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On Yau's uniformization conjecture*

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Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth, we prove that M is biholomorphic to $\mathbb{C}^n$. This confirms the uniformization conjecture of Yau when M has maximal volume growth.

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1. Introduction

The classical uniformization theorem states that a simply connected Riemann surface is isomorphic to the Riemann sphere $\mathbb{CP}^1$,

the Poincare disk $\mathbb{D}^2$ or the complex plane $\mathbb{C}$. A geometric consequence is that a complete orientable Riemannian surface of positive curvature is necessarily conformal to $\mathbb{CP}^1$ or $\mathbb{C}$. An orientable Riemannian surface can be regarded as a Kähler manifold of complex dimension 1. A natural question is to generalize such result to higher dimensional Kähler manifolds. The curvature we adopt here is the so-called (holomorphic) bisectional curvature.

Definition 1.1. *[32] [54] On a Kähler manifold M^n, we say the bisectional curvature is greater than or equal to K ($BK \geq K$), if*

$$\frac{R(X,\overline{X},Y,\overline{Y})}{||X||^2||Y||^2+|\langle X,\overline{Y}\rangle|^2} \geq K \tag{1.1}$$

for any two nonzero vectors $X, Y \in T^{1,0}M$.

Observe that the equality holds for complex space forms. The bisectional curvature lower bound condition is weaker than the sectional curvature lower bound, while stronger than the Ricci curvature lower bound.

So the question above can be refined as the classification of Kähler manifolds with positive bisectional curvature. In the compact case, the famous

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Frankel conjecture, solved by Mori [42] and Siu-Yau [53] independently, states that a compact Kähler manifold of positive bisectional curvature is biholomorphic to $\mathbb{CP}^n$ (In fact, Mori proved a stronger result). The noncompact analogue was proposed by Yau [57] in 1970s:

Conjecture 1. *Let M^n be a complete noncompact Kähler manifold with positive bisectional curvature. Then M is biholomorphic to $\mathbb{C}^n$.*

Conjecture 1 is open so far. However, there has been much important progress. In earlier works, Mok-Siu-Yau [41] and Mok [40] considered embedding by using holomorphic functions of polynomial growth. Later, with the Kähler-Ricci flow, results were improved significantly. For example, in [1], A. Chau and L. F. Tam proved that a complete noncompact Kähler manifold with bounded nonnegative bisectional curvature and maximal volume growth is biholomorphic to $\mathbb{C}^n$. See also [50][51][11][44][1][2][3][10][25][23][29] for related works.

In [34]-[38], we introduced a new method to study the conjecture. The basic strategy is to consider the Gromov-Hausdorff limit of Kähler manifolds with bisectional curvature lower bound. For instance, in [36], it was proved that if a complete noncompact Kähler manifold has nonnegative bisectional curvature and maximal volume growth, then it is biholomorphic to an affine algebraic variety. In this paper, we shall continue to study the conjecture by this strategy. The main theorem is the following:

Theorem 1.1. *Let M^n be a complete noncompact Kähler manifold with nonnegative bisectional curvature. Assume M has maximal volume growth, then M is biholomorphic to $\mathbb{C}^n$. In fact, we can find n polynomial growth holomorphic functions $f_1, .., f_n$ which serve as the biholomorphism.*

Remark 1.1. *Note theorem 1.1 was also proved in a very recent preprint [30] by M. C. Li and L. F. Tam. The proof is completely different from ours.*

Let f be a polynomial growth holomorphic function on M. We define the degree of f be the infimum of $d > 0$ so that $f \in \mathcal{O}_d(M)$.

Corollary 1.1. *Under the same assumption as above, if f is a nonconstant polynomial growth holomorphic function on M with minimal degree, then $df \neq 0$ at any point.*

We shall also study the case when the bisectional curvature has a lower bound.

Theorem 1.2. *Let (M_i^n, p_i) be a sequence of complete (compact or noncompact) Kähler manifolds with bisectional curvature lower bound -1. Assume*

$vol(B(p_i, 1)) \geq v > 0$. Suppose (X, p) is the pointed Gromov-Hausdorff limit of (M_i, p_i). Then

- *(X, p) is homeomorphic to a normal complex analytic space with singularity of complex codimension at least 4. In particular, if $n \leq 3$, X is a complex manifold.*
- *X is homeomorphic to a manifold. If the diameters are uniformly bounded, X is homeomorphic to M_i for all large i.*

Remark 1.2. *In the topological sense, the second part is the complex analogue of Perelman's stability theorem [48][27] for Riemannian manifolds with sectional curvature lower bound. If the bisectional curvature lower bound is replaced by two side bounds of Ricci curvature, then X could have singularity of complex codimension 2. See for example, [15][55].*

Remark 1.3. *Our original approach to theorem 1.1 is to prove sufficient regularity for the tangent cone. For instance, by using the the regularity result of theorem 1.2, we can prove theorem 1.1 for $n \leq 3$. It is interesting to note that the regularity in theorem 1.2 is very closely related with a conjecture of Shokurov ([52], conjecture 2) in algebraic geometry. For instance, if Shokurov conjecture is true, then the limit space X is complex analytically smooth. So far, the Shokurov conjecture is only solved for dimension less than or equal to three (this is responsible for the dimension restriction $n \leq 3$ for theorem 1.1). For some details on the Shokurov conjecture, one can refer to [39]. In the current version, we bypass the difficulty in algebraic geometry by using a different method.*

Now let us explain the basic strategy to theorem 1.1. We follow [36][37] [15][16] closely. Recall under the same assumption of theorem 1.1, it was proved in [36] that the manifold is biholomorphic to an affine algebraic variety. How to prove the affine variety is in fact $\mathbb{C}^n$? If $n = 2$, Ramanujam's result says an algebraic surface homeomorphic to $\mathbb{R}^4$ is necessarily isomorphic to $\mathbb{C}^2$. Unfortunately, there is no such criteria in higher dimensions. Moreover, argument in [36] does not provide information about topology of the manifold.

Consider a tangent cone V of M at infinity. That is, there exists $r_i \to \infty$ so that the sequence $(M_i, p_i, d_i) = (M, p, \frac{d}{r_i})$ converges to V in the pointed Gromov-Hausdorff sense. Cheeger-Colding theory asserts that V is a metric cone. Let r be the distance to the vertex. Then the vector field $-r\frac{\partial}{\partial r}$ retracts V to the vertex. A key new idea in this paper is to solve $\overline{\partial}$ equation on the holomorphic tangent bundle. More precisely, we construct holomorphic vector fields Z_i on $B(p_i, 1)$ so that in a natural sense, ReZ_i converges to the $-r\frac{\partial}{\partial r}$. By using some complex analytic techniques, we manage to prove that the flow generated by ReZ_i contracts a domain containing $B(p_i, \frac{1}{2})$ to a point. Since $B(p_i, \frac{1}{2})$ exhausts M, we see M is in fact exhausted by topological balls. Then by Stalling's result, the manifold is diffeomorphic to $\mathbb{R}^{2n}$. As we see before, if $n = 2$, the manifold is biholomorphic to $\mathbb{C}^2$.

Recall the domain of Z_i exhausts M. However, it seems very difficult to glue these Z_i together. A technical reason is that the unique zero point of Z_i might diverge to infinity.

There are two possible ways to get around this difficulty. One is to prove that the tangent cone V is complex analytically smooth. Eventually, by using powerful algebro-geometric tools (Mumford criteria, for example), we can prove that if $n \leq 3$, then V is complex analytically smooth. Then it is relatively easily to prove that M is biholomorphic to $\mathbb{C}^n$. Unfortunately, the algebro-geometric method fails for higher dimensions.

Another approach is to construct a nice *global* holomorphic vector field on M. This is how we prove theorem 1.1. A key point is to study a linear space Z consisting of holomorphic vector fields on M so that the action (derivative) on any polynomial growth holomorphic function preserves the degree. It turns out that Z has finite dimension. With an argument by contradiction, we managed to prove that in Z, there exists a global holomorphic vector field which contracts M to a point. This gives us the desired biholomorphism from M to $\mathbb{C}^n$. A detailed analysis also gives "canonical" holomorphic coordinate on M.

This paper is organized as follows. Section 2 is some basic preliminary results. In section 3, we solve the $\overline{\partial}$ equation on the holomorphic tangent bundle. The outcome is a sequence retracting holomorphic vector fields on exhaustion domains of M. With the preparation in section 3, we present the proof of theorem 1.1 in section 4. In the last section, we prove theorem 1.2.

2. Preliminaries

For the basic theory of Gromov-Hausdorff convergence, we refer to [18]. See also preliminaries of [34]-[38].

Hörmander L^2 theory: The following result can be found on page 37-38 of [14].

Proposition 2.1. *Let (X, ω) be a connected Kähler manifold which is not necessarily complete. Assume X is Stein. Let (F, h) be a Hermitian holomorphic vector bundle over X (h is the metric). Assume the curvature operator $A = [\sqrt{-1}\Theta_{F,h}, \Lambda]$ is positive definite everywhere on $\Lambda^{n,1}T^*_X \otimes F$. Then for* *any form* $g \in L^2(M, \Lambda^{n,1}T_X^* \otimes F)$ *satisfying* $\overline{\partial} g = 0$ *and* $\int_X \langle A^{-1}g, g\rangle \omega^n < +\infty$, *there exists* $f \in L^2(X, \Lambda^{n,0}T_X^* \otimes F)$ *such that* $\overline{\partial} f = g$ *and* $\int_X |f|^2\omega^n \leq \int_X \langle A^{-1}g, g\rangle\omega^n$.

Let (F, h) be a Hermitian holomorphic vector bundle. Let $z_1, ..., z_n$ be a local holomorphic chart and $e_1, .., e_m$ be an orthonormal frame of F. Let

$$\sqrt{-1}\Theta_{F,h} = \sum_{j,k,\lambda,\mu} c_{jk\lambda\mu} dz_j \wedge d\overline{z_k} \otimes e_\lambda^* \otimes e_\mu. \tag{2.1}$$

By using the metric h, we identify the curvature tensor Θ with a Hermitian form

$$\tilde{\Theta}_{F,h}(\xi, v) = c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda \overline{v}_\mu \tag{2.2}$$

on $T^{1,0}X \otimes F$. The next definition appears on page 27-28 of [14].

Definition 2.1. *We say* (F, h) *is*

(a) Nakano positive if $\tilde{\Theta}_{F,h}(\tau) > 0$ *for all nonzero tensors* $\tau = \sum \tau_{j\lambda}\frac{\partial}{\partial z_j} \otimes e_\lambda \in T_X \otimes F$.

(b) Griffiths positive if $\tilde{\Theta}_{F,h}(\xi \otimes v) > 0$ *for all nonzero decomposable tensors* $\xi \otimes v \in T_X \otimes F$.

The computation on page 35 of [14] shows that if F is Nakano positive, then $[\sqrt{-1}\Theta_{F,h}, \Lambda]$ is positive on $(n, 1)$ forms with values in F. More explicitly, by equation (4.8) on page 35 of [14], we have that if $\tilde{\Theta}_{F,h}(\tau) \geq c|\tau|^2$, then

$$\langle[\sqrt{-1}\Theta_{F,h}, \Lambda]u, u\rangle \geq c|u|^2 \tag{2.3}$$

for any $u \in \Lambda^{n,1}T_M^* \otimes F$. We also need the following

Proposition 2.2. *For any Hermitian holomorphic vector bundle* E, *if* E *is Griffiths positive, then* $E \otimes \det(E)$ *is Nakano positive. In fact, if* $\tilde{\Theta}_{E,h}(\xi \otimes v) \geq c|\xi \otimes v|^2 \geq 0$, *then* $\tilde{\Theta}_{E\otimes\det E,h}(\tau) \geq c|\tau|^2 \geq 0$.

The proof can be found on page 93 of [14]. Notice the proof there also gives the second statement. More precisely, check the last equation of the proof which appears on page 95.

Next we introduce a gluing technique:

Definition 2.2. *Let* χ *be a strictly increasing continuous function over* $\mathbb{R}^+$ *and* $\chi(0) = 0$. *A metric space* X *is* χ*-connected if for any two points* $x_1, x_2 \in X$, *we can find a curve* γ *connecting* x_1, x_2 *so that the diameter of* γ *is bounded by* $\chi(d(x_1, x_2))$.

