Morse index and multiplicity of min-max minimal hypersurfaces

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The Min-max Theory for the area functional, started by Almgren in the early 1960s and greatly improved by Pitts in 1981, was left incomplete because it gave no Morse index estimate for the min-max minimal hypersurface.

We advance the theory further and prove the first general Morse index bounds for minimal hypersurfaces produced by it. We also settle the multiplicity problem for the classical case of one-parameter sweepouts.

1. Introduction

The Min-max Theory for the area functional was started by Almgren in the early 1960s [4, 5] and greatly improved by Pitts in 1981, resulting in the proof of the Almgren-Pitts Min-max Theorem [25]. The subject can be regarded as a deep higher-dimensional generalization of the study of closed geodesics and uses the regularity results of Schoen-Simon [27].

Despite the many applications of this theory found recently, no general information about the Morse index was known. The difficulty is mainly caused by the weak notions of convergence (in the sense of Geometric Measure Theory) and the possibility of multiplicity. There is no Hilbert space structure or a Palais-Smale condition to check.

In this paper we prove the first general Morse index bounds for minimal hypersurfaces produced by this theory. Inspired by finite-dimensional Morse theory, one expects that the Morse index of the critical point being produced should be less than or equal to the number of parameters (dimension of the families used in the min-max process). Here we confirm this heuristic for the index of the support of the minimal hypersurface, i.e. the sum of the indices of the components.

We also prove the first general multiplicity one theorem in min-max theory, in the setting of one-parameter sweepouts. These sweepouts have been considered since the work of Birkhoff [6]. We show that for manifolds with generic metrics, the min-max minimal hypersurface either has Morse index one with a multiplicity one two-sided unstable component, or is stable (i.e. has Morse index zero) and contains an one-sided component with even multiplicity and unstable double cover. We do not know whether the second case can be perturbed away in general, except of course if there are no one-sided closed hypersurfaces like in the case of manifolds M^{n+1} with $H_n(M, \mathbb{Z}_2) = 0$. Nonetheless, the fact that a two-sided min-max minimal hypersurface must be unstable generically was not known prior to our work.

Motivated by our finding, we propose the following conjecture for the k-parameter min-max setting:

Multiplicity One Conjecture. *For generic metrics on M^{n+1}, $3 \le (n + 1) \le 7$, two-sided unstable components of closed minimal hypersurfaces obtained by min-max methods must have multiplicity one.*

Theorem 1.7 of this paper proves this conjecture in the case of min-max with one parameter.

The one-parameter case in manifolds with positive Ricci curvature had been studied before by Zhou [35, 36], who did important work extending to high dimensions previous work of the authors [19]. These results have been improved recently by Ketover and the authors in [16], where new index characterizations and multiplicity one theorems are proven as a consequence of the catenoid estimate. Related results have been proven for the least-area closed minimal hypersurface by Mazet and Rosenberg [23] and later by Song [30].

Remark 1.1. For surfaces the conjecture is not true as it is shown in Aiex [2]. This case is special because when $n = 1$ the Almgren-Pitts Min-max Theorem provides the existence of a stationary geodesic network, not necessarily smooth.

We now state the main theorems and some applications.

Let (M^{n+1}, g) be an $(n + 1)$-dimensional closed Riemannian manifold, with $3 \le (n + 1) \le 7$. Let X be a simplicial complex of dimension k and $\Phi : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ be a continuous map. Here G is either the group $\mathbb{Z}$ or $\mathbb{Z}_2$, while $\mathcal{Z}_n(M; G)$ denotes the space of n-dimensional flat chains T in M with coefficients in G and $\partial T = 0$. These are called *flat cycles*. The notation $\mathcal{Z}_n(M; \mathbf{F}; G)$ indicates the space $\mathcal{Z}_n(M; G)$ endowed with the F-metric, to be defined later. Basically this means Φ is continuous in both the flat and the varifold topologies.

We let Π be the class of all continuous maps $\Phi' : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ such that Φ and Φ' are homotopic to each other in the flat topology. The *width* of Π is defined to be the min-max invariant:

$$\mathbf{L}(\Pi) = \inf_{\Phi' \in \Pi} \sup_{x \in X} \{\mathbf{M}(\Phi'(x))\},$$

where $\mathbf{M}(T)$ denotes the mass of T (or n-dimensional area).

Given a sequence $\{\Phi_i\}_{i \in \mathbb{N}}$ of continuous maps from X into $\mathcal{Z}_n(M; \mathbf{F}; G)$, we set

$$\mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}}) := \limsup_{i \to \infty} \sup_{x \in X} \mathbf{M}(\Phi_i(x)).$$

When $\mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}}) = \mathbf{L}(\Pi)$, we say $\{\Phi_i\}_{i \in \mathbb{N}}$ is a *min-max sequence* in Π.

The Almgren-Pitts min-max theory is actually formulated in terms of sequences of maps defined on the vertices of finer and finer subdivisions of X and that become finer and finer in the mass topology. This imposes quite a bit of notation and technicalities. In Section 3, we explain how to use the original discretized setting to obtain a min-max theory for maps that are continuous in the $\mathbf{F}$-topology as above. A similar construction for the mass topology was done by the authors in [22].

The next theorem establishes general upper Morse index bounds for the min-max minimal hypersurface. It is a consequence of Deformation Theorem A of Section 5 and its proof is left to Section 6.

Theorem 1.2. *Let (M^{n+1}, g) be an $(n+1)$-dimensional closed Riemannian manifold, with $3 \leq (n+1) \leq 7$. There exists an integral stationary varifold $\Sigma \in \mathcal{V}_n(M)$ with the following properties:*

- $||\Sigma||(M) = \mathbf{L}(\Pi)$,
- *the support of Σ is a smooth closed embedded hypersurface in M,*
- $index(support\ of\ \Sigma) \leq k$.

Simon-Smith variant 1.3. For three-dimensional manifolds M^3, one can restrict to sweepouts by smooth surfaces (or by level sets of Morse functions) and consider the class Π of all sweepouts that differ from a fixed one by the action of ambient isotopies. The isotopies vary smoothly in the parameter space. This is the Simon-Smith variant of Almgren-Pitts theory (see Smith [29] and a survey by Colding-De Lellis [11]). This variant is specially convenient if one wants to control the topology of the min-max minimal surface. Genus bounds were proven recently by Ketover [15] (see also De Lellis-Pellandini [12]), as conjectured by Pitts and Rubinstein [26]. We note

that another variant of min-max theory, based on phase transition partial differential equations, has been recently proposed by Guaraco [14].

If one follows the proof of Theorem 1.2, which is based on Deformation Theorem A, one will see that all deformations are by isotopies. By mollifying these deformations so that they depend smoothly of the parameters, we get that Theorem 1.2 is also true in the Simon-Smith setting. We leave the details to the reader. As an application, the main result of Ketover-Zhou [17] about the entropy of closed surfaces can be extended to surfaces with genus different than one.

Willmore and Freedman-He-Wang conjectures 1.4. We proved the Willmore conjecture in [20] by introducing certain five-parameter sweepouts of closed surfaces in the three-sphere and using min-max theory. Later we proved jointly with I. Agol [1] the Freedman-He-Wang conjecture about links following a similar strategy. The basic intuition was that the min-max minimal surface produced would have index less than or equal to five (the number of parameters) and then it could be only an equatorial sphere or the Clifford torus by a theorem of Urbano [31]. The possibility of an equatorial sphere is ruled out by topological arguments. The index bound was not available at the time of those papers, and we had to proceed indirectly. With Theorem 1.2 we can argue directly and conclude the min-max surface is a Clifford torus. This simplifies a bit the work involved in the proofs of the main statements of [20] and [1].

The space of cycles 1.5. The calculation by Almgren [4] of the homotopy groups of the space of n-cycles mod 2 in M^{n+1} gives that this space is weakly homotopically equivalent to $\mathbb{RP}^\infty$ (see [21]).

We used this structure in [21] to prove Yau's conjecture [34] about the existence of infinitely many closed minimal hypersurfaces, in the case the manifold has positive Ricci curvature. The minimal hypersurfaces are constructed by applying min-max theory to the class of k-sweepouts, which are multiparameter families defined by a cohomological condition, for each k. We now explain how to deduce from Theorem 1.2 that the support of the corresponding min-max minimal hypersurface must have index less than or equal to k. There has been considerable activity after our paper [21] regarding compactness properties of minimal hypersurfaces with bounded index ([8], [9], [10], [18]).

We have $H^1(\mathcal{Z}_n(M;\mathbb{Z}_2);\mathbb{Z}_2) = \mathbb{Z}_2 = \{0,\bar{\lambda}\}$ and, for every $k \in \mathbb{N}$, we consider the set $\mathcal{P}_k$ of all continuous maps $\Phi : X \to \mathcal{Z}_n(M;\mathbf{F};\mathbb{Z}_2)$ such that $\Phi^*(\bar{\lambda})^k$ does not vanish in $H^k(X;\mathbb{Z}_2)$, where X is some finite dimensional

compact simplicial complex. We defined in [21]

$$\omega_k := \inf_{\Phi \in \mathcal{P}_k} \sup_{x \in dmn(\Phi)} \mathbf{M}(\Phi(x)).$$

The first remark is that if $\tilde{\mathcal{P}}_k$ denotes those elements $\Phi \in \mathcal{P}_k$ whose domain of definition is a compact simplicial complex of dimension k then

$$\omega_k = \inf_{\Phi \in \tilde{\mathcal{P}}_k} \sup_{x \in dmn(\Phi)} \mathbf{M}(\Phi(x)).$$

Indeed, given $\Phi \in \mathcal{P}_k$ consider $X^{(k)}$ the k-skeleton of $X := dmn(\Phi)$. Then $H^k(X, X^{(k)}; \mathbb{Z}_2) = 0$ and so the long exact cohomology sequence gives that the natural pullback map from $H^k(X; \mathbb{Z}_2)$ into $H^k(X^{(k)}; \mathbb{Z}_2)$ is injective, which means that $\Phi_{|X^{(k)}} \in \mathcal{P}_k$ and so $\Phi_{|X^{(k)}} \in \tilde{\mathcal{P}}_k$.

Consider a min-max sequence $\{\Phi_i\}_{i \in \mathbb{N}} \subset \tilde{\mathcal{P}}_k$ for ω_k. Each Φ_i induces a homotopy class Π_i, $i \in \mathbb{N}$. From Theorem 1.2 we obtain that $\mathbf{L}(\Pi_i) = ||\Sigma_i||(M)$, where the support of Σ_i has index $\leq k$. Note that $\mathbf{L}(\Pi_i)$ tends to ω_k. The Compactness Theorem proven by Sharp [28] implies that, after passing to a subsequence, $\{\Sigma_i\}$ converges in the varifold sense to an integral stationary varifold Σ with support a minimal embedded hypersurface with index $\leq k$ and $||\Sigma||(M) = \omega_k$.

One-parameter case 1.6. Next we assume that X is the interval $[0, 1]$, for simplicity. We further assume that the continuous maps satisfy $\Phi(0) = \Phi(1) = 0$ and the homotopy class is relative to $\partial[0, 1] = \{0, 1\}$. The case $X = S^1$ is also considered in the paper.

The metric g is called *bumpy* if there is no smooth immersed minimal hypersurface with a non-trivial Jacobi field. White showed in [32, 33] that bumpy metrics are generic in the Baire sense.

The next theorem establishes the instability of the min-max minimal hypersurface, and proves that the multiplicity of an unstable two-sided component must be exactly equal to one. The proof uses the Deformation Theorems B and C and is left to Section 6.

Theorem 1.7. *Suppose (M, g) is a bumpy metric, and $X = [0, 1]$. For any min-max sequence $\{\Phi_i\}_{i \in \mathbb{N}}$ in Π, there exists a subsequence $i_j \to \infty$ such that, for some $x_{i_j} \in X$, $\Phi_{i_j}(x_{i_j})$ converges in varifold sense to an integral stationary varifold $\Sigma \in \mathcal{V}_n(M)$ with $||\Sigma||(M) = \mathbf{L}(\Pi)$ and satisfying:*

   (a) *the support of Σ is a smooth closed embedded minimal hypersurface in M;*

and either

(b1) *index (support of Σ) = 1, and the unstable component of Σ has multiplicity one;*

or

(b2) *index (support of Σ) = 0, and at least one component of Σ is one-sided, has even multiplicity and unstable double cover.*

The theorem has the following corollary:

Corollary 1.8. *Suppose (M, g) contains no one-sided embedded hypersurfaces, and $X = [0, 1]$. There exists an integral stationary varifold $\Sigma \in \mathcal{V}_n(M)$ with $||\Sigma||(M) = \mathbf{L}(\Pi)$ and satisfying the following properties:*

(a) *the support of Σ is a smooth closed embedded minimal hypersurface in M;*

(b) *we have*

$$index(support\, of\, \Sigma) \leq 1$$
$$\leq index(support\, of\, \Sigma) + nullity(support\, of\, \Sigma);$$

(c) *any unstable component (necessarily unique) must have multiplicity one;*

(d) *if the metric g is bumpy and $\{\Phi_i\}_{i \in \mathbb{N}}$ is a min-max sequence, Σ can be chosen so that for some subsequence $i_j \to \infty$ there is some $x_{i_j} \in X$ such that $\Phi_{i_j}(x_{i_j})$ converges in varifold sense to Σ.*

Notice that the topological assumption being required in Corollary 1.8 is automatically satisfied if $H_{n-1}(M, \mathbb{Z}_2) = 0$.

This paper is organized as follows. In Section 2, we set up some basic notation for the rest of the article. In Section 3, we describe how to formulate a min-max theory that makes sense in a continuous setting. In Section 4, we define instability of stationary varifolds and derive some consequences. In Section 5, we prove Deformation Theorems A, B and C which are the fundamental tools in our analysis. In Section 6, we prove the main theorems on the Morse index and multiplicity of min-max minimal hypersurfaces.

2. Basic notation

Let (M^{n+1}, g) be an $(n + 1)$-dimensional closed Riemannian manifold. We assume, for convenience, that (M, g) is isometrically embedded in some Euclidean space $\mathbb{R}^L$.

Let G be either the group $\mathbb{Z}$ or $\mathbb{Z}_2$. The spaces we will work with in this paper are:

- the space $\mathbf{I}_n(M; G)$ of n-dimensional flat chains in $\mathbb{R}^L$ with coefficients in G and support contained in M;
- the space $\mathcal{Z}_n(M; G)$ of flat chains $T \in \mathbf{I}_n(M; G)$ with $\partial T = 0$. These are called *flat cycles*;
- the space $\mathcal{Z}_n(U; G)$ of flat cycles with support contained in the open set U;
- the closure $\mathcal{V}_n(M)$, in the weak topology, of the space of n-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support contained in M.

Given $T \in \mathbf{I}_n(M; G)$, we denote by $|T|$ and $||T||$ the integral varifold and the Radon measure in M associated with $|T|$, respectively; given $V \in \mathcal{V}_n(M)$, $||V||$ denotes the Radon measure in M associated with V.

The above spaces come with several relevant metrics. The *flat metric* and the *mass* of $T \in \mathbf{I}_n(M; G)$, denoted by $\mathcal{F}(T)$ and $\mathbf{M}(T)$, are defined in [13], respectively. The $\mathbf{F}$-*metric* is defined in Pitts book [25, page 66] and induces the varifold weak topology on $\mathcal{V}_n(M) \cap \{V : ||V||(M) \leq a\}$ for any a. We denote by $\overline{\mathbf{B}}_\delta^{\mathbf{F}}(V)$ and $\mathbf{B}_\delta^{\mathbf{F}}(V)$ the closed and open metric balls, respectively, with radius δ and center $V \in \mathcal{V}_n(M)$. Similarly, we denote by $\overline{\mathbf{B}}_\delta^{\mathcal{F}}(T)$ and $\mathbf{B}_\delta^{\mathcal{F}}(T)$ the corresponding balls with center $T \in \mathcal{Z}_n(M; G)$ in the flat metric. Finally, the $\mathbf{F}$-*metric* on $\mathbf{I}_n(M; G)$ is defined by

$$\mathbf{F}(S, T) = \mathcal{F}(S - T) + \mathbf{F}(|S|, |T|).$$

We assume that $\mathbf{I}_n(M; G)$ and $\mathcal{Z}_n(M; G)$ have the topology induced by the flat metric. When endowed with the topology of the $\mathbf{F}$-metric or the mass norm, these spaces will be denoted by $\mathbf{I}_n(M; \mathbf{F}; G)$, $\mathcal{Z}_n(M; \mathbf{F}; G)$, $\mathbf{I}_n(M; \mathbf{M}; G)$, $\mathcal{Z}_n(M; \mathbf{M}; G)$, respectively. The space $\mathcal{V}_n(M)$ is considered with the weak topology of varifolds. Given $\mathcal{A}, \mathcal{B} \subset \mathcal{V}_n(M)$, we also define

$$\mathbf{F}(\mathcal{A}, \mathcal{B}) = \inf\{\mathbf{F}(V, W) : V \in \mathcal{A}, W \in \mathcal{B}\}.$$

We take X to be a cubical complex of dimension k, i.e. a subcomplex of dimension k of the m-dimensional cube $I^m = [0, 1]^m$ for some m. Each p-cell of I^m is of the form $\alpha_1 \otimes \cdots \otimes \alpha_m$, where $\alpha_i \in \{0, 1, [0, 1]\}$ for every i and $\sum \dim(\alpha_i) = p$. By Construction 4.8 of [7], any k-dimensional simplicial complex is homeomorphic to a cubical complex of dimension k. If Y is a subcomplex of X, we denote by Y_0 the set of vertices of Y.

Throughout this paper we will make use of results contained in [20] and [21]. In Section 2 of [21] we described modifications to the Almgren-Pitts Min-Max Theory so that the k-dimensional cube I^k is replaced by X as the parameter space. We refer the reader to Section 2 of [21] for more details of the Almgren-Pitts discrete formulation of min-max theory. In that formulation, continuous maps Φ defined on X are replaced by sequences of maps ϕ_i defined on the vertices of finer and finer subdivisions of X, so that $M(\phi_i(x) - \phi_i(y))$ becomes arbitrarily small whenever x, y are adjacent vertices as $i \to \infty$.

3. Min-max theory in continuous setting

As mentioned in the last section, the Almgren-Pitts min-max theory is formulated in terms of sequences of maps defined on the vertices of finer and finer grids. This requires quite a bit of notation and a lot of technical work. Besides, one works with the mass topology ([25]) which is rather unusual and very strong. The families that appear naturally in applications tend to be defined on the full parameter space X and are continuous with respect to either the flat or the $\mathbf{F}$ topologies. The use of discretizations for families in $\mathcal{Z}_n(M; G)$ seems to be essential in order to prove regularity, but it can be restricted to the proofs so that a more reasonable continuous theory can be formulated. This is what we accomplish in this section.

Let $\Phi : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ be a continuous map. We let Π be the class of all continuous maps $\Phi' : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ such that Φ and Φ' are homotopic to each other in the flat topology.

Definition 3.1. The *width* of Π is defined by:

$$\mathbf{L}(\Pi) = \inf_{\Phi \in \Pi} \sup_{x \in X} \{\mathbf{M}(\Phi(x))\}.$$

If $X = I^k$, we can also restrict to maps that are zero at ∂I^k and homotopies are taken relative to ∂I^k.

Example 3.2. Let $f : M \to \mathbb{R}$ be a Morse function with $f(M) = [0, 1]$ and set $\Phi(t) = \partial (\{x \in M : f(x) < t\})$, $t \in [0, 1]$. If M is non-orientable we must necessarily choose $G = \mathbb{Z}_2$, otherwise we have both possibilities $G = \mathbb{Z}$ or $G = \mathbb{Z}_2$. Then $\Phi : I \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ is continuous, with $\Phi(0) = \Phi(1) = 0$. We denote by Π_1 the class of all maps $\Phi' : I \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ such that $\Phi'(0) = \Phi'(1) = 0$ and such that Φ' is homotopic to Φ in the flat topology relative to $\partial I = \{0, 1\}$. Almgren's work [4] (see also Section 3 of [21]) implies $\mathbf{L}(\Pi) > 0$.

