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Local ε-isomorphisms for rank two p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a functional equation of Kato's Euler system

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In this article, we prove many parts of the rank two case of the Kato's local ε-conjecture using the Colmez's p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. We show that a Colmez's pairing defined in his study of locally algebraic vectors gives us the conjectural ε-isomorphisms for (almost) all the families of p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ of rank two, which satisfy the desired interpolation property for the de Rham and trianguline case. For the de Rham and non-trianguline case, we also show this interpolation property for the "critical" range of Hodge-Tate weights using the Emerton's theorem on the compatibility of classical and p-adic local Langlands correspondence. As an application, we prove that the Kato's Euler system associated to any Hecke eigen new form which is supercuspidal at p satisfies a functional equation which has the same form as predicted by the Kato's global ε-conjecture.

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1. Introduction

1.0.1. Background. By the ground breaking work of Colmez [Co10b] and many other important works by Berger, Breuil, Dospinescu, Kisin and Paskunas, the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is now a theorem ([Pa13], [CDP14b]). This gives us a correspondence between absolutely irreducible two dimensional p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and absolutely irreducible unitary Banach admissible non-ordinary representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ via the so called the Montreal functor. An important feature of this functor is that it also gives us a correspondence between representations with torsion coefficients. From this property, the Colmez's theory is expected to have many applications to problems in number theory concerning the relationship between the p-adic variations of the Galois side and those of the automorphic side. For example, Emerton [Em] and Kisin [Ki09] independently applied his theory to the Fontaine-Mazur conjecture on the modularity of two dimensional geometric p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

In the present article, we give another application to the rank two case of a series of Kato's conjectures in [Ka93a], [Ka93b] on the p-adic interpolations of special values of L-functions and local (L-and ε-) constants. There are two main theorems in the article, the first one concerns with the p-adic local ε-conjecture, where the Colmez's theory crucially enters in, and the second concerns with the global ε-conjecture, which we roughly explain now (see §2.1, §3.1, §4.1 for more details).

In this introduction, we assume $p \neq 2$ (for simplicity), fix an isomorphism $\iota_{\infty,p} : \overline{\mathbb{C}} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$, and set $\zeta^{(l)} := (\iota_{\infty,p}(\exp(\frac{2\pi i}{l^n}))_{n\geqq 1} \in \mathbb{Z}_l(1) := \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_l(1))$ for each prime l. Set $\Gamma := \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$.

1.0.2. The p-adic local ε-conjecture. Fix a prime l. Let R be a commutative noetherian semi-local $\mathbb{Z}_p$-algebra such that $R/\mathrm{Jac}(R)$ is a finite ring with a p-power order, or a finite extension of $\mathbb{Q}_p$. To any R-representation T of $G_{\mathbb{Q}_l}$, one can functorially attach a (graded) invertible R-module $\Delta_R(T)$ using the determinant of the perfect complex $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T)$ of continuous cochains of $G_{\mathbb{Q}_l}$ with values in T. For a pair $(R, T) = (L, V)$ such that L is a finite extension of $\mathbb{Q}_p$ and V is arbitrary (resp. de Rham) L-representation of $G_{\mathbb{Q}_l}$ when $l \neq p$ (resp. $l = p$), one can define a representation $W(V)$ of the Weil-Deligne group $'W_{\mathbb{Q}_l}$ of $\mathbb{Q}_l$ by the Grothendieck local monodromy theorem (resp. the p-adic monodromy theorem) when $l \neq p$ (resp. $l = p$). Using the local (L-and ε-) constants associated to $W(V)$ (and the Bloch-Kato fundamental exact sequence when $l = p$), one can define a canonical L-linear isomorphism which we call the de Rham ε-isomorphism

$$\varepsilon^{\mathrm{dR}}_L(V)(= \varepsilon^{\mathrm{dR}}_{L,\zeta^{(l)}}(V)) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(V)$$

depending on the choice of $\zeta^{(l)}$, where, for any R, $\mathbf{1}_R := (R, 0)$ is the trivial graded invertible R-module of degree zero. The l-adic local ε-conjecture [Ka93b] predicts the existence of a canonical isomorphism

$$\varepsilon_R(T)(= \varepsilon_{R,\zeta^{(l)}}(T)) : \mathbf{1}_R \xrightarrow{\sim} \Delta_R(T)$$

(also depending on the choice of $\zeta^{(l)}$) for any pair (R, T) as above which interpolates the de Rham ε-isomorphisms (see Conjecture 2.1 for the precise formulation). The $l \neq p$ case of this conjecture has been already proved by Yasuda [Ya09]. The first main theorem of the present article concerns with the rank two case of the p-adic local ε-conjecture (see Theorem 3.1 for more details).

Theorem 1.1. *Assume $l = p$. For (almost) all the pairs (R, T) as above such that T are of rank one or two, one can canonically define R-linear isomorphisms*

$$\varepsilon_R(T) : \mathbf{1}_R \xrightarrow{\sim} \Delta_R(T)$$

which are compatible with arbitrary base changes, and satisfy the following: for any pair (L, V) such that V is de Rham of rank one or two satisfying at least one of the following two conditions (i) and (ii),

(i) V is trianguline,
(ii) the set of the Hodge-Tate weights of V is $\{k_1, k_2\}$ such that $k_1 \leqq 0$, $k_2 \geqq 1$,

then we have

$$\varepsilon_L(V) = \varepsilon_L^{\mathrm{dR}}(V).$$

Remark 1.2. For $l = p$, this conjecture is much more difficult than that for $l \neq p$, and has been proved only in some special cases before the present article. For the rank one case, it is proved by Kato [Ka93b] (see also [Ve13]). For the cyclotomic deformation, or more general abelian twists of crystalline representations, it is proved by Benois-Berger [BB08] and Loeffler-Venjakob-Zerbes [LVZ13]. For the trianguline case, it is proved by the author [Na14b]. More precisely, in [Na14b], we generalized the p-adic local ε-conjecture for rigid analytic families of (φ, Γ)-modules over the Robba ring, and proved this generalized version of the conjecture for all the trianguline families of (φ, Γ)-modules. Since the rigid analytic family of (φ, Γ)-modules associated to any abelian twist of any crystalline representation is a trianguline family, the result in [Na14b] seems to be the most general one on the p-adic local ε-conjecture before the present article. In the theorem above, in particular, in the condition (ii), we don't need to assume that V is trianguline. Therefore, the theorem in the case of the condition (ii) seems to be an essentially new result in the literatures on the p-adic local ε-conjecture.

1.0.3. The global ε-conjecture. We next explain the second main result of the article. Let S be a finite set of primes containing p. Let $\mathbb{Q}_S$ be the maximal Galois extension of $\mathbb{Q}$ which is unramified outside $S \cup \{\infty\}$, and set $G_{\mathbb{Q},S} := \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. For an R-representation T of $G_{\mathbb{Q},S}$, one can also define a graded invertible R-module $\Delta_{R,S}(T)$ using $C^{\bullet}_{\mathrm{cont}}(G_{\mathbb{Q},S}, T)$. Set $\Delta_{R,(l)}(T) := \Delta_R(T|_{G_{\mathbb{Q}_l}})$ for each $l \in S$.

The generalized Iwasawa main conjecture [Ka93a] predicts the existence of the canonical isomorphism

$$z_{R,S}(T) : \mathbf{1}_R \xrightarrow{\sim} \Delta_{R,S}(T)$$

for any pair (R, T) as above which interpolates the special values of the L-functions of the motives over $\mathbb{Q}$ with good reduction outside S (see [Ka93a] for the precise formulation).

Ser $T^* := \mathrm{Hom}_R(T, R)(1)$ be the Tate dual of T. By the Poitou-Tate duality, one has a canonical isomorphism

$$\Delta_{R,S}(T^*) \xrightarrow{\sim} \left(\boxtimes_{l\in S}\Delta_{R,(l)}(T)\right) \boxtimes \Delta_{R,S}(T).$$

Then, the global ε-conjecture [Ka93b] asserts that one has the equality

$$z_{R,S}(T^*) = \left(\boxtimes_{l\in S}\varepsilon_{R,(l)}(T)\right) \boxtimes z_{R,S}(T),$$

where $\varepsilon_{R,(l)}(T) := \varepsilon_R(T|_{G_{\mathbb{Q}_l}}) : \mathbf{1}_R \xrightarrow{\sim} \Delta_{R,(l)}(T)$ is the local ε-isomorphism defined by [Ya09] for $l \neq p$ and the conjectural local ε-isomorphism for $l = p$.

To state the second main theorem, we need to recall the notion of cyclotomic deformations. Set $\Lambda_R := R[[\Gamma]]$, and define a Λ_R-representation $\mathbf{Dfm}(T) := T \otimes_R \Lambda_R$ on which $G_{\mathbb{Q},S}$ acts by $g(x \otimes \lambda) := g(x) \otimes [\bar{g}]^{-1}\lambda$. We set

$$\Delta^{\mathrm{Iw}}_{R,*}(T) := \Delta_{\Lambda_R,*}(\mathbf{Dfm}(T))$$

for $* = S, (l)$, and set (conjecturally)

$$z^{\mathrm{Iw}}_{R,S}(T) := z_{\Lambda_R,S}(\mathbf{Dfm}(T)),\ \ \varepsilon^{\mathrm{Iw}}_{R,(l)}(T) := \varepsilon_{\Lambda_R,(l)}(\mathbf{Dfm}(T)).$$

Define an involution $\iota : \Lambda_R \xrightarrow{\sim} \Lambda_R : [\gamma] \mapsto [\gamma]^{-1}$, and denote M^ι for the base change by ι for any Λ_R-module M. Then, one has a canonical isomorphism

$$\Delta^{\mathrm{Iw}}_{R,S}(T^*)^\iota \xrightarrow{\sim} \Delta_{\Lambda_R,S}((\mathbf{Dfm}(T))^*).$$

The second main theorem of the present article concerns with the global ε-conjecture for $\mathbf{Dfm}(T_f)$ for Hecke eigen new forms f (see §4.2 for the precise statement). Let

$$f(\tau) := \sum_{n\geqq 1} a_n(f)q^n \in S_{k+1}(\Gamma_1(N))^{\mathrm{new}}$$

be a normalized Hecke eigen new form of level N and of weight $k+1$ for some $N, k \in \mathbb{Z}_{\geqq 1}$. Set

$$f^*(\tau) := \sum_{n\geqq 1} \overline{a_n(f)}q^n,$$

where $\overline{(-)}$ is the complex conjugation, then $f^*(\tau)$ is also a Hecke eigen new form in $S_{k+1}(\Gamma_1(N))^{\mathrm{new}}$ by the theory of new form. Set $S := \{l|N\} \cup \{p\}$, $L := \mathbb{Q}_p(\{\iota_{\infty,p}(a_n(f))\}_{n\geqq 1}) \subseteq \overline{\mathbb{Q}}_p$, and $\mathcal{O} := \mathcal{O}_L$ the ring of integers in L. For $f_0 = f, f^*$, let T_{f_0} be the $\mathcal{O}$-representation of $G_{\mathbb{Q},S}$ of rank two associated to f_0 defined by Deligne [De69]. In [Ka04], Kato defined an Euler system associated to f_0 which interpolates the critical values of the twisted L-functions associated to f_0^*. Denote by $Q(\Lambda)$ for the total fraction ring of $\Lambda := \Lambda_{\mathcal{O}}$. As a consequence of Kato's theorem proved in §12 of [Ka04], we define in §4.2 a canonical $Q(\Lambda)$-linear isomorphism

$$\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)) : \mathbf{1}_{Q(\Lambda)} \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)) \otimes_{\Lambda} Q(\Lambda)$$

for any $r \in \mathbb{Z}$, which should be the base change to $Q(\Lambda)$ of the conjectural zeta isomorphism

$$z^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)) : \mathbf{1}_{\Lambda} \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)).$$

We remark that one has a canonical isomorphism $T_{f^*}(1)[1/p] \xrightarrow{\sim} (T_f(k))^*[1/p]$ and one has a canonical isomorphism

$$\Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f^*}(1))^{\iota} \xrightarrow{\sim} \Delta_{\Lambda,S}((\mathbf{Dfm}(T_f(k)))^*).$$

The second main theorem is the following.

Theorem 1.3. *Assume that $V := T_f[1/p]|_{G_{\mathbb{Q}_p}}$ is non-trianguline. Then one has the following equality*

$$\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f^*}(1))^{\iota} = \left(\boxtimes_{l\in S}(\varepsilon^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k)) \otimes \mathrm{id}_{Q(\Lambda)})\right) \boxtimes \widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_f(k))$$

under the base change to $Q(\Lambda)$ of the canonical isomorphism

$$\Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f^*}(1))^{\iota} \xrightarrow{\sim} \left(\boxtimes_{l\in S}\Delta^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k))\right) \boxtimes \Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_f(k))$$

defined by the Poitou-Tate duality, where the isomorphism

$$\varepsilon^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k)) : \mathbf{1}_{\Lambda} \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k))$$

in the above equality is the local ε-isomorphism for the pair $(\Lambda, \mathbf{Dfm}(T_f(k))|_{G_{\mathbb{Q}_l}})$ defined by [Ya09] when $l \neq p$, by Theorem 1.1 when $l = p$.

Remark 1.4. In many cases where V is trianguline, we can obtain the same result by almost the same proof. However, when V is ordinary, or is in the exceptional zero case (special cases of the trianguline case), we need some additional arguments. Since this additional arguments makes article a little bit long, we will treat the trianguline case in our next article.

The following conjecture is a part of the generalized Iwasawa main conjecture for the pair $(\Lambda, \mathbf{Dfm}(T_{f_0}(r))$.

Conjecture 1.5. *For any $r \in \mathbb{Z}$, the isomorphism $\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r))$ comes, by extension of scalar, from a Λ-linear isomorphism*

$$z^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)) : \mathbf{1}_\Lambda \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)),$$

i.e. one has $\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)) = z^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)) \otimes \mathrm{id}_{Q(\Lambda)}$.

We remark that such $z^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r))$ is unique if it exists since the natural map $\Lambda \to Q(\Lambda)$ is injective, and, if the conjecture is true for one $r \in \mathbb{Z}$, then it is true for all $r \in \mathbb{Z}$.

As an immediate corollary of the theorem, we obtain the following.

Corollary 1.6. *Assume that $V := T_f[1/p]|_{G_{\mathbb{Q}_p}}$ is non-trianguline. Then, the conjecture 1.5 is true for f if and only if it is true for f^*.*

1.0.4. Contents of the article. Now, we briefly describe the contents of different sections.

In §2, §3, we study the p-adic local ε-conjecture. We first remark that many results in these sections heavily depend on many deep results in the theory of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([Co10a], [Co10b], [Do11], [Em]). In particular, our local ε-isomorphism defined in Theorem 1.1 is nothing else but the Colmez's pairing defined in VI.6 of [Co10b]. Our contributions are to find the relation between the Colmez's pairing and the local ε-isomorphism, and to show that this pairing satisfies the interpolation property (i.e. the condition (i), (ii) in the theorem).

Section 2 is mainly for preliminaries. In §2.1, we first recall the l-adic and the p-adic local ε-conjecture. In §2.2, we recall the theory of (φ, Γ)-modules and re-state the p-adic local ε-conjecture in terms of (φ, Γ)-modules. In §2.3, we propose a conjecture (Conjecture 2.11) on a conjectural definition of the local ε-isomorphism for any (φ, Γ)-modules of any rank using the Colmez's multiplicative convolution defined in [Co10a].

Section 3 is devoted to the proof of Theorem 1.1, in particular, we prove Conjecture 2.11 for the rank two case. In §3.1, we state the main theorem. In §3.2, we define our local ε-isomorphism using the Colmez's pairing defined in VI.6 of [Co10b], and prove Conjecture 2.11 for the rank two case, which is essentially a consequence of the $\mathrm{GL}_2(\mathbb{Q}_p)$-compatibility (a notion defined in §III of [CD14]) of the (φ, Γ)-modules of rank two. The subsections §3.3 and §3.4 are the technical hearts of this article, where we show that our ε-isomorphisms satisfy the conditions (i) and (ii) in Conjecture 1.1. In §3.3, we show the interpolation property for the trianguline case (i.e. the condition (i) in the theorem) by comparing the local ε-isomorphism defined in §3.2 with that defined in our previous work [Na14b], where we use a result of Dospinescu [Do11] on the explicit description of the action of $w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$ on the locally analytic vectors (see Theorem 3.9). In §3.4, we show the interpolation property for the non-trianguline case (i.e. the condition (ii) in the theorem). For a (φ, Γ)-module D of rank two such that $V(D)$ is de Rham and non-trianguline with distinct Hodge-Tate weights $\{k_1, k_2\}$ $(k_1 \leqq 0, k_2 \geqq 1)$, using the Colmez's theory of Kirillov model of locally algebraic vectors in VI of [Co10b], we prove two explicit formulas (Proposition 3.16, Proposition 3.18) which respectively (essentially) describe $\varepsilon_L(V)$ and $\varepsilon_L^{\mathrm{dR}}(V)$. Finally, using the Emerton's theorem [Em] on the compatibility of the p-adic and the classical local Langlands correspondence and the classical explicit formula of the action of w on the (classical) Kirillov model, we prove the condition (ii) in Theorem 1.1 for the non-trianguline case.

The final section §4 is devoted to the proof of Theorem 1.3. In §4.1, we recall the definition of the global fundamental lines and give a general set up. In §4.2, we (re-)state our second main theorem (Theorem 4.2) and define our (candidate of) zeta isomorphism $\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_f(r))$ using the (p-th layer of) Kato's Euler system [Ka04] associated to f. In the final subsection §4.3, we prove Theorem 1.3 (Theorem 4.2), where we reduce the theorem to the classical functional equation of the (twisted) L-function of f using the Kato's explicit reciprocity law and Theorem 1.1.

Notation 1.7. Let p be a prime number. For a field F, set $G_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ the absolute Galois group of F. For each prime l, let $W_{\mathbb{Q}_l} \subseteq G_{\mathbb{Q}_l}$ be the Weil group of $\mathbb{Q}_l$, $I_l \subseteq W_{\mathbb{Q}_l}$ be the inertia subgroup. Let $\mathrm{rec}_{\mathbb{Q}_l} : \mathbb{Q}_l^{\times} \stackrel{\sim}{\to} W_{\mathbb{Q}_l}^{\mathrm{ab}}$ be the reciprocity map of the local class field theory normalized so that $\mathrm{rec}_{\mathbb{Q}_l}(l)$ is a lift of the geometric Frobenius $\mathrm{Fr}_l \in G_{\mathbb{F}_l}$. Throughout the article, we fix a $\mathbb{Z}_l$-basis $\zeta = (\zeta_{l^n})_{n \geqq 0} \in \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_l(1))$. Here, we remark that many objects defined in the main body of the article depend on this choice of ζ. We will usually omit the notation ζ, but we will sometimes add the notation ζ when we consider the dependence of ζ.

Set $\Gamma := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \stackrel{\sim}{\to} \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, and let $\chi : \Gamma \stackrel{\sim}{\to} \mathbb{Z}_p^{\times}$ be the p-adic cyclotomic character which we also see as a character of $G_{\mathbb{Q}}$ or $G_{\mathbb{Q}_l}$ for any l. For $l = p$, set $H_{\mathbb{Q}_p} := \mathrm{Ker}(\chi) \subseteq G_{\mathbb{Q}_p}$. For each $b \in \mathbb{Z}_p^{\times}$, define $\sigma_b \in \Gamma$ such that $\chi(\sigma_b) = b$. For a perfect field k of characteristic p, we denote $W(k)$ for the ring of Witt vectors, on which the lift φ of the p-th power Frobenius on k acts. Let $[-] : k \to W(k)$ be the Teichmüller lift. Set $\widetilde{\mathbf{E}}^+ := \varprojlim_{n \geqq 0} \mathcal{O}_{\mathbb{C}_p}/p$ where the projective limit with respect to p-th power map, $\widetilde{\mathbf{E}} := \mathrm{Frac}(\widetilde{\mathbf{E}}^+)$, $\widetilde{\mathbf{A}}^+ := W(\widetilde{\mathbf{E}}^+)$, $\widetilde{\mathbf{A}} := W(\widetilde{\mathbf{E}})$, $\widetilde{\mathbf{B}}^+ := \widetilde{\mathbf{A}}^+[1/p]$ and $\widetilde{\mathbf{B}} := \widetilde{\mathbf{A}}[1/p]$. Let $\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}$ be the continuous $\mathbb{Z}_p$-algebra homomorphism defined by $\theta([(\bar{x}_n)_{n\geqq 0}]) := \lim_{n\to\infty} x_n^{p^n}$ for any $(\bar{x}_n)_{n\geqq 0} \in \widetilde{\mathbf{E}}^+$, where $x_n \in \mathcal{O}_{\mathbb{C}_p}$ is a lift of $\bar{x}_n \in \mathcal{O}_{\mathbb{C}_p}/p$. Set $\mathbf{B}^+_{\mathrm{dR}} := \varprojlim_{n\geqq 1} \widetilde{\mathbf{A}}^+[1/p]/\mathrm{Ker}(\theta)^n[1/p]$. Using the fixed $\mathbb{Z}_p$-basis $\zeta = (\zeta_{p^n})_{n\geqq 0} \in \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_p(1))$ for $l = p$, define $t(:= t_\zeta) := \log([(\bar{\zeta}_{p^n})_{n\geqq 0}]) \in \mathbf{B}^+_{\mathrm{dR}}$, which is a uniformizer of $\mathbf{B}^+_{\mathrm{dR}}$. Set $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}^+_{\mathrm{dR}}[1/t]$.

For a commutative ring R, we denote by $\mathbf{D}^-(R)$ the derived category of bounded below complex of R-modules, by $\mathbf{D}_{\mathrm{perf}}(R)$ the full subcategory of perfect complexes of R-modules. We denote by $\mathbf{P}_{\mathrm{fg}}(R)$ the category of finite projective R-modules. For any $P \in \mathbf{P}_{\mathrm{fg}}(R)$, we denote by r_P its R-rank, by $P^\vee := \mathrm{Hom}_R(P, R)$ its dual. For $P_1, P_2 \in \mathbf{P}_{\mathrm{fg}}(R)$ and $\langle , \rangle : P_1 \times P_2 \to R$ a perfect pairing of R-modules, we always identify P_2 with $P_1^\vee$ by the isomorphism $P_2 \xrightarrow{\sim} P_1^\vee : x \mapsto [y \mapsto \langle y, x\rangle]$.

2. Preliminaries and conjectures

In this section, we first recall the l-adic and the p-adic local ε-conjectures. Then, after reviewing the theory of Iwasawa cohomology of (φ,Γ)-modules, we formulate a conjecture on a conjectural definition of the p-adic local ε-isomorphism using a multivariable version of the Colmez's multiplicative convolution.

2.1. Review of the local ε-conjecture

In this subsection, we quickly recall the local ε-conjecture. See the original articles [Ka93b], [FK06] (the latter one includes the non-commutative version) or [Na14b] for more details.

2.1.1. Knudsen-Mumford's determinant functor. The local ε-conjecture is formulated using the theory of the determinant functor, for which we use the Knudsen-Mumford's one [KM76], which we briefly recall here (see also §3.1 of [Na14b]).

Let R be a commutative ring. We define a category $\mathcal{P}_R$, whose objects are the pairs (L, r) where L is an invertible R-module and $r : \mathrm{Spec}(R) \to \mathbb{Z}$ is a locally constant function, whose morphisms are defined by $\mathrm{Mor}_{\mathcal{P}_R}((L, r), (M, s)) := \mathrm{Isom}_R(L, M)$ if $r = s$, or empty otherwise. We call the objects of this category graded invertible R-modules. For $(L, r), (M, s)$, define its product by $(L, r) \boxtimes (M, s) := (L \otimes_R M, r + s)$ with the natural associativity constraint and the commutativity constraint $(L, r) \boxtimes (M, s) \xrightarrow{\sim} (M, s) \boxtimes$

$(L,r) : l\otimes m \mapsto (-1)^{rs} m\otimes l$. We always identify $(L,r)\boxtimes(M,s) = (M,s)\boxtimes(L,r)$ by this constraint isomorphism. The unit object for the product is $\mathbf{1}_R := (R,0)$. For each (L,r), we set $(L,r)^{-1} := (L^\vee, -r)$, which is the inverse of (L,r) by the isomorphism $i_{(L,r)} : (L,r)\boxtimes(L^\vee,-r) \xrightarrow{\sim} \mathbf{1}_R$ induced by the evaluation map $L\otimes_R L^\vee \xrightarrow{\sim} R : x\otimes f \mapsto f(x)$. For a ring homomorphism $f : R\to R'$, one has a base change functor $(-)\otimes_R R' : \mathcal{P}_R \to \mathcal{P}_{R'}$ defined by $(L,r)\mapsto (L,r)\otimes_R R' := (L\otimes_R R', r\circ f^*)$ where $f^* : \mathrm{Spec}(R')\to\mathrm{Spec}(R)$.

For a category $\mathcal{C}$, denote by $(\mathcal{C},\mathrm{is})$ the category such that the objects are the same as $\mathcal{C}$ and the morphisms are all the isomorphisms in $\mathcal{C}$. Define a functor

$$\mathrm{Det}_R : (\mathbf{P}_{\mathrm{fg}}(R),\mathrm{is}) \to \mathcal{P}_R : P \mapsto (\det_R P, r_P)$$

where we set $\det_R P := \wedge_R^{r_P} P$. Note that $\mathrm{Det}_R(0) = \mathbf{1}_R$ is the unit object. For a short exact sequence $0\to P_1\to P_2\to P_3\to 0$ in $\mathbf{P}_{\mathrm{fg}}(R)$, we always identify $\mathrm{Det}_R(P_1)\boxtimes\mathrm{Det}_R(P_3)$ with $\mathrm{Det}_R(P_2)$ by the following functorial isomorphism (put $r_i := r_{P_i}$)

$$\mathrm{Det}_R(P_1)\boxtimes\mathrm{Det}_R(P_3) \xrightarrow{\sim} \mathrm{Det}_R(P_2) \tag{1}$$

induced by

$$(x_1\wedge\cdots\wedge x_{r_1})\otimes(\overline{x_{r_1+1}}\wedge\cdots\wedge\overline{x_{r_2}}) \mapsto x_1\wedge\cdots\wedge x_{r_1}\wedge x_{r_1+1}\wedge\cdots\wedge x_{r_2}$$

where $x_1,\dots,x_{r_1}$ (resp. $\overline{x_{r_1+1}},\dots,\overline{x_{r_2}}$) are local sections of P_1 (resp. P_3) and $x_i\in P_2$ $(i = r_1+1,\dots,r_2)$ is a lift of $\overline{x_i}\in P_3$. For a bounded complex $P^\bullet$ in $\mathbf{P}_{\mathrm{fg}}(R)$, define $\mathrm{Det}_R(P^\bullet)\in\mathcal{P}_R$ by

$$\mathrm{Det}_R(P^\bullet) := \boxtimes_{i\in\mathbb{Z}}\mathrm{Det}_R(P^i)^{(-1)^i}.$$

By the result of [KM76], Det_R naturally extends to a functor

$$\mathrm{Det}_R : (\mathbf{D}_{\mathrm{perf}}(R),\mathrm{is}) \to \mathcal{P}_R$$

such that the isomorphism (1) extends to the following situation: for any exact sequence $0\to P_1^\bullet\to P_2^\bullet\to P_3^\bullet\to 0$ of bounded below complexes of R-modules such that each $P_i^\bullet$ is a perfect complex, then there exists a canonical isomorphism

$$\mathrm{Det}_R(P_1^\bullet)\boxtimes\mathrm{Det}_R(P_3^\bullet) \xrightarrow{\sim} \mathrm{Det}_R(P_2^\bullet).$$

If $P^\bullet\in\mathbf{D}_{\mathrm{perf}}(R)$ satisfies that $\mathrm{H}^i(P^\bullet)$ are finite projective for all i, there exists a canonical isomorphism

$$\mathrm{Det}_R(P^\bullet) \xrightarrow{\sim} \boxtimes_{i\in\mathbb{Z}}\mathrm{Det}_R(\mathrm{H}^i(P^\bullet))^{(-1)^i}.$$

For $(L,r) \in \mathcal{P}_R$, define $(L,r)^\vee := (L^\vee, r) \in \mathcal{P}_R$, which induces an anti-equivalence $(-)^\vee : \mathcal{P}_R \xrightarrow{\sim} \mathcal{P}_R$. For $P \in \mathbf{P}_{\mathrm{fg}}(R)$ and an R-basis $\{e_1, \ldots, e_{r_P}\}$, we denote by $\{e_1^\vee, \ldots, e_{r_P}^\vee\}$ its dual basis of $P^\vee$. Then one has a canonical isomorphism $\mathrm{Det}_R(P^\vee) \xrightarrow{\sim} \mathrm{Det}_R(P)^\vee$ defined by the isomorphism

$$\det_R(P^\vee) \xrightarrow{\sim} (\det_R P)^\vee : e_1^\vee \wedge \cdots \wedge e_{r_P}^\vee \mapsto (e_1 \wedge \cdots \wedge e_{r_P})^\vee.$$

This isomorphism naturally extends to $(\mathbf{D}_{\mathrm{perf}}(R), \mathrm{is})$, i.e. for any $P^\bullet \in \mathbf{D}_{\mathrm{perf}}(R)$, there exists a canonical isomorphism

$$\mathrm{Det}_R(\mathbf{R}\mathrm{Hom}_R(P^\bullet, R)) \xrightarrow{\sim} \mathrm{Det}_R(P^\bullet)^\vee. \tag{2}$$

2.1.2. The local fundamental line. Now, we start to recall the local ε-conjecture. Fix a prime p. From now on until the end of the article, we use the notation R to represent a commutative topological $\mathbb{Z}_p$-algebra satisfying one of the following conditions (i) or (ii).

- (i) R is a $\mathrm{Jac}(R)$-adically complete noetherian semi-local ring such that $R/\mathrm{Jac}(R)$ is a finite ring (equipped with the $\mathrm{Jac}(R)$-adic topology), where $\mathrm{Jac}(R)$ is the Jacobson radical of R,
- (ii) R is a finite extension of $\mathbb{Q}_p$ (equipped with the topology defined by the p-adic valuation).

We note that a ring R satisfying (i) or (ii) satisfies (i) if and only if $p \notin R^\times$. We use the notation L instead of R if we consider only the case (ii).

In this article, we mainly treat representations (of $G_{\mathbb{Q}_p}$ or $\mathrm{GL}_2(\mathbb{Q}_p)$, etc.) defined over such a ring R. Let G be a topological group. We say that T is an R-representation of G if T is a finite projective R-module with a continuous R-linear G-action. For a continuous homomorphism $\delta : G \to R^\times$, we set $R(\delta) := R\mathbf{e}_\delta$ the R-representation of rank one with a fixed basis $\mathbf{e}_\delta$ on which G acts by $g(\mathbf{e}_\delta) := \delta(g)\mathbf{e}_\delta$. We always identify $R(\delta^{-1})$ with the R-dual $R(\delta)^\vee$ by $R(\delta^{-1}) \xrightarrow{\sim} R(\delta)^\vee : \mathbf{e}_{\delta^{-1}} \mapsto \mathbf{e}_\delta^\vee$, and identify $R(\delta_1) \otimes_R R(\delta_2) \xrightarrow{\sim} R(\delta_1\delta_2)$ by $\mathbf{e}_{\delta_1} \otimes \mathbf{e}_{\delta_2} \xrightarrow{\sim} \mathbf{e}_{\delta_1\delta_2}$ for any $\delta_1, \delta_2 : G \to R^\times$. We set $T(\delta) := T \otimes_R R(\delta)$. For an R-representation T of G, we set $T(\delta) := T \otimes_R R(\delta)$ and denote by $C^\bullet_{\mathrm{cont}}(G, T)$ the complex of continuous cochains of G with values in T, i.e. defined by $C^i_{\mathrm{cont}}(G,T) := \{c : G^{\times i} \to T : \text{ continuous maps } \}$ for each $i \geqq 0$ with the usual boundary map. We also regard $C^\bullet_{\mathrm{cont}}(G,T)$ as an object of $\mathbf{D}^-(R)$.

Now, we fix another prime l (we don't assume $l \neq p$). Let T be an R-representation of $G_{\mathbb{Q}_l}$. We set $\mathrm{H}^i(\mathbb{Q}_l, T) := \mathrm{H}^i(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T))$. For each $r \in \mathbb{Z}$, we set $T(r) := T \otimes_{\mathbb{Z}_p} \Gamma(\overline{\mathbb{Q}}_l, \mathbb{Z}_p(1))^{\otimes r}$. We denote by $T^* := T^\vee(1)$

the Tate dual of T. By the classical theory of the Galois cohomology of local fields, it is known that one has $C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T) \in \mathbf{D}_{\mathrm{perf}}(R)$. Using the determinant functor, we define the following graded invertible R-module

$$\Delta_{R,1}(T) := \mathrm{Det}_R(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T)),$$

which is of degree $-r_T$ (resp. of degree 0) when $l = p$ (resp. when $l \neq p$) by the Euler-Poincaré formula.

For $a \in R^\times$ ($a \in \mathcal{O}^\times$ if $R = L$), we set

$$R_a := \{x \in W(\overline{\mathbb{F}}_p)\hat{\otimes}_{\mathbb{Z}_p} R | (\varphi \otimes \mathrm{id}_R)(x) = (1 \otimes a)x\},$$

which is an invertible R-module. For T as above, we freely regard $\det_R T$ as a continuous homomorphism $\det_R T : G^{\mathrm{ab}}_{\mathbb{Q}_l} \to R^\times$. Define a constant

$$a_l(T) := \det_R T(\mathrm{rec}_{\mathbb{Q}_l}(p)) \in R^\times,$$

and define another graded invertible R-module

$$\Delta_{R,2}(T) := \begin{cases} (R_{a_l(T)}, 0) & (l \neq p) \\ (\det_R T \otimes_R R_{a_p(T)}, r_T) & (l = p). \end{cases}$$

Finally, we set

$$\Delta_R(T) := \Delta_{R,1}(T) \boxtimes \Delta_{R,2}(T)$$

which we call the local fundamental line.

The local fundamental line is compatible with the functorial operations, i.e. for any $R \to R'$, one has a canonical isomorphism

$$\Delta_R(T) \otimes_R R' \xrightarrow{\sim} \Delta_{R'}(T \otimes_R R'),$$

for any exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of R-representations of $G_{\mathbb{Q}_l}$, one has a canonical isomorphism

$$\Delta_R(T_2) \xrightarrow{\sim} \Delta_R(T_1) \boxtimes \Delta_R(T_3),$$

and one has the following canonical isomorphism

$$\Delta_R(T) \xrightarrow{\sim} \begin{cases} \Delta_R(T^*)^\vee & (l \neq p) \\ \Delta_R(T^*)^\vee \boxtimes (L(r_T), 0) & (l = p) \end{cases}$$

defined as the product of the following two isomorphisms

$$\Delta_{R,1}(T) \xrightarrow{\sim} \Delta_{R,1}(T^*)^\vee,$$

which is induced by the Tate duality $C^{\bullet}_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_R(C^{\bullet}_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T^*), R)$, and

$$\Delta_{R,2}(T) \xrightarrow{\sim} \begin{cases} \Delta_{R,2}(T^*)^{\vee} & (l \neq p) \\ \Delta_{R,2}(T^*)^{\vee} \boxtimes (L(r_T), 0) & (l = p) \end{cases}$$

which is defined by $x \mapsto [y \mapsto x \otimes y]$ for $x \in R_{a_l(T)}, y \in R_{a_l(T^*)}$ when $l \neq p$, by $x \otimes y \mapsto [z \otimes w \mapsto y \otimes w] \otimes z(x)$ for $x \in \det_R T, y \in R_{a_p(T)}, z \in \det_R(T^*) = (\det_R T)^{\vee}(r_T), w \in R_{a_p(T^*)}$ when $l = p$ (remark that one has $R_{a_l(T)} \otimes_R R_{a_l(T^*)} = R$ since one has $a_l(T)a_l(T^*) = 1$ for any l).

2.1.3. The de Rham $\boldsymbol{\varepsilon}$-isomorphism. The local ε-conjecture concerns with the existence of a compatible family of trivializations $\varepsilon_R(T)(= \varepsilon_{R,\zeta}(T)) : \mathbf{1}_R \xrightarrow{\sim} \Delta_R(T)$, (depending on the fixed choice $\zeta \in \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_l(1))$) which we call the local ε-isomorphisms, for all the pairs (R, T) as above which interpolate the trivializations $\varepsilon^{\mathrm{dR}}_L(V)(= \varepsilon^{\mathrm{dR}}_{L,\zeta}(V)) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(V)$, which we call the de Rham ε-isomorphisms, for all the pairs $(L, V) = (R, T)$ such that V is de Rham (resp. arbitrary) if $l = p$ (resp. if $l \neq p$), whose definition we briefly recall now.