We will need the gluing theorem which appears in [27][48]:

Proposition 2.3. *[Gluing theorem] Let X be a compact topological manifold which is also a metric space. Let U_α be a finite open covering of X. Given a function χ_0, there exists $\delta = \delta(X, \chi_0, \{U_\alpha\},)$ so that the following holds: Given a χ_0-connected topological manifold $\tilde{X}$ (metric space), an open cover of $\tilde{X}$ $\{\tilde{U}_\alpha\}$, a δ-Hausdorff approximation $\varphi : X \to \tilde{X}$ and a family of homeomorphisms φ_α: $U_\alpha \to \tilde{U}_\alpha$, δ-close to φ, then there exists a homeomorphism $\overline{\varphi} : X \to \tilde{X}$, $\chi(\delta)$-close to φ.*

In this paper, we will denote by $\Phi(u_1, ..., u_k|....)$ any nonnegative functions depending on $u_1, ..., u_k$ and some additional parameters such that when these parameters are fixed,

$$\lim_{u_k \to 0} \cdots \lim_{u_1 \to 0} \Phi(u_1, ..., u_k|...) = 0.$$

Let $C(n), C(n, v)$ be large positive constants depending only on n or n, v; $c(n), c(n, v)$ be small positive constants depending only on n or n, v. The values might change from line to line. Let $⨍$ be the average of an integral.

3. Construction of retracting holomorphic vector fields

In this section, we shall construct retracting holomorphic vector fields on geodesic balls which are Gromov-Hausdorff close to metric cones. The argument is crucial for all the results in this paper.

Proposition 3.1. *Give any $n \in \mathbb{N}$ and $v > 0$, there exists $\epsilon = \epsilon(n, v) > 0$ so that the following holds: Let (M^n, p) be a complete Kähler manifold with nonnegative bisectional curvature. If $vol(B(p, \frac{1}{\epsilon})) \geq v\frac{1}{\epsilon^{2n}}$ and $d_{GH}(B(p, \frac{1}{\epsilon}), B_W(o, \frac{1}{\epsilon})) < \epsilon$ for some metric cone (W, o), then there exists a holomorphic vector field Z on some open set $U \supset B(p, 1)$ so that the flow σ_t generated by $-Z$ retracts to a point $\tilde{p}$ where $d(p, \tilde{p}) = \Phi(\epsilon|n, v)$. Furthermore, $\sigma_t(B(p, \frac{1}{2})) \subset B(p, 1)$ for all $t \geq 0$.*

Proof. Assume the proposition is not true. Then there exist some n, v so that for any $i \in \mathbb{N}$, we can find a complete Kähler manifold (M_i^n, p_i) which does not satisfy the proposition. Furthermore, M_i has nonnegative bisectional curvature and for metric cones (V_i, o'_i),

$$vol(B(p_i, i)) \geq vi^{2n}, d_{GH}(B(p_i, i), B_{V_i}(o'_i, i)) < \frac{1}{i}. \tag{3.1}$$

By passing to a subsequence if necessary, we may assume (M_i, p_i, d_i) pointed converges to a metric cone $(M_\infty, p_\infty, d_\infty)$. Let R be a large number depending only on n, v. The value will be determined later. Let r_i be the distance function to p_i. According to Cheeger-Colding theory [4] and claim 5.1 in [36], for sufficiently large i, we can find smooth functions u_i such that

$$\int_{B(p_i,4R)} |\nabla u_i - \nabla \frac{1}{2} r_i^2|^2 + |\nabla^2 u_i - g_i|^2 < \Phi(\frac{1}{i}|R) \tag{3.2}$$

$$|u_i - \frac{r_i^2}{2}| < \Phi(\frac{1}{i}|R) \tag{3.3}$$

$$|\nabla u_i| \leq C(n) r_i \tag{3.4}$$

on $B(p_i, 4R)$. Now define a $(1,0)$ type vector field $\tilde{Z}_i = \nabla u_i - \sqrt{-1} J \nabla u_i$. Then we have that

$$\int_{B(p_i,4R)} |\overline{\partial} \tilde{Z}_i|^2 < \Phi(\frac{1}{i}|R). \tag{3.5}$$

The idea is to perturb $\tilde{Z}_i$ so that it becomes holomorphic. According to proposition 5.1 of [36], we can find a smooth function v_i on $B(p_i, \frac{R}{2})$ such that

$$0 \leq v_i \leq C(R,n), \sqrt{-1}\partial\overline{\partial} v_i \geq c(n,v)\omega_i > 0. \tag{3.6}$$

$$\min_{y \in \partial B(p_i, \frac{R}{20})} v_i(y) > 4 \sup_{y \in B(p_i, \epsilon_0 \frac{R}{20})} v_i(y). \tag{3.7}$$

for some $\epsilon_0(n,v) > 0$. Now we fix $R = R(n,v)$ so that $R\epsilon_0 > 2000$. Let Ω_i be the connected component of $\{y \in B(p_i, \frac{R}{20}) | v_i(y) < 2 \sup_{B(p_i, \frac{\epsilon_0 R}{20})} v_i\}$ containing $B(p_i, \frac{\epsilon_0 R}{20})$. Then Ω_i is a Stein manifold containing $B(p_i, 100)$. Consider the metric

$$g_i' = e^{-v_i} g_i \tag{3.8}$$

on the tangent bundle $T^{1,0}M_i$. The curvature of g_i' satisfies $\Theta_{g_i'} = \Theta_{g_i} + \sqrt{-1}\partial\overline{\partial} v_i \otimes Id$. Then we find that

$$\tilde{\Theta}(\xi \otimes u) = \langle \Theta_{g_i'}(\xi \otimes u), \overline{\xi \otimes u} \rangle \geq \frac{1}{2} c(n,v) |\xi \otimes u|^2. \tag{3.9}$$

We have used that the bisectional curvature is nonnegative. This implies that $(T^{1,0}M, g_i')$ is Griffiths positive.

g_i' induces a metric on the anti-canonical line bundle $K^{-1}M_i$. Take $F = T^{1,0}_{M_i} \otimes K^{-1}(M_i)$. Let the metric h on F be induced by g_i' on both the tangent bundle and $K^{-1}M_i$. Let the metric $\tilde{h}$ on F be induced by the Kähler metric. According to (3.6) and (3.8),

$$c(n,v)\tilde{h} \leq h \leq C(n,v)\tilde{h} \tag{3.10}$$

on $B(p_i, \frac{R}{2})$. By proposition 2.2, F is Nakano positive. Therefore (2.3) holds for (F,h). Write $T^{1,0}(M_i) = \Lambda^{n,0}T^*M_i \otimes F$. By applying proposition 2.1 to Stein manifold Ω_i and (F,h), we obtain a $(1,0)$ type vector field Y_i satisfying

$$\overline{\partial} Y_i = \overline{\partial}\tilde{Z}_i, \tag{3.11}$$

$$\int_{\Omega_i} |Y_i|^2 \leq \frac{1}{c(n,v)} \int_{\Omega_i} |\overline{\partial}\tilde{Z}_i|^2 < \Phi(\frac{1}{i}). \tag{3.12}$$

In (3.12), the norms are induced by the Kähler metric of M_i (we have used (3.10)). Therefore,

$$Z_i = \tilde{Z}_i - Y_i \tag{3.13}$$

is a holomorphic vector field. The idea is to study the flow generated by the real part of $-Z_i$.

For any point $q \in \partial B(p_\infty, 1)$, take a tangent cone (V, o'). Cheeger-Colding [4] says (V, o') is a metric cone. According to [4][8], V splits off $\mathbb{R}^2$. Take $M_i \ni q_i \to q$. Given any $\epsilon > 0$, we may take $\delta = \delta(\epsilon, v)$ so small that $d_{GH}(B(q_i, \frac{1}{\epsilon}\delta), B_V(o', \frac{1}{\epsilon}\delta)) < \epsilon\delta$ for all large i. By lemma 6.15 of [4], we find harmonic functions h_i on $B(q_i, \delta)$ such that

$$|h_i(x) - (r_i(x) - r_i(q_i))| \leq \Phi(\delta)\delta. \tag{3.14}$$

Furthermore, lemma 6.25 of [4] says

$$⨍_{B(q_i,\delta)} |\nabla h_i - \nabla r_i|^2 < \Phi(\delta). \tag{3.15}$$

By the argument in [36], we may find a holomorphic function f_i in $B(q_i, \delta)$ such that

$$|Ref_i(x) - h_i(x)| < \Phi(\epsilon)\delta, |\nabla(Ref_i(x) - h_i(x))| \leq \Phi(\epsilon) \tag{3.16}$$

in $B(q_i, \frac{1}{2}\delta)$. Given a function w on $B(q_i, \delta)$, define a norm $||w|| = (⨍_{B(q_i,\frac{1}{2}\delta)} |w|^2)^{\frac{1}{2}}$.

Now by the estimates above, for sufficiently large i, we have

(3.17)
$$\begin{aligned} ||\langle ReZ_i, \nabla f_i\rangle - 1|| &\leq ||\langle Re(Z_i - \tilde{Z}_i), \nabla f_i\rangle|| + ||\langle \nabla u_i, \nabla f_i\rangle - 1|| \\ &\leq \frac{\Phi(\frac{1}{i})}{\delta^n} + C(n)|||\nabla u_i - r_i\nabla r_i||| + ||\langle r_i\nabla r_i, \nabla f_i\rangle - 1|| \\ &\leq \frac{\Phi(\frac{1}{i})}{\delta^n} + ||r_i - 1|| + ||\langle r_i\nabla r_i, \nabla f_i - (\nabla r_i - \sqrt{-1}J\nabla r_i)\rangle| \\ &\leq \frac{\Phi(\frac{1}{i})}{\delta^n} + 2\delta + 10|||\nabla Ref_i - \nabla r_i||| \\ &\leq \frac{\Phi(\frac{1}{i})}{\delta^n} + 2\delta + \Phi(\epsilon|n, v) + \Phi(\delta) \\ &< \Phi(\epsilon, \delta|n, v). \end{aligned}$$

Note that $\langle ReZ_i, \nabla f_i\rangle - 1$ is holomorphic on $B(q_i, \delta)$. By the mean value inequality,

$$|\langle ReZ_i, \nabla f_i\rangle - 1| \leq \Phi(\epsilon, \delta|n, v) \tag{3.18}$$

on $B(q_i, \frac{\delta}{4})$. Let σ_t be the flow generated by $-ReZ_i$. Then

$$|\frac{df_i(\sigma_t(q_i))}{dt} + 1| \leq \Phi(\epsilon, \delta|n, v) \tag{3.19}$$

as long as $\sigma_t(q_i) \in B(q_i, \frac{\delta}{4})$. By applying proposition 6.1 in [36], we find $N = N(n, v)$, $\frac{1}{2} > \gamma_1 = \gamma_1(n, v) > 5\gamma_2 = 5\gamma_2(n, v) > 0$, holomorphic functions g_i^j on $B(q_i, \delta)$ such that the following holds: $g_i^j(q_i) = 0$;

$$\delta = \min_{x\in\partial B(q_i,\frac{\gamma_1\delta}{3})} \sum_{j=1}^{N} |g_i^j(x)|^2 > 2 \sup_{x\in B(q_i,\gamma_2\delta)} \sum_{j=1}^{N} |g_i^j(x)|^2. \tag{3.20}$$

$$\frac{\sup_{x\in B(q_i,\frac{1}{2}\gamma_1\delta)} |g_i^j(x)|^2}{\sup_{x\in B(q_i,\frac{1}{3}\gamma_1\delta)} |g_i^j(x)|^2} \leq C(n, v). \tag{3.21}$$

According to three circle theorem in [34], $\sup_{x\in\partial B(q_i,\frac{\gamma_1}{2}\delta)} |g_i^j(x)| \leq C(n, v)\delta.$

Thus $|dg_i^j| \le C(n,v)$ on $B(q_i, \frac{5\gamma_1}{12}\delta)$. Then for sufficiently large i,

$$\begin{aligned}(3.22)\qquad \fint_{B(q_i,\frac{5}{12}\gamma_1\delta)} |\langle ReZ_i, dg_i^j\rangle|^2 &\le C(n,v)\fint_{B(q_i,\frac{5}{12}\gamma_1\delta)} |Z_i|^2 \\ &\le C(n,v) + \frac{\Phi(\frac{1}{i})}{\delta^{2n}} \le C(n,v).\end{aligned}$$

By similar arguments as above, if $\sigma_t(q_i) \in B(q_i, \frac{1}{3}\gamma_1\delta)$,

$$(3.23)\qquad |\frac{dg_i^j(\sigma_t(q_i))}{dt}| \le C(n,v).$$

Combining this with (3.20), we find $c(n,v) > 0$ such that if $|t| \le c(n,v)\delta$,

$$(3.24)\qquad \sigma_t(q_i) \in B(q_i, \frac{1}{3}\gamma_1\delta) \subset B(q_i, \frac{1}{4}\delta).$$

Applying (3.19), we find

$$(3.25)\qquad Ref_i(\sigma_{c(n,v)\delta}(q_i)) \le Ref_i(q_i) - (1 - \Phi(\epsilon, \delta|n,v))c(n,v)\delta.$$

If ϵ, δ are sufficiently small depending only on n and v, (3.14) and (3.16) imply

$$(3.26)\qquad r_i(\sigma_{c(n,v)\delta}(q_i)) \le r_i(q_i) - \frac{1}{2}c(n,v)\delta.$$

We conclude that

$$(3.27)\qquad \sigma_{c(n,v)\delta}(B(p_i,1)) \subset B(p_i, 1 - \frac{1}{4}c(n,v)\delta).$$

Also, for any $0 < t < c(n,v)\delta$, we may require

$$(3.28)\qquad \sigma_t(B(p_i, 1 - \frac{1}{4}c(n,v)\delta)) \subset B(p_i, 1 - \frac{1}{8}c(n,v)\delta).$$

Now fix $\epsilon = \epsilon(n,v), \delta = \delta(n,v)$ small such that the all inequalities above hold.