Definition 3.3. A sequence $\{\Phi_i\}_i \subset \Pi$ is called a *min-max sequence* if

$$\mathbf{L}(\Phi_i) := \sup_{x \in X} \mathbf{M}(\Phi_i(x))$$

satisfies $\mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}}) := \limsup_{i \to \infty} \mathbf{L}(\Phi_i) = \mathbf{L}(\Pi)$.

Definition 3.4. The *image set* of $\{\Phi_i\}_{i \in \mathbb{N}}$ is defined by

$$\begin{aligned}\mathbf{\Lambda}(\{\Phi_i\}_{i \in \mathbb{N}}) \quad = \quad &\{V \in \mathcal{V}_n(M) : \exists \, \text{sequences} \, \{i_j\} \to \infty, x_{i_j} \in X \\ &\text{such that} \lim_{j \to \infty} \mathbf{F}(|\Phi_{i_j}(x_{i_j})|, V) = 0\}.\end{aligned}$$

Definition 3.5. If $\{\Phi_i\}_{i \in \mathbb{N}}$ is a min-max sequence in Π such that $L = \mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}})$, the *critical set* of $\{\Phi_i\}_{i \in \mathbb{N}}$ is defined by

$$\mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}}) = \{V \in \mathbf{\Lambda}(\{\Phi_i\}) : ||V||(M) = L\}.$$

Pull-tight 3.6. Following Pitts ([25] p. 153, see also [11, 20]), we can define a continuous map

$$\begin{aligned}H : I \times (\mathcal{Z}_n(M^{n+1}; \mathbf{F}; G) \quad \cap \quad &\{T : \mathbf{M}(T) \le 2\mathbf{L}(\Pi)\}) \\ \to \quad &(\mathcal{Z}_n(M^{n+1}; \mathbf{F}; G) \cap \{T : \mathbf{M}(T) \le 2\mathbf{L}(\Pi)\})\end{aligned}$$

such that:

- $H(0, T) = T$ for all T;
- $H(t, T) = T$ for all $t \in [0, 1]$ if $|T|$ is stationary;
- $\mathbf{M}(H(1, T)) < \mathbf{M}(T)$ if $|T|$ is not stationary.

Given a min-max sequence $\{\Phi_i^*\} \subset \Pi$, we define $\Phi_i(x) = H(1, \Phi_i^*(x))$ for every $x \in X$. Then $\{\Phi_i\} \subset \Pi$ is also a min-max sequence. It follows from the construction that $\mathbf{C}(\{\Phi_i\}) \subset \mathbf{C}(\{\Phi_i^*\})$ and that every element of $\mathbf{C}(\{\Phi_i\})$ is stationary.

Definition 3.7. Any min-max sequence $\{\Phi_i\} \subset \Pi$ such that every element of $\mathbf{C}(\{\Phi_i\})$ is stationary is called *pulled-tight*.

Min-max Theorem 3.8. *Suppose* $\mathbf{L}(\Pi) > 0$, *and let* $\{\Phi_i\}_i$ *be a min-max sequence for* Π. *Then there exists a stationary integral varifold* $V \in \mathbf{C}(\{\Phi_i\})$ *(hence* $||V||(M) = \mathbf{L}(\Pi)$*), with support a closed minimal hypersurface that is smooth embedded outside a set of dimension* $n - 7$.

Remark 3.9. A slightly different version of this theorem was proven in Theorem 2.4 of [22].

Proof. It follows from the pull-tight procedure that we can assume every element of $\mathbf{C}(\{\Phi_i\})$ is stationary. Given $\Phi_i : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$, the continuity in the $\mathbf{F}$-metric and a compactness argument imply there is no concentration of mass:

$$\lim_{r \to 0} \sup_{x \in X, p \in M} \|\Phi_i(x)\|(B_r(p)) = 0,$$

where $B_r(p)$ denotes the open geodesic ball of radius r and center $p \in M$. Theorem 3.9 of [21] (the analogous statement holds for $G = \mathbb{Z}$) implies there exist a sequence of maps

$$\phi_i^j : X(k_j^i)_0 \to \mathcal{Z}_n(M; G),$$

with $k_j^i < k_{j+1}^i$, and a sequence of positive numbers $\{\delta_j^i\}_{j \in \mathbb{N}}$ converging to zero such that

(i)
$$S_i = \{\phi_i^j\}_{j \in \mathbb{N}}$$

   is an $(X, \mathbf{M})$-homotopy sequence of mappings into $\mathcal{Z}_n(M; \mathbf{M}; G)$ with $\mathbf{f}(\phi_i^j) < \delta_j^i$;

(ii)
$$\sup\{\mathcal{F}(\phi_i^j(x) - \Phi_i(x)) : x \in X(k_j^i)_0\} \leq \delta_j^i;$$

(iii)
$$\sup\{\mathbf{M}(\phi_i^j(x)) : x \in X(k_j^i)_0\} \leq \sup\{\mathbf{M}(\Phi_i(x)) : x \in X\} + \delta_j^i.$$

We can also guarantee that (see Theorem 13.1 (i) of [20] for the case $X = I^m$)

(iv) for some $l_j^i \to \infty$ and every $y \in X(k_j^i)_0$,

$$\mathbf{M}(\phi_i^j(y)) \leq \sup\{\mathbf{M}(\Phi_i(x)) : \alpha \in X(l_j^i), x, y \in \alpha\} + \delta_j^i.$$

Because $x \in X \mapsto \mathbf{M}(\Phi_i(x))$ is continuous, we get from property (iv) above that for every $y \in X(k_j^i)_0$,

$$\mathbf{M}(\phi_i^j(y)) \leq \mathbf{M}(\Phi_i(y)) + \eta_j^i$$

with $\eta_j^i \to 0$ as $j \to \infty$. If we apply Lemma 4.1 of [20] with $\mathcal{S} = \Phi_i(X)$, property [(ii)] above implies

$$\sup\{\mathbf{F}(\phi_i^j(x), \Phi_i(x)) : x \in X(k_j^i)_0\} \to 0$$

as $j \to \infty$.

We can choose $j(i) \to \infty$ as $i \to \infty$ (note that necessarily $k^i_{j(i)} \to \infty$) such that $\varphi_i = \phi^{j(i)}_i : X(k^i_{j(i)})_0 \to \mathcal{Z}_n(M; G)$ satisfies:

- $\sup\{\mathbf{F}(\varphi_i(x), \Phi_i(x)) : x \in X(k^i_{j(i)})_0\} \le a_i$ with $\lim_{i \to \infty} a_i = 0$;
- $\sup\{\mathbf{F}(\Phi_i(x), \Phi_i(y)) : x, y \in \alpha, \alpha \in X(k^i_{j(i)})\} \le a_i$;
- the fineness $\mathbf{f}(\varphi_i)$ tends to zero as $i \to \infty$;
- the Almgren extension $\Phi^{j(i)}_i : X \to \mathcal{Z}_n(M; \mathbf{M}; G)$ is homotopic to Φ_i in the flat topology (by Corollary 3.12 (ii) of [21] and its analogue for $G = \mathbb{Z}$).

Following the notation of Subsection 2.12 of [21], if $S = \{\varphi_i\}_i$ we have $\mathbf{L}(S) = \mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}})$ and $\mathbf{C}(S) = \mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}})$. In particular, every element of $\mathbf{C}(S)$ is stationary.

By Theorem 2.13 of [21] (and its analogue for $G = \mathbb{Z}$), if no element $V \in \mathbf{C}(S)$ is G-almost minimizing in annuli we can find a sequence $S^* = \{\varphi^*_i\}$ of maps

$$\varphi^*_i : X(k^i_{j(i)} + l_i)_0 \to \mathcal{Z}_n(M; G)$$

such that

- φ_i and φ^*_i are homotopic to each other with finesses tending to zero as $i \to \infty$;
- $\mathbf{L}(S^*) < \mathbf{L}(S)$.

By Corollary 3.12 (i) of [21] (and its analogue for $G = \mathbb{Z}$), the Almgren extensions of φ_i, φ^*_i:

$$\Phi^{j(i)}_i, \Phi^*_i : X \to \mathcal{Z}_n(M; \mathbf{M}; G),$$

respectively, are homotopic to each other in the flat topology for sufficiently large i. Hence Φ^*_i is homotopic to Φ_i in the flat topology and so $\Phi^*_i \in \Pi$ for sufficiently large i. It follows from Theorem 3.10 of [21] (and its analogue for $G = \mathbb{Z}$) that

$$\limsup_{i \to \infty} \sup\{\mathbf{M}(\Phi^*_i(x) : x \in X\} \le \mathbf{L}(S^*) < \mathbf{L}(S) = \mathbf{L}(\Pi).$$

This is a contradiction, hence some element $V \in \mathbf{C}(S)$ is G-almost minimizing in annuli. Since $\mathbf{C}(S) = \mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}})$, the theorem follows by the regularity theory of Pitts [25, Theorem 3.13 and Section 7] and Schoen-Simon [27, Theorem 4] (see also Theorem 2.11 of [21]). □

The following remark will be useful in Section 6 where we handle the case in which the min-max hypersurfaces are one-sided.

Remark 3.10. The proof actually gives more. It was observed by X. Zhou [35] that Pitts contradiction arguments (Theorem 4.10 of [25]) imply that some element V of $\mathbf{C}(S) = \mathbf{C}(\{\Phi_i\}_{i\in\mathbb{N}})$ must be G-almost minimizing in annuli in the following stronger sense (compared to Pitts notion): for every $p \in M$ there exists $r > 0$ so that for any $0 < s < r$ and any $\varepsilon > 0$ there exists $T \in \mathcal{Z}_n(M;G)$ and $\delta > 0$ such that $T \in \mathfrak{A}_n(B_r(p) \setminus \bar{B}_s(p), \varepsilon, \delta)$ (Definition 6.3 of [35], Definition 3.1 of [25]) and $\mathbf{F}(V, |T|) < \varepsilon$. If $\Phi(x)$ is a boundary for any $\Phi \in \Pi$ and $x \in X$, we can further require that T is a boundary too, i.e., there exists $U \in \mathbf{I}_{n+1}(M;G)$ such that $T = \partial U$. In that case we say V is G-almost minimizing of *boundary type* in annuli.

4. Unstable variations and the Morse index

If Σ is a smooth embedded closed minimal hypersurface, its *Morse index* is the maximal dimension of a subspace of normal variations restricted to which the second variation of area quadratic form $Q(\cdot, \cdot)$ is negative definite. Here we have to consider the general case of stationary varifolds.

Definition 4.1. Let Σ be a stationary varifold in $\mathcal{V}_n(M)$ and $\varepsilon \geq 0$. We say that Σ is *k-unstable in an ε-neighborhood* if there exist $0 < c_0 < 1$ and a smooth family $\{F_v\}_{v\in\overline{B}^k} \subset \mathrm{Diff}(M)$ with $F_0 = \mathrm{Id}$, $F_{-v} = F_v^{-1}$ for all $v \in \overline{B}^k$ such that, for any $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma)$, the smooth function

$$A^V : \overline{B}^k \to [0, \infty), \qquad A^V(v) = ||(F_v)_\# V||(M)$$

satisfies:

- A^V has a unique maximum at $m(V) \in B^k_{c_0/\sqrt{10}}(0)$;
- $-\frac{1}{c_0} \mathrm{Id} \leq D^2 A^V(u) \leq -c_0 \mathrm{Id}$ for all $u \in \overline{B}^k$.

Here $(F_v)_\#$ denotes the push-forward operation. Also, because Σ is stationary, necessarily $m(\Sigma) = 0$.

The map $V \mapsto m(V)$ is continuous on $\overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma)$ and

$$(1) \quad A^V(m(V)) - \frac{1}{2c_0}|u - m(V)|^2 \leq A^V(u) \leq A^V(m(V)) - \frac{c_0}{2}|u - m(V)|^2$$

for all $u \in \overline{B}^k$.

If V_i tends to V in the $\mathbf{F}$-topology then A^{V_i} tends to A^V in the smooth topology [25, Section 2.3]. Thus if Σ is stationary and k-unstable in a 0-neighborhood then it is k-unstable in an ε-neighborhood for some $\varepsilon > 0$.

Definition 4.2. We say that Σ is *k-unstable* if it is stationary and k-unstable in an ε-neighborhood for some $\varepsilon > 0$.

Proposition 4.3. *If Σ is smooth embedded closed and minimal, then Σ is k-unstable if and only if* index$(\Sigma) \geq k$.

Proof. Denote by $Q(\cdot, \cdot)$ the second variation of area quadratic form, acting on normal vector fields along Σ. Suppose index$(\Sigma) \geq k$. Then there exist normal vector fields $X_1, \ldots, X_k$ along Σ such that

$$Q(\sum_i a_i X_i, \sum_i a_i X_i) < 0$$

for any $(a_i) \neq 0 \in \mathbb{R}^k$. These vector fields can be extended to ambient vector fields $X_1, \ldots, X_k$ in M. Denote by ϕ_t^X the flow of X and, for some δ to be chosen later, define

$$F_v(x) = F(v, x) = \phi_\delta^{\sum_i v_i X_i}(x)$$

for $v \in \overline{B}^k$ and $x \in M$. Then $F_0 = \mathrm{Id}$, $F_{-v} = F_v^{-1}$ for all $v \in \overline{B}^k$, $DA^\Sigma(0) = 0$. Moreover, since

$$F(tv, x) = \phi_\delta^{t \sum_i v_i X_i}(x) = \phi_t^{\delta \sum_i v_i X_i}(x),$$

we have

$$\frac{\partial}{\partial t}_{|t=0} F(tv, x) = \delta \sum_i v_i X_i(x).$$

Hence $D^2 A^\Sigma(0) < 0$. By choosing δ sufficiently small, we conclude that Σ is k-unstable.

Now suppose Σ is k-unstable, and let $\{F_v\}_{v \in \overline{B}^k}$ be the corresponding family of diffeomorphisms. Define $Y_i(x) = \frac{\partial}{\partial t}_{|t=0} F(te_i, x) = \frac{\partial F}{\partial v_i}(0, x)$. Then

$$0 > \frac{d^2}{dt^2}_{|t=0} \mathrm{area}(F_{tv}(\Sigma)) = Q(\sum_i v_i Y_i^\perp, \sum_i v_i Y_i^\perp),$$

where $X_i = Y_i^\perp$ is the normal component of Y_i along Σ. This means $\{X_1, \ldots, X_k\}$ is linearly independent and spans a subspace of normal vector fields restricted to which Q is negative definite. Therefore index$(\Sigma) \geq k$. $\square$

In what follows we assume that Σ is a stationary n-varifold that is k-unstable in an ε-neighborhood, $\varepsilon > 0$, and derive two basic lemmas. The family $\{F_v\}_{v \in \overline{B}^k}$ is as in Definition 4.1.

Lemma 4.4. *There exist* $\bar{\eta} = \bar{\eta}(\varepsilon, \Sigma, \{F_v\}) > 0$, *so that if* $V \in \mathcal{V}_n(M)$ *with* $\mathbf{F}(V, \Sigma) \geq \varepsilon$ *satisfies*

$$||(F_v)_\# V||(M) \leq ||V||(M) + \bar{\eta}$$

for some $v \in \overline{B}^k$, *then* $\mathbf{F}((F_v)_\# V, \Sigma) \geq 2\bar{\eta}$.

Proof. We argue by contradiction and assume that there is a sequence $V_i \in \mathcal{V}_n(M)$ with $\mathbf{F}(V_i, \Sigma) \geq \varepsilon$,

$$||(F_{v_i})_\# V_i||(M) \leq ||V_i||(M) + \frac{1}{i}$$

for some $v_i \in \overline{B}^k$, and $(F_{v_i})_\# V_i$ tending to Σ in the $\mathbf{F}$-topology. After passing to a subsequence we can assume that v_i tends to $v \in \overline{B}^k$. Hence V_i converges to $(F_{-v})_\# \Sigma$ and $||\Sigma||(M) \leq ||(F_{-v})_\# \Sigma||(M)$, which forces $v = 0$. Thus V_i tends to Σ, which is impossible because $\mathbf{F}(V_i, \Sigma) \geq \varepsilon$ for all $i \in \mathbb{N}$. $\qquad\square$

For each $V \in \overline{\mathbf{B}}_{2\varepsilon}^{\mathbf{F}}(\Sigma)$, consider the one-parameter flow $\{\phi^V(\cdot, t)\}_{t \geq 0} \subset \mathrm{Diff}(\overline{B}^k)$ generated by the vector field

$$u \mapsto -(1 - |u|^2)\nabla A^V(u), \quad u \in \overline{B}^k.$$

With $u \in \overline{B}^k$ fixed we have that $t \mapsto A^V(\phi^V(u, t))$ is non-increasing.

Lemma 4.5. *For all* $\delta < 1/4$ *there is* $T = T(\delta, \varepsilon, \Sigma, \{F_v\}, c_0) \geq 0$ *so that for any* $V \in \overline{\mathbf{B}}_{2\varepsilon}^{\mathbf{F}}(\Sigma)$ *and* $v \in \overline{B}^k$ *with* $|v - m(V)| \geq \delta$ *we have*

$$A^V(\phi^V(v, T)) < A^V(0) - \frac{c_0}{10} \quad and \quad |\phi^V(v, T)| > \frac{c_0}{4}.$$

Proof. Because $m(V) \in B_{c_0/\sqrt{10}}^k(0)$, we have by inequality (1) that

$$\sup_{|u| \leq 1} A^V(u) = A^V(m(V)) \leq A^V(0) + \frac{c_0}{20}.$$

So, to prove the first inequality in Lemma 4.5 it suffices to show the existence of T so that

$$(2) \quad V \in \overline{\mathbf{B}}_{2\varepsilon}^{\mathbf{F}}(\Sigma), |v - m(V)| \geq \delta \implies A^V(\phi^V(v, T)) < \sup_{|u| \leq 1} A^V(u) - \frac{c_0}{5}.$$

We argue by contradiction and assume that for all $i \in \mathbb{N}$ there exist $v_i \in \overline{B}^k$ and $V_i \in \overline{\mathbf{B}}_{2\varepsilon}^{\mathbf{F}}(\Sigma)$ so that $|v_i - m(V_i)| \geq \delta$ but

$$A^{V_i}(\phi^{V_i}(v_i, i)) \geq \sup_{|u| \leq 1} A^{V_i}(u) - \frac{c_0}{5}.$$

This inequality implies

$$A^{V_i}(\phi^{V_i}(v_i, t)) \geq \sup_{|u| \leq 1} A^{V_i}(u) - \frac{c_0}{5} \quad \text{for all } 0 \leq t \leq i.$$

Since both $\overline{\mathbf{B}}_{2\varepsilon}^{\mathbf{F}}(\Sigma)$ and $\overline{B}^k$ are compact, we obtain subsequential limits $V \in \overline{\mathbf{B}}_{2\varepsilon}^{\mathbf{F}}(\Sigma)$ and $v \in \overline{B}^k \setminus B_\delta^k(m(V))$ so that

$$A^V(\phi^V(v, t)) \geq \sup_{|u| \leq 1} A^V(u) - \frac{c_0}{5} \quad \text{for all } t \geq 0.$$

Because $|v - m(V)| \geq \delta$, we have $\lim_{t\to\infty} |\phi^V(v, t)| = 1$ and thus we deduce from the inequality above that

$$\sup_{|u|=1} A^V(u) \geq \sup_{|u| \leq 1} A^V(u) - \frac{c_0}{5}.$$

On the other hand, $|u - m(V)| > 2/3$ for all $u \in \overline{B}^k$ with $|u| = 1$. Hence, by inequality (1),

$$\sup_{|u|=1} A^V(u) \leq \sup_{|u| \leq 1} A^V(u) - \frac{c_0}{2}\left(\frac{2}{3}\right)^2 < \sup_{|u| \leq 1} A^V(u) - \frac{c_0}{5},$$

which gives us the desired contradiction.