We first recall the ε-constants defined for the representations of the Weil-Deligne group $'W_{\mathbb{Q}_l}$ of $\mathbb{Q}_l$. Let K be a field of characteristic zero which contains all the l-power roots of unity. For a $\mathbb{Z}_l$-basis $\zeta = \{\zeta_{l^n}\}_{n \geqq 0} \in \Gamma(K, \mathbb{Z}_l(1)) := \varprojlim_{n \geqq 0} \mu_{l^n}(K)$, define an additive character

$$\psi_\zeta : \mathbb{Q}_l \to K^{\times} \text{ by } \psi_\zeta(\frac{1}{l^n}) = \zeta_{l^n}$$

for any $n \geqq 0$. By the theory of local constants [De73], one can attach a constant

$$\varepsilon(\rho, \psi, dx) \in K^{\times}$$

to any smooth K-representation $\rho = (M, \rho)$ of $W_{\mathbb{Q}_l}$ (i.e. M is a finite dimensional K-vector space with a K-linear smooth action ρ of $W_{\mathbb{Q}_l}$), which depends on the choices of an additive character $\psi : \mathbb{Q}_l^{\times} \to K^{\times}$ and a (K-valued) Haar measure dx on $\mathbb{Q}_l$. In this article, we consider this constant only for the pair (ψ_ζ, dx) such that $\int_{\mathbb{Z}_l} dx = 1$, which we denote by

$$\varepsilon(\rho, \zeta) := \varepsilon(\rho, \psi_\zeta, dx)$$

for simplicity. For a K-representation $M = (M, \rho, N)$ of the Weil-Deligne group $'W_{\mathbb{Q}_l}$ (i.e. $\rho := (M, \rho)$ is a smooth K-representation of $W_{\mathbb{Q}_l}$ with a K-linear endomorphism $N : M \to M$ such that $\widetilde{\mathrm{Fr}}_l \circ N = l^{-1} N \circ \widetilde{\mathrm{Fr}}_l$ for any lift $\widetilde{\mathrm{Fr}}_l \in W_{\mathbb{Q}_l}$ of the geometric Frobenius $\mathrm{Fr}_l \in G_{\mathbb{F}_l}$), its ε-constant is defined by

$$\varepsilon(M,\zeta) := \varepsilon(\rho,\zeta)\mathrm{det}_K(-\mathrm{Fr}_l | M^{I_l}/(M^{N=0})^{I_l}).$$

Now we recall the definition of de Rham ε-isomorphism $\varepsilon_L^{\mathrm{dR}}(V) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(V)$ for arbitrary (resp. de Rham) L-representation V of $G_{\mathbb{Q}_l}$ when $l \neq p$ (resp. $l = p$). By the Grothendieck's local monodromy theorem (resp. the p-adic local monodromy theorem [Be02] and the Fontaine's functor $\mathbf{D}_{\mathrm{pst}}(-)$ [FP94]) when $l \neq p$ (resp. $l = p$), one can functorially define an L-representation

$$W(V) = (W(V), \rho, N)$$

of $'W_{\mathbb{Q}_l}$. Set $L_\infty := L \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\mu_{l^\infty})$, and decompose it $L_\infty = \prod_\tau L_\tau$ into the product of fields L_τ. Then, we define a constant

$$\varepsilon_L(W(V)) \in L_\infty^\times$$

(depending on the choice of the fixed $\zeta \in \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_l(1))$) as the product of the ε-constants $\varepsilon(W(V)_\tau, \zeta_\tau) \in L_\tau^\times$ of $W(V)_\tau := W(V) \otimes_L L_\tau$ for all τ, where $\zeta_\tau \in \Gamma(L_\tau, \mathbb{Z}_l(1))$ is the natural image of the fixed $\zeta \in \Gamma(\mathbb{Q}_p(\mu_{l^\infty}), \mathbb{Z}_l(1))$.

We set

$$\mathbf{D}_{\mathrm{st}}(V) := W(V)^{I_l}, \quad \mathbf{D}_{\mathrm{cris}}(V) := \mathbf{D}_{\mathrm{st}}(V)^{N=0}$$

on which the Frobenius $\varphi_l := \mathrm{Fr}_l$ naturally acts. Remark that one has

$$\mathbf{D}_{\mathrm{cris}}(V) = V^{I_l}$$

if $l \neq p$. Set

$$\mathbf{D}_{\mathrm{dR}}(V) := (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}, \quad \mathbf{D}^i_{\mathrm{dR}}(V) := (t^i \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$$

and

$$t_V := \mathbf{D}_{\mathrm{dR}}(V)/\mathbf{D}^0_{\mathrm{dR}}(V)$$

(resp. $\mathbf{D}_{\mathrm{dR}}(V) = \mathbf{D}^i_{\mathrm{dR}}(V) = t_V := 0$) when $l = p$ (resp. $l \neq p$).

Using these preliminaries, we first define an isomorphism

$$\theta_L(V) : \mathbf{1}_L \xrightarrow{\sim} \Delta_{L,1}(V) \boxtimes \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(V))$$

which is naturally induced by the following exact sequence of L-vector spaces

$$(3) \qquad 0 \to \mathrm{H}^0(\mathbb{Q}_l, V) \to \mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{(a)} \mathbf{D}_{\mathrm{cris}}(V) \oplus t_V \xrightarrow{(b)} \mathrm{H}^1(\mathbb{Q}_l, V)$$
$$\xrightarrow{(c)} \mathbf{D}_{\mathrm{cris}}(V^*)^\vee \oplus \mathbf{D}^0_{\mathrm{dR}}(V) \xrightarrow{(d)} \mathbf{D}_{\mathrm{cris}}(V^*)^\vee \to \mathrm{H}^2(\mathbb{Q}_l, V) \to 0,$$

where the map (a) is the sum of $1-\varphi_l : \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{D}_{\mathrm{cris}}(V)$ and the canonical map $\mathbf{D}_{\mathrm{cris}}(V) \to t_V$, and the maps (b) and (c) are defined by using the Bloch-Kato's exponential and its dual when $l=p$, and the map (d) is the dual of (a) for V^* (see [Ka93b], [FK06] and [Na14b] for the precise definition).

Define a constant $\Gamma(V) \in \mathbb{Q}^\times$ by

$$\Gamma(V) := \begin{cases} 1 & (l \neq p) \\ \prod_{r\in\mathbb{Z}} \Gamma^*(r)^{-\dim_L \mathrm{gr}^{-r}\mathbf{D}_{\mathrm{dR}}(V)} & (l=p), \end{cases}$$

where we set

$$\Gamma^*(r) := \begin{cases} (r-1)! & (r \geqq 1) \\ \frac{(-1)^r}{(-r)!} & (r \leqq 0). \end{cases}$$

We next define an isomorphism

$$\theta_{\mathrm{dR},L}(V) : \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(V)) \xrightarrow{\sim} \Delta_{L,2}(V)$$

which is induced by the isomorphism

$$\det_L \mathbf{D}_{\mathrm{dR}}(V) = L \xrightarrow{\sim} L_{a_l(V)} : x \mapsto \varepsilon_L(W(V))x$$

when $l \neq p$ (remark that one has $\varepsilon_L(W(V)) \in L_{a_l(V)}$ when $l \neq p$), by the inverse of the isomorphism

$$L_{a_p(T)} \otimes_L \det_L V \xrightarrow{\sim} \det_L \mathbf{D}_{\mathrm{dR}}(V) (\subseteq \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} \det_L V) : x \mapsto \frac{1}{\varepsilon_L(W(V))t^{h_V}} x$$

when $l=p$ (Lemma 3.4 [Na14b]), where we set $h_V := \sum_{r\in\mathbb{Z}} r \cdot \dim_L \mathrm{gr}^{-r}\mathbf{D}_{\mathrm{dR}}(V)$.

Finally, we define the de Rham ε-isomorphism

$$\varepsilon_L^{\mathrm{dR}}(V) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(V)$$

as the following composites

$$\begin{aligned} \varepsilon_L^{\mathrm{dR}}(V) : \mathbf{1}_L &\xrightarrow{\Gamma(V)\theta_L(V)} \Delta_{L,1}(V) \boxtimes \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(V)) \\ &\xrightarrow{\mathrm{id}\boxtimes\theta_{\mathrm{dR},L}(V)} \Delta_{L,1}(V) \boxtimes \Delta_{L,2}(V) = \Delta_L(V). \end{aligned}$$

As we remarked above, the isomorphism $\varepsilon_L^{\mathrm{dR}}(V)$ depends on the choice of ζ. If we'd like to consider this dependence, we use the notation $\varepsilon_{L,\zeta}^{\mathrm{dR}}(V) := \varepsilon_L^{\mathrm{dR}}(V)$.

2.1.4. The local ε-conjecture. The local ε-conjecture (Conjecture 1.8 [Ka93b], Conjecture 3.4.3 [FK06], and Conjecture 3.8 [Na14b]) is the following, which is now a theorem when $l \neq p$ by [Ya09].

Conjecture 2.1. *Fix a prime l. Then, there exists a unique compatible family of isomorphisms*

$$\varepsilon_R(T)(=\varepsilon_{R,\zeta}(T)) : \mathbf{1}_R \xrightarrow{\sim} \Delta_R(T)$$

for all the pairs (R,T) such that T is an R-representation of $G_{\mathbb{Q}_l}$, which satisfies the following properties.

(1) For any continuous $\mathbb{Z}_p$-algebra homomorphism $R \to R'$, one has

$$\varepsilon_R(T) \otimes \mathrm{id}_{R'} = \varepsilon_{R'}(T \otimes_R R')$$

under the canonical isomorphism

$$\Delta_R(T) \otimes_R R' \xrightarrow{\sim} \Delta_{R'}(T \otimes_R R').$$

(2) For any exact sequence $0 \to T_1 \to T_2 \to T_3 \to 0$ of R-representations of $G_{\mathbb{Q}_l}$, one has

$$\varepsilon_R(T_2) = \varepsilon_R(T_1) \boxtimes \varepsilon_R(T_3)$$

under the canonical isomorphism

$$\Delta_R(T_2) \xrightarrow{\sim} \Delta_R(T_1) \boxtimes \Delta_R(T_3).$$

(3) (dependence on ζ) For any $a \in \mathbb{Z}_l^\times$, one has

$$\varepsilon_{R,a\zeta}(T) = \det_R T(\mathrm{rec}_{\mathbb{Q}_l}(a))\varepsilon_{R,\zeta}(T).$$

(4) One has the following commutative diagrams

$$\begin{tikzcd} \Delta_R(T) \arrow[r, "\mathrm{can}"] & \Delta_R(T^*)^\vee \arrow[d, "\varepsilon_{R,\zeta}(T^*)^\vee"] \\ \mathbf{1}_R \arrow[u, "\varepsilon_{R,\zeta}(T)"] \arrow[r, "\det_R T(\mathrm{rec}_{\mathbb{Q}_l}(-1))\cdot \mathrm{id}"] & \mathbf{1}_R \end{tikzcd}$$

when $l \neq p$, and

$$\begin{tikzcd} \Delta_R(T) \arrow[r, "\mathrm{can}"] & \Delta_R(T^*)^\vee \boxtimes (R(r_T), 0) \arrow[d, "\varepsilon_{R,\zeta}(T^*)^\vee \boxtimes [\mathbf{e}_{r_T} \mapsto 1]"] \\ \mathbf{1}_R \arrow[u, "\varepsilon_{R,-\zeta}(T)"] \arrow[r, "\det_R T(\mathrm{rec}_{\mathbb{Q}_l}(-1))\cdot \mathrm{can}"] & \mathbf{1}_R \boxtimes \mathbf{1}_R \end{tikzcd}$$

when $l = p$.

(5) For any pair (L, V) such that V is arbitrary (resp. de Rham) if $l \neq p$ (resp. if $l = p$), one has

$$\varepsilon_L(V) = \varepsilon_L^{\mathrm{dR}}(V).$$

Remark 2.2. In the conjecture, the conditions (2), (3) and (4) should follow from the other conditions (1) and (5). In fact, it is known that $\varepsilon_L^{\mathrm{dR}}(V)$ satisfies the similar conditions (2), (3), (4) (e.g. Remark 3.5, Lemma 3.7 [Na14b]). Hence, assuming the density of de Rham representations in the universal deformation, which is known in many cases, the conditions (1) and (5) induce the conditions (2), (3) and (4).

Remark 2.3. There exists a non-commutative version of this conjecture, but we only consider the commutative case in this article. See [FK06] for the non-commutative version.

Remark 2.4. When $l \neq p$, this conjecture has been already proved by Yasuda [Ya09]. More precisely, he proved that the correspondence

$$(L, V) \mapsto \varepsilon_{0,L}(V) := \det_L(-\varphi_l | V^{I_l})\varepsilon_L(W(V)) \in L_{a_l(V)}$$

defined for all the pairs (L, V) as in the condition (5) (for $l \neq p$) in Conjecture 2.1 uniquely extends to a correspondence

$$(R, T) \mapsto \varepsilon_{0,R}(T) \in R_{a_l(T)}$$

for all the pairs (R, T) as in the conjecture, which satisfies the similar properties (1)-(5) in the conjecture. Then, the isomorphism $\varepsilon_R(T) : \mathbf{1}_R \xrightarrow{\sim} \Delta_R(T)$ is defined as the product of the isomorphism $\mathbf{1}_R \xrightarrow{\sim} \Delta_{R,2}(T)$ induced by the isomorphism

$$R \xrightarrow{\sim} R_{a_l(T)} : x \mapsto \varepsilon_{0,R}(T)x$$

with the isomorphism $\mathbf{1}_R \xrightarrow{\sim} \Delta_{R,1}(T)$ defined by

$$\mathbf{1}_R \xrightarrow{\sim} \mathrm{Det}_R(C^{\bullet}_{\mathrm{cont}}(I_l, T)) \boxtimes \mathrm{Det}_R(C^{\bullet}_{\mathrm{cont}}(I_l, T))^{-1} \xrightarrow{\sim} \mathrm{Det}_R(C^{\bullet}_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T)),$$

where the first isomorphism is the canonical one and the second isomorphism is induced by the canonical quasi-isomorphism

$$C^{\bullet}_{\mathrm{cont}}(G_{\mathbb{Q}_l}, T) \xrightarrow{\sim} [C^{\bullet}_{\mathrm{cont}}(I_l, T) \xrightarrow{1-\varphi_l^{-1}} C^{\bullet}_{\mathrm{cont}}(I_l, T)].$$

2.1.5. The cyclotomic deformation. Before proceeding to the next subsection, we recall here the notion of the cyclotomic deformations of R-representations, which will play an important role in this article.

For any R such that $p \notin R^\times$, we set $\Lambda_R := R[[\Gamma]]$ the Iwasawa algebra of Γ with coefficients in R, and set $\Lambda_L := \Lambda_{\mathcal{O}_L}[1/p]$ for any L. For an R-representation T of $G_{\mathbb{Q}_l}$, we define a Λ_R-representation $\mathbf{Dfm}(T)$ which we call the cyclotomic deformation of T by

$$\mathbf{Dfm}(T) := T \otimes_R \Lambda_R$$

on which $G_{\mathbb{Q}_l}$ acts by

$$g(x \otimes \lambda) := g(x) \otimes [\bar{g}]^{-1}\lambda$$

for $g \in G_{\mathbb{Q}_l}$, $x \in T$, $\lambda \in \Lambda_R$, where $\bar{g} \in \Gamma$ is the image of g by the natural restriction map $G_{\mathbb{Q}_l} \to \Gamma$. We set

$$\Delta^{\mathrm{Iw}}_{R,*}(T) := \Delta_{\Lambda_R,*}(\mathbf{Dfm}(T)) \text{ and } \mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Q}_l, T) := \mathrm{H}^i(\mathbb{Q}_l, \mathbf{Dfm}(T))$$

for $* = 1, 2$, or $* = \phi$ (the empty set).

For a continuous homomorphism $\delta : \Gamma \to R^\times$, define a continuous R-algebra homomorphism

$$f_\delta : \Lambda_R \to R : [\gamma] \mapsto \delta(\gamma)^{-1}$$

for any $\gamma \in \Gamma$. Then, one has a canonical isomorphism of R-representations of $G_{\mathbb{Q}_l}$

$$\mathbf{Dfm}(T) \otimes_{\Lambda_R, f_\delta} R \xrightarrow{\sim} T(\delta) : (x \otimes \lambda) \otimes a \mapsto a f_\delta(\lambda) x \otimes \mathbf{e}_\delta$$

for $x \in T, \lambda \in \Lambda_R, a \in R$. By the compatibility with base changes, this isomorphism induces a canonical isomorphism

$$\Delta^{\mathrm{Iw}}_R(T) \otimes_{\Lambda_R, f_\delta} R \xrightarrow{\sim} \Delta_R(T(\delta)).$$

Let $\iota : \Lambda_R \xrightarrow{\sim} \Lambda_R$ be the involution of the topological R-algebra defined by $\iota([\gamma]) = [\gamma]^{-1}$ for any $\gamma \in \Gamma$. For any Λ_R-module M, we set $M^\iota := M \otimes_{\Lambda_R, \iota} \Lambda_R$, i.e. $M^\iota = M$ as R-module on which Λ_R acts by $\lambda \cdot_\iota x := \iota(\lambda) \cdot x$ for $\lambda \in \Lambda_R, x \in M$, where $\cdot_\iota$ is the action on M^ι and $\cdot$ is the usual action on M. One has a canonical isomorphism of Λ_R-representations of $G_{\mathbb{Q}_l}$

$$\mathbf{Dfm}(T^*)^\iota \xrightarrow{\sim} \mathbf{Dfm}(T)^* : x \otimes \lambda \mapsto [y \otimes \lambda' \mapsto \iota(\lambda)\lambda' \otimes x(y)]$$

for $x \in T^*$, $y \in T$, $\lambda, \lambda' \in \Lambda_R$, which naturally induces a canonical isomorphism

$$\Delta_{R,*}^{\mathrm{Iw}}(T^*)^{\iota} \xrightarrow{\sim} \Delta_{\Lambda_R,*}(\mathbf{Dfm}(T)^*) \tag{4}$$

for $* = 1, 2$, or $* = \phi$.

2.2. Review of the theory of étale (φ, Γ)-modules

From now on until the end of §3, we concentrate on the case where $l = p$. We set $\mathbf{e}_1 := \zeta \in \mathbb{Z}_p(1) := \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_p(1))$ and $\mathbf{e}_r := \mathbf{e}_1^{\otimes r} \in \mathbb{Z}_p(r)$ for $r \in \mathbb{Z}$.

2.2.1. Étale (φ, Γ)-modules. For R as in §2.1 such that $p \notin R^{\times}$, we set $\mathcal{E}_R := \varprojlim_{n \geqq 1}(R/\mathrm{Jac}(R)^n[[X]][1/X])$, and set $\mathcal{E}_L := \mathcal{E}_{\mathcal{O}}[1/p]$ for $R = L$, on which φ and Γ acts as continuous R-algebra homomorphism by $\varphi(X) := (1+X)^p - 1$, $\gamma(X) := (1+X)^{\chi(\gamma)} - 1$ for $\gamma \in \Gamma$.

For R such that $p \notin R^{\times}$, we say that D is an étale (φ, Γ)-module over $\mathcal{E}_R$ if D is a finite projective $\mathcal{E}_R$-module equipped with a Frobenius structure $\varphi : \varphi^* D := D \otimes_{\mathcal{E}_R, \varphi} \mathcal{E}_R \xrightarrow{\sim} D$ and a commuting continuous semi-linear action $\Gamma \times D \to D : (\gamma, x) \mapsto \gamma(x)$. For $R = L$, we say that D is an étale (φ, Γ)-module over $\mathcal{E}_L$ if D is the base change to $\mathcal{E}_L$ of an étale (φ, Γ)-module over $\mathcal{E}_{\mathcal{O}}$. We denote by $D^{\vee} := \mathrm{Hom}_{\mathcal{E}_R}(D, \mathcal{E}_R)$ the dual (φ, Γ)-module of D, by $D(r) := D \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$ the r-th Tate twist of D (for $r \in \mathbb{Z}$), by $D^* := D^{\vee}(1)$ the Tate dual of D.

One has the Fontaine's equivalence $T \mapsto D(T)$ between the category of R-representations of $G_{\mathbb{Q}_p}$ and that of étale (φ, Γ)-modules over $\mathcal{E}_R$. In the construction of this equivalence, we need to embed the ring $\mathcal{E}_{\mathbb{Z}_p}$ ($\mathcal{E}_R$ for $R = \mathbb{Z}_p$) into the Fontaine ring $\widetilde{\mathbf{A}}^+ (:= W(\widetilde{\mathbf{E}}^+))$ by $X \mapsto X_{\zeta} := [(\bar{\zeta}_{p^n})_{n \geqq 0}] - 1 \in \widetilde{\mathbf{A}}^+$, which depends on the choice of the fixed basis $\zeta = (\zeta_{p^n})_{n \geqq 0} \in \mathbb{Z}_p(1)$. We remark that, for a different choice $a\zeta$ of a basis for $a \in \mathbb{Z}_p^{\times}$, one has $X_{a\zeta} = (X_{\zeta} + 1)^a - 1$.

Define a left inverse $\psi : D \to D$ of φ by

$$\psi : D = \sum_{i=1}^{p-1} (1+X)^i \varphi(D) \to D : \sum_{i=0}^{p-1} (1+X)^i \varphi(x_i) \mapsto x_0.$$

2.2.2. Cohomology of étale (φ, Γ)-modules. We next recall the cohomology theory of (φ, Γ)-modules. Let $\Gamma_{\mathrm{tor}} \subseteq \Gamma$ denote the torsion subgroup of Γ. Define a finite subgroup $\Delta \subseteq \Gamma_{\mathrm{tor}}$ by $\Delta := \{1\}$ when $p > 2$ and $\Delta := \Gamma_{\mathrm{tor}}$ when $p = 2$. Then Γ/Δ has a topological generator $\bar{\gamma}$, which we fix. We also fix a lift $\gamma \in \Gamma$ of $\bar{\gamma}$.

Remark 2.5. When $p=2$ and $p\notin R^{\times}$, the cohomology theory of (φ,Γ)-modules over $\mathcal{E}_R$ is a little more subtle than that in other cases since one has $|\Gamma_{\mathrm{tor}}|=p$ in this case. To avoid this subtlety, we treat (φ,Γ)-modules over the rings of the form $R=R_0[1/p]$, where R_0 is a topological $\mathbb{Z}_p$-algebra satisfying the condition (i) in §2.1. For such R, we set $\mathcal{E}_R:=\mathcal{E}_{R_0}[1/p]$, and say that an $\mathcal{E}_R$-module D is an étale (φ,Γ)-module over $\mathcal{E}_R$ if it is the base change to $\mathcal{E}_R$ of an étale (φ,Γ)-module over $\mathcal{E}_{R_0}$. From now on until the end of this article, we use the notation R to represent topological $\mathbb{Z}_p$-algebras of the form R_0 or $R_0[1/p]$ (resp. $R_0[1/2]$) as above when $p\geqq 3$ (resp. $p=2$), and we only consider the R-representations of $G_{\mathbb{Q}_l}$ or $G_{\mathbb{Q},S}$, and étale (φ,Γ)-modules over $\mathcal{E}_R$ for such R.

Definition 2.6. For any étale (φ,Γ)-module D over $\mathcal{E}_R$, define complexes $C^{\bullet}_{\varphi,\gamma}(D)$ and $C^{\bullet}_{\psi,\gamma}(D)$ of R-modules concentrated in degree $[0,2]$, and define a morphism Ψ_D between them as follows:

$$\begin{tikzcd} C^{\bullet}_{\varphi,\gamma}(D)= \arrow[d, "\Psi_D"'] & {[}D^{\Delta} \arrow[r, "{(\gamma-1,\varphi-1)}"] \arrow[d, "\mathrm{id}"] & D^{\Delta}\oplus D^{\Delta} \arrow[r, "(\varphi-1)\oplus(1-\gamma)"] \arrow[d, "\mathrm{id}\oplus-\psi"] & D^{\Delta}{]} \arrow[d, "-\psi"] \\ C^{\bullet}_{\psi,\gamma}(D)= & {[}D^{\Delta} \arrow[r, "{(\gamma-1,\psi-1)}"] & D^{\Delta}\oplus D^{\Delta} \arrow[r, "(\psi-1)\oplus(1-\gamma)"] & D^{\Delta}{]}. \end{tikzcd} \tag{5}$$

For $i\in\mathbb{Z}_{\geqq 0}$, we denote by $\mathrm{H}^i_{\varphi,\gamma}(D)$ (resp. $\mathrm{H}^i_{\psi,\gamma}(D)$) the i-th cohomology of $C^{\bullet}_{\varphi,\gamma}(D)$ (resp.$C^{\bullet}_{\psi,\gamma}(D)$). It is known that the map $\Psi_D:C^{\bullet}_{\varphi,\gamma}(D)\to C^{\bullet}_{\psi,\gamma}(D)$ is quasi-isomorphism by (for example) Proposition I.5.1 and Lemme I.5.2 of [CC99]. In this article, we freely identify $C^{\bullet}_{\varphi,\gamma}(D)$ (resp. $\mathrm{H}^i_{\varphi,\gamma}(D)$) with $C^{\bullet}_{\psi,\gamma}(D)$ in $\mathbf{D}^-(R)$ (resp. $\mathrm{H}^i_{\psi,\gamma}(D)$) via the quasi-isomorphism Ψ_D.

For étale (φ,Γ)-modules D_1,D_2 over $\mathcal{E}_R$, one has an R-bilinear cup product pairing

$$C^{\bullet}_{\varphi,\gamma}(D_1)\times C^{\bullet}_{\varphi,\gamma}(D_2)\to C^{\bullet}_{\varphi,\gamma}(D_1\otimes D_2),$$

which induces the cup product pairing

$$\cup:\mathrm{H}^i_{\varphi,\gamma}(D_1)\times \mathrm{H}^j_{\varphi,\gamma}(D_2)\to \mathrm{H}^{i+j}_{\varphi,\gamma}(D_1\otimes D_2).$$

For example, this pairing is explicitly defined by the formulae

$$x\cup[y]:=[x\otimes y] \text{ for } i=0,j=2,$$
$$[x_1,y_1]\cup[x_2,y_2]:=[x_1\otimes\gamma(y_2)-y_1\otimes\varphi(x_2)] \text{ for } i=j=1.$$

Definition 2.7. Using the cup product, the evaluation map $\mathrm{ev}:D^*\otimes D\to\mathcal{E}_R(1):f\otimes x\mapsto f(x)$, the comparison isomorphism $\mathrm{H}^2(\mathbb{Q}_p,R(1))\xrightarrow{\sim}$

$\mathrm{H}^2_{\varphi,\gamma}(\mathcal{E}_R(1))$ (see below), and the Tate's trace map $\mathrm{H}^2(\mathbb{Q}_p, R(1)) \xrightarrow{\sim} R$, one gets the Tate duality pairings

$$C^\bullet_{\varphi,\gamma}(D^*) \times C^\bullet_{\varphi,\gamma}(D) \to R[-2]$$

and

$$\langle -, - \rangle_{\mathrm{Tate}} : \mathrm{H}^i_{\varphi,\gamma}(D^*) \times \mathrm{H}^{2-i}_{\varphi,\gamma}(D) \to R.$$

Let T be an R-representation of $G_{\mathbb{Q}_p}$. By the result of [He98], one has a canonical functorial isomorphism

$$C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_p}, T) \xrightarrow{\sim} C^\bullet_{\varphi,\gamma}(D(T))$$

in $\mathbf{D}^-(R)$ and a canonical functorial R-linear isomorphism

$$\mathrm{H}^i(\mathbb{Q}_p, T) \xrightarrow{\sim} \mathrm{H}^i_{\varphi,\gamma}(D(T)).$$

In particular, we obtain a canonical isomorphism

$$\Delta_{R,1}(T) \xrightarrow{\sim} \mathrm{Det}_R(C^\bullet_{\varphi,\gamma}(D(T))) =: \Delta_{R,1}(D(T)).$$

For an étale (φ,Γ)-module D. We freely regard the rank one (φ,Γ)-module $\det_{\mathcal{E}_R} D$ as a character $\det_{\mathcal{E}_R} D : G^{\mathrm{ab}}_{\mathbb{Q}_p} \to R^\times$ by the Fontaine's equivalence. Then, the (φ,Γ)-module $\det_{\mathcal{E}_R} D$ has a basis $\mathbf{e}$ on which φ and Γ act by

$$\varphi(\mathbf{e}) = \det_{\mathcal{E}_R} D(\mathrm{rec}_{\mathbb{Q}_p}(p))\mathbf{e}, \quad \gamma'(\mathbf{e}) = \det_{\mathcal{E}_R} D(\gamma')\mathbf{e}$$

for $\gamma' \in \Gamma$, where we regard Γ as a subgroup of $G^{\mathrm{ab}}_{\mathbb{Q}_p}$ by the canonical isomorphism $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p^{\mathrm{ur}}) \xrightarrow{\sim} \Gamma$. Using the character $\det_{\mathcal{E}_R} D$, we set

$$\mathcal{L}_R(D) := R\mathbf{e},$$

which is a free R-module of rank one, and define the following graded invertible R-modules

$$\Delta_{R,2}(D) := (\mathcal{L}_R(D), r_D) \text{ and } \Delta_R(D) := \Delta_{R,1}(D) \boxtimes \Delta_{R,2}(D).$$

By (the proof of) Lemma 3.1 of [Na14b], there exists a canonical isomorphism

$$\Delta_{R,2}(T) \xrightarrow{\sim} \Delta_{R,2}(D(T))$$

for any R-representation T of $G_{\mathbb{Q}_p}$. Therefore, we obtain a canonical isomorphism

$$\Delta_R(T) \xrightarrow{\sim} \Delta_R(D(T)),$$

by which we identify the both sides.

2.2.3. Iwasawa cohomology of étale (φ,Γ)-modules. We next recall the theory of the Iwasawa cohomology of étale (φ,Γ)-modules. For an étale (φ,Γ)-module D over $\mathcal{E}_R$, we define the cyclotomic deformation $\mathbf{Dfm}(D)$ which is an étale (φ,Γ)-module over $\mathcal{E}_{\Lambda_R}$ by

$$\mathbf{Dfm}(D) := D\otimes_{\mathcal{E}_R}\mathcal{E}_{\Lambda_R}$$

as $\mathcal{E}_{\Lambda_R}$-module on which φ and Γ act by

$$\varphi(x\otimes y) := \varphi(x)\otimes\varphi(y),\ \gamma'(x\otimes y) := \gamma'(x)\otimes[\gamma']^{-1}\gamma'(y)$$

for $x\in D, y\in\mathcal{E}_{\Lambda_R}$, $\gamma'\in\Gamma$. Then, one has a canonical isomorphism

$$D(\mathbf{Dfm}(T))\xrightarrow{\sim}\mathbf{Dfm}(D(T)).$$

Hence, if we set

$$\mathrm{H}^i_{\mathrm{Iw},\varphi,\gamma}(D) := \mathrm{H}^i_{\varphi,\gamma}(\mathbf{Dfm}(D)) \text{ and } \Delta^{\mathrm{Iw}}_R(D) := \Delta_{\Lambda_R}(\mathbf{Dfm}(D)),\ \text{etc.},$$

then we obtain the following canonical isomorphisms

$$\mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Q}_p,T)\xrightarrow{\sim}\mathrm{H}^i_{\mathrm{Iw},\varphi,\gamma}(D(T)) \text{ and } \Delta^{\mathrm{Iw}}_R(T)\xrightarrow{\sim}\Delta^{\mathrm{Iw}}_R(D(T)),\ \text{etc.}$$

for any R-representation T of $G_{\mathbb{Q}_p}$. For any continuous homomorphism $\delta:\Gamma\to R^\times$, the base change with respect to $f_\delta:\Lambda_R\to R:[\gamma']\mapsto\delta(\gamma')^{-1}$ induces canonical isomorphisms

$$\mathbf{Dfm}(D)\otimes_{\Lambda_R,f_\delta}R\xrightarrow{\sim}D(\delta):(x\otimes y)\otimes 1\mapsto f_\delta(y)x\otimes\mathbf{e}_\delta$$

and

$$\Delta^{\mathrm{Iw}}_R(D)\otimes_{\Lambda_R,f_\delta}R\xrightarrow{\sim}\Delta_R(D(\delta)),$$

and induces a canonical specialization map

$$\mathrm{sp}_\delta:\mathrm{H}^i_{\mathrm{Iw},\varphi,\gamma}(D)\to\mathrm{H}^i_{\varphi,\gamma}(D(\delta)).$$

We remark that the continuous action of Γ on D uniquely extends to a Λ_R-module structure on D. We define a complex $C^\bullet_\psi(D)$ of Λ_R-modules which concentrated in degree $[1,2]$ by

$$C^\bullet_\psi(D):[D\xrightarrow{\psi-1}D].$$

By the result of [CC99], there exists a canonical isomorphism

$$C^\bullet_\psi(D)\xrightarrow{\sim}C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(D))$$

in $\mathbf{D}^{-}(\Lambda_R)$. In particular, there exists a canonical isomorphism

$$\mathrm{can} : D^{\psi=1} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{Iw},\psi,\gamma}(D)$$

of Λ_R-modules which is explicitly defined by

$$x \mapsto \left[\left(\frac{p-1}{p}\log(\chi(\gamma))p_\Delta \cdot (x \otimes 1), 0\right)\right],$$

where $p_\Delta := \frac{1}{|\Delta|}\sum_{\sigma\in\Delta}[\sigma] \in \mathbb{Z}[1/2][\Delta]$ (remark that we have $\frac{p-1}{p}\log(\chi(\gamma))p_\Delta \in \mathbb{Z}_p[\Delta]$ for any p). Hence, if we define a specialization map

$$\iota_\delta : D^{\psi=1} \to \mathrm{H}^1_{\psi,\gamma}(D(\delta)) : x \mapsto x_\delta := \left[\left(\frac{p-1}{p}\log(\chi(\gamma))p_\Delta \cdot (x \otimes \mathbf{e}_\delta), 0\right)\right]$$

for any continuous homomorphism $\delta : \Gamma \to R^\times$, then it makes the diagram

$$\begin{array}{ccc} D^{\psi=1} & \xrightarrow{\mathrm{can}} & \mathrm{H}^1_{\mathrm{Iw},\psi,\gamma}(D) \\ {\scriptstyle \iota_\delta}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{sp}_\delta} \\ \mathrm{H}^1_{\psi,\gamma}(D(\delta)) & \xrightarrow{\mathrm{id}} & \mathrm{H}^1_{\psi,\gamma}(D(\delta)) \end{array} \tag{6}$$

commutative.