Proposition 3.2. *There exists a point o_i with $d_i(o_i, p_i) = \Phi(\frac{1}{i})$ and $\lim_{t\to\infty} \sigma_t(B(p_i,1)) = o_i$. The convergence is uniform on $B(p_i,1)$.*

Proof.

Claim 3.1. *$\sigma_t(B(p_i,1)) \subset B(p_i, 1-\frac{1}{8}c(n,v)\delta)$ for all $t \geq c(n,v)\delta$.*

Proof. Indeed, we may write $t = kc(n,v)\delta + t'$ where $0 \leq t' < c(n,v)\delta$ and k is an integer.

$$\begin{aligned} \sigma_t(B(p_i,1)) &= \sigma_{t'}\sigma_{c(n,v)\delta}\cdot\cdot\sigma_{c(n,v)\delta}(B(p_i,1)) \\ &\subset \sigma_{t'}(B(p_i, 1-\frac{1}{4}c(n,v)\delta)) \\ &\subset B(p_i, 1-\frac{1}{8}c(n,v)\delta). \end{aligned} \tag{3.29}$$

□

Recall $B(p_i,2)$ is contained in the Stein manifold Ω_i. Thus, we may embed $B(p_i,2)$ in some $\mathbb{C}^{N_i}$. In particular, we have bounded holomorphic functions $z_1,..,z_{N_i}$ which separate points on $B(p_i,1)$. Consider any sequence $t_j \to \infty$. Then by passing to subsequences if necessary, we may assume $z_k(\sigma_{t_j}(x))$ converges uniformly on $B(p_i,1)$ for all $1 \leq k \leq N_i$. Note there is no problem for the uniform convergence close to boundary, according to claim 3.1. Let $F_i(x) = \lim_{j\to\infty}\sigma_{t_j}(x)$ for $x \in B(p_i,1)$. Then F_i is a holomorphic map. It is clear that $F_i(B(p_i,1)) \subset B(p_i, 1-\frac{1}{16}c(n,v)\delta)$.

Claim 3.2. *$F_i(B(p_i,1))$ is a compact subset of $B(p_i, 1-\frac{1}{8}c(n)\delta)$.*

Proof. It suffices to show that $F_i(B(p_i,1)) = F_i(\overline{B(p_i, 1-\frac{1}{16}c(n,v)\delta)})$, since this implies that $F_i(B(p_i,1)) = F_i(\overline{B(p_i, 1-\frac{1}{16}c(n,v)\delta)})$. For any $x \in B(p_i,1)$, $F_i(x) = \lim_{j\to\infty}\sigma_{t_j}(x) = \lim_{j\to\infty}\sigma_{t_{j-1}}(\sigma_{t_j-t_{j-1}}(x))$. We may assume $t_j - t_{j-1} > c(n,v)\delta$ for all j. Then $\sigma_{t_j-t_{j-1}}(x) \in B(p_i, 1-\frac{1}{8}c(n,v)\delta)$. By taking subsequence if necessary, we may assume that $\lim_{j\to\infty}\sigma_{t_j-t_{j-1}}(x) = y$. Since the convergence of $\sigma_{t_j}(x)$ is uniform on $B(p_i,1)$, $\lim_{j\to\infty}\sigma_{t_{j-1}}(\sigma_{t_j-t_{j-1}}(x)) = F_i(y) \in F_i(B(p_i, 1-\frac{1}{16}c(n,v)\delta))$. □

Claim 3.3. *$F_i(B(p_i,1))$ is an analytic set in $B(p_i, 1-\frac{1}{16}c(n,v)\delta)$.*

Proof. Since $F_i(B(p_i,1)) = F_i(\overline{B(p_i, 1-\frac{1}{16}c(n,v)\delta)})$, the claim is a direct consequence of the following proposition, which is the generalization of the proper mapping theorem:

Proposition 3.3. *[28] Let M and N be connected complex manifolds and f is a holomorphic map from M to N. Suppose that for any compact set $L \subset N$, there exists a compact set $K \subset M$ with $L \cap f(M) \subset f(K)$, then $f(M)$ is an analytic set in N.*

Remark 3.1. *We thank Professor Yum-Tong Siu for pointing this result.*

□

Since $B(p_i, 1)$ is connected, $F_i(B(p_i, 1))$ is a connected compact analytic set which is contained in a Stein manifold. Thus it must be a point. Let us say $F_i(B(p_i, 1)) = o_i$. That is, σ_{t_j} converges uniformly to o_i on $B(p_i, 1)$. Pick an arbitrary sequence $t'_j \to \infty$. By passing to subsequence if necessary, we may assume that $t'_j > 2t_j$ for all j. Then $\lim_{j\to\infty} \sigma_{t'_j}(x) = \lim_{j\to\infty} \sigma_{t_j}(\sigma_{t'_j - t_j}(x)) = o_i$. This proves that $\lim_{t\to\infty} \sigma_t(B(p_i, 1)) = o_i$ and the convergence is uniform. Finally, given any $\rho > 0$, by using the same argument as before, we can prove that for sufficiently large i, $F_i(\overline{B(p_i, \rho)}) \subset B(p_i, \rho)$. Thus $o_i \in B(p_i, \rho)$ for sufficiently large i. This implies $d_i(o_i, p_i) = \Phi(\frac{1}{i})$. The proof of proposition 3.2 is complete. □

From the argument above, we see that (M_i, p_i) satisfies proposition 3.1 for large i. This is a contradiction. The proof of proposition 3.1 is complete.

□

By using the argument in [37] and a rescaling, we obtain the following result.

Corollary 3.1. *Give any $n \in \mathbb{N}$ and $v > 0$, there exist $\epsilon = \epsilon(n, v) > 0, \delta = \delta(n, v) > 0$ so that the following holds: Let (M^n, p) be a complete Kähler manifold with $BK \geq -\epsilon$. If $vol(B(p, 1)) \geq v$ and $d_{GH}(B(p, 1), B_W(o, 1)) < \epsilon$ for some metric cone (W, o), then there exists a holomorphic vector field Z on some open set $U \supset B(p, 2\delta)$ so that the flow σ_t generated by $-Z$ retracts to a point $\tilde{p}$ where $d(p, \tilde{p}) = \Phi(\epsilon|n, v)\delta$. Furthermore, $\sigma_t(B(p, \delta)) \subset B(p, 2\delta)$ for all $t \geq 0$.*

The next result is suggested by Nan Li. This should be compared with a result in [19][20]. See for example, page 206 and 212 of [20].

Corollary 3.2. *(Uniform contractibility) Let M^n be a compact Kähler manifold with $BK \geq -1$ and $Vol(M) \geq v$, $diam(M) \leq d$. Then there exists $r_0 = r_0(n, v, d) > 0$, $C = C(n, v, d)$ so that for any $r < r_0$, $p \in M$, $B(p, r)$ is contractible in $B(p, Cr)$.*

Proof. The manifold is noncollapsed uniformly. According to Cheeger-Colding [4] and volume comparison theorem, given any $\delta' > 0$, $\epsilon' > 0$, we can find $N = N(\delta', \epsilon', n, v, d)$, $r_0 = r_0(\delta', \epsilon', n, v, d)$ so that for any $r < r_0$, there exists some integer m between 1 and N and $d_{GH}(B(p, \frac{r}{\delta'^m})\backslash B(p, \frac{r}{\delta'^{m-1}}), B_V(o, \frac{r}{\delta'^m})\backslash B_V(o, \frac{r}{\delta'^{m-1}})) < \epsilon' \frac{r}{\delta'^m}$, where (V, o) is a metric cone. This implies that $d_{GH}(B(p, \frac{r}{\delta'^m}), B_V(o, \frac{r}{\delta'^m})) < (\epsilon' + 100\delta')\frac{r}{\delta'^m}$. Define $(M', g', p') = (M, (\frac{\delta'^m}{r})^2 g, p)$. Thus

$$d_{GH}(B(p', 1), B_V(o, 1)) < \epsilon' + 100\delta' \tag{3.30}$$

$$BK(M') \geq -\frac{r^2}{\delta'^{2m}}, \tag{3.31}$$

$$Vol(B(p', 1)) > c(n, v, d) > 0. \tag{3.32}$$

Let $\delta = \delta(n, v, d)$ be the constant in corollary 3.1 (we have to use the volume lower bound as in (3.32)). Now fix $\epsilon' = \epsilon'(n, v, d), \delta' = \delta'(n, v, d) < \delta(n, v, d)$ be sufficiently small so that the right hand side of (3.30) is sufficiently small. Then fix r_0 be sufficiently small so that the right hand side of (3.31) is sufficiently small. We may assume the condition of corollary 3.1 is satisfied. Therefore, $B(p', \delta)$ is contractible in $B(p', 2\delta)$. In particular, $B(p, r) \subset B(p, \delta\frac{r}{\delta'^m})$ is contractible in $B(p, \frac{r}{\delta'^m}) \subset B(p, \frac{r}{\delta'^N})$. This concludes the proof of corollary 3.2. □

Let (M_i^n, p_i) be a sequence of complete Kähler manifolds with $BK(M_i) \geq -1, vol(B(p_i, 1)) \geq v > 0$. Assume (M^i, p_i) converges in the pointed Gromov-Hausdorff sense to a metric space (X, p). Let q be a point on X and let (V, o) be a tangent cone at q. Take a sequence $r_i \to 0$ such that the rescaled metrics $(M_i', q_i') = (M_i, q_i, \frac{g_i}{r_i^2}) \to (V, o)$ in the pointed Gromov-Hausdorff sense. By corollary 3.1, for sufficiently large i, we can define a holomorphic vector field Z_i on $B(q_i', 100)$. By shifting q_i' a little bit, we may assume the flow generated by $-ReZ_i$ converges to q_i'.

Proposition 3.4. *Let σ_t^i be the flow generated by $-ReZ_i$, ρ_t be the flow on (V, o) generated by $-r_V\frac{\partial}{\partial r_V}$ where r_V is the distance to o. Then σ_t^i converges to ρ_t uniformly. More precisely, if $y_i \in B(q_i', 100)$ and $y_i \to y \in V$, then $\sigma_t^i(y_i) \to \rho_t(y)$ for any $t > 0$.*

Proof. We need two lemmas.

Lemma 3.1. *Let x be a regular point (in metric sense) around $o \in V$. Take a sequence $M_i' \ni x_i \to x$. Then $\sigma_t^i(x_i)$ converges to $\rho_t(x)$.*

Proof. For simplicity, we assume $d(x,o)=1$. The general case easily follows from a rescaling argument. Let C be a large constant, to be determined. Take a small geodesic ball $B(x,Cr)$ such that we have a holomorphic chart $U=(z_1,...,z_n)$. According to lemma 7.1 in [37], if C is large enough, we also have holomorphic charts $U_i=(z_1^i,...,z_n^i)$ around $B(x_i,r)$ and $z_j^i \to z_j$. In view of (3.22), by passing to subsequence if necessary, we may assume the holomorphic vector field Z_i converges to a holomorphic vector field Z on U. Then σ_t^i converges on $B(x_i,\frac{r}{10})$ to a holomorphic map σ_t for $|t|$ small.