We now show the second inequality of Lemma 4.5. If $w \in \overline{B}^k$ has $|w| \leq c_0/4$, then $|w - m(V)| \leq c_0(1/4 + 1/\sqrt{10}) \leq c_0(1/4 + 1/3)$ and so, using inequality (1) again,

$$A^V(w) \geq \sup_{|u| \leq 1} A^V(u) - \frac{c_0}{2}\left(\frac{1}{4} + \frac{1}{3}\right)^2 > \sup_{|u| \leq 1} A^V(u) - \frac{c_0}{5}.$$

Combining this inequality with (2) we obtain the desired statement. $\qquad\square$

5. Deformation theorems

Let $\{\Phi_i\}_{i\in\mathbb{N}}$ be a sequence of continuous maps from X into $\mathcal{Z}_n(M; \mathbf{F}; G)$. We set

$$L = \mathbf{L}(\{\Phi_i\}_{i\in\mathbb{N}}) := \limsup_{i\to\infty} \sup_{x\in X} \mathbf{M}(\Phi_i(x)).$$

If $X = I^k$, we can also consider maps that are zero at ∂I^k with homotopies relative to ∂I^k.

In this section we will prove some deformation theorems that will be fundamental in the proof of the index estimates. The basic idea is to produce homotopic families $\{\Psi_i\}_{i \in \mathbb{N}}$ to $\{\Phi_i\}_{i \in \mathbb{N}}$ that do not increase the area and avoid minimal hypersurfaces with large index or multiplicity.

Deformation Theorem A 5.1. *Suppose that*

  (a) $\Sigma \in \mathcal{V}_n(M)$ *is stationary and* $(k + 1)$-*unstable;*
  (b) $K \subset \mathcal{V}_n(M)$ *is a compact set so that* $\Sigma \notin K$ *and* $|\Phi_i|(X) \cap K = \emptyset$ *for all* $i \geq i_0$;
  (c) $\|\Sigma\|(M) = L$.

There exist $\bar{\varepsilon} > 0$, $j_0 \in \mathbb{N}$, *and another sequence* $\{\Psi_i\}_{i \in \mathbb{N}}$ *so that*

  (i) Ψ_i *is homotopic to* Φ_i *in the* **F**-*topology for all* $i \in \mathbb{N}$;
  (ii) $\mathbf{L}(\{\Psi_i\}_{i \in \mathbb{N}}) \leq L$;
  (iii) for all $i \geq j_0$, $|\Psi_i|(X) \cap (\overline{\mathbf{B}}_{\bar{\varepsilon}}^{\mathbf{F}}(\Sigma) \cup K) = \emptyset$.

Proof. Set $d = \min\{\mathbf{F}(\Sigma, Z) : Z \in K\} > 0$.

By assumption, Σ satisfies the conditions of Definition 4.1 with $(k + 1)$ in place of k for some $\varepsilon > 0$. Let $\{F_v\}_{v \in \overline{B}^{k+1}}$ and c_0 be the corresponding family and constant. Without loss of generality (by changing ε, $\{F_v\}$, c_0) we can assume that

$$(3) \quad \min\{\mathbf{F}((F_v)_\# V, Z) : Z \in K, v \in \overline{B}^{k+1}\} > d/2 \quad \text{for all } V \in \overline{\mathbf{B}}_{2\varepsilon}^{\mathbf{F}}(\Sigma).$$

Let $X(k_i)$ be a sufficiently fine subdivision of X (see Section 2 of [21]) so that $\mathbf{F}(|\Phi_i(x)|, |\Phi_i(y)|) < \delta_i$ for any x, y belonging to the same cell in $X(k_i)$, with $\delta_i = \min\{2^{-(i+k+2)}, \varepsilon/4\}$. We can also assume

$$|m(|\Phi_i(x)|) - m(|\Phi_i(y)|)| < \delta_i$$

for any x, y with $\mathbf{F}(|\Phi_i(x)|, \Sigma) \leq 2\varepsilon$, $\mathbf{F}(|\Phi_i(y)|, \Sigma) \leq 2\varepsilon$, and belonging to the same cell in $X(k_i)$. For $\eta > 0$, let $U_{i,\eta}$ be the union of all cells $\sigma \in X(k_i)$ so that $\mathbf{F}(|\Phi_i(x)|, \Sigma) < \eta$ for all $x \in \sigma$. Then $U_{i,\eta}$ is a subcomplex of $X(k_i)$. If $\beta \notin U_{i,\eta}$, then $\mathbf{F}(|\Phi_i(x')|, \Sigma) \geq \eta$ for some $x' \in \beta$. Consequently, $\mathbf{F}(|\Phi_i(x)|, \Sigma) \geq \eta - \delta_i$ for all $x \in \beta$. For each $i \in \mathbb{N}$ and $x \in U_{i,2\varepsilon}$ we use the notation $A_i^x = A^{|\Phi_i(x)|}$, $m_i(x) = m(|\Phi_i(x)|)$ and $\phi_i^x = \phi^{|\Phi_i(x)|}$.

The function $m_i : U_{i,2\varepsilon} \to \overline{B}^{k+1}$ is continuous. We would like to define a continuous homotopy

$$\hat{H}_i : U_{i,2\varepsilon} \times [0,1] \to B_{1/2^i}^{k+1}(0) \quad \text{so that } \hat{H}_i(x,0) = 0 \text{ for all } x \in U_{i,2\varepsilon}$$

and

$$(4) \qquad \inf_{x \in U_{i,2\varepsilon}} |m_i(x) - \hat{H}_i(x,1)| \geq \eta_i > 0$$

for some $\eta_i > 0$. Hypothesis (a) is crucial for the construction of $\hat{H}_i$ and, roughly speaking, the motivation is the following: the subspaces $A_i = \{(x,0) : x \in U_{i,2\varepsilon}\}$ and $B_i = \{(x, m_i(x)) : x \in U_{i,2\varepsilon}\}$ have both dimension k and are contained in a space of dimension $2k + 1$. Hence one should be able to perturb A_i slightly so that it does not intersect B_i and thus show (4). We now proceed to describe the construction of $\hat{H}_i$ carefully.

Because $B^{k+1}_{1/2^i}(0)$ is convex it is enough to define $a_i(x) = \hat{H}_i(x,1)$. We will write $a_i(x) = m_i(x) + b_i(x)$ and define $b_i(x)$ for $x \in U_{i,2\varepsilon}$. Hence we need $b_i(x) \neq 0$ and $b_i(x) \in B^{k+1}_{1/2^i}(-m_i(x))$ for all $x \in U_{i,2\varepsilon}$.

We will define the function b_i inductively on the skeletons of $U_{i,2\varepsilon}$. Let $U^{(j)}_{i,2\varepsilon}$ denote the j-dimensional skeleton, and recall that X is k-dimensional. For every $x \in U^{(0)}_{i,2\varepsilon}$ it is easy to choose $b_i(x) \neq 0$ and $b_i(x) \in B^{k+1}_{2^{-(i+k)}}(-m_i(x))$. Suppose we have defined b_i on $U^{(j)}_{i,2\varepsilon}$ so that $b_i(x) \neq 0$ and

$$b_i(x) \in B^{k+1}_{2^{-(i+k-j)}}(-m_i(x)) \text{ for all } x \in U^{(j)}_{i,2\varepsilon}.$$

Let $\sigma \in U^{(j+1)}_{i,2\varepsilon}$, and let x_σ be its center. Then $b_i(x) \in B^{k+1}_{2^{-(i+k-j)}+\delta_i}(-m_i(x_\sigma))$ for every $x \in \partial\sigma$. Because $\dim(\partial\sigma) \leq j \leq (k-1)$ and $\pi_j(B^{k+1} \setminus \{0\}) = 0$, we can find a continuous extension of b_i to σ so that $b_i(x) \neq 0$ and $b_i(x) \in B^{k+1}_{2^{-(i+k-j)}+\delta_i}(-m_i(x_\sigma))$ for every $x \in \sigma$. Then $b_i(x) \in B^{k+1}_{2^{-(i+k-j)}+2\delta_i}(-m_i(x))$ for every $x \in \sigma$. But $2^{-(i+k-j)} + 2\delta_i < 2^{-(i+k-j-1)}$, completing the induction step.

Let $c : [0, \infty] \to [0, 1]$ be a cutoff function which is non-increasing, one in a neighborhood of $[0, 3\varepsilon/2]$, and zero in a neighborhood of $[7\varepsilon/4, +\infty)$. Note that if $y \notin U_{i,2\varepsilon}$, then $\mathbf{F}(|\Phi_i(y)|, \Sigma) \geq 2\varepsilon - \delta_i \geq 7\varepsilon/4$. Hence

$$c(\mathbf{F}(|\Phi_i(y)|, \Sigma)) = 0 \text{ for all } y \notin U_{i,2\varepsilon}.$$

Consider $H_i : X \times [0, 1] \to B^{k+1}_{2^{-i}}(0)$ given by

$$H_i(x,t) = \hat{H}_i(x, c(\mathbf{F}(|\Phi_i(x)|, \Sigma))t) \text{ if } x \in U_{i,2\varepsilon}$$

and

$$H_i(x,t) = 0 \quad \text{if } x \in X \setminus U_{i,2\varepsilon}.$$

The map H_i is continuous.

With η_i given by (4), consider $T_i = T(\eta_i, \varepsilon, \Sigma, \{F_v\}, c_0) \geq 0$ given by Lemma 4.5. We now set

$$D_i : X \to \overline{B}^{k+1}, \quad D_i(x) = \phi_i^x(H_i(x,1), c(\mathbf{F}(|\Phi_i(x)|, \Sigma))T_i) \text{ if } x \in U_{i,2\varepsilon}$$

and

$$D_i(x) = 0 \quad \text{if } x \in X \setminus U_{i,2\varepsilon}.$$

The map D_i is continuous. Define

$$\Psi_i : X \to \mathcal{Z}_n(M; \mathbf{F}; G), \quad \Psi_i(x) = (F_{D_i(x)})_\#(\Phi_i(x)).$$

In particular,

$$\Psi_i(x) = \Phi_i(x) \quad \text{if } x \in X \setminus U_{i,2\varepsilon}.$$

The map D_i is homotopic to the zero map in $\overline{B}^{k+1}$, so Ψ_i is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$.

Claim 1. $\mathbf{L}(\{\Psi_i\}_{i\in\mathbb{N}}) \leq L$.

From the mass non-increasing property of ϕ_i^x we have that for all $x \in X$

$$||\Psi_i(x)||(M) \leq ||(F_{H_i(x,1)})_\#(\Phi_i(x))||(M).$$

Note that $||F_v - \mathrm{Id}||_{C^2}$ tends to zero uniformly as $v \in B^{k+1}$ tends to zero. Thus, using the fact that $H_i(x,1) \subset B^{k+1}_{1/2^i}(0)$ for all $x \in X$, we have that

$$(5) \qquad \lim_{i\to\infty} \sup_{x\in X} \big| \, ||\Phi_i(x)||(M) - ||(F_{H_i(x,1)})_\#(\Phi_i(x))||(M) \, \big| = 0$$

and this finishes the proof.

Claim 2. *There exists $\bar{\varepsilon} > 0$ so that, for all sufficiently large i, $|\Psi_i|(X) \cap \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}}(\Sigma) = \emptyset$.*

There are three cases to consider. If $x \in X \setminus U_{i,2\varepsilon}$, then $\Psi_i(x) = \Phi_i(x)$ and so $\mathbf{F}(|\Psi_i(x)|, \Sigma) \geq 7\varepsilon/4$.

If $x \in U_{i,2\varepsilon} \setminus U_{i,5\varepsilon/4}$, then $\mathbf{F}(|\Phi_i(x)|, \Sigma) \geq \varepsilon$. The mass non-increasing property of ϕ_i^x implies

$$||\Psi_i(x)||(M) = ||(F_{D_i(x)})_\#(\Phi_i(x))||(M) \leq ||(F_{H_i(x,1)})_\#(\Phi_i(x))||(M).$$

From (5) we have that for all i sufficiently large

$$||(F_{H_i(y,1)})_\#(\Phi_i(y))||(M) \leq ||\Phi_i(y)||(M) + \bar{\eta} \quad \text{for all } y \in X,$$

where $\bar{\eta} = \bar{\eta}(\varepsilon, \Sigma, \{F_v\}) > 0$ is given by Lemma 4.4. Combining the two inequalities above with Lemma 4.4 applied to $V = |\Phi_i(x)|$ and $v = D_i(x)$ we have that $\mathbf{F}(|\Psi_i(x)|, \Sigma) \geq 2\bar{\eta}$.

The last case to consider is when $x \in U_{i,5\varepsilon/4}$. Then $c(\mathbf{F}(|\Phi_i(x)|, \Sigma)) = 1$. Note that there exists $\bar{\gamma} = \bar{\gamma}(\Sigma, c_0) > 0$ so that

$$(6) \qquad ||V||(M) \leq ||\Sigma||(M) - \frac{c_0}{20} \implies \mathbf{F}(V, \Sigma) \geq 2\bar{\gamma}.$$

Choose i large enough so that

$$\sup_{x \in X} ||\Phi_i(x)||(M) \leq ||\Sigma||(M) + \frac{c_0}{20}.$$

From Lemma 4.5 (with $\delta = \eta_i$, $V = |\Phi_i(x)|$, $v = H_i(x, 1)$) we have

$$||\Psi_i(x)||(M) = A_i^x(\phi_i^x(H_i(x, 1), T_i)) < A_i^x(0) - \frac{c_0}{10} = ||\Phi_i(x)||(M) - \frac{c_0}{10}$$

and so

$$||\Psi_i(x)||(M) \leq ||\Sigma||(M) - \frac{c_0}{20}.$$

From (6) we obtain that $\mathbf{F}(|\Psi_i(x)|, \Sigma) \geq 2\bar{\gamma}$. This ends the proof of Claim 2.

Claim 3. *For all i, $|\Psi_i|(X) \cap K = \emptyset$.*

If $x \in X \setminus U_{i,2\varepsilon}$, then $|\Psi_i(x)| = |\Phi_i(x)| \notin K$. If $x \in U_{i,2\varepsilon}$, then $\mathbf{F}(|\Phi_i(x)|, \Sigma) \leq 2\varepsilon$ and we have from (3) that $|\Psi_i(x)| \notin K$. We have proved the theorem. $\qquad \square$

Remark 5.2. In Deformation Theorem A, if there is a subcomplex $Y \subset X$ such that $(\Phi_i)_{|Y} \equiv 0$ then $(\Psi_i)_{|Y} \equiv 0$ and the homotopy is relative to Y.

Throughout the rest of the section the space X can be either S^1 or $[0, 1]$, and as before G is either $\mathbb{Z}$ or $\mathbb{Z}_2$. Let $\{\Phi_i\}_{i \in \mathbb{N}}$ be a sequence of continuous maps from X into $\mathcal{Z}_n(M; \mathbf{F}; G)$ and let $L = \mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}})$. If $X = [0, 1]$, assume further that $\Phi_i(0) = \Phi_i(1) = 0$ and that all homotopies are relative to $\partial[0, 1] = \{0, 1\}$.

Deformation Theorem B 5.3. *Suppose that $\Sigma \in \mathcal{V}_n(M)$ is a stationary integral varifold such that:*

(a) *the support of Σ is a smooth embedded closed minimal hypersurface with a connected component S of Morse index one;*

(b) *for any sufficiently small tubular neighborhood $\Omega \subset M$ of S we have $||\Sigma||(\Omega) = m||S||(\Omega)$ for some integer $m \geq 2$;*

(c) *there exists a compact set $K \subset \mathcal{V}_n(M)$ so that $\Sigma \notin K$ and $|\Phi_i|(X) \cap K = \emptyset$ for all $i \geq i_0$;*

(d) $||\Sigma||(M) = L$.

Then there exist $\xi > 0$, $j_0 \in \mathbb{N}$ so that for all $i \geq j_0$ one can find $\Psi_i : X \to \mathcal{Z}_n(M; \mathbf{F}; G)$ such that

(i) Ψ_i *is homotopic to Φ_i in the flat topology;*

(ii) $\mathbf{L}(\{\Psi_i\}_{i \in \mathbb{N}}) \leq L$;

(iii) $|\Psi_i|(X) \cap \overline{\mathbf{B}}_\xi^{\mathbf{F}}(\Sigma) = \emptyset$;

(iv) $|\Psi_i|(X) \cap K \cap \{V : ||V||(M) \geq L - \xi\} = \emptyset$.

Proof. Set $d = \min\{\mathbf{F}(\Sigma, Z) : Z \in K\} > 0$. Let $S_1, \ldots, S_P$ be the connected components of Σ, with $S_1 = S$. Then, as a stationary integral varifold,

$$\Sigma = m_1 \cdot |S_1| + \cdots + m_P \cdot |S_P|$$

with $m_p \in \mathbb{N}$ and $m_1 \geq 2$.

Let $\mathcal{S}_0$ be the finite set of all $T = \sum_p a_p \cdot S_p \in \mathcal{Z}_n(M; G)$ with $a_p \in G$ and $|a_p| \leq m_p$. If $G = \mathbb{Z}$ and S_p is non-orientable, we set $a_p = 0$. We choose a constant $\delta > 0$ so that for all $T \in \mathcal{S}_0$, the balls $\mathbf{B}_{2\delta}^{\mathcal{F}}(T)$ are mutually disjoint and any two paths in $\mathbf{B}_{2\delta}^{\mathcal{F}}(T)$ with common endpoints are homotopic (not necessarily in $\mathbf{B}_{2\delta}^{\mathcal{F}}(T)$) to each other relative to the boundary in the flat topology. The existence of such constant follows from the work of Almgren [4] (see also Proposition 3.5 or Appendix A of [21]).

The fact that S has index one implies that Σ is 1-unstable in an ε-neighborhood for some $\varepsilon > 0$ (Definition 4.1). Let $\{F_v\}_{v \in [-1,1]}$ and $c_0 > 0$ be the corresponding family and constant. Note that we can choose the family of diffeomorphisms $\{F_v\}$ such that they are all equal to the identity map outside some small tubular neighborhood of S. We let $J = [-1, 1]$ and denote

$$\mathcal{S}_1 = \{(F_v)_\# T : v \in J, T \in \mathcal{S}_0\} \subset \mathcal{Z}_n(M; G).$$

By changing $\varepsilon > 0$, $\{F_v\}$ and c_0 if necessary, we can assume that

(h_0) every $T \in \mathcal{S}_0$ with $|T| \neq \Sigma$ satisfies $||(F_v)_\#(|T|)||(M) \leq ||\Sigma||(M) - c_0$ for all $v \in J$;

(h_1) for all $T \in \mathcal{S}_1$ and $v \in J$, $(F_v)_\# T \in \mathbf{B}_{\delta/(2m_1)}^{\mathcal{F}}(T)$;

(h_2) for all $V \in \overline{\mathbf{B}}_{2\varepsilon}^{\mathbf{F}}(\Sigma)$

$$\min\{\mathbf{F}((F_v)_\# V, Z) : Z \in K, v \in J\} > d/2;$$

(h_3) for all $V \in \overline{\mathbf{B}}^{\mathbf{F}}_{2\varepsilon}(\Sigma)$, $||V||(M) \leq ||\Sigma||(M) + c_0/20$.

We need to prove some lemmas.

