Geodesic period integrals of eigenfunctions on Riemannian surfaces and the Gauss-Bonnet Theorem[*]

CHRISTOPHER D. SOGGE, YAKUN XI, AND CHENG ZHANG

We use the Gauss-Bonnet theorem and the triangle comparison theorems of Rauch and Toponogov to show that on compact Riemannian surfaces of negative curvature period integrals of eigenfunctions e_λ over geodesics go to zero at the rate of $O((\log \lambda)^{-1/2})$ if λ are their frequencies. As discussed in [4], no such result is possible in the constant curvature case if the curvature is ≥ 0. Notwithstanding, we also show that these bounds for period integrals are valid provided that integrals of the curvature over all geodesic balls of radius $r \leq 1$ are pinched from above by $-\delta r^N$ for some fixed N and $\delta > 0$. This allows, for instance, the curvature to be nonpositive and to vanish of finite order at a finite number of isolated points. Naturally, the above results also hold for the appropriate type of quasi-modes.

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1. Introduction and preliminaries

Using Kuznecov formulae, Good [6] and Hejhal [7] showed that if γ_{per} is a periodic geodesic on a compact hyperbolic surface M then

$$(1.1) \qquad \Big| \int_{\gamma_{per}} e_\lambda \, ds \Big| \leq C_{\gamma_{per}},$$

with ds denoting arc length measure on γ_{per} and with e_λ denoting the L^2-normalized eigenfunction on M, i.e.,

$$-\Delta_g e_\lambda = \lambda^2 e_\lambda, \quad \text{and} \quad \int_M |e_\lambda|^2 \, dV_g = 1.$$

Here Δ_g denotes the Laplace-Beltrami operator on (M, g) and dV_g is the volume element.

This result was generalized by Zelditch [15], who showed that if λ_j are the eigenvalues of $\sqrt{-\Delta_g}$ on an compact Riemannian surface and if $a_j(\gamma_{per})$ denote the period integrals in (1.1) for an orthonormal basis of eigenfunctions with eigenvalues λ_j then

$$\sum_{\lambda_j \leq \lambda} |a_j(\gamma_{per})|^2 = c_{\gamma_{per}}\lambda + O(1),$$

which implies (1.1). Further work for hyperbolic surfaces giving more information about the lower order terms in terms of geometric data for γ_{per} was done by Pitt [9]. Since the number of eigenvalues that are smaller than λ is $O(\lambda^2)$, this asymptotic formula implies that, on average, one can do much better than (1.1). The problem of improving this upper bound was raised and discussed in Pitt [9] and Reznikov [10].

In an earlier joint paper of Chen and the first author [4], it was pointed out that no improvement of (1.1) is possible on compact two-dimensional manifolds of constant non-negative curvature. For instance, on S^2, the integrals in (1.1) have unit size if γ_{per} is the equator and e_λ is an L^2-normalized zonal function of even degree. Also on $\mathbb{T}^2$, for every periodic geodesic, γ_{per}, one can find a sequence of eigenvalues λ_k and eigenfunctions e_{λ_k} so that $e_{\lambda_k} \equiv 1$ on γ_{per} and $\|e_{\lambda_k}\|_{L^2(\mathbb{T}^2)} \approx 1$.

Despite this, in [4], it was shown that the period integrals in (1.1) are $o(1)$ as $\lambda \to \infty$ if (M, g) has strictly negative curvature. The proof exploited the fact that, in this case, quadrilaterals always have their four interior angles summing to a value strictly smaller than 2π. This "defect" (see Figure 2) allowed the authors to obtain $o(1)$ decay for period integrals using a stationary phase argument involving reproducing kernels for the eigenfunctions.

The purpose of this paper is to improve this result in two ways. First, even though there can be no decay for period integrals for the flat two-torus, we shall obtain decay if the curvature $K = K_g$ of (M, g) is assumed to be non-positive but allowed to vanish at an averaged rate of finite type in the sense that whenever $B_r \subset M$ is a geodesic ball of radius $r \leq 1$ (and arbitrary center) we have that

$$(1.2) \qquad \int_{B_r} K \, dV_g \leq -\delta r^N, \quad r \leq 1,$$

for some fixed $\delta > 0$ and $N < \infty$. Of course if $K \leq -\delta$ everywhere then we can take $N = 2$ in (1.2) (and possibly have to replace δ by a multiple of itself). Condition (1.2) holds, for instance, if the curvature is negative

off of a finite collection of points where it vanishes to finite order. Besides this improvement, we shall also show that, under the assumption (1.2), the period integrals in (1.1) are $O((\log \lambda)^{-1/2})$.

To be more specific, our main result is the following.

Theorem 1.1. *Let (M, g) be a compact two-dimensional boundaryless manifold. Assume that its curvature satisfies (1.2). Then if $\gamma(t)$ is a geodesic in M parametrized by arc length and if $b \in C_0^\infty((-1/2, 1/2))$ we have for $\lambda \gg 1$*

$$(1.3) \qquad \left| \int b(t)\, e_\lambda(\gamma(t))\, dt \right| \le C_{M,b} (\log \lambda)^{-1/2},$$

where $C_{M,b}$ depends on M and b, but not on γ. Additionally, if γ_{per} is a periodic geodesic and if $|\gamma_{per}|$ denotes its length then for $\lambda \gg 1$

$$(1.4) \qquad \left| \int_{\gamma_{per}} e_\lambda\, ds \right| \le C_M\, |\gamma_{per}|\, (\log \lambda)^{-1/2},$$

where C_M depends only on (M, g).

If one uses a partition of unity argument, it is clear that (1.3) implies (1.4). So we only need to prove the former.

The broad strategy will be similar to the earlier work of Chen and the first author [4]. We shall need to refine the stationary phase arguments used there a bit and use the Gauss-Bonnet theorem to exploit the aforementioned "defects" of quadrilaterals that arise in these arguments, which allow one to obtain favorable control of lower bounds for first and second derivatives of the phase functions occurring in the stationary phase arguments (unlike in the case of the two-torus).

This paper is organized as follows. In the next section we shall show that we can prove (1.3) by estimating integrals over geodesics in the universal cover of (M, g) that arise from reproducing kernels for eigenfunctions. We shall also see here that (1.3) also holds when the eigenfunctions are replaced by appropriate types of quasi-modes. In §3, using the Gauss-Bonnet theorem and triangle comparison theorems, we shall collect the geometric facts that we shall need for our estimates. In §4 we shall derive some simple one-dimensional stationary phase estimates that will be needed for our proof. In the next section, we shall use the Hadamard parametrix to show that the oscillatory integrals that we need to estimate lend themselves to these stationary phase estimates. We shall also show that we can get favorable bounds for first and second derivatives of the phase functions using the aforementioned geometric facts. In the final section we put things together and finish the proof of our main estimate (1.3).

In what follows, as we may, we shall assume that the injectivity radius of (M, g) is ten or more and that its nonpositive curvature is pinched below by -1, i.e., $-1 \leq K \leq 0$.

2. Hadamard's theorem and a standard reduction

To prove (1.3) let us first fix a real-valued function $\rho \in \mathcal{S}(\mathbb{R})$ satisfying

$$\rho(0) = 1 \quad \text{and} \quad \hat{\rho}(\tau) = 0, \quad |\tau| \geq 1/4.$$

Then since $\rho(T(\lambda - \sqrt{-\Delta_g}))e_\lambda = e_\lambda$, for any $T > 0$, in order to prove (1.3) it suffices to show that we can choose $T = T(\lambda)$ so that for $\lambda \gg 1$ we have the uniform bounds

$$(2.1) \quad \left| \int b(t) \big(\rho(T(\lambda - \sqrt{-\Delta_g}))f\big)(\gamma(t)) \, dt \right| \leq C_{M,b} \, (\log \lambda)^{-1/2} \, \|f\|_{L^2(M)}.$$

To do this we shall take

$$(2.2) \qquad\qquad\qquad T = c \log \lambda,$$

where $c = c_M > 0$ is a small constant depending on (M, g).

Let $\{e_j\}$ be an orthonormal basis of eigenfunctions with eigenvalues $\{\lambda_j\}$, and let

$$E_j f = \langle f, e_j \rangle e_j,$$

denote the projection of $f \in L^2(M)$ onto the eigenspace with eigenvalue λ_j. Then since $\rho(\tau) \geq 1/2$ for $|\tau| \leq \delta$, some $\delta > 0$, clearly (2.1)-(2.2) imply that

$$(2.3) \quad \left| \int b(t) \big(\rho(T(\lambda - \sqrt{-\Delta_g}))\chi_{[\lambda - (\log \lambda)^{-1}, \lambda + (\log \lambda)^{-1}]}f\big)(\gamma(t)) \, dt \right|$$
$$\leq C_{M,b} \, (\log \lambda)^{-1/2} \, \|f\|_{L^2(M)},$$

if

$$\chi_{[\lambda - (\log \lambda)^{-1}, \lambda + (\log \lambda)^{-1}]}f = \sum_{|\lambda - \lambda_j| \leq (\log \lambda)^{-1}} E_j f$$

denotes the projection of f onto a spectral band of width $(\log \lambda)^{-1}$ about λ. Using standard arguments (see [14]) one sees from this that we have

$$(2.4) \qquad\qquad \left| \int b(t) \, \Psi_\lambda(\gamma(t)) \, dt \right| \leq C_{b,M} (\log \lambda)^{-1/2},$$

for quasi-modes Ψ_λ satisfying

$$(2.5) \qquad (\log \lambda/\lambda)\|(\Delta_g + \lambda^2)\Psi_\lambda\|_{L^2(M)} + \|\Psi_\lambda\|_{L^2(M)} \leq 1$$

with $\lambda \gg 1$. Of course (2.4) implies that when (2.5) holds we also have the following analog of (1.4)

$$(2.6) \qquad \left| \int_{\gamma_{per}} \Psi_\lambda \, ds \right| \leq C_M |\gamma_{per}| (\log \lambda)^{-1/2}$$

if γ_{per} is a periodic geodesic in M.