Claim 3.4. *For any sequence $t_k \to 0$, $\lim\limits_{k\to\infty} \frac{z_j(\sigma_{t_k}(x))-z_j(\rho_{t_k}(x))}{t_k}=0$.*

Proof. We first blow up $x \in V$. Then $(V_k, x_k'', \tilde{d}_k)=(V,x,\frac{d}{t_k}) \to (\mathbb{R}^{2n},0)$. Let Φ_k be the Gromov-Hausdorff approximation from (M_k', q_k') to (V,o). Below we shall pass to subsequence of M_k' which is still denoted by M_k'. That is to say, for each k, M_k' is arbitrarily close to V as we want. Let us say Φ_k is a t_k-Gromov-Hausdorff approximation from $B_{N_k}(x_k',\frac{1}{t_k})$ to $B(x_k'',\frac{1}{t_k})$, where $(N_k,x_k',d_k')=(M_k',x_k,\frac{d_k}{t_k})$. As before, consider the holomorphic coordinates $(w_1^k,...,w_n^k)$ around x_k' satisfying $w_s^k(x_k')=0$, $(w_1^k,...,w_n^k)$ is a $\Phi(\frac{1}{k})$-Gromov-Hausdorff approximation to the Euclidean ball $B(0,100)$. We may further assume that $w_1^k=f_k$ as constructed in (3.16). We may regard Z_k as a holomorphic vector field on N_k. Define $Z_k'=t_kZ_k$. Let σ_t' be the flow generated by $-ReZ_k'$ on (N_k,x_k'). We have

$$\fint_{B(x_k',100)} |\langle dw_i^k, \overline{dw_j^k}\rangle - 2\delta_{ij}|^2 < \Phi(\frac{1}{k}). \tag{3.33}$$

As $d(x,o)=1$, by using the same argument as in (3.17),

$$|\langle ReZ_k', dw_i^k\rangle - \delta_{i1}| < \Phi(\frac{1}{k}). \tag{3.34}$$

Therefore

$$|w_i^k(\sigma_1'(x_k'))+\delta_{i1}| < \Phi(\frac{1}{k}). \tag{3.35}$$

On the other hand, we may require that for any $z \in B(x_k',2)$,

$$|Rew_1^k(z) - \frac{d_{M_k'}(q_k',z)}{t_k} + \frac{d(o,x)}{t_k}| < \Phi(\frac{1}{k}). \tag{3.36}$$

By definition of ρ_t,

$$|w_i^k(\overline{\rho_{t_k}(x)}) + \delta_{i1}| < \Phi(\frac{1}{k}), \tag{3.37}$$

where $\overline{\rho_{t_k}(x)}$ is a preimage of $\rho_{t_k}(x)$ on N_k under Φ_k. We have

$$|w_i^k(\sigma_1'(x_k')) - w_i^k(\overline{\rho_{t_k}(x)})| < \Phi(\frac{1}{k}). \tag{3.38}$$

The gradient estimate says $|dz_j^k| \leq C$ on $B(x_k, \frac{r}{10})$, where C is independent of k, j. By Cauchy estimate, $|\frac{\partial z_j^k}{\partial w_s^k}| \leq Ct_k$. Then

$$|z_j^k(\sigma_1'(x_k')) - z_j^k(\overline{\rho_{t_k}(x)})| < t_k\Phi(\frac{1}{k}). \tag{3.39}$$

If M_k' is sufficiently close to V, we can ensure that

$$\frac{z_j^k(\sigma_1'(x_k')) - z_j(\sigma_{t_k}(x))}{t_k} = \Phi(\frac{1}{k}), \frac{z_j^k(\overline{\rho_{t_k}(x)}) - z_j(\rho_{t_k}(x))}{t_k} = \Phi(\frac{1}{k}). \tag{3.40}$$

(3.39) and (3.40) give the proof of claim 3.4. □

Observe that the argument in claim 3.4 works for any regular point. Since $\rho_t(x)$ are all regular points for $t > 0$, $\frac{d\rho_t(x)}{dt} = (-ReZ)(\rho_t(x))$ for all t. By definition, $\frac{d\sigma_t(y)}{dt} = (-ReZ)(\sigma_t(y))$ for all t, y. Then by the uniqueness of the integral curve generated by $-ReZ$, we find $\sigma_t(x) = \rho_t(x)$ for all small $t > 0$. This completes the proof of lemma 3.1. □

Corollary 3.3. *Let f_i be a sequence of holomorphic functions on $B(q_i', 10)$ so that $f_i \to f_\infty$ uniformly on each compact set. Then $(-ReZ_i)(f_i) \to (-r_V\frac{\partial}{\partial r_V})f_\infty$ uniformly on $B(q_i', 2)$.*

Proof. Claim 3.4 says we have the convergence on the regular points on the limit space. Note that $|(-ReZ_i)(f_i)|$ has uniform L^2 bounds on $B(q_i', 5)$, as Z_i has uniform L^2 bound. Mean value inequality gives uniform bounds for $|(-ReZ_i)(f_i)|$. Hence its gradient bound is bounded. As regular points are dense on limit space, we complete the proof. □

Lemma 3.2. *Let $y_i \in M_i'$, $y \in V$ and $y_i \to y$. Then there exist $N > 0, r > 0, r' > 0$ which are independent of i such that for all sufficiently large i, the following holds:*

1. there exist a sequence of holomorphic embeddings F_i*:* $B(y, r') \to \mathbb{C}^N$*,* F*:* $B(y, r') \to \mathbb{C}^N$*.*

2. $F_i \to F$*, if we compose with the Gromov-Hausdorff approximation. Thus, the image of* $B(y_i, r')$ *converges to the image of* $B(y, r')$ *in the Hausdorff topology of* $\mathbb{C}^N$*.*

3. $F_i(y_i) = F(y) = 0 \in \mathbb{C}^N$*.*

4. $B(y_i, r) \subset F_i^{-1}(B_{\mathbb{C}^N}(0, 1)) \subset\subset B(y_i, r')$ *and* $B(y, r) \subset F^{-1}(B_{\mathbb{C}^N}(0, 1)) \subset\subset B(y, r')$*.*

Proof. According to the main theorem in [37], (V, o) is a normal complex analytic space. Thus for some small $a, b > 0$ and some large $C > 0$, we may holomorphically embed $B(y, Ca)$ in some $(\mathbb{C}^{N'}, 0)$ by map F' (Ca is still small). Say $F'(y) = 0 \in \mathbb{C}^{N'}$. Also we may assume $B(y, 2b) \subset F'^{-1}(B(0, 1)) \subset\subset B(y, a)$. Let the coordinates on $\mathbb{C}^{N'}$ be $z_1, ..., z_{N'}$. According to lemma 7.1 in [37], we may find holomorphic functions $z_1^i, ..., z_{N'}^i$ on $B(y_i, 2a)$ so that $z_s^i \to z_s$ for all s. By shrinking a if necessary, we may assume for each $B(y_i, 2a)$, there exists a holomorphic flow σ_t^i retracting to $y_i^{''}$. Consider the local holomorphic coordinates $U_i = (w_1^i, ..., w_n^i)$ near $y_i^{''}$ with $w_j^i(y_i^{''}) = 0$ for all j. By scaling, we may assume $|w_j^i| \leq \frac{1}{2^i}$ for all i, j. Note that the size of U_i could go to zero. For each i, there exists large t_i so that $\sigma_{t_i}^i(B(y_i, 2a)) \subset U_i$. We can pull the coordinate back to $B(y_i, 2a)$ via $\sigma_{t_i}^i$. Say the new coordinate on $B(y_i, 2a)$ is still denoted by $(w_1^i, .., w_n^i)$. Take $r = b$, $r' = 2a$, $N = N' + n$. One can verify that the holomorphic maps $F_i = (z_1^i, ..., z_{N'}^i, w_1^i, .., w_n^i)$ and $F = (z_1, ..., z_{N'}, 0, 0, 0, .., 0)$ satisfy the lemma. □

Let $z_1, .., z_N$ be coordinates of $\mathbb{C}^N$. We identify $B(y, r)$ and $B(y_i, r)$ with their images in $\mathbb{C}^N$. By (3.24), for each t small, σ_t^i is bounded. Thus $z_j(\sigma_t^i)$ is uniformly bounded for $j = 1, ..., N$. We can extract a subsequence such that $z_j(\sigma_t^i)$ all converge uniformly on $B(y_i, r)$. Lemma 3.2 says that $\sigma_t^i(B(y_i, r))$ converges uniformly. Notice y is arbitrary. Since regular points are dense, by lemma 3.1, we conclude the proof of proposition 3.4. □

4. Proof of theorem 1.1

In this section, let us assume (M, p) is a complete noncompact Kähler manifold with nonnegative bisectional curvature and maximal volume growth. Define $(M_i, p_i, g_i) = (M, p, \frac{g}{r_i^2})$ where $r_i \to \infty$. Cheeger-Colding theory says (M_i, p_i) is getting closer and closer to metric cones. We may apply proposition 3.1 to (M_i, p_i). Let us say the holomorphic vector field is Z_i and the flow

σ_t generated by $-ReZ_i$ converges to o_i. Since M_i is smooth Kähler manifold, at each point o_i, we may take a holomorphic chart on $B(o_i, \rho_i)$ which is also diffeomorphic to an Euclidean ball. We may assume $\lim_{i\to\infty} \rho_i = 0$. When t is sufficiently large, $\sigma_t(B(p_i, 1)) \subset B(o_i, \rho_i)$. This proves that $B(p_i, 1)$ is biholomorphic to a domain in $\mathbb{C}^n$.

Claim 4.1. *There exists some open set $B(p_i, \frac{1}{2}) \subset U \subset B(p_i, \frac{3}{4})$ such that U is diffeomorphic to $\mathbb{R}^{2n}$.*

Proof. Now we consider the inverse flow σ_{-t}. The hope is that for some large t, $\sigma_{-t}(B(o_i, \rho_i))$ would be the desired open set. However, in general, this might not be true. The problem is that some point might touch the boundary much earlier than other points. To overcome this difficulty, we cut off the holomorphic vector field ReZ_i. More precisely, let $f(u)$ be a smooth function with $f(u) = 1$ for $0 \leq u \leq \frac{1}{2}$; $0 < f(u) \leq 1$ for $\frac{1}{2} \leq u < \frac{3}{4}$; $f(u) = 0$ for $u \geq \frac{3}{4}$. Let σ'_{-t} be the flow generated by $ReZ_i(x)f(r_i(x))$ (r_i is the distance to p_i). Then σ'_{-t} is not holomorphic, but it induces diffeomorphism. Since $d_i(p_i, o_i) = \Phi(\frac{1}{i})$ and $\rho_i \to 0$, $B(o_i, \rho_i) \subset B(p_i, \frac{1}{10})$ for large i. Then σ'_{-t} exists for all $t > 0$ on $B(o_i, \rho_i)$ and $\sigma'_{-t}(B(o_i, \rho_i)) \subset B(p_i, \frac{3}{4})$ for all $t > 0$. Each orbit of σ'_{-t} belongs to an orbit of σ_{-t}. According to the uniform convergence of σ_t, there exists some $T > 0$ such that if $t > T$, $\sigma'_{-t}(\partial B(o_i, \rho_i)) \cap B(p_i, \frac{5}{8}) = \emptyset$. Let $U = \sigma'_{-2T}(B(o_i, \rho_i))$. Then U satisfies the claim. □

We may pull the open set U back to the original manifold M. We obtain an exhaustion of M by Euclidean balls. According to a theorem of Stallings, M is homeomorphic to $\mathbb{R}^{2n}$. If $n \neq 2$, then M is diffeomorphic to $\mathbb{R}^{2n}$. According to [36], M is biholomorphic to an affine algebraic variety. If $n = 2$, by a theorem of Ramanujam, M is biholomorphic to $\mathbb{C}^2$.

Next we move to general dimension. Let $\mathcal{O}_d(M)$ denote polynomial growth holomorphic functions on M with degree bounded by d. Let $\mathcal{O}_P(M) = \cup_{d>0} \mathcal{O}_d(M)$.

Assume the blow down sequence $(M_i, p_i, d_i) = (M, p, \frac{d}{r_i}) \to (M_\infty, p_\infty, d_\infty)$ in the Gromov-Hausdorff sense. Here r_i is a sequence increasing to infinity.