Lemma 5.4. *There exists $\varepsilon_1 > 0$ so that if $V \in \mathcal{V}_n(M)$ is such that $\mathbf{F}(V, U) \leq \varepsilon_1$ for some $U \in \mathcal{S}_1$ then $||V||(M) \leq ||U||(M) + c_0/20$.*

Proof. The result is a direct consequence of the fact that $|\mathcal{S}_1|$ is compact in $\mathcal{V}_n(M)$. $\qquad\square$

Lemma 5.5. *For all $0 < \eta < 2\delta$ there exists $\varepsilon_2 = \varepsilon_2(\eta, \Sigma) > 0$ so that for any continuous map in the flat topology $\Psi : [0, 1] \to \mathcal{Z}_n(M; G)$ with $|\Psi|([0, 1]) \subset \mathbf{B}^{\mathbf{F}}_{\varepsilon_2}(\Sigma)$ there exists $T \in \mathcal{S}_0$ so that $\Psi([0, 1]) \subset \mathbf{B}^{\mathcal{F}}_{\eta}(T)$.*

Proof. First we prove that there exists $\varepsilon_2 = \varepsilon_2(\eta, \Sigma) > 0$ such that if $T' \in \mathcal{Z}_n(M; G)$ satisfies $\mathbf{F}(|T'|, \Sigma) < \varepsilon_2$, then there exists $T \in \mathcal{S}_0$ with $T' \in \mathbf{B}^{\mathcal{F}}_{\eta}(T)$.

Suppose this is not true. Then for some $\eta > 0$ we can find a sequence $T'_i \in \mathcal{Z}_n(M; G)$ satisfying $\mathbf{F}(|T'_i|, \Sigma) < 1/i$ and $\mathcal{F}(T'_i, T) \geq \eta$ for every $T \in \mathcal{S}_0$. We have $\mathbf{M}(T'_i) \leq C$ for all i and some $C > 0$, hence by compactness a subsequence T'_j converges to $\hat{T} \in \mathcal{Z}_n(M; G)$ in the flat topology. The convergence of $|T'_j|$ to Σ in the $\mathbf{F}$-metric implies $||T'_j||(M \setminus \text{supp}(\Sigma)) \to 0$ and for appropriate tubular neighborhoods U_p of S_p, $||T'_j||(U_p) \to m_p \cdot |S_p|$. Then $\text{supp}(\hat{T}) \subset \text{supp}(\Sigma)$ and by the Constancy Theorem $\hat{T} = \sum_p a_p \cdot S_p$ for $a_p \in G$, where we set $a_p = 0$ if S_p is unorientable and $G = \mathbb{Z}$. Lower semicontinuity of mass implies $|a_p| \leq m_p$ and thus $\hat{T} \in \mathcal{S}_0$. Contradiction because we were assuming $\mathcal{F}(T'_i, T) \geq \eta$ for every $T \in \mathcal{S}_0$.

If $\Psi : [0, 1] \to \mathcal{Z}_n(M; G)$ is a continuous map in the flat topology with $|\Psi|([0, 1]) \subset \mathbf{B}^{\mathbf{F}}_{\varepsilon_2}(\Sigma)$, then by what we just proved $\{\Psi^{-1}(B^{\mathcal{F}}_{\eta}(T))\}_{T \in \mathcal{S}_0}$ is a family of disjoint open sets covering $[0, 1]$. By connectedness, there must exist a single $T \in \mathcal{S}_0$ so that $\Psi([0, 1]) \subset \mathbf{B}^{\mathcal{F}}_{\eta}(T)$. $\qquad\square$

Finally, there is a constant $C \geq 1$ so that for all $S, T \in \mathcal{Z}_n(M; G)$ we have

$$\sup_{u \in J} \mathcal{F}((F_u)_\# S, (F_u)_\# T) \leq C\mathcal{F}(S, T).$$

Fix $\tilde{\delta} < \{\delta/2, c_0/40\}$. Consider $\eta = \eta(\mathcal{S}_1, \tilde{\delta}, L) < \tilde{\delta}$ given by Proposition A.2, $\varepsilon_1 > 0$ given by Lemma 5.4, $\varepsilon_2 = \varepsilon_2(\eta/C, \Sigma) > 0$ given by Lemma 5.5, and choose $0 < \bar{\varepsilon} \leq \min\{\varepsilon, \varepsilon_1/2, \varepsilon_2/2\}$.

If $|\Phi_i|(X) \subset \mathbf{B}^{\mathbf{F}}_{\varepsilon_2}(\Sigma)$, Lemma 5.5 and the choice of δ imply in this case Φ_i is homotopic to a constant path in the flat topology. Then Deformation Theorem B automatically holds by choosing $\Psi_i \equiv 0$. Likewise, if $|\Phi_i|(X) \cap$

$\mathbf{B}_{\bar{\varepsilon}}^{\mathbf{F}}(\Sigma) = \emptyset$ we can just take $\Psi_i = \Phi_i$ and so we assume that this case does not occur as well.

Elementary considerations allow us to find a closed set $V_{i,\bar{\varepsilon}} \subset X$ with finitely many connected components which are homeomorphic to closed intervals, $\partial V_{i,\bar{\varepsilon}}$ a finite union of points, and such that

(7) $|\Phi_i|(V_{i,\bar{\varepsilon}}) \subset \mathbf{B}_{2\bar{\varepsilon}}^{\mathbf{F}}(\Sigma)$ and $|\Phi_i|(X \setminus V_{i,\bar{\varepsilon}}) \subset \mathcal{V}_n(M) \setminus \mathbf{B}_{\bar{\varepsilon}}^{\mathbf{F}}(\Sigma)$.

Consequently, $|\Phi_i|(\partial V_{i,\bar{\varepsilon}}) \subset \mathbf{B}_{2\bar{\varepsilon}}^{\mathbf{F}}(\Sigma) \setminus \mathbf{B}_{\bar{\varepsilon}}^{\mathbf{F}}(\Sigma)$. It also follows that $\mathbf{M}(\Phi_i(y)) \leq \mathbf{M}(\Sigma) + c_0/20$ for every $y \in V_{i,\bar{\varepsilon}}$ by property (h_3) above.

Choose a continuous map

$$H_i^1 : \partial V_{i,\bar{\varepsilon}} \times [0,1] \to B_{1/i}^1(0) \quad \text{so that } H_i^1(x,0) = 0 \text{ for all } x \in \partial V_{i,\bar{\varepsilon}}$$

and

(8) $$\inf_{x \in \partial V_{i,\bar{\varepsilon}}} |m_i(x) - H_i^1(x,1)| \geq \eta_i > 0$$

for some $\eta_i > 0$. Consider

$$D_i : \partial V_{i,\bar{\varepsilon}} \times [0,1] \to J, \quad D_i(x,t) = \phi_i^x(H_i^1(x,1), tT_i),$$

where $T_i = T(\eta_i, \bar{\varepsilon}, \Sigma, \{F_v\}, c_0) \geq 0$ is given by Lemma 4.5. We have $|D_i(x,1)| \geq c_0/4$ for $x \in \partial V_{i,\bar{\varepsilon}}$.

Let $\tilde{V}_{i,\bar{\varepsilon}}$ be one of the connected components of $V_{i,\bar{\varepsilon}}$. From Lemma 5.5 we obtain the existence of $T_i \in \mathcal{S}_0$ so that $\Phi_i(\tilde{V}_{i,\bar{\varepsilon}}) \subset \mathbf{B}_{\eta/C}^{\mathcal{F}}(T_i)$ and thus

(9) $$\mathcal{F}((F_v)_\# \Phi_i(x), (F_v)_\# T_i) < \eta \quad \text{for all } x \in \tilde{V}_{i,\bar{\varepsilon}}, v \in J.$$

In particular this holds for $x \in \partial \tilde{V}_{i,\bar{\varepsilon}}$. Therefore, we can invoke Proposition A.2 and derive the existence of a continuous map $H_i^2 : \partial \tilde{V}_{i,\bar{\varepsilon}} \times [0,1] \to \mathcal{Z}_n(M; \mathbf{M}; G)$ such that

  (a) $H_i^2(x,0) = (F_{D_i(x,1)})_\# \Phi_i(x)$, $H_i^2(x,1) = (F_{D_i(x,1)})_\# T_i$;
  (b) $H_i^2(x,[0,1]) \subset \mathbf{B}_{\tilde{\delta}}^{\mathcal{F}}((F_{D_i(x,1)})_\# T_i)$;
  (c) $\mathbf{M}(H_i^2(x,t)) \leq \mathbf{M}((F_{D_i(x,1)})_\# \Phi_i(x)) + \tilde{\delta}$ for all $0 \leq t \leq 1$.

From the third property and Lemma 4.5 we obtain that for all $x \in \partial \tilde{V}_{i,\bar{\varepsilon}}$ and $0 \leq t \leq 1$

(10) $$\mathbf{M}(H_i^2(x,t)) \leq \mathbf{M}(\Phi_i(x)) - \frac{c_0}{10} + \tilde{\delta} \leq \mathbf{M}(\Phi_i(x)) - 3\frac{c_0}{40}.$$

If $|T_i| \neq \Sigma$, we have from property (h_0) that

$$\mathbf{M}((F_v)_\# T_i) \leq \mathbf{M}(\Sigma) - c_0 \quad \text{for all } v \in J.$$

Hence there exists a continuous map $\Xi_i : \tilde{V}_{i,\bar{\varepsilon}} \to \mathcal{S}_1$ so that Ξ_i equals $H_i^2(\cdot, 1)$ on $\partial \tilde{V}_{i,\bar{\varepsilon}}$, $\Xi_i(\tilde{V}_{i,\bar{\varepsilon}}) \subset \{(F_v)_\# T_i; v \in J\}$ and

(11)
$$\sup_{x \in \tilde{V}_{i,\bar{\varepsilon}}} \mathbf{M}(\Xi_i(x)) \leq ||\Sigma||(M) - c_0.$$

If $|T_i| = |\Sigma|$, then $T_i = \sum_p n_p \cdot S_p$ with $|n_p| = |m_p|$. Because $m_1 \geq 2$ we must necessarily have $G = \mathbb{Z}$ and S_1 orientable. By possibly changing the orientation of S_1, we can assume $n_1 = m_1 \geq 2$. We proceed as follows. Set

$$A = \{(u, v) \in J \times J : |u| \geq c_0/4 \text{ or } |v| \geq c_0/4\}$$

and write $\tilde{V}_{i,\bar{\varepsilon}} = [a_i, b_i]$, where $\partial \tilde{V}_{i,\bar{\varepsilon}} = \{a_i, b_i\}$. Denote by $\gamma_i(t) = (u_i(t), v_i(t))$, $a_i \leq t \leq b_i$, a path in A with $\gamma_i(a_i) = (D_i(a_i, 1), D_i(a_i, 1))$ and $\gamma_i(b_i) = (D_i(b_i, 1), D_i(b_i, 1))$.

Note that $(F_v)_\#(T_i) = m_1 \cdot (F_v)_\#(S_1) + \sum_{p \geq 2} n_p \cdot S_p$. Define $\Xi_i : \tilde{V}_{i,\bar{\varepsilon}} \to \mathcal{Z}_n(M; \mathbf{F}; G)$ as

$$\Xi_i(t) = (F_{u_i(t)})_\# S_1 + (m_1 - 1)(F_{v_i(t)})_\# S_1 + \sum_{p \geq 2} n_p \cdot S_p.$$

The map Ξ_i equals $H_i^2(\cdot, 1)$ on $\partial \tilde{V}_{i,\bar{\varepsilon}}$ and

$$
\begin{aligned}
\mathbf{M}(\Xi_i(t)) \quad \leq \quad & \mathbf{M}((F_{u_i(t)})_\# S_1) + (m_1 - 1)\mathbf{M}(F_{v_i(t)})_\# S_1) + \sum_{p \geq 2} |m_p| \cdot \mathbf{M}(S_p) \\
\leq \quad & |S_1| - \frac{c_0}{2m_1}|u(t)|^2 + (m_1 - 1)|S_1| - (m_1 - 1)\frac{c_0}{2m_1}|v(t)|^2 \\
& + \sum_{p \geq 2} |m_p| \cdot \mathbf{M}(S_p) \\
\leq \quad & ||\Sigma||(M) - \frac{c_0^3}{32}
\end{aligned}
$$

(12)

because $\max\{|u(p,t)|, |v(p,t)|\} \geq c_0/4$. We have used the inequality (1) applied to $V = \Sigma$ and the fact that the diffeomorphisms F_v differ from the identity only in a small neighborhood of S_1.

We define a map $\bar{\Psi}_i$ on the space $\left(\partial \tilde{V}_{i,\bar{\varepsilon}} \times [0,1]\right) \cup \left(\tilde{V}_{i,\bar{\varepsilon}} \times \{1\}\right)$ by

$$
\bar{\Psi}_i(w) = \begin{cases} (F_{H_i^1(x,3t)})_\# \Phi_i(x) & \text{if } w = (x,t) \in \partial \tilde{V}_{i,\bar{\varepsilon}} \times [0,1/3], \\ (F_{D_i(x,3t-1)})_\# \Phi_i(x) & \text{if } w = (x,t) \in \partial \tilde{V}_{i,\bar{\varepsilon}} \times [1/3,2/3], \\ H_i^2(x,3t-2) & \text{if } w = (x,t) \in \partial \tilde{V}_{i,\bar{\varepsilon}} \times [2/3,1], \\ \Xi_i(x) & \text{if } w = (x,1) \in \tilde{V}_{i,\bar{\varepsilon}} \times \{1\}. \end{cases}
$$

The map $\Psi_i : \tilde{V}_{i,\bar{\varepsilon}} \to \mathcal{Z}_n(M; \mathbf{F}; G)$ is defined by composing $\bar{\Psi}_i$ with a homeomorphism from $\tilde{V}_{i,\bar{\varepsilon}}$ to $\left(\partial \tilde{V}_{i,\bar{\varepsilon}} \times [0,1] \right) \cup \left(\tilde{V}_{i,\bar{\varepsilon}} \times \{1\} \right)$.

Since $\Psi_i(y) = \Phi_i(y)$ for every $y \in \partial \tilde{V}_{i,\bar{\varepsilon}}$, we can apply the constructions above to every connected component of $V_{i,\bar{\varepsilon}}$ and get a well-defined continuous map $\Psi_i : X \to \mathcal{Z}_n(M; \mathbf{F}; G)$ with $\Psi_i = \Phi_i$ on $\overline{X \setminus V_{i,\bar{\varepsilon}}}$.

Claim 1. Ψ_i *is homotopic to* Φ_i *in the flat topology for all* $i \in \mathbb{N}$.

It suffices to show that $(\Psi_i)_{|\tilde{V}_{i,\bar{\varepsilon}}}$ is homotopic to $(\Phi_i)_{|\tilde{V}_{i,\bar{\varepsilon}}}$ relative to $\partial \tilde{V}_{i,\bar{\varepsilon}}$ in the flat topology for all $i \in \mathbb{N}$ and every connected component $\tilde{V}_{i,\bar{\varepsilon}}$. From (9) and property (b) of H_i^2 we have that

$$
\bar{\Psi}_i \left((\partial \tilde{V}_{i,\bar{\varepsilon}} \times [0,1]) \cup (\tilde{V}_{i,\bar{\varepsilon}} \times \{1\}) \right) \subset \mathbf{B}_{2\delta}^{\mathcal{F}}(T_i).
$$

Recall that $\Phi_i(\tilde{V}_{i,\bar{\varepsilon}}) \subset \mathbf{B}_{\eta/C}^{\mathcal{F}}(T_i) \subset \mathbf{B}_{\delta/2}^{\mathcal{F}}(T_i)$ and so the result follows from the choice of δ.

Claim 2. $\limsup_{i \to \infty} \sup\{\mathbf{M}(\Psi_i(x)) : x \in V_{i,\bar{\varepsilon}}\} \leq L$.

The image of H_i^1 is contained in $B_{1/i}^1(0)$ and thus, because $||F_v - \mathrm{Id}||_{C^2}$ tends to zero uniformly as $v \in J$ tends to zero, we have that

$$
\limsup_{i \to \infty} \sup\{\mathbf{M}(\bar{\Psi}_i(y)) : y \in \partial V_{i,\bar{\varepsilon}} \times [0,1/3]\}
$$
$$
= \limsup_{i \to \infty} \sup\{\mathbf{M}((F_{H_i^1(x,t)})_\#(\Phi_i(x))) : (x,t) \in \partial V_{i,\bar{\varepsilon}} \times [0,1]\} \leq L.
$$

From the mass non-increasing property of ϕ_i^x, we have that for all $(x,t) \in \partial V_{i,\bar{\varepsilon}} \times [0,1]$,

$$
(13) \qquad ||(F_{D_i(x,t)})_\#(\Phi_i(x))||(M) \leq ||(F_{H_i^1(x,1)})_\#(\Phi_i(x))||(M)
$$

and so

$$
\limsup_{i \to \infty} \sup\{\mathbf{M}(\bar{\Psi}_i(y)) : y \in \partial V_{i,\bar{\varepsilon}} \times [1/3,2/3]\} \leq L.
$$

Thus the claim follows from (10), (11), and (12).

Claim 3. *There exists $0 < \xi < \min\{c_0/40, c_0^3/32\}$ so that $|\Psi_i|(V_{i,\bar{\varepsilon}} \cap \overline{\mathbf{B}}_\xi^{\mathbf{F}}(\Sigma) = \emptyset$ for all i sufficiently large.*

Again because the image of H_i^1 is contained in $B_{1/i}^1(0)$, we have that

$$\limsup_{i \to \infty}\{\mathbf{F}(|\Phi_i(x)|, (F_{H_i^1(x,t)})_\#(|\Phi_i(x)|)) : (x,t) \in \partial V_{i,\bar{\varepsilon}} \times [0,1]\} = 0.$$

Hence from (7) we have that, for all i sufficiently large, $\mathbf{F}(|\bar{\Psi}_i(y)|, \Sigma) \geq \bar{\varepsilon}/2$ for all $y \in \partial V_{i,\bar{\varepsilon}} \times [0, 1/3]$.

From (13) we have that, for all sufficiently large i,

$$||(F_{D_i(x,t)})_\#(\Phi_i(x))||(M) \leq ||\Phi_i(x)||(M) + \bar{\eta} \quad \text{for all } (x,t) \in \partial V_{i,\bar{\varepsilon}} \times [0,1],$$

where $\bar{\eta} = \bar{\eta}(\bar{\varepsilon}, \Sigma, \{F_v\}) > 0$ is given by Lemma 4.4. Hence, we obtain from Lemma 4.4 and (7) that $\mathbf{F}(|\bar{\Psi}_i(y)|, \Sigma) \geq 2\bar{\eta}$ for all $y \in \partial V_{i,\bar{\varepsilon}} \times [1/3, 2/3]$.

We have by (10) and property (h_3) that

$$\mathbf{M}(H_i^2(x,t)) \leq \mathbf{M}(\Phi_i(x)) - 3\frac{c_0}{40} \leq \mathbf{M}(\Sigma) - c_0/40.$$

for all $(x,t) \in \partial V_{i,\bar{\varepsilon}} \times [0,1]$. Moreover (independently if $|T_i| \neq \Sigma$ or $|T_i| = \Sigma$ in each connected component) we have

$$\sup_{x \in V_{i,\varepsilon}} \mathbf{M}(\Xi_i(x)) \leq ||\Sigma||(M) - \min\{c_0, c_0^3/32\}.$$

Thus, we have that $\mathbf{F}(|\Psi_i(y)|, \Sigma) \geq \bar{\gamma}$ for all y in $(\partial V_{i,\bar{\varepsilon}} \times [2/3, 1]) \cup (V_{i,\bar{\varepsilon}} \times 1)$ and some $\bar{\gamma} = \bar{\gamma}(\Sigma, c_0) > 0$. This concludes the proof of Claim 3.

Claim 4. *For all sufficiently large i, $|\Psi_i|(V_{i,\bar{\varepsilon}}) \cap K \cap \{V : ||V||(M) \geq L - \xi\} = \emptyset$.*

We see that if $x \in V_{i,\bar{\varepsilon}}$ and $\mathbf{M}(\Psi_i(x)) \geq L - \xi$, then $\Psi_i(x) = (F_v)_\# V$ for some $v \in J$, $V \in \overline{\mathbf{B}}_{2\bar{\varepsilon}}^{\mathbf{F}}(\Sigma)$. The claim then follows from (h_2).