2.2.4. The Iwasawa pairing. We next consider the Tate dual of $\mathbf{Dfm}(D)$. For this, we first remark that the involution $\iota : \Lambda_R \xrightarrow{\sim} \Lambda_R : [\gamma'] \mapsto [\gamma']^{-1}$ naturally induces an $\mathcal{E}_R$-linear involution $\iota : \mathcal{E}_{\Lambda_R} \xrightarrow{\sim} \mathcal{E}_{\Lambda_R}$. For an étale (φ, Γ)-module D over $\mathcal{E}_R$, define an étale (φ, Γ)-module $D \otimes_{\mathcal{E}_R} \widetilde{\mathcal{E}_{\Lambda_R}}$ over $\mathcal{E}_{\Lambda_R}$ by

$$D \otimes_{\mathcal{E}_R} \widetilde{\mathcal{E}_{\Lambda_R}} = D \otimes_{\mathcal{E}_R} \mathcal{E}_{\Lambda_R}$$

as $\mathcal{E}_{\Lambda_R}$-module on which φ and Γ act by

$$\varphi(x \otimes y) = \varphi(x) \otimes \varphi(y) \text{ and } \gamma'(x \otimes y) = \gamma'(x) \otimes [\gamma']\gamma'(y)$$

for $x \in D$, $y \in \mathcal{E}_{\Lambda_R}, \gamma' \in \Gamma$. Then, the isomorphism

$$D \otimes_{\mathcal{E}_R} \mathcal{E}_{\Lambda_R} \xrightarrow{\sim} D \otimes_{\mathcal{E}_R} \mathcal{E}_{\Lambda_R} : x \otimes y \mapsto x \otimes \iota(y)$$

induces an isomorphism

$$D \otimes_{\mathcal{E}_R} \widetilde{\mathcal{E}_{\Lambda_R}} \xrightarrow{\sim} \mathbf{Dfm}(D)^\iota$$

of (φ, Γ)-modules over $\mathcal{E}_{\Lambda_R}$. Since one has a canonical isomorphism

$$(D \otimes_{\mathcal{E}_R} \widetilde{\mathcal{E}_{\Lambda_R}}) \otimes_{\Lambda_R, f_\delta} R \xrightarrow{\sim} D(\delta^{-1}) : (x \otimes y) \otimes 1 \mapsto f_\delta(y) x \otimes \mathbf{e}_{\delta^{-1}}$$

for any $\delta : \Gamma \to R^\times$, we obtain a canonical specialization map

$$\tilde{\iota}_\delta : D^{\psi=1,\iota} \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D)^\iota) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(D \otimes_{\mathcal{E}_R} \widetilde{\mathcal{E}_{\Lambda_R}}) \to \mathrm{H}^1_{\psi,\gamma}(D(\delta^{-1}))$$

which is explicitly defined by

$$x \mapsto \left[\left(\frac{p-1}{p}\log(\chi(\gamma)) p_\Delta \cdot (x \otimes \mathbf{e}_{\delta^{-1}}), 0\right)\right].$$

We apply this to the Tate dual D^* of D. Since one has a canonical isomorphism

$$\mathbf{Dfm}(D^*)^\iota \xrightarrow{\sim} D^* \otimes_{\mathcal{E}_R} \widetilde{\mathcal{E}_{\Lambda_R}} \xrightarrow{\sim} \mathbf{Dfm}(D)^*,$$

we obtain canonical isomorphisms

$$\Delta_R^{\mathrm{Iw}}(D^*)^\iota \xrightarrow{\sim} \Delta_{\Lambda_R}(\mathbf{Dfm}(D)^*)$$

and

$$\mathrm{can} : (D^*)^{\psi=1,\iota} \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D^*)^\iota) \xrightarrow{\sim} \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D)^*),$$

which makes the diagram

$$\begin{array}{ccc} (D^*)^{\psi=1,\iota} & \xrightarrow{\mathrm{can}} & \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D)^*) \\ {\scriptstyle \tilde{\iota}_\delta}\downarrow & & \downarrow{\scriptstyle \mathrm{sp}_\delta} \\ \mathrm{H}^1_{\psi,\gamma}(D(\delta)^*) & \xrightarrow{\mathrm{id}} & \mathrm{H}^1_{\psi,\gamma}(D(\delta)^*) \end{array} \tag{7}$$

for any $\delta : \Gamma \to R^\times$ commutative, where the right vertical arrow is the specialization map with respect to the base change

$$\mathbf{Dfm}(D)^* \otimes_{\Lambda_R, f_\delta} R \xrightarrow{\sim} (\mathbf{Dfm}(D) \otimes_{\Lambda_R, f_\delta} R)^* \xrightarrow{\sim} D(\delta)^*.$$

Using these preliminaries, we define a Λ_R-bilinear pairing

$$\{-,-\}_{\mathrm{Iw}} : (D^*)^{\psi=1,\iota} \times D^{\psi=1} \to \Lambda_R$$

which we call the Iwasawa pairing by the following commutative diagram

$$
\begin{tikzcd}
(D^*)^{\psi=1,\iota} \times D^{\psi=1} \arrow[r, "\mathrm{can}\times\mathrm{can}"] \arrow[d, "\{-,-\}_{\mathrm{Iw}}"'] & \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D)^*) \times \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D)) \arrow[d, "\langle -,-\rangle_{\mathrm{Tate}}"] \\
\Lambda_R \arrow[r, "\mathrm{id}"] & \Lambda_R.
\end{tikzcd}
\tag{8}
$$

From the arguments above, we obtain the commutative diagram

$$
\begin{tikzcd}
(D^*)^{\psi=1,\iota} \times D^{\psi=1} \arrow[r, "\tilde{\iota}_\delta\times\iota_\delta"] \arrow[d, "\{-,-\}_{\mathrm{Iw}}"'] & \mathrm{H}^1_{\psi,\gamma}(D(\delta)^*) \times \mathrm{H}^1_{\psi,\gamma}(D(\delta)) \arrow[d, "\langle -,-\rangle_{\mathrm{Tate}}"] \\
\Lambda_R \arrow[r, "f_\delta"] & R
\end{tikzcd}
\tag{9}
$$

for any $\delta : \Gamma \to R^\times$.

Remark 2.8. We remark that the pairing $\{-,-\}_{\mathrm{Iw}}$ coincides with the Colmez's Iwasawa pairing which is defined in §VI.1 of [Co10a] in a different way.

For a continuous homomorphism $\delta : \Gamma \to R^\times$, we define a continuous R-algebra automorphism $g_\delta : \Lambda_R \xrightarrow{\sim} \Lambda_R$ by $g_\delta([\gamma']) = \delta(\gamma')^{-1}[\gamma']$ for $\gamma' \in \Gamma$.

Lemma 2.9. *For any $\delta : \Gamma \to R^\times$, one has the following commutative diagram*

$$
\begin{tikzcd}
(D^*)^{\psi=1,\iota} \times D^{\psi=1} \arrow[r, "\{-,-\}_{\mathrm{Iw}}"] \arrow[d, "(x,y)\mapsto(x\otimes\mathbf{e}_{\delta^{-1}},y\otimes\mathbf{e}_\delta)"'] & \Lambda_R \arrow[d, "g_\delta"] \\
(D(\delta)^*)^{\psi=1,\iota} \times D(\delta)^{\psi=1} \arrow[r, "\{-,-\}_{\mathrm{Iw}}"] & \Lambda_R.
\end{tikzcd}
\tag{10}
$$

Proof. For a Λ_R-module M, we define a Λ_R-module $g_{\delta *}(M) := M$ on which Λ_R-acts by g_δ. Then, we have isomorphisms $\mathbf{Dfm}(D) \xrightarrow{\sim} g_{\delta *}(\mathbf{Dfm}(D(\delta)))$: $x \otimes y \mapsto (x \otimes \mathbf{e}_\delta) \otimes g_\delta(y)$ and $\mathbf{Dfm}(D^*)^\iota \xrightarrow{\sim} g_{\delta *}(\mathbf{Dfm}(D(\delta)^*)^\iota)$: $x \otimes y \mapsto (x \otimes \mathbf{e}_{\delta^{-1}}) \otimes g_{\delta^{-1}}(y)$, and these induce the following commutative diagram

$$
\begin{tikzcd}
\mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D)^*) \times \mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D)) \arrow[r, "\sim"] \arrow[d, "\langle -,-\rangle_{\mathrm{Tate}}"'] & g_{\delta *}(\mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D(\delta)^*))) \times g_{\delta *}(\mathrm{H}^1_{\psi,\gamma}(\mathbf{Dfm}(D(\delta)))) \arrow[d, "\langle -,-\rangle_{\mathrm{Tate}}"] \\
\Lambda_R \arrow[r, "g_\delta"] & g_{\delta *}(\Lambda_R).
\end{tikzcd}
\tag{11}
$$

By definition of $\{-,-\}_{\mathrm{Iw}}$, the lemma follows from this commutative diagram. $\square$

2.2.5. The local fundamental line over $\mathcal{E}_R(\Gamma)$. Take an isomorphism $\Gamma \xrightarrow{\sim} \Gamma_{\mathrm{tor}} \times \mathbb{Z}_p$ of topological groups. Let $\gamma_0 \in \Gamma$ be the element corresponding to $(e, 1)$ by this isomorphism, where $e \in \Gamma_{\mathrm{tor}}$ is the identity element. For R such that $p \notin R^\times$, define $\mathcal{E}_R(\Gamma) := \Lambda_R[\frac{1}{[\gamma_0]-1}]^\wedge$ the $\mathrm{Jac}(R)$-adic completion of $\Lambda_R[\frac{1}{[\gamma_0]-1}]$, which does not depend on the choice of the decomposition $\Gamma \xrightarrow{\sim} \Gamma_{\mathrm{tor}} \times \mathbb{Z}_p$. For $R = R_0[1/p]$ such that $p \notin R_0^\times$, define $\mathcal{E}_R(\Gamma) := \mathcal{E}_{R_0}(\Gamma)[1/p]$.

Here, we recall some properties of the base changes to $\mathcal{E}_R(\Gamma)$ of $\Delta_R(D)$ and $\{-,-\}_{\mathrm{Iw}}$, which are proved in [Co10a]. By III.4 of [Co10a], $\gamma_0 - 1$ acts on $D^{\psi=0}$ as a topological automorphism and the induced action of $\Lambda_R[\frac{1}{[\gamma_0]-1}]$ on $D^{\psi=0}$ uniquely extends to an action of $\mathcal{E}_R(\Gamma)$, which makes $D^{\psi=0}$ a finite projective $\mathcal{E}_R(\Gamma)$-module of rank r_D. By VI.1 of [Co10a], the Λ_R-linear homomorphism $D^{\psi=1} \xrightarrow{1-\varphi} D^{\psi=0}$ induces an isomorphism

$$D^{\psi=1} \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) \xrightarrow{\sim} D^{\psi=0}$$

of $\mathcal{E}_R(\Gamma)$-modules, and one has

$$D^{\varphi=1} \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) = (D/(\psi-1)D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) = 0$$

(since $D^{\varphi=1}$ and $D/(\psi-1)D$ are finite generated R-modules). In particular, we obtain a canonical isomorphism

$$C^\bullet_\psi(D) \otimes^{\mathbf{L}}_{\Lambda_R} \mathcal{E}_R(\Gamma) \xrightarrow{\sim} D^{\psi=0}[-1]$$

in $\mathbf{D}^-(\mathcal{E}_R(\Gamma))$, and this induces a canonical isomorphism

$$\Delta^{\mathrm{Iw}}_{R,1}(D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) \xrightarrow{\sim} (\det_{\mathcal{E}_R(\Gamma)} D^{\psi=0}, r_D)^{-1}.$$

Moreover, since we have $\mathcal{L}_{\Lambda_R}(\mathbf{Dfm}(D)) = \mathcal{L}_R(D) \otimes_R \Lambda_R$, we obtain the following canonical isomorphism

$$\Delta^{\mathrm{Iw}}_R(D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) \xrightarrow{\sim} (\det_{\mathcal{E}_R(\Gamma)} D^{\psi=0} \otimes_R \mathcal{L}_R(D)^\vee, 0)^{-1}. \tag{12}$$

Using the isomorphism $\Delta^{\mathrm{Iw}}_R(D^*)^\iota \xrightarrow{\sim} \Delta_{\Lambda_R}(\mathbf{Dfm}(D)^*)$, we similarly obtain the following canonical isomorphism

$$\Delta_{\Lambda_R}(\mathbf{Dfm}(D)^*) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) \xrightarrow{\sim} (\det_{\mathcal{E}_R(\Gamma)} (D^*)^{\psi=0,\iota} \otimes_R \mathcal{L}_R(D^*)^\vee, 0)^{-1}.$$

Finally, by Proposition VI.1.2 of [Co10a], the Iwasawa pairing $\{-,-\}_{\mathrm{Iw}} : (D^*)^{\psi=1,\iota} \times D^{\psi=1} \to \Lambda_R$ uniquely extends to an $\mathcal{E}_R(\Gamma)$-bilinear perfect pairing

$$\{-,-\}_{0,\mathrm{Iw}} : (D^*)^{\psi=0,\iota} \times D^{\psi=0} \to \mathcal{E}_R(\Gamma)$$

such that $\{(1-\varphi)x,(1-\varphi)y\}_{0,\mathrm{Iw}} = \{x,y\}_{\mathrm{Iw}}$ for any $x \in (D^*)^{\psi=1}, y \in D^{\psi=1}$.

Remark 2.10. As we mentioned in Remark 1.2, Conjecture 2.1 is known for the rank one case by [Ka93b]. Using the isomorphism (12), the base change to $\mathcal{E}_R(\Gamma)$ of the local ε-isomorphism $\varepsilon_R^{\mathrm{Iw}}(D(T)) := \varepsilon_R^{\mathrm{Iw}}(T)$ for any R-representation T of $G_{\mathbb{Q}_p}$ of rank one defined in [Ka93b] is explicitly described as follows, which will play an important role in this article. Let $\delta : \mathbb{Q}_p^\times \to R^\times$ be a continuous homomorphism corresponding to a character $\delta : G_{\mathbb{Q}_p}^{\mathrm{ab}} \to R^\times$ by the local class field theory. Then, the (φ,Γ)-module $D(R(\delta))$ corresponding to $R(\delta)$ is isomorphic to $\mathcal{E}_R(\delta) := \mathcal{E}_R\mathbf{e}_\delta$ on which (φ,Γ) acts by $\varphi(\mathbf{e}_\delta) = \delta(p)\mathbf{e}_\delta, \gamma'(\mathbf{e}_\delta) = \delta(\gamma')\mathbf{e}_\delta$ $(\gamma' \in \Gamma)$. For $\mathcal{E}_R(\delta)$, one has an $\mathcal{E}_R(\Gamma)$-linear isomorphism

$$\mathcal{E}_R(\Gamma) \xrightarrow{\sim} \mathcal{E}_R(\delta)^{\psi=0} : \lambda \mapsto \lambda \cdot ((1+X)^{-1}\mathbf{e}_\delta),$$

and, under the isomorphism (12) for $D = \mathcal{E}_R(\delta)$, the base change to $\mathcal{E}_R(\Gamma)$ of the local ε-isomorphism $\varepsilon_R^{\mathrm{Iw}}(\mathcal{E}_R(\delta)) : \mathbf{1}_{\Lambda_R} \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(\mathcal{E}_R(\delta))$ which is defined in [Ka93b] is the natural one induced by the isomorphism

$$\mathcal{E}_R(\Gamma) \xrightarrow{\sim} \mathcal{E}_R(\delta)^{\psi=0} \otimes_R (R\mathbf{e}_\delta)^\vee : \lambda \mapsto \lambda \cdot ((1+X)^{-1}\mathbf{e}_\delta) \otimes \mathbf{e}_\delta^\vee.$$

This fact easily follows from the another definition of $\varepsilon_R^{\mathrm{Iw}}(\mathcal{E}_R(\delta))$ given in §4.1 (and Remark 4.9 and Lemma 4.10) of [Na14b].

2.3. A conjectural definition of the local ε-isomorphism

In this subsection, we first recall the definition of (a multivariable version of) the Colmez's multiplicative convolution. After that, we propose a conjectural definition of the local ε-isomorphism using the multiplicative convolution.

2.3.1. Colmez's multiplicative convolution. Let $D_1, \ldots, D_{n+1}$ be étale (φ,Γ)-modules over $\mathcal{E}_R$, and let

$$M : D_1 \times D_2 \times \cdots \times D_n \to D_{n+1}$$

be an $\mathcal{E}_R$-multilinear pairing compatible with φ and Γ, i.e. we have

$$M(\varphi(x_1), \ldots, \varphi(x_n)) = \varphi(M(x_1, \ldots, x_n))$$

and

$$M(\gamma'(x_1), \ldots, \gamma'(x_n)) = \gamma'(M(x_1, \ldots, x_n))$$

for any $x_i \in D_i$ and $\gamma' \in \Gamma$. For such a data, we define a map (depending on the choice of ζ)

$$M_{\mathbb{Z}_p^\times}(:= M^{(\zeta)}_{\mathbb{Z}_p^\times}) : D_1^{\psi=0} \times \cdots \times D_n^{\psi=0} \to D_{n+1}^{\psi=0} : (x_1, \ldots, x_n) \\ \mapsto M_{\mathbb{Z}_p^\times}(x_1, \ldots, x_n) =: (*)$$

by the formula

$$(*) := \lim_{n\to\infty} \sum_{i_1,\ldots,i_n \in \mathbb{Z}_p^\times \bmod p^n} (1+X)^{i_1\cdots i_n}\varphi^n \\ \times (M(\sigma_{j_1}(\psi^n((1+X)^{-i_1}x_1)), \ldots, \sigma_{j_n}(\psi^n((1+X)^{-i_n}x_n)))),$$

where we set $j_k := \prod_{k'\neq k} i_{k'}$. This is a multivariable version of the Colmez's multiplicative convolution defined V.4 of [Co10a], whose well-definedness can be proved in the same way as in Proposition V.4.1 of [Co10a]. We remark that this pairing $M^{(\zeta)}_{\mathbb{Z}_p^\times}$ depends on the choice of the parameter $X = X_\zeta$, i.e. the choice of $\zeta \in \mathbb{Z}_p(1)$. We can easily check that this dependence can be written by the formula

$$M^{(a\zeta)}_{\mathbb{Z}_p^\times} = [\sigma_a]^{-(n-1)} M^{(\zeta)}_{\mathbb{Z}_p^\times}$$

for any $a \in \mathbb{Z}_p^\times$. Moreover, we have

$$M_{\mathbb{Z}_p^\times}(x_1, \ldots, \gamma'(x_i), \ldots, x_n) = \gamma'(M_{\mathbb{Z}_p^\times}(x_1, \ldots, x_i, \ldots, x_n))$$

for any i and $\gamma' \in \Gamma$, in particular, $M_{\mathbb{Z}_p^\times}$ is $\mathcal{E}_R(\Gamma)$-multilinear.

2.3.2. A conjectural definition of the local ε-isomorphisms. We next formulate a conjecture on a conjectural definition of the local ε-isomorphisms using the multiplicative convolution. Let D be an étale (φ, Γ)-module over $\mathcal{E}_R$. Applying the multiplicative convolution to the highest wedge product

$$\wedge : D^{\times r_D} \to \det_{\mathcal{E}_R} D : (x_1, \ldots, x_{r_D}) \mapsto x_1 \wedge \cdots \wedge x_{r_D},$$

we obtain an $\mathcal{E}_R(\Gamma)$-multilinear pairing

$$\wedge_{\mathbb{Z}_p^\times}(= \wedge^{(\zeta)}_{\mathbb{Z}_p^\times}) : (D^{\psi=0})^{\times r_D} \to (\det_{\mathcal{E}_R} D)^{\psi=0}.$$

It is easy to see that this map is alternating. Hence this induces an $\mathcal{E}_R(\Gamma)$-linear morphism

$$\wedge_{\mathbb{Z}_p^\times} : \det_{\mathcal{E}_R(\Gamma)} D^{\psi=0} \to (\det_{\mathcal{E}_R} D)^{\psi=0}.$$

Concerning the relationship between this map with the local ε-isomorphism, we propose the following conjecture, which grew out from discussions with S. Yasuda. Recall that we have canonical isomorphisms

$$\Delta_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) \xrightarrow{\sim} (\det_{\mathcal{E}_R(\Gamma)} D^{\psi=0} \otimes_R \mathcal{L}_R(D)^\vee, r_D)^{-1}$$

and $\mathcal{L}_R(D) = \mathcal{L}_R(\det_{\mathcal{E}_R} D)$.

Conjecture 2.11. *(1) For any D, the map $\wedge_{\mathbb{Z}_p^\times} : \det_{\mathcal{E}_R(\Gamma)} D^{\psi=0} \to (\det_{\mathcal{E}_R} D)^{\psi=0}$ is isomorphism.*

(2) If (1) *holds for D, then the isomorphism*

$$\wedge_{\mathbb{Z}_p^\times} : \Delta_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma)$$

induced by the isomorphism

$$\det_{\mathcal{E}_R(\Gamma)} D^{\psi=0} \otimes_R \mathcal{L}_R(D)^\vee \xrightarrow{\sim} (\det_{\mathcal{E}_R} D)^{\psi=0} \otimes_R \mathcal{L}_R(\det_{\mathcal{E}_R} D)^\vee$$

defined by

$$(x_1 \wedge \cdots \wedge x_{r_D}) \otimes y \mapsto \wedge_{\mathbb{Z}_p^\times}(x_1 \wedge \cdots \wedge x_{r_D}) \otimes y$$

uniquely descends to a Λ_R-linear isomorphism

$$\wedge_{\mathbb{Z}_p^\times} : \Delta_R^{\mathrm{Iw}}(D) \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D).$$

(3) If (2) *holds for D, then the conjectural ε-isomorphism*

$$\varepsilon_R^{\mathrm{Iw}}(D) : \mathbf{1}_{\Lambda_R} \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(D)$$

satisfies the commutative diagram

$$\begin{tikzcd}
\Delta_R^{\mathrm{Iw}}(D) \arrow[r, "\wedge_{\mathbb{Z}_p^\times}"] & \Delta_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D) \\
\mathbf{1}_{\Lambda_R} \arrow[u, "\varepsilon_R^{\mathrm{Iw}}(D)"] \arrow[r, "\mathrm{id}"'] & \mathbf{1}_{\Lambda_R}, \arrow[u, "\varepsilon_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D)"']
\end{tikzcd}$$

where the isomorphism $\varepsilon_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D)$ is the ε-isomorphism defined by Kato [Ka93b] (*or Remark 2.10*).

Remark 2.12. The condition (3) in the conjecture above says that, if (2) is true for D, then the composite $(\wedge_{\mathbb{Z}_p^\times})^{-1} \circ \varepsilon_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D) : \mathbf{1}_{\Lambda_R} \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(D)$ satisfies all the conditions $(1), \ldots, (5)$ in Conjecture 2.1. For example, since one has

$$\wedge^{(a\zeta)}_{\mathbb{Z}_p^\times} = [\sigma_a]^{r_D-1}\wedge^{(\zeta)}_{\mathbb{Z}_p^\times} : \Delta^{\mathrm{Iw}}_R(D) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_R(\det_{\mathcal{E}_R} D)$$

(which follows from $\wedge^{(a\zeta)}_{\mathbb{Z}_p^\times} = [\sigma_a]^{-(r_D-1)}\wedge^{(\zeta)}_{\mathbb{Z}_p^\times} : \det_{\mathcal{E}_R(\Gamma)}(D^{\psi=0}) \to (\det_{\mathcal{E}_R} D)^{\psi=0})$ and

$$\begin{aligned}\varepsilon^{\mathrm{Iw}}_{R,a\zeta}(\det_{\mathcal{E}_R} D) &= (\det_{\mathcal{E}_{\Lambda_R}}\mathbf{Dfm}(\det_{\mathcal{E}_R} D)(a))\varepsilon^{\mathrm{Iw}}_{R,\zeta}(\det_{\mathcal{E}_R} D)\\ &= \det_{\mathcal{E}_R} D(a)[\sigma_a]^{-1}\varepsilon^{\mathrm{Iw}}_{R,\zeta}(\det_{\mathcal{E}_R} D)\end{aligned}$$

for any $a \in \mathbb{Z}_p^\times$, we obtain

$$\begin{aligned}&(\wedge^{(a\zeta)}_{\mathbb{Z}_p^\times})^{-1} \circ \varepsilon^{\mathrm{Iw}}_{R,a\zeta}(\det_{\mathcal{E}_R} D)\\ &\quad = [\sigma_a]^{-r_D+1}(\det_{\mathcal{E}_R} D(a)[\sigma_a]^{-1})(\wedge^{(\zeta)}_{\mathbb{Z}_p^\times})^{-1} \circ \varepsilon^{\mathrm{Iw}}_{R,\zeta}(\det_{\mathcal{E}_R} D)\\ &\quad = \det_{\mathcal{E}_{\Lambda_R}}\mathbf{Dfm}(D)(a)(\wedge^{(\zeta)}_{\mathbb{Z}_p^\times})^{-1} \circ \varepsilon^{\mathrm{Iw}}_{R,\zeta}(\det_{\mathcal{E}_R} D),\end{aligned}$$

i.e. the isomorphism $(\wedge_{\mathbb{Z}_p^\times})^{-1} \circ \varepsilon^{\mathrm{Iw}}_R(\det_{\mathcal{E}_R} D)$ satisfies the condition (3) in Conjecture 2.1.

Remark 2.13. In the next section, we prove almost all the parts of the conjecture above for the rank two case. In fact, we can prove many parts of the conjecture even for the higher rank case. However, we do not pursue this problem in the present article since the main theme of this article is to pursue the connection between the local ε-conjecture with the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. In the next article [Na], we will prove (1), (almost all the parts of) (2) for the higher rank case, and prove that the isomorphism $(\wedge_{\mathbb{Z}_p^\times})^{-1} \circ \varepsilon^{\mathrm{Iw}}_R(\det_{\mathcal{E}_R} D) : \mathbf{1}_{\Lambda_R} \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_R(D)$ (obtained by (2)) satisfies the conditions $(1), \ldots, (4)$ in Conjecture 2.1. Moreover, we will prove that this isomorphism satisfies the condition (5) for the crystabelline case.

3. Local ε-isomorphisms for rank two p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

In this section, using the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, we prove many parts of Conjecture 2.1 and Conjecture 2.11 for the rank two case.

3.1. Statement of the main theorem on the local ε-conjecture

We start this section by stating our main result concerning the local ε-conjecture for the rank two case. We say that an étale (φ, Γ)-module D over $\mathcal{E}_L$ is de Rham, trianguline, etc. if the corresponding $V(D) := T(D)$

is so. If D is de Rham, we set $\varepsilon_L^{\mathrm{dR}}(D) := \varepsilon_L^{\mathrm{dR}}(V(D))$, which we regard as an isomorphism $\mathbf{1}_L \xrightarrow{\sim} \Delta_L(D)$ by the canonical isomorphism $\Delta_L(D) \xrightarrow{\sim} \Delta_L(V(D))$.

Theorem 3.1. *(1) Conjecture 2.11* (1) *is true for all the* (φ,Γ)*-modules of rank two.*

(2) Conjecture 2.11 (2) *is true for "almost all" the* (φ,Γ)*-modules of rank two.*

(3) For D as in (2) (*then we can define an isomorphism*

$$\varepsilon_R^{\mathrm{Iw}}(D) := (\wedge_{\mathbb{Z}_p^\times}^{-1} \circ \varepsilon_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D) : \mathbf{1}_{\Lambda_R} \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(D)),$$

we define

$$\varepsilon_R(D) : \mathbf{1}_R \xrightarrow{\sim} \Delta_R(D)$$

to be the base change of $\varepsilon_R^{\mathrm{Iw}}(D)$ *by* $f_{\mathbf{1}} : \Lambda_R \to R : [\gamma'] \mapsto 1$ $(\gamma' \in \Gamma)$*. Then the set of isomorphisms* $\{\varepsilon_R(D)\}_{(R,D)}$*, where D run through all the D of rank one or rank two as in* (2), *satisfies the conditions* (1), …, (4) *in Conjecture 2.1 and satisfies the following:*
For any pair (L,D) such that D is de Rham of rank one or two satisfying at least one of the following conditions (i) *and* (ii),

(i) D is trianguline,

(ii) the set of the Hodge-Tate weights of D is $\{k_1,k_2\}$ such that $k_1 \leqq 0, k_2 \geqq 1$,

then we have

$$\varepsilon_L(D) = \varepsilon_L^{\mathrm{dR}}(D).$$

We will prove this theorem in the next subsections:(1) is proved in Proposition 3.2, (2) is proved in Proposition 3.4 (see this proposition and Remark 3.5 for the precise meaning of "almost all" in the theorem above), (3) for (i) is proved in §3.3, (3) for (ii) is proved in §3.4.

3.2. Definition of the ε-isomorphisms

In [Co10b], Colmez constructed a correspondence $D \mapsto \Pi(D)$ from (almost all) étale (φ,Γ)-modules of rank two to representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. In the construction of $\Pi(D)$, he introduced a mysterious involution $w_{\delta_D} : D^{\psi=0} \xrightarrow{\sim} D^{\psi=0}$ (whose definition we recall below) which is intimately related with the action of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$ on $\Pi(D)$. Moreover, he proved a formula describing the multiplicative convolution $\wedge_{\mathbb{Z}_p^\times}$ using the involution w_{δ_D} and the Iwasawa pairing $\{-,-\}_{0,\mathrm{Iw}} : (D^*)^{\psi=0,\iota} \times D^{\psi=0} \to \mathcal{E}_R(\Gamma)$, which we also recall below. Since the ε-constant of an irreducible smooth admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ can be described using the action of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ by the classical theory of Kirillov model, this formula is crucial for our application to the local ε-conjecture.

3.2.1. Analytic operations on $D^{\psi=0}$. We start this subsection by recalling the definitions of some of analytic operations on $D^{\psi=0}$ defined in [Co10a], [Co10b]. We remark that these operations also depend on the choice of the parameter $X = X_\zeta \in \mathcal{E}_R$, i.e., the choice of $\mathbf{e}_1 := \zeta \in \mathbb{Z}_p(1)$, which we have fixed.

For a continuous homomorphism $\delta : \Gamma \to R^\times$, Colmez defined in V of [Co10a] the following map

$$m_\delta : D^{\psi=0} \to D^{\psi=0} : x \mapsto \lim_{n\to\infty} \sum_{i\in\mathbb{Z}_p^\times \mathrm{mod} p^n} \delta(i)(1+X)^i\varphi^n\psi^n((1+X)^{-i}x).$$

We remark that this map satisfies $m_{\mathbf{1}} = \mathrm{id}_{D^{\psi=0}}$ for the trivial homomorphism $\mathbf{1} : \Gamma \to R^\times$, $m_{\delta_1} \circ m_{\delta_2} = m_{\delta_1\delta_2}$ for any δ_1, δ_2, and $\sigma_a \circ m_\delta = \delta(a)^{-1} m_\delta \circ \sigma_a$ for $a \in \mathbb{Z}_p^\times$. In particular, the map

$$m_\delta \otimes \mathbf{e}_\delta : D^{\psi=0} \xrightarrow{\sim} D(\delta)^{\psi=0} : x \mapsto m_\delta(x) \otimes \mathbf{e}_\delta$$

is an isomorphism of $\mathcal{E}_R(\Gamma)$-modules. In V of [Co10a], he also defined an involution

$$w_* : D^{\psi=0} \xrightarrow{\sim} D^{\psi=0}$$

by the formula

$$w_*(x) := \lim_{n\to+\infty} \sum_{i\in\mathbb{Z}_p^\times \mathrm{mod}\, p^n} (1+X)^{1/i}\sigma_{-1/i^2}(\varphi^n\psi^n((1+X)^{-i}x))$$

and also defined in II of [Co10b] an involution

$$w_\delta := m_{\delta^{-1}} \circ w_* : D^{\psi=0} \xrightarrow{\sim} D^{\psi=0}$$

for any $\delta : \Gamma \to R^\times$. By definition, the latter satisfies the equalities $w_\delta(\sigma_a(x)) = \delta(a)\sigma_{a^{-1}}(w_\delta(x))$ for any $a \in \mathbb{Z}_p^\times$. In particular, this induces an $\mathcal{E}_R(\Gamma)$-linear isomorphism

$$w_\delta \otimes \mathbf{e}_{\delta^{-1}} : D^{\psi=0} \xrightarrow{\sim} D(\delta^{-1})^{\psi=0,\iota} : x \mapsto w_\delta(x) \otimes \mathbf{e}_{\delta^{-1}}.$$

3.2.2. The definition of the local ε-isomorphism over $\mathcal{E}_R(\Gamma)$. Now we assume that D is of rank two. Set

$$\delta_D := \chi^{-1}\det_{\mathcal{E}_R} D : \mathbb{Q}_p^\times \to R^\times.$$

Using the canonical isomorphism $\mathcal{E}_R \otimes_R \mathcal{L}_R(D) \xrightarrow{\sim} \det_{\mathcal{E}_R} D : f \otimes x \mapsto fx$, we obtain a canonical isomorphism

$$\det_{\mathcal{E}_R} D \otimes_R \mathcal{L}_R(D)^\vee \xrightarrow{\sim} \mathcal{E}_R \otimes_R \mathcal{L}_R(D) \otimes_R \mathcal{L}_R(D)^\vee \xrightarrow{\sim} \mathcal{E}_R.$$

Using this isomorphism, we define the following canonical isomorphism of (φ, Γ)-modules

$$D \otimes_R \mathcal{L}_R(D)^\vee \xrightarrow{\sim} D^\vee : x \otimes z^\vee \mapsto [y \mapsto (y \wedge x) \otimes z^\vee]$$

for $x, y \in D, z \in \mathcal{L}_R(D)^\times$ (and recall that $z^\vee \in \mathcal{L}_R(D)^\vee$ is the dual base of z), by which we identify both sides. By these isomorphisms, we also obtain the following canonical isomorphism

$$D^{\psi=0} \otimes_R \mathcal{L}_R(D)^\vee \xrightarrow{\sim} (D^*)^{\psi=0,\iota} : x \otimes z^\vee \mapsto w_{\delta_D}(x) \otimes z^\vee \otimes \mathbf{e}_1$$

of $\mathcal{E}_R(\Gamma)$-modules.

Using these preliminaries, we define the following $\mathcal{E}_R(\Gamma)$-bilinear perfect pairing

$$\begin{aligned}[-,-]_{\mathrm{Iw}} : D^{\psi=0} \otimes \mathcal{L}_R(D)^\vee \times D^{\psi=0} \to \mathcal{E}_R(\Gamma) : (x \otimes z^\vee, y) \\ \mapsto \{w_{\delta_D}(x) \otimes z^\vee \otimes \mathbf{e}_1, y\}_{0,\mathrm{Iw}}\end{aligned}$$

which is a modified version of the Colmez's pairing defined in Corollaire VI.6.2 of [Co10b]. This pairing is related with the multiplicative convolution $\wedge_{\mathbb{Z}_p^\times} : D^{\psi=0} \times D^{\psi=0} \to (\det_{\mathcal{E}_R} D)^{\psi=0}$ as follows. Let us consider the R-linear map $d : \mathcal{E}_R \to \mathcal{E}_R(1) : f(X) \mapsto (1+X)\frac{df(X)}{dX} \otimes \mathbf{e}_1$. It is easy to see that this does not depend on the choice of $\zeta \in \mathbb{Z}_p(1)$, and satisfies $\sigma_a \circ d = d \circ \sigma_a$ ($a \in \mathbb{Z}_p^\times$) and $\varphi \circ d = pd \circ \varphi$, and induces an $\mathcal{E}_R(\Gamma)$-linear isomorphism $d : \mathcal{E}_R^{\psi=0} \xrightarrow{\sim} \mathcal{E}_R(1)^{\psi=0}$. We note that one has $d|_{\mathcal{E}_R^{\psi=0}} = m_\chi \otimes \mathbf{e}_1$ since both are $\mathcal{E}_R(\Gamma)$-linear and one has $d(1+X) = (1+X) \otimes \mathbf{e}_1 = m_\chi(1+X) \otimes \mathbf{e}_1$.

As a consequence of Colmez's generalized reciprocity law (see Théorème VI.2.1 of [Co10a]), he proved, in the proof of Corollaire VI.6.2 [Co10b], that $[-,-]_{\mathrm{Iw}}$ satisfies the following equality

$$d([x \otimes z^\vee, y]_{\mathrm{Iw}} \cdot (1+X)) = -\delta_D(-1) m_{\delta_D^{-1}}(\wedge_{\mathbb{Z}_p^\times}(x,y)) \otimes z^\vee \otimes \mathbf{e}_1 \tag{13}$$

in $\mathcal{E}_R(1)^{\psi=0} \xrightarrow{\sim} (\det_{\mathcal{E}_R} D)^{\psi=0} \otimes_R \mathcal{L}_R(D)^\vee(1)$.

Since $\wedge_{\mathbb{Z}_p^\times}$ is anti-symmetric, this formula implies that the perfect pairing $[-,-]_{\mathrm{Iw}}$ is also anti-symmetric, i.e. we have $[x \otimes z^\vee, y]_{\mathrm{Iw}} = -[y \otimes z^\vee, x]_{\mathrm{Iw}}$

for any $x, y \in D^{\psi=0}$ and $z \in \mathcal{L}_R(D)$. Therefore, this induces an $\mathcal{E}_R(\Gamma)$-linear isomorphism

$$\det_{\mathcal{E}_R(\Gamma)} D^{\psi=0} \otimes_R \mathcal{L}_R(D)^\vee \xrightarrow{\sim} \mathcal{E}_R(\Gamma) : (x \wedge y) \otimes z^\vee \mapsto [\sigma_{-1}][x \otimes z^\vee, y]_{\mathrm{Iw}}.$$

The last isomorphism, together with (12), naturally induces an $\mathcal{E}_R(\Gamma)$-linear isomorphism (which we denote by)

$$\eta_R(D) : \mathbf{1}_{\mathcal{E}_R(\Gamma)} \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma).$$

3.2.3. Proof of (1) of Conjecture 2.11. We first prove the following proposition concerning the alternative description of our conjectural ε-isomorphism, in particular, which proves Conjecture 2.11 (1) for the rank two case.