Proposition 4.1. *For any $d > 0$, $dim(\mathcal{O}_d(M)) = dim(\mathcal{O}_d(M_\infty))$.*

Proof. The proof is in fact contained in [36][35]. We only give a sketch. First we prove $dim(\mathcal{O}_d(M)) \leq dim(\mathcal{O}_d(M_\infty))$. Define an inner product $\langle f, g\rangle = \fint_{B(p_i,1)} f\overline{g}$ on $\mathcal{O}_d(M)$. Apply the three circle theorem and pass to limit for these functions. This concludes the proof of the first inequality. For details, see lemma 2 of [35].

For the reverse inequality, define a norm on $\mathcal{O}_d(M_\infty)$ by $\langle f, g\rangle = \fint_{B(p_\infty,1)} f\bar{g}$. Also define a norm on $\mathcal{O}_d(M)$ by $\langle u, v\rangle = \fint_{B(p,1)} u\bar{v}$. Let $f_1, .., f_s$ be a basis of $\mathcal{O}_d(M_\infty)$. For sufficiently large i, we may lift $f_j(1 \leq j \leq s)$ to $B(p_i, 1)$, say f^i_j. It is clear that f^i_j are linearly independent. We can find constants c_{ijk} so that $F_{ik} = \sum_j c_{ijk} f^i_j$ satisfies $\fint_{B(p,1)} F_{ik}\overline{F_{il}} = \delta_{kl}$. We look at the quotient

$$\frac{\sup_{B(p,\frac{r_i}{2})} |F_{ik}|}{\sup_{B(p,\frac{r_i}{3})} |F_{ik}|} = \frac{\sup_{B(p_i,\frac{1}{2})} |F_{ik}|}{\sup_{B(p_i,\frac{1}{3})} |F_{ik}|} = \frac{\sup_{B(p,\frac{r_i}{2})} |\sum_j c_{ijk} f^i_j|}{\sup_{B(p,\frac{r_i}{3})} |\sum_j c_{ijk} f^i_j|}. \tag{4.1}$$

By dividing by the supremum of c_{ijk} (fix i, k), we may assume that the maximal coefficient in the last part of (4.1) is equal to 1. As f^i_j are linearly independent, by a compactness argument, we find that for sufficiently large i, (4.1) is bounded by $d + \epsilon$ for any $\epsilon > 0$. Let $i \to \infty$ and apply the three circle theorem, the functions F_{ik} converge to linearly independent functions on $\mathcal{O}_d(M)$. □

By choosing a large D, we may assume $\mathcal{O}_D(M)$ embeds M to $\mathbb{C}^{N-1}$ and $\mathcal{O}_D(M_\infty)$ embeds M_∞ to $\mathbb{C}^{N-1}$. Here $N = dim(\mathcal{O}_D(M)) = dim(\mathcal{O}_D(M_\infty))$. That is, we ignore the constant function in the holomorphic embedding.

Let us apply some argument in [16]. By dimension estimate for $\mathcal{O}_d(M)$, we can find a strictly increasing sequence $d_1, d_2, d_3, ...$ so that for any d satisfying $d_s \leq d < d_{s+1}$, $\mathcal{O}_{d_s}(M) = \mathcal{O}_d(M) \neq \mathcal{O}_{d_{s+1}}(M)$. Let us choose $f_{s,l} \in \mathcal{O}_{d_s}(M)(l = 1, 2, .., l_s)$ so that they form a basis of $\mathcal{O}_{d_s}(M)/\mathcal{O}_{d_{s-1}}(M)$ as quotient of vector spaces. Set W_s =span$\{f_{s,1}, .., f_{s,l_s}\}$. Let $f^i_{s,l}$ be an orthonormal basis of W_s, with respect to the L^2 integration on $B(p_i, 1)$. After taking subsequences, we may assume $f^i_{s,l} \to f^\infty_{s,l}$ uniformly on each compact set of M_∞.

Claim 4.2. *$f^\infty_{s,l}$ is homogeneous of degree d_s.*

Proof. First, $f^\infty_{s,l} \in \mathcal{O}_{d_s}(M_\infty)$ by three circle theorem. Second, for any $\epsilon > 0$, there exists $R > 0$ depending on ϵ so that for any function $u \in W_s$, $N(R, u) = \frac{\sup_{B(p,2R)} |u|}{\sup_{B(p,R)} |u|} \geq 2^{d_s} - \epsilon$. To prove this, write $u = c \sum a_l f_{s,l}$ where $\sup |a_l| = 1$. We may assume $c = 1$ by scaling. By definition of W_s, for each u, we can find R_u so that $N(R_u, u) \geq 2^{d_s} - \frac{\epsilon}{2}$. By continuity, if $v = \sum b_l f_{s,l}$ and $|a_l - b_l|$ is sufficiently small, $N(R_u, v) \geq 2^{d_s} - \epsilon$. Three circle theorem implies that $N(r, u)$ is monotonic increasing. Now the second point follows from the compactness of $\mathbb{CP}^{l_s-1}$ (we are thinking the coefficients lives in $\mathbb{CP}^{l_s-1}$). We can apply the argument of proposition 5 in [38] to complete the proof of the claim. □

Let Z be the vector space of holomorphic vector fields X on M so that $X(\mathcal{O}_d(M)) \subset \mathcal{O}_d(M)$ for all d. This means for any $f \in \mathcal{O}_d(M)$, the derivative $X(f) \in \mathcal{O}_d(M)$. Finite generation of $\mathcal{O}_P(M)$ and linear algebra imply that Z has finite dimension. Similarly, let Z_∞ be the vector space of holomorphic vector fields Y on M_∞ so that $Y(\mathcal{O}_d(M_\infty)) \subset \mathcal{O}_d(M_\infty)$ for all d.

Let $f_1, .., f_N$ be a basis for $\mathcal{O}_D(M)$. Assume $\{f_1^i, .., f_N^i\}$ is a new basis so that the functions are orthonormal with respect to the L^2 integration on $B(p_i, 1)$. Let $X_1, ..., X_k$ be a basis of Z. We can find new basis $X_1^i, ..., X_k^i$ so that they are orthnormal with respect to the Hermitian inner product defined by $\langle X_a, \overline{X}_b\rangle_i = ⨍_{B(p_i,1)} \sum_{j=1}^{N} \langle X_a(f_j^i), \overline{X_b(f_j^i)}\rangle$. We can similarly define a Hermitian inner product on Z_∞ by $\langle Y_a, \overline{Y}_b\rangle_\infty = ⨍_{B(p_\infty,1)} \sum_{j=1}^{N} \langle Y_a(f_j^\infty), \overline{Y_b(f_j^\infty)}\rangle$. Here f_j^∞ is the limit of f_j^i. By proposition 4.1, they form a basis of $\mathcal{O}_D(M_\infty)$. Also, by the choice of D, these functions embed M_∞ in $\mathbb{C}^{N-1}$.

Definition 4.1. *Let $R > 0$. Let X_i be a holomorphic vector field on $B(p_i, 2R)$. We say X_i converges to X_∞ on $B(p_\infty, R)$, if for any $d > 0$, any $f_i \in \mathcal{O}_d(M_i)$ with $f_i \to f_\infty$ on $B(p_\infty, 2R)$, $X_i(f_i) \to X_\infty(f_\infty)$ uniformly on $B(p_\infty, R)$.*

We assert that after taking subsequence, $X_1^i, .., X_k^i$ converge to $X_1^\infty, X_2^\infty, ..., X_k^\infty$ on M_∞. Moreover, $X_j^\infty(\mathcal{O}_d(M_\infty)) \subset \mathcal{O}_d(M_\infty)$ for all d. To see this, assume $g_i \in \mathcal{O}_d(M_i)$, $g_i \to g_\infty$ uniformly on each compact set of M_∞. Take a complex analytic regular point $q \in M_\infty$. Then we can find a subset $\{a_s | 1 \leq s \leq n\} \subset \{1, .., N\}$ so that $f_{a_1}^\infty, ..., f_{a_n}^\infty$ form a local holomorphic chart near q. For sufficiently large i, we may find $q_i \in M_i$ and $q_i \to q$ so that $f_{a_1}^i, ..., f_{a_n}^i$ form a local holomorphic chart near q_i and the charts have uniform size independent of i. Around q_i, we can write $g_i = g_i(f_{a_1}^i, ..., f_{a_n}^i)$. By standard Cauchy estimates, around q_i, we find uniform bounds for $X_j^i(g_i)$. As $X_j^i(g_i) \in \mathcal{O}_d(M_i)$, we can apply three circle theorem to find uniform bound of $X_j^i(g_i)$ on any compact set. Arzela-Ascoli argument proves the assertion.

Corollary 3.3 states $-ReZ_i \to -r\frac{\partial}{\partial r}$ (recall Z_i was defined in (3.13)). It is clear from the definition that the Hermitian inner product $\langle\cdot,\cdot\rangle_i$ on Z converges to the inner product $\langle\cdot,\cdot\rangle_\infty$ on Z_∞. That is, $X_1^\infty, X_2^\infty, ..., X_k^\infty$ are orthogonal unit vectors on Z_∞. Note we do not assert that $X_1^\infty, X_2^\infty, ..., X_k^\infty$ form a basis of Z_∞. A priori, $dim(Z_\infty)$ could be strictly greater than $dim(Z)$.

The following claim is crucial. The argument is in the same spirit as claim 6.1 of [36].

Claim 4.3. *The complexification of $-r\frac{\partial}{\partial r}$ is in the span of $X_1^\infty, ..., X_k^\infty$.*

Proof. It is clear that the complexification of $-r\frac{\partial}{\partial r} \in Z_\infty$. Assume the claim is not true. After orthogonalization via $\langle\cdot,\cdot\rangle_i$, we can find a basis $X_1^i, .., X_k^i, X_{k+1}^i$ of span of $X_1, .., X_k, Z_i$ so that $X_j^i \to X_j^\infty$ for all $j = 1, .., k+1$. Here we may require $X_j^i (1 \leq j \leq k)$ be the same as defined above the claim. We further require that $X_1^\infty, .., X_{k+1}^\infty$ be linearly independent.

We can also diagonalize the span of $X_1, ..., X_k, Z_i$ on $B(p, 1)$. This is just given by the L^2 integration on $B(p, 1)$. Say the new basis is given by $Z_1^i, .., Z_{k+1}^i$ and $Z_j^i = \sum_{s=1}^{k+1} a_{ijs} X_s^i$. We assert that by taking subsequence, $Z_1^i, ..., Z_{k+1}^i$ converge uniformly, as holomorphic vectors on each compact set of M, to holomorphic vector fields in Z. If this is proved, we have a contradiction with that $dim(Z) = k$.

To prove the assertion, let $f \in \mathcal{O}_d(M)$. Let $s_j^i = \frac{\max_{B(p_i,2)} |Z_j^i(f)|}{\max_{B(p_i,1)} |Z_j^i(f)|} = \frac{\max_{B(p_i,2)} |(\sum a_{ijs} X_s^i)(f_i)|}{\max_{B(p_i,1)} |(\sum a_{ijs} X_s^i)(f_i)|}$. Here $f_i = c_i f$ where c_i is a constant so that the L^2 norm of f_i is 1 on $B(p_i, 1)$. Assume $f_i \to f_\infty$. We can also find a constant c_i' so that $b_{ijs} = c_i' a_{ijs}$ satisfy $\max_s |b_{ijs}| = 1$. Say $b_{ijs} \to b_{\infty js}$.

Case 1:

If $\sum b_{\infty js} X_s^\infty (f_\infty) \neq 0$, $s_j^i \leq 2^d + \epsilon$ for all sufficiently large i. Three circle theorem implies that $Z_j^i(f)$ converges to a function in $\mathcal{O}_d(M)$.

Case 2:

If $\sum b_{\infty js} X_s^\infty (f_\infty) = 0$, then for some large e, we can find $g_i \in W_e$ so that $\int_{B(p,1)} |g_i|^2 = 1$. Furthermore, we can require that after normalization of g_i on $B(p_i, 1)$, $\hat{g}_i \to g_\infty$ on $B(p_\infty, 1)$ and $\sum b_{\infty js} X_s^\infty (g_\infty) \neq 0$. Here $\hat{g}_i = h_i g_i$ for constants h_i. Also $\int_{B(p_i,1)} |\hat{g}_i|^2 = 1$. By using the argument in case 1, after passing to subsequence, we may assume $g_i \to g \in W_e$, $Z_j^i(g_i) \to u_j \in \mathcal{O}_{d_e}(M)$, $Z_j^i(g_i f) \to v_j \in \mathcal{O}_{d+d_e}(M)$. Now $Z_j^i(f) = \frac{Z_j^i(fg_i) - Z_j^i(g_i) f}{g_i} \to \frac{v_j - f u_j}{g}$. Note the convergence is uniform on each compact set. There is no problem near the zero of g_i or g (just apply the Cauchy estimate).