To set up the proof of (2.1) we first note that the kernel of the operator there is given by

$$\rho\big(T(\lambda - \sqrt{-\Delta_g})\big)(x,y) = \sum_j \rho(T(\lambda - \lambda_j))e_j(x)\overline{e_j(y)}.$$

By Schwarz's inequality, we would have (2.1) if we could show that

$$\int_M \left| \int b(t) \sum_j \rho\big(T(\lambda - \lambda_j)\big) \, e_j(\gamma(t))\overline{e_j(y)} \, dt \right|^2 dV_g(y) \leq C_{b,M}(\log \lambda)^{-1},$$

$$\lambda \gg 1.$$

By orthogonality, if $\chi(\tau) = (\rho(\tau))^2$, this is equivalent to showing that if

$$b(t,s) = b(t)b(s) \in C_0^\infty((-1/2, 1/2)^2),$$

then

$$(2.7) \quad \left| \iint b(t,s) \sum_j \chi(T(\lambda - \lambda_j))e_j(\gamma(t))\overline{e_j(\gamma(s))} \, dtds \right| \leq C_{b,M}(\log \lambda)^{-1},$$

if $\lambda \gg 1$.

Note that

$$\sum_j \chi(T(\lambda - \lambda_j))e_j(x)\overline{e_j(y)} = \frac{1}{2\pi T} \int \hat{\chi}(\tau/T)e^{i\tau\lambda}\big(e^{-i\tau\sqrt{-\Delta_g}}\big)(x,y) \, d\tau.$$

As a first step in the proof of (2.7) fix a bump function $\beta \in C_0^\infty(\mathbb{R})$ satisfying

$$\beta(\tau) = 1, \ |\tau| \leq 3 \quad \text{and} \quad \beta(\tau) = 0, \ |\tau| \geq 4.$$

Then the proof of Lemma 5.1.3 in [11] shows that, because of our assumption that the injectivity radius of (M, g) is ten or more, we can write

$$(2.8) \quad \frac{1}{2\pi T} \int \beta(\tau)\hat{\chi}(\tau/T)e^{i\tau\lambda}\big(e^{-i\tau\sqrt{-\Delta_g}}\big)(x, y)\, d\tau$$

$$= \frac{\lambda^{1/2}}{T} \sum_{\pm} a_{\pm}(\lambda; d_g(x, y))e^{\pm i\lambda d_g(x,y)} + O(1/T),$$

if d_g denotes the Riemannian distance on (M, g), where

$$(2.9) \qquad\qquad \left| \frac{d^j}{dr^j}a_{\pm}(\lambda; r) \right| \leq C_j r^{-j-1/2} \quad \text{if} \ \ r \geq \lambda^{-1},$$

and

$$(2.10) \qquad\qquad |a_{\pm}(\lambda; r)| \leq C\lambda^{1/2} \quad \text{if} \ \ 0 \leq r \leq \lambda^{-1}.$$

Since $d_g(\gamma(t), \gamma(s)) = |t-s|$, we conclude from (2.8) that we would have, for a given $c > 0$,

$$(2.11) \quad \frac{1}{2\pi T}\left| \iiint b(t, s)\beta(\tau)\hat{\chi}(\tau/T)e^{i\tau\lambda}\big(e^{-i\tau\sqrt{-\Delta_g}}\big)(\gamma(t), \gamma(s))\, d\tau dt ds \right|$$

$$\leq C_{b,M}(\log \lambda)^{-1}, \quad \text{if} \ \ T = c \log \lambda,$$

if

$$\lambda^{1/2}\left| \iint b(t, s)e^{\pm i\lambda|t-s|}a_{\pm}(\lambda; |t-s|)\, dt ds \right| \leq C_{b,M}.$$

Since the latter estimate is a simple consequence of (2.9) and (2.10), we obtain (2.11).

In view of (2.11), we conclude that we would have (2.7) if we could obtain the following bounds for the remaining part of $\chi(T(\lambda - \sqrt{-\Delta_g}))$:

$$\frac{1}{2\pi T}\left| \iiint b(t, s)\big(1 - \beta(\tau)\big)\, \hat{\chi}(\tau/T)e^{i\tau\lambda}\big(e^{-i\tau\sqrt{-\Delta_g}}\big)(\gamma(t), \gamma(s))\, d\tau ds dt \right|$$

$$\leq C_{b,M}(\log \lambda)^{-1},$$

if T is as in (2.2). Note that for $T \geq 1$ we have the uniform bounds

$$\frac{1}{2\pi T}\left| \int \big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\tau(\lambda+\lambda_j)}\, d\tau \right| \leq C_N(1 + |\lambda + \lambda_j|)^{-N}, \ \ N = 1, 2, \ldots,$$

and so, since $\lambda_j \geq 0$ and $\lambda \gg 1$,

$$(2.12) \quad \frac{1}{2\pi T}\Big| \int \big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\tau\lambda}\big(e^{i\tau\sqrt{-\Delta_g}}\big)(\gamma(t),\gamma(s))\,d\tau\,\Big|$$
$$\leq C_N(1+\lambda)^{-N}, \quad N = 1, 2, \ldots.$$

Thus, by Euler's formula, to prove (2.12), it suffices to show that if T is as in (2.2) (for an appropriate choice of $c = c_M > 0$) we have

$$(2.13) \quad \Big| \iiint b(t,s)\big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\tau\lambda}\big(\cos\tau\sqrt{-\Delta_g}\big)(\gamma(t),\gamma(s))\,d\tau dt ds \,\Big|$$
$$\leq C_{b,M}.$$

Here $\big(\cos\tau\sqrt{-\Delta_g}\big)(x,y)$ is the kernel for the map $C^\infty(M) \ni f \to u \in C^\infty(\mathbb{R} \times M)$ solving the Cauchy problem with initial data $(f,0)$, i.e.,

$$\big(\partial_\tau^2 - \Delta_g\big)u = 0, \quad u(0,\cdot) = f, \quad \partial_\tau u(0,\cdot) = 0.$$

To be able to compute the integral in (2.13) we need to relate this wave kernel to the corresponding one in the universal cover for (M,g). Recall that by a theorem of Hadamard (see [5, Chapter 7]) for every point $P \in M$, the exponential map at P, $\exp_P : T_P M \to M$ is a covering map. We might as well take $P = \gamma(0)$ to be the midpoint of the geodesic segment$\{\gamma(t) : |t| \leq \frac{1}{2}\}$. If we identify $T_P M$ with $\mathbb{R}^2$, and let κ denote this exponential map then $\kappa : \mathbb{R}^2 \to M$ is a covering map. We also will denote by $\tilde{g}$ the metric on $\mathbb{R}^2$ which is the pullback via κ of the metric g on M. Also, let Γ denote the group of deck transformations, which are the diffeomorphisms α from $\mathbb{R}^2$ to itself preserving κ, i.e., $\kappa = \kappa \circ \alpha$. Next, let

$$D_{Dir} = \{\tilde{y} \in \mathbb{R}^2 : d_{\tilde{g}}(0,\tilde{y}) < d_{\tilde{g}}(0,\alpha(\tilde{y})), \forall \alpha \in \Gamma, \ \alpha \neq Identity\}$$

be the Dirichlet domain for $(\mathbb{R}^2, \tilde{g})$, where $d_{\tilde{g}}(\cdot,\cdot)$ denotes the Riemannian distance function for $\mathbb{R}^2$ corresponding to the metric $\tilde{g}$. We can then add to D_{Dir} a subset of $\partial D_{Dir} = \overline{D_{Dir}}\backslash\text{Int}\,(D_{Dir})$ to obtain a natural fundamental domain D, which has the property that $\mathbb{R}^2$ is the disjoint union of the $\alpha(D)$ as α ranges over Γ and $\{\tilde{y} \in \mathbb{R}^2 : d_{\tilde{g}}(0,\tilde{y}) < 10\} \subset D$ since we are assuming that the injectivity radius of (M,g) is more than ten. It then follows that we can identify every point $x \in M$ with the unique point $\tilde{x} \in D$ having the property that $\kappa(\tilde{x}) = x$. Let also $\tilde{\gamma}(t), |t| \leq \frac{1}{2}$ similarly denote those points in D corresponding to our geodesic segment $\gamma(t), |t| \leq \frac{1}{2}$ in M. Then

$\{\tilde\gamma(t) : |t| \leq \frac{1}{2}\}$ is a line segment of unit length whose midpoint is the origin, and we shall denote just by $\tilde\gamma$ the line through the origin containing this segment. Note that $\tilde\gamma$ then is a geodesic in $\mathbb{R}^2$ for the metric $\tilde g$, and the Riemannian distance between two points on $\tilde\gamma$ agrees with their Euclidean distance. Finally, if $\Delta_{\tilde g}$ denotes the Laplace-Beltrami operator associated to $\tilde g$ then since solutions of the above Cauchy problem for (M, g) correspond exactly to periodic (i.e. Γ-invariant) solutions of the corresponding Cauchy problem associated to $\partial_t^2 - \Delta_{\tilde g}$, we have the following important formula relating the wave kernel on (M, g) to the one for the universal cover $(\mathbb{R}^2, \tilde g)$:

$$(2.14) \qquad \big(\cos \tau \sqrt{-\Delta_g}\big)(x, y) = \sum_{\alpha \in \Gamma} \big(\cos \tau \sqrt{-\Delta_{\tilde g}}\big)(\tilde x, \alpha(\tilde y)).$$

Due to this formula, we would have (2.13) if we could show that for T as in (2.2),

$$(2.15)$$
$$\sum_{\alpha \in \Gamma} \Big| \iiint b(t, s)\big(1 - \beta(\tau)\big)\hat\chi(\tau/T)e^{i\tau\lambda}\big(\cos \tau \sqrt{-\Delta_{\tilde g}}\big)(\tilde\gamma(t), \alpha(\tilde\gamma(s)))\, d\tau dt ds \Big|$$
$$\leq C_{b,M}.$$

By Huygens principle, $\big(\cos \tau \sqrt{-\Delta_{\tilde g}}\big)(\tilde x, \tilde y) = 0$ if $d_{\tilde g}(\tilde x, \tilde y) > \tau$, where $d_{\tilde g}$ denotes the Riemannian distance on $(\mathbb{R}^2, \tilde g)$. Since $\chi = \rho^2$ our assumption that $\rho(\tau) = 0$ for $|\tau| \geq 1/4$ means that the integrand in (2.15) vanishes when $|\tau| \geq T/2$. Therefore, since there are $O(\exp(C_M T))$ "translates" of D satisfying $d_{\tilde g}(D, \alpha(D)) < T$, we conclude that the sum in (2.15) involves $O(\exp(C_M T))$ nonzero terms. Based on this, we conclude that we would have (2.15) if we could prove the following.

