p$-linear		0		0		1		1		1		1		2
Zeta-like	1	0	1	1	1	1	1	0	0	0	0	0	0	0

17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
14	14	16	16	17	17	18	18	19	19	20	21	22	22	24
	2		2		2		3		3		3		3	
2	0	0	0	0	0	2	0	3	2	2	0	0	0	0

32	33	34	35	36	37	38	39	40	41	42	43	44	45	46
24	25	25	26	26	28	28	29	29	30	30	31	31	32	33
4		4		4		4		5		5		5		5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
34	34	35	35	36	36	37	37	39	39	40	40	41	41	42
	6		6		6		6		7		7		7	
0	0	0	0	0	0	2	0	0	0	0	0	0	0	0

62	63	64	65	66	67	68	69	70	71	72	73	74	75	76
42	43	44	45	45	46	46	47	47	48	48	50	50	51	51
7		8		8		8		8		9		9		9
0	0	0	0	0	0	0	0	0	2	0	0	0	0	0

77	78	79	80	81	82	83	84	85	86	87	88	89	90	91
52	52	53	53	54	55	56	56	58	58	59	59	60	60	62
	9		10		10		10		10		11		11	
4	0	3	2	0	0	0	0	0	0	0	0	0	0	0

92	93	94	95	96	97	98	99	100	101	102	103	104	105	106
62	63	63	64	64	65	65	66	67	68	68	69	69	70	70
11		11		12		12		12		12		13		13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

107	108	109	110	111	112	113	114	115	116	117	118	119	120	121
71	71	73	73	74	74	75	75	76	76	77	78	79	79	80
	13		13		14		14		14		14		15	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

122	123	124	125	126	127	128	129	130	131	132	133	134	135	136
80	81	81	82	82	84	84	85	85	86	86	87	87	88	89
15		15		15		16		16		16		16		17
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

137	138	139	140	141	142	143	144	145	146	147	148	149	150
90	90	91	91	92	92	93	93	95	95	96	96	97	97
	17		17		17		18		18		18		18
0	0	0	0	0	0	0	0	0	0	0	0	0	0

References

[A86]   G. W. Anderson, *t-motives*, Duke Math. J. **53** (1986), no. 2, 457–502.

[ABP04] G. W. Anderson, W. D. Brownawell and M. A. Papanikolas, *Determination of the algebraic relations among special Γ-values in positive characteristic*, Ann. of Math. (2) **160** (2004), no. 1, 237–313.

[AT90]   G. W. Anderson and D. S. Thakur, *Tensor powers of the Carlitz module and zeta values*, Ann. of Math. (2) **132** (1990), no. 1, 159–191.

[AT09]   G. W. Anderson and D. S. Thakur, *Multizeta values for $\mathbb{F}_q[t]$, their period interpretation, and relations between them,* Int. Math. Res. Not. IMRN (2009), no. 11, 2038–2055.

[André04]   Y. André,   *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthéses, **17**. Société Mathématique de France, Paris, 2004.

[B12]   F. Brown, *Mixed Tate motives over $\mathbb{Z}$*, Ann. of Math. (2) **175** (2012), no. 2, 949–976.

[BP]   W. D. Brownawell and M. A. Papanikolas, *A rapid introduction to Drinfeld modules, t-modules, and t-motives*, available at http://www.math.tamu.edu/˜map/BanffSurvey.pdf, 2011.

[Ca35]   L. Carlitz, *On certain functions connected with polynomials in a Galois field*, Duke Math. J. **1** (1935), no. 2, 137–168.

[Cart02]   P. Cartier, *Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents*, Séminaire Bourbaki, Vol. 2000/2001. Astérisque No. 282 (2002), Exp. No. 885, viii, 137–173.

[C14]   C.-Y. Chang, *Linear independence of monomials of multizeta values in positive characteristic*, Compos. Math. **150** (2014), no. 11, 1789–1808.

[CP12]   C.-Y. Chang and M. A. Papanikolas, *Algebraic independence of periods and logarithms of Drinfeld modules. With an appendix by Brian Conrad.* J. Amer. Math. Soc. **25** (2012), no. 1, 123–150.

[CPY14]   C.-Y. Chang, M. A. Papanikolas and J. Yu, *An effective criterion for Eulerian multizeta values in positive characteristic*, to appear in J. Eur. Math. Soc. (JEMS), arXiv:1411.0124.

[CY07]   C.-Y. Chang and J. Yu, *Determination of algebraic relations among special zeta values in positive characteristic*, Adv. Math. **216** (2007), no. 1, 321–345.

[Chen15]   H.-J. Chen, *On shuffle of double zeta values for $\mathbb{F}q[t]$*, J. Number Theory **148** (2015), 153–163.

[Chen16]   H.-J. Chen, *On shuffle of double Eisenstein series in positive characteristic*, in preparation.

[DG05]   P. Deligne and A. B. Goncharov, *Groupes fondamentaux motiviques de Tate mixte*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 1, 1–56.