Lemma 4.1. *$\mu_j = \frac{v_j - f u_j}{g} \in \mathcal{O}_d(M)$.*

Proof. Observe the numerator has order $d + d_e$. We need the following result in [40].

Proposition 4.2 (Mok)**.** *Let f, g be polynomial growth holomorphic functions on a complete Kähler manifold M with Ric ≥ 0. Suppose $h = \frac{f}{g}$ is holomorphic, then h is of polynomial growth.*

Proof. Let us say $f(p), g(p) \neq 0$. Set $F_1(x) = \log|f(x)|^2 + \int_{B(p,R)} G_R(x, y)\Delta \log|f(y)|^2, F_2(x) = \log|g(x)|^2 + \int_{B(p,R)} G_R(x,y)\Delta \log|g(y)|^2$.

Lemma 4.2. *For large R and $i = 1, 2$, on $B(p, \frac{R}{2})$, $-C\log R \leq F_i(x) \leq C\log R$.*

Proof. It is clear that $F_i(x)$ is harmonic on $B(p, R)$. Now maximum principle says that $F_i(x) \leq C\log R$ on $B(p, R)$. Let $H_i = C\log R - F_i \geq 0$. Then gradient estimate implies that on $B(p, \frac{3}{4}R)$, $|\nabla \log H_i| \leq \frac{C_1}{R}$. Observe $H_i(p) \leq C\log R$. Then the harnack inequality implies that $H_i \leq C_2 \log R$ on $B(p, \frac{R}{2})$. This completes the proof of the claim. □

It is clear that on $B(p, \frac{R}{2})$, $\log|h(x)|^2 \leq F_1(x) - F_2(x) \leq C\log R$ (C is independent of R). The proof of the proposition is complete. □

We come back to the proof of the lemma. Let us assume a is the smallest number so that $\mu_j \in \mathcal{O}_a(M)$. Then three circle theorem says $\lim_{r\to\infty} \frac{M(\mu_j, 2r)}{M(\mu_j, r)} = 2^a$ where $M(\mu_j, r) = \sup_{B(p,r)} |\mu_j|$. Assume the lemma is not true. Then $a > d$. As $g \in W_e$, we can apply claim 4.2. After normalizing the functions μ_j and g on $B(p_i, 1)$ and taking limits, we find their product of the limits, converges to a homogeneous function of degree $a + d_e$ on M_∞. However, this contradicts that $v_j - f u_j \in \mathcal{O}_{d+d_e}$.

We conclude that $Z^i_j(f)$ converges to a holomorphic function of degree d, for any $f \in \mathcal{O}_d(M)$. This implies that Z^i_j converges to an element in Z. The assertion is proved. □

The proof of claim 4.3 is complete. □

Recall $\mathcal{O}_D(M)$ embeds M in $\mathbb{C}^{N-1}$. By applying claim 4.2, we can find basis f^j_i of $\mathcal{O}_D(M)$ so that they are almost orthonormal on $B(p_i, 1)$ and the limits are all homogeneous. Let us take for granted that $f^N_i = 1$ (this is the constant function in $\mathcal{O}_D(M)$). Given claim 4.3, we can find $X_i \in Z$ so that X_i converges to the complexification of $-r\frac{\partial}{\partial r}$ on any compact set of M_∞. In particular, $X_i(f^j_i) \to 2(-r\frac{\partial}{\partial r})f^j_\infty = -2d(f^j_\infty)f^j_\infty$. In the last equality, we have used that f_∞^j are homogeneous. By using the basis f_i^j of $\mathcal{O}_D(M)$, we find the action of X_i on $\mathcal{O}_D(M)$ is given by

$$X_i(\vec{P_i}) = A_i\vec{P_i} + \vec{C_i}, \tag{4.2}$$

where $\vec{P_i}$ is the column vector $(f_i^1, .., f_i^{N-1})^T$, $\vec{C_i}$ is a constant $(N-1)\times 1$ vector. A_i is a constant $(N-1)\times(N-1)$ matrix (depending on i) which satisfies that all real parts of diagonal elements are less than or equal to $-\frac{1}{2}$ (recall that the degree of a nonzero homogeneous holomorphic function on M_∞ is at least 1), also the off diagonal elements are very small. In particular, the real part of each eigenvalue of the matrix is strictly negative. Let us fix a sufficiently large i_0. Set $X = X_{i_0} \in Z$.

Claim 4.4. *X is an integrable vector field on M. Moreover, X retracts M to a point.*

Proof. This is just linear algebra. Indeed, the action of X on M could be seen from (4.2). We extend the vector field in the natural way to $\mathbb{C}^{N-1}$ which we still call X. Since the real part of each eigenvalue is strictly negative, the flow σ_t generated by X must retract $\mathbb{C}^{N-1}$ to a point, say o. Then $o \in M$. □

Now from a result in [49], we see that M is biholomorphic to $\mathbb{C}^n$. Below we shall analyze the coordinates of M in more details. Let us consider the Poincare-Dulac normal coordinate [26]. The result (page 1190 of [26]) says that we can find a local holomorphic chart $U = U(z_1, ..., z_n)$ near o (the unique fixed point) so that U is the unit ball in $\mathbb{C}^n$ (measured in Euclidean coordinate $(z_1, ..., z_n)$) and $X = -\sum\limits_{j=1}^{n}(\lambda_j z_j + g_j(z))\frac{\partial}{\partial z_j}$, where

- $0 < Re\lambda_1 \leq Re\lambda_2 \leq \cdots \leq Re\lambda_n$
- $g_1 \equiv 0$
- For every $j \in \{2, ..., n\}$, $g_j(z)$ is a polynomial of $z_1, .., z_{j-1}$ only, vanishing at the origin. If the identity $\lambda_j = \sum\limits_{k=1}^{j-1} m_k\lambda_k$ holds for some nonnegative integers m_k, then the condition $g_j(e^{\lambda_1 t}z_1, ..., e^{\lambda_{j-1}t}z_{j-1}) = e^{\lambda_j t}g_j(z_1, ..., z_{j-1})$. If $\lambda_j = \sum\limits_{k=1}^{j-1} m_k\lambda_k$ never hold for nonnegative integers m_k, $g_j = 0$.

On $M' = \mathbb{C}^n$, we can define a holomorphic vector $\hat{X} = -\sum\limits_{j=1}^{n}(\lambda_j z_j + g_j(z))\frac{\partial}{\partial z_j}$, where g_j is the same polynomial as in X. By ode, one can prove that $\hat{X}$ is integrable. Let $\hat{\sigma}_t(z)$ be the flow generated by $\hat{X}$ on $\mathbb{C}^n$. Then one can verify that $\hat{\sigma}_t$ is a retracting holomorphic vector field on M' with the origin as the unique fixed point.

U is an open set of M. Let us identify it with the unit ball in $M' = \mathbb{C}^n$. Define a map $F : M' = \mathbb{C}^n \to M$ as follows: Given any $z \in M' = \mathbb{C}^n$, we can find sufficiently large t so that $\hat{\sigma}_t(z)$ is contained in the unit ball. Then define $F(z) = \lim_{t\to+\infty} \sigma_{-t}(\hat{\sigma}_t(z))$. Since the vector field $\hat{X}$ on the unit ball of M' is the same as X in $\mathbb{C}^n$, we obtain that $F(z)$ is well defined (independent of the value of t, as t is sufficiently large). It is clear that F is holomorphic and invertible: $F^{-1}(y) = \lim_{t\to+\infty} \hat{\sigma}_{-t}(\sigma_t(y))$.

Now let us check that these coordinate functions $z_1, ..., z_n$ are of polynomial growth on M. We use induction on the degree of $Re\lambda_s$. Assume z_s are all of polynomial growth for $Re\lambda_s \leq h$. Let $h_1 > h$ be so that there exists some j with $Re\lambda_j = h_1$ while there is no $Re\lambda_j$ between h and h_1. Assume for $j = j_1, .., j_1 + k - 1$, z_j satisfy $Re\lambda_j = h_1$.

Let $\mathcal{O}'_D(M)$ be the subset of $\mathcal{O}_D(M)$ which vanish at o (recall this is the unique fixed point of the flow generated by X). Then $X(\mathcal{O}'_D(M)) \subset \mathcal{O}'_D(M)$. Let $f_1, .., f_{N-1}$ be the basis of the Jordan form for the action of X on $\mathcal{O}'_D(M)$. We claim that each f_s is a polynomial of $z_1, .., z_n$. Given a monomial $z_1^{i_1} \cdots z_n^{i_n}$, define the weight w as $\lambda_1 i_1 + \cdots + \lambda_n i_n$. Since $Re\lambda_s > 0$, given any $c \in \mathbb{R}$, there are at most finitely many monomials (up to a factor) so that the real part of w is no greater than c. Note the action of X on monomials preserves the weight. Let V_w be the span of monomials with weight w. Then each V_w is finite dimensional.

Assume f_s (generalized eigenvector) corresponds to eigenvalue λ. By Taylor expansion at o and Cayley-Hamilton theorem, we see $f_s \in V_\lambda$. In particular, f_s is a polynomial of $z_1, .., z_n$. Since $f_1, .., f_{N-1}$ gives the embedding of M in $\mathbb{C}^{N-1}$, we can always find $f_{l_1}, ..., f_{l_k}$ so that $\det(\frac{\partial f_{l_s}}{\partial z_j})|_{s=1,..,k}^{j=j_1,...,j_1+k-1} \neq 0$ at 0. In particular, these f_{l_s} must satisfy that the real part of the eigenvalue is equal to h_1. According to induction, there exists an invertible $k \times k$ matrix A so that $f_{l_s} = \sum_j A_{sj} z_j + B_s$, where each B_s has polynomial growth. Thus z_j has polynomial growth. The induction is completed.

As any function in $\mathcal{O}_D(M)$ is a polynomial of $z_1, .., z_n$, we see that $\mathcal{O}_P(M)$ is generated by n polynomial growth holomorphic functions $z_1, .., z_n$. We can say in this way, M is isomorphic to $\mathbb{C}^n$. Thus theorem 1.1 is proved.

Now we give a "canonical" global coordinate on M. For any $m \in \mathbb{N}$, pick a maximal linearly independent vectors $f_{m,1}, .., f_{m,k_m}$ of $\mathcal{O}_{d_m}(M)$ so that no element in the span is given by polynomials of $\mathcal{O}_{d_{m-1}}(M)$. Then the first n functions $\{f_{1,1}, ..., f_{1,k_1}, f_{2,1}, ...\}$ form a global coordinate system on M. The proof uses that the eigenvalues λ_j are close to the degrees of holomorphic functions in $\mathcal{O}_D(M)$. Basically we can show that $z_1, .., z_n$ are polynomials of these functions.

Finally, notice that corollary 1.1 follows, since f with minimal degree can serve as one of the global coordinate functions on M.

5. Proof of theorem 1.2

Proof. Let us adopt the assumptions in theorem 1.2. According to the main theorem in [37], X is homeomorphic to an irreducible normal complex analytic space. Take $q \in X$ and a tangent cone V at q. Let ϵ be a very small number. Then there exists $r > 0$ such that $d_{GH}(B_X(q, 100r), B_V(o, 100r)) < \epsilon r$. We may assume r is sufficiently small. First we give a separate proof for the case when $n = 2$, since the argument is more instructive. Then as X is normal with dimension 2, the possible singularities are all isolated. Without loss of generality, assume q is an analytic singular point and a small punctured ball $B(q, 100r)\backslash\{q\}$ is analytically smooth. Take a closed curve γ in the small punctured ball $B(q, r)\backslash\{q\}$. We may assume that there exists some $\epsilon_1 > 0$ so that $\gamma \subset B(q, r)\backslash B(q, \epsilon_1 r)$.