Proposition 2.1. *Given our (M, g) satisfying* (1.2) *we can fix $c = c_M > 0$ so that we have for $\lambda \gg 1$*

$$(2.16)$$
$$\Big| \iiint b(t, s)\big(1 - \beta(\tau)\big)\hat\chi(\tau/T)e^{i\tau\lambda}\big(\cos \tau \sqrt{-\Delta_{\tilde g}}\big)(\tilde\gamma(t), \alpha(\tilde\gamma(s)))\, d\tau dt ds \Big|$$
$$\leq C_{b,M}\lambda^{-\delta_M} \quad \textit{if} \ \ T = c \log \lambda,$$

for some $\delta_M > 0$ which depends on M but not on b or λ.

The power δ_M in (2.16) depends on the power N in our assumption (1.2). As we shall see we can take it to be $1/10N$.

3. Geometric tools

In this section we are working with $\mathbb{R}^2$ equipped with the metric $\tilde{g}$ which is the pullback of the metric g on M via the covering map. Thus, if K denotes the Gaussian curvature on $(\mathbb{R}^2, \tilde{g})$ and if $B_r(\tilde{x})$ denotes a geodesic ball of radius r centered at some $\tilde{x} \in \mathbb{R}^2$, our curvature assumption (1.2) on (M, g) lifts to

$$(3.1) \quad \fint_{B_r(\tilde{x})} K \, dV \le -\delta r^N, \quad \text{if } r < 1, \text{ and } \tilde{x} \in \mathbb{R}^2,$$

$$\text{for some } \delta > 0 \text{ and } N \ge 2.$$

To prove our estimates for period integrals over geodesics we shall require a couple consequences of elementary results from Riemannian geometry. One is based on (3.1) and the Gauss-Bonnet theorem. As we pointed this assumption is valid when the curvature on M is pinched from above by a negative constant but allows situations where the curvature is nonpositive and vanishes on lower dimensional sets. The other result is based on Togonogov's theorem and the fact that we are assuming that the curvature on (M, g) and hence on $(\mathbb{R}^2, \tilde{g})$ is pinched below by -1.

Let us now state the two geometric results that will play a key role in our analysis.

Proposition 3.1. *Let $\tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(s)$, $|s|, |t| \le 1/2$ be two unit length geodesics in $(\mathbb{R}^2, \tilde{g})$ parameterized by arc length satisfying $d_{\tilde{g}}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(s)) \ge 1$, $|t|, |s| \le 1/2$. Suppose that there is a $(t_0, s_0) \in [-1/2, 1/2] \times [-1/2, 1/2]$ so that the geodesic through $\tilde{\gamma}_1(t_0)$ and $\tilde{\gamma}_2(s_0)$ intersects $\tilde{\gamma}_1$ with angle θ_{t_0} and $\tilde{\gamma}_2$ with angle θ_{s_0} (see Figure 1) and suppose further that*

$$(3.2) \qquad\qquad \theta_{t_0}, \theta_{s_0} \in [\pi/2 - \lambda^{-1/3}, \pi/2].$$

Then if

$$(3.3) \qquad\qquad\qquad \varepsilon_0 = 1/5N$$

where N is as in (3.1) and if λ is larger than a fixed constant

$$(3.4) \quad \max(\pi/2 - \theta_t, \pi/2 - \theta_s) \ge \lambda^{-1/4},$$
$$\text{if } t, s \in [-1/2, 1/2] \text{ and } \max(|t - t_0|, |s - s_0|) \ge \lambda^{-\varepsilon_0},$$

if θ_t denotes the intersection angle of $\tilde{\gamma}_1$ and the geodesic through $\tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(s)$ and θ_s denotes the intersection angle of this geodesic and $\tilde{\gamma}_2$.
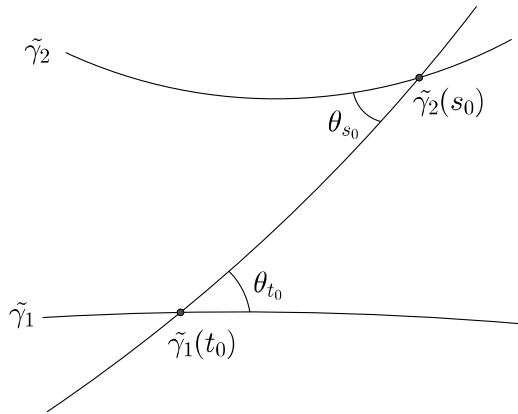
Figure 1

The other proposition that we need is the following simple consequence of Toponogov's theorem which was used in earlier joint work of the first author and Blair [2].

Proposition 3.2. *As above assume that the Gaussian curvature of $(\mathbb{R}^2, \tilde{g})$ satisfies*

$$K \geq -1.$$

Let $\tilde{\gamma}(t)$, $t \in \mathbb{R}$, be a geodesic with $\tilde{\gamma}(0) = P_0$. Given $T \gg 1$, let $C(\theta; T)$, $\theta \ll 1$, denote the set of points $Q \in B_T(P_0)$ which lie on a geodesic though P_0 which intersects $\tilde{\gamma}$ of angle $\leq \theta$. Thus, $C(\theta; T)$ is the intersection of the geodesic ball $B_T(P_0)$ of radius T about P_0 with the cone of aperture θ about $\tilde{\gamma}$ with vertex P_0. Then if $0 < r \leq 1$ and, if

$$\mathcal{T}_r(\tilde{\gamma}) = \{x \in \mathbb{R}^2 : d_{\tilde{g}}(x, \tilde{\gamma}) \leq r\}$$

denotes the tube of radius r about $\tilde{\gamma}$, we have that

$$(3.5) \qquad C(\theta_{T,r}; T) \subset \mathcal{T}_r(\tilde{\gamma}), \quad \text{if} \ \ \sin \tfrac{1}{2}\theta_{T,r} = \frac{\sinh \tfrac{1}{2}r}{\sinh T}, \quad \text{if} \ T > 0.$$

To prove Proposition 3.1 we shall use a couple of special cases for the Gauss-Bonnet theorem (see [5]) concerning the sum of the interior angles α_j for geodesic quadrilaterals Q and geodesic triangles $\mathcal{T}$ in $(\mathbb{R}^2, g)$. In the first case we define the "defect" of Q, Defect Q, to be 2π minus the sum of the four interior angles at the vertices, and in the case of $\mathcal{T}$, we define Defect $\mathcal{T}$ to be π minus the sum of its three interior angles, as shown in Figure 2.
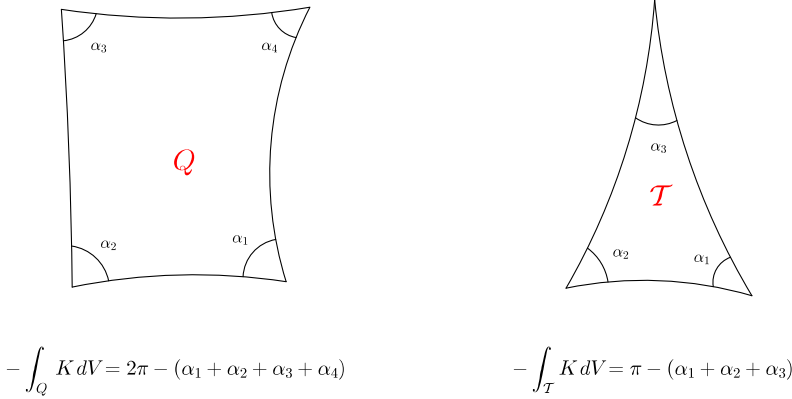
$$-\int_Q K\, dV = 2\pi - (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \qquad -\int_{\mathcal{T}} K\, dV = \pi - (\alpha_1 + \alpha_2 + \alpha_3)$$

Figure 2: Gauss-Bonnet Theorem

Then, by the Gauss-Bonnet theorem we have

$$\text{Defect } Q = -\int_Q K\, dV$$

$$\text{Defect } \mathcal{T} = -\int_{\mathcal{T}} K\, dV.$$

Proof of Proposition 3.1. Suppose that, for a given $(t_0, s_0) \in [-1/2, 1/2] \times [-1/2, 1/2]$, (3.2) is valid. By symmetry it suffices to show that the conclusion in (3.4) is valid if we assume that $t \in [-1/2, 1/2] \setminus (t_0 - \lambda^{-\varepsilon_0}, t_0 + \lambda^{-\varepsilon_0})$ and $|s| \leq 1/2$. If $s \neq s_0$ there are two cases as shown in Figure 3: Either the geodesic segment connecting $\tilde{\gamma}_1(t_0)$ and $\tilde{\gamma}_2(s_0)$ and the one connecting $\tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(s)$ do not intersect or intersect. In the first case we obtain a geodesic quadrilateral Q with vertices $\tilde{\gamma}_1(t_0)$, $\tilde{\gamma}_1(t)$, $\tilde{\gamma}_2(s_0)$ and $\tilde{\gamma}_2(s)$, while in the other case we obtain two geodesic triangles using those four points and the intersection point of the aforementioned geodesic segments. To reach this conclusion we are using the fact that since we are assuming $K \leq 0$, two geodesics in $(\mathbb{R}^2, \tilde{g})$ are disjoint or intersect at exactly one point by the Cartan-Hadamard theorem.