Proposition 3.2. *The map $\wedge_{\mathbb{Z}_p^\times} : \det_{\mathcal{E}_R(\Gamma)} D^{\psi=0} \to (\det_{\mathcal{E}_R} D)^{\psi=0}$ is isomorphism, and the isomorphism $\eta_R(D)$ fits into the following commutative diagram:*

$$\begin{array}{ccc} \Delta_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) & \xrightarrow{\wedge_{\mathbb{Z}_p^\times}} & \Delta_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma) \\ {\scriptstyle \eta_R(D)} \uparrow & & \uparrow {\scriptstyle \varepsilon_R^{\mathrm{Iw}}(\det_{\mathcal{E}_R} D) \otimes \mathrm{id}_{\mathcal{E}_R(\Gamma)}} \\ \mathbf{1}_{\mathcal{E}_R(\Gamma)} & \xrightarrow[\mathrm{id}]{} & \mathbf{1}_{\mathcal{E}_R(\Gamma)}. \end{array}$$

Proof. By Remark 2.10, it suffices to show the equality

$$[\sigma_{-1}][x \otimes z^\vee, y]_{\mathrm{Iw}} \cdot ((1+X)^{-1} z) = \wedge_{\mathbb{Z}_p^\times}(x, y) \tag{14}$$

for any $x, y \in D^{\psi=0}$, $z \in \mathcal{L}_R(D)^\times = \mathcal{L}_R(\det_{\mathcal{E}_R} D)^\times$.

We prove this equality as follows. We first remark that, since one has $d = m_\chi \otimes \mathbf{e}_1$, the equality (13) is equivalent to the equality

$$m_\chi([x \otimes z^\vee, y]_{\mathrm{Iw}} \cdot (1+X)) \otimes \mathbf{e}_1 = -\delta_D(-1) m_{\delta_D^{-1}}(\wedge_{\mathbb{Z}_p^\times}(x, y)) \otimes z^\vee \otimes \mathbf{e}_1. \tag{15}$$

Applying the $\mathcal{E}_R(\Gamma)$-linear isomorphism $m_{\delta_D} \otimes z \otimes \mathbf{e}_{-1} : \mathcal{E}_R(1)^{\psi=0} \xrightarrow{\sim} (\det_{\mathcal{E}_R} D)^{\psi=0}$ to this equality, the right hand side is equal to

$$\begin{aligned} &- \delta_D(-1) m_{\delta_D}(m_{\delta_D^{-1}}(\wedge_{\mathbb{Z}_p^\times}(x, y)) \otimes z^\vee \otimes \mathbf{e}_1) \otimes z \otimes \mathbf{e}_{-1} \\ &= -\delta_D(-1) m_{\delta_D}(m_{\delta_D^{-1}}(\wedge_{\mathbb{Z}_p^\times}(x, y))) \otimes z^\vee \otimes \mathbf{e}_1 \otimes z \otimes \mathbf{e}_{-1} = -\delta_D(-1) \wedge_{\mathbb{Z}_p^\times}(x, y) \end{aligned}$$

since one has $m_\delta(x \otimes \mathbf{e}_{\delta'}) = m_\delta(x) \otimes \mathbf{e}_{\delta'}$ and $m_\delta \circ m_{\delta'} = m_{\delta\delta'}$ for any D and δ, δ', and the left hand side is equal to (set $\delta_0 := \det_{\mathcal{E}_R} D|_{\mathbb{Z}_p^\times}$)

$$\begin{aligned} &m_{\delta_D}(m_\chi([x\otimes z^\vee, y]_{\mathrm{Iw}}\cdot(1+X))\otimes \mathbf{e}_1)\otimes z\otimes \mathbf{e}_{-1}\\ &\qquad = m_{\delta_D}(m_\chi([x\otimes z^\vee, y]_{\mathrm{Iw}}\cdot(1+X)))\otimes \mathbf{e}_1\otimes z\otimes \mathbf{e}_{-1}\\ &= m_{\delta_0}([x\otimes z^\vee, y]_{\mathrm{Iw}}\cdot(1+X))\otimes z = [x\otimes z^\vee, y]_{\mathrm{Iw}}\cdot(m_{\delta_0}(1+X)\otimes z)\\ &= [x\otimes z^\vee, y]_{\mathrm{Iw}}\cdot((1+X)z) = -\delta_D(-1)[\sigma_{-1}][x\otimes z^\vee, y]_{\mathrm{Iw}}\cdot((1+X)^{-1}z), \end{aligned}$$

where the third equality follows from the $\mathcal{E}_R(\Gamma)$-linearity of $m_{\delta_0}\otimes z$, and the fourth follows from $m_{\delta_0}(1+X) = 1+X$, from which the equality (14) follows. □

3.2.4. Proof of (4) of Conjecture 2.11. Before proving (2) of Conjecture 2.11, we show the isomorphism $\eta_R(D)$ satisfies the condition similar to (4) in Conjecture 2.1 (over the ring $\mathcal{E}_R(\Gamma)$).

Lemma 3.3. *Let D be an étale (φ,Γ)-module over $\mathcal{E}_R$ of rank two. Then the isomorphisms $\eta_R(D)$ and $\eta_R(D^*)$ fit into the following commutative diagram:*

$$\begin{array}{ccc} \Delta_R^{\mathrm{Iw}}(D)\otimes_{\Lambda_R}\mathcal{E}_R(\Gamma) & \longrightarrow & (\Delta_R^{\mathrm{Iw}}(D^*)^\iota\otimes_{\Lambda_R}\mathcal{E}_R(\Gamma))^\vee\boxtimes(\mathcal{E}_R(\Gamma)(r_T),0)\\ {\scriptstyle \eta_R(D)}\big\uparrow & & \big\downarrow{\scriptstyle (\eta_R(D^*)^\iota)^\vee\boxtimes[\mathbf{e}_{r_T}\mapsto 1]}\\ \mathbf{1}_{\mathcal{E}_R(\Gamma)} & \xrightarrow{\det_{\mathcal{E}_R}D(\sigma_{-1})\cdot\mathrm{can}} & \mathbf{1}_{\mathcal{E}_R(\Gamma)}\boxtimes\mathbf{1}_{\mathcal{E}_R(\Gamma)}. \end{array}$$

Here the upper horizontal arrow is the base change to $\mathcal{E}_R(\Gamma)$ of the isomorphism $\Delta_R^{\mathrm{Iw}}(D)\xrightarrow{\sim}(\Delta_R^{\mathrm{Iw}}(D^*)^\iota)^\vee\boxtimes(\Lambda_R(r_T),0)$ defined by the Tate duality.

Before starting the proof, let us introduce the following notation. In the proof we will use the pairings $[-,-]_{\mathrm{Iw}}$ and $\{-,-\}_{\mathrm{Iw},0}$ for D and those for D^* simultaneously. In order to distinguish the pairings for D with those for D^*, we will denote, for any étale (φ,Γ)-module D_1 of rank two, the pairings $[-,-]_{\mathrm{Iw}}$ and $\{-,-\}_{\mathrm{Iw},0}$ for D_1 by $[-,-]_{\mathrm{Iw},D_1}$ and $\{-,-\}^{D_1}_{\mathrm{Iw},0}$, respectively.

Proof. Fix $z\in\mathcal{L}_R(D)^\times$. Then we have $z^\vee\otimes\mathbf{e}_2\in\mathcal{L}_R(D^*)^\times$. By definition, it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} \det_{\mathcal{E}_R(\Gamma)}D^{\psi=0} & \xrightarrow{(a)} & (\det_{\mathcal{E}_R(\Gamma)}(D^*)^{\psi=0,\iota})^\vee\\ {\scriptstyle (b)}\big\downarrow & & \big\uparrow{\scriptstyle (c)\boxtimes[\mathbf{e}_2\mapsto 1]}\\ \mathcal{E}_R(\Gamma)\otimes_R\mathcal{L}_R(D) & \xrightarrow{(d)} & (\mathcal{E}_R(\Gamma)\otimes_R\mathcal{L}_R(D^*))^\vee\otimes_R R(2), \end{array}$$

where the horizontal arrows are the natural one defined by the Tate duality, and (b) is defined by $x\wedge y\mapsto \det_{\mathcal{E}_R}D(\sigma_{-1})[\sigma_{-1}][x\otimes z^\vee, y]_{\mathrm{Iw},D}\otimes z$ for $x,y\in D^{\psi=0}$ and (c) is the dual of the map $x'\wedge y'\mapsto\iota([\sigma_{-1}][x'\otimes(z\otimes\mathbf{e}_{-2}),y']_{\mathrm{Iw},D^*})\otimes(z^\vee\otimes\mathbf{e}_2)$ for $x',y'\in(D^*)^{\psi=0}$. We prove this commutativity as follows.

Take a basis $\{x, y\}$ of $D^{\psi=0}$. Since we have an isomorphism $w_{\delta_D}\otimes z^\vee\otimes\mathbf{e}_1 : D^{\psi=0}\xrightarrow{\sim}(D^*)^{\psi=0,\iota}$, then $\{w_{\delta_D}(x)\otimes z^\vee\otimes\mathbf{e}_1, w_{\delta_D}(y)\otimes z^\vee\otimes\mathbf{e}_1\}$ is a basis of $(D^*)^{\psi=0,\iota}$. Then, (a) sends $x\wedge y$ to $f\in(\det_{\mathcal{E}_R(\Gamma)}(D^*)^{\psi=0,\iota})^\vee$ defined by

$$\begin{aligned}
&f((w_{\delta_D}(x)\otimes z^\vee\otimes\mathbf{e}_1)\wedge(w_{\delta_D}(y)\otimes z^\vee\otimes\mathbf{e}_1))\\
&\quad=[x\otimes z^\vee, x]_{\mathrm{Iw},D}[y\otimes z^\vee, y]_{\mathrm{Iw},D}-[x\otimes z^\vee, y]_{\mathrm{Iw},D}[y\otimes z^\vee, x]_{\mathrm{Iw},D}\\
&\quad=([x\otimes z^\vee, y]_{\mathrm{Iw},D})^2,
\end{aligned}$$

where the first equality is by definition and the second follows since $[\ ,\ ]_{\mathrm{Iw}}$ is anti-symmetric. By definition, the composite $((c)\boxtimes[\mathbf{e}_1\mapsto 1])\circ(d)\circ(b)$ sends $x\wedge y$ to $f'\in(\det_{\mathcal{E}_R(\Gamma)}(D^*)^{\psi=0,\iota})^\vee$ defined by

$$\begin{aligned}
&f'((w_{\delta_D}(x)\otimes z^\vee\otimes\mathbf{e}_1)\wedge(w_{\delta_D}(y)\otimes z^\vee\otimes\mathbf{e}_1))\\
&\quad=\det_{\mathcal{E}_R}D(\sigma_{-1})[x\otimes z^\vee, y]_{\mathrm{Iw},D}\\
&\qquad\cdot\iota([(w_{\delta_D}(x)\otimes z^\vee\otimes\mathbf{e}_1)\otimes(z\otimes\mathbf{e}_{-2}), w_{\delta_D}(y)\otimes z^\vee\otimes\mathbf{e}_1)]_{\mathrm{Iw},D^*}).
\end{aligned}$$

Therefore, it suffices to show the equality

$$\begin{aligned}
&[x\otimes z^\vee, y]_{\mathrm{Iw},D}\\
&\quad=\det_{\mathcal{E}_R}D(\sigma_{-1})\iota([(w_{\delta_D}(x)\otimes z^\vee\otimes\mathbf{e}_1)\otimes(z\otimes\mathbf{e}_{-2}), w_{\delta_D}(y)\otimes z^\vee\otimes\mathbf{e}_1)]_{\mathrm{Iw},D^*}).
\end{aligned}\tag{16}$$

To show this equality, we first remark that one has

$$\begin{aligned}
&w_{\delta_{D^*}}(w_{\delta_D}(x)\otimes z^\vee\otimes\mathbf{e}_1)\\
&\quad=w_{\delta_D^{-1}}(w_{\delta_D}(x)\otimes z^\vee\otimes\mathbf{e}_1)\\
&\quad=m_{\delta_D}\circ w_*(m_{\delta_D^{-1}}\circ w_*(x)\otimes z^\vee\otimes\mathbf{e}_1)\\
&\quad=\delta_D(-1)(m_{\delta_D}\circ m_{\delta_D^{-2}}\circ w_*\circ m_{\delta_D^{-1}}\circ w_*(x))\otimes z^\vee\otimes\mathbf{e}_1\\
&\quad=\delta_D(-1)(m_{\delta_D}\circ m_{\delta_D^{-2}}\circ m_{\delta_D}\circ w_*\circ w_*(x))\otimes z^\vee\otimes\mathbf{e}_1\\
&\quad=\delta_D(-1)x\otimes z^\vee\otimes\mathbf{e}_1,
\end{aligned}$$

where the third equality follows from $w_*(x\otimes\mathbf{e}_\delta)=\delta(-1)m_{\delta^2}\circ w_*(x)\otimes\mathbf{e}_\delta$ (Corollaire V.5.2. of [Co10a]) and the fourth follows from $m_{\delta^{-1}}\circ w_*=w_*\circ m_\delta$ (Proposition V.2.4 of [Co10a]) for any δ. Hence, the right hand side of (16) is equal to

$$\begin{aligned}
&\det_{\mathcal{E}_R}D(\sigma_{-1})\delta_D(-1)\iota(\{(x\otimes z^\vee\otimes\mathbf{e}_1)\otimes(z\otimes\mathbf{e}_{-2})\otimes\mathbf{e}_1, w_{\delta_D}(y)\otimes z^\vee\otimes\mathbf{e}_1\}^{D^*}_{\mathrm{Iw},0})\\
&\quad=-\det_{\mathcal{E}_R}D(\sigma_{-1})\delta_D(-1)\iota(\{x, w_{\delta_D}(y)\otimes z^\vee\otimes\mathbf{e}_1\}^{D^*}_{\mathrm{Iw},0})
\end{aligned}$$

$$\begin{aligned}
&= \iota(\{x, w_{\delta_D}(y)\otimes z^\vee \otimes \mathbf{e}_1\}^{D^*}_{\mathrm{Iw},0})\\
&= -\{w_{\delta_D}(y)\otimes z^\vee \otimes \mathbf{e}_1, x\}^{D}_{\mathrm{Iw},0}\\
&= -[y\otimes z^\vee, x]_{\mathrm{Iw},D}\\
&= [x\otimes z^\vee, y]_{\mathrm{Iw},D},
\end{aligned}$$

where the first equality follows from the fact that the composite of the canonical isomorphisms $(D\otimes_R\mathcal{L}_R(D)^\vee)\otimes_R\mathcal{L}_R(D^\vee)^\vee \xrightarrow{\sim} D^\vee\otimes_R\mathcal{L}_D(D^\vee)^\vee \xrightarrow{\sim} (D^\vee)^\vee$ is given by $x\otimes z^\vee\otimes z \mapsto [f\mapsto -f(x)]$ for any $z\in\mathcal{L}_R(D)^\times$, which shows the equality (16), hence finishes to prove the lemma. □

3.2.5. Proof of (2) of Conjecture 2.11. We next prove (2) of Conjecture 2.11 under the following assumption. Let $\overline{V}$ be an $\mathbb{F}$-representation of $G_{\mathbb{Q}_p}$ of over a finite field $\mathbb{F}$ of characteristic p. We denote by $R_{\overline{V}}$ the universal deformation ring of $\overline{V}$ (resp. a versal deformation ring or the universal framed deformation ring) if it exists (resp. the universal deformation ring does not exist), and denote by V^{univ} the universal deformation (resp. a versal deformation or the underlying representation of the universal framed deformation) of $\overline{V}$ over $R_{\overline{V}}$.

Let R_0 be a topological $\mathbb{Z}_p$-algebra satisfying the condition (i) in §2.1. Let R be either R_0 or $R_0[1/p]$ (resp. $R=R_0[1/p]$) when $p\geqq 3$ (resp. $p=2$). Let V be an R-representation of $G_{\mathbb{Q}_p}$. Set $V_0:=V$ (resp. V_0 a $G_{\mathbb{Q}_p}$-stable R_0-lattice of V) if $R=R_0$ (resp. $R=R_0[1/p]$). Since R_0 is a finite product of local rings, we may assume that R_0 is local and denote by $\mathfrak{m}_{R_0}$ the maximal ideal of R_0. If we set $\overline{V}:=V_0\otimes_{R_0}R_0/\mathfrak{m}_{R_0}$, then there exists a homomorphism $R_{\overline{V}}\to R$ such that $V^{\mathrm{univ}}\otimes_{R_{\overline{V}}}R\xrightarrow{\sim} V$. Set $\mathcal{X}:=\mathrm{Spec}(R_{\overline{V}}[1/p])$, and denote by $\mathcal{X}_0$ the subset of all the closed points in $\mathcal{X}$.

Proposition 3.4. *Let D be an étale (φ,Γ)-module over $\mathcal{E}_R$ of rank two. Set $V:=V(D)$ and $\overline{V}:=V_0\otimes_{R_0}R_0/\mathfrak{m}_{R_0}$ for an R_0-lattice V_0 of V. Assume one of the following conditions* (1) *and* (2)*:*

(1) $p\geqq 3$.

(2) $p=2$ *and, for an $R_{\overline{V}}$ as above,*

$$\mathcal{X}_{\mathrm{cris}} := \{x\in\mathcal{X}_0 \,|\, V_x := x^*(V^{\mathrm{univ}}) \textit{ is absolutely irreducible and crystalline}\}$$

is Zariski dense in $\mathcal{X}$,

then the isomorphism $\eta_R(D)$ descends to Λ_R, which we denote by

$$\varepsilon^{\mathrm{Iw}}_R(D): \mathbf{1}_{\Lambda_R}\xrightarrow{\sim}\Delta^{\mathrm{Iw}}_R(D).$$

Proof. We first remark that, when $p=3$, the second condition in (2) (i.e. density of $\mathcal{X}_{\mathrm{cris}}$) always holds for the universal (or a versal) deformation ring $R_{\overline{V}}$ and it is known to be an integral domain (in particular p-torsion free) by the results of [Co08], [Ki10], [Bo10] and [BJ14].

By the compatibility with the base change, it suffices to show the proposition for V^{univ} (for $V^{\mathrm{univ}}[1/p]$ if $p=2$). Set $R := R_{\overline{V}}$ and $V := V^{\mathrm{univ}}$ for simplicity. The p-torsion freeness of R when $p \geqq 3$ implies that we have $\Lambda_R = \mathcal{E}_R(\Gamma) \cap \Lambda_{R[1/p]}(\Gamma)$. Therefore, it suffices to show the theorem for $R[1/p]$ (for any p). Moreover, since we have $\Lambda_{R[1/p]}(\Gamma) = \mathrm{Ker}(\mathcal{E}_{R[1/p]}(\Gamma) \to \prod_{x\in\mathcal{X}_{\mathrm{cris}}} \mathcal{E}_{L_x}(\Gamma)/\Lambda_{L_x}(\Gamma))$ (here, L_x is the residue field at x) by the assumption on the density, it suffices to show the proposition for $V_x := x^*(V)$ for any $x \in \mathcal{X}_{\mathrm{cris}}$.

Let V be an absolutely irreducible L-representation for a finite extension L of $\mathbb{Q}_p$ corresponding to a point in $\mathcal{X}$. Set $D := D(V)$. Then, one has $D^{\varphi=1} = D/(\psi-1)D = 0$ and $D^{\psi=1} \xrightarrow{\sim} (1-\varphi)D^{\psi=1} =: \mathcal{C}(D)$ is a free $\Lambda_L(\Gamma)$-module of rank two by §II, §VI of [Co10a], and the same results hold for D^*. Hence, as in the case of $\Delta_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R} \mathcal{E}_R(\Gamma)$, we obtain a canonical isomorphism

$$\Delta_L^{\mathrm{Iw}}(D) \xrightarrow{\sim} (\det_{\Lambda_L(\Gamma)}\mathcal{C}(D) \otimes_L \mathcal{L}_L(D)^\vee, 0)^{-1}.$$

Moreover, the Iwasawa pairing $\{-,-\}_{0,\mathrm{Iw}} : \mathcal{C}(D^*)^\iota \times \mathcal{C}(D) \to \Lambda_L(\Gamma)$ is perfect by Proposition VI.1.2 of [Co10a], and, if we fix $z \in \mathcal{L}_L(D)^\times$, one has an isomorphism

$$\mathcal{C}(D) \xrightarrow{\sim} \mathcal{C}(D^*) : x \mapsto w_{\delta_D}(x) \otimes z^\vee \otimes \mathbf{e}_1$$

by Proposition V.2.1 of [Co10b]. Therefore, we obtain an isomorphism

$$\det_{\Lambda_L(\Gamma)}\mathcal{C}(D) \otimes_L \mathcal{L}_L(D)^\vee \xrightarrow{\sim} \Lambda_L(\Gamma) : (x\wedge y)\otimes z^\vee \mapsto \{w_{\delta_D}(x)\otimes z^\vee \otimes \mathbf{e}_1, y\}_{0,\mathrm{Iw}},$$

which proves the proposition for D by definition of $\eta_R(D)$. □

Remark 3.5. Even when $p=2$, the assumption in the proposition holds for almost all the cases (the author does not know any example which does not satisfies the assumption). For example, for any L-representation V for a finite extension L of $\mathbb{Q}_p$, there exists an $\mathcal{O}$-lattice V_0 of V such that its $R_{\overline{V}}$ satisfies the assumption (see [CDP14a]).

3.2.6. The definition of the local ε-isomorphisms. From now on, we only treat the (φ,Γ)-modules of rank two which satisfy the assumption in Proposition 3.4 without any comment, which gives no restriction to the results proved in the next sections since any L-representations of $G_{\mathbb{Q}_p}$ of rank two satisfies the assumption by Remark 3.5.

Specializing the ε-isomorphism above, we define the ε-isomorphism $\varepsilon_R(D)$ as follows.

Definition 3.6. Let D be an étale (φ, Γ)-module over $\mathcal{E}_R$ of rank two. We define the isomorphism $\varepsilon_R(D)$ to be the base change

$$\varepsilon_R(D) := \varepsilon_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R, f_1} \mathrm{id}_R : \mathbf{1}_R \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R, f_1} R \xrightarrow{\sim} \Delta_R(D)$$

by the morphism $f_{\mathbf{1}} : \Lambda_R \to R$ defined by $f_{\mathbf{1}}([\gamma']) := 1$ for arbitrary $\gamma' \in \Gamma$.

Corollary 3.7. *Our local ε-isomorphism $\varepsilon_R(D)$ defined in Definition 3.6 satisfies the conditions* $(1), (3), (4)$ *of Conjecture 2.1.*

Proof. That $\varepsilon_R(D)$ satisfies the condition (1) is trivial by definition. To show that $\varepsilon_R(D)$ satisfies the conditions (3) and (4), it suffices to show that $\varepsilon_R^{\mathrm{Iw}}(D)$ satisfies (3) and (4). Since the canonical map $\Lambda_R \to \mathcal{E}_R(\Gamma)$ is injective, this claim follows from Remark 2.12 and Lemma 3.3. $\square$

Remark 3.8. By definition and Lemma 2.9, we also have

$$\varepsilon_R(D(\delta)) = \varepsilon_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R, f_\delta} R$$

for any $\delta : \Gamma \to R^\times$ under the canonical isomorphism

$$\Delta_R(D(\delta)) \xrightarrow{\sim} \Delta_R^{\mathrm{Iw}}(D) \otimes_{\Lambda_R, f_\delta} R.$$

3.3. The verification of the de Rham condition: the trianguline case

This and the next subsections are the technical hearts of this article, where we prove that our ε-isomorphism defined in Definition 3.6 satisfies the condition (5) in Conjecture 2.1, which we call the de Rham condition. In this subsection, we verify this condition in the trianguline case by comparing the local ε-isomorphism defined in Definition 3.6 with that defined in the previous article [Na14b].

3.3.1. Recall of the local ε-conjecture for (φ, Γ)-modules over the Robba ring. In [Na14b], we generalized the p-adic local ε-conjecture for rigid analytic families of (φ, Γ)-modules over the Robba ring, and proved this generalized version of conjecture for families of trianguline (φ, Γ)-modules, (a special case of) which we briefly recall now. For details, see [KPX14] for the general results on the cohomology theory of (φ,Γ)-modules over the Robba ring, and [Na14b] for the generalized version of the local ε-conjecture.

We denote by $|-|:\overline{\mathbb{Q}}_p^\times\to\mathbb{Q}_{>0}$ the absolute value normalized by $|p|:=1/p$. Define topological L-algebras $\mathcal{R}_L^{(n)}$ $(n\geqq 1)$ and $\mathcal{R}_L$ by

$$\mathcal{R}_L^{(n)}:=\left\{\sum_{m\in\mathbb{Z}}a_mX^m\,\middle|\,a_m\in L,\sum_{m\in\mathbb{Z}}a_mX^m\right.$$
$$\left.\text{is convergent on } |\zeta_{p^n}-1|\leqq|X|<1\right\}$$

and $\mathcal{R}_L:=\bigcup_{n\geqq 1}\mathcal{R}_L^{(n)}$ on which φ and Γ act by $\varphi(X)=(1+X)^p-1$ and $\gamma'(X)=(1+X)^{\chi(\gamma')}-1$ $(\gamma'\in\Gamma)$. For $n\geqq 1$, we say that $M^{(n)}$ is a (φ,Γ)-module over $\mathcal{R}_L^{(n)}$ if $M^{(n)}$ is a finite free $\mathcal{R}_L^{(n)}$-module with a Frobenius structure

$$\varphi^*M^{(n)}:=M^{(n)}\otimes_{\mathcal{R}_L^{(n)},\varphi}\mathcal{R}_L^{(n+1)}\xrightarrow{\sim}M^{(n+1)}:=M^{(n)}\otimes_{\mathcal{R}_L^{(n)}}\mathcal{R}_L^{(n+1)}$$

and a continuous semi-linear action of Γ which commutes with the Frobenius structure. We say that an $\mathcal{R}_L$-module M is a (φ,Γ)-module over $\mathcal{R}_L$ if it is the base change of a (φ,Γ)-module $M^{(n)}$ over $\mathcal{R}_L^{(n)}$ for some $n\geqq 1$. We denote by $n(M)\geqq 1$ the smallest such n, and set $M^{(n)}:=M^{(n(M))}\otimes_{\mathcal{R}_L^{(n(M))}}\mathcal{R}_L^{(n)}$.

By the theorems of Cherbonnier-Colmez [CC98] and Kedlaya [Ke04], one has an exact fully faithful functor

$$D\mapsto D_{\mathrm{rig}}:=D^\dagger\otimes_{\mathcal{E}_L^\dagger}\mathcal{R}_L$$

from the category of étale (φ,Γ)-modules over $\mathcal{E}_L$ to that of (φ,Γ)-modules over $\mathcal{R}_L$, where $D^\dagger$ is the largest étale (φ,Γ)-submodule of D defined over the following ring

$$\mathcal{E}_L^\dagger:=\{f(X)\in\mathcal{E}_L|f(X)\text{ is convergent on }r\leqq|X|<1\text{ for some }r<1\}.$$

For any (φ,Γ)-module M over $\mathcal{R}_L$, we can similarly define $C^\bullet_{\varphi,\gamma}(M)$, $\Delta_L(M)$, $\mathbf{Dfm}(M)$ and $\Delta_L^{\mathrm{Iw}}(M)$, etc. as follows. First, we define

$$C^\bullet_{\varphi,\gamma}(M),\ C^\bullet_{\psi,\gamma}(M)\text{ and }\Delta_{L,1}(M)$$

in the same way as in the étale (φ,Γ)-case. To define $\Delta_{L,2}(D)$, we first recall that the rank one (φ,Γ)-modules over $\mathcal{R}_L$ are classified by continuous homomorphisms $\delta:\mathbb{Q}_p^\times\to L^\times$, i.e. the rank one (φ,Γ)-module corresponding to δ is defined by

$$\mathcal{R}_L(\delta) := \mathcal{R}_L \mathbf{e}_\delta$$

on which φ and Γ act by

$$\varphi(\mathbf{e}_\delta) = \delta(p)\mathbf{e}_\delta \quad \text{and} \quad \gamma'(\mathbf{e}_\delta) = \delta(\chi(\gamma'))\mathbf{e}_\delta \quad \text{for } \gamma' \in \Gamma.$$

For any (φ, Γ)-module M over $\mathcal{R}_L$, we also regard $\det_{\mathcal{R}_L} M$ as continuous homomorphisms $\det_{\mathcal{R}_L} M : \mathbb{Q}_p^\times \to L^\times$ or $\det_{\mathcal{R}_L} M : W_{\mathbb{Q}_p}^{\mathrm{ab}} \to L^\times$ by this correspondence and the local class field theory. Using the homomorphism $\det_{\mathcal{R}_L} M$, we define

$$\mathcal{L}_L(M), \quad \Delta_{L,2}(M) \quad \text{and} \quad \Delta_L(M)$$

in the same way as in the étale case. To define $\mathbf{Dfm}(M)$ and $\Delta_L^{\mathrm{Iw}}(M)$, we first define $\Lambda_L(\Gamma)$-algebras $\mathcal{R}_L^+(\Gamma)$ and $\mathcal{R}_L(\Gamma)$ as follows. Fix a decomposition $\Gamma \xrightarrow{\sim} \Gamma_{\mathrm{tor}} \times \mathbb{Z}_p$ and set $\gamma_0 \in \Gamma$ corresponding to $(e, 1)$ on the right hand side. Then, we set

$$\begin{aligned} \mathcal{R}_L^+(\Gamma) &:= \mathbb{Z}_p[\Gamma_{\mathrm{tor}}] \otimes_{\mathbb{Z}_p} \mathcal{R}_L^+([\gamma_0] - 1) \quad \text{and} \\ \mathcal{R}_L(\Gamma) &:= \mathbb{Z}_p[\Gamma_{\mathrm{tor}}] \otimes_{\mathbb{Z}_p} \mathcal{R}_L([\gamma_0] - 1), \end{aligned}$$

where we set

$$\mathcal{R}_L([\gamma_0] - 1) := \left\{ \sum_{m \in \mathbb{Z}} a_m([\gamma_0] - 1)^m \,\middle|\, \sum_{m \in \mathbb{Z}} a_m X^m \in \mathcal{R}_L \right\}$$

and

$$\mathcal{R}_L^+([\gamma_0] - 1) := \left\{ \sum_{m \geqq 0} a_m([\gamma_0] - 1)^m \in \mathcal{R}_L([\gamma_0] - 1) \right\}.$$

We define a (φ, Γ)-module $\mathbf{Dfm}(M)$ over $\mathcal{R}_L \hat{\otimes}_L \mathcal{R}_L^+(\Gamma)$ (which is the relative Robba ring with coefficients in $\mathcal{R}_L^+(\Gamma)$) to be

$$\mathbf{Dfm}(M) := M \hat{\otimes}_L \mathcal{R}_L^+(\Gamma)$$

as an $\mathcal{R}_L \hat{\otimes}_L \mathcal{R}_L^+(\Gamma)$-module on which φ and Γ act by

$$\varphi(x \hat{\otimes} y) := \varphi(x) \hat{\otimes} y \quad \text{and} \quad \gamma'(x \hat{\otimes} y) := \gamma'(x) \hat{\otimes} [\gamma']^{-1} y$$

for $x \in M, y \in \mathcal{R}_L^+(\Gamma)$ and $\gamma' \in \Gamma$. By [KPX14], one similarly has the following canonical quasi-isomorphisms

$$C^\bullet_{\varphi,\gamma}(\mathbf{Dfm}(M)) \xrightarrow{\sim} C^\bullet_{\psi,\gamma}(\mathbf{Dfm}(M)) \xrightarrow{\sim} C^\bullet_{\psi}(M)$$

of complexes of $\mathcal{R}^+_L(\Gamma)$-modules, and it is known that these are perfect complexes of $\mathcal{R}^+_L(\Gamma)$-modules. One also has the (extended) Iwasawa pairing

$$\{-,-\}_{0,\mathrm{Iw}} : (M^*)^{\psi=0,\iota} \times M^{\psi=0} \to \mathcal{R}_L(\Gamma)$$

by §4.2 of [KPX14]. Therefore, we can similarly define the following graded invertible $\mathcal{R}^+_L(\Gamma)$-modules

$$\Delta^{\mathrm{Iw}}_{L,1}(M) := \mathrm{Det}_{\mathcal{R}^+_L(\Gamma)}(C^\bullet_{\varphi,\gamma}(\mathbf{Dfm}(M)))$$

and

$$\Delta^{\mathrm{Iw}}_{L}(M) := \Delta^{\mathrm{Iw}}_{L,1}(M) \boxtimes_{\mathcal{R}^+_L(\Gamma)} (\Delta_{L,2}(M) \otimes_L \mathcal{R}^+_L(\Gamma))$$

(remark that we have $\Delta_{\mathcal{R}^+_L(\Gamma),2}(\mathbf{Dfm}(M)) \xrightarrow{\sim} \Delta_{L,2}(M) \otimes_L \mathcal{R}^+_L(\Gamma))$, and we can similarly obtain a canonical isomorphism

$$\Delta^{\mathrm{Iw}}_{L}(M) \otimes_{\mathcal{R}^+_L(\Gamma)} \mathcal{R}_L(\Gamma) \xrightarrow{\sim} (\det_{\mathcal{R}_L(\Gamma)} M^{\psi=0} \otimes_L \mathcal{L}_L(M)^\vee, 0)^{-1}$$

using Proposition 4.3.8 (3) [KPX14] (precisely, this proposition is proved under the assumption that $M/(\psi-1) = M^*/(\psi-1) = 0$, but we can easily prove the statement (3) of this proposition for general M in a similar way).

One can also generalize the p-adic Hodge theory for (φ,Γ)-modules over $\mathcal{R}_L$. For a field F of characteristic zero and $n \in \mathbb{Z}_{\geqq 1}$, we set $F_n := F \otimes_{\mathbb{Q}} \mathbb{Q}(\mu_{p^n})$ and $F_\infty := \bigcup_{n\geqq 1} F_n$. Set $t := \log(1+X) \in \mathcal{R}_L$. Set $\mathbf{D}_{\mathrm{cris}}(M) := M[1/t]^\Gamma$. For $n \geqq 1$, one has the following Γ-equivariant injection

$$\iota_n : \mathcal{R}^{(n)}_L \hookrightarrow L_n[[t]] : f(X) \mapsto f(\zeta_{p^n}\exp(t/p^n) - 1).$$

Using this map, we set, for $n \geqq n(M)$,

$$\mathbf{D}^+_{\mathrm{dif},n}(M) := M^{(n)} \otimes_{\mathcal{R}^{(n)}_L,\iota_n} L_n[[t]], \quad \mathbf{D}_{\mathrm{dif},n}(M) := \mathbf{D}^+_{\mathrm{dif},n}(M)[1/t]$$

and

$$\mathbf{D}^*_{\mathrm{dif},\infty}(M) := \varinjlim_{n\geqq n(M)} \mathbf{D}^*_{\mathrm{dif},n}(M)$$

for $* = +$ or $* = \phi$ (the empty set), where the transition maps are defined by $\mathbf{D}^+_{\mathrm{dif},n}(M) \to \mathbf{D}^+_{\mathrm{dif},n+1}(M) : x \otimes y \mapsto \varphi(x) \otimes y$. Using these, we set

$$\mathbf{D}_{\mathrm{dR}}(M) := \mathbf{D}_{\mathrm{dif},\infty}(M)^\Gamma, \quad \mathbf{D}^i_{\mathrm{dR}}(M) := (t^i\mathbf{D}^+_{\mathrm{dif},\infty}(M))^\Gamma$$

and

$$t_M := \mathbf{D}_{\mathrm{dR}}(M)/\mathbf{D}^0_{\mathrm{dR}}(M)$$

for $i \in \mathbb{Z}$. Using these (and those for $M \otimes_{\mathcal{R}_L} \mathcal{R}_L(K)$, where $\mathcal{R}_L(K)$ is the Robba ring of a finite extension K of $\mathbb{Q}_p$), one can define the notions of a crystalline (φ,Γ)-module, a de Rham (φ,Γ)-module, etc. over the Robba ring. In particular, for any de Rham M (which is also known to be potentially semi-stable), one can define $\mathbf{D}_{\mathrm{pst}}(M)$ and its associated L-representation $W(M)$ of $'W_{\mathbb{Q}_p}$, as we usually do for de Rham representations of $G_{\mathbb{Q}_p}$. By §2 of [Na14b], one can also generalize the Bloch-Kato's fundamental exact sequence

$$(17)\quad \begin{aligned} 0 \to \mathrm{H}^0_{\varphi,\gamma}(M) \to \mathbf{D}_{\mathrm{cris}}(M) \xrightarrow{(a)} \mathbf{D}_{\mathrm{cris}}(M) \oplus t_M \xrightarrow{(b)} \mathrm{H}^1_{\varphi,\gamma}(M) \\ \xrightarrow{(c)} \mathbf{D}_{\mathrm{cris}}(M^*)^\vee \oplus \mathbf{D}^0_{\mathrm{dR}}(M) \xrightarrow{(d)} \mathbf{D}_{\mathrm{cris}}(M^*)^\vee \to \mathrm{H}^2_{\varphi,\gamma}(M) \to 0, \end{aligned}$$

as in the exact sequence (3) in §2.1 for any de Rham M. Using this, one can define the de Rham ε-isomorphism

$$\varepsilon^{\mathrm{dR}}_L(M) : \mathbf{1}_L \xrightarrow{\sim} \Delta_L(M)$$

for any de Rham M (see §3.3 of [Na14b] for the precise definition) in the same way as that for de Rham V.