Lemma 5.1. *γ is contractible in $B(q, 10r)\backslash\{q\}$, if ϵ is small enough.*

Proof. Consider $M_i \ni q_i \to q$ in the Gromov-Hausdorff sense. Then according to the argument in [37], we may assume that the Gromov-Hausdorff convergence is in fact smooth in the complex analytic sense (not necessarily in metric sense) from $B(q_i, 50r)\backslash B(q_i, \frac{1}{100}\epsilon_1 r)$ to $B(q, 50r)\backslash B(q, \frac{1}{100}\epsilon_1 r)$. In particular, there exists a diffeomorphism from $\gamma \subset B(q, r)\backslash B(q, \epsilon_1 r)$ to $U \subset\subset B(q_i, 50r)\backslash B(q_i, \frac{1}{100}\epsilon_1 r)$. We may life the curve γ to U. It suffices to prove the image of γ is contractible on U. This can be done by using the same argument as in claim 4.1. Basically we prove that the image of γ lies between two topological balls. The details are omitted. $\square$

Now we apply Mumford criteria [43] to obtain that q is in fact a smooth point for the normal variety X. Thus X is complex analytically smooth. Let z_1, z_2 be a holomorphic chart around $q \in X$. According to lemma 7.1 in [37], we may find holomorphic functions z_1^i, z_2^i on fixed size neighborhood of q_i with $z_1^i \to z_1$, $z_2^i \to z_2$. By using a degree argument, one can verify that z_1^i, z_2^i form a holomorphic chart on some fixed size neighborhood of q_i. The stability for dimension 2 follows from a standard gluing argument.

Next we consider the general n dimensional case.

From proposition 3.4, we see $-r_V\frac{\partial}{\partial r_V}$ is a holomorphic vector field on V. Given any holomorphic function f around $o\in V$, we may write f as an infinite sum of homogeneous harmonic functions (basically we just do the spectral decomposition on the cross section). We claim that each homogeneous function appeared must be holomorphic. For instance, to show the lowest degree harmonic function (say degree a) is holomorphic, one verifies that it is the limit of $f(\sigma_t(x))e^{at}$ as $t\to+\infty$. By subtracting the first function, one can show the remaining homogeneous harmonic functions are all holomorphic.

Let $z_1,...,z_{N'}$ be holomorphic functions on V which give a local holomorphic embedding near o. Let us say $z_s(o)=0$ for all s. Now we use some argument in [16]. Consider the restriction of $z_1,...,z_{N'}$ on V. We write $z_s=\sum z_s^\alpha$ where z_s^α are all homogeneous holomorphic functions as in the last paragraph. Then we extend each z_s^α to a holomorphic function on $B(0,1)$ of $\mathbb{C}^N$. We may require that the sum is still equal to z_s on $B(0,\frac{1}{2})$, since there is a bounded extension of holomorphic functions. See for example, corollary 4 on page 157 of [21]. Then for each s, we can find some z_s^α so that $\det(\frac{\partial z_s^\alpha}{\partial z_s})\neq 0$ at 0. According to the implicit function theorem, these z_s^α form a local holomorphic chart in a small neighborhood of 0 in $\mathbb{C}^N$. Thus we obtain a global holomorphic embedding from V to $\mathbb{C}^{N'}$ by these z_s^α. For notational convenience, we still denote the homogeneous coordinates by z_s. Consider the integral ring R generated by functions z_s on V. By using the three circle theorem in [34], we can prove the dimension estimate $dim(\mathcal{O}_d(V))\leq Cd^n$ as in the smooth case. Here $\mathcal{O}_d(V)$ denotes polynomial growth holomorphic functions with degree bounded by d. By a dimension counting argument, we see that the affine algebraic variety defined by R has dimension n. Then we can verify that V is biholomorphic to the affine algebraic variety defined by R. Since the argument is very similar to section 7 in [36], we skip the details.

We may find $C>0$ so that $(z_1,..,z_{N'})^{-1}B_{\mathbb{C}^N}(0,10)\subset\subset B(o,C)$. Moreover, $(z_1,..,z_{N'})$ is a holomorphic embedding on $B(o,2C)$. Next we lift these z_s to M_i', say $z_s^i\to z_s$ uniformly on $B(q_i',C)$. We add coordinate functions $w_k^i(k=1,..,n)$ as introduced in the proof of lemma 3.2. Set $N=N'+n$. Then we can define embeddings $F_i\to F$ as $(z_1^i,..,z_{N'}^i,w_1^i,...,w_n^i)$ and $(z_1,..,z_{N'},0,0,..,0)$. We identify V and $B(q_i',C)\subset M_i'$ with their image in $\mathbb{C}^N$. Let $(z_1,..,z_N)$ be the coordinate on $\mathbb{C}^N$. Define a holomorphic vector field Y on $\mathbb{C}^N$ by $Y=-\sum_j\alpha_jz_j\frac{\partial}{\partial z_j}$ where $\alpha_j\geq 0$ is the degree of z_j. For $j>N'$, we set $\alpha_j=0$. Then ReY coincides with the vector field $-r_V\frac{\partial}{\partial r_V}$

on V. Observe on the intersection of the unit sphere in $\mathbb{C}^N$ and V,

$$ReY(\sum |z_j|^2) < 0. \tag{5.1}$$

Note also that $\alpha_j \geq 1$ if z_j is not zero. Otherwise, there exists some homogeneous function which is of sublinear growth. Then by gradient estimate, it must be constant, hence, identically zero. Since $-ReZ_i(z_j)$ (holomorphic) is uniformly convergent to $-ReY(z_j)$ on $B(o, \frac{C}{2})$, we have for sufficiently large i,

$$-ReZ_i(\sum |z_j|^2) < -\frac{1}{10} \tag{5.2}$$

on the intersection of the unit sphere in $\mathbb{C}^N$ and M_i'. We have used $\alpha_j \geq 1$.

Proposition 5.1. *Given any $n \in \mathbb{N}, v > 0$, there exist $\epsilon = \epsilon(n, v) > 0, \delta = \delta(n, v) > 0$ so that the following holds: let (M^n, p) be a complete Kähler manifold with bisectional curvature lower bound $-\epsilon^3$. Assume $vol(B(p, \frac{1}{\epsilon})) \geq \frac{v}{\epsilon^{2n}} > 0$ and $B(p, \frac{1}{\epsilon})$ is ϵ-Gromov-Hausdorff close to a geodesic ball centered at the vertex of $(\mathbb{R}^{2k}, 0) \times (Y, o)$. Here (Y, o) is a metric cone. Then there exists an open set U and a map F satisfying $B(p, \delta) \subset U \subset B(p, 1)$, $F : \mathbb{D}^k \times Z \to U$ is a biholomorphism. Here $\mathbb{D}^k$ is a polydisk in $\mathbb{C}^k$ and Z is a complex manifold. Finally, for any two points $y_1, y_2 \in B(p, \delta) \cap F((0, 0, .., 0) \times Z)$, there exists a curve $l \subset B(p, g(\delta)) \cap F((0, 0, .., 0) \times Z)$ connecting y_1, y_2. Here g is a continuous increasing function on $\mathbb{R}$ depending only on n, v and $g(0) = 0$. Also, $g(\delta) < \frac{1}{10}$.*

Proof. We will assume ϵ is as small as we want. Eventually we see its value depends only on n, v. By proposition 2.1 and similar arguments in section 3, we find holomorphic functions $z_1, .., z_k$ and holomorphic vector fields $X_1, .., X_k$ on $B(p, 1)$ (X_s is obtained by perturbing the gradient of the harmonic functions) such that $(z_1, .., z_k)$ almost gives the splitting of the factor $\mathbb{R}^{2k}$. Also,

$$|X_i(z_j) - \delta_{ij}| < \Phi(\epsilon|n, v). \tag{5.3}$$

Let us assume $z_s(p) = 0$ for all s. Now we use induction. When $k = 0$, the conclusion is trivial. Assume the proposition is true for $k = m - 1$ and $U_{m-1} = \mathbb{D}^{m-1} \times Z_{m-1}$ where Z_{m-1} is given by the zeros of $z_1, ..., z_{m-1}$. We shall prove it $k = m$. Let $\epsilon_{m-1}, \delta_{m-1}$ be the corresponding constants in proposition 5.1 for $k = m - 1$.

Claim 5.1. *There exist holomorphic functions* $c_1, ..., c_{m-1}, c_m$ *with* $|c_i|(1 \leq i \leq m-1)$ *and* $|1-c_m|$ *very small such that if* $W_m = c_m X_m - \sum_{i=1}^{m-1} c_i X_i$, *then* $W_m(z_j) = \delta_{jm}$.

Proof. Just use linear algebra, in view of (5.3). □

The claim says W_m is a holomorphic vector field on Z_{m-1}. Define Z_m be the zeros of $z_1, ..., z_m$. Then Z_m is smooth by claim 5.1. Let σ_t be the flow on Z_{m-1} generated by W_m for $t \in \mathbb{C}$. Define a holomorphic map $\sigma : Z_m \times \Delta_m$ ($\Delta_m = \{t \in \mathbb{C} | |t| < \gamma\}$) to Z_{m-1} as $\sigma(x, t) = \sigma_t(x)$. Here $\gamma = \gamma(n, v)$ is small, to be determined.

Claim 5.2. *If* γ *is small,* σ *defines a biholomorphic map onto its image which contains* $B(p, \delta_m) \cap Z_{m-1}$ *for some* $\delta_m = \delta_m(n, v) > 0$.

Proof. By using the same argument as in section 3, we see that if $\gamma = \gamma(v, n)$ is small and $x \in Z_m \cap B(p, \frac{1}{10})$, $\sigma(t, x) \in B(x, \frac{1}{10})$ for $|t| < \gamma$. By claim 5.1, $z_j(\sigma(x, t)) = t\delta_{jm}$. Therefore $\sigma(x, t)$ is injective for $x \in Z_m \cap B(p, \frac{1}{10})$, $|t| < \gamma$. If δ_m small and $y \in B(p, \delta_m) \cap Z_{m-1}$, $\sigma(y, -z_m(y)) \in Z_m$. □

Let $y_1, y_2 \in B(p, \delta_m) \cap Z_m$. We may assume $\delta_m < \delta_{m-1}$. By induction hypothesis, there exists a curve $l \subset B(p, g_{m-1}(\delta_m)) \cap Z_{m-1}$ connecting y_1, y_2. Here g_{m-1} is a continuous increasing function with $g_{m-1}(0) = 0$. If $\delta_m = \delta_m(n, v)$ is small enough, we can project l to Z_m via σ. The image lies in $B(p, g_m(\delta_m))$ for some continuous increasing function g_m depending only on n, v. Of course, we may assume $g_m(\delta_m) < \frac{1}{10}$. The last assertion is verified. The proof of proposition 5.1 is complete. □

Remark 5.1. *In the proposition above, we have identified* $\mathbb{D}^k = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_k$ *where* Δ_k *is defined right above claim 5.2. Let us endow* $\mathbb{D}^k$ *with the product metric on the right hand side. Let the distance on* Z_m *be induced by the distance on* M. *Then by a limiting argument, we can prove that the biholomorphic map* F *is a* $\Phi(\epsilon|n, v)$*-Gromov-Hausdorff approximation.*

Proposition 5.2. *Let* (M_i^n, p_i) *be a sequence of pointed Kähler manifolds converging to* $X = \mathbb{C}^k \times (V, o)$ *in the Gromov-Hausdorff sense, where* (V, o) *is a metric cone. Assume the bisectional curvature of* M_i^n *has lower bound* $-\Phi(\frac{1}{i})$ *and* $vol(B(p_i, r)) \geq cr^{2n}$ *for any* $0 < r < R_i$, *where* c *is a positive constant and* $R_i \to \infty$. *Then* V *is homeomorphic to an irreducible normal complex analytic variety.*

Proof. By applying proposition 5.1 to M_i, we construct holomorphic vector fields $W^i_j(1 \leq j \leq k)$ and holomorphic functions z^i_s so that $W^i_j(z^i_s) = \delta_{js}$. Let $\sigma_{ij}(t)$ be the biholomorphisms induced by W^i_j. Then after passing to subsequences, $\sigma_{ij}(t) \to \sigma_j(t)$ which induces biholomorphism on the limit space. Also $z^i_s \to z_s$. Now set Σ be the zero set of $z_1, .., z_k$. We should regard Σ as a closed subscheme induced by the ideal generated by $z_1, .., z_k$. Since X is irreducible, by using projections as in the last part of the proof of proposition 5.1, we see that the regular points of Σ are connected. By claim 5.1, we see Σ is reduced. Therefore Σ is integral. We can also verify that X is isomorphic to $\Sigma \times \mathbb{C}^k$ as a complex space. Since X is normal, Σ must be normal. Also X is isometric to $\Sigma \times \mathbb{C}^k$ (the metric on Σ is induced from X), since the coordinate functions $z_1, ..., z_k$ are Euclidean splitting factors. □

Proposition 5.3. *In proposition 5.2, if $k \geq n - 3$, then X is in fact a complex manifold.*

Proof. If $k = n$ or $n - 1$, the conclusion follows from sec 6 of [37]. Let Z_i be the holomorphic vector field in (3.13). By shifting p_i a little bit if necessary, we may assume that the flow generated by $-ReZ_i$ converges to p_i.