In the first case, let $\alpha_{t_0}, \alpha_t, \alpha_{s_0}$ and α_s denote the interior angles of the geodesic quadrilateral Q at vertices $\tilde{\gamma}_1(t_0)$, $\tilde{\gamma}_1(t)$, $\tilde{\gamma}_2(s_0)$ and $\tilde{\gamma}_2(s)$, respectively. Note that $\alpha_t = \theta_t$ if $0 < \alpha_t \leq \pi/2$ and $\theta_t = \pi - \alpha_t$ if $\alpha_t \in (\pi/2, \pi)$, etc.

As we mentioned before, by the Gauss-Bonnet theorem

$$(3.6) \qquad \text{Defect } Q = 2\pi - \left(\alpha_{t_0} + \alpha_t + \alpha_{s_0} + \alpha_s\right) = -\int_Q K\, dV.$$

As in Figure 3, if we consider the geodesic ball, B_r, $r = \lambda^{-\varepsilon_0}/100$, which is tangent to $\tilde{\gamma}_1$ at $\tilde{\gamma}_1((t + t_0)/2)$ and on the same side of $\tilde{\gamma}_1$ as Q, it follows that, if λ is larger than a fixed constant depending on the metric, we have $B_r \subset Q$ if $\alpha_t \notin (0, \pi/4) \cup (3\pi/4, \pi)$. We may make this assumption since otherwise we have $\pi/2 - \theta_t \geq \pi/4 \gg \lambda^{-1/4}$. Thus, in the nontrivial case where $\alpha_t \notin (0, \pi/4) \cup (3\pi/4, \pi)$, since $K \leq 0$, we have for large enough λ

$$\text{Defect } Q \geq - \int_{B_r} K \, dV \geq \delta \lambda^{-N\varepsilon_0} = \delta \lambda^{-1/5},$$

for some $\delta > 0$ by (3.1). Since we are assuming (3.2) we must have $|\pi/2 - \alpha_{s_0}|$, $|\pi/2 - \alpha_{t_0}| \leq \lambda^{-1/3}$ and therefore

$$(\pi/2 - \alpha_t) + (\pi/2 - \alpha_s) \geq \delta \lambda^{-1/5} - 2\lambda^{-1/3} \geq \tfrac{\delta}{2} \lambda^{-1/5} \quad \text{if} \quad \lambda \gg 1,$$

which of course implies that

$$(3.7) \qquad\qquad \max\big(\pi/2 - \theta_t, \pi/2 - \theta_s \big) \geq \lambda^{-1/4},$$

if λ is larger than a fixed constant which is independent of our two geodesic segments $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.

As noted before, the other case where $s \neq s_0$ and $|t - t_0| \geq \lambda^{-\varepsilon_0}$ is where the geodesics connecting $\tilde{\gamma}_1(t_0)$ and $\tilde{\gamma}_2(s_0)$ and the one connecting $\tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(s)$ intersect at a point P. Then as in the second case Figure 3 we shall consider the geodesic triangle $\mathcal{T}$ with vertices $\tilde{\gamma}_1(t_0)$, $\tilde{\gamma}_1(t)$ and P. If α_{t_0}, α_t and α_P are the corresponding interior angles for $\mathcal{T}$, as before, we may assume that $\alpha_t \notin (0, \pi/4) \cup (3\pi/4, \pi)$, for, if not, (3.7) trivially holds. Then, as in Figure 3, if λ is large enough the geodesic ball B_r, $r = \lambda^{-\varepsilon_0}/100$, which is tangent to $\tilde{\gamma}_1$ at $\tilde{\gamma}_1((t + t_0)/2)$ and on the same side as $\mathcal{T}$ must be contained in $\mathcal{T}$ if λ is larger than a fixed constant depending on the metric. Therefore, by the Gauss-Bonnet theorem

$$\pi - (\alpha_{t_0} + \alpha_t + \alpha_P) = - \int_{\mathcal{T}} K \, dV \geq - \int_{B_r} K \, dV \geq \delta \lambda^{-N\varepsilon_0} = \delta \lambda^{-1/5}.$$

Therefore, by our assumption (3.2) and a variation of the earlier argument

$$\pi/2 - (\alpha_t + \alpha_P) \geq \tfrac{\delta}{2} \lambda^{-1/5} \quad \text{if} \quad \lambda \gg 1.$$

Since $\alpha_P > 0$ this means that we must have

$$\theta_t = \alpha_t \in (0, \tfrac{\pi}{2} - \tfrac{\delta}{2} \lambda^{-1/5}) \subset (0, \tfrac{\pi}{2} - \lambda^{-1/4}) \quad \text{if} \quad \lambda \gg 1,$$

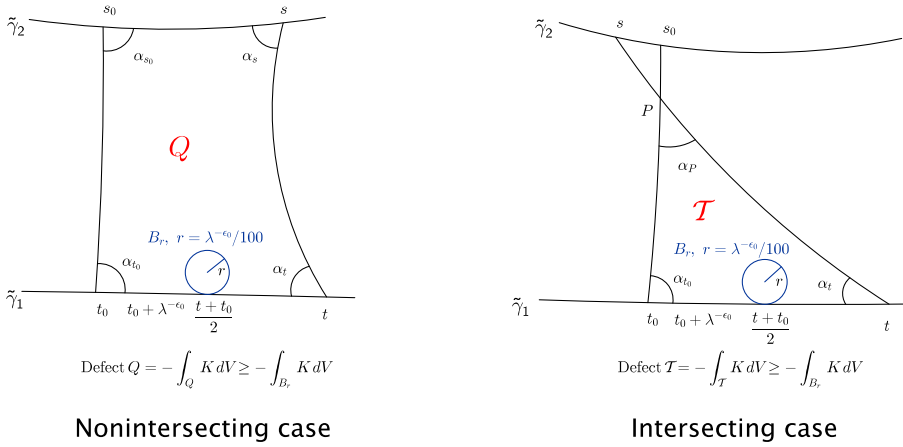and so (3.7) is valid in this case as well.

Figure 3: Two cases

The one remaining case to consider is where $s = s_0$ and $|t - t_0| \geq \lambda^{-\varepsilon_0}$. One obtains (3.7) for this case as well by using this argument but with $\mathcal{T}$ now being the geodesic triangle with vertices $\tilde{\gamma}_1(t_0)$, $\tilde{\gamma}_1(t)$ and $\tilde{\gamma}_2(s_0)$, which completes the proof. $\qquad\square$

Even though Proposition 3.2 was proved in [2], for the sake of completeness we shall give its simple proof now.

Proof of Proposition 3.2. Recall that we are trying to show that

$$C(\theta_{T,r}; T) \subset \mathcal{T}_r(\tilde{\gamma}), \quad \text{if } \sin \tfrac{1}{2}\theta_{T,r} = \frac{\sinh \tfrac{1}{2}r}{\sinh T}.$$

We shall work in geodesic normal coordinates about P_0 and we may assume that, in these coordinates, $\tilde{\gamma} = \{(t, 0) : t \in \mathbb{R}\}$. $C(\theta; T)$ then is the intersection of the geodesic ball of radius $T > 0$ about our origin with the cone of aperture θ about $\tilde{\gamma}$. Also, $\mathcal{T}_r(\tilde{\gamma})$ denotes the closed tube of fixed radius $0 < r < 1$ about $\tilde{\gamma}$.

Since, for fixed r, $T \to \theta_{T,r}$ is monotonically decreasing, it suffices to show that a point Q with coordinates $T\omega$, $\omega \in S^{n-1}$, belongs to $\mathcal{T}_r(\tilde{\gamma})$ if the angle, $\sphericalangle(\omega, (1, 0))$, is $\leq \theta_{T,r}$. In other words, to obtain (3.5), it suffices to show that

(3.8) $$\Sigma(T; \theta_{T,r}) \subset \mathcal{T}_r(\tilde{\gamma}),$$

if $\Sigma(T, \theta)$ denotes all points Q with coordinates $T\omega$ satisfying $\sphericalangle(\omega, \mathbf{1}) \leq \theta$, with $\mathbf{1} = (1, 0)$.

Clearly $\Sigma(T; \theta) \subset \mathcal{T}_r(\tilde{\gamma})$ when θ is very small (depending on T). So choose the maximal $\Theta_{T,r} \leq \pi/2$ so that $\Sigma(T; \theta) \subset \mathcal{T}_r(\tilde{\gamma})$ when $0 < \theta < \Theta_{T,r}$. It follows that there must be a point Q with coordinates, $T\omega_0$, satisfying $\sphericalangle(\omega_0, \mathbb{1}) = \Theta_{T,r}$ and $d_{\tilde{g}}(Q, \tilde{\gamma}) = r$. Also, (3.8) is valid when $\theta_{T,r}$ is replaced by $\Theta_{T,r}$. So we would have (3.5) and be done if we could show that

$$(3.9) \qquad\qquad \Theta_{T,r} \geq \theta_{T,r}.$$

At this point, we shall use Toponogov's theorem. First consider the geodesic triangle, $\triangle_{\Theta_{T,r}}^{\tilde{g}}$, in $(\mathbb{R}^2, \tilde{g})$ with vertices Q and the point with coordinates 0 and the point P with coordinates $(T, 0)$. It is an isosceles triangle since the geodesics connecting the point with coordinates 0 with P and Q, respectively, each have length T. The point P lies on $\tilde{\gamma}$ and hence if $\tilde{\gamma}_{opp}$ is the third side of our geodesic triangle, which connects P and Q, we must have that its length, $\ell(\tilde{\gamma}_{opp})$ satisfies

$$\ell(\tilde{\gamma}_{opp}) = d_{\tilde{g}}(P, Q) \geq r,$$

since, as we pointed out before, we must have $d_{\tilde{g}}(Q, \tilde{\gamma}) = r$. The angle at the vertex whose coordinates are the origin, by construction, is $\Theta_{T,r}$, and the two sides passing through it each have length T. The third side of our isosceles triangle, $\tilde{\gamma}_{opp}$, is called a "Rauch hinge".