This finishes the proof of Deformation Theorem B.   □

Deformation Theorem C 5.6. *Suppose the sequence $\{\Phi_i\}$ is pulled-tight, i.e., every varifold $V \in \mathbf{\Lambda}(\{\Phi_i\})$ with $||V||(M) = L$ is stationary. Let $\{\Sigma^{(1)}, \ldots, \Sigma^{(Q)}\} \subset \mathcal{V}_n(M)$ be a collection of stationary integral varifolds such that:*

(a) *the support of $\Sigma^{(q)}$ is a strictly stable two-sided closed embedded minimal hypersurface $S^{(q)}$ for every $1 \leq q \leq Q$;*

(b) $L = ||\Sigma^{(q)}||(M)$ for every $1 \leq q \leq Q$.

Then there exist $\xi > 0$, $j_0 \in \mathbb{N}$ so that for all $i \geq j_0$ one can find $\Psi_i : X \to \mathcal{Z}_n(M; \mathbf{F}; G)$ such that

(i) Ψ_i is homotopic to Φ_i in the flat topology;
(ii) $\limsup_{i \to \infty} \sup\{\mathbf{M}(\Psi_i(x)) : x \in X\} \leq L$;
(iii)

$$\mathbf{\Lambda}(\{\Psi_i\}) \subset \left(\mathbf{\Lambda}(\{\Phi_i\}) \setminus \cup_{q=1}^Q \mathbf{B}_\xi^{\mathbf{F}}(\Sigma^{(q)})\right) \cup \left(\mathcal{V}_n(M) \setminus \mathbf{B}_\xi^{\mathbf{F}}(\Gamma)\right),$$

where Γ is the collection of all stationary integral varifolds $V \in \mathcal{V}_n(M)$ with $L = ||V||(M)$.

Proof. We are going to consider the case $Q = 1$. The general case follows similarly with simple modifications.

Let $\Sigma = \Sigma^{(1)}, S = S^{(1)}$. Let $S_1, \ldots, S_P$ be the connected components of S. Then, as an integral varifold,

$$\Sigma = m_1 \cdot |S_1| + \cdots + m_P \cdot |S_P|$$

with $m_p \in \mathbb{N}$.

Let $\mathcal{S}_0$ be the finite set of all $T = \sum_p a_p \cdot S_p \in \mathcal{Z}_n(M; G)$ with $a_p \in G$ and $|a_p| \leq m_p$. If $G = \mathbb{Z}$ and S_p is non-orientable, we set $a_p = 0$. We choose a constant $\delta > 0$ so that for all $T \in \mathcal{S}_0$, the balls $\mathbf{B}_{2\delta}^{\mathcal{F}}(T)$ are mutually disjoint and any two paths in $\mathbf{B}_{2\delta}^{\mathcal{F}}(T)$ with common endpoints are homotopic (not necessarily in $\mathbf{B}_{2\delta}^{\mathcal{F}}(T)$) to each other relative to the boundary in the flat topology. The existence of such constant follows from the work of Almgren [4] (see also Proposition 3.5 or Appendix 2 of [21]).

The fact that S is strictly stable and two-sided implies that a neighborhood of S can be foliated by hypersurfaces whose mean curvature vectors point towards S. This can be done by using the first eigenfunction of the Jacobi operator of S. As a consequence, we can define a real function f defined on a neighborhood of S with $\nabla f \neq 0$, such that $S(s) = f^{-1}(s)$ are smooth embedded hypersurfaces for all $|s| \leq 1$, $S_0 = S$, and $S(s)$ is strictly mean convex for all $0 < |s| \leq 1$, i.e., $s\langle \nabla f, \vec{H}(S(s))\rangle < 0$. Denoting by $X = \nabla f / |\nabla f|^2$, we consider the diffeomorphism onto its image

$$\phi : S \times [-1, 1] \to M \quad \text{such that} \quad \frac{\partial \phi}{\partial s}(x, s) = X(\phi(x, s)),$$

where $\phi(S, s) = S(s)$ for all $|s| \leq 1$.

Set $\Omega_r = \phi(S \times (-r, r))$ and consider

$$P : \Omega_1 \times [0, 1] \to \Omega_1 \quad \text{such that } P(\phi(x, s), t) = \phi(x, (1-t)s).$$

Write $P_t(x) = P(x, t)$.

Proposition 5.7. *There exists $r_0 > 0$ such that $P : \Omega_{r_0} \times [0, 1] \to \Omega_{r_0}$ satisfies*

  (i) $P(x, 0) = x$ for all $x \in \Omega_{r_0}$, $P(x, t) = x$ for all $x \in S$ and $0 \le t \le 1$;
  (ii) $P(\Omega_r, t) \subset \Omega_r$ for all $0 \le t \le 1$, $r \le r_0$, and $P(\Omega_{r_0}, 1) = S$;
  (iii) the map $P_t : \Omega_{r_0} \to \Omega_{r_0}$, $x \mapsto P(x, t)$, is a diffeomorphism onto its image for all $0 \le t < 1$;
  (iv) for all $V \in \mathcal{V}_n(\Omega_{r_0})$ and every connected component Ω of Ω_{r_0}, the function $t \mapsto ||(P_t)_\# V||(\Omega)$ has strictly negative derivative unless $\operatorname{supp} V \cap \Omega \subset S \cap \Omega$, in which case it is constant.

Proof. Directly from the definition we get that for every $r \le 1$ the map P restricted to $\Omega_r \times [0, 1]$ satisfies conditions (i), (ii), and (iii) of Proposition 5.7.

For every $0 \le t < 1$, consider the vector field $Z_t = -\frac{f}{1-t}X$. Given $y = \phi(x, s) \in \Omega_1$, we have $P_t(y) = P(y, t) = P(\phi(x, s), t) = \phi(x, (1-t)s)$. Hence $f(P_t(y)) = (1-t)s$ and

$$\frac{d}{dt}P_t(y) = -s\frac{\partial \phi}{\partial s}(x, (1-t)s) = -\frac{f(P_t(y))}{(1-t)}X(P_t(y)) = Z_t(P_t(y)).$$

Given $x \in \Omega_1$ and an n-dimensional subspace $\sigma \subset T_x\Omega_1$, we use the notation

$$\operatorname{div}_\sigma Z_t(x) = \sum_{i=1}^n \langle D_{v_i} Z_t, v_i \rangle \text{ where } \{v_i\}_{i=1}^n \text{ is an orthonormal basis for } \sigma.$$

The first variation formula says that for all $V \in \mathcal{V}_n(\Omega_1)$,

$$\frac{d}{dt}||(P_t)_\# V||(M) = \int \operatorname{div}_\sigma Z_t \, d\mu_W(x, \sigma), \quad \text{where } W = (P_t)_\# V.$$

Given $y = \phi(x, s) \in \Omega_1$, we can choose local coordinates $(x_1, \ldots, x_n)$ in a neighborhood of $x \in S$. Then $e_i = \frac{\partial \phi}{\partial x_i}(x, s)$, $i = 1, \ldots, n$, span $T_y S(s)$. Let $N = \nabla f/|\nabla f|$, and let A be the second fundamental form of $S(s)$. We have

$$\langle D_{e_i} Z_t, e_j \rangle = -\langle Z_t, A(e_i, e_j) \rangle,$$

$$\langle D_{e_i} Z_t, N \rangle = -\frac{f}{1-t} \left\langle D_{\frac{\partial \phi}{\partial x_i}} \frac{\partial \phi}{\partial s}, N \right\rangle = \frac{f}{1-t} \left\langle D_{\frac{\partial \phi}{\partial s}} N, \frac{\partial \phi}{\partial x_i} \right\rangle = -\langle D_N Z_t, e_i \rangle,$$

$$\langle D_N Z_t, N \rangle = -\frac{1}{1-t}(1 + f\langle D_N X, N \rangle).$$

Choose an orthonormal basis $\{v_i\}_{i=1}^n$ for σ so that $v_1, \ldots, v_{n-1}$ are orthogonal to N. Then $v_n = \cos\theta u + \sin\theta N$ for some $u \in T_y S(s)$, $|u| = 1$, and angle θ. The expressions above show that $\sum_{i=1}^n \langle D_{v_i} Z_t, v_i \rangle$ is equal to

$$\frac{1}{1-t} \left(|\nabla f|^{-1} s \langle N, \vec{H}(S_l) \rangle - \sin^2\theta(1 + s\langle D_N X, N \rangle + s\langle X, A(u, u) \rangle) \right).$$

Thus there is r_0 so that if $|s| \le r_0$ we obtain that $\sum_{i=1}^n \langle D_{v_i} Z_t, v_i \rangle$ is always non-positive and zero only if $s = 0$. This shows property (iv). □

We deduce the following corollary:

Corollary 5.8. *There exists $\varepsilon_0 = \varepsilon_0(\Sigma) > 0$ so that every stationary integral varifold $V \in \mathcal{V}_n(M)$ in $\mathbf{B}_{\varepsilon_0}^{\mathbf{F}}(\Sigma)$ coincides with Σ.*

The above result does not hold if one does not require the stationary varifold V to be integral.

Proof. Suppose there is a sequence of stationary integral varifolds $\{V_j\}_{j \in \mathbb{N}}$ in $\mathcal{V}_n(M)$ converging to Σ in the varifold norm with $V_j \ne \Sigma$. Thus $\|V_j\|(M \setminus \Omega_{r_0/2})$ tends to zero as j tends to infinity and so the monotonicity formula implies that $\operatorname{supp} V_j \subset \Omega_{r_0}$ for all j sufficiently large. From Proposition 5.7 (iv) we deduce that $\operatorname{supp} V_j \subset S$, which when combined with the Constancy Theorem for stationary varifolds implies that $V_j = \sum_p n_{j,p} \cdot |S_p|$ for $n_{j,p} \in \mathbb{Z}_+$. Since $\mathbf{F}(V_j, \Sigma) \to 0$, we necessarily have $V_j = \Sigma$ for all j sufficiently large. Contradiction. □

From Proposition A.5 we see that we can assume, without loss of generality, that Φ_i is continuous in the mass topology for every i.

Let $X(k_i)$ be a sufficiently fine subdivision of X so that $\mathbf{M}(\Phi_i(x) - \Phi_i(y)) < 1/i$ and $\mathbf{F}(|\Phi_i(x)|, |\Phi_i(y)|) < 1/i$ for any x, y belonging to the same 1-cell in $X(k_i)$. For the purpose of this proof we redefine $V_{i,\alpha}$ to be the union of all 1-cells $\sigma \in X(k_i)$ so that $\mathbf{F}(|\Phi_i(x)|, \Sigma) < 2\alpha$ for all $x \in \sigma$. If $\beta \notin V_{i,\alpha}$, then $\mathbf{F}(|\Phi_i(x')|, \Sigma) \ge 2\alpha$ for some $x' \in \beta$. Consequently, $\mathbf{F}(|\Phi_i(x)|, \Sigma) \ge 2\alpha - 1/i$ for all $x \in \beta$.

First construction 5.9. The next Proposition deforms those cycles $\Phi_i(x)$ that are very close to Σ in varifold norm to cycles supported in a tubular neighborhood of S. We use the deformation given by Lemma 7.1 of Montezuma [24].

Proposition 5.10. *There exists a constant $C_1 = C_1(M) > 0$ so that for all $\eta > 0$ one can find $\tilde{\varepsilon} = \tilde{\varepsilon}(\Sigma, r_0, \eta) > 0$ with the property that for all $0 < \varepsilon \leq \tilde{\varepsilon}$, there exists $i_1 \in \mathbb{N}$ such that if $i \geq i_1$, there is a continuous map in the flat topology*

$$H_i^1 : V_{i,\varepsilon} \times [0,1] \to \mathcal{Z}_n(M; \mathcal{F}; G)$$

such that

(i) *$H_i^1(x,0) = \Phi_i(x)$ and $\mathrm{supp}\, H_i^1(x,1) \subset \Omega_{r_0/2}$ for all $x \in V_{i,\varepsilon}$;*
(ii) *for all $(x,t) \in V_{i,\varepsilon} \times [0,1]$ we have $\mathbf{M}(H_i^1(x,t) - \Phi_i(x)) \leq C_1 \eta$ and in particular $\mathbf{M}(H_i^1(x,t)) \leq \mathbf{M}(\Phi_i(x)) + C_1 \eta$;*
(iii) *for all $(x,t) \in \partial V_{i,\varepsilon} \times [0,1]$ we have $\mathbf{M}(H_i^1(x,t)) \leq \mathbf{M}(\Phi_i(x)) + C_1/i$ and $H_i^1(x,t) \llcorner \Omega_{r_0/4} = \Phi_i(x) \llcorner \Omega_{r_0/4}$;*
(iv) *H_i^1 restricted to $(\partial V_{i,\varepsilon} \times [0,1]) \cup (V_{i,\varepsilon} \times \{1\})$ is continuous in the mass topology.*

Proof. Let $U = M \setminus \bar{\Omega}_{r_0/3}$ and $W = M \setminus \bar{\Omega}_{2r_0/5}$. Consider $\varepsilon_1 = \varepsilon_1(U, W) > 0$ given by [24, Lemma 7.1]. In what follows let $\eta > 0$ be such that $5\eta < \min\{\delta_0, \varepsilon_1\}$, where δ_0 is the constant provided by Theorem 3.10 of [21] (the same statement holds if $G = \mathbb{Z}$). There exists $\bar{\varepsilon} = \bar{\varepsilon}(\Sigma, r_0, \eta) > 0$ so that

$$\mathbf{F}(|\Phi_i(x)|, \Sigma) \leq 2\bar{\varepsilon} \implies \|\Phi_i(x)\|(U) < \eta.$$

We assume that $0 < \varepsilon \leq \min\{\bar{\varepsilon}, \eta\}$.

Applying Lemma 7.1 of [24] to each image of Φ_i restricted to $(V_{i,\varepsilon})_0$ and observing that the deformations are local, we obtain $\hat{H}_i^1 : (V_{i,\varepsilon})_0 \times I(1, l_i)_0 \to \mathcal{Z}_n(M; G)$ for some $l_i \in \mathbb{N}$ sufficiently large such that

(a1) $\hat{H}_i^1(x,0) = \Phi_i(x)$ and $\mathrm{supp}\, \hat{H}_i^1(x,1) \subset \Omega_{2r_0/5}$ for all $x \in (V_{i,\varepsilon})_0$;
(b1) $\mathrm{supp}(\hat{H}_i^1(x,t) - \Phi_i(x)) \subset U$ for all $(x,t) \in (V_{i,\varepsilon})_0 \times I(1, l_i)_0$;
(c1) $\mathbf{M}(\hat{H}_i^1(x,t) - \hat{H}_i^1(x,t')) < 1/i$ if $(x,t), (x,t')$ are adjacent vertices in $(V_{i,\varepsilon})_0 \times I(1, l_i)_0$;
(d1) $\mathbf{M}(\hat{H}_i^1(x,t) \llcorner U) \leq \mathbf{M}(\Phi_i(x) \llcorner U) + 1/i$ for all $(x,t) \in (V_{i,\varepsilon})_0 \times I(1, l_i)_0$.

We have

$$\begin{aligned}
\mathbf{M}(\hat{H}_i^1(x,t) - \Phi_i(x)) &\leq \mathbf{M}(\hat{H}_i^1(x,t) \llcorner U) + \mathbf{M}(\Phi_i(x) \llcorner U) \\
&\leq 2\mathbf{M}(\Phi_i(x) \llcorner U) + 1/i \\
&\leq 2\eta + 1/i,
\end{aligned}$$

for all $(x,t) \in (V_{i,\varepsilon})_0 \times I(1, l_i)_0$. Hence, for all $(x,t), (x',t)$ adjacent vertices in $(V_{i,\varepsilon})_0 \times I(1, l_i)_0$ and i sufficiently large, we have

$$\begin{aligned}
\mathbf{M}(\hat{H}_i^1(x,t) - \hat{H}_i^1(x',t)) &\leq \mathbf{M}(\hat{H}_i^1(x,t) - \Phi_i(x)) \\
&\quad + \mathbf{M}(\Phi_i(x) - \Phi_i(x')) + \mathbf{M}(\hat{H}_i^1(x',t) - \Phi_i(x')) \\
&\leq 4\eta + 3/i \\
&< 5\eta.
\end{aligned}$$

This implies the map $\hat{H}_i^1$ has fineness smaller than $5\eta < \delta_0$. Using Theorem 3.10 of [21] (the same statement holds for $G = \mathbb{Z}$), we obtain a continuous map $\bar{H}_i^1 : V_{i,\varepsilon} \times [0,1] \to \mathcal{Z}_n(M; \mathbf{M}; G)$ in the mass topology (the *Almgren extension* of $\hat{H}_i^1$) and $C_0 = C_0(M) > 0$ such that

(a2) $\bar{H}_i^1(x,t) = \hat{H}_i^1(x,t)$ if $(x,t) \in (V_{i,\varepsilon})_0 \times I(1,l_i)_0$ and $\operatorname{supp} \bar{H}_i^1(x,1) \subset \Omega_{r_0/2}$ for all $x \in V_{i,\varepsilon}$;

(b2) $\lim_{i \to \infty} \sup\{\mathbf{M}(\Phi_i(x) - \bar{H}_i^1(x,0)) : x \in V_{i,\varepsilon}\} = 0$;

(c2) $\mathbf{M}(\bar{H}_i^1(x,t) - \Phi_i(x)) \leq (5C_0 + 3)\eta$ for all $(x,t) \in V_{i,\varepsilon} \times I$;

(d2) $\mathbf{M}(\bar{H}_i^1(x,t)) \leq \mathbf{M}(\Phi_i(x)) + (C_0 + 1)/i$ and

$$\bar{H}_i^1(x,t) \llcorner \Omega_{r_0/4} = \Phi_i(x) \llcorner \Omega_{r_0/4}$$

for all $(x,t) \in \partial V_{i,\varepsilon} \times I$.

Almgren interpolation results use a triangulation of M. The claims about the supports above follow by choosing the triangulation sufficiently fine with respect to r_0 (see proof of Theorem 14.1 of [20]).

Note that $\bar{H}_i^1(x,0) = \Phi_i(x)$ for $x \in \partial V_{i,\varepsilon}$. Using Proposition A.1 of the Appendix and property (b2), we find a homotopy of $\Phi_{i|V_{i,\varepsilon}}$ and $\bar{H}_i^1(\cdot, 0)$. We can glue that homotopy with $\bar{H}_i^1$ to construct H_i^1 satisfying properties (i), (ii), (iii) and (iv). □

Second construction 5.11. From the Constancy Theorem, there is $\eta_1 = \eta_1(\Sigma) > 0$ so that any $T \in \mathcal{Z}_n(M; G)$ with $\operatorname{supp} T \subset S_p$ for some $1 \leq p \leq P$ and $\mathbf{M}(T) \leq m_p \|S_p\|(M) + 2\eta_1$ must satisfy $T = m \cdot S_p$ with $m \in G$ and $|m| \leq m_p$.

From the Constancy Theorem for stationary varifolds, there is also $\alpha > 0$ (depending only on Σ) so that for every stationary integral varifold Z with $\operatorname{supp} Z \subset S$, $\mathbf{M}(Z) = \mathbf{M}(\Sigma)$, but $\Sigma \neq Z$, there is Ω^Z a connected component of Ω_{r_0} so that $\|Z\|(\Omega^Z) > \|\Sigma\|(\Omega^Z) + 2\alpha$. Moreover there is $\beta > 0$ (depending on Σ and α) so that every $W \in \mathbf{B}_\beta^{\mathbf{F}}(\Sigma)$ has $\|W\|(\Omega) \leq \|\Sigma\|(\Omega) + \alpha/2$ for every connected component Ω of Ω_{r_0}.