Let D be an étale (φ,Γ)-module over $\mathcal{E}_L$. Then, one has the following canonical comparison isomorphisms

$$\Delta_L(D) \xrightarrow{\sim} \Delta_L(D_{\mathrm{rig}})$$

and

$$\Delta^{\mathrm{Iw}}_L(D) \otimes_{\Lambda_L(\Gamma)} \mathcal{R}^+_L(\Gamma) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_L(D_{\mathrm{rig}})$$

by Proposition 2.7 [Li08] and Theorem 1.9 of [Po13] respectively. For any de Rham L-representation V of $G_{\mathbb{Q}_p}$, one has canonical isomorphisms

$$\mathbf{D}_{\mathrm{dR}}(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{dR}}(D(V)_{\mathrm{rig}}), \quad \mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris}}(D(V)_{\mathrm{rig}})$$

and

$$W(V) \xrightarrow{\sim} W(D(V)_{\mathrm{rig}}).$$

Moreover, under these identifications, one also has

$$(18)\qquad \varepsilon^{\mathrm{dR}}_L(V) = \varepsilon^{\mathrm{dR}}_L(D(V)_{\mathrm{rig}})$$

under the isomorphism $\Delta_L(V) \xrightarrow{\sim} \Delta_L(D(V)) \xrightarrow{\sim} \Delta_L(D(V)_{\mathrm{rig}})$ (see [Na14b]), by which we freely identify the both sides of (18) with each other.

3.3.2. The comparison of our local ε-isomorphism with that defined in [Na14b]. Now, let us go back to our situation. Let D be an étale (φ,Γ)-module over $\mathcal{E}_L$ of rank two. We assume that D is trianguline, which means that there exist a finite extension L' of L and continuous homomorphisms $\delta_1,\delta_2:\mathbb{Q}_p^\times \to (L')^\times$ such that $D_{\mathrm{rig}}\otimes_L L'$ sits in an exact sequence of the following form

$$0\to\mathcal{R}_{L'}(\delta_1)\to D_{\mathrm{rig}}\otimes_L L'\to\mathcal{R}_{L'}(\delta_2)\to 0. \tag{19}$$

Since scalar extensions do not affect our results, we assume that $L'=L$ from now on. In our previous article [Na14b], we defined an ε-isomorphism

$$\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)):\mathbf{1}_{\mathcal{R}_L^+(\Gamma)}\xrightarrow{\sim}\Delta_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta))$$

for any continuous homomorphism $\delta:\mathbb{Q}_p^\times\to L^\times$, and showed that this satisfies the same conditions (1), (3), (4) and (5) in Conjecture 2.1. The main result of this subsection is the following theorem.

Theorem 3.9. *Under the situation above, one has an equality*

$$\varepsilon_L^{\mathrm{Iw}}(D)\otimes\mathrm{id}_{\mathcal{R}_L^+(\Gamma)}=\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta_1))\boxtimes\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta_2))$$

under the canonical isomorphisms

$$\Delta_L^{\mathrm{Iw}}(D)\otimes_{\Lambda_L(\Gamma)}\mathcal{R}_L^+(\Gamma)\xrightarrow{\sim}\Delta_L^{\mathrm{Iw}}(D_{\mathrm{rig}})\xrightarrow{\sim}\Delta_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta_1))\boxtimes\Delta_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta_2)),$$

where the latter is induced by the short exact sequence (19).

Before proving this theorem, we first show the equality $\varepsilon_L(D)=\varepsilon_L^{\mathrm{dR}}(D)$ for trianguline and de Rham D as a corollary of this theorem.

Corollary 3.10. *Let D be an étale (φ,Γ)-module over $\mathcal{E}_L$ of rank two which is de Rham and trianguline, then we have*

$$\varepsilon_L(D)=\varepsilon_L^{\mathrm{dR}}(D).$$

Proof. Specializing the equality $\varepsilon_L^{\mathrm{Iw}}(D)\otimes\mathrm{id}_{\mathcal{R}_L^+(\Gamma)}=\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta_1))\boxtimes\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta_2))$ in the above theorem by the continuous L-algebra morphism $f_{\mathbf{1}}:\mathcal{R}_L^+(\Gamma)\to L:[\gamma']\mapsto 1$ $(\forall\gamma'\in\Gamma)$, we obtain an equality

$$\varepsilon_L(D)=\varepsilon_L(\mathcal{R}_L(\delta_1))\boxtimes\varepsilon_L(\mathcal{R}_L(\delta_2)).$$

Then, the corollary follows from the equalities

$$\varepsilon_L(\mathcal{R}_L(\delta_i)) = \varepsilon_L^{\mathrm{dR}}(\mathcal{R}_L(\delta_i))$$

for $i = 1, 2$ (Theorem 3.13 of [Na14b]) and

$$\varepsilon_L^{\mathrm{dR}}(D) = \varepsilon_L^{\mathrm{dR}}(D_{\mathrm{rig}}) = \varepsilon_L^{\mathrm{dR}}(\mathcal{R}_L(\delta_1)) \boxtimes \varepsilon_L^{\mathrm{dR}}(\mathcal{R}_L(\delta_2))$$

(Lemma 3.9 of [Na14b]). □

To show the theorem above, we first prove a lemma concerning the explicit description of the extended Iwasawa pairing

$$\{-,-\}_{0,\mathrm{Iw}} : (\mathcal{R}_L(\delta)^*)^{\psi=0,\iota} \times \mathcal{R}_L(\delta)^{\psi=0} \to \mathcal{R}_L(\Gamma)$$

for $M = \mathcal{R}_L(\delta)$. We identify $\mathcal{R}_L(\delta^{-1})$ with $\mathcal{R}_L(\delta)^\vee$ via the isomorphism

$$\mathcal{R}_L(\delta^{-1}) \xrightarrow{\sim} \mathcal{R}_L(\delta)^\vee : f\mathbf{e}_{\delta^{-1}} \mapsto [g\mathbf{e}_\delta \mapsto fg].$$

Lemma 3.11. *The extended Iwasawa pairing*

$$\{-,-\}_{0,\mathrm{Iw}} : (\mathcal{R}_L(\delta)^*)^{\psi=0,\iota} \times \mathcal{R}_L(\delta)^{\psi=0} \to \mathcal{R}_L(\Gamma)$$

satisfies the equality

$$\{\lambda_1 \cdot_\iota ((1+X)^{-1}\mathbf{e}_{\delta^{-1}} \otimes \mathbf{e}_1), \lambda_2 \cdot ((1+X)\mathbf{e}_\delta)\}_{0,\mathrm{Iw}} = \lambda_1\lambda_2$$

for any $\lambda_1, \lambda_2 \in \mathcal{R}_L(\Gamma)$.

Proof. We first remark that the isomorphism

$$\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)) \otimes \mathrm{id}_{\mathcal{R}_L(\Gamma)} : \mathbf{1}_{\mathcal{R}_L(\Gamma)} \xrightarrow{\sim} \Delta_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)) \otimes_{\mathcal{R}_L^+(\Gamma)} \mathcal{R}_L(\Gamma)$$

is equal to the one induced by the isomorphism

$$\theta(\mathcal{R}_L(\delta)) : \mathcal{R}_L(\Gamma) \otimes_L L(\delta) \xrightarrow{\sim} \mathcal{R}_L(\delta)^{\psi=0} : \lambda \otimes \mathbf{e}_\delta \mapsto \lambda \cdot ((1+X)^{-1}\mathbf{e}_\delta)$$

and the isomorphism $\Delta_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} (\mathcal{R}_L(\delta)^{\psi=0} \otimes_L L(\delta)^\vee, 0)^{-1}$. This follows easily from the definition of $\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta))$ given in §4.1 of [Na14b].

Since one has

$$\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)^*)^\iota = \varepsilon_{\mathcal{R}_L^+(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_L(\delta))^*)$$

under the canonical isomorphism

$$\Delta_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)^*)^\iota \xrightarrow{\sim} \Delta_{\mathcal{R}_L^+(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_L(\delta))^*),$$

the isomorphism $\varepsilon_{\mathcal{R}_L^+(\Gamma)}(\mathbf{Dfm}(\mathcal{R}_L(\delta))^*)\otimes\mathrm{id}_{\mathcal{R}_L(\Gamma)}$ is equal to the one induced by the isomorphism

$$\begin{aligned}\theta(\mathcal{R}_L(\delta)^*)^\iota : \mathcal{R}_L(\Gamma)\otimes_L L(\delta^{-1})(1) \xrightarrow{\sim} (\mathcal{R}_L(\delta)^*)^{\psi=0,\iota} &: \lambda\otimes(\mathbf{e}_{\delta^{-1}}\otimes\mathbf{e}_1)\\ &\mapsto \lambda\cdot_\iota((1+X)^{-1}\mathbf{e}_{\delta^{-1}}\otimes\mathbf{e}_1).\end{aligned}$$

Under the canonical isomorphism

$$\Delta_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} (\Delta_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)^*)^\iota)^\vee \boxtimes (\mathcal{R}_L^+(\Gamma)\otimes_L L(1),0)$$

defined by the Tate duality (see §3.2 of [Na14b]), one has

$$\delta(-1)[\sigma_{-1}]\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta))^{-1} = (\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta)^*)^\iota)^\vee \boxtimes [\mathbf{e}_1\mapsto 1]$$

by the condition (4) of Conjecture 2.1 for $\mathbf{Dfm}(\mathcal{R}_L(\delta))$ (which is proved in Theorem 3.13 of [Na14b]). Using the isomorphisms $\theta(\mathcal{R}_A(\delta))$ and $\theta(\mathcal{R}_A(\delta)^*)^\iota$, we obtain from this equality the following commutative diagram of $\mathcal{R}_L(\Gamma)$-bilinear pairings:

$$\begin{array}{ccc}\mathcal{R}_L(\Gamma)\otimes_L L(\delta^{-1})(1)\times\mathcal{R}_L(\Gamma)\otimes_L L(\delta) & \xrightarrow{\theta(\mathcal{R}_L(\delta)^*)^\iota\times\theta(\mathcal{R}_L(\delta))} & (\mathcal{R}_L(\delta)^*)^{\psi=0,\iota}\times\mathcal{R}_L(\delta)^{\psi=0}\\ \Big\downarrow{\scriptstyle (1\otimes\mathbf{e}_{\delta^{-1}}\otimes\mathbf{e}_1,1\otimes\mathbf{e}_\delta)\mapsto 1\otimes\mathbf{e}_1} & & \Big\downarrow{\scriptstyle \{-,-\}_{0,\mathrm{Iw}}}\\ \mathcal{R}_L(\Gamma)\otimes_L L(1) & \xrightarrow{\mathbf{e}_1\mapsto\delta(-1)[\sigma_{-1}]} & \mathcal{R}_L(\Gamma).\end{array}$$

The lemma follows from the commutativity of this diagram. □

Using this lemma, we prove the theorem as follows. As we show in the proof, a result of Dospinescu [Do11] on the explicit description of the action of w_{δ_D} on $D_{\mathrm{rig}}^{\psi=0}$, which is intimately related with the action of $w=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ on the locally analytic vectors, is crucial for the proof.

Proof. (of Theorem 3.9) We first show the theorem when D is absolutely irreducible. Since the canonical map $\mathcal{R}_L^+(\Gamma)\to\mathcal{R}_L(\Gamma)$ is injective, it suffices to show the equality after the base change to $\mathcal{R}_L(\Gamma)$.

By the results in V.2 of [Co10b], the involution $w_{\delta_D}: D^{\psi=0}\xrightarrow{\sim} D^{\psi=0}$ (first descends to $w_{\delta_D}: D^{\dagger,\psi=0}\xrightarrow{\sim} D^{\dagger,\psi=0}$, and then) uniquely extends to $w_{\delta_D}: D_{\mathrm{rig}}^{\psi=0}\xrightarrow{\sim} D_{\mathrm{rig}}^{\psi=0}$, and the isomorphism

$$\begin{aligned}&\varepsilon_L^{\mathrm{Iw}}(D)\otimes\mathrm{id}_{\mathcal{R}_L(\Gamma)} : \mathbf{1}_{\mathcal{R}_L(\Gamma)}\\ &\quad\xrightarrow{\sim}\Delta_L^{\mathrm{Iw}}(D)\otimes_{\Lambda_L(\Gamma)}\mathcal{R}_L(\Gamma)(\xrightarrow{\sim}(\det_{\mathcal{R}_L(\Gamma)}D_{\mathrm{rig}}^{\psi=0}\otimes_L\mathcal{L}_L(D)^\vee,0)^{-1})\end{aligned}$$

is the one which is naturally induced by the isomorphism

$$\theta(D_{\mathrm{rig}}) : \det_{\mathcal{R}_L(\Gamma)} D_{\mathrm{rig}}^{\psi=0} \xrightarrow{\sim} \mathcal{R}_L(\Gamma) \otimes_L \mathcal{L}_L(D_{\mathrm{rig}}) : \\ (x \wedge y) \mapsto [\sigma_{-1}] \cdot \{w_{\delta_D}(x) \otimes z^{\vee} \otimes \mathbf{e}_1, y\}_{0,\mathrm{Iw}} \otimes z$$

for any $z \in \mathcal{L}_L(D_{\mathrm{rig}})^{\times}$. By the explicit descriptions of $\varepsilon_L^{\mathrm{Iw}}(\mathcal{R}_L(\delta_i)) \otimes \mathrm{id}_{\mathcal{R}_L(\Gamma)}$ and $\varepsilon_L^{\mathrm{Iw}}(D) \otimes \mathrm{id}_{\mathcal{R}_L(\Gamma)}$, it suffices to show that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{R}_L(\delta_1)^{\psi=0} \otimes_{\mathcal{R}_L(\Gamma)} \mathcal{R}_L(\delta_2)^{\psi=0} \arrow[r, "x\otimes y \mapsto x \wedge \widetilde{y}"] \arrow[d, "\theta(\mathcal{R}_L(\delta_1))^{-1} \otimes \theta(\mathcal{R}_L(\delta_2))^{-1}"'] & \det_{\mathcal{R}_L(\Gamma)} D_{\mathrm{rig}}^{\psi=0} \arrow[d, "\theta(D_{\mathrm{rig}})"] \\ (\mathcal{R}_L(\Gamma) \otimes_L L(\delta_1)) \otimes_{\mathcal{R}_L(\Gamma)} (\mathcal{R}_L(\Gamma) \otimes_L L(\delta_2)) \arrow[r, "\mathbf{e}_{\delta_1} \otimes \mathbf{e}_{\delta_2} \mapsto \mathbf{e}_{\delta_1} \wedge \widetilde{\mathbf{e}}_{\delta_2}"] & \mathcal{R}_L(\Gamma) \otimes_L \mathcal{L}_L(D_{\mathrm{rig}}). \end{tikzcd}$$

Here $\widetilde{y} \in D_{\mathrm{rig}}^{\psi=0}$ (resp. $\widetilde{\mathbf{e}}_{\delta_2} \in D_{\mathrm{rig}}$) is a lift of $y \in \mathcal{R}_L(\delta_2)^{\psi=0}$ (resp. $\mathbf{e}_{\delta_2} \in \mathcal{R}_L(\delta_2)$).

By definitions of $\theta(\mathcal{R}_L(\delta_i))$ and $\theta(D_{\mathrm{rig}})$, and the $\mathcal{R}_L(\Gamma)$-bilinearity of the pairings in the diagram above, it suffices to show the equality

(20)
$$[\sigma_{-1}] \cdot \{w_{\delta_D}((1+X)^{-1}\mathbf{e}_{\delta_1}) \otimes (\mathbf{e}_{\delta_1} \wedge \widetilde{\mathbf{e}}_{\delta_2})^{\vee} \otimes \mathbf{e}_1, \delta_2(p)^{-1}(1+X)^{-1}\varphi(\widetilde{\mathbf{e}}_{\delta_2})\}_{0,\mathrm{Iw}} = 1.$$

Since one has an equality

$$w_{\delta_D}((1+X)\mathbf{e}_{\delta_1}) = \delta_1(-1)(1+X)\mathbf{e}_{\delta_1}$$

by (the proof of) Proposition 3.2 of [Do11], one also has

$$\begin{aligned} w_{\delta_D}((1+X)^{-1}\mathbf{e}_{\delta_1}) &= \delta_1(-1) w_{\delta_D}(\sigma_{-1}((1+X)\mathbf{e}_{\delta_1})) \\ &= \delta_D(-1)\delta_1(-1)\sigma_{-1}(w_{\delta_D}((1+X)\mathbf{e}_{\delta_1})) \\ &= \delta_D(-1)\delta_1(-1)\sigma_{-1}(\delta_1(-1)(1+X)\mathbf{e}_{\delta_1}) \\ &= \delta_D(-1)\delta_1(-1)(1+X)^{-1}\mathbf{e}_{\delta_1} \end{aligned} \tag{21}$$

since one has $w_{\delta_D} \circ \sigma_a = \delta_D(a)\sigma_a^{-1} \circ w_{\delta_D}$ $(a \in \mathbb{Z}_p^{\times})$. Using this equality and the equality $\mathbf{e}_{\delta_1} \otimes (\mathbf{e}_{\delta_1} \wedge \widetilde{\mathbf{e}}_{\delta_2})^{\vee} = -\mathbf{e}_{\delta_2^{-1}}$ in $\mathcal{R}_L(\delta_2^{-1}) \subseteq D_{\mathrm{rig}}^{\vee}$, the left hand side of (20) is equal to

$$\begin{aligned} &- \delta_D(-1)\delta_1(-1)[\sigma_{-1}] \cdot \{(1+X)^{-1}\mathbf{e}_{\delta_2^{-1}} \otimes \mathbf{e}_1, \delta_2(p)^{-1}(1+X)^{-1}\varphi(\widetilde{\mathbf{e}}_{\delta_2})\}_{0,\mathrm{Iw}} \\ &\quad = -\delta_D(-1)\delta_1(-1)[\sigma_{-1}] \cdot \{(1+X)^{-1}\mathbf{e}_{\delta_2^{-1}} \otimes \mathbf{e}_1, (1+X)^{-1}\mathbf{e}_{\delta_2}\}_{0,\mathrm{Iw}} \end{aligned}$$

$$= -\delta_D(-1)\delta_1(-1)\delta_2(-1)[\sigma_{-1}]\cdot\{(1+X)^{-1}\mathbf{e}_{\delta_2^{-1}}\otimes\mathbf{e}_1,[\sigma_{-1}]\cdot((1+X)\mathbf{e}_{\delta_2})\}_{0,\mathrm{Iw}}$$
$$= \{(1+X)^{-1}\mathbf{e}_{\delta_2^{-1}}\otimes\mathbf{e}_1,(1+X)\mathbf{e}_{\delta_2}\}_{0,\mathrm{Iw}} = 1,$$

where the last equality follows from the equality $-\delta_D(-1)\delta_1(-1)\delta_2(-1)=1$ and Lemma 3.11.

When D is not absolutely irreducible, then (after extending scalars L), we have an exact sequence

$$0\to\mathcal{E}_L(\delta_1)\to D\to\mathcal{E}_L(\delta_2)\to 0$$

for some continuous homomorphisms $\delta_1,\delta_2:\mathbb{Q}_p^\times\to\mathcal{O}^\times$. Then, the involution w_{δ_D} acts on $\mathcal{E}_L(\delta_1)^{\psi=0}$ (in fact, it acts on any étale (φ,Γ)-modules), and one can directly check that one has $w_{\delta_D}((1+X)\mathbf{e}_{\delta_1}))=\delta_1(-1)(1+X)\mathbf{e}_{\delta_1}$. Then, the theorem follows by the same argument as in the absolutely irreducible case. □

Remark 3.12. In the last paragraph of the proof above, for any exact sequence $0\to\mathcal{E}_R(\delta_1)\to D\to\mathcal{E}_R(\delta_2)\to 0$ of étale (φ,Γ)-modules over $\mathcal{E}_R$, we show the equality

$$\varepsilon_R^{\mathrm{Iw}}(D)=\varepsilon_R^{\mathrm{Iw}}(\mathcal{E}_R(\delta_1))\boxtimes\varepsilon_R^{\mathrm{Iw}}(\mathcal{E}_R(\delta_2))$$

under the canonical isomorphism $\Delta_R^{\mathrm{Iw}}(D)\xrightarrow{\sim}\Delta_R^{\mathrm{Iw}}(\mathcal{E}_R(\delta_1))\boxtimes\Delta_R^{\mathrm{Iw}}(\mathcal{E}_R(\delta_2))$, which shows that our ε-isomorphism satisfies the condition (2) in Conjecture 2.1.

3.4. The verification of the de Rham condition: the non-trianguline case

By the results in previous subsection, it remains to show the case (ii) of Theorem 3.1 (3) for non-trianguline ones. Precisely, it suffices to show the following theorem, whose proof will be given in the last part of this section.

Theorem 3.13. *Let D be an étale (φ,Γ)-module over $\mathcal{E}_L$ of rank two which is de Rham and non-trianguline. Assume that the Hodge-Tate weights of D are $\{k_1,k_2\}$ such that $k_1\leqq 0$ and $k_2\geqq 1$. Then, we have*

$$\varepsilon_L(D)=\varepsilon_L^{\mathrm{dR}}(D).$$

We first reduce the proof of this theorem to Proposition 3.14 below by explicitly describing the both sides of the equality in the theorem, then, in the last part of this subsection, we prove this key proposition using the Colmez's theory of Kirillov model of locally algebraic vectors $\Pi(D)^{\mathrm{alg}}$ of

$\Pi(D)$ and the Emerton's theorem on the compatibility of the p-adic and the classical local Langlands correspondence.

Hence, we first explicitly describe the ε-isomorphism and the de Rham ε-isomorphism under the assumption in the theorem above.

3.4.1. Explicit description of $\boldsymbol{\varepsilon_L^{\mathrm{dR}}(D)}$. From now on, let D be an étale (φ, Γ)-module over $\mathcal{E}_L$ of rank two which is de Rham and non-trianguline with the Hodge-Tate weights $\{k_1, k_2\}$ such that $k_1 \leqq 0$ and $k_2 \geqq 1$. We set

$$\mathbf{D}_{\mathrm{dR}}(D) := \mathbf{D}_{\mathrm{dR}}(V(D)), \quad W(D) := W(V(D)), etc.$$

We remark that, under the assumption that D is non-trianguline, D is absolutely irreducible. One has

$$\dim_L \mathbf{D}^i_{\mathrm{dR}}(D) = 1 \quad \text{ if and only if } -(k_2-1) \leqq i \leqq -k_1.$$

In particular, one has $\dim_L \mathbf{D}^0_{\mathrm{dR}}(D) = 1$. We fix a basis $\{f_1, f_2\}$ of $\mathbf{D}_{\mathrm{dR}}(D)$ over L such that $f_1 \in \mathbf{D}^0_{\mathrm{dR}}(D)$. Then, we have

$$t_D := \mathbf{D}_{\mathrm{dR}}(D)/\mathbf{D}^0_{\mathrm{dR}}(D) = L\overline{f}_2$$

where $\overline{f}_2 \in t_D$ is the image of f_2. Since D is absolutely irreducible, we have

$$\mathrm{H}^0_{\varphi,\gamma}(D) = \mathrm{H}^2_{\varphi,\gamma}(D) = 0, \ \dim_L \mathrm{H}^1_{\varphi,\gamma}(D) = 2$$

and the canonical specialization maps

$$\iota_D : D^{\psi=1} \to \mathrm{H}^1_{\varphi,\gamma}(D)$$

and

$$\iota_{D^*} : (D^*)^{\psi=1} \to \mathrm{H}^1_{\varphi,\gamma}(D^*)$$

are surjective since the cokernel is contained in $D/(\psi-1)$ which is zero by the absolutely irreducibility. Hence, we obtain a canonical isomorphism

$$\Delta_{L,1}(D) \xrightarrow{\sim} (\det_L \mathrm{H}^1_{\varphi,\gamma}(D), 2)^{-1},$$

and the Bloch-Kato's exact sequence for D is just the following short exact sequence

$$0 \to t_D \xrightarrow{\exp} \mathrm{H}^1_{\varphi,\gamma}(D) \xrightarrow{\exp^*} \mathbf{D}^0_{\mathrm{dR}}(D) \to 0.$$

Hence, the determinant $\det_L \mathrm{H}^1_{\varphi,\gamma}(D)$ has a basis of the form $y \wedge \exp(\overline{f}_2)$ such that $\exp^*(y) \neq 0$.

We also fix a base $\mathbf{e}_D$ of $\mathcal{L}_L(D)$, and set $h_D := k_1 + k_2$. Since we have

$$\det_L \mathbf{D}_{\mathrm{dR}}(D) = \mathbf{D}_{\mathrm{dR}}(\det_{\mathcal{R}_L} D) = (L_\infty t^{-h_D} \mathbf{e}_D)^\Gamma,$$

there exists a unique $\Omega \in L_\infty^\times$ such that

$$f_1 \wedge f_2 = \frac{1}{\Omega t^{h_D}} \mathbf{e}_D =: f_D.$$

If we define a map

$$\alpha : D^{\psi=1} \xrightarrow{\iota_D} \mathrm{H}^1_{\varphi,\gamma}(D) \to L \tag{22}$$

by the formula

$$\exp^*(\iota_D(x)) := \alpha(x) f_1$$

for $x \in D^{\psi=1}$, the de Rham ε-isomorphism $\varepsilon_L^{\mathrm{dR}}(D)$ is defined as the composite of the following isomorphisms

$$\begin{aligned} \varepsilon_L^{\mathrm{dR}}(D) : \mathbf{1}_L &\xrightarrow{\theta_1(D)} \Delta_{L,1}(D) \boxtimes \mathrm{Det}_L(\mathbf{D}_{\mathrm{dR}}(D)) \\ &\xrightarrow{\mathrm{id}\boxtimes\theta_2(D)} \Delta_{L,1}(D) \boxtimes \Delta_{L,2}(D) = \Delta_L(D) \end{aligned}$$

where the isomorphisms $\theta_1(D)$ and $\theta_2(D)$ are respectively induced by the isomorphisms defined by, for $x \in D^{\psi=1}$ such that $\alpha(x) \neq 0$,

$$\begin{aligned} \theta_1(D) : \det_L \mathrm{H}^1_{\varphi,\gamma}(D) \xrightarrow{\sim} \det_L \mathbf{D}_{\mathrm{dR}}(D) &: \iota_D(x) \wedge \exp(f_2) \\ &\mapsto \Gamma(D) \exp^*(\iota_D(x)) \wedge f_2 = \Gamma(D)\alpha(x) f_1 \wedge f_2 \end{aligned}$$

and

$$\theta_2(D)^{-1} : \mathcal{L}_L(D) \xrightarrow{\sim} \det_L \mathbf{D}_{\mathrm{dR}}(D) : \mathbf{e}_D \mapsto \frac{1}{\varepsilon_L(W(D)) t^{h_D}} \mathbf{e}_D.$$

Here, we remark that we have $\Gamma(D) = \frac{(-1)^{k_1}}{(-k_1)!}(k_2 - 1)!$. Hence, using α and Ω, the isomorphism

$$\eta(D) := \theta_2(D) \circ \theta_1(D) : \det_L \mathrm{H}^1_{\varphi,\gamma}(D) \xrightarrow{\sim} \mathcal{L}_L(D)$$

is explicitly described as follows:

$$\eta(D)(\iota_D(x) \wedge \exp(f_2)) = \Gamma(D)\varepsilon_L(W(D))\Omega^{-1}\alpha(x)\mathbf{e}_D. \tag{23}$$

3.4.2. Explicit description of $\varepsilon_L(D)$. We next consider the isomorphism $\varepsilon_L(D)$. Let

$$[-,-]_{\mathrm{dR}} : \mathbf{D}_{\mathrm{dR}}(D^*) \times \mathbf{D}_{\mathrm{dR}}(D) \to L$$

be the canonical dual pairing. We remarked in the proof of Proposition 3.4 that, under the assumption that D is absolutely irreducible, the natural map $1-\varphi : D^{\psi=1} \to D^{\psi=0}$ is injective, by which we identify $D^{\psi=1}$ with $\mathcal{C}(D)$ (and similarly for D^*), and one has $\omega_{\delta_D}(\mathcal{C}(D)) \otimes_L \mathcal{L}_L(D)^\vee = \mathcal{C}(D^\vee)$ under the canonical isomorphism $D \otimes_L \mathcal{L}_L(D)^\vee \xrightarrow{\sim} D^\vee$. By this fact and the definition of $\varepsilon_L(D)$, $\varepsilon_L(D)$ is the isomorphism which is naturally induced by the isomorphism

$$\eta'(D) : \det_L \mathrm{H}^1_{\varphi,\gamma}(D) \xrightarrow{\sim} \mathcal{L}_L(D) = ((L\mathbf{e}_D)^\vee)^\vee$$

defined by the following formula, for $x \in D^{\psi=1}$ such that $\alpha(x) \neq 0$,

(24)
$$\begin{aligned}\eta'(D)(\iota_D(x) \wedge \exp(f_2))((\mathbf{e}_D)^\vee) &= \langle \iota_{D^*}(\sigma_{-1}(\omega_{\delta_D}(x) \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1)), \exp(f_2)\rangle_{\mathrm{Tate}} \\ &= -[\exp^*(\iota_{D^*}(\sigma_{-1}(\omega_{\delta_D}(x) \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1))), f_2]_{\mathrm{dR}} =: (*),\end{aligned}$$

where the second equality follows from Proposition 2.16 of [Na14a]. Using the canonical isomorphism (identification)

$$\mathbf{D}^0_{\mathrm{dR}}(D^*) = Lf_1 \otimes f_D^\vee \otimes t^{-1}\mathbf{e}_1$$

induced by the canonical isomorphism $D \otimes_L \mathcal{L}_L(D)^\vee \xrightarrow{\sim} D^\vee$ (remark that we have $\Omega t^{h_D} \mathbf{e}_D^\vee = f_D^\vee$), we define a map

$$\beta : D^{\delta_D(p)\psi=1} \xrightarrow{y \mapsto \iota_{D^*}(\sigma_{-1}(y \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1))} \mathrm{H}^1_{\varphi,\gamma}(D^*) \to L \tag{25}$$

by the formula

$$\exp^*(\iota_{D^*}(\sigma_{-1}(y \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1))) := \beta(y) f_1 \otimes f_D^\vee \otimes t^{-1}\mathbf{e}_1$$

for $y \in D^{\delta_D(p)\psi=1}$. Using this β, the last term $(*)$ in the equalities (24) is equal to

$$(*) = -[\beta(\omega_{\delta_D}(x)) f_1 \otimes f_D^\vee \otimes t^{-1}\mathbf{e}_1, f_2]_{\mathrm{dR}} = \beta(w_{\delta_D}(x)). \tag{26}$$

We see from the formulae (24) and (26) that the isomorphism

$$\eta'(D) : \det_L \mathrm{H}^1_{\varphi,\gamma}(D) \xrightarrow{\sim} \mathcal{L}_L(D)$$

is explicitly described as follows:

$$\eta'(D)(\iota_D(x) \wedge \exp(f_2)) = \beta(w_{\delta_D}(x))\mathbf{e}_D. \tag{27}$$

The formulae (23) and (27) show that the equality $\varepsilon_L(D) = \varepsilon_L^{\mathrm{dR}}(D)$ follows from the following key proposition. Thus the proof of Theorem 3.13 is reduced to this proposition.

Proposition 3.14. *For any $x \in D^{\psi=1}$, we have*

$$\beta(w_{\delta_D}(x)) = \Gamma(D)\varepsilon_L(W(D))\Omega^{-1}\alpha(x).$$

In the rest of this subsection, we prove this key proposition. The proof will be given in the last part of this subsection. Our proof heavily depends on the Colmez's theory of Kirillov model of locally algebraic vectors $\Pi(D)^{\mathrm{alg}}$ of $\Pi(D)$ [Co10b], which we recall in details below.

3.4.3. Recall of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Set $G := \mathrm{GL}_2(\mathbb{Q}_p)$, $B := \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G\right\}$, $P := \begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}$, $P^+ := \begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ and $Z := \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{Q}_p^\times\right\}$. We identify Z with $\mathbb{Q}_p^\times$ via the isomorphism $\mathbb{Q}_p^\times \xrightarrow{\sim} Z : a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

Let us briefly recall the construction of the representation $\Pi(D)$ of G for our D. Let the monoid P^+ act on D by the rule $\begin{pmatrix} p^n a & b \\ 0 & 1 \end{pmatrix} \cdot x := (1+X)^b\varphi^n(\sigma_a(x))$ for $n \geqq 0$, $a \in \mathbb{Z}_p^\times$, $b \in \mathbb{Z}_p$ and $x \in D$. Using the involution $w_{\delta_D} : D^{\psi=0} \xrightarrow{\sim} D^{\psi=0}$, we define a topological L-vector space

$$D \boxtimes_{\delta_D} \mathbf{P}^1 := \{(z_1, z_2) \in D \times D \mid w_{\delta_D}((1-\varphi\psi)z_1) = (1-\varphi\psi)z_2\}$$

and an L-linear map

$$\mathrm{Res}_{\mathbb{Z}_p} : D \boxtimes_{\delta_D} \mathbf{P}^1 \to D : (z_1, z_2) \mapsto z_1.$$

By the recipe in [Co10b] II, one can define a continuous action of G on $D \boxtimes_{\delta_D} \mathbf{P}^1$ with the central character δ_D such that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot (z_1, z_2) = (z_2, z_1)$ and the map $\mathrm{Res}_{\mathbb{Z}_p}$ is P^+-equivariant. We denote by $D \boxtimes \mathbb{Q}_p$ the topological L-vector space consisting of the sequences $(z_n)_{n \geqq 0}$ such that $\psi(z_{n+1}) = z_n$ for all $n \geqq 0$. One can define a continuous action of P on $D \boxtimes \mathbb{Q}_p$ by

$$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot (z_n)_{n \geqq 0} := (\sigma_a(x_n))_{n \geqq 0}, \ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot (x_n)_{n \geqq 0} := (x_{n+1})_{n \geqq 0}$$

and

$$\begin{pmatrix} 1 & b/p^m \\ 0 & 1 \end{pmatrix} \cdot (z_n)_{n \geqq 0} := (\psi^m((1+X)^{p^n b} z_{n+m}))_{n \geqq 0}$$

for $a \in \mathbb{Z}_p^\times, b \in \mathbb{Z}_p$ and $m \geqq 0$.