Consider as well, an isosceles triangle, $\triangle_{\Theta_{T,r}}^{\mathbb{H}^2}$, in two-dimensional hyperbolic space, $\mathbb{H}^2$, having two sides of equal length T, angle $\Theta_{T,r}$ at the associated vertex and "Rauch hinge" γ_{opp}, with length $\ell(\gamma_{opp})$. By Toponogov's theorem (see [3, Theorem 2.2 (B)]), since we are assuming that the sectional curvatures of $(\mathbb{R}^2, \tilde{g})$ satisfy $-1 \leq K \leq 0$, we must have

$$\ell(\gamma_{opp}) \geq \ell(\tilde{\gamma}_{opp}) \geq r.$$

By properties of isosceles triangles in $\mathbb{H}^2$, the ray bisecting the triangle at the vertex spanned by the two sides of equal length T must intersect the Rauch hinge, $\gamma_{opp} \in \triangle_{\Theta_{T,r}}^{\mathbb{H}^2}$, orthogonally at its midpoint. Consequently, by hyperbolic trigonometry, we must have

$$\sin \tfrac{1}{2}\Theta_{T,r} = \frac{\sinh(\ell(\gamma_{opp})/2)}{\sinh T} \geq \frac{\sinh \frac{1}{2}r}{\sinh T} = \sin \tfrac{1}{2}\theta_{T,r}.$$

Thus, (3.9) is valid and the proof of Proposition 3.2 is complete. $\qquad\qquad\square$

4. Stationary phase bounds

Let us now collect the bounds for oscillatory integrals that we shall use to prove our bounds for smoothly localized integrals over geodesic segments and period integrals. These are more precise variations of the ones used in the earlier work of Chen and the first author in [4].

The first concerns estimates for one-dimensional oscillatory integrals with natural lower bounds for first derivatives of the phase function.

Lemma 4.1. *Suppose that $\phi \in C^\infty((-1,1))$, is real valued and that $a \in C_0^\infty(\mathcal{I})$, where $\mathcal{I} \subset (-1,1)$ is an open interval and set*

$$(4.1) \qquad I(\lambda) = \int e^{i\lambda\phi(t)}\, a(t)\, dt, \quad \lambda \geq 1.$$

Suppose that

$$(4.2) \qquad |\partial_t \phi| \geq \lambda^{-1/2+\delta}, \quad \text{on } \mathcal{I}$$

and suppose further that for $0 \leq j \leq N = \lceil 4\delta^{-1} \rceil$.

$$(4.3) \qquad |\partial_t^j \phi'| \leq \lambda^{\delta/2}, \quad \text{and } |\partial_t^j a| \leq C_j \lambda^{j/2} \text{ on } \mathcal{I}.$$

Then if $0 < \delta < 1/2$

$$(4.4) \qquad |I(\lambda)| \leq C\lambda^{-2},$$

where C depends only on δ and the C_j.

As the following result says, we also can obtain favorable estimates for one-dimensional oscillatory integrals if we do not have the above hypothesis concerning lower bounds for the first derivatives of the phase, but rather have related lower bounds for second derivatives.

Lemma 4.2. *Set*

$$(4.5) \qquad J(\lambda) = \int e^{i\lambda\varphi(t)}\, b(t)\, dt, \quad \lambda \geq 1,$$

where $b \in C_0^\infty(\mathcal{I})$, where $\mathcal{I}$ is as above, and that $\varphi \in C^\infty((-1,1))$ is real valued. Suppose further that $0 \in \mathcal{I}$,

$$(4.6) \qquad |\varphi'(0)| \leq \lambda^{-1/2+\delta}, \quad \text{and } \lambda^{-\delta/2} \leq |\varphi''(t)| \leq \lambda^{\delta/2}, \ t \in \mathcal{I},$$

and that

(4.7) $$|b| \leq 1, \quad |b'| \leq \lambda^{1/2}.$$

Then if $0 < \delta \leq 1/4$ there is a constant $C = C_\delta$ so that

(4.8) $$|J(\lambda)| \leq C\lambda^{-1/2+2\delta}.$$

Proof of Lemma 4.1. Note that

$$Le^{i\lambda\phi(t)} = e^{i\lambda\phi(t)}, \quad \text{if} \;\; L = \frac{1}{i\lambda\phi'(t)}\frac{d}{dt}.$$

Therefore, if L^* denotes the adjoint of L, for every $N = 1, 2, 3, \ldots,$

(4.9) $$I(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda\phi(t)}\big((L^*)^N a\big)(t)\, dt.$$

By a simple induction argument, one shows that $(L^*)^N a$ is a finite linear combination of terms of the form

$$\lambda^{-N}(\phi')^{-2N+j}\left(\frac{d}{dt}\right)^j a \cdot \left(\frac{d}{dt}\right)^{\beta_1}\phi' \cdots \left(\frac{d}{dt}\right)^{\beta_K}\phi',$$
$$\text{where} \quad j, \; \beta_\nu \in \{0, 1, 2, \ldots, N\}, \quad \text{and} \;\; K \leq N.$$

Therefore by (4.2) and (4.3) each of these terms is bounded by

$$C_N\lambda^{-N}\lambda^{(2N-j)(1/2-\delta)}\lambda^{j/2}\lambda^{N\delta/2} \leq C_N\lambda^{-N\delta/2}.$$

This and (4.9) gives us (4.4).   □

Proof of Lemma 4.2. Fix $\rho \in C^\infty(\mathbb{R})$ satisfying

(4.10) $$|\rho| \leq 1, \;\; \rho(t) = 1, \; |t| \leq 1, \;\; \text{and} \;\; \rho(t) = 0, \; |t| \geq 2.$$

Clearly

$$\left|\int e^{i\lambda\varphi(t)}\, b(t)\, \rho(\lambda^{1/2-2\delta}t)\, dt\right| \leq 4\lambda^{-1/2+2\delta},$$

and so it suffices to show that

$$\widetilde{J}(\lambda) = \int e^{i\lambda\varphi(t)}\, b(t)(1 - \rho(\lambda^{1/2-2\delta}t))\, dt$$

satisfies the bounds in (4.8).

If we integrate by parts as in the proof of Lemma 4.1 and use (4.7) and (4.10), we see that

$$(4.11) \quad |\widetilde{J}(\lambda)| \leq \lambda^{-1} \int_{\lambda^{-1/2+2\delta} \leq |t| \leq 1} \Big(|\varphi'(t)|^{-1} \big| \tfrac{d}{dt}\big[b(t)(1 - \rho(\lambda^{1/2-2\delta}t)) \big] \big| $$
$$+ |b(t)|\, |\varphi'(t)|^{-2} |\varphi''(t)| \Big) \, dt.$$

By the mean value theorem and (4.6), for large enough $\lambda \geq 1$, we have

$$|\varphi'(t)| \geq \lambda^{-\delta/2}|t| - \lambda^{-1/2+\delta} \geq \frac{1}{2}\lambda^{-\delta/2}|t|, \quad \text{if } t \in \mathcal{I} \cap \{ t : \lambda^{-1/2+2\delta} \leq |t| \leq 1 \}.$$

Therefore, since we are assuming supp $b \subset \mathcal{I}$, by the second part of (4.6) and by (4.7)

$$|\widetilde{J}(\lambda)| \lesssim \lambda^{-1} \int_{\lambda^{-1/2+2\delta}}^{1} \big(\lambda^{\delta/2} t^{-1} \big[\lambda^{1/2} + t^{-1} \big] + \lambda^{3\delta/2} t^{-2} \big) \, dt$$
$$\lesssim \lambda^{-1/2+\delta/2} \ln \lambda + \lambda^{-1/2-3\delta/2} + \lambda^{-1/2-\delta/2}$$
$$\lesssim \lambda^{-1/2+2\delta},$$

as desired if $\lambda \gg 1$. □

We can combine Lemma 4.1 and Lemma 4.2 to obtain the following.

Proposition 4.3. *Suppose that $\phi \in C^\infty(\mathbb{R})$ is real valued and that $a \in C_0^\infty(\mathcal{I})$, where $\mathcal{I} \subset (-1/2, 1/2)$ is an open interval. Suppose that for some $0 < \delta < 1/2$*

$$(4.12) \qquad\qquad \lambda^{-\delta/2} \leq |\phi''(t)|, \quad t \in \mathcal{I}.$$

Suppose further that for $0 \leq j \leq N = \lceil 4\delta^{-1} \rceil$

$$(4.13) \qquad\qquad |\partial_t^j a(t)| \leq C_j \lambda^{j/2}$$

and that

$$(4.14) \qquad\qquad |\partial_t^j \phi'| \leq \lambda^{\delta/2}, \quad t \in \mathcal{I}.$$

Then

$$(4.15) \qquad\qquad \left| \int e^{i\lambda\phi(t)} a(t) \, dt \right| \leq C\lambda^{-1/2+2\delta},$$

where C depends only on δ and the above constants C_j, $j \leq \lceil 4\delta^{-1} \rceil$.

Proof. To see this, we note that if (4.2) is valid we can replace (4.12) by the stronger bounds in (4.4). For the other case, where (4.2) is not valid, there must be a point $t_0 \in \mathcal{I}$ where $|\phi'(t_0)| \leq \lambda^{-1/2+\delta}$. We then get (4.15) from (4.8) if we let the phase function φ in Lemma 4.2 be $\phi(t - t_0)$ and the bump function $b \in C_0^\infty(\mathcal{I} - \{t_0\})$ be $a(t - t_0)$, completing the proof. $\qquad\square$

5. Kernel bounds

To be able to use the results from the last two sections to prove Proposition 2.1 and thus complete the proof of Theorem 1.1 we need to calculate the kernels in (2.16), i.e,

$$(5.1) \qquad K_{T,\lambda}(x,y) = \int \big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\tau\lambda}\big(\cos\tau\sqrt{-\Delta}\big)(x,y)\, d\tau.$$

Here, since all the calculations from now on will be taking place in the universal cover, to simplify the notation, we are setting $\Delta = \Delta_{\tilde{g}}$. Also, in what follows Δ_x^N and Δ_y^N will denote N powers of the $\Delta_{\tilde{g}}$ with respect to the x and y variables, respectively.