Consider $\varepsilon_0 > 0$ given by Corollary 5.8. We choose $\eta > 0$ (as in Proposition 5.10) and $\varepsilon > 0$ so that $\eta < \min\{(2\eta_1)/(C_1 + 1), \varepsilon_0/(6C_1)\}$, $\varepsilon <$

$\min\{\tilde{\varepsilon}(\Sigma, r_0, \eta), \varepsilon_2/2, \varepsilon_0/4, \beta/2\}$, where $\varepsilon_2 = \varepsilon_2(\delta, \Sigma)$ is as in Lemma 5.5. We can also require

$$\mathbf{F}(V, \Sigma) \leq 2\varepsilon \implies ||V||(\Omega^{(p)}_{r_0/2}) \leq m_p ||S_p||(M) + \eta,$$

where $\Omega^{(p)}_{r_0/2}$ denotes the connected component of $\Omega_{r_0/2}$ that contains S_p, $1 \leq p \leq P$.

Like in the proof of Deformation Theorem B we can assume that $|\Phi_i|(X)$ is not contained in $\mathbf{B}^{\mathbf{F}}_{\varepsilon_2}(\Sigma)$ and that intersects $\mathbf{B}^{\mathbf{F}}_{\varepsilon}(\Sigma)$. Thus $V_{i,\varepsilon}$ has finitely many connected components which are homeomorphic to closed intervals and $\partial V_{i,\varepsilon}$ is a non-empty union of vertices of $X(k_i)$.

For all $i \geq i_1$ consider the continuous map in the flat topology

$$H_i^2 : V_{i,\varepsilon} \times [0,1] \to \mathcal{Z}_n(M; G), \quad H_i^2(x,t) = (P_t)_\#(H_i^1(x,1)).$$

Denote by $\tilde{V}_{i,\varepsilon}$ a connected component of $V_{i,\varepsilon}$.

Lemma 5.12. *There exists $i_2 \in \mathbb{N}$ so that for all $i \geq i_2$:*

    (i) $\operatorname{supp} H_i^2(x,t) \subset \Omega_{r_0/2}$ *for all* $(x,t) \in \tilde{V}_{i,\varepsilon} \times [0,1]$ *and* H_i^2 *is continuous in the $\mathbf{F}$-topology on* $\tilde{V}_{i,\varepsilon} \times [0,1)$;

   (ii) $||H_i^2(x,t)||(\Omega) \leq ||H_i^1(x,1)||(\Omega)$ *for every* $(x,t) \in \tilde{V}_{i,\varepsilon} \times [0,1]$ *and every connected component Ω of Ω_{r_0};*

  (iii) *there exists $\Sigma_i \in \mathcal{S}_0$ so that $H_i^2(x,1) = \Sigma_i$ for all $x \in \tilde{V}_{i,\varepsilon}$.*

Proof. The first and second properties follow from Proposition 5.7 and Proposition 5.10 (i).

From Proposition 5.10 (ii) we obtain the existence of $i_2 \in \mathbb{N}$ so that for all $i \geq i_2$ and $1 \leq p \leq P$ we have

$$\begin{aligned}
\mathbf{M}(H_i^2(x,1) \llcorner \Omega^{(p)}_{r_0/2}) &\leq \mathbf{M}(H_i^1(x,1) \llcorner \Omega^{(p)}_{r_0/2}) \\
&\leq \mathbf{M}(\Phi_i(x) \llcorner \Omega^{(p)}_{r_0/2}) + C_1 \eta \\
&\leq m_p ||S_p||(M) + (C_1 + 1)\eta \\
&\leq m_p ||S_p||(M) + 2\eta_1,
\end{aligned}$$

From the choice of η_1 we conclude that $H_i^2(x,1) \in \mathcal{S}_0$ for every $x \in \tilde{V}_{i,\varepsilon}$. The third property follows from the facts that $\mathcal{S}_0$ is finite, H_i^2 is continuous in the flat topology and $\tilde{V}_{i,\varepsilon}$ is connected. $\qquad\square$

Third construction 5.13. For $i \in \mathbb{N}$ sufficiently large we have $1/i < \min\{\delta/P, \nu_3\}$, where P is the number of connected components of S and ν_3 is

the constant of Corollary 1.14 of [4]. If $\mathcal{F}(T') < \nu_3$ for some $T' \in \mathcal{Z}_n(M, G)$, then there exists $D' \in I_{n+1}(M, G)$ with $\partial D' = T'$ and $\mathbf{M}(D') = \mathcal{F}(T')$. Consider $\eta_i = \eta_i(\mathcal{S}_0, 1/i, L, W, U)$ given by Proposition A.2 of the Appendix, where $W = \Omega_{r_0/2}$ and $U = \Omega_{r_0}$.

Let $V_{i,\varepsilon}^{(l)}$, $l = 1, \ldots, N_i$, be the connected components of $V_{i,\varepsilon}$ and $\Sigma_i^{(l)}$ be the corresponding cycles given by Lemma 5.12 (iii). We obtain the existence of $0 < q_i < 1/i$ so that

$$H_i^2(V_{i,\varepsilon}^{(l)} \times [1 - q_i, 1]) \subset \mathbf{B}_{\eta_i}^{\mathcal{F}}(\Sigma_i^{(l)}) \quad \text{for all } l \leq N_i, i \geq i_2.$$

Note that supp $H_i^2(x, 1-q_i) \subset \Omega_{r_0/2}$ for all $(x, t) \in V_{i,\varepsilon} \times [0, 1]$, and supp $T \subset S \subset \Omega_{r_0/2}$ for all $T \in \mathcal{S}_0$. Then (by the choice of ν_3 and the Constancy Theorem) we have $\mathcal{F}(H_i^2(x, 1 - q_i) \llcorner \Omega, \Sigma_i^{(l)} \llcorner \Omega) < \eta_i$ for every connected component Ω of Ω_{r_0} and $x \in \partial V_{i,\varepsilon}^{(l)}$. Therefore, we can invoke Proposition A.2 (since $\Sigma_i^{(l)} \llcorner \Omega \in \mathcal{S}_0$) and, for all $i \geq i_2$, conclude the existence of a continuous map $H_i^3 : \partial V_{i,\varepsilon}^{(l)} \times [0, 1] \to \mathcal{Z}_n(M; \mathbf{M}; G)$ such that

(a) $H_i^3(x, 0) = H_i^2(x, 1 - q_i)$, $H_i^3(x, 1) = \Sigma_i^{(l)}$;

(b) $H_i^3(x, t) \subset \mathbf{B}_{P/i}^{\mathcal{F}}(\Sigma_i^{(l)})$ and supp $H_i^3(x, t) \subset \Omega_{r_0}$ for all $(x, t) \in \partial V_{i,\varepsilon}^{(l)} \times [0, 1]$;

(c) $\|H_i^3(x, t)\|(\Omega) \leq \|H_i^2(x, 1-q_i)\|(\Omega) + 1/i$ for every $(x, t) \in \partial V_{i,\varepsilon}^{(l)} \times [0, 1]$ and every connected component Ω of Ω_{r_0}.

We set $H_i^3(x, 1) = \Sigma_i^{(l)}$ for $x \in V_{i,\varepsilon}^{(l)}$. By the choice of δ, the maps $H_i^2(\cdot, 1 - q_i)_{|V_{i,\varepsilon}^{(l)}}$ and $(H_i^3)_{|(\partial V_{i,\varepsilon}^{(l)} \times [0,1]) \cup (V_{i,\varepsilon}^{(l)} \times \{1\})}$ are homotopic to each other in the flat topology.

Final construction 5.14. For each $l = 1, \ldots, N_i$, we define a continuous map in the $\mathbf{F}$-metric $\bar{\Psi}_i$ on $(\partial V_{i,\varepsilon} \times [0, 1]) \cup (V_{i,\varepsilon} \times \{1\})$ as

$$\bar{\Psi}_i(w) = \begin{cases} H_i^1(x, 3t) & \text{if } w = (x, t) \in \partial V_{i,\varepsilon}^{(l)} \times [0, 1/3], \\ H_i^2(x, 3(1 - q_i)t - 1 + q_i) & \text{if } w = (x, t) \in \partial V_{i,\varepsilon}^{(l)} \times [1/3, 2/3], \\ H_i^3(x, 3t - 2) & \text{if } w = (x, t) \in \partial V_{i,\varepsilon}^{(l)} \times [2/3, 1], \\ H_i^3(x, 1) & \text{if } w = (x, 1) \in V_{i,\varepsilon}^{(l)} \times \{1\} \end{cases}$$

The map $\Psi_i : V_{i,\varepsilon}^{(l)} \to \mathcal{Z}_n(M; \mathbf{F}; G)$ is defined by composing $\bar{\Psi}_i$ with a homeomorphism from $V_{i,\varepsilon}^{(l)}$ to $(\partial V_{i,\varepsilon^{(l)}} \times [0, 1]) \cup (V_{i,\varepsilon}^{(l)} \times \{1\})$. Since $\Psi_i(x) = \Phi_i(x)$ for $x \in \partial V_{i,\varepsilon}^{(l)}$, this defines a map $\Psi_i : X \to \mathcal{Z}_n(M; \mathbf{F}; G)$.

Property (i) of Deformation Theorem C follows from the construction of $\bar{\Psi}_i$. Property (ii) follows from Proposition 5.10 (iii), Proposition 5.12 (ii), and property (c) in Subsection 5.13.

We now prove Deformation Theorem C (iii).

Claim 1. *There exist $\varepsilon_3 = \varepsilon_3(\Sigma, r_0, \eta, \{H_i^1\}_i) > 0$ and $i_3 \in \mathbb{N}$ so that for all $i \geq i_3$ we have*

$$|H_i^1|(\partial V_{i,\varepsilon} \times [0,1]) \cap \mathbf{B}_{\varepsilon_3}^{\mathbf{F}}(V) = \emptyset$$

for every stationary integral varifold $V \in \mathcal{V}_n(M)$ with $L = \|V\|(M)$.

It suffices to show that there exists no subsequence $\{j\} \subset \{i\}$ and $\{(x_j, t_j)\}_j$ in $\partial V_{j,\varepsilon} \times [0,1]$ such that for some stationary integral varifold V_j with $L = \|V_j\|(M)$ we have $\mathbf{F}(|H_j^1(x_j, t_j)|, V_j) \to 0$ as $j \to \infty$. We can further assume V_j converges to V in the $\mathbf{F}$-metric.

If such a sequence exists, we have from Proposition 5.10 (ii) and the fact that $\mathbf{F}(|S|, |T|) \leq \mathbf{M}(S - T)$ for any $S, T \in \mathcal{Z}_n(M; G)$ (see Pitts [25], Section 2.1),

$$\mathbf{F}(|H_j^1(x_j, t_j)|, \Sigma) < 2\varepsilon + C_1\eta \leq 2\varepsilon_0/3.$$

Hence $V_j \in \mathbf{B}_{\varepsilon_0}^{\mathbf{F}}(\Sigma)$ for sufficiently large j and so $V = V_j = \Sigma$ by Corollary 5.8. Thus

$$\lim_{j \to \infty} \mathbf{M}(H_j^1(x_j, t_j) \llcorner \Omega_{r_0/8}) = L, \quad \lim_{j \to \infty} \mathbf{M}(H_j^1(x_j, t_j) \llcorner (M \setminus \Omega_{r_0/8})) = 0.$$

We obtain from Proposition 5.10 (iii) that

$$\lim_{j \to \infty} \mathbf{M}(\Phi_j(x_j) \llcorner \Omega_{r_0/8}) = L.$$

Since $\limsup_{j \to \infty} \mathbf{M}(\Phi_j(x_j)) \leq L$, we also get

$$\lim_{j \to \infty} \mathbf{M}(\Phi_j(x_j) \llcorner (M \setminus \Omega_{r_0/8})) = 0.$$

Therefore

$$\lim_{j \to \infty} \mathbf{M}(H_j^1(x_j, t_j) - \Phi_j(x_j)) \leq \lim_{j \to \infty} \big(\mathbf{M}((H_j^1(x_j, t_j) - \Phi_j(x_j)) \llcorner \Omega_{r_0/4})$$
$$+ \mathbf{M}(\Phi_j(x_j) \llcorner (M \setminus \Omega_{r_0/8})) + \mathbf{M}(H_j^1(x_j, t_j) \llcorner (M \setminus \Omega_{r_0/8})) \big) = 0$$

and so $|\Phi_j(x_j)|$ converges to Σ in varifold sense as j tends to infinity. This is impossible because $x_j \in \partial V_{j,\varepsilon}$ for all j.

Claim 2. *There is $\varepsilon_4 = \varepsilon_4(\Sigma, r_0, \eta, \{H_i^2\}_i) > 0$ and $i_4 \in \mathbb{N}$ so that for all $i \geq i_4$ we have*

$$|H_i^2|(\partial V_{i,\varepsilon} \times [0,1)) \cap \mathbf{B}_{\varepsilon_4}^{\mathbf{F}}(V) = \emptyset$$

for every stationary integral varifold $V \in \mathcal{V}_n(M)$ with $L = ||V||(M)$.

Like before, it suffices to show that there exists no subsequence $\{j\} \subset \{i\}$ and $\{(x_j, t_j)\}_j$ in $\partial V_{j,\varepsilon} \times [0,1)$ such that for some stationary integral varifold V_j with $L = ||V_j||(M)$ we have $\mathbf{F}(|H_j^2(x_j, t_j)|, V_j) \to 0$ as $j \to \infty$. We can further assume V_j converges to V in the $\mathbf{F}$-metric.

If such a sequence exists, we have from Lemma 5.12 (i) that $\operatorname{supp} V \subset \bar{\Omega}_{r_0/2}$ and thus $\operatorname{supp} V \subset S$ by Proposition 5.7 (iv). After passing to a subsequence if necessary we can assume that, as j tends to infinity, t_j tends to $\bar{t} \in [0,1]$ and $|H_j^1(x_j, 1)|$ tends to $\tilde{V}$, where $||\tilde{V}||(M) \le L$. From the definition of H_i^2 and Lemma 5.12 (ii) we see that $V = (P_{\bar{t}})_{\#}(\tilde{V})$ and $||V||(M) \le ||\tilde{V}||(M)$. Thus we have from Proposition 5.7 (i) and (iv) that $\tilde{V} = V$. This contradicts Claim 1.

Claim 3. *There is $\delta_1 = \delta_1(\Sigma, r_0, \eta, \{H_i^2\}_i) > 0$ and $i_5 \in \mathbb{N}$ so that for all $i \ge i_5$, $x \in \partial V_{i,\varepsilon}$ one can find Ω_i, a connected component of Ω_{r_0}, so that*

$$||(P_1)_{\#}(|H_i^1(x,1)|)||(\Omega_i) \le ||\Sigma||(\Omega_i) - 2\delta_1.$$

We argue by contradiction and assume there is a subsequence $\{j\} \subset \{i\}$ and $\{x_j\}_j$ in $\partial V_{j,\varepsilon}$ so that for every connected component Ω of Ω_{r_0} we have

$$\limsup_{j \to \infty} ||(P_1)_{\#}(|H_j^1(x_j,1)|)||(\Omega) \ge ||\Sigma||(\Omega).$$

We can assume $(P_1)_{\#}(|H_j^1(x_j,1)|)$ converges, in varifold sense, to $V \in \mathcal{V}_n(M)$ as j tends to infinity, where $||V||(M) = L$ and $\operatorname{supp} V \subset S$. Arguing like in the proof of Claim 2, we conclude that $\{|H_j^1(x_j,1)|\}_j$ converges to $V \in \mathcal{V}_n(M)$ as well. Thus

$$\lim_{j \to \infty} \mathbf{M}(H_j^1(x_j,1) \llcorner (M \setminus \Omega_{r_0/8})) = 0 \quad \text{and} \quad \lim_{j \to \infty} \mathbf{M}(H_j^1(x_j,1) \llcorner \Omega_{r_0/8}) = L.$$

We can argue like in the proof of Claim 1 to conclude that

$$\lim_{j \to \infty} \mathbf{M}(H_j^1(x_j,1) - \Phi_j(x_j)) = 0,$$

from which it follows that $\{|\Phi_j(x_j)|\}_j$ converges to V. Hence $\mathbf{F}(V, \Sigma) \ge \varepsilon$.

On the other hand, $||V||(M) = L$ and hence V is stationary because the sequence $\{\Phi_i\}$ is pulled-tight. Since we have

$$||V||(\Omega) \ge ||\Sigma||(\Omega) \quad \text{for every connected component } \Omega \text{ of } \Omega_{r_0},$$

the Constancy Theorem for stationary varifolds implies that $V = \Sigma$, which is a contradiction.

From Claim 3 we see that we can choose $0 < q_i < 1/i$ in Section 5.13 so that for every $i \geq i_5$ and $x \in \partial V_{i,\varepsilon}$ we have $||H_i^2(x, 1 - q_i)||(\Omega_i(x)) \leq ||\Sigma||(\Omega_i(x)) - \delta_1$.

Claim 4. *There exist $\varepsilon_5 = \varepsilon_5(\Sigma, r_0, \eta, \{H_i^3\}_i) > 0$ and $i_6 \in \mathbb{N}$ so that for all $i \geq i_6$ we have*

$$|H_i^3|(\partial V_{i,\varepsilon} \times [0,1]) \cap \mathbf{B}_{\varepsilon_5}^{\mathbf{F}}(V) = \emptyset$$

for every stationary integral $V \in \mathcal{V}_n(M)$ with $L = ||V||(M)$.

Like before, suppose there is a subsequence $\{j\} \subset \{i\}$ and $\{(x_j, t_j)\}_j$ in $\partial V_{j,\varepsilon} \times [0,1]$ such that for some stationary integral varifold V_j with $L = ||V_j||(M)$ we have $\mathbf{F}(|H_j^3(x_j, t_j)|, V_j) \to 0$ as $j \to \infty$. We can further assume V_j converges to V in the $\mathbf{F}$-metric. Then $\mathrm{supp}\, V \subset S$. By Allard's compactness theorem [3] V is also integral.

From property (c) in Section 5.13 we have that

$$||H_j^3(x_j, t_j)||(\Omega_j(x_j)) \leq ||\Sigma||(\Omega_j(x_j)) - \delta_1/2$$

for all j sufficiently large. Hence $||V||(\Omega) < ||\Sigma||(\Omega)$ for some connected component Ω of Ω_{r_0}.

In particular $V \neq \Sigma$ and so, for some connected component Ω^V of Ω_{r_0}, we have that $||V||(\Omega^V) > ||\Sigma||(\Omega^V) + 2\alpha$. Thus, we obtain from Proposition 5.12 (ii) and property (c) in Section 5.13 that, for all j sufficiently large, $||H_j^1(x_j, 1)||(\Omega^V) > ||\Sigma||(\Omega^V) + 3\alpha/2$. By passing to a subsequence, we can assume $|H_j^1(x_j, 1)|$ converges to $\bar{V} \in \mathcal{V}_n(M)$ with $||\bar{V}||(M) \leq L$, $\mathrm{supp}\, \bar{V} \subset \bar{\Omega}_{r_0/2}$. Since $\lim_{j\to\infty} ||H_j^3(x_j, t_j)||(M) = L$, we necessarily have $||\bar{V}||(M) = L$. It follows that $\lim_{j\to\infty} ||H_j^2(x_j, 1/2)||(M) = L$. Since $|H_j^2(x_j, 1/2)| \to (P_{1/2})_\#(\bar{V})$ as $j \to \infty$, we have $||(P_{1/2})_\#(\bar{V})||(M) = ||\bar{V}||(M) = L$. Therefore $\mathrm{supp}\, \bar{V} \subset S$. We again have

$$\lim_{j\to\infty} \mathbf{M}(H_j^1(x_j, 1) \llcorner (M \setminus \Omega_{r_0/8})) = 0 \quad \text{and} \quad \lim_{j\to\infty} \mathbf{M}(H_j^1(x_j, 1) \llcorner \Omega_{r_0/8}) = L.$$

Arguing like in Claim 1 we deduce that $\mathbf{M}(H_j^1(x_j, 1) - \Phi_j(x_j)) \to 0$, so $\bar{V}$ is stationary because the sequence $\{\Phi_i\}_i$ is pulled-tight. But this also implies that $||\Phi_j(x_j)||(\Omega^V) > ||\Sigma||(\Omega^V) + \alpha/2$ for all j sufficiently large. This is impossible because, since $x_j \in V_{j,\varepsilon}$ we have $\mathbf{F}(\Phi_j(x_j), \Sigma) < 2\varepsilon < \beta$.