Hypothesis: V is complex analytically smooth away from o.

We first assume the hypothesis above. According to the analysis right above proposition 5.1, we find local embeddings of (M_i, p_i) and X in $\mathbb{C}^N$. We have assumed all coordinate functions are homogeneous on X. Also the embedding maps p_i to the origin of $\mathbb{C}^N$. Furthermore, M_i (local part containing p_i) converges to X in the Hausdorff topology in $\mathbb{C}^N$. Now we use the same notations as in proposition 5.2. Since the holomorphic functions z^i_s converge to z_s, the Hausdorff limit of Z^i_k is contained in Σ. Here Z^i_k is the common zeroes of $z^i_1, .., z^i_k$. For simplicity, we may assume $z_s(1 \leq s \leq k)$ are contained in the coordinate functions of $\mathbb{C}^N$ (if not, we just add them to the coordinate. Notice that these z_s are homogeneous of degree 1 on X). On the other hand, given any point on Σ, we first find a point on M_i which is very close to that point. Then by using the flow introduced in proposition 5.1, one can find a nearby point so that the holomorphic functions z^i_s all vanish. This proves that Z^i_k converges to Σ in the Hausdorff sense. One also verifies that the limit of Z^i_k has multiplicity 1 by claim 5.1. We identify V and Σ. Now pick a point $p \in \Sigma \backslash 0$. By the hypothesis above, p is a regular point on Σ. Let A be the intersection of Σ and the unit sphere of $\mathbb{C}^N$. Observe that Σ is invariant under the flow generated by the real holomorphic vector field $-r\nabla r$ (r is the distance to the vertex $(0, o) \in X = \mathbb{C}^k \times V$). According to (5.1), A is transverse to the $-r\nabla r$ on Σ. Therefore, A is smooth. Let A_i be the intersection of the unit sphere in $\mathbb{C}^N$ and Z^i_k for large i. Then

A_i is diffeomorphic to A for large A, since A is compact and smooth. A and A_i admit natural contact structure induced by the Levi form. For this part, please refer to section 1 and section 2 of [39]. Due to the stability of contact structure (theorem 2.2 in [39]), A is in fact contactomorphic to A_i for sufficiently large i.

Let W^i_j be the holomorphic vector field so that $W^i_j(z^i_s) = \delta_{js}$. The argument is the same as in claim 5.1. Define a holomorphic vector field $H_i = Z_i - \sum_{j=1}^{k} c_{ij} W^i_j$. Here c_{ij} are holomorphic functions so that H_i is tangential to Z^i_k.

Claim 5.3. $\lim_{i\to\infty} \langle Z_i, dz^i_s\rangle = 0$ *for* $1 \leq s \leq k$ *on* A_i.

Proof. Observe $\langle Z_i, dz^i_s\rangle$ is holomorphic. If $i \to \infty$, by passing to subsequence, we have uniform convergence on A_i. Notice in the limit case, $\langle r\frac{\partial}{\partial r}, dz_s\rangle = 0$ on A. This concludes the proof. □

Claim 5.3 implies that c_{ij} are small functions on A_i. Combining this with (5.2), we find $-ReH_i(\sum |z_j|^2) < 0$ on A_i. On Z^i_k, let Φ^i_t be the biholomorphism generated by $-ReH_i$. Notice that the open set on Z^i_k bounded by A_i is connected: given any two points a, b there, connect them by a shortest geodesic L on M_i. We can project L to Z^i_k by using the flow σ as in proposition 5.1. Say the image of L is L'. Notice L' is contained in a uniformly bounded set of Z^i_k. For any $R > 0$, if i is large, by using the same argument as above, we may assume that $-ReH_i(\sum |z_j|^2) < 0$ on $(B_E(0, R)\backslash B_E(0, 1))\cap Z^i_k$ (E is the Euclidean metric on $\mathbb{C}^N$). Thus for large t, $\Phi^i_t(L')$ is contained in the domain bounded by A_i on Z^i_k. This proves the connectness. Then by the same argument as in proposition 3.2, the flow Φ^i_t converges to a point. According to our assumption in the beginning of proposition 5.3, the flow generated by $-ReZ_i$ converges to p_i which is $0 \in \mathbb{C}^N$. Thus Z_i vanishes at 0. Therefore, c_{ij} all vanish at 0. This implies that Φ^i_t is retracting to 0. Now we freeze i for a moment. Let $B_i = \frac{d\Phi^i_t}{dt}|_{t=0}$ on $T_0 Z^i_k$. By Schwarz lemma, the real part of the eigenvalues of B_i are all negative. Consider a coordinate U given by $(\tilde{z}_1, ..., \tilde{z}_{n-k})$ around 0 of Z^i_k. We may assume that B_i has the Jordan normal form. In particular, B_i is an upper triangular matrix. Then the real part of entries of the main diagonal are all negative. By rescaling each $\tilde{z}_j$ by some positive factors, we may assume that the absolute values of entries off the main diagonal are all very small. We still denote the new coordinate system by $(\tilde{z}_1, ..., \tilde{z}_{n-k})$. Now for any $z \in U$, define $Q(z, \overline{z}) = \sum |\tilde{z}_j(z)|^2$. For very small $\epsilon > 0$, define $D = \{z \in U | Q(z, \overline{z}) = \epsilon\}$ which is contactomorphic to the standard sphere $\mathbb{S}^{2(n-k)-1}$.

Lemma 5.2. *For any* $z \in D$, $\frac{dQ(\Phi_t^i(z),\overline{\Phi_t^i(z)})}{dt}|_{t=0} < 0$. *Therefore,* $-ReH_i$ *is pointing inside the sphere* D *transversely.*

Proof. According to the assumptions of B_i, for any $z \in \mathbb{C}^{n-k}$ (we consider z as a column vector), $z^T B_i^T \overline{z} + z^T \overline{B}_i \overline{z} < -\delta |z|^2$ for some $\delta > 0$. Note that near the origin, by using the coordinate $(\tilde{z}_1, ..., \tilde{z}_{n-k})$, we have the vector field $H_i(z) = -2B_i z + O(|z|^2)$. The factor -2 comes from the assumption that Φ_t^i is generated by $-ReH_i$. Therefore

$$\frac{dQ(\Phi_t^i(z),\overline{\Phi_t^i(z)})}{dt}|_{t=0} = z^T B_i^T \overline{z} + z^T \overline{B}_i \overline{z} + o(|z|^2) < 0. \tag{5.4}$$

□

Recall that ReH_i is transverse to A_i and inside the domain bounded by A_i on Z_k^i, ReH_i is nonvanishing except at 0. According to lemma 5.2, A_i and A are diffeomorphic to $\mathbb{S}^{2(n-k)-1}$.

Claim 5.4. *The hypothesis is satisfied for* $k = n - 2$.

Proof. For any point $a \in \Sigma \backslash \{0\}$, take a tangent cone W at a. Then W splits off $\mathbb{R}^{2n-2}$. Then we apply the result for $k = n - 1$ to see that W is in fact smooth. We can pull the holomorphic chart back to a small neighborhood of a. By a degree argument, we verify that this is a holomorphic chart around a. □

If $k = n - 2$, from the Mumford criteria [43], we see that Σ is in fact smooth. This concludes the proof for $k = n - 2$. For $k = n - 3$, we can use the same argument as in claim 5.4 to show that the hypothesis is satisfied.

The following result is elementary. However, since we cannot find it in a reference, let us present a proof.

Proposition 5.4. *Let* V_1, V_2 *be strictly pseudoconvex domains in* $\mathbb{C}^n$ *with smooth boundary. Assume the closure of* V_2 *is contained in* V_1. *Let* X *be a real holomorphic vector field (*X *induces biholomorphism) defined in a neighborhood of* V_1 *which points inward the boundary of* V_1 *and the boundary of* V_2. *Assume* X *has only one zero point* p *inside* V_2 *and the flow generated by* X *is retracting to* p *on* V_1. *Then the boundary of* V_1 *is contactomorphic to the boundary of* V_2.

Proof. For notational convenience, we simplify plurisubharmonic function as psh function. It suffices to construct a strictly psh function f without critical point such that the boundary of V_1 and boundary of V_2 are all regular level sets.

Let $G(t)$ be the flow generated by X. By using $G(t)$, we can biholomorphically push V_1 and V_2 sufficiently close to the attraction point p. So without loss of generality, we may assume there exists some $t_0 < 0$ such that $G(-t_0)(V_2)$ is well-defined and the closure of V_1 is contained in $G(-t_0)V_2$. Let S_1 be the boundary of V_1, S_2 be the boundary of V_2. Consider the sets $G(-t)(S_2)$ for $0 \leq t \leq t_0$. Then it is clear that they are all strictly pseudoconvex. On $G(-t_0)B\backslash V_2$, we can find a smooth function g_1 satisfying

- $g_1 = 0$ on S_2.
- g_1 is constant on the sets $G(-t)(S_2)$ for each t.
- g_1 is strictly decreasing along the vector field X.

We can find an increasing convex function u so that $u(g_1)$ is strictly psh. Set $f_1 = u(g_1)$. Similarly, we can construct a strictly psh function f_2 which satisfies

- f_2 is constant on sets $G(t)(S_1)$ for each $t \geq 0$.
- f_2 is strictly greater than the maximum of f_1 on S_1.
- f_2 is strictly less than f_1 on S_2.
- f_2 is strictly decreasing along X.

More precisely, there exists $\delta > 0$ so that $G(\delta)(V_1)$ contains the closure of V_2. Define a function g_2 which is constant on sets $G(t)(S_1)$ for each $t \geq 0$ and g_2 is strictly decreasing along X. Then we find an increasing convex function v so that $v(g_2)$ is strictly psh. By subtracting a large number, we may assume $v(g_2) < -100$ for all $t > \delta$. Now let w be an increasing smooth convex function satisfying that $w(y) = y$ for $y \leq v(g_2(G(\delta)(S_1)))$; w is increasing sufficiently fast so that $w(v(g_2(G(\frac{\delta}{2})S_1))) > \sup f_1$ on V_1. Set $f_2 = w(v(g_2))$. Then f_2 satisfies the conditions above. Now let f be the max of f_1 and f_2. We can mollify the function f so that it is strictly psh and has no critical point, since the derivatives of f_1, f_2 along X are all strictly negative. For this part, check corollary 3.20 of [13]. This concludes the proof of proposition 5.4. □

We can apply proposition 5.4 to A_i and D which are introduced above lemma 5.2. As D is contactomorphic to the standard sphere, A_i and A are all contactomorphic to the standard sphere. We need the following result of Mclean:

Proposition 5.5. *[39] Let V be a normal variety of dimension 3 with isolated singularity p. Assume the link of V is contactomorphic to the standard sphere $\mathbb{S}^5$, then p is a smooth point.*

We just apply Mclean's theorem for the case $k = n - 3$. This concludes the proof of proposition 5.3. □

Remark 5.2. *Corollary* 1.4 *of [39] states that if the so called Shokurov conjecture in algebraic geometry (we skip the statement) is true, then proposition 5.5 holds for any dimension. As a consequence, if the Shokurov conjecture is true, X is a complex manifold, i.e., no singularity appears.*

Now we consider the setting as in theorem 1.2. Pick a point q in the limit space X. If a tangent cone splits off $\mathbb{R}^{2k}$ where $k \geq n-3$, then according to proposition 5.3, the tangent cone is complex analytically smooth. By lifting the holomorphic chart to M_i as in claim 5.4, we find that q is a complex analytically smooth point on X. According to the dimension estimate of singular set in Cheeger-Colding [5], metric singularities whose tangent cones do not split $\mathbb{R}^{2n-6}$ have Hausdorff dimension at most $2n-8$. Since the distance function induced by holomorphic coordinates are bounded by the metric (each coordinate function has locally bounded gradient), we find that the complex analytic singularity of X has complex codimension at least 4. This concludes the first part of theorem 1.2.

For the second part of theorem 1.2, we use a similar argument as in Perelman's proof of the stability theorem [48][28].

Proposition 5.6. *Given any $n