Take an étale (φ, Γ)-submodule $D_0 \subseteq D$ over $\mathcal{E}_{\mathcal{O}}$ such that $D_0[1/p] = D$. By [Co10a], there exists the smallest ψ-stable compact $\mathcal{O}[[X]]$-submodules of D_0, which we denote by $D_0^\natural \subseteq D_0$. One also has the largest ψ-stable compact $\mathcal{O}[[X]]$-submodule $D_0^\sharp \subseteq D_0$ on which ψ is surjective. We set $D^\natural := D_0^\natural[1/p]$ and $D^\sharp := D_0^\sharp[1/p]$, which are independent of the choice of D_0. We note that one has $D^\natural = D^\sharp$ under our assumption that D is absolutely irreducible (Corollaire II.5.21 of [Co10a]). One also has $(D^\natural)^{\psi=1} = (D^\sharp)^{\psi=1} = D^{\psi=1}$, where the second equality follows from Proposition II.5.6 of [Co10a]. Define a sub $L[B]$-module $D^\natural \boxtimes_{\delta_D} \mathbf{P}^1$ of $D \boxtimes_{\delta_D} \mathbf{P}^1$ by

$$D^\natural \boxtimes_{\delta_D} \mathbf{P}^1 := \left\{ z \in D \boxtimes_{\delta_D} \mathbf{P}^1 \;\middle|\; \mathrm{Res}_{\mathbb{Z}_p}\left(\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \cdot z \right) \in D^\natural \text{ for all } n \geqq 0 \right\}.$$

One of the deepest results in the theory of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ is that the pair (D, δ_D) is G-compatible, which means that $D^\natural \boxtimes_{\delta_D} \mathbf{P}^1$ is stable under the action of G (Théorème II.3.1 of [Co10b], Proposition 10.1 of [CDP14a]). Finally, one defines

$$\Pi(D) := D \boxtimes_{\delta_D} \mathbf{P}^1 / D^\natural \boxtimes_{\delta_D} \mathbf{P}^1$$

which is a topologically irreducible unitary L-Banach admissible representation of G.

3.4.4. Recall of the Kirillov model of the locally algebraic vectors. We next recall in details the Colmez's theory of the Kirillov model of the locally algebraic vectors $\Pi(D)^{\mathrm{alg}}$ of $\Pi(D)$. We set $L_\infty[[t]] := \bigcup_{n \geqq 1} L_n[[t]]$. For the fixed $\zeta = \{\zeta_{p^n}\}_{n \geqq 1} \in \mathbb{Z}_p(1)$, we define a homomorphism

$$[\overline{\zeta}^{(-)}] : \mathbb{Q}_p \to ((\widetilde{\mathbf{B}}^+)^\times)^{H_{\mathbb{Q}_p}} : a \mapsto [\overline{\zeta}^a].$$

For $V := V(D)$, we set

$$\widetilde{D}^+ := (V \otimes_{\mathbb{Q}_p} \widetilde{\mathbf{B}}^+)^{H_{\mathbb{Q}_p}},\ \widetilde{D} := (V \otimes_{\mathbb{Q}_p} \widetilde{\mathbf{B}})^{H_{\mathbb{Q}_p}} \text{ and } \widetilde{D}^+_{\mathrm{dif}} := (V \otimes_{\mathbb{Q}_p} \mathbf{B}^+_{\mathrm{dR}})^{H_{\mathbb{Q}_p}}.$$

One has a canonical isomorphism

$$\widetilde{D}^+_{\mathrm{dif}} \xrightarrow{\sim} \mathbf{D}^+_{\mathrm{dif},\infty}(D_{\mathrm{rig}}) \otimes_{\mathbb{Q}_{p,\infty}[[t]]} (\mathbf{B}^+_{\mathrm{dR}})^{H_{\mathbb{Q}_p}}.$$

The natural inclusion $\iota_0 : \widetilde{\mathbf{B}}^+ \hookrightarrow \mathbf{B}^+_{\mathrm{dR}}$ induces a canonical Γ-equivariant inclusion

$$\iota_0 : \widetilde{D}^+ \hookrightarrow \widetilde{D}^+_{\mathrm{dif}}.$$

The group B acts on $\widetilde{D}^+$, $\widetilde{D}$ and $\widetilde{D}/\widetilde{D}^+$ by the rule, for $z \in \widetilde{D}^+, \widetilde{D}, \widetilde{D}/\widetilde{D}^+$,

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot z := \delta_D(a) z,\ \begin{pmatrix} p^n b & 0 \\ 0 & 1 \end{pmatrix} \cdot z := \varphi^n(\sigma_b(z)),\ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \cdot z := [\overline{\zeta}^c] z$$

for $a \in \mathbb{Q}_p^\times$, $b \in \mathbb{Z}_p^\times$, $n \in \mathbb{Z}$ and $c \in \mathbb{Q}_p$.

We set $k := k_2 - k_1 \geqq 1$. We denote by

$$\mathrm{LP}\left(\mathbb{Q}_p^\times, \frac{1}{t^k}\widetilde{D}^+_{\mathrm{dif}}/\widetilde{D}^+_{\mathrm{dif}}\right)^\Gamma$$

the L-vector space consisting of functions $\phi : \mathbb{Q}_p^\times \to \frac{1}{t^k}\widetilde{D}^+_{\mathrm{dif}}/\widetilde{D}^+_{\mathrm{dif}}$ such that the support is compact in $\mathbb{Q}_p$ (i.e. $\phi(\frac{1}{p^n}\mathbb{Z}_p^\times) = 0$ for any sufficiently large n) and $\sigma_a(\phi(x)) = \phi(ax)$ for any $a \in \mathbb{Z}_p^\times$ and $x \in \mathbb{Q}_p^\times$. We equip this space with an action of B by

$$\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \phi\right)(x) := \delta_D(a)\phi(x),\ \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \phi\right)(x) := \phi(ax),$$
$$\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \phi\right)(x) := \iota_0([\overline{\zeta}^{bx}])\phi(x)$$

for $a \in \mathbb{Q}_p^\times$ and $b \in \mathbb{Q}_p$. Remark that, for $a = \frac{b}{p^n} \in \mathbb{Q}_p^\times$ such that $b \in \mathbb{Z}_p$, $n \geqq 0$, one has $\iota_0([\overline{\zeta}^a]) = \zeta_{p^n}^b \exp(at) \in L_\infty[[t]]^\times$.

For $z \in \bigcup_{n \geqq 0} \frac{1}{\varphi^n(X)^k}\widetilde{D}^+/\widetilde{D}^+$ (this is a B-stable subspace of $\widetilde{D}/\widetilde{D}^+$), define a function $\phi_z \in \mathrm{LP}\left(\mathbb{Q}_p^\times, \frac{1}{t^k}\widetilde{D}^+_{\mathrm{dif}}/\widetilde{D}^+_{\mathrm{dif}}\right)^\Gamma$ by

$$\phi_z(x) := \iota_0\left(\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \cdot z\right)$$

for $x \in \mathbb{Q}_p^\times$. By Lemme VI.5.4 (i) of [Co10b], this correspondence induces a B-equivariant inclusion

$$\bigcup_{n \geqq 0} \frac{1}{\varphi^n(X)^k}\widetilde{D}^+/\widetilde{D}^+ \hookrightarrow \mathrm{LP}\left(\mathbb{Q}_p^\times, \frac{1}{t^k}\widetilde{D}^+_{\mathrm{dif}}/\widetilde{D}^+_{\mathrm{dif}}\right)^\Gamma : z \mapsto \phi_z.$$

Let us write

$$\mathbf{N}^{+}_{\mathrm{dif},*}(D_{\mathrm{rig}}) := \mathbf{D}_{\mathrm{dR}}(D_{\mathrm{rig}}) \otimes_L L_*[[t]]$$

for $* = n \geqq n(D_{\mathrm{rig}})$ or $* = \infty$. We set

$$X^{-}_{\infty} := (t^{k_1}\mathbf{N}^{+}_{\mathrm{dif},\infty}(D))/\mathbf{D}^{+}_{\mathrm{dif},\infty}(D).$$

Since we have $\mathbf{D}^{+}_{\mathrm{dif},\infty}(D_{\mathrm{rig}}) = L_\infty[[t]](t^{k_1}f_1) \oplus L_\infty[[t]](t^{k_2}f_2)$, one has

$$X^{-}_{\infty} = (L_\infty[[t]]/t^k L_\infty[[t]]) \otimes_L L(t^{k_1}\overline{f}_2) \subseteq \frac{1}{t^k}\widetilde{D}^{+}_{\mathrm{dif}}/\widetilde{D}^{+}_{\mathrm{dif}}.$$

We denote by

$$\mathrm{LP}(\mathbb{Q}_p^\times, X^{-}_{\infty})^{\Gamma}$$

the B-stable L-subspace of $\mathrm{LP}\left(\mathbb{Q}_p^\times, \frac{1}{t^k}\widetilde{D}^{+}_{\mathrm{dif}}/\widetilde{D}^{+}_{\mathrm{dif}}\right)^{\Gamma}$ consisting of functions ϕ with values in X^{-}_{∞}, in other words, consisting of functions

$$\phi : \mathbb{Q}_p^\times \to X^{-}_{\infty} : x \mapsto \sum_{i=0}^{k-1} \phi_i(x)(xt)^{i+k_1} \otimes \overline{f}_2$$

such that, for any $0 \leqq i \leqq k-1$, the function $\phi_i : \mathbb{Q}_p^\times \to L_\infty$ is locally constant with compact support in $\mathbb{Q}_p$ and $\phi_i(ax) = \sigma_a(\phi_i(x))$ for any $a \in \mathbb{Z}_p^\times$ and $x \in \mathbb{Q}_p^\times$. We denote by

$$\mathrm{LP}_{\mathrm{c}}(\mathbb{Q}_p^\times, X^{-}_{\infty})^{\Gamma}$$

the B-stable L-subspace of $\mathrm{LP}(\mathbb{Q}_p^\times, X^{-}_{\infty})^{\Gamma}$ consisting of functions ϕ with compact support in $\mathbb{Q}_p^\times$, i.e. $\phi_i(p^{\pm n}\mathbb{Z}_p^\times) = 0$ for sufficiently large n.

By Corollaire II.2.9 (ii) of [Co10b], one has a canonical B-equivariant topological isomorphism

$$\widetilde{D}/\widetilde{D}^{+} \xrightarrow{\sim} \Pi(D)$$

(under the assumption that D is absolutely irreducible), by which we identify the both sides with each other. We denote by $\Pi(D)^{\mathrm{alg}}$ the G-stable L-subspace of $\Pi(D)$ consisting of locally algebraic vectors, which is non zero due to Théorème VI.6.18 of [Co10b]. By Lemme VI.5.3, Corollaire VI.5.9, and Théorème VI.6.30 of [Co10b], one has

$$\Pi(D)^{\mathrm{alg}} \subseteq \bigcup_{n\geqq 0} \frac{1}{\varphi^n(X)^k}\widetilde{D}^{+}/\widetilde{D}^{+},$$

and the map $z \mapsto \phi_z$ defined above induces a B-equivariant isomorphism

$$\Pi(D)^{\mathrm{alg}} \xrightarrow{\sim} \mathrm{LP}_{\mathrm{c}}(\mathbb{Q}_p^\times, X_\infty^-)^\Gamma \tag{28}$$

under our assumption that D is non-trianguline.

We denote by

$$\mathrm{LC}_{\mathrm{c}}(\mathbb{Q}_p^\times, L_\infty)^\Gamma$$

the L-vector space consisting of locally constant functions $\phi : \mathbb{Q}_p^\times \to L_\infty$ such that the support of ϕ is compact in $\mathbb{Q}_p^\times$ and that $\sigma_a(\phi(x)) = \phi(ax)$ for any $a \in \mathbb{Z}_p^\times$ and $x \in \mathbb{Q}_p^\times$. We similarly define an action of B on this space by the rule

$$\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \cdot \phi\right)(x) := \frac{\delta_D(a)}{a^{k-1}}\phi(x), \ \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \cdot \phi\right)(x) := \phi(ax),$$
$$\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \phi\right)(x) := \psi_\zeta(bx)\phi(x)$$

for $a \in \mathbb{Q}_p^*$ and $b \in \mathbb{Q}_p$, where $\psi_\zeta : \mathbb{Q}_p \to L_\infty^\times$ is the additive character associated to ζ (i.e. we define $\psi_\zeta(a) := \zeta_{p^n}^b \in L_\infty^\times$ for $a = \frac{b}{p^n} \in \mathbb{Q}_p$ with $b \in \mathbb{Z}_p$ and $n \geqq 0$). Let $\mathrm{Sym}^{k-1}L^2$ be the $(k-1)$-th symmetric power of the standard representation L^2 of G. Set $\mathrm{Sym}^{k-1}L^2 \otimes \det^{k_1} := \oplus_{i=0}^{k-1} L e_1^i e_2^{k-1-i}$ on which G acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot e_1^i e_2^{k-1-i} := (ad-bc)^{k_1}(ae_1 + ce_2)^i(be_1 + de_2)^{k-1-i}.$$

Then, one has a canonical (up to the choice of f_2) B-equivariant isomorphism

$$\mathrm{LC}_{\mathrm{c}}(\mathbb{Q}_p^\times, L_\infty)^\Gamma \otimes_L \mathrm{Sym}^{k-1}L^2 \otimes \det{}^{k_1} \xrightarrow{\sim} \mathrm{LP}_{\mathrm{c}}(\mathbb{Q}_p^\times, X_\infty^-)^\Gamma : \tag{29}$$
$$\sum_{i=0}^{k-1} \phi_i \otimes e_1^i e_2^{k-1-i} \mapsto \left[x \mapsto \sum_{i=0}^{k-1}(k-1-i)!\phi_i(x)(xt)^{i+k_1} \otimes \overline{f}_2\right].$$

Therefore, as the composite of isomorphism (28) and the inverse of (29), one obtains a B-equivariant isomorphism

$$\Pi(D)^{\mathrm{alg}} \xrightarrow{\sim} \mathrm{LC}_{\mathrm{c}}(\mathbb{Q}_p^\times, L_\infty)^\Gamma \otimes_L \mathrm{Sym}^{k-1}L^2 \otimes \det{}^{k_1}. \tag{30}$$

Using the map $z \mapsto \phi_z$, we define a $\Gamma \xrightarrow{\sim} \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}$-equivariant map

$$\iota_i^- : \Pi(D)^{\mathrm{alg}} \to X_\infty^- : z \mapsto \phi_z(p^{-i})(= \iota_0(\varphi^{-i}(z)))$$

for each $i \in \mathbb{Z}$.

Set

$$X_n^+ := \mathbf{D}^+_{\mathrm{dif},n}(D_{\mathrm{rig}})/t^{k_2}\mathbf{N}^+_{\mathrm{dif},n}(D_{\mathrm{rig}}) \xrightarrow{\sim} (L_n[[t]]/t^k L_n[[t]]) \otimes_L L(t^{k_1}\overline{f}_1)$$

for each $n \geqq n(D_{\mathrm{rig}})$, and set

$$X^+ \boxtimes \mathbb{Q}_p := \varprojlim_n X_n^+$$

where the transition maps are the maps induced by

$$\begin{aligned}&\frac{1}{p}\mathrm{Tr}_{L_{n+1}/L_n} : L_{n+1}((t))f_i\\ &\qquad \to L_n((t))f_i : \sum_{m\in\mathbb{Z}} a_m t^m f_i \mapsto \sum_{m\in\mathbb{Z}} \frac{1}{p}\mathrm{Tr}_{L_{n+1}/L_n}(a_m)t^m f_i\end{aligned}$$

for $i = 1, 2$. Set $g_p := \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}$. For each $i \in \mathbb{Z}$ and $n \geqq n(D_{\mathrm{rig}})$, define a $\Gamma \xrightarrow{\sim} \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1\end{pmatrix}$-equivariant map

$$\iota_{i,n}^+ : D^\natural \boxtimes_{\delta_D} \mathbf{P}^1 \to X_n^+ : z \mapsto \iota_n(\mathrm{Res}_{\mathbb{Z}_p}(g_p^{n-i}\cdot z)) \in X_n^+,$$

where $\iota_n : D^{(n)}_{\mathrm{rig}} \hookrightarrow \mathbf{D}^+_{\mathrm{dif},n}(D_{\mathrm{rig}})$ is the canonical map (remark that we have $D^\natural \subseteq D_{\mathrm{rig}}$ by Corollaire II.7.2 of [Co10a]), which also induces a Γ-equivariant map

$$\iota_i^+ : D^\natural \boxtimes \mathbf{P}^1 \to X_\infty^+ : z \mapsto (\iota_{i,n}^+(z))_{n\geqq n(D_{\mathrm{rig}})}.$$

Let

$$\langle -, - \rangle : D^* \times D \to \mathcal{E}_L(1)$$

be the canonical $\mathcal{E}_L$-bilinear pairing. Since we have $\mathbf{D}^+_{\mathrm{dif},n}(\mathcal{R}_L(1)) = L_n[[t]]\mathbf{e}_1$, this pairing also induces an $L_n((t))$-bilinear pairing

$$\langle -, - \rangle : \mathbf{D}_{\mathrm{dif},n}(D^*_{\mathrm{rig}}) \times \mathbf{D}_{\mathrm{dif},n}(D_{\mathrm{rig}}) \to L_n((t))\mathbf{e}_1,$$

by which we identify $\mathbf{D}^+_{\mathrm{dif},n}(D^*_{\mathrm{rig}})$ with $\mathrm{Hom}_{L_n[[t]]}(\mathbf{D}^+_{\mathrm{dif},n}(D_{\mathrm{rig}}), L_n[[t]]\mathbf{e}_1)$. Then, using the canonical isomorphism $\mathbf{D}^+_{\mathrm{dif},n}(\det_{\mathcal{R}_L} D_{\mathrm{rig}}) \xrightarrow{\sim} \mathcal{L}_L(D) \otimes_L L_n[[t]]$, we define a canonical isomorphism

(31)
$$\mathbf{D}^+_{\mathrm{dif},n}(D_{\mathrm{rig}})\otimes_L\mathcal{L}_L(D)^\vee\otimes_L L(1) \xrightarrow{\sim} \mathbf{D}^+_{\mathrm{dif},n}(D^*_{\mathrm{rig}}) : x\otimes z\otimes\mathbf{e}_1 \mapsto [y \mapsto z(y\wedge x)\mathbf{e}_1].$$

Using this isomorphism and the fixed basis $\mathbf{e}_D \in \mathcal{L}_L(D)$, we define a pairing

$[-,-]_{\mathrm{dif}} : \mathbf{D}_{\mathrm{dif},n}(D_{\mathrm{rig}}) \times \mathbf{D}_{\mathrm{dif},n}(D_{\mathrm{rig}}) : (x,y) \mapsto \mathrm{res}_L(\langle \sigma_{-1}(x \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1), y\rangle),$

where res_L is the map

$$\mathrm{res}_L : L_n((t))\mathbf{e}_1 \to L : \sum_{m\in\mathbb{Z}} a_m t^m \mathbf{e}_1 \mapsto \frac{1}{[\mathbb{Q}_p(\zeta_{p^n}) : \mathbb{Q}_p]} \mathrm{Tr}_{L_n/L}(a_{-1}).$$

We remark that one has

$$[\sigma_a(x), \sigma_a(y)]_{\mathrm{dif}} = \delta_D(a)[x,y]_{\mathrm{dif}}$$

for any $a \in \mathbb{Z}_p^\times$. This pairing also induces a pairing

$$[-,-]_{\mathrm{dif}} : X_n^+ \times X_n^- \to L,$$

and, by taking limits, one also obtains a pairing

$$[-,-]_{\mathrm{dif}} : X^+ \boxtimes \mathbb{Q}_p \times X_\infty^- \to L.$$

Similarly, using the canonical isomorphism $D \otimes_L \mathcal{L}_L(D)^\vee \otimes_L L(1) \xrightarrow{\sim} D^*$, we define a pairing

$$[-,-] : D \times D \to L : (x,y) \mapsto \mathrm{res}_0(\langle \sigma_{-1}(x \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1), y\rangle)$$

using the residue map

$$\mathrm{res}_0 : \mathcal{E}_L(1) \to L : f(X)\mathbf{e}_1 \mapsto \mathrm{Res}_{X=0}\left(\frac{f(X)}{1+X}\right).$$

This pairing also induces a pairing

$$\begin{aligned} [-,-]_{\mathbf{P}^1} : D \boxtimes_{\delta_D} \mathbf{P}^1 &\times D \boxtimes_{\delta_D} \mathbf{P}^1 \\ &\to L : ((z_1,z_2),(z_1',z_2')) \mapsto [z_1,z_1'] + [\varphi\psi(z_2), \varphi\psi(z_2')], \end{aligned}$$

which satisfies

$$[g\cdot x, g\cdot y]_{\mathbf{P}^1} = \delta_D(\det(g))[x,y]_{\mathbf{P}^1}$$

for any $x,y \in D \boxtimes_{\delta_D} \mathbf{P}^1$ and $g \in G$ by Théorème II.1.13 of [Co10b]. By Théorème II.3.1 of [Co10b], this pairing $[-,-]_{\mathbf{P}^1}$ satisfies that $[x,y]_{\mathbf{P}^1} = 0$ for any $x,y \in D^\natural \boxtimes_{\delta_D} \mathbf{P}^1$ and induces a G-equivariant topological isomorphism

$$D^\natural \boxtimes_{\delta_D} \mathbf{P}^1 \xrightarrow{\sim} \Pi(D)^\vee \otimes_L (\delta_D \circ \det) : x \mapsto [y \in \Pi(D) \mapsto [x,y]_{\mathbf{P}^1}], \tag{32}$$

where we set $\Pi(D)^{\vee} := \mathrm{Hom}_L^{\mathrm{cont}}(\Pi(D), L)$. The following proposition is crucial for the proof of the key proposition

Proposition 3.15. (Proposition VI.5.12.(ii) of [Co10b]) *For any $x \in D^{\natural}\boxtimes_{\delta_D} \mathbf{P}^1$ and $y \in \Pi(D)^{\mathrm{alg}}$, one has*

$$[x,y]_{\mathbf{P}^1} = \sum_{i\in\mathbb{Z}} \delta_D(p^i)[\iota_i^+(x), \iota_i^-(y)]_{\mathrm{dif}}.$$

3.4.5. Explicit formulae of the maps $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Using these preliminaries, we prove two propositions below (Proposition 3.16 and Proposition 3.18) which explicitly describe the maps α and β introduced in (22) and (25) in terms of the pairing $[-,-]_{\mathbf{P}^1}$.

Since D is absolutely irreducible, one has $D^{\natural} = D^{\sharp}$, and then one has a natural P-equivariant isomorphism

$$D^{\natural}\boxtimes_{w_{\delta_D}} \mathbf{P}^1 \xrightarrow{\sim} D^{\natural}\boxtimes \mathbb{Q}_p : z \mapsto \left(\mathrm{Res}_{\mathbb{Z}_p}(g_p^n\cdot z)\right)_{n\geqq 0}.$$

This isomorphism and the inverse of the natural isomorphism

$$D^{\psi=1} = (D^{\sharp})^{\psi=1} = (D^{\natural})^{\psi=1} \xrightarrow{\sim} (D^{\natural}\boxtimes\mathbb{Q}_p)^{g_p=1} : z \mapsto (z_n)_{n\geqq 0},$$

where $z_n := z$ for any n, induce an isomorphism

$$(D^{\natural}\boxtimes_{w_{\delta_D}} \mathbf{P}^1)^{g_p=1} \xrightarrow{\sim} D^{\psi=1} : z \mapsto \mathrm{Res}_{\mathbb{Z}_p}(z).$$

For $x \in D^{\psi=1}$, we denote by $\widetilde{x} \in (D^{\natural}\boxtimes_{w_{\delta_D}} \mathbf{P}^1)^{g_p=1}$ the element which corresponds to x via the last isomorphism.

For each $m\in\mathbb{Z}$, define a function

$$\phi_m \in \mathrm{LP}_c(\mathbb{Q}_p^{\times}, X_{\infty}^-)^{\Gamma}$$

by

$$\phi_m(p^n a) := \begin{cases} \sigma_a(\Omega t^{h_D-1})\overline{f}_2 & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

for $n\in\mathbb{Z}$ and $a\in\mathbb{Z}_p^{\times}$. Since we have $\frac{1}{\Omega t^{h_D}}\mathbf{e}_D = f_1\wedge f_2 \in \mathbf{D}_{\mathrm{dR}}(\det_{\mathcal{E}_L} D)$, we have $\sigma_a(\Omega) = \frac{\det D(\sigma_a)}{a^{h_D}}\Omega = \det W(D)(\sigma_a)\Omega$ for any $a\in\mathbb{Z}_p^{\times}$. Hence, we have

$$\sigma_a\left(\Omega t^{h_D-1}\right)\overline{f}_2 = \det W(D)(\sigma_a)\Omega(at)^{h_D-1}\overline{f}_2.$$

Proposition 3.16. *For any $x \in D^{\psi=1}$ and $m \in \mathbb{Z}$, we have*

$$\alpha(x) = -\frac{p-1}{p}\delta_D(-p^m)[\widetilde{x}, \phi_m]_{\mathbf{P}^1}.$$

Proof. Since $\tilde{x}$ is fixed by g_p, for any $i \in \mathbb{Z}$ and $n \geqq n(D_{\mathrm{rig}})$, we have

$$\iota^+_{i,n}(\widetilde{x}) = \iota_n\left(\mathrm{Res}_{\mathbb{Z}_p}(g_p^{n-i} \cdot \widetilde{x})\right) = \iota_n(\mathrm{Res}_{\mathbb{Z}_p}(\widetilde{x})) = \iota_n(x).$$

Hence, we have

$$\iota^+_i(\widetilde{x}) = (\iota^+_{i,n}(\tilde{x}))_{n \geqq n(D_{\mathrm{rig}})} = (\iota_n(x))_{n \geqq n(D_{\mathrm{rig}})} \in X^+ \boxtimes \mathbb{Q}_p.$$

Then, by Proposition 3.15, we have

$$\begin{aligned}[\widetilde{x}, \phi_m]_{\mathbf{P}^1} &= \sum_{i \in \mathbb{Z}} \delta_D(p^i)[(\iota_n(x))_{n \geqq n(D_{\mathrm{rig}})}, \phi_m(p^{-i})]_{\mathrm{dif}} \\ &= \delta_D(p^{-m})[(\iota_n(x))_{n \geqq n(D_{\mathrm{rig}})}, \Omega t^{h_D - 1} f_2]_{\mathrm{dif}} = \delta_D(p^{-m})[\iota_n(x), \Omega t^{h_D-1} f_2]_{\mathrm{dif}}\end{aligned}$$

for any $n \geqq n(D_{\mathrm{rig}})$.

For an $L[\Gamma]$-module N, we set $\mathrm{H}^1_\gamma(N) := N^\Delta/(\gamma-1)N^\Delta$ using the fixed $\Delta \subseteq \Gamma_{\mathrm{tor}}$ and $\gamma \in \Gamma$ in §2.2.

By Proposition 2.16 of [Na14a], one has a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1_{\varphi,\gamma}(D) & \longrightarrow & \mathrm{H}^1_\gamma(\mathbf{D}_{\mathrm{dif},\infty}(D)) \\ {\scriptstyle \mathrm{id}}\downarrow & & \uparrow{\scriptstyle x \mapsto \log(\chi(\gamma))[x]} \\ \mathrm{H}^1_{\varphi,\gamma}(D) & \xrightarrow{\exp^*} & \mathbf{D}_{\mathrm{dR}}(D), \end{array}$$

where the upper horizontal arrow is the map defined by $[(x,y)] \mapsto [\iota_n(x)]$ for any sufficiently large $n \geqq n(D_{\mathrm{rig}})$ (which is independent of n). We remark that the right vertical arrow is isomorphism since D is de Rham. Hence, we have

$$[\iota_n(\iota_D(x))] = [\log(\chi(\gamma))\alpha(x) f_1] \in \mathrm{H}^1_\gamma(\mathbf{D}_{\mathrm{dif},\infty}(D))$$

for any $n \geqq n(D_{\mathrm{rig}})$ by definition of α. Since we have

$$\iota_n(\iota_D(x)) = \frac{p-1}{p}\log(\chi(\gamma)) p_\Delta \cdot \iota_n(x),$$

we have

$$[\iota_n(x), \Omega t^{h_D-1} f_2]_{\mathrm{dif}} = \frac{p}{p-1}[\alpha(x) f_1, \Omega t^{h_D-1} f_2]_{\mathrm{dif}}$$

$$
\begin{aligned}
&= \frac{p}{p-1}\mathrm{Res}_L(\langle \sigma_{-1}(\alpha(x)f_1\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1), \Omega t^{h_D-1}f_2\rangle)\\
&= \frac{p}{p-1}\delta_D(-1)\alpha(x)\mathrm{Res}_L(\langle f_1\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1, \Omega t^{h_D-1}f_2\rangle)\\
&= -\frac{p}{p-1}\delta_D(-1)\alpha(x),
\end{aligned}
$$

where the first equality follows form Lemma 3.17 below. Hence, we obtain the equality

$$\alpha(x) = -\frac{p-1}{p}\delta_D(-p^m)[\tilde{x}, \phi_m]_{\mathbf{P}^1}. \qquad \square$$

Recall that we put $p_\Delta := \frac{1}{|\Delta|}\sum_{\sigma\in\Delta}[\sigma] \in L[\Delta]$.

Lemma 3.17. *The following hold.*

(i) For $y \in \mathbf{D}_{\mathrm{dif},n}(D)$, we have

$$[y, \Omega t^{h_D-1}f_2]_{\mathrm{dif}} = [p_\Delta\cdot y, \Omega t^{h_D-1}f_2]_{\mathrm{dif}}.$$

(ii) For $y \in \mathbf{D}_{\mathrm{dif},n}(D)$, we have

$$[(\gamma-1)\cdot y, \Omega t^{h_D-1}f_2]_{\mathrm{dif}} = 0.$$

Proof. We first remark that, for any $x, y \in \mathbf{D}_{\mathrm{dif},n}(D_{\mathrm{rig}})$ and $a \in \mathbb{Z}_p^\times$, we have

$$[\sigma_a(x), y]_{\mathrm{dif}} = \delta_D(a)[x, \sigma_a^{-1}(y)]_{\mathrm{dif}}.$$

Using these, for y and y_1 as in (i), we have

$$[p_\Delta\cdot y, \Omega t^{h_D-1}f_2]_{\mathrm{dif}} = [y, p_\Delta^{\delta_D}\cdot\left(\Omega t^{h_D-1}\right)f_2]_{\mathrm{dif}} = [y, \Omega t^{h_D-1}f_2]_{\mathrm{dif}},$$

where we set $p_\Delta^{\delta_D} := \frac{1}{|\Delta|}\sum_{\sigma\in\Delta}\delta_D(\chi(\sigma))[\sigma]^{-1} \in L[\Delta]$, and the third equality follows from the fact that $\sigma_a(\Omega t^{h_D-1}) = \delta_D(a)\Omega t^{h_D-1}$ for any $a \in \mathbb{Z}_p^\times$.

Similarly, we have

$$[(\gamma-1)\cdot y, \Omega t^{h_D-1}f_2]_{\mathrm{dif}} = [y, (\delta_D(\chi(\gamma))\gamma^{-1}-1)\cdot(\Omega t^{h_D-1})f_2]_{\mathrm{dif}} = 0. \qquad \square$$

We next consider the map $\beta : D^{\delta_D(p)\psi=1} \to L$. We first recall that, under the canonical inclusions $1-\varphi : D^{\psi=1} \hookrightarrow D^{\psi=0}$ and $1-\delta_D(p)^{-1}\varphi : D^{\delta_D(p)\psi=1} \hookrightarrow D^{\psi=0}$, one has $w_{\delta_D}(D^{\psi=1}) = D^{\delta_D(p)\psi=1}$ by Proposition V.2.1 of [Co10b].

Similarly, for the case $D^{\psi=1}$, one has the following isomorphism

$$(D^\natural\boxtimes_{w_{\delta_D}}\mathbf{P}^1)^{g_p=\delta_D(p)} \xrightarrow{\sim} D^{\delta_D(p)\psi=1} : z \mapsto \mathrm{Res}_{\mathbb{Z}_p}(z),$$

which induces the commutative diagram (set $g_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G$)

$$\begin{tikzcd}
(D^{\natural} \boxtimes_{w_{\delta_D}} \mathbf{P}^1)^{g_p=1} \arrow[r, "z\mapsto \mathrm{Res}_{\mathbb{Z}_p}(z)"] \arrow[d, "z\mapsto w\cdot z"'] & D^{\psi=1} \arrow[d, "z\mapsto w_{\delta_D}(z)"] \\
(D^{\natural} \boxtimes_{w_{\delta_D}} \mathbf{P}^1)^{g_p=\delta_D(p)} \arrow[r, "z\mapsto \mathrm{Res}_{\mathbb{Z}_p}(z)"] & D^{\delta_D(p)\psi=1}
\end{tikzcd}$$

in which all the arrows are isomorphism. For $x \in D^{\delta_D(p)\psi=1}$, we denote by $\widetilde{x} \in (D^{\natural} \boxtimes_{w_{\delta_D}} \mathbf{P}^1)^{g_p=\delta_D(p)}$ the element such that $\mathrm{Res}_{\mathbb{Z}_p}(\widetilde{x}) = x$.

For each $m \in \mathbb{Z}$, define a function

$$\psi_m \in \mathrm{LP}_{\mathrm{c}}(\mathbb{Q}_p^{\times}, X_{\infty}^{-})^{\Gamma}$$

by

$$\psi_m(p^m a) := \begin{cases} \overline{f}_2 & \text{if } n = m \\ 0 & \text{if } n \neq 0 \end{cases}$$

and for $n \in \mathbb{Z}$ and $a \in \mathbb{Z}_p^{\times}$.

Proposition 3.18. *For any $x \in D^{\delta_D(p)\psi=1}$, $m \in \mathbb{Z}$, we have*

$$\beta(x) = -\frac{p-1}{p}[\widetilde{x}, \psi_m]_{\mathbf{P}^1}.$$

Proof. By Proposition 3.15, we have

$$[\widetilde{x}, \psi_m]_{\mathbf{P}^1} = \sum_{i \in \mathbb{Z}} \delta_D(p^i)[\iota_i^+(\widetilde{x}), \psi_m(p^i)]_{\mathrm{dif}} = \delta_D(p)^m[\iota_m^+(\widetilde{x}), f_2]_{\mathrm{dif}}.$$

Since we have

$$\iota_{m,n}^+(\widetilde{x}) = \iota_n\left(\mathrm{Res}_{\mathbb{Z}_p}(g_p^{n-m} \cdot \widetilde{x})\right) = \delta_D(p)^{n-m}\iota_n(x)$$

for any $n \geqq n(D_{\mathrm{rig}})$, we have

$$[\widetilde{x}, \psi_m]_{\mathbf{P}^1} = \delta_D(p)^n[\iota_n(x), f_2]_{\mathrm{dif}}.$$

On the other hand, since we have

$$\exp^*(\iota_{D^*}(\sigma_{-1}(x \otimes \mathbf{e}_D^{\vee} \otimes \mathbf{e}_1))) = \beta(x) f_1 \otimes f_D^{\vee} \otimes t^{-1}\mathbf{e}_1$$

by definition of β, we obtain

$$[\iota_n(\iota_{D^*}(\sigma_{-1}(x\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1)))] \\ = \log(\chi(\gamma))\beta(x)[f_1\otimes f_D^\vee\otimes t^{-1}\mathbf{e}_1]\in \mathrm{H}^1_\gamma(\mathbf{D}_{\mathrm{dif},\infty}(D^*))$$

by Proposition 2.16 of [Na14a]. Since we have

$$\begin{aligned}\iota_n(\iota_{D^*}(\sigma_{-1}(x\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1))) &= \frac{p-1}{p}\log(\chi(\gamma))p_\Delta\cdot(\iota_n(\sigma_{-1}(x\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1))) \\ &= \frac{p-1}{p}\log(\chi(\gamma))\delta_D(p)^n p_\Delta\cdot(\iota_n(x)\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1),\end{aligned}$$

we obtain

$$\begin{aligned}[\iota_n(x), f_2]_{\mathrm{dif}} &= \delta_D(p)^{-n}\frac{p}{p-1}\beta(x)[\Omega t^{h_D-1}f_1, f_2]_{\mathrm{dif}} \\ &= \delta_D(p)^{-n}\frac{p}{p-1}\beta(x)\mathrm{Res}_L(\langle\sigma_{-1}(f_1\otimes f_D^\vee\otimes t^{-1}\mathbf{e}_1), f_2\rangle) \\ &= \delta_D(p)^{-n}\frac{p}{p-1}\beta(x)\mathrm{Res}_L(\langle f_1\otimes f_D^\vee\otimes t^{-1}\mathbf{e}_1, f_2\rangle) = -\delta_D(p)^{-n}\frac{p}{p-1}\beta(x),\end{aligned}$$

where the first equality follows from Lemma 3.19 below. By this equality, we obtain

$$[\widetilde{x}, \psi_m]_{\mathbf{P}^1} = -\frac{p}{p-1}\beta(x),$$

which proves the proposition. □

Lemma 3.19. *The following hold.*

(i) For $y\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1\in \mathbf{D}_{\mathrm{dif},n}(D^*)$*, set*

$$p_\Delta\cdot(y\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1) = y_1\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1,$$

then we have

$$[y, f_2]_{\mathrm{dif}} = [y_1, f_2]_{\mathrm{dif}}.$$

(ii) For $y\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1\in \mathbf{D}_{\mathrm{dif},n}(D^*)$*, set*

$$(\gamma-1)\cdot(y\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1) = y_2\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1,$$

then we have

$$[y_2, f_2]_{\mathrm{dif}} = 0.$$

Proof. We first remark that we have

$$[x\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1, y\otimes \mathbf{e}_D^\vee\otimes \mathbf{e}_1]_{\mathrm{dif}} = \delta_D(-1)[x, y]_{\mathrm{dif}}$$

and

$$\sigma_a(f_2 \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1) = \delta_{D^*}(a) f_2 \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1$$

for any $a \in \mathbb{Z}_p^\times$

For y, y_1 as in (i), then we have

$$\begin{aligned}[y_1, f_2]_{\mathrm{dif}} &= \delta_D(-1)[y_1 \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1, f_2 \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1]_{\mathrm{dif}} \\ &= \delta_D(-1)[p_\Delta \cdot (y \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1), f_2 \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1]_{\mathrm{dif}} \\ &= \delta_D(-1)[y \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1, p_\Delta^{\delta_{D^*}} \cdot (f_2 \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1)]_{\mathrm{dif}} \\ &= \delta_D(-1)[y \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1, f_2 \otimes \mathbf{e}_D^\vee \otimes \mathbf{e}_1]_{\mathrm{dif}} \\ &= [y, f_2]_{\mathrm{dif}}.\end{aligned}$$

We can also prove (ii) in the same way, hence we omit the proof. □

3.4.6. The compatibility of the p-adic and the classical local Langlands correspondence. We next recall the theorem of Emerton on the compatibility of the p-adic and the classical local Langlands correspondence for the non-trianguline D. Fix an isomorphism $\iota : \overline{L} \xrightarrow{\sim} \mathbb{C}$.