Recall that the bump function β in (5.1) is supported in $(-4, 4)$ and equals one on $[-3, 3]$ and that $\hat{\chi}(\tau) = 0$ for $|\tau| \geq 1/2$. Also recall that we are assuming, as in (2.2), that $T = c\log\lambda$ where $c = c_M$ is a small positive constant that will be specified later on. Using this and the Hadamard parametrix we shall obtain the following useful result.

Proposition 5.1. *If $d_{\tilde{g}} \geq 1$ and $\lambda \gg 1$ we can write*

$$(5.2) \qquad K_{T,\lambda}(x,y) = \lambda^{1/2}\sum_{\pm} a_{\pm}(T,\lambda;x,y)e^{\pm i\lambda d_{\tilde{g}}(x,y)} + R_{T,\lambda}(x,y),$$

where

$$(5.3) \qquad\qquad\qquad |a_{\pm}(T,\lambda;x,y)| \leq C,$$

and if $\ell = 1, 2, 3, \ldots$ is fixed

$$(5.4) \qquad\qquad \Delta_x^\ell a_{\pm}(T,\lambda;x,y) = O(\exp(C_\ell d_{\tilde{g}}(x,y)))$$

or

$$(5.5) \qquad\qquad \Delta_y^\ell a_{\pm}(T,\lambda;x,y) = O(\exp(C_\ell d_{\tilde{g}}(x,y))),$$

and

$$|R_{T,\lambda}(x,y)| \le \lambda^{-1}, \tag{5.6}$$

provided that the constant $c > 0$ in (2.2) is sufficiently small. Also, in this case we also have

$$K_{T,\lambda}(x,y) = O(\lambda^{-1}), \quad if \;\; d_{\tilde{g}}(x,y) \le 1. \tag{5.7}$$

Let us first handle the case were $d_{\tilde{g}}(x,y) \ge 1$ since proving (5.7) will be much easier than proving the first part of the Proposition. Since $\cos \tau \sqrt{-\Delta}$ is self-adjoint, we only need to show that $K_{T,\lambda}$ can be written as in (5.2) where the amplitudes satisfy (5.5) and the remainder term is as in (5.6).

To prove this we shall use the Hadamard parametrix as in Bérard [1]. As was shown there we can write for $|\tau| \ge 1$

$$\left(\cos \tau \sqrt{-\Delta}\right)(x,y) = \sum_{j=0}^{m} \alpha_j(x,y) \int_{-\infty}^{\infty} e^{i\theta(d^2-|\tau|^2)}|\tau|\,|\theta|^{1/2-j}\,d\theta + R(\tau,x,y), \tag{5.8}$$

where $d = d_{\tilde{g}}(x,y)$,

$$\alpha_0(x,y) = O(1) \quad \text{and} \quad \alpha_j = O(\exp(C_j d)), \; j = 1,2,\ldots, \tag{5.9}$$

and

$$|\Delta_y^N \alpha_j| = O(\exp(C_N d)), \quad j = 0,1,\ldots, \; N = 1,2,\ldots, \tag{5.10}$$

and, if m is large enough[1],

$$|\partial_\tau^j R(\tau,x,y)| = O(\exp(Cd)), \quad j = 0,1,2. \tag{5.11}$$

In the above Fourier integrals we regularize the powers of $|\theta|$ near the origin at the expense of smooth errors that can be absorbed in the remainder term.

The fact that the first coefficient, α_0, in the Hadamard parametrix is bounded here is well known (see [12]) and was used, for instance, by the first author and Zelditch in the related work [13]. It is a consequence of the

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[1]Strictly speaking Bérard [1] only stated this sort of bound for R itself in (42) on p. 263. The proof of this particular pointwise estimate for the remainder was based on energy estimates. If one includes sufficiently many terms in (5.8) and uses higher order energy estimates one can obtain bounds like (5.11). (See also [12].)

Günther comparison theorem and our assumption that $K \leq 0$. Bérard [1] proved the other bounds (5.9)–(5.10) and used them, along with energy estimates, to obtain bounds of the form (5.11) for the remainder term in the parametrix.

If we change variables we can rewrite (5.8) in the more useful form

$$(5.12) \qquad \big(\cos \tau \sqrt{-\Delta}\,\big)(x,y) = \int_{-\infty}^{\infty} e^{i(d-|\tau|)\theta} q(\tau,x,y,\theta)\, d\theta + R(\tau,x,y),$$

where the remainder term is as before and where

$$(5.13) \quad \big|\partial_\tau^j \partial_\theta^k q(\tau,x,y,\theta)\big| \leq C_{jk}\big[(1+|\theta|)^{1/2-k}(1+d+|\tau|)^{-j}$$
$$+ \exp(Cd)(1+|\theta|)^{-1/2-k}(1+d+|\tau|)^{-j}\big], \ \text{if} \ \ d+|\tau| \geq 1,$$

as well as

$$(5.14) \quad \big|\Delta_y^\ell \partial_\tau^j \partial_\theta^k q(\tau,x,y,\theta)\big| \leq C_{jk\ell} \exp(C_\ell d)(1+|\theta|)^{1/2-k}(1+d+|\tau|)^{-j},$$
$$\text{if} \ \ d+|\tau| \geq 1.$$

Since $\hat{\chi}(\tau/T) = 0$ if $|\tau| > T/2$, it is clear that by (5.11) and an integration by parts argument

$$\int \big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\tau\lambda} R(\tau,x,y)\, d\tau = O(\lambda^{-2}\exp(CT)),$$

and thus this term can be made to satisfy the bounds in (5.6) if $T = c \log \lambda$ with $c > 0$ sufficiently small.

On account of this, if we plug the main term in (5.12) into (5.1), we would have the first part of the proposition if we could show that
(5.15)
$$\int_0^{\infty} \int_{-\infty}^{\infty} \big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\tau\lambda}e^{i\theta(d-\tau)} q(\tau,x,y,\theta)\, d\theta d\tau = \lambda^{1/2} e^{i\lambda d} a_+,$$

and
(5.16)
$$\int_{-\infty}^0 \int_{-\infty}^{\infty} \big(1 - \beta(\tau)\big)\hat{\chi}(\tau/T)e^{i\tau\lambda}e^{i\theta(d+\tau)} q(\tau,x,y,\theta)\, d\theta d\tau = \lambda^{1/2} e^{-i\lambda d} a_-,$$

where $a_\pm$ satisfy the bounds in (5.3) and (5.4).

To see this for (5.15) we note that the left side can be written as

$$\lambda^{1/2}e^{i\lambda d}\left[\lambda^{-1/2}\int_0^\infty\int_{-\infty}^\infty\big(1-\beta(\tau)\big)\hat{\chi}(\tau/T)e^{i(\theta-\lambda)(d-\tau)}q(\tau,x,y,\theta)\,d\theta d\tau\right].$$

Thus, if we set a_+ to be the term inside the square brackets, we can use (5.12) and integration by parts argument to see that

$$|a_+|\leq C\lambda^{-1/2}\iint(1+|\theta-\lambda|)^{-2}(1+|d-\tau|)^{-2}(1+|\theta|)^{1/2}\,d\tau d\theta$$
$$+\,C\exp(Cd)\lambda^{-1/2}\iint(1+|\theta-\lambda|)^{-2}(1+|d-\tau|)^{-2}(1+|\theta|)^{-1/2}\,d\tau d\theta$$
$$\leq C\big(1+\exp(Cd)\lambda^{-1}\big).$$

This yields the bounds in (5.3) for a_+ if $T=c\log\lambda$ with $c>0$ small enough since $K(x,y)=0$ if $d_{\tilde{g}}(x,y)>T$. If we repeat this argument and use (5.14) we also obtain the bounds in (5.4) for a_+ since

$$(5.17)\qquad\qquad\Delta_y^\ell d_{\tilde{g}}(x,y)=O(\exp(C_\ell d_{\tilde{g}}(x,y)))$$

(which also follows from estimates in the appendix in Bérard [1]). Since the same argument shows that (5.16) is valid with a_- satisfying these two bounds, the proof of the first part of Proposition 5.1 is complete.

To prove (5.7) we recall that the factor $(1-\beta(\tau))=0$ if $|\tau|\leq 3$ and so the bounds in (5.13) and (5.14) hold on the support of the integrals in (5.15)–(5.16). Since $|d-\tau|\geq 1$ as well on the support if as in (5.7), $d\leq 1$, we conclude that (5.7) follows from a simple integration by parts argument. $\square$

Note that (5.7) implies that the estimate in Proposition 2.1 is valid when α is the identity map. To handle the other nonzero summands in (2.16) we note that the kernel coming from the τ-integration is $K_{T,\lambda}(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))$ with $|t|,|s|\leq 1/2$. Our assumption that the injectivity radius of M is ten or more insures that $d_{\tilde{g}}(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))\geq 1$ in this case if $\alpha\neq Id$ and so we can use (5.2)–(5.6).

We shall need more information about the phase functions

$$(5.18)\qquad\qquad\phi(\alpha;t,s)=d_{\tilde{g}}(\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))$$

that arise from (5.2). Specifically, we shall require the following.