[De91]   L. Denis, *Géométrie diophantienne sur les modules de Drinfel'd*, The arithmetic of function fields (Columbus, OH, 1991), 285–302, Ohio State Univ. Math. Res. Inst. Publ., 2, de Gruyter, Berlin, 1992.

[De92]   L. Denis, *Hauteurs canoniques et modules de Drinfeld*, Math. Ann. **294**, 213–223 (1992).

[GKZ06]   H. Gangl, M. Kazeko and D. Zagier, *Double zeta values and modular forms*, Automorphic forms and zeta functions, 71–106, World Sci. Publ., Hackensack, NJ, 2006.

[Ge88]   E.U. Gekeler, *On the coefficients of Drinfeld modular forms*, Invent. Math. **93** (1988), no. 3, 667–700.

[Gon97]   A. B. Goncharov, *The double logarithm and Manin's complex for modular curves*, Math. Res. Lett. **4** (1997), 617–636.

[Go80]   D. Goss, *Modular forms for $\mathbb{F}_r[T]$*, J. Reine Angew. Math. **317** (1980), 16–39.

[Go96]   D. Goss, *Basic structures of function field arithmetic*, Springer-Verlag, Berlin, 1996.

[HJ16]   U. Hartl and A.-K. Juschka, *Pink's theory of Hodge structures over function fields*, available at www.math.uni-muenster.de/u/urs.hartl/Publikat/Hodge2.pdf, 2016.

[HP04]   U. Hartl and R. Pink, *Vector bundles with a Frobenius structure on the punctured unit disc*, Compos. Math. **140** (2004), no. 3, 689–716.

[IKZ06]   L. Ihara, M. Kaneko and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compos. Math. **142** (2006), no. 2, 307–338.

[KL15]   Y.-L. Kuan and Y.-H. Lin, *Criterion for deciding zeta-like multi-zeta values in positive characteristic*, Exp. Math. **25** (2016), no. 3, 246–256.

[LRT14]   J. A. Lara Rodríguez and D. S. Thakur, *Zeta-like multizeta values for $\mathbb{F}_q[t]$*, Indian J. Pure Appl. Math. **45** (2014), no. 5, 785–798.

[M14]   Y. Mishiba, *On algebraic independence of certain multizeta values in characteristic p*, arXiv:1401.3628.

[P08]   M. A. Papanikolas, *Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms*, Invent. Math. **171** (2008), no. 1, 123–174.

[P14]   M. A. Papanikolas, *Log-algebraicity on tensor powers of the Carlitz module and special values of Goss L-function*, in preparation.

[PR03]   M. A. Papanikolas and N. Ramachandran, *A Weil-Barsotti formula for Drinfeld modules*, J. Number Theory **98** (2003), no. 2, 407–431.

[S97]   S. K. Sinha, *Periods of t-motives and transcendence*, Duke Math. J. **88** (1997), no. 3, 465–535.

[Ta10]   L. Taelman, *1-t-motifs*, arXiv:0908.1503, 2010.

[Te02]   T. Terasoma, *Mixed Tate motives and multiple zeta values*, Invent. Math. **149** (2002), no. 2, 339–369.

[T04]   D. S. Thakur, *Function field arithmetic*, World Scientific Publishing, River Edge NJ, 2004.

[T09a]   D. S. Thakur, *Power sums with applications to multizeta and zeta zero distribution for $\mathbb{F}_q[t]$*, Finite Fields Appl. **15** (2009), no. 4, 534–552.

[T09b]   D. S. Thakur, *Relations between multizeta values for $\mathbb{F}_q[T]$*, Int. Math. Res. Notices IMRN (2009), no. 12, 2318–2346

[T10]   D. S. Thakur, *Shuffle relations for function field multizeta values*, Int. Math. Res. Not. IMRN (2010), no. 11, 1973–1980.

[To15]   G. Todd, *Linear relations between multizeta values*, Ph.D. Thesis, University of Arizona (2015).

[Yu91]   J. Yu, *Transcendence and special zeta values in characteristic p*, Ann. of Math. (2) **134** (1991), no. 1, 1–23.

[Z93]   D. Zagier, *Periods of modular forms, traces of Hecke operators, and multiple zeta values*, Research into automorphic forms and L functions (Japanese) (Kyoto, 1992). Sūrikaisekikenkyūsho Kōkyūroku No. 843 (1993), 162–170.

[Z94]   D. Zagier, *Values of zeta functions and their applications*, First European Congress of Mathematics, Vol. II (Paris, 1992), Progress in Math. 120, Birkhäuser-Verlag, Basel, (1994) 497–512.

[Zh16]   J. Zhao, *Multiple zeta functions, multiple polylogarithms and their special values*, Series on Number Theory and its Applications, 12. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2016.

CHIEH-YU CHANG
DEPARTMENT OF MATHEMATICS
NATIONAL TSING HUA UNIVERSITY
HSINCHU CITY 30042
TAIWAN R.O.C.
E-mail address: cychang@math.nthu.edu.tw