Let $\pi_p'(D)$ be the irreducible smooth admissible representation of G defined over $\mathbb{C}$ corresponding to the absolutely irreducible representation $W(D) \otimes_{L,\iota} \mathbb{C}$ of $W_{\mathbb{Q}_p}$ over $\mathbb{C}$ of rank two via the unitary normalized local Langlands correspondence. We remark that, under this normalization, the local L- and ε-factors attached to $W(D) \otimes_{L,\iota} \mathbb{C}$ coincide with those for $\pi_p'(D)$, and the central character of $\pi_p'(D)$ is equal to

$$\iota \circ \det{}_L W(D) : Z(\xrightarrow{\sim} \mathbb{Q}_p^\times) \to \mathbb{C}^\times,$$

where we regard $\det_L W(D)$ as a character $\det_L W(D) : \mathbb{Q}_p^\times \to L^\times$ via local class field theory.

For our purpose, we need another normalization called Tate's normalization, which we define by

$$\pi_p(D)_{\overline{L}} := (\pi_p'(D) \otimes |\det|_p^{-1/2}) \otimes_{\mathbb{C},\iota^{-1}} \overline{L}.$$

Then, it is known that $\pi_p(D)_{\overline{L}}$ does not depend on the choice of ι, and is defined over L. Then, we denote $\pi_p(D)$ for the model of $\pi_p(D)_{\overline{L}}$ defined over L. Let $\omega_{\pi_p(D)} : \mathbb{Q}_p^\times \to L^\times$ be the central character of $\pi_p(D)$. Since we have $\det_L W(D) = \det_{\mathcal{E}_L} D \cdot x^{-h_D}$, then one has an equality

$$\omega_{\pi_p(D)} = \det{}_L W(D) \cdot |-|_p^{-1} = \delta_D \cdot x^{-(h_D - 1)}, \tag{33}$$

where we set $x^i : \mathbb{Q}_p^\times \to L^\times : y \mapsto y^i$ for $i \in \mathbb{Z}$.

Theorem 3.20. *Under the situation above, there exists a G-equivariant isomorphism*

$$\Pi(D)^{\mathrm{alg}} \xrightarrow{\sim} \pi_p(D) \otimes_L \mathrm{Sym}^{k-1} L^2 \otimes (\det)^{k_1}. \tag{34}$$

Proof. Theorem 3.3.22 of [Em] (for the non-trianguline case). □

Remark 3.21. The proof of Theorem 3.3.22 of [Em] is done by a global method using the complete cohomology of modular curves. No purely local proof of this theorem has been known (at least to the author) until now. As we can easily see from the proof below, this theorem is in fact equivalent to Proposition 3.14. Hence, our proof of Proposition 3.14 given below also depends on the global method.

3.4.7. The Kirillov model of supercuspidal representations. We next recall a formula of the action of $w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G$ on the Kirillov model of the supercuspidal representation $\pi_p(D)$ of G following the book of Bushnell-Henniart [BH].

Under our assumption that D is non-trianguline, $\pi_p(D)$ is a supercuspidal representation of G. By the classical theory of Kirillov model, then there exists a B-equivariant isomorphism

$$\pi_p(D) \xrightarrow{\sim} \mathrm{LC_c}(\mathbb{Q}_p, L_\infty)^\Gamma$$

which is unique up to $L^\times$ (see, for example, VI.4 of [Co10b]). Using this isomorphism, we can uniquely extend the action of B on $\mathrm{LC_c}(\mathbb{Q}_p, L_\infty)^\Gamma$ to that of G such that this isomorphism is G-equivariant, which we denote by $\pi_p(g) \cdot f$ for $g \in G$ and $f \in \mathrm{LC_c}(\mathbb{Q}_p^\times, L_\infty)^\Gamma$.

We now recall a formula on the action of $\pi_p(w)$ $\left(w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ on $\mathrm{LC_c}(\mathbb{Q}_p, L_\infty)^\Gamma$ using the ε-factor associated to $\pi_p(D)$. Decompose $L_\infty = \prod_\tau L_\tau$ into a finite product of fields L_τ. For each τ, fix an isomorphism $\iota_\tau : \overline{L}_\tau \xrightarrow{\sim} \mathbb{C}$. Let

$$\varepsilon(\pi_p(D) \otimes_{L,\iota_\tau} \mathbb{C}, s, \iota_\tau \circ \psi_\zeta) \quad (s \in \mathbb{C})$$

be the ε-factor associated to $\pi_p(D) \otimes_{L,\iota_\tau} \mathbb{C}$ with respect to the additive character $\iota_\tau \circ \psi_\zeta : \mathbb{Q}_p \to \mathbb{C}^\times$. Since $(\pi_p(D) \otimes_{L,\iota_\tau} \mathbb{C}) \otimes |\det|^{1/2}$ corresponds to $W(D) \otimes_{L,\iota_\tau} \mathbb{C}$ via the unitary local Langlands correspondence, we have

$$\varepsilon(\pi_p(D)\otimes_{L,\iota_\tau}\mathbb{C},\frac{1}{2},\iota_\tau\circ\psi_\zeta)=\varepsilon((\pi_p(D)\otimes_{L,\iota_\tau}\mathbb{C})\otimes|\det|^{1/2},0,\iota_\tau\circ\psi_\zeta)$$
$$=\varepsilon(W(D)\otimes_{L,\iota_\tau}\mathbb{C},\iota_\tau\circ\psi_\zeta)=\varepsilon(W(D)\otimes_L\overline{L}_\tau,\psi_\zeta)\otimes 1\in\overline{L}_\tau\otimes_{\overline{L}_\tau,\iota_\tau}\mathbb{C}.$$

Hence, $\varepsilon_L(W(D))=\prod_\tau\varepsilon(W(D)\otimes_L\overline{L}_\tau,\psi_\zeta)\in(L_\infty)^\times$ satisfies the equality

$$\varepsilon_L(W(D))\otimes_{L_\infty,\iota_\tau}1=\varepsilon(\pi_p(D)\otimes_{L,\iota_\tau}\mathbb{C},\frac{1}{2},\iota_\tau\circ\psi_\zeta)\tag{35}$$

for arbitrary τ.

For $m\in\mathbb{Z}$ and a locally constant homomorphism $\eta:\mathbb{Q}_p^\times\to L^\times$, define a locally constant function $\xi_{\eta,m}:\mathbb{Q}_p^\times\to L$ with compact support by

$$\xi_{\eta,m}(x):=\begin{cases}\eta(x) & \text{if } x\in p^m\mathbb{Z}_p^\times\\ 0 & \text{otherwise}\end{cases}.$$

We remark that we have $\alpha_\eta^{-1}\xi_{\eta,k}\in\mathrm{LC}_{\mathrm{c}}(\mathbb{Q}_p^\times,L_\infty)^\Gamma$ if we take a base $\alpha_\eta\mathbf{e}_\eta\in L_\infty(\eta)^\Gamma$ since we have $\sigma_a(\alpha_\eta)=\eta(a)^{-1}\alpha_\eta$ for any $a\in\mathbb{Z}_p^\times$.

Under these preliminaries, we have the following formula.

Theorem 3.22. ([BH] 37.3) *For any locally constant homomorphism $\eta:\mathbb{Q}_p^\times\to L^\times$ and any $m\in\mathbb{Z}$, we have*

$$\pi_p(w)\cdot(\alpha_\eta^{-1}\xi_{\eta,m})=\eta(-1)\alpha_\eta^{-1}\varepsilon_L(W(D)(\eta^{-1}))\xi_{\eta^{-1}w_{\pi_p(D)},-a(W(D)(\eta^{-1}))-m},$$

where $a(W(D)(\eta^{-1}))$ is the exponent of the Artin conductor of $W(D)(\eta^{-1})$ (see [De73] for the definition).

Proof. We first remark that the right hand side in the theorem is contained in $\mathrm{LC}_{\mathrm{c}}(\mathbb{Q}_p^\times,L_\infty)^\Gamma$ since we have

$$\sigma_a(\varepsilon_L(W(D)(\eta^{-1})))=\det{}_LW(D)(a)\eta(a)^{-2}\varepsilon_L(W(D)(\eta^{-1}))$$

for any $a\in\mathbb{Z}_p^\times$. Since we have

$$\varepsilon_L(W(D)(\eta^{-1}))\otimes_{L_\infty,\iota_\tau}1=\varepsilon((\pi_p(D)\otimes(\eta^{-1}\circ\det))\otimes_{L,\iota_\tau}\mathbb{C},\frac{1}{2},\iota_\tau\circ\psi_\zeta)$$

for any τ, the theorem follows from Theorem 37.3 of [BH]. □

Remark 3.23. We will apply this theorem only in the most simple case, i.e. when $\eta=\mathbf{1}$ is the trivial homomorphism (and $\alpha_\eta=1$).

3.4.8. Proof of the key proposition. We recall that one has a canonical B-equivariant isomorphism

$$\Pi(D)^{\mathrm{alg}} \xrightarrow{\sim} \mathrm{LP}_{\mathrm{c}}(\mathbb{Q}_p^\times, X_\infty^-)^\Gamma$$

(under the assumption that D is non-trianguline), by which we extend the action of B on the right hand side to that of G such that this isomorphism becomes G-equivariant. We denote this action by $\Pi(g)\cdot f$ for $g\in G$ and $f\in \mathrm{LP}_{\mathrm{c}}(\mathbb{Q}_p^\times, X_\infty^-)^\Gamma$.

To show the key proposition, we need the following corollary of Theorem 3.22.

Corollary 3.24.

$$\Pi(w)\cdot\psi_m = \Gamma(D)\frac{\varepsilon_L(W(D))}{(w_{\pi_p(D)}(p)p^{h_D-1})^{a(W(D))+m}}\Omega^{-1}\phi_{-a(W(D))-m}$$

Proof. We first remark that, under the B-equivariant isomorphism

$$\begin{aligned}(36)\quad \mathrm{LC}_c(\mathbb{Q}_p^\times, L_\infty)^\Gamma\otimes_L \mathrm{Sym}^{k-1}L^2\otimes(\det)^{k_1} &\xrightarrow{\sim} \mathrm{LP}_c(\mathbb{Q}_p^\times, X_\infty^-)^\Gamma :\\ \phi_i\otimes e_1^i e_2^{k-1-i} &\mapsto [x\mapsto (k-1-i)!\phi_i(x)(xt)^{i+k_1}]\overline{f}_2,\end{aligned}$$

$\psi_m\in \mathrm{LP}_c(\mathbb{Q}_p^\times, X_\infty^-)^\Gamma$ corresponds to

$$\frac{1}{(k_2-1)!}\xi_{\mathbf{1},m}\otimes e_1^{-k_1}e_2^{k_2-1}\in \mathrm{LC}_c(\mathbb{Q}_p^\times, L_\infty)^\Gamma\otimes_L \mathrm{Sym}^{k-1}L^2\otimes(\det)^{k_1}$$

for the trivial homomorphism $\mathbf{1}:\mathbb{Q}_p^\times\to L^\times : a\mapsto 1$. Applying Theorem 3.22 to $\xi_{\mathbf{1},m}$, then we obtain

$$\begin{aligned}\Pi(w)\cdot\psi_m &= \frac{(-1)^{k_1}}{(k_2-1)!}(\pi_p(w)\cdot(\xi_{\mathbf{1},m})\otimes(w\cdot(e_1^{-k_1}e_2^{k_2-1}))\\ &= \frac{(-1)^{k_1}}{(k_2-1)!}\varepsilon_L(W(D))\xi_{w_{\pi_p(D)},-a(W(D))-m}\otimes e_1^{k_2-1}e_2^{-k_1}\\ &= \Gamma(D)\frac{\varepsilon_L(W(D))}{(w_{\pi_p(D)}(p)p^{h_D-1})^{a(W(D))+m}}\Omega^{-1}\phi_{-a(W(D))-m},\end{aligned}$$

where the third equality follows from the fact that ϕ_m corresponds to

$$\frac{(w_{\pi_p(D)}(p)p^{h_D-1})^{-m}}{(-k_1)!}\Omega\xi_{w_{\pi_p(D)},m}\otimes e_1^{k_2-1}e_2^{-k_1}$$

by the isomorphism (36). $\square$

Finally, we prove Proposition 3.14.

Proof. (of Proposition 3.14) Take $x \in D^{\psi=1}$. Take $\widetilde{x} \in (D^{\natural} \boxtimes_{w_{\delta_D}} \mathbf{P}^1)^{g_p=1}$ such that $\mathrm{Res}_{\mathbb{Z}_p}(\widetilde{x}) = x$. Then, $w \cdot \tilde{x} \in (D^{\natural} \boxtimes_{w_{\delta_D}} \mathbf{P}^1)^{g_p=\delta_D(p)}$ satisfies that

$$\mathrm{Res}_{\mathbb{Z}_p}(w \cdot \widetilde{x}) = w_{\delta_D}(x) \in D^{\delta_D(p)\psi=1}.$$

By Proposition 3.16 for $m = -a(W(D))$, Proposition 3.18 for $m = 0$ and Corollary 3.24, then we have

$$\begin{aligned}\beta(w_{\delta_D}(x)) = -\frac{p-1}{p}[w \cdot \tilde{x}, \psi_0]_{\mathbf{P}^1} = -\frac{p-1}{p}\delta_D(\det(w))[\tilde{x}, \Pi(w) \cdot \psi_0]_{\mathbf{P}^1}\\ = -\frac{(p-1)}{p}\delta_D(-1)\Gamma(D)\frac{\varepsilon_L(W(D))}{(w_{\pi_p(D)}(p)p^{(h_D-1)})^{a(W(D))}}\Omega^{-1}[\tilde{x}, \phi_{-a(W(D))}]_{\mathbf{P}^1}\\ = \Gamma(D)\frac{\varepsilon_L(W(D))}{(w_{\pi_p(D)}(p)p^{(h_D-1)})^{a(W(D))}}\Omega^{-1}\delta_D(p)^{a(W(D))}\alpha(x)\\ = \Gamma(D)\varepsilon_L(W(D))\Omega^{-1}\alpha(x),\end{aligned}$$

where the last equality follows from the equality $\delta_D = w_{\pi_p(D)}x^{h_D-1}$. $\square$

4. A functional equation of Kato's Euler system

Throughout this section, we fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and fix an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$ such that $\iota \circ \iota_\infty = \iota_p$. Using this isomorphism, we identify $\Gamma(\mathbb{C}, \mathbb{Z}_l(1)) \xrightarrow{\sim} \Gamma(\overline{\mathbb{Q}}_p, \mathbb{Z}_l(1)) =: \mathbb{Z}_l(1)$, and set $\zeta^{(l)} := \{\iota(\exp(\frac{2\pi i}{l^n}))\}_{n \geqq 1} \in \mathbb{Z}_l(1)$ for each prime l. Let S be a finite set of primes containing p. Let $\mathbb{Q}_S(\subseteq \overline{\mathbb{Q}})$ be the maximal Galois extension of $\mathbb{Q}$ which is unramified outside $S \cup \{\infty\}$, and set $G_{\mathbb{Q},S} := \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. Set $c \in G_{\mathbb{Q},S}$ be the restriction by ι_∞ of the complex conjugation. For each $\mathbb{Z}[G_{\mathbb{R}}]$-module M and $k \in \mathbb{Z}$, we define a canonical $G_{\mathbb{R}}$-equivariant map $M(k) := M \otimes_{\mathbb{Z}} \mathbb{Z}(2\pi i)^k \to M_{\mathbb{Z}_p}(k) := (M \otimes_{\mathbb{Z}} \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k)$ by $x \otimes (2\pi i)^k \mapsto x \otimes (\zeta^{(p)})^{\otimes k}$ using the basis $\zeta^{(p)} \in \mathbb{Z}_p(1)$. We set $M^{\pm} := M^{c=\pm 1}$.

4.1. The global fundamental lines and its compatibility with the Poitou-Tate duality

In this subsection, we recall, for global Galois representations, the definition of the global fundamental lines and its compatibility with the Poitou-Tate duality, which we need to formulate our second main theorem.

4.1.1. The global fundamental lines. Let T be an R-representation of $G_{\mathbb{Q},S}$. We set

$$\mathrm{H}^i(\mathbb{Z}[1/S],T) := \mathrm{H}^i(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q},S},T))$$

for $i \geqq 0$. For each $l \in S$ and $i = 1, 2$, we set

$$\Delta_{R,i,(l)}(T) := \Delta_{R,i}(T|_{G_{\mathbb{Q}_l}}) \text{ and } \Delta_{R,(l)}(T) := \Delta_R(T|_{G_{\mathbb{Q}_l}})$$

which are defined in §2.1. Set $c_T := \mathrm{rank}_R T(-1)^+$ (remark that $T(-1)^+$ is a finite projective R-module since we have assumed that $2 \in R^\times$). We set

$$\Delta_{R,1,S}(T) := \mathrm{Det}_R(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q},S},T))^{-1}, \quad \Delta_{R,2,S}(T) := (\det_R(T(-1)^+), c_T)^{-1}$$

and

$$\Delta_{R,S}(T) := \Delta_{R,1,S}(T) \boxtimes \Delta_{R,2,S}(T).$$

We remark that $\Delta_{R,S}(T)$ is a graded invertible R-module of degree zero by the global Euler-Poincaré characteristic formula.

4.1.2. Compatibility with the Poitou-Tate duality. We next recall the definition of the isomorphism

$$\Delta_{R,S}(T^*) \xrightarrow{\sim} \boxtimes_{l\in S}\Delta_{R,(l)}(T) \boxtimes \Delta_{R,S}(T) \tag{37}$$

induced by the Poitou-Tate duality.

By the Poitou-Tate duality, one has a canonical quasi-isomorphism

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_R(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q},S},T^*),R)[-2]\\ &\quad\xrightarrow{\sim} \mathrm{Cone}(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q},S},T) \to \oplus_{l\in S}C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q}_l},T|_{G_{\mathbb{Q}_l}}))[-1],\end{aligned}$$

from which we obtain a canonical isomorphism

$$\begin{aligned}(\Delta_{R,1,S}(T^*)^{-1})^\vee &\xrightarrow{\sim} \mathrm{Det}_R(\mathbf{R}\mathrm{Hom}_R(C^\bullet_{\mathrm{cont}}(G_{\mathbb{Q},S},T^*),R))\\ &\xrightarrow{\sim} \boxtimes_{l\in S}\Delta_{R,1,(l)}(T) \boxtimes \Delta_{R,1,S}(T).\end{aligned} \tag{38}$$

We next define the following isomorphism

(39)

$$(\Delta_{R,2,S}(T^*)^{-1})^\vee \xrightarrow{\sim} (\det_R T, r_T) \boxtimes \Delta_{R,2,S}(T) \xrightarrow{\sim} \boxtimes_{l\in S}\Delta_{R,2,(l)}(T) \boxtimes \Delta_{R,2,S}(T),$$

where the first isomorphism is naturally induced by the isomorphism

$$T^+ \oplus T(-1)^+ \xrightarrow{\sim} T : (x,y) \mapsto 2x + \frac{1}{2}y \otimes \zeta^{(p)}$$

and the canonical isomorphism

$$T^+ \xrightarrow{\sim} ((T^\vee)^\vee)^+ = (T^*(-1)^\vee)^+ \xrightarrow{\sim} (T^*(-1)^+)^\vee$$

(where the last map is defined by $f \mapsto f|_{T^*(-1)^+}$), and the second isomorphism is induced by the isomorphism

$$(\det_R T, r_T) \xrightarrow{\sim} \boxtimes_{l\in S}\Delta_{R,2,(l)}(T)$$

defined by (the inverse of) the isomorphism

$$\otimes_{l\in S} R_{a_l(T)} \otimes \det_R T \xrightarrow{\sim} \det_R T : (\otimes_{l\in S} x_l)\otimes y \mapsto \left(\prod_{l\in S} x_l\right)\cdot y$$

for $x_l \in R_{a_l(T)}$ and $y \in \det_R T$ (remark that one has $\otimes_{l\in S} R_{a_l(T)} = R$ since one has $\prod_{l\in S} a_l(T) = 1$ by the global class field theory).

Finally, the isomorphism (37) is defined as the product of the isomorphisms (38) and (39) (remark that one has $(\Delta_{R,S}(T^*)^{-1})^\vee = \Delta_{R,S}(T^*)$ since $\Delta_{R,S}(T^*)$ is of degree zero).

4.2. Statement of the main theorem on the global ε-conjecture

4.2.1. Setting. Let $k, N \geqq 1$ be positive integers. Let $f(\tau) = \sum_{n=1}^{\infty} a_n(f)q^n \in S_{k+1}(\Gamma_1(N))^{\mathrm{new}}$ be a normalized Hecke eigen new form of level N, weight $k+1$, where $\tau \in \mathbb{C}$ such that $\mathrm{Im}(\tau) > 0$, $q := \exp(2\pi i\tau)$ and

$$\Gamma_1(N) := \left\{ g \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

Set $f^*(\tau) := \sum_{n=1}^{\infty} \overline{a_n(f)} q^n$ ($\overline{a_n(f)}$ is the complex conjugation of $a_n(f)$), which is also a Hecke eigen new form in $S_{k+1}(\Gamma_1(N))^{\mathrm{new}}$ by the theory of new forms.

For each homomorphism $\delta : \mathbb{Z}_p^\times \to \mathbb{C}^\times$ with finite image (which we naturally regard as a Dirichlet character $\delta : (\mathbb{Z}/p^{n(\delta)})^\times \to \mathbb{C}^\times$ ($n(\delta)$ is the conductor of δ), or a Hecke character $\delta : \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$), set

$$L(f,\delta,s) := \sum_{n\geqq 1} \frac{a_n(f)\delta(n)}{n^s} \quad \text{and} \quad L_{\{p\}}(f,\delta,s) := \sum_{n\geqq 1,(n,p)=1} \frac{a_n(f)\delta(n)}{n^s}.$$

These functions absolutely converge when $\mathrm{Re}(s) > \frac{k}{2} + 1$. The L-function $L(f,\delta,s)$ is analytically continued to the whole $\mathbb{C}$, and, if we denote by

$\pi_f = \otimes'_{v:\text{place of } \mathbb{Q}} \pi_{f,v}$ the automorphic cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to f, then it satisfies the following functional equation

$$\text{(40)}\quad \Gamma_{\mathbb{C}}(s)L(f,\delta,s) = \varepsilon(f,\delta,s)\Gamma_{\mathbb{C}}(k+1-s)L(f^*,\delta^{-1},k+1-s) \quad (s\in\mathbb{C}),$$

where we set $\Gamma_{\mathbb{C}}(s) := \frac{\Gamma(s)}{(2\pi)^s}$ and $\varepsilon(f,\delta,s)$ is the global ε-factor associated to $\pi_f \otimes (\delta\circ\det)$, which is defined as the product of the local ε-factors

$$\varepsilon(f,\delta,s) = \varepsilon_\infty(f,\delta,s)\prod_{l\in S}\varepsilon_l(f,\delta,s),$$

where, for $v \in S\cup\{\infty\}$, $\varepsilon_v(f,\delta,s)$ is the local ε-factor associated to the v-th component $\pi_{f,v}\otimes(\delta\circ\det)$ with respect to the additive character $\psi_v : \mathbb{Q}_v \to \mathbb{C}^\times$ and the Haar measure dx_v on $\mathbb{Q}_v$ which are uniquely characterized by $\psi_\infty(a) := \exp(-2\pi i\cdot a)$ $(a\in\mathbb{R})$, $\psi_l(\frac{1}{l^n}) = \exp(\frac{2\pi i}{l^n})$ $(n\in\mathbb{Z})$, $\int_{\mathbb{Z}_l} dx_l = 1$ and dx_∞ is the standard Lebesgue measure on $\mathbb{R}$. We remark that one has

$$\text{(41)}\qquad \varepsilon_\infty(f,\delta,s) = i^{k+1}.$$

Set $F := \mathbb{Q}(\{\iota_\infty^{-1}(a_n(f))\}_{n\geqq 1}) \subseteq \overline{\mathbb{Q}}$, $L := \mathbb{Q}_p(\{\iota_p(\iota_\infty^{-1}(a_n(f)))\}_{n\geqq 1}) \subseteq \overline{\mathbb{Q}}_p$ and $S := \{l|N\}\cup\{p\}$. Let denote by $\mathcal{O}_F$, $\mathcal{O} := \mathcal{O}_L$ the rings of integers of F, L respectively. For $f_0 = f, f^*$, let T_{f_0} be the $\mathcal{O}$-representation of $G_{\mathbb{Q},S}$ of rank two associated to f_0 which is obtained as a quotient of the étale cohomology (with coefficients) of a modular curve (this is denoted by $V_{\mathcal{O}_\lambda}(f_0)$ in § 8.3 of [Ka04]). Set $V_{f_0} = T_{f_0}[1/p]$. By the Poincaré duality of the étale cohomology of a modular curve, one has a canonical $G_{\mathbb{Q},S}$-equivariant isomorphism

$$V_{f^*}(1) \xrightarrow{\sim} (V_f(k))^*,$$

which induces a canonical isomorphism $\Delta^{\mathrm{Iw}}_{L,S}(V_{f^*}(1)) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{L,S}((V_f(k))^*)$. Since the sub $\Lambda := \Lambda_{\mathcal{O}}$-module $\Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T)$ of $\Delta^{\mathrm{Iw}}_{L,S}(V)$ is independent of the choice of $G_{\mathbb{Q},S}$-stable lattice T of V for any L-representation V of $G_{\mathbb{Q},S}$ (because $\Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T)$ is of grade zero), the latter also induces a canonical isomorphism

$$\Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f^*}(1)) \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},S}((T_f(k))^*).$$

Therefore, we obtain a canonical isomorphism

$$\text{(42)}\qquad \Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f^*}(1))^\iota \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},S}((T_f(k))^*)^\iota \xrightarrow{\sim} \Delta^S_\Lambda(\mathbf{Dfm}(T_f(k))^*),$$

where the second isomorphism is defined in the same way as in the last part of §2.1.

We denote by $Q(\Lambda)$ the total fraction ring of Λ. For a Λ-module or a graded invertible Λ-module M, we set

$$M_Q := M \otimes_\Lambda Q(\Lambda)$$

to simplify the notation.

4.2.2. Statement of the main theorem. Using (the p-th layer of) the Kato's Euler system associated to f_0, we define below a candidate of the zeta-isomorphism

$$\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r)) : \mathbf{1}_{Q(\Lambda)} \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r))_Q$$

for $f_0 = f, f^*$ and $r \in \mathbb{Z}$.

Before defining this isomorphism, we propose the following conjecture, and state the second main theorem of this article concerning the global ε-conjecture, whose proof is given in the next subsection.

Conjecture 4.1. *One has the equality*

$$\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f^*}(1))^\iota = \boxtimes_{l\in S}\left(\varepsilon^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k)) \otimes \mathrm{id}_{Q(\Lambda)}\right) \boxtimes \widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_f(k))$$

under the isomorphism obtained by the base change to $Q(\Lambda)$ of the canonical isomorphism

$$\Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f^*}(1))^\iota \xrightarrow{\sim} \boxtimes_{l\in S}\Delta^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k)) \boxtimes \Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_f(k))$$

defined by (37) for $(R,T) = (\Lambda, \mathbf{Dfm}(T_f(k)))$ and (42), where the isomorphism

$$\varepsilon^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k)) := \varepsilon^{\mathrm{Iw}}_{\mathcal{O},\zeta^{(l)}}(T_f(k)|_{G_{\mathbb{Q}_l}}) : \mathbf{1}_\Lambda \xrightarrow{\sim} \Delta^{\mathrm{Iw},(l)}_{\mathcal{O}}(T_f(k))$$

is the local ε-isomorphism defined by Theorem 3.1 (resp. [Ya09]) for $l = p$ (resp.$l \neq p$) for the pair $(\Lambda, \mathbf{Dfm}(T_f(k)|_{G_{\mathbb{Q}_l}}))$.

Theorem 4.2. *Assume that $V_f|_{G_{\mathbb{Q}_p}}$ is non-trianguline. Then, the conjecture 4.1 is true.*

Remark 4.3. Assuming Conjecture 1.5 which state that the isomorphism $\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_f(k))$ comes from the conjectural zeta isomorphism for $\mathbf{Dfm}(T_f(k))$ defined over Λ, then Conjecture 4.1 is equivalent to the global ε-conjecture in [Ka93b] and [FK06] for $\mathbf{Dfm}(T_f(k))$.

4.2.3. Definition of the zeta isomorphism. In the rest of this subsection, we define our zeta isomorphism $\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_{f_0}(r))$ using the p-th layer of the

Kato's Euler system which we recall now. Since the definitions for f and f^* are the same, we only define it for $f_0 = f$.

For an $\mathcal{O}$-representation T of $G_{\mathbb{Q},S}$ (which we also regard as a smooth $\mathcal{O}$-sheaf on the étale site over $\mathrm{Spec}(\mathbb{Z}[1/S])$), we define a Λ-module

$$\mathbf{H}^i(T) := \mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Z}[1/p], T) := \varprojlim_{n \geqq 0} \mathrm{H}^i(\mathbb{Z}[1/p, \zeta_{p^n}], T)$$

for $i \geqq 0$, where we define for $n \geqq 0$

$$\mathrm{H}^i(\mathbb{Z}[1/p, \zeta_{p^n}], T) := \mathrm{H}^i_{\text{ét}}(\mathrm{Spec}(\mathbb{Z}[1/p, \zeta_{p^n}]), (j_n)_*(T|_{\mathrm{Spec}(\mathbb{Z}[1/S, \zeta_{p^n}])}))$$

for the canonical inclusion $j_n : \mathrm{Spec}(\mathbb{Z}[1/S, \zeta_{p^n}]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[1/p, \zeta_{p^n}])$. For $V = T[1/p]$, we set $\mathbf{H}^i(V) := \mathbf{H}^i(T)[1/p]$.

For the eigen form f, Kato defined in Theorem 12.5 [Ka04] an L-linear map

$$V_f \to \mathbf{H}^1(V_f) : \gamma \mapsto \mathbf{z}^{(p)}_{\gamma}(f)$$

which interpolates the critical values of the L-functions $L(f^*, \delta, s)$ for all δ, whose precise meaning we explain in the next subsection. By Theorem 12.4 of [Ka04], $\mathbf{H}^1(T_f)$ is torsion free over Λ, and $\mathbf{H}^1(V_f)$ is a free $\Lambda_L := \Lambda[1/p]$-module of rank one, and $\mathbf{H}^2(T_f)$ is a torsion Λ-module (and $\mathbf{H}^i(T_f) = 0$ for $i \neq 1, 2$). The restriction map $\mathbf{H}^i(T_f) \to \mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Z}[1/S], T_f)$ induces an isomorphism

$$\mathbf{H}^1(T_f) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Z}[1/S], T_f)$$

and an exact sequence

$$0 \to \mathbf{H}^2(T_f) \to \mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/S], T_f) \to \bigoplus_{l \in S \setminus \{p\}} \mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Q}_l, T_f) \to 0,$$

which follow from (for example) the proof of Lemma 8.5 of [Ka04]. Since $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Q}_l, T)$ is a torsion Λ-module for any l by Proposition A.2.3 of [Pe95], $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/S], T_f)$ is also a torsion Λ-module by the above exact sequence.

By these facts, we obtain a canonical $Q(\Lambda)$-linear isomorphism

$$\Delta^{\mathrm{Iw}}_{\mathcal{O},1,S}(T_f(r))_Q \xrightarrow{\sim} (\mathbf{H}^1(T_f(r))_Q, 1) \tag{43}$$

for $r = 0$. For general $r \in \mathbb{Z}$, we also define the isomorphism above induced by that for $r = 0$ using the canonical (not Λ-linear) isomorphism

$$\mathbf{H}^i(T_f) \xrightarrow{\sim} \mathbf{H}^i(T_f(r)) : \mathbf{z} \mapsto \mathbf{z}(r)$$

which is induced by the isomorphism

$$T_f \otimes_{\mathcal{O}} \Lambda \xrightarrow{\sim} T_f(r) \otimes_{\mathcal{O}} \Lambda : x \otimes y \mapsto (x \otimes \mathbf{e}_r) \otimes g_{\chi^r}(y)$$

defined in the same way as in the proof of Lemma 2.9.

For each $l \in S \setminus \{p\}$ and $r \in \mathbb{Z}$, we set

$$L^{(l)}_{\mathrm{Iw}}(T_f(r)) := \det_\Lambda(1 - \varphi_l | \mathbf{Dfm}(T_f(r))^{I_l}) = 1 - a_l(f) l^{-r}[\sigma_l] \in \Lambda \cap Q(\Lambda)^\times$$

(remark that $\mathbf{Dfm}(T_f(r))^{I_l} = T_f^{I_l}(r) \otimes_{\mathcal{O}} \Lambda$ is free over Λ), where the second equality follows from the global-local compatibility of the Langlands correspondence proved by [La73], [Ca86].

Denote the sign of $(-1)^r$ by $\mathrm{sgn}(r) \in \{\pm\}$. Set $\Lambda^\pm := \{\lambda \in \Lambda | [\sigma_{-1}] \cdot \lambda = \pm\lambda\}$.

Using these, we define an isomorphism

$$\widetilde{z}^{\mathrm{Iw}}_{\mathcal{O},S}(T_f(r)) : \mathbf{1}_{Q(\Lambda)} \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},S}(T_f(r))_Q$$

which corresponds to the isomorphism

$$\Theta_r(f) : \Delta^{\mathrm{Iw}}_{\mathcal{O},2,S}(T_f(r))_Q^{-1} \xrightarrow{\sim} (\mathbf{H}^1(T_f(r))_Q, 1)$$

defined as the base change to $Q(\Lambda)$ of the Λ-linear morphism

(44)
$$\begin{aligned}
\Theta_r(f) &: (\mathbf{Dfm}(T_f(r))(-1))^+ \\
&= T_f^{\mathrm{sgn}(r-1)}(r-1) \otimes_{\mathcal{O}} \Lambda^+ \oplus T_f^{\mathrm{sgn}(r)}(r-1) \otimes_{\mathcal{O}} \Lambda^- \\
&\to \mathbf{H}^1(V_f(r)) : (\gamma \otimes \mathbf{e}_{r-1} \otimes \lambda^+, \gamma' \otimes \mathbf{e}_{r-1} \otimes \lambda^-) \\
&\mapsto \prod_{l \in S \setminus \{p\}} L^{(l)}_{\mathrm{Iw}}(T_{f^*}(1+k-r))^\iota (\lambda^+ \cdot (\mathbf{z}^{(p)}_\gamma(f)(r)) + \lambda^- \cdot (\mathbf{z}^{(p)}_{\gamma'}(f)(r))),
\end{aligned}$$

where we set $\lambda^\iota := \iota(\lambda)$ for $\lambda \in Q(\Lambda)$, and the fact that the base change to $Q(\Lambda)$ of this morphism is isomorphism follows from Theorem 12.5 of [Ka04].