Proposition 5.2. *Let $\phi(\alpha; t, s)$ be as in* (5.18) *with $\alpha \neq Id$. Then for each $j = 1, 2, 3, \ldots$ there is a constant C_j so that*

$$(5.19) \quad |\partial_t^j \phi(\alpha; t, s)| + |\partial_s^j \phi(\alpha; t, s)| \leq \exp(C_j T),$$
$$\text{if} \ \ \max\{d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))) : |t|, |s| \leq 1/2\} \leq T.$$

Moreover, we have the uniform bounds

$$(5.20) \qquad\qquad\qquad |\partial_t \partial_s \phi(\alpha; t, s)| \leq C.$$

Additionally,

$$(5.21) \qquad \partial_t^2 \phi(\alpha; t_0, s_0) \geq \exp(-CT), \quad \text{if} \ \ d_{\tilde{g}}(\tilde{\gamma}(t_0), \alpha(\tilde{\gamma}(s_0))) \leq T$$

for some C if

$$(5.22) \qquad\qquad\qquad |\partial_t \phi(\alpha; t_0, s_0)| \leq 1/4.$$

The bound in (5.19) for $\partial_t^j \phi$ follows from (5.17). Since $\phi(\alpha; t, s) = d_{\tilde{g}}(\tilde{\gamma}(s), \alpha^{-1}(\tilde{\gamma}(t)))$, the bound for $\partial_s^j \phi$ also follows from (5.17).

To prove (5.20) we may work in geodesic normal coordinates about $\tilde{\gamma}(t)$, with $\tilde{\gamma}$ being the first coordinate axis in these coordinates. Write $\alpha(\tilde{\gamma}(s)) = (x_1(s), x_2(s))$ in these coordinates. Then

$$\frac{\partial \phi}{\partial t}(\alpha; t, s) = -\frac{x_1(s)}{|x(s)|}.$$

Thus

$$\frac{\partial^2 \phi}{\partial t \partial s} = \frac{x_1(s)\left[\frac{d}{ds}(x_1^2(s) + x_2^2(s))\right]}{|x(s)|^2} - \frac{\dot{x}_1(s)}{|x(s)|} = O(|\dot{x}(s)|)$$

since our assumptions give $|x(s)| \geq 1$. Consequently we would have (5.20) if we could show that

$$(5.23) \qquad\qquad\qquad |\dot{x}(s)| = O(1).$$

To do this we note that since α is an isometric mapping $\alpha(\tilde{\gamma}(s)) = x(s)$ must be a geodesic. We recall that if

$$p(x, \xi) = \sqrt{\sum_{j,k=1}^{2} g^{jk}(x)\xi_j \xi_k}$$

where g^{jk} is the cometric, i.e., $g^{jk} = (g_{jk})^{-1}$, then by the Hamilton-Jacobi formulation of unit speed geodesic flow (see, e.g. §2.3 in [12]) we must have that

$$\dot{x}(s) = \frac{\partial p}{\partial \xi}\big(x(s), \xi(s)\big), \quad \text{where} \quad p(x(s), \xi(s)) = 1.$$

Therefore, to get (5.23) it suffices to show that in our geodesic coordinate system

$$(5.24) \qquad \Big|\sum_{j=1}^{2} g^{jk}(x)\xi_j\Big| \leq 1 \quad \text{if} \quad \sum_{j,k=1}^{2} g^{jk}(x)\xi_j\xi_k = 1.$$

After a rotation U

$$U^t g^{jk}(x) U = \operatorname{diag}(1, |g|^{-1}),$$

where $|g| = \det(g_{jk}(x))$. By volume comparison estimates since we are assuming that $K \leq 0$ in (M, g) and hence in $(\mathbb{R}^2, \tilde{g})$ we must have that $|g| \geq 1$ since we are working in geodesic normal coordinates about $\gamma(t)$.

If $\xi = U\eta$ then

$$(5.25) \qquad \sum_{j,k=1}^{2} g^{jk}(x)\xi_j\xi_k = 1 \iff \eta_1^2 + |g|^{-1}\eta_2^2 = 1.$$

Thus, since $|g| \geq 1$

$$\big|\big(g^{jk}(x)\big)\xi\big|^2 = \big|\big(g^{jk}(x)\big)U\eta\big|^2 = \big|U^t g^{jk}(x)U\eta\big|^2$$
$$= \eta_1^2 + |g|^{-2}\eta_2^2 \leq \eta_1^2 + |g|^{-1}\eta_2^2 \leq 1,$$

as desired, which completes the proof of (5.23) and (5.20).

To prove (5.21) we shall again work in geodesic normal coordinates, this time about $\tilde{\gamma}(t_0)$, again with $\tilde{\gamma}$ being the first coordinate axis. Then, as before

$$\frac{\partial \phi}{\partial t}(\alpha; t_0, s_0) = -\frac{x_1(s)}{|x(s)|} = -\cos\theta_{s_0}(t_0),$$

where $\theta_{s_0}(t_0) \in [0, \pi)$ denotes the intersection angle of the geodesic ray $\tilde{\gamma}(t)$, $t \geq t_0$ with the geodesic ray starting at $\tilde{\gamma}(t_0)$ and passing through $x(s_0) = \alpha(\tilde{\gamma}(s_0))$. See Figure 4. For $\Delta t > 0$ small, as in this figure, consider the angle $\theta_{s_0}(t_0 + \Delta t)$ formed by the geodesic ray $\tilde{\gamma}(t)$, $t > t_0 + \Delta t$ and the

geodesic ray passing through $\tilde{\gamma}(t_0 + \Delta t)$ and $\alpha(\tilde{\gamma}(s_0))$ as in the Figure. Then

$$\frac{\partial^2 \phi}{\partial t^2}(\alpha; t_0, s_0) = \sin(\theta_{s_0}(t_0)) \lim_{\Delta t \searrow 0} \frac{\theta_{s_0}(t_0 + \Delta t) - \theta_{s_0}(t_0)}{\Delta t}.$$

Our assumption (5.22) means that $\sin \theta_{s_0}(t_0) \geq 1/10$ and so

$$\frac{\partial^2 \phi}{\partial t^2}(\alpha; t_0, s_0) \approx \lim_{\Delta t \searrow 0} \frac{\theta_{s_0}(t_0 + \Delta t) - \theta_{s_0}(t_0)}{\Delta t}.$$

Since $K \leq 0$, by the Rauch comparison theorem (see [8]), if $\Delta \theta$ denotes the angle of the aforementioned rays through $\alpha(\tilde{\gamma}(s_0))$ as in Figure 4, then we must have

$$\Delta \theta + \theta_{s_0}(t_0) + \left(\pi - \theta_{s_0}(t_0 + \Delta t)\right) \leq \pi,$$

since $\pi - \theta_{s_0}(t_0 + \Delta t)$, $\theta_{s_0}(t_0)$ and $\Delta \theta$ are the three interior angles for the triangle with vertices $\tilde{\gamma}(t_0)$, $\tilde{\gamma}(t_0 + \Delta t)$ and $\alpha(\tilde{\gamma}(s_0))$. Thus,

$$\frac{\partial^2 \phi}{\partial t^2}(\alpha; t_0, s_0) \approx \lim_{\Delta t \searrow 0} \frac{\theta_{s_0}(t_0 + \Delta t) - \theta_{s_0}(t_0)}{\Delta t} \geq \lim_{\Delta t \searrow 0} \frac{\Delta \theta}{\Delta t}.$$

By Proposition 3.2, we must have that $\Delta \theta \geq \sinh(\Delta t/2)/\sinh T$, which leads to (5.21) and completes the proof of Proposition 5.2.   □

We also need the following simple consequence of Proposition 3.1, which was based on our assumption (1.2).

Proposition 5.3. *Let* $\phi(\alpha; t, s)$ *be is as above with* $\alpha \neq \mathrm{Id}$. *Then if for* $t_0, s_0 \in [-1/2, 1/2]$ *and* $\lambda \gg 1$

$$(5.26) \qquad\qquad \left| \nabla_{t,s} \phi(\alpha; t_0, s_0) \right| \leq \frac{1}{2} \lambda^{-1/3},$$

it follows that if ε_0 *is as in* (3.3), *we have*

$$(5.27) \quad \left| \nabla_{t,s} \phi(\alpha; t, s) \right| \geq \frac{1}{2} \lambda^{-1/4},$$
$$\text{if} \quad \max\left(|t - t_0|, |s - s_0| \right) \geq \lambda^{-\varepsilon_0} \text{ and } |t|, |s| \leq 1/2.$$

By the above arguments $|\partial_t \phi(\alpha; t, s)| = \cos \theta_t$ and $|\partial_s \phi(\alpha; t, s)| = \cos \theta_s$ where θ_t and θ_s are as in Proposition 3.1. From this one immediately sees that Proposition 5.3 follows from Proposition 3.1.   □
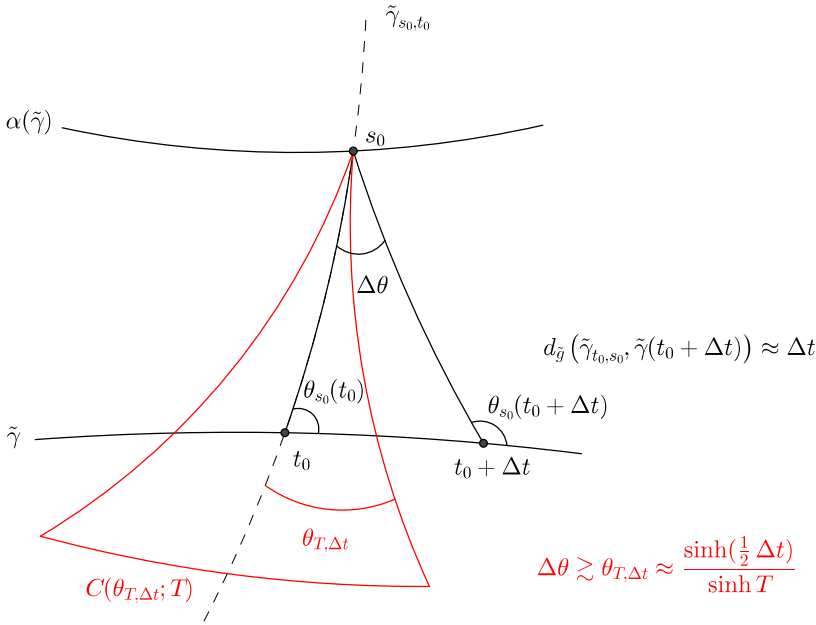
Figure 4: Bounding Hessian from below

6. End of proof of period integral estimates

In this section we shall complete the proof of Proposition 2.1 and hence that of Theorem 1.1. We need to verify that we can fix $c = c_M > 0$ so that for $\lambda \gg 1$ (2.16) is valid for some $\delta_M > 0$. We shall take

$$\delta_M = \varepsilon_0/2$$

where $0 < \varepsilon_0 < 1/10$ is as in (3.3) and (5.27).