Deformation Theorem C (iii) follows from Claim 1, Claim 2 and Claim 4, together with the fact that $\Psi_i = \Phi_i$ in $X \setminus V_{i,\varepsilon}$ and hence $\mathbf{F}(\Psi_i(x), \Sigma) \geq \varepsilon$ for all $x \notin V_{i,\varepsilon}$. $\qquad\square$

6. Index bounds

We assume in this section that $3 \leq (n+1) \leq 7$. Let X be a k-dimensional cubical complex and $\Phi : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ be a continuous map. We let Π be the class of all continuous maps $\Phi' : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ such that Φ and Φ' are homotopic to each other in the flat topology.

The metric (M, g) is said to be *bumpy* if no smooth immersed closed minimal hypersurface has a non-trivial Jacobi vector field. White showed [32, 33] that bumpy metrics are generic in the Baire sense.

Theorem 6.1. *Assume (M, g) is a bumpy metric and let $\{\Phi_i\}_{i \in \mathbb{N}}$ be a sequence in Π such that $L = \mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}}) = \mathbf{L}(\Pi)$.*

There is $\Sigma \in \mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}})$ with support a smooth embedded closed minimal hypersurface such that

$$\mathbf{L}(\Pi) = ||\Sigma||(M) \text{ and index (support of } \Sigma) \leq k.$$

Proof. It suffices to show that, for every $r > 0$, there is a stationary integral varifold $\tilde{\Sigma} \in \mathcal{V}_n(M)$ whose support is a smooth embedded closed minimal hypersurface such that $\mathbf{F}(\tilde{\Sigma}, \mathbf{C}(\{\Phi_i\})_{i \in \mathbb{N}}) < r$,

$$\mathbf{L}(\Pi) = ||\tilde{\Sigma}||(M) \text{ and index (support of } \tilde{\Sigma}) \leq k.$$

If we prove this we obtain a sequence $\Sigma_j \in \mathcal{V}_n(M)$ whose supports are smooth embedded closed minimal hypersurfaces of index less than or equal to k and such that $\mathbf{F}(\Sigma_j, \mathbf{C}(\{\Phi_i\})_{i \in \mathbb{N}}) < 1/j$. By the monotonicity formula and the mass bound we get that the number of connected components of the support of Σ_j and their multiplicities are uniformly bounded. The Compactness Theorem proven by Sharp [28] and the assumption that the metric is bumpy imply that there is a subsequence $\{m\} \subset \{j\}$ such that $\{\Sigma_m\} = \{\Sigma\}$. This follows because otherwise we can find an embedded minimal surface of index $\leq k$ that is a nontrivial graphical limit away from finitely many points and with finite multiplicity of embedded minimal surfaces of index $\leq k$. This must induce a nontrivial Jacobi field in the limit surface (see [28], Claim 6).

Denote by $\mathcal{W}$ the set of all stationary integral varifolds with mass L whose support is a smooth embedded closed minimal hypersurface and by $\mathcal{W}(r)$ the set

$$\{V \in \mathcal{W} : \mathbf{F}(V, \mathbf{C}(\{\Phi_i\})_{i \in \mathbb{N}}) \geq r\}.$$

Lemma 6.2. *There exist $i_0 \in \mathbb{N}$ and $\bar{\varepsilon}_0 > 0$ so that $|\Phi_i|(X) \cap \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r)) = \emptyset$ for all $i \geq i_0$.*

Proof. If not there is a subsequence $\{j\} \subset \{i\}$, and $x_j \in X$, $V_j \in \mathcal{W}(r)$ so that

$$\lim_{j \to \infty} \mathbf{F}(|\Phi_j(x_j)|, V_j) = 0.$$

Thus $\lim_{i \to \infty} ||\Phi_j(x_j)||(M) = L$ and hence a subsequence of $|\Phi_j(x_j)|$ will converge to some $V \in \mathbf{C}(\{\Phi_i\})$. This is a contradiction. $\qquad\square$

Denote by $\mathcal{W}^{k+1}$ the collection of elements in $\mathcal{W}$ whose support has index greater than or equal to $(k+1)$. Because the metric is bumpy this set is countable and thus we can write $\mathcal{W}^{k+1} \setminus \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{W}(r)) = \{\Sigma_1, \Sigma_2, \ldots\}$.

Using Deformation Theorem A with $K = \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{W}(r))$ and $\Sigma = \Sigma_1$, we find $\bar{\varepsilon}_1 > 0$, $i_1 \in \mathbb{N}$, and $\{\Phi_i^1\}_{i \in \mathbb{N}}$ in $\mathcal{Z}_n(M; \mathbf{F}; G)$ so that

- Φ_i^1 is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$;
- $\mathbf{L}(\{\Phi_i^1\}_{i \in \mathbb{N}}) \leq L$;
- for all $i \geq i_1$, $|\Phi_i^1|(X) \cap (\overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\Sigma_1) \cup \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{W}(r))) = \emptyset$;
- no Σ_j belongs to $\partial\overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\Sigma_1)$.

We now look at Σ_2. If $\Sigma_2 \notin \overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\Sigma_1)$, we apply Deformation Theorem A and find $\bar{\varepsilon}_2 > 0$, $i_2 \in \mathbb{N}$, and $\{\Phi_i^2\}_{i \in \mathbb{N}}$ in $\mathcal{Z}_n(M; \mathbf{F}; G)$ so that

- Φ_i^2 is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$;
- $\mathbf{L}(\{\Phi_i^2\}_{i \in \mathbb{N}}) \leq L$;
- for all $i \geq i_2$, $|\Phi_i^2|(X) \cap (\overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\Sigma_1) \cup \overline{\mathbf{B}}_{\bar{\varepsilon}_2}^{\mathbf{F}}(\Sigma_2) \cup \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{W}(r))) = \emptyset$;
- no Σ_j belongs to $\partial\overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\Sigma_1) \cup \partial\overline{\mathbf{B}}_{\bar{\varepsilon}_2}^{\mathbf{F}}(\Sigma_2)$.

If $\Sigma_2 \in \mathbf{B}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\Sigma_1)$, we skip it and repeat the procedure with Σ_3.

Proceeding inductively there are two possibilities. We can find for all $l \in \mathbb{N}$ a sequence $\{\Phi_i^l\}_{i \in \mathbb{N}}$, $\bar{\varepsilon}_l > 0$, $i_l \in \mathbb{N}$, and $\Sigma_{j_l} \in \mathcal{W}^{k+1} \setminus \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{W}(r))$ for some $j_l \in \mathbb{N}$ so that

- Φ_i^l is homotopic to Φ_i in the $\mathbf{F}$-topology for all $i \in \mathbb{N}$;
- $\mathbf{L}(\{\Phi_i^l\}_{i \in \mathbb{N}}) \leq L$;
- for all $i \geq i_l$, $|\Phi_i^l|(X) \cap (\cup_{q=1}^l \overline{\mathbf{B}}_{\bar{\varepsilon}_q}^{\mathbf{F}}(\Sigma_{j_q}) \cup \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{W}(r))) = \emptyset$;
- $\{\Sigma_1, \ldots, \Sigma_l\} \subset \cup_{q=1}^l \mathbf{B}_{\bar{\varepsilon}_q}^{\mathbf{F}}(\Sigma_{j_q})$;
- no Σ_j belongs to $\partial\overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\Sigma_1) \cup \cdots \cup \partial\overline{\mathbf{B}}_{\bar{\varepsilon}_l}^{\mathbf{F}}(\Sigma_{j_l})$.

Or the process ends in finitely many steps, which means we can find some $m \in \mathbb{N}$, a sequence $\{\Phi_i^m\}_{i \in \mathbb{N}}$, $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_m > 0$, $i_m \in \mathbb{N}$, and $\Sigma_{j_1}, \ldots, \Sigma_{j_m} \in \mathcal{W}^{k+1} \setminus \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{W}(r))$ for some $j_l \in \mathbb{N}$ so that

- Φ_i^m is homotopic to Φ_i in the **F**-topology for all $i \in \mathbb{N}$;
- $\mathbf{L}(\{\Phi_i^m\}_{i\in\mathbb{N}}) \leq L$;
- for all $i \geq i_m$, $|\Phi_i^l|(X) \cap (\cup_{q=1}^m \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_q}(\Sigma_{j_q}) \cup \overline{\mathbf{B}}^{\mathbf{F}}_{\bar{\varepsilon}_0}(\mathcal{W}(r))) = \emptyset$;
- $\{\Sigma_j : j \geq 1\} \subset \cup_{q=1}^m \mathbf{B}^{\mathbf{F}}_{\bar{\varepsilon}_q}(\Sigma_{j_q})$.

In the first case we choose an increasing sequence $p_l \geq i_l$ so that

$$\sup_{x\in X} ||\Phi_{p_l}^l||(M) \leq L + \frac{1}{l},$$

and set $\Psi_l = \Phi_{p_l}^l$. In the second case we set $p_l = l$ and $\Psi_l = \Phi_l^m$. The sequence $\{\Psi_l\}_{l\in\mathbb{N}}$ is such that

(i) Ψ_l is homotopic to Φ_{p_l} in the **F**-topology for all $l \in \mathbb{N}$;
(ii) $\mathbf{L}(\{\Psi_l\}_{l\in\mathbb{N}}) \leq L$;
(ii) $\mathbf{C}(\{\Psi_l\}_{l\in\mathbb{N}}) \cap (\mathcal{W}^{k+1} \cup \mathcal{W}(r)) = \emptyset$.

The Min-max Theorem 3.8 implies $\mathcal{W} \setminus (\mathcal{W}^{k+1} \cup \mathcal{W}(r))$ is nonempty and this is what we wanted to prove. $\quad\square$

Remark 6.3. Again the proof gives more. One can choose Σ to be G-almost minimizing as in Remark 3.10.

We deduce Theorem 1.2 from the previous theorem.

Theorem 1.2. *Let (M^{n+1}, g) be an $(n+1)$-dimensional closed Riemannian manifold, with $3 \leq (n+1) \leq 7$. There exists an integral stationary varifold $\Sigma \in \mathcal{V}_n(M)$ with the following properties:*

- $||\Sigma||(M) = \mathbf{L}(\Pi)$,
- *the support of Σ is a smooth closed embedded hypersurface in M,*
- $index(support\, of\, \Sigma) \leq k$.

Proof. Consider a sequence of bumpy metrics $\{g_j\}_{j\in\mathbb{N}}$ tending smoothly to g [33]. If L_j is the width of Π with respect to g_j, the previous theorem gives a sequence $\{\Phi_i^j\}_{i\in\mathbb{N}}$ in Π such that $L_j = \mathbf{L}(\{\Phi_i^j\}_{i\in\mathbb{N}})$ and $\Sigma_j \in \mathbf{C}(\{\Phi_i^j\}_{i\in\mathbb{N}})$ with support a smooth embedded minimal hypersurface of index $\leq k$ such that $L_j = ||\Sigma_j||(M)$ (with respect to the metric g_j). The sequence $\{L_j\}_{j\in\mathbb{N}}$ tends to $L = \mathbf{L}(\Pi)$ and so the result follows by using the Compactness Theorem A.6 of [28] (for changing ambient metrics). $\quad\square$

Next we assume that X is either S^1 or $[0, 1]$, G is either $\mathbb{Z}$ or $\mathbb{Z}_2$, $\Phi : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ is a continuous map, and Π is the class of all continuous maps $\Phi' : X \to \mathcal{Z}_n(M^{n+1}; \mathbf{F}; G)$ such that Φ and Φ' are homotopic to each

other in the flat topology. If $X = [0, 1]$, assume further that $\Phi'(0) = \Phi'(1) = 0$ for all $\Phi' \in \Pi$ and that all homotopies are relative to $\partial[0, 1] = \{0, 1\}$.

We first prove Theorem 1.7 when the manifold M does not contain one-sided embedded closed hypersurfaces. This is guaranteed if, for instance, $H_n(M; \mathbb{Z}_2) = 0$.

Theorem 6.4. *We assume that M contains no one-sided embedded closed hypersurfaces and that (M, g) is a bumpy metric. Let $\{\Phi_i\}_{i \in \mathbb{N}}$ be a sequence in Π such that $L = \mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}}) = \mathbf{L}(\Pi)$. There is $\Sigma \in \mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}})$ with support a smooth embedded closed minimal hypersurface such that*

$$\mathbf{L}(\Pi) = ||\Sigma||(M), \; index\,(support\ of\ \Sigma) = 1.$$

Moreover, the unstable component of Σ has multiplicity one.

Proof. Let $\mathcal{W}_j$ be the set of all stationary integral varifolds with mass L whose support is a smooth embedded closed minimal hypersurface with index j, where $j = 0, 1$. Because the metric is bumpy, we obtain from [28, Theorem 2.3] (or more precisely its proof) that $\mathcal{W}_0 \cup \mathcal{W}_1$ is finite. There can be at most one unstable component in each element in $\mathcal{W}_1$. We set $\mathcal{A}$ to be those elements of $\mathcal{W}_0 \cup \mathcal{W}_1$ that do not belong to $\mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}})$ and set $\mathcal{U}$ to be those elements of $\mathcal{W}_1 \cap \mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}})$ for which its unstable component appears with multiplicity higher than one. We want to show that $\mathcal{W}_1 \setminus (\mathcal{A} \cup \mathcal{U})$ is non-empty.

There exist $i_0 \in \mathbb{N}$ and $\bar{\varepsilon}_0 > 0$ so that $|\Phi_i|(X) \cap \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{A}) = \emptyset$ for all $i \geq i_0$. Write $\mathcal{U} = \{\Sigma_1, \ldots, \Sigma_k\}$. Applying Deformation Theorem B to $\{\Phi_i\}_{i \in \mathbb{N}}$, with $K = \overline{\mathbf{B}}_{\bar{\varepsilon}_0}^{\mathbf{F}}(\mathcal{A})$ and $\Sigma = \Sigma_1$, we find $\bar{\varepsilon}_1 > 0$, $i_1 \in \mathbb{N}$, and $\{\Phi_i^1\}_{i \in \mathbb{N}}$ in $\mathcal{Z}_n(M; \mathbf{F}; G)$ so that

- Φ_i^1 is homotopic to Φ_i in the flat topology for all $i \in \mathbb{N}$;
- $\mathbf{L}(\{\Phi_i^1\}_{i \in \mathbb{N}}) \leq L$;
- for all $i \geq i_1$, $|\Phi_i^1|(X) \cap \overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\{\Sigma_1\} \cup \mathcal{A}) = \emptyset$;
- no Σ_j belongs to $\partial\overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\{\Sigma_1\} \cup \mathcal{A})$.

If $\Sigma_2 \in \mathbf{C}(\{\Phi_i^1\}_{i \in \mathbb{N}})$, we apply Deformation Theorem B to a suitable subsequence of $\{\Phi_i^1\}_{i \in \mathbb{N}}$, with $K = \overline{\mathbf{B}}_{\bar{\varepsilon}_1}^{\mathbf{F}}(\{\Sigma_1\} \cup \mathcal{A})$ and $\Sigma = \Sigma_2$, to find $\bar{\varepsilon}_2 > 0$, $i_2 \in \mathbb{N}$, and $\{\Phi_i^2\}_{i \in \mathbb{N}}$ in $\mathcal{Z}_n(M; \mathbf{F}; G)$ so that

- Φ_i^2 is homotopic to Φ_i in the flat topology for all $i \in \mathbb{N}$;
- $\mathbf{L}(\{\Phi_i^2\}_{i \in \mathbb{N}}) \leq L$;
- for all $i \geq i_2$, $|\Phi_i^2|(X) \cap \overline{\mathbf{B}}_{\bar{\varepsilon}_2}^{\mathbf{F}}(\{\Sigma_1, \Sigma_2\} \cup \mathcal{A}) = \emptyset$;
- no Σ_j belongs to $\partial\overline{\mathbf{B}}_{\bar{\varepsilon}_2}^{\mathbf{F}}(\{\Sigma_1, \Sigma_2\} \cup \mathcal{A})$.

If Σ_2 is not in $\mathbf{C}(\{\Phi_i^1\}_{i\in\mathbb{N}})$, then the above properties hold with $\Phi_i^2 = \Phi_i^1$ and for some small $\bar{\varepsilon}_2$.

After a finite number of times we obtain $\{\Psi_i\}_{i\in\mathbb{N}}$ in $\mathcal{Z}_n(M;\mathbf{F};G)$, $i_k \in \mathbb{N}$, and $\delta > 0$ so that

- Ψ_i is homotopic to Φ_i in the flat topology for all $i \in \mathbb{N}$;
- $\mathbf{L}(\{\Psi_i\}_{i\in\mathbb{N}}) \leq L$;
- for all $i \geq i_3$, $|\Psi_i|(X) \cap \overline{\mathbf{B}}_\delta^{\mathbf{F}}(\mathcal{U} \cup \mathcal{A}) = \emptyset$.

Let $\hat{\Psi}_i$ be a pulled-tight sequence obtained from $\{\Psi_i\}_{i\in\mathbb{N}}$. This means $\mathbf{C}(\{\hat{\Psi}_i\}) \subset \mathbf{C}(\{\Psi_i\})$ and every element of $\mathbf{C}(\{\hat{\Psi}_i\})$ is stationary. One can then find another $\delta' > 0$ and $i_{k+1} \in \mathbb{N}$ so that $|\hat{\Psi}_i|(X) \cap \overline{\mathbf{B}}_{\delta'}^{\mathbf{F}}(\mathcal{U} \cup \mathcal{A} \cup \mathcal{B}) = \emptyset$ for all $i \geq i_{k+1}$, where $\mathcal{B}$ is the collection of elements of $\mathcal{W}_0$ that are not in $\mathbf{C}(\{\hat{\Psi}_i\}_{i\in\mathbb{N}})$.

Write $\mathcal{W}_0 \setminus (\mathcal{A} \cup \mathcal{B}) = \{\tilde{\Sigma}_1, \ldots, \tilde{\Sigma}_l\}$. Every element in this set belongs to $\mathbf{C}(\{\hat{\Psi}_i\}_{i\in\mathbb{N}})$. Then by Deformation Theorem C there exist $\xi > 0$, $j_0 \in \mathbb{N}$ so that for all $i \geq j_0$ one can find $\bar{\Psi}_i : X \to \mathcal{Z}_n(M;\mathbf{F};G)$ such that

(i) $\bar{\Psi}_i$ is homotopic to $\hat{\Psi}_i$ in the flat topology;
(ii) $\mathbf{L}(\bar{\Psi}_i\}) \leq L$;
(iii)
$$\mathbf{\Lambda}(\{\bar{\Psi}_i\}) \subset \left(\mathbf{\Lambda}(\{\hat{\Psi}_i\}) \setminus \cup_{q=1}^l \mathbf{B}_\xi^{\mathbf{F}}(\tilde{\Sigma}_q)\right) \cup \left(\mathcal{V}_n(M) \setminus \mathbf{B}_\xi^{\mathbf{F}}(\Gamma)\right),$$

where Γ is the collection of all stationary integral varifolds $V \in \mathcal{V}_n(M)$ with $L = \|V\|(M)$.

In particular, $\mathbf{C}(\{\bar{\Psi}_i\}_{i\in\mathbb{N}}) \cap (\mathcal{U} \cup \mathcal{W}_0 \cup \mathcal{A}) = \emptyset$. Theorem 6.1 applied to $\{\bar{\Psi}_i\}$ implies that $(\mathcal{W}_0 \cup \mathcal{W}_1) \setminus (\mathcal{U} \cup \mathcal{W}_0 \cup \mathcal{A})$ is nonempty and this is what we wanted to prove. $\square$

The next proposition is similar to arguments of X. Zhou [35], who considered $G = \mathbb{Z}$ and the multiplicity of non-orientable components.