4.3. Proof of Theorem 4.2

In this subsection, we give a proof of Theorem 4.2. We first precisely recall the interpolation property of the Kato's Euler system which is so called the explicit reciprocity law (Theorem 12.5 (1) of [Ka04]), which is crucial in our proof of the theorem.

4.3.1. Kato's explicit reciprocity law. Using the comparison between the Betti and the étale cohomologies, one has a canonical $\mathcal{O}_F$-lattice $T_{f,\mathcal{O}_F}$ of T_f which is stable by the action of $G_{\mathbb{R}}(\subseteq G_{\mathbb{Q},S})$. Set $V_{f,F} := T_{f,\mathcal{O}_F} \otimes_{\mathcal{O}_F} F$, which is a $G_{\mathbb{R}}$-stable F-lattice of V_f. Using the comparison theorem in the p-adic Hodge theory, one has a canonical F-lattice $S(f) = Ff$ of $\mathbf{D}^1_{\mathrm{dR}}(V_f) = \mathbf{D}^k_{\mathrm{dR}}(V_f)$.

By the theory of Eichler-Shimura, one has a canonical F-linear map

$$\mathrm{per}_f : S(f) \hookrightarrow V_{f,\mathbb{C}} := V_{f,F} \otimes_{F,\iota_\infty} \mathbb{C}.$$

For each pair (r,δ) such that $0 \leqq r \leqq k-1$ and $\delta : \Gamma \to \overline{\mathbb{Q}}^\times$ a homomorphism with finite image (which we regard as a homomorphism with values in $\mathbb{C}^\times$ or $\overline{\mathbb{Q}}_p^\times$ by the fixed embeddings ι_∞ or ι_p), we set $V_f(k-r)(\delta) := V_f(k-r) \otimes_L \overline{\mathbb{Q}}_p(\delta)$, and define an L-linear map

$$\mathbf{H}^1(V_f) \to \mathbf{D}^0_{\mathrm{dR}}(V_f(k-r)(\delta)) = L(f \otimes \frac{1}{t^{k-r}}\mathbf{e}_{k-r}) \otimes_L (\overline{\mathbb{Q}}_{p,\infty}(\delta))^\Gamma \tag{45}$$

as the composites of the following morphisms

$$\mathbf{H}^1(V_f) \xrightarrow{\mathrm{can}} \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f) \xrightarrow{\mathrm{sp}_{\chi^{k-r}\delta}} \mathrm{H}^1(\mathbb{Q}_p, V_f(k-r)(\delta)) \xrightarrow{\exp^*} \mathbf{D}^0_{\mathrm{dR}}(V_f(k-r)(\delta)),$$

For $\gamma \in V_{f,F}$, we decompose $\gamma = \gamma^+ + \gamma^-$ such that $\gamma^\pm \in V^\pm_{f,F}$. For each (r,δ) as above, we denote by $\mathrm{sgn}(r,\delta) \in \{\pm\}$ the sign of $\delta(-1)(-1)^r$.

Under these preliminaries, the interpolation property of $\mathbf{z}^{(p)}_\gamma(f)$ can be described as follows. By Theorem 12.5 (1) of [Ka04], the image of $\mathbf{z}^{(p)}_\gamma(f)$ by the map (45) is contained in the sub F-vector space

$$F(f \otimes \frac{1}{t^{k-r}}\mathbf{e}_{k-r}) \otimes_{\mathbb{Q}} (\overline{\mathbb{Q}}_\infty(\delta))^\Gamma$$

of $\mathbf{D}^0_{\mathrm{dR}}(V_f(k-r)(\delta))$, and is sent to

$$(2\pi i)^{k-r-1} L_{\{p\}}(f^*, \delta^{-1}, r+1)\gamma^{\mathrm{sgn}(k-r-1,\delta)} \in V^{\mathrm{sgn}(k-r-1,\delta)}_{f,\mathbb{C}}. \tag{46}$$

by the injection map defined by the following composite

$$\mathrm{per}^{(k-r,\delta)}_f : F(f \otimes \frac{1}{t^{k-r}}\mathbf{e}_{k-r}) \otimes_F (\overline{\mathbb{Q}}_\infty(\delta))^\Gamma \to V_{f,\mathbb{C}} \to V^{\mathrm{sgn}(k-r-1,\delta)}_{f,\mathbb{C}}$$

where the first map is defined by

$$(f \otimes \frac{1}{t^{k-r}}\mathbf{e}_{k-r}) \otimes (\sum_{i\in I} b_i \otimes c_i\mathbf{e}_\delta) \mapsto \iota_\infty(\sum_{i\in I} b_ic_i)\mathrm{per}_f(f)$$

for $b_i \in \overline{\mathbb{Q}}, c_i \in \mathbb{Q}(\zeta_{p^\infty})(\subseteq \overline{\mathbb{Q}})$, and the second map is the canonical projection $V_{f,\mathbb{C}} \to V^{\pm}_{f,\mathbb{C}} : x \mapsto \frac{1}{2}(x \pm c(x))$.

4.3.2. Comparison of $\mathbf{z}^{(p)}_\gamma(f)(k)$ with $\mathbf{z}^{(p)}_{\gamma'}(f^*)(1)$. We next reduce Theorem 4.2 to the following Theorem 4.6 below concerning the equality of the two elements in $D_{f^*}(1)^{\psi=1}$ respectively defined by using $\mathbf{z}^{(p)}_\gamma(f)(k)$ and $\mathbf{z}^{(p)}_{\gamma'}(f^*)(1)$. We denote by D_{f_0} the étale (φ,Γ)-module over $\mathcal{E}_L$ associated to $V_{f_0}|_{G_{\mathbb{Q}_p}}$ for $f_0 = f, f^*$. By the following canonical morphisms (for $r \in \mathbb{Z}$)

$$\mathbf{H}^1(V_{f_0}(r)) \xrightarrow{\mathrm{can}} \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_{f_0}(r)) \xrightarrow{\sim} D_{f_0}(r)^{\psi=1} \xrightarrow{1-\varphi} D_{f_0}(r)^{\psi=0},$$

we freely regard $\mathbf{z}^{(p)}_\gamma(f_0)(r)$ as an element in these modules.

Fix an $\mathcal{O}_F$-basis $\gamma^\pm$ of $T^\pm_{f,\mathcal{O}_F}$ for each $\pm$, and set

$$\gamma := \gamma^+ + \gamma^- \in T_{f,\mathcal{O}_F} \text{ and } \mathbf{f}_\gamma := (\gamma^{\mathrm{sgn}(k)}\mathbf{e}_k) \wedge (\gamma^{\mathrm{sgn}(k-1)}\mathbf{e}_k) \in \det_{\mathcal{O}_F} T_{f,\mathcal{O}_F}(k).$$

We take the basis $\gamma^\pm_*$ of $V^\pm_{f^*,F}$ such that the ordered pair $\{\gamma^+_*, \gamma^-_*\}$ is the dual basis of $\{\gamma^{\mathrm{sgn}(k)}\mathbf{e}_k, \gamma^{\mathrm{sgn}(k-1)}\mathbf{e}_k\}$ under the canonical F-bilinear perfect pairing

$$V_{f^*,F} \times V_{f,F}(k) \to F$$

induced by the Poincaré duality. We also set

$$\gamma_* := \gamma^+_* + \gamma^-_* \in V_{f^*,F}.$$

For each $l \in S \setminus \{p\}$, set

$$\varepsilon_{0,(l)}(T_f(k)) := \varepsilon_{0,\mathcal{O}}(T_f(k)|_{G_{\mathbb{Q}_l}}, \zeta^{(l)}) \in (\mathcal{O})^\times_{a_l(T_f(k))}$$

the ε_0-constant associated to the triple $(\mathcal{O}, T_f(k)|_{G_{\mathbb{Q}_l}}, \zeta^{(l)})$ defined in Remark 2.4. Using the canonical isomorphism

$$\otimes_{l\in S}(\mathcal{O})_{a_l(T_f(k))} \xrightarrow{\sim} \mathcal{O} : \otimes_{l\in S} x_l \mapsto \prod_{l\in S} x_l,$$

we set

$$\varepsilon_0 := \otimes_{l\in S\setminus\{p\}}\varepsilon_{0,(l)}(T_f(k))^{-1} \in (\otimes_{l\neq p\in S}(\mathcal{O})_{a_l(T_f(k))^{-1}})^{\times} = (\mathcal{O})^{\times}_{a_p(T_f(k))}.$$

Using this, we also set

$$\mathbf{e}_\gamma := \mathbf{f}_\gamma \otimes \varepsilon_0 \in \mathcal{L}_{\mathcal{O}}(T_f(k)) = \det_{\mathcal{O}} T_f(k) \otimes_{\mathcal{O}} (\mathcal{O})_{a_p(T_f(k))}.$$

For $l \in S$ and an L-representation V of $G_{\mathbb{Q},S}$ such that $V|_{G_{\mathbb{Q}_p}}$ is de Rham, we denote by $a^{(l)}(V)$ the exponent of the Artin conductor of $W(V|_{G_{\mathbb{Q}_l}})$ defined in 8.12 of [De73], and set $L^{(l)}(V) := \det_L(1-\varphi_l|\mathbf{D}_{\mathrm{cris}}(V|_{G_{\mathbb{Q}_l}})) \in L$ and $\varepsilon_{L,(l)}(V) := \varepsilon_L(W(V|_{G_{\mathbb{Q}_l}}), \zeta^{(l)}) \in L_\infty$, which we also regard as elements in $\mathbb{C}$ by the injection $L \subseteq \overline{\mathbb{Q}}_p \xrightarrow{\iota^{-1}} \mathbb{C}$ and the projection $L_\infty = L\otimes_{\mathbb{Q}}\mathbb{Q}(\zeta_{p^\infty}) \to \mathbb{C} : a\otimes b \mapsto \iota^{-1}(a)\iota_\infty(b)$.

Conjecture 4.4. *One has the equality*

$$\prod_{l\in S\setminus\{p\}} \frac{[\sigma_l]^{-a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f(k)^{I_l})}(w_{\delta_{D_f(k)}}(\mathbf{z}^{(p)}_\gamma(f)(k)) \otimes \mathbf{e}^\vee_\gamma \otimes \mathbf{e}_1) = -\mathbf{z}^{(p)}_{\gamma_*}(f^*)(1)$$

under the canonical isomorphism

$$w_{\delta_{D_f(k)}}(D_f(k)^{\psi=1})_Q \otimes_L \mathcal{L}_L(D_f(k))^\vee(1) \xrightarrow{\sim} ((D_f(k)^*)^{\psi=1})_Q \xrightarrow{\sim} (D_{f^*}(1)^{\psi=1})_Q,$$

where the first isomorphism is defined in §3 and the second one is induced by the canonical isomorphism $D_f(k)^* \xrightarrow{\sim} D_{f^*}(1)$ defined by the Poincaré duality.

Remark 4.5. Since one has

$$[\sigma_{-1}]\cdot \mathbf{z}^{(p)}_{\gamma^\pm}(f) = \mp \mathbf{z}^{(p)}_{\gamma^\pm}(f)$$

(and similarly for $\mathbf{z}^{(p)}_{\gamma^\pm_*}(f^*)$) by Theorem 12.5 of [Ka04], the equality in the conjecture above is equivalent to the equation

$$\prod_{l\in S\setminus\{p\}} \frac{[\sigma_l]^{-a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f(k)^{I_l})}[\sigma_{-1}]\cdot(w_{\delta_{D_f(k)}}(\mathbf{z}^{(p)}_{\gamma'}(f)(k)) \otimes \mathbf{e}^\vee_\gamma \otimes \mathbf{e}_1) = \pm \mathbf{z}^{(p)}_{\gamma'_*}(f^*)(1)$$

for each $(\gamma', \gamma'_*, \pm) \in \{(\gamma^{\mathrm{sgn}(k)}, \gamma^-_*, +), (\gamma^{\mathrm{sgn}(k-1)}, \gamma^+_*, -)\}$.

We prove this conjecture for the non-trianguline case.

Theorem 4.6. *Assume that $V_f|_{G_{\mathbb{Q}_p}}$ is non-trianguline. Then, the conjecture 4.4 is true.*

4.3.3. Reduction of Theorem 4.2 to Theorem 4.6. Before proceeding to the proof of Theorem 4.6, we first prove Theorem 4.2 using this Theorem. For this purpose, it suffices to show the following proposition.

Proposition 4.7. *The conjecture 4.1 for f is equivalent to the conjecture 4.4 for f.*

Proof. We first explicitly describe the base change to $Q(\Lambda)$ of the l-adic ε-isomorphism

$$\varepsilon^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k)) : \mathbf{1}_\Lambda \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k))$$

for $l \in S \setminus \{p\}$ which is defined in [Ya09] (and Remark 2.4). We first remark that, if we set $\varepsilon^{\mathrm{Iw}}_{0,(l)}(T_f(k)) := \varepsilon_{0,\Lambda}(\mathbf{Dfm}(T_f(k))|_{G_{\mathbb{Q}_l}}, \zeta^{(l)})$, then we have

$$\varepsilon^{\mathrm{Iw}}_{0,(l)}(T_f(k)) = [\sigma_l]^{(a^{(l)}(V_f(k))+\dim_L V_f(k)^{I_l})}\varepsilon_{0,(l)}(T_f(k)) \tag{47}$$

since one has

$$\varepsilon_{0,\overline{\mathbb{Q}}_p}(V_f(k)(\delta)|_{G_{\mathbb{Q}_l}}, \zeta^{(l)}) = \delta(\sigma_l)^{-(a^{(l)}(V_f(k))+\dim_L V_f(k)^{I_l})}\varepsilon_{0,(l)}(T_f(k)) \tag{48}$$

for any continuous homomorphism $\delta : \Gamma \to \overline{\mathbb{Q}}_p^\times$ by (5.5.1) and (8.12.1) of [De73]. Since $\mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Q}_l, T_f(k))$ is a torsion Λ-module for any i by Proposition A.2.3 of [Pe95], we have

$$\Delta^{\mathrm{Iw}}_{\mathcal{O},1,(l)}(T_f(k))_Q = \mathbf{1}_{Q(\Lambda)}.$$

Then, the base change to $Q(\Lambda)$ of the isomorphism $\mathbf{1}_\Lambda \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},1,(l)}(T_f(k))$ defined in Remark 2.4 is explicitly defined by

$$\begin{aligned} &\mathbf{1}_{Q(\Lambda)} \xrightarrow{\sim} \Delta^{\mathrm{Iw}}_{\mathcal{O},1,(l)}(T_f(k))_Q = \mathbf{1}_{Q(\Lambda)} : \\ &1 \mapsto \frac{\det_\Lambda(1-\varphi_l^{-1}|\mathbf{Dfm}(T_f(k))^{I_l})}{\det_\Lambda(1-\varphi_l|\mathbf{Dfm}(T_f^*(1))^{I_l})^\iota} = \frac{\det_L(-\varphi_l^{-1}|V_f^{I_l}(k))}{[\sigma_l]^{\dim_L V_f(k)^{I_l}}}\frac{L^{(l)}_{\mathrm{Iw}}(T_f(k))}{L^{(l)}_{\mathrm{Iw}}(T_{f^*}(1))^\iota}. \end{aligned} \tag{49}$$

Since we have

$$(\Lambda)_{a_l(\mathbf{Dfm}(T))} = (\mathcal{O})_{a_l(T)} \otimes_{\mathcal{O}} \Lambda$$

for any $\mathcal{O}$-representation T of $G_{\mathbb{Q}_l}$, (47) and (49) imply that the base change to $Q(\Lambda)$ of $\varepsilon^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k))$ can be explicitly defined by

(50) $$\mathbf{1}_{Q(\Lambda)} \overset{\sim}{\to} \Delta^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k))_Q = \Delta_{\mathcal{O},2,(l)}(T_f(k)) \otimes_{\mathcal{O}} Q(\Lambda) :$$
$$1 \mapsto \varepsilon_{0,(l)}(T_f(k)) \otimes \frac{[\sigma_l]^{a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f^{I_l}(k))} \frac{L^{(l)}_{\mathrm{Iw}}(T_f(k))}{L^{(l)}_{\mathrm{Iw}}(T_{f^*}(1))^{\iota}}.$$

Using this explicit expression, we next remark that Conjecture 4.1 is equivalent to the commutativity of the following diagram.

(51)
$$\begin{array}{ccc} \mathbf{Dfm}(T_f(k))(-1)^+_Q \otimes_{Q(\Lambda)} (\mathbf{Dfm}(T_{f^*}(1))^{\iota}(-1)^+_Q)^{\vee} & \xrightarrow{(a)} & (\det_{\Lambda}\mathbf{Dfm}(T_f(k)))_Q \\ {\scriptstyle \Theta_k(f)\otimes((\Theta_1(f^*)^{\iota})^{-1})^{\vee}}\big\downarrow & & \big\uparrow{\scriptstyle (c)} \\ \mathbf{H}^1(T_f(k))_Q \otimes_{Q(\Lambda)} (\mathbf{H}^1(T_{f^*}(1))^{\iota}_Q)^{\vee} & \xrightarrow[(b)]{} & \det_{Q(\Lambda)}\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_f(k))_Q. \end{array}$$

Here the arrows (a), (b) and (c) in the diagram above is defined as follows.

First, the isomorphism (a) is the base change to $Q(\Lambda)$ of the canonical isomorphism

(52) $$\mathbf{Dfm}(V_f(k))(-1)^+ \otimes_{\Lambda_L(\Gamma)} (\mathbf{Dfm}(V_{f^*}(1))^{\iota}(-1)^+)^{\vee}$$
$$\to (\det_{\Lambda_L(\Gamma)}\mathbf{Dfm}(V_f(k))) : (\lambda_1^+ \cdot \gamma^{\mathrm{sgn}(k-1)}\mathbf{e}_{k-1} + \lambda_1^- \cdot \gamma^{\mathrm{sgn}(k)}\mathbf{e}_{k-1})$$
$$\otimes (\lambda_2^+ \cdot_{\iota} (\gamma_*^+)^{\vee} + \lambda_2^- \cdot_{\iota} (\gamma_*^-)^{\vee}) \mapsto (\lambda_1^-\lambda_2^- - \lambda_1^+\lambda_2^+) \cdot \mathbf{f}_{\gamma}$$

for $\lambda_i^{\pm} \in \Lambda^{\pm}$ $(i = 1, 2)$ (remark that we have $(\gamma_*^+)^{\vee} = \gamma^{\mathrm{sgn}(k)}\mathbf{e}_k, (\gamma_*^-)^{\vee} = \gamma^{\mathrm{sgn}(k-1)}\mathbf{e}_k$).

The isomorphism (b) is the isomorphism naturally induced by the short exact sequence

(53) $$0 \to \mathbf{H}^1(T_f(k))_Q \to \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_f(k))_Q \to (\mathbf{H}^1(T_{f^*}(1))^{\iota})^{\vee}_Q \to 0,$$

which is obtained by the base change to $Q(\Lambda)$ of the Poitou-Tate exact sequence for the pair $(\Lambda, \mathbf{Dfm}(T_f(k)))$ (and the Λ-torsionness of $\mathrm{H}^2_{\mathrm{Iw}}(\mathbb{Z}[1/S], T_{f_0}(r))$ for $f_0 = f, f^*$, $r \in \mathbb{Z}$, and that of $\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_l, T_f(k))$ for $l \in S \setminus \{p\}$).

Finally, the isomorphism (c)

$$(\det_{\Lambda}\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, T_f(k)))_Q \overset{\sim}{\to} (\det_{\Lambda}\mathbf{Dfm}(T_f(k)))_Q$$

is defined by sending $(x \wedge y) \otimes \lambda$ for $x, y \in \mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p, V_f(k)) \overset{\sim}{\to} D_f(k)^{\psi=1}$, $\lambda \in Q(\Lambda)$ to

$$\lambda\{[\sigma_{-1}] \cdot (w_{\delta_{D_f(k)}}(x) \otimes \mathbf{e}_{\gamma}^{\vee} \otimes \mathbf{e}_1), y\}_{\mathrm{Iw}} \prod_{l \in S\setminus\{p\}} \frac{[\sigma_l]^{a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f^{I_l}(k))} \frac{L^{(l)}_{\mathrm{Iw}}(T_f(k))}{L^{(l)}_{\mathrm{Iw}}(T_{f^*}(1))^{\iota}} \mathbf{f}_{\gamma}$$

(remark that here we use the definition of $\varepsilon^{\mathrm{Iw}}_{\mathcal{O},(p)}(T_f(k))$ given in §3 and the explicit description (50) of $\varepsilon^{\mathrm{Iw}}_{\mathcal{O},(l)}(T_f(k))$ for $\in S\setminus\{p\}$).

By definition of these maps, the commutativity of the diagram (51) is equivalent to the equalities

$$\prod_{l\in S\setminus\{p\}}\frac{[\sigma_l]^{a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f^{I_l}(k))}$$
$$\times\{[\sigma_{-1}]\cdot(w_{\delta_{D_f(k)}}(\mathbf{z}^{(p)}_{\gamma'}(f)(k))\otimes\mathbf{e}_f^{\vee}\otimes\mathbf{e}_1),\widetilde{(\mathbf{z}^{(p)}_{\gamma'_*}(f^*)(1))^{\vee}}\}_{\mathrm{Iw}}=\pm 1$$

for each $(\gamma',\gamma'_*,\pm)\in\{(\gamma^{\mathrm{sgn}(k)},\gamma_*^-,+),(\gamma^{\mathrm{sgn}(k-1)},\gamma_*^+,-)\}$, where we denote by $\widetilde{y}\in\mathrm{H}^1_{\mathrm{Iw}}(\mathbb{Q}_p,T_f(k))_Q$ an arbitrary lift of $y\in(\mathbf{H}^1(T_{f^*}(1))^{\iota}_Q)^{\vee}$ by the surjection in (53). This equality (for arbitrary lifts $\widetilde{(\mathbf{z}^{(p)}_{\gamma'_*}(f^*)(1))^{\vee}}$) is equivalent to Conjecture 4.4 by Remark 4.5. □

4.3.4. Proof of Theorem 4.6. From now on untile the end, we assume that $V_f|_{G_{\mathbb{Q}_p}}$ is non-trianguline. Finally, we prove Theorem 4.6.

Proof. (of Theorem 4.6) In this proof, we freely use the notations which are used in §3.3. To simplify the notation, we set $D:=D_f(k)$. We identify $D^*\xrightarrow{\sim}D_{f^*}(1)$ by the canonical isomorphism induced the Poincaré duality. For $x\in D^{\psi=1},y\in(D^*)^{\psi=1}$ and a continuous character $\delta:\Gamma\to\overline{L}^{\times}$, we define $x_\delta\in\mathrm{H}^1_{\varphi,\gamma}(D(\delta)),y_{\delta^{-1}}\in\mathrm{H}^1_{\varphi,\gamma}(D^*(\delta^{-1}))$ to be the images of x and y by the canonical specialization maps $D^{\psi=1}\to\mathrm{H}^1_{\varphi,\gamma}(D(\delta))$ and $(D^*)^{\psi=1}\to\mathrm{H}^1_{\varphi,\gamma}(D^*(\delta^{-1}))$.

By Théorème A [Be05], it suffices to show the equality

$$\prod_{l\in S\setminus\{p\}}\frac{\delta(\sigma_l)^{-a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f(k)^{I_l})}\exp^*((w_{\delta_D}(\mathbf{z}^{(p)}_{\gamma}(f)(k))\otimes\mathbf{e}^{\vee}_{\gamma}\otimes\mathbf{e}_1)_{\delta^{-1}})$$
$$=-\exp^*((\mathbf{z}^{(p)}_{\gamma_*}(f^*)(1))_{\delta^{-1}})\tag{54}$$

in $\mathbf{D}^0_{\mathrm{dR}}(D^*(\delta^{-1}))$ for all the characters $\delta:\Gamma\to\overline{\mathbb{Q}}_p^{\times}$ with finite images.

Take any character δ as above. We first remark that we have

$$\mathbf{D}^0_{\mathrm{dR}}(D^*(\delta^{-1}))=\mathbf{D}^0_{\mathrm{dR}}(V_{f^*}(1)(\delta^{-1}))\subseteq\overline{\mathbb{Q}}_{p,\infty}f^*\otimes\frac{1}{t}\mathbf{e}_1\otimes\mathbf{e}_{\delta^{-1}},$$

and also have

$$\mathbf{D}^0_{\mathrm{dR}}(D(\delta))=\mathbf{D}^0_{\mathrm{dR}}(V_f(k)(\delta))\subseteq\overline{\mathbb{Q}}_{p,\infty}f\otimes\frac{1}{t^k}\mathbf{e}_k\otimes\mathbf{e}_{\delta}.$$

By the interpolation property (46) of $\mathbf{z}_{\gamma_*}^{(p)}(f^*)$, if we set

$$\exp^*((\mathbf{z}_{\gamma_*}^{(p)}(f^*)(1))_{\delta^{-1}}) =: a_\delta f^* \otimes \frac{1}{t}\mathbf{e}_1 \otimes \mathbf{e}_{\delta^{-1}} \in \overline{\mathbb{Q}}_{p,\infty} f^* \otimes \frac{1}{t}\mathbf{e}_1 \otimes \mathbf{e}_{\delta^{-1}},$$

then we have $a_\delta \in \overline{\mathbb{Q}}_\infty$ and

$$\mathrm{per}_{f^*}^{(1,\delta^{-1})}(a_\delta f^* \otimes \frac{1}{t}\mathbf{e}_1 \otimes \mathbf{e}_{\delta^{-1}}) = L(f,\delta,k)\gamma_*^{\mathrm{sgn}(0,\delta^{-1})} =: \alpha_\delta \gamma_*^{\mathrm{sgn}(0,\delta^{-1})}$$

(remark that one has $L_{\{p\}}(f,\delta,s) = L(f,\delta,s)$ when $V_f|_{G_{\mathbb{Q}_p}}$ is non-trianguline).

By (46) for $\mathbf{z}_\gamma^{(p)}(f)$, if we set

$$\exp^*((\mathbf{z}_\gamma^{(p)}(f)(k)))_\delta) = b_\delta f \otimes \frac{1}{t^k}\mathbf{e}_k \otimes \mathbf{e}_\delta \in \overline{\mathbb{Q}}_{p,\infty} f \otimes \frac{1}{t^k}\mathbf{e}_k \otimes \mathbf{e}_\delta,$$

then we have $b_\delta \in \overline{\mathbb{Q}}_\infty$ and

$$\mathrm{per}_f^{(k,\delta)}(b_\delta f \otimes \frac{1}{t^k}\mathbf{e}_k \otimes \mathbf{e}_\delta) = (2\pi i)^{k-1} L(f^*,\delta^{-1},1)\gamma^{\mathrm{sgn}(k-1,\delta)} =: \beta_\delta \gamma^{\mathrm{sgn}(k-1,\delta)}.$$

By Proposition 3.14, if we set

(55)
$$\exp^*((w_{\delta_D}(\mathbf{z}_\gamma^{(p)}(f)(k)) \otimes \mathbf{e}_\gamma^\vee \otimes \mathbf{e}_1)_{\delta^{-1}}) =: c_\delta f \otimes t^k \mathbf{e}_\gamma^\vee \otimes \frac{1}{t^{k+1}}\mathbf{e}_{k+1} \otimes \mathbf{e}_{\delta^{-1}}$$
$$\in \mathbf{D}_{\mathrm{dR}}^0(D^*(\delta^{-1})) \subseteq \overline{\mathbb{Q}}_{p,\infty}(f \otimes \frac{1}{t^k}\mathbf{e}_k) \otimes t^k \mathbf{e}_\gamma^\vee \otimes \frac{1}{t}\mathbf{e}_1 \otimes \mathbf{e}_{\delta^{-1}},$$

then we have

$$c_\delta = \frac{\delta(-1)}{(k-1)!}\varepsilon_{L,(p)}(V_f(k)(\delta))b_\delta =: d_\delta b_\delta$$

(remark that we have

$$(w_{\delta_D}(\mathbf{z}_\gamma^{(p)}(f)(k)) \otimes \mathbf{e}_\gamma^\vee \otimes \mathbf{e}_1)_{\delta^{-1}} = \delta(-1)([\sigma_{-1}]\cdot(w_{\delta_D}(\mathbf{z}_\gamma^{(p)}(f)(k)) \otimes \mathbf{e}_\gamma^\vee \otimes \mathbf{e}_1))_{\delta^{-1}}).$$

Therefore, it suffices to show the equality

(56)
$$\prod_{l \in S\setminus\{p\}} \frac{\delta(\sigma_l)^{-a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f(k)^{I_l})} d_\delta \beta_\delta \gamma^{\mathrm{sgn}(k-1,\delta)} \mathbf{e}_k \otimes \mathbf{e}_\gamma^\vee = -\alpha_\delta \gamma_*^{\mathrm{sgn}(0,\delta^{-1})}.$$

Remark that $\gamma^{\mathrm{sgn}(k)}\mathbf{e}_k \otimes \mathbf{f}_\gamma^\vee$ (resp. $\gamma^{\mathrm{sgn}(k-1)}\mathbf{e}_k \otimes \mathbf{f}_\gamma^\vee$) is sent to $-\gamma_*^-$ (resp. γ_*^+) under the canonical isomorphism

$$T_{f,\mathcal{O}_F}(k)\otimes_{\mathcal{O}_F}(\det_{\mathcal{O}_F}(T_{f,\mathcal{O}_F}(k)))^{\vee}\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}_F}(T_{f,\mathcal{O}_F}(k),\mathcal{O}_F)\xrightarrow{\sim}T_{f^*,\mathcal{O}_F},$$

where the first isomorphism is defined by

$$x\otimes\mathbf{f}_{\gamma}^{\vee}\mapsto[y\mapsto\mathbf{f}_{\gamma}^{\vee}(y\wedge x)],$$

and the second isomorphism is defined by the Poincaré duality.

Hence, we obtain

$$\gamma^{\mathrm{sgn}(k-1,\delta)}\mathbf{e}_k\otimes\mathbf{e}_{\gamma}^{\vee}=\delta(-1)\varepsilon_0^{-1}\gamma_*^{\mathrm{sgn}(0,\delta^{-1})}.$$

Therefore, the equality (56) is equivalent to the equality

$$\delta(-1)\prod_{l\in S\setminus\{p\}}\frac{\delta(\sigma_l)^{-a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f(k)^{I_l})}\varepsilon_0^{-1}d_{\delta}\beta_{\delta}=-\alpha_{\delta}. \tag{57}$$

Since we have

$$\begin{aligned}\frac{\delta(\sigma_l)^{-a^{(l)}(V_f(k))}}{\det_L(-\varphi_l|V_f(k)^{I_l})}\varepsilon_{0,(l)}(T_f(k))&=\frac{\delta(\sigma_l)^{-(a^{(l)}(V_f(k))+\dim_L V_f(k)^{I_l})}}{\det_L(-\varphi_l|V_f(k)(\delta)^{I_l})}\varepsilon_{0,(l)}(T_f(k))\\&=\frac{\varepsilon_{0,(l)}(T_f(k)(\delta))}{\det_L(-\varphi_l|V_f(k)(\delta)^{I_l})}=\varepsilon_{L,(l)}(V_f(k)(\delta)),\end{aligned}$$

the left hand side of (57) is equal to

$$\frac{1}{(k-1)!}\prod_{l\in S}\varepsilon_{L,(l)}(V_f(k)(\delta))(2\pi i)^{k-1}L(f^*,\delta^{-1},1). \tag{58}$$

Since we have

$$\frac{(k-1)!}{(2\pi)^k}L(f,\delta,k)=i^{k+1}\prod_{l\in S}\varepsilon_l(f,\delta,k)\frac{1}{2\pi}L(f^*,\delta^{-1},1)$$

by evaluating at $s=k$ of the functional equation (40) of $L(f,\delta,s)$ and the ε-and L-constants for V_f correspond to those for π_f by the global-local compatibility ([Ca86] for $l\neq p$, and [Sa97] for $l=p$), the value (58) is equal to

$$\frac{(2\pi)^k}{i^2(k-1)!}\left(i^{k+1}\prod_{l\in S}\varepsilon_l(f,\delta,k)\frac{1}{2\pi}L(f^*,\delta^{-1},1)\right)$$

$$= \frac{(2\pi)^k}{i^2(k-1)!}\left(\frac{(k-1)!}{(2\pi)^k}L(f,\delta,k)\right) = \frac{1}{i^2}L(f,\delta,k)$$
$$= -L(f,\delta,k) = -\alpha_\delta,$$

which shows the equality (57), hence finishes to prove the theorem. $\square$

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List of notations

p. 288: $G_{\mathbb{Q}_l}, W_{\mathbb{Q}_l}, I_l, \mathrm{rec}_{\mathbb{Q}_l}, \mathrm{Fr}_l, \zeta, \Gamma, \chi, H_{\mathbb{Q}_p}, \sigma_b, \widetilde{\mathbf{E}}^+$.
p. 289: $\widetilde{\mathbf{E}}, \widetilde{\mathbf{A}}, \widetilde{\mathbf{B}}^+, \widetilde{\mathbf{B}}, \mathbf{B}_{\mathrm{dR}}, t, \mathbf{D}_{\mathrm{perf}}(R), \mathbf{P}_{\mathrm{fg}}(R), P^\vee, \mathcal{P}_R, \boxtimes$.
p. 290: $\mathrm{Det}_R(-), \boxtimes, \mathbf{1}_R$.
p. 291: $R, L, R(\delta), \mathbf{e}_\delta, T(\delta), C^\bullet_{\mathrm{cont}}(G,T), \mathrm{H}^i(\mathbb{Q}_l,T), T(r), T^*$.
p. 292: $\Delta_{R,1}(T), R_a, a_l(T), \Delta_{R,2}(T), \Delta_R(T)$.
p. 293: $W_{\mathbb{Q}_l}, \psi_\zeta, \varepsilon(\rho,\psi,dx), \varepsilon(\rho,\zeta)$.
p. 294: $\varepsilon(M,\zeta),' W_{\mathbb{Q}_l}, W(V), L_\infty, \varepsilon_L(W(V)), \mathbf{D}_{\mathrm{pst}}(V), \mathbf{D}_{\mathrm{cris}}(V), \mathbf{D}_{\mathrm{dR}}(V),$
$\mathbf{D}^i_{\mathrm{dR}}(V), t_V, \theta_L(V)$.
p. 295: $\Gamma(V), \theta_{\mathrm{dR},L}(V), h_V, \varepsilon^{\mathrm{dR}}_L(V)$.
p. 297: $\varepsilon_{0,L}(V), \varepsilon_{0,R}(T)$.
p. 298: $\Lambda_R, \mathbf{Dfm}(T), \Delta^{\mathrm{Iw}}_{R,*}(T), \mathrm{H}^i_{\mathrm{Iw}}(\mathbb{Q}_l,T), f_\delta, \iota$.
p. 299: $\mathbf{e}_r, \mathcal{E}_R, D^*, D(T), X_\zeta, \psi, \Delta, \gamma$.
p. 300: $C^\bullet_{\varphi,\gamma}(D), C^\bullet_{\psi,\gamma}(D), \mathrm{H}^i_{\varphi,\gamma}(D), \mathrm{H}^i_{\psi,\gamma}(D)$.
p. 301: $\langle -,-\rangle_{\mathrm{Tate}}, \Delta_{R,*}(D), \mathcal{L}_R(D)$.
p. 302: $\mathbf{Dfm}(D), \Delta^{\mathrm{Iw}}_R(D), \mathrm{sp}_\delta, C^\bullet_\psi(D)$.
p. 303: ι_δ.
p. 304: $\tilde{\iota}_\delta, \{-,\}_{\mathrm{Iw}}$.
p. 305: g_δ.
p. 306: $\gamma_0, \mathcal{E}_R(\Gamma), \{-,-\}_{0,\mathrm{Iw}}$.
p. 306: $M_{\mathbb{Z}_p^\times}, \wedge_{\mathbb{Z}_p^\times}$.
p. 310: $V(D)$.
p. 311: $\varepsilon^{\mathrm{dR}}_L(D)$.
p. 312: m_δ, w_*, w_δ.
p. 313: $\delta_D, d, [-,-]_{\mathrm{Iw}}$.
p. 314: $\eta_R(D)$.

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