As we pointed out earlier, we know that (2.16) is valid when $\alpha = Id$. Hence it suffices to show that all the other nonzero terms there satisfy

$$(6.1) \qquad \left| \iint b(t,s) K_{T,\lambda}\big(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))\big) \, dtds \right| \leq C_{b,M} \lambda^{-\varepsilon_0/2},$$

if $T = c_M \log \lambda$ with $c_M > 0$ small enough. Recall that $b \in C_0^\infty((-1/2, 1/2)^2)$.

In view of the estimate (5.6) for the remainder term in (5.2), it suffices to show that

$$(6.2) \quad \lambda^{1/2} \left| \iint b(t,s) a_\pm(T, \lambda; \tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) \, e^{\pm i\lambda \phi(\alpha; t, s)} \, dtds \right| \leq C_{b,M} \lambda^{-\varepsilon_0/2}$$

under the above assumptions with $\phi(\alpha; t, s) = d_{\tilde{g}}(\tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))$ as in Propositions 5.2 and 5.3. As noted before, we have $\phi(\alpha; t, s) = d_{\tilde{g}}(\alpha^{-1}(\tilde{\gamma}(t)), \tilde{\gamma}(s))$. Also since $\left(\cos \tau \sqrt{-\Delta_{\tilde{g}}}\right)(x, y) = \left(\cos \tau \sqrt{-\Delta_{\tilde{g}}}\right)(\alpha^{-1}(x), \alpha^{-1}(y))$, we have that $K_{T,\lambda}(x, y) = K_{T,\lambda}(\alpha^{-1}(x), \alpha^{-1}(y))$ and so

$$a_{\pm}(T, \lambda; \tilde{\gamma}(t), \alpha(\tilde{\gamma}(s))) = a_{\pm}(T, \lambda, \alpha^{-1}(\tilde{\gamma}(t)), \tilde{\gamma}(s)).$$

Therefore, by (5.4), (5.5) and (5.17) if $T = c \log \lambda$ with $c = c_M > 0$ sufficiently small and if $K_{T,\lambda}$ does not vanish identically we have

$$(6.3) \quad |\partial_t^j \phi(\alpha; s, t)| + |\partial_s^j \phi(\alpha; s, t)| + |\partial_t^j a(t, \lambda; \tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))|$$
$$+ |\partial_s^j a(t, \lambda; \tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))| \le \lambda^{\varepsilon_0/8}, \quad \text{with} \quad 1 \le j \le \lceil 8\varepsilon_0^{-1} \rceil.$$

We use $\varepsilon_0/8$ here since we shall eventually want to apply Proposition 4.3 with $\delta = \varepsilon_0/4$.

To apply Proposition 5.3 and the stationary phase bounds from §4, we shall consider two cases:

$$(6.4) \qquad\qquad |\nabla_{t,s} \phi(\alpha; s, t)| \ge \frac{1}{2} \lambda^{-1/3}, \quad |t|, |s| < 1/2,$$

and the complementary case where

$$(6.5) \quad |\nabla_{t,s} \phi(\alpha; t_0, s_0)| \le \frac{1}{2} \lambda^{-1/3} \quad \text{for some} \quad (t_0, s_0) \in (-1/2) \times (-1/2).$$

To show that (6.2) is valid under the assumption (6.4) we shall use a partition of unity argument to exploit (6.3). Specifically, choose $\rho \in C_0^\infty((-1, 1))$ satisfying

$$\sum_{j=-\infty}^{\infty} \rho(t - j) \equiv 1, \quad t \in \mathbb{R}.$$

Then for $m = (m_1, m_2) \in \mathbb{Z}^2$ set

$$\rho_m(t, s) = \rho(\lambda^{1/2} t - m_1) \rho(\lambda^{1/2} s - m_2).$$

It follows that $\sum_{m \in \mathbb{Z}^2} \rho_m(t, s) \equiv 1$ and that $|\partial_t^j \rho_m| + |\partial_s^j \rho_m| \le C_j \lambda^{j/2}$. Also, ρ_m is supported in a $O(\lambda^{-1/2})$ size neighborhood about $(t_m, s_m) = (\lambda^{-1/2} m_1, \lambda^{-1/2} m_2)$. Assuming that this neighborhood intersects $(-1/2, 1/2) \times (-1/2, 1/2)$ and that $(\overline{t_m}, \overline{s_m})$ is in the intersection, by (6.4) we must have that

$$|\partial_t \phi(\alpha; \overline{t_m}, \overline{s_m})| \ge \frac{1}{4} \lambda^{-1/3}, \quad \text{or} \quad |\partial_s \phi(\alpha; \overline{t_m}, \overline{s_m})| \ge \frac{1}{4} \lambda^{-1/3}.$$

Let us assume the former since the argument for the latter is similar. By (5.20) and (6.3) we must have that

$$|\partial_t \phi| \geq \frac{1}{4}\lambda^{-1/3} - O(\lambda^{-1/2}\lambda^{\varepsilon_0/4}) - O(\lambda^{-1/2}) \geq \frac{1}{8}\lambda^{-1/3} \quad \text{on supp } \rho_m$$

for large λ since $\varepsilon_0 < 1/10$. Therefore, by Lemma 4.1, we have that

$$\lambda^{1/2}\left| \int b(t,s)\rho_m(t,s)a_\pm(T,\lambda;\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))\, e^{\pm i\lambda\phi(\alpha;t,s)}\, dt \right| \leq C_{b,\varepsilon_0}\lambda^{-3/2}.$$

Since $\rho_m(t,s) = 0$ if $|s - s_m| \geq C\lambda^{-1/2}$, this in turn gives the bounds

$$\lambda^{1/2}\left| \iint b(t,s)\rho_m(t,s)a_\pm(T,\lambda;\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))\, e^{\pm i\lambda\phi(\alpha;t,s)}\, dtds \right| \leq C_{b,\varepsilon_0}\lambda^{-2}.$$

Since there are $O(\lambda)$ such terms which are nonzero, we conclude that when (6.4) holds we obtain a stronger version of (6.2) where $\lambda^{-\varepsilon_0/2}$ is replaced by λ^{-1}.

To complete the proof, we must show that (6.2) is valid when we assume (6.5). We shall use Proposition 5.3 for this (which makes use of our curvature assumption (1.2)). To this end, let $\beta \in C_0^\infty(\mathbb{R})$ be as above, i.e.,

$$\beta(t) = 1 \text{ on } [-3,3] \quad \text{and} \quad \beta(t) = 0 \text{ for } |t| \geq 4.$$

We then obtain

$$\left| \iint \big(1 - \beta(\lambda^{\varepsilon_0}|t-t_0|)\beta(\lambda^{\varepsilon_0}|s-s_0|)\big)\, a_\pm(T,\lambda;\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))\, e^{\pm i\lambda\phi(\alpha;t,s)}\, dtds \right|$$
$$\leq C_{b,\varepsilon_0}\lambda^{-3/2}$$

by the previous argument since by (5.27)

$$|\nabla_{t,s}\phi(\alpha;t,s)| \geq \frac{1}{2}\lambda^{-1/4} \quad \text{if} \quad \big(1 - \beta(\lambda^{\varepsilon_0}|t-t_0|)\beta(\lambda^{\varepsilon_0}|s-s_0|)\big) \neq 0.$$

Thus, our proof would be complete if we could show that

(6.6)
$$\left| \iint \beta(\lambda^{\varepsilon_0}|t-t_0|)\beta(\lambda^{\varepsilon_0}|s-s_0|)b(t,s)\, a_\pm(T,\lambda;\tilde{\gamma}(t),\alpha(\tilde{\gamma}(s)))\, e^{\pm i\lambda\phi(\alpha;t,s)}dtds \right|$$
$$\leq C_{b,M}\lambda^{-\varepsilon_0/2-1/2}.$$

To do this, we note that, by (5.21), we have that when $T = c \log \lambda$ with $c = c_M > 0$ small,

$$|\partial_t^2 \phi(\alpha; t, s)| \geq \lambda^{-\varepsilon_0/8} \quad \text{if} \quad \beta(\lambda^{\varepsilon_0}|t - t_0|)\beta(\lambda^{\varepsilon_0}|s - s_0|) \neq 0,$$

since our assumption (6.5) along with (6.3) and (5.20) ensure that (5.22) is valid for such (t, s). Therefore, by Proposition 4.3 with $\delta = \varepsilon_0/4$, we have that for each $|s| \leq 1/2$

$$\lambda^{1/2}\left| \int \beta(\lambda^{\varepsilon_0}|t-t_0|)\beta(\lambda^{\varepsilon_0}|s-s_0|)b(t, s)\, a_\pm(T, \lambda; \tilde{\gamma}(t), \alpha(\tilde{\gamma}(s)))e^{\pm i\lambda\phi(\alpha:t,s)}dt\right|$$
$$\leq C_{b,M}\lambda^{\varepsilon_0/2}.$$

Since $s \to \beta(\lambda^{\varepsilon_0}|s-s_0|)$ is supported in an interval of size $\approx \lambda^{-\varepsilon_0}$ this implies (6.6), which completes the proof. $\qquad\square$

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CHRISTOPHER D. SOGGE
DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218
USA
E-mail address: sogge@jhu.edu

YAKUN XI
DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218
USA
E-mail address: ykxi@math.jhu.edu

CHENG ZHANG
DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218
USA
E-mail address: czhang67@jhu.edu