Proposition 6.5. *Suppose $\Sigma \in \mathcal{V}_n(M)$ is stationary and G-almost minimizing of boundary type (see Remark 3.10) in annuli. Write*

$$\Sigma = m_1|S_1| + \cdots + m_P|S_P|,$$

where $m_i \in \mathbb{N}$ and $S_1, \ldots, S_P$ are the connected components of $\operatorname{supp}\Sigma$ (smooth embedded closed minimal hypersurface). For any one-sided component S_i, m_i is an even number.

Proof. As in the proof of Proposition 6.1 of [35], we can find $p \in S_i$, $s > 0$, a sequence $T_j \in \mathcal{Z}_n(M;G)$, $T_j = \partial U_j$, such that $\mathbf{F}(|T_j|, \Sigma) \to 0$ as $j \to \infty$

and supp $T_j \llcorner B_s(p)$ is a smooth embedded stable minimal hypersurface. By compactness, we can assume $T_j \to T = \partial U$ in the flat topology. For some tubular neighborhood of S_i, $T \llcorner \Omega \in \mathcal{Z}_n(M, M \setminus \Omega; G)$ is a relative boundary. But supp $(T \llcorner \Omega) \subset S_i$. Since S_i is one-sided, the Constancy Theorem implies $T \llcorner \Omega = k_i S_i$ with k_i even (if $G = \mathbb{Z}$ and S_i is non-orientable, necessarily $k_i = 0$). By stability and Schoen-Simon regularity theory [27], the convergence of T_j to Σ is smooth and locally graphical near p. Since $|a| \equiv a \bmod 2$ for $a \in G$, we have $m_i \equiv k_i \bmod 2$. □

We can now prove Theorem 1.7 and Corollary 1.8.

Theorem 1.7. *Suppose (M, g) is a bumpy metric, and $X = [0, 1]$. We assume $\{\Phi_i\}_{i \in \mathbb{N}}$ is a sequence in Π such that $L = \mathbf{L}(\{\Phi_i\}_{i \in \mathbb{N}}) = \mathbf{L}(\Pi)$. There exists $\Sigma \in \mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}})$ with support a smooth embedded closed minimal hypersurface such that $\mathbf{L}(\Pi) = ||\Sigma||(M)$ and either:*

(a) *index (support of Σ) = 1, and the unstable component of Σ has multiplicity one;*
(b) *index (support of Σ) = 0, and at least one component of Σ is one-sided, has even multiplicity and unstable double cover.*

Proof. Let $\bar{\Sigma}$ be an embedded one-sided closed minimal hypersurface. Since the metric is bumpy, the double cover is either unstable or strictly stable. If it is strictly stable, one can construct $P_t : \Omega_{r_0} \to \Omega_{r_0}$ just like in Proposition 5.7, where Ω_{r_0} is a neighborhood of $\bar{\Sigma}$ in M. This means that embedded one-sided closed minimal hypersurfaces with strictly stable double covers can be treated just like strictly stable two-sided minimal hypersurfaces if one follows the proof of Theorem 6.4 and Deformation Lemma C. With the appropriate modifications, we get the existence of $\Sigma \in \mathbf{C}(\{\Phi_i\}_{i \in \mathbb{N}})$ with support a smooth embedded closed minimal hypersurface such that $\mathbf{L}(\Pi) = ||\Sigma||(M)$ and either:

(a) index (support of Σ) = 1, and the unstable component of Σ has multiplicity one;
(b) index (support of Σ) = 0, and at least one component of Σ is one-sided, and has unstable double cover.

Since Σ can be chosen to be G-almost minimizing of boundary type in annuli, any one-sided component of it has even multiplicity by Proposition 6.5. This proves Theorem 1.7. □

Corollary 1.8. *Suppose (M, g) contains no one-sided embedded hypersurface, and $X = [0, 1]$. There exists an integral stationary varifold $\Sigma \in \mathcal{V}_n(M)$ with $||\Sigma||(M) = \mathbf{L}(\Pi)$ and satisfying the following properties:*

(a) *the support of Σ is a smooth closed embedded minimal hypersurface in M;*

(b) *we have*

$$index(support\, of\, \Sigma) \leq 1$$
$$\leq index(support\, of\, \Sigma) + nullity(support\, of\, \Sigma);$$

(c) *any unstable component (necessarily unique) of Σ must have multiplicity one;*

(d) *if the metric g is bumpy and $\{\Phi_i\}_{i\in\mathbb{N}}$ is a min-max sequence, Σ can be chosen so that for some subsequence $i_j \to \infty$ there is some $x_{i_j} \in X$ such that $\Phi_{i_j}(x_{i_j})$ converges in varifold sense to Σ.*

Proof. For bumpy metrics the result follows at once from Theorem 1.7. For a general metric g, consider a sequence of bumpy metrics $\{g_j\}_{j\in\mathbb{N}}$ tending smoothly to g (by [33]). If L_j is the width of Π with respect to g_j, we obtain a sequence $\{\Sigma_j\}_{j\in\mathbb{N}} \in \mathcal{V}_n(M)$ satisfying

- the support of Σ_j is a smooth closed embedded minimal hypersurface in M;
- $index(support\, of\, \Sigma_j) = 1$;
- the unstable component of Σ_j has multiplicity one;
- $L_j = ||\Sigma_j||(M)$ (with respect to the metric g_j) for all $j \in \mathbb{N}$.

The sequence $\{L_j\}_{j\in\mathbb{N}}$ tends to $L = \mathbf{L}(\Pi)$ and the Compactness Theorem A.6 of Sharp [28] (for changing ambient metrics) implies that, after passing to a subsequence, $\{\Sigma_j\}_{j\in\mathbb{N}}$ converges in the varifold sense to an integral stationary varifold $\Sigma \in \mathcal{V}_n(M)$ satisfying (a) and such that $||\Sigma||(M) = \mathbf{L}(\Pi)$. Moreover, it also follows from the same result that (after passing to a further subsequence):

- the union of the stable components of Σ_j with their respective multiplicities converges in the varifold sense to an integral stationary varifold Σ_s with support a smooth stable closed embedded minimal hypersurface in M;
- the unstable component of Σ_j with multiplicity one converges in the varifold sense to an integral stationary varifold Σ_u with support a smooth closed embedded minimal hypersurface in M.

Either the convergence to Σ_u is smooth and graphical, in which case Σ_u has multiplicity one and satisfies (b) and (c), or the convergence is not smooth. In the latter case Σ_u must be stable with nullity (see [28], Claim 6), and therefore also satisfies (b). Since $\Sigma = \Sigma_s + \Sigma_u$, we are done. $\square$

Appendix A. Various interpolation results

Proposition A.1. *There exist constants $\alpha = \alpha(M) > 0$ and $A_1 = A_1(M) > 0$ such that, if $\Phi_1, \Phi_2 : [0,1] \to \mathcal{Z}_n(M; \mathcal{F}; G)$ are continuous maps in the flat topology with $\Phi_1(0) = \Phi_2(0)$, $\Phi_1(1) = \Phi_2(1)$ and*

$$\sup\{\mathbf{M}(\Phi_1(x) - \Phi_2(x)) : x \in [0,1]\} < \alpha,$$

then there exists a homotopy H of Φ_1 and Φ_2, continuous in the flat topology and relative to $\partial[0,1] = \{0,1\}$, such that

$$\sup\{\mathbf{M}(H(x,t) - \Phi_i(x)) : x \in [0,1], t \in [0,1]\} < A_1\alpha,$$

where $i = 1,2$.

Proof. Let $\Psi(x) = \Phi_2(x) - \Phi_1(x)$. Note that $\mathcal{F}(\Psi(x)) \leq \mathbf{M}(\Psi(x))$, $\Psi(0) = \Psi(1) = 0$. By Theorem 8.2 of Almgren [4], if α is sufficiently small there exists a homotopy $h(x,t)$ between Ψ and 0 such that

$$\sup\{\mathbf{M}(h(x,t)) : x \in [0,1], t \in [0,1]\} \leq \mu_1 \sup\{\mathbf{M}(\Psi(x)) : x \in [0,1]\}.$$

The result follows by defining $H(x,t) = \Phi_2(x) - h(x,t)$. $\square$

A related result to the next proposition was also proven by Zhou [36].

Proposition A.2. *Given $\delta > 0$, $L > 0$, let $\mathcal{S} \subset \mathcal{Z}_n(M; \mathbf{F}; G)$ be a compact set with $\mathbf{M}(R) \leq 2L$ for every $R \in \mathcal{S}$. Then there exists $\eta = \eta(\mathcal{S}, \delta, L) < \delta$ so that for all $S \in \mathcal{Z}_n(M; G)$ with*

$$\mathbf{M}(S) \leq 2L \quad and \quad \mathcal{F}(S, R) < \eta \quad for\ some\ R \in \mathcal{S},$$

there exists $H : [0,1] \to \mathcal{Z}_n(M; \mathbf{M}; G)$ continuous in the mass topology and such that

- *$H(0) = S$, $H(1) = R$, and $H([0,1]) \subset \mathbf{B}_\delta^{\mathcal{F}}(R)$;*
- *$\mathbf{M}(H(x)) \leq \mathbf{M}(S) + \delta$ for all $0 \leq x \leq 1$.*

Moreover, if $\operatorname{supp} R \subset W \subset\subset U$ for all $R \in \mathcal{S}$ and some open sets U, W, we can choose $\eta = \eta(\mathcal{S}, \delta, L, W, U) < \delta$ such that if S is as above and $\operatorname{supp} S \subset W$ we can choose H with $\operatorname{supp} H(t) \subset U$ for every $t \in [0,1]$.

Proof. The next two auxiliary lemmas were essentially proven in [25].

Lemma A.3. *Let $W \subset\subset U$ be open subsets of M. If $V \in \mathcal{V}_n(M), T, T_1, \ldots,$ are elements in $\mathcal{Z}_n(W; G)$ with*

$$\lim_{i\to\infty}\left(\mathcal{F}(T,T_i)+\mathbf{F}(V,|T_i|)\right)=0,$$

then for every $\bar{\delta}>0$ there exist a sequence $S_1,S_2,\ldots,$ in $\mathcal{Z}_n(U;G)$ such that $\lim_{i\to\infty}\mathbf{F}(T,S_i)=0$ and the following property holds:

For each $i\in\mathbb{N}$ there is $m_i\in\mathbb{N}$ and a finite sequence $R_0^i,\ldots,R_{m_i}^i$ in $\mathcal{Z}_n(U;G)$ such that $R_0^i=T_i$, $R_{m_i}^i=S_i$,

$$\mathbf{M}(R_j^i)\le\mathbf{M}(T_i)+\bar{\delta},\quad\mathbf{M}(R_j^i-R_{j-1}^i)\le\bar{\delta}\quad\text{for all }j=1,\ldots,m_i$$

and $\lim_{i\to\infty}\sup_{0\le j_i\le m_i}\mathcal{F}(R_{j_i}^i,T)=0$.

Proof. In Lemma 3.7 of [25] (using $T^*=0$ and $K=\bar{V}$ in his notation), all properties are proven except for the fact that $R_{j_i}^i$ tends to T uniformly in the flat topology as $i\to\infty$. While Pitts did not state it, this property follows from the proof of Lemma 3.7, as we now explain. We refer to the notation used by Pitts. The currents R_j^i are constructed either in [25, page 117], and one sees at once that $\mathcal{F}(R_j^i,T)\le\mathbf{M}(Q_i)=\mathcal{F}(T_i,T)$ for all $i\in\mathbb{N}$ and $0\le j\le m_i$, or in [25, page 121] and one sees that, for all $0\le j\le m_i$, R_j^i coincides with T_i on $M\setminus B_{r_i}(p_i)$, where r_i tends to zero and p_i tends to p, as $i\to\infty$. In both cases $R_{j_i}^i$ tends to T uniformly in j_i as $i\to\infty$. □

Lemma A.4. *Let $W\subset\subset U$ be open subsets of M. Given $\bar{L}>0$ and $\bar{\delta}>0$, there exists $\varepsilon=\varepsilon(W,U,\bar{L},\bar{\delta})>0$ such that whenever $S_1,S_2\in\mathcal{Z}_n(W;G)$ have $\mathcal{F}(S_1,S_2)\le\varepsilon$ and $\mathbf{M}(S_1)\le\bar{L}$, $\mathbf{M}(S_2)\le\bar{L}$, there is a finite sequence $T_0,\ldots,T_m$ in $\mathcal{Z}_n(U;G)$ such that $T_0=S_1$, $T_m=S_2$,*

$$\mathbf{M}(T_j)\le\bar{L}+\bar{\delta},\quad\mathbf{M}(T_j-T_{j-1})\le\bar{\delta}\quad\text{for all }j=1,\ldots,m$$

and $\mathcal{F}(T_j,S_1)<\bar{\delta}$ for all $0\le j\le m$.

Proof. In Lemma 3.8 of [25] (using $T=0$ and $K=\bar{W}$ in his notation), all properties are proven except for the fact that $\mathcal{F}(T_j,S_1)<\bar{\delta}$ for all $0\le j\le m$. To prove this last property it suffices to do the following modification of the proof of Lemma 3.8 of [25] (we now use his notation): On page 122 of [25], use Lemma A.3 instead of Lemma 3.7. The whole argument carries over but one needs to check that the currents T_j defined at the bottom of page 123 can be made arbitrarily close to S_i in the flat metric and this is indeed the case because $\mathcal{F}(S_0,T_j)\le\mathbf{M}(Q)=\mathcal{F}(S_0,S_i)$. □

Let $C_0=C_0(M,1)>0$ and $\delta_0=\delta_0(M)>0$ be as in Theorem 3.10 of [21]. We can assume $\delta<\delta_0$ and set $\bar{\delta}=\delta/(3+C_0)$.

We argue by contradiction and suppose there is a sequence $S_i \in \mathcal{Z}_n(M; G)$, $\mathbf{M}(S_i) \leq 2L$, such that $\mathcal{F}(S_i, R_i) < 1/i$ for some $R_i \in \mathcal{S}$ and for which the conclusion of the proposition does not hold. After passing to a subsequence if necessary, we find $V \in \mathcal{V}_n(M)$ and $R \in \mathcal{S}$ such that

$$\lim_{i \to \infty} (\mathbf{F}(R_i, R) + \mathbf{F}(|S_i|, V)) = 0 \quad \text{and} \quad \mathbf{M}(R) \leq ||V||(M).$$

Set $\bar{L} = ||V||(M) + \bar{\delta}$ and consider $\varepsilon = \varepsilon(M, M, \bar{L}, \bar{\delta})$ given by Lemma A.4. Choose i sufficiently large so that

$$||S_i||(M) \leq \bar{L} \leq ||S_i||(M) + 2\bar{\delta}, \ \mathbf{M}(R_i) \leq \bar{L}, \ \text{and} \ \mathcal{F}(S_i, R_i) < \varepsilon.$$

Applying Lemma A.4 to S_i, R_i and using Theorem 3.10 of [21] we obtain $H : [0, 1] \to \mathcal{Z}_n(M; \mathbf{M}; G)$ continuous in the mass topology such that $H(0) = S_i$, $H(1) = R_i$,

$$\sup_{x \in [0,1]} \mathbf{M}(H(x)) \leq \bar{L} + (C_0 + 1)\bar{\delta} \leq \mathbf{M}(S_i) + (C_0 + 3)\bar{\delta} \leq \mathbf{M}(S_i) + \delta,$$

and $H([0, 1]) \subset \mathbf{B}^{\mathcal{F}}_{(C_0+1)\bar{\delta}}(R_i) \subset \mathbf{B}^{\mathcal{F}}_{\delta}(R_i)$. This is a contradiction and finishes the first part of the proof of the Proposition.

To prove the last statement one needs to localize Theorem 3.10 of [21] so that the deformations take place near U. This follows because Almgren interpolation results and the proof of Theorem 3.10 of [21] use a triangulation of M and so one chooses a triangulation sufficiently fine with respect to $d(U, M \setminus W)$ (see proof of Theorem 14.1 of [20]). □

Proposition A.5. *Let Y be a cubical subcomplex of some $I(m, l)$ and let $\Phi : Y \to \mathcal{Z}_n(M; \mathbf{F}; G)$ be a continuous map in the $\mathbf{F}$-topology. There exist a sequence of maps continuous in the mass topology*

$$\Phi_i : Y \to \mathcal{Z}_n(M; \mathbf{M}; G), \quad j \in \mathbb{N},$$

homotopic to Φ in the flat topology, and such that

$$\sup\{\mathbf{F}(\Phi_i(y), \Phi(y)) : y \in Y\} \to 0$$

as $i \to \infty$. If Z is a subcomplex of Y such that $\Phi_{|Z} \equiv 0$, then the sequence $\{\Phi_i\}_{i \in \mathbb{N}}$ can be chosen so that $(\Phi_i)_{|Z} \equiv 0$ for all $i \in \mathbb{N}$ and the homotopies can be taken relative to Z.

Proof. Given $\Phi : Y \to \mathcal{Z}_n(M; \mathbf{F}; G)$, Theorem 3.9 of [21] (the analogous statement holds for $G = \mathbb{Z}$) implies there exist a sequence of maps

$$\phi_i : Y(k_i)_0 \to \mathcal{Z}_n(M; G),$$

with $k_i < k_{i+1}$, and a sequence of positive numbers $\{\delta_i\}_{i \in \mathbb{N}}$ converging to zero such that

(i)
$$S = \{\phi_i\}_{i \in \mathbb{N}}$$

is an $(Y, \mathbf{M})$-homotopy sequence of mappings into $\mathcal{Z}_n(M; \mathbf{M}; G)$ with $\mathbf{f}(\phi_i) < \delta_i$;

(ii)
$$\sup\{\mathcal{F}(\phi_i(x) - \Phi(x)) : x \in Y(k_i)_0\} \le \delta_i;$$

(iii)
$$\sup\{\mathbf{M}(\phi_i(x)) : x \in X(k_i)_0\} \le \sup\{\mathbf{M}(\Phi(x)) : x \in Y\} + \delta_i.$$

We can also guarantee that (see Theorem 13.1 (i) of [20] for the case $Y = I^m$)

(iv) for some $l_i \to \infty$ and every $y \in X(k_i)_0$,

$$\mathbf{M}(\phi_i(y)) \le \sup\{\mathbf{M}(\Phi(x)) : \alpha \in X(l_i), x, y \in \alpha\} + \delta_i.$$

Because $x \in Y \mapsto \mathbf{M}(\Phi(x))$ is continuous, we get from property (iv) above that for every $y \in X(k_i)_0$,

$$\mathbf{M}(\phi_i(y)) \le \mathbf{M}(\Phi(y)) + \eta_i$$

with $\eta_i \to 0$ as $i \to \infty$. If we apply Lemma 4.1 of [20] with $\mathcal{S} = \Phi(X)$, property [(ii)] above implies

(14) $$\sup\{\mathbf{F}(\phi_i(x), \Phi(x)) : x \in X(k_i)_0\} \to 0$$

as $i \to \infty$.

For sufficiently large i, by Theorem 3.10 of [21] we can take the Almgren extension of ϕ_i, $\Phi_i : Y \to \mathcal{Z}_n(M; \mathbf{M}; G)$. By Corollary 3.12 of [21], Φ_i and Φ are homotopic to each other in the flat topology. By Theorem 3.10 (iii) [21], (14) and since Φ is continuous in the F-metric, we have that

$$\sup\{\mathbf{F}(\Phi_i(y), \Phi(y)) : y \in Y\} \to 0$$

as $i \to \infty$.

If $\Phi_{|Z} \equiv 0$, then naturally $(\Phi_i)_{|Z} \equiv 0$ and the fact that the homotopies can be taken relative to Z follows from the proof of Proposition 3.5 in the Appendix of [21]. $\qquad\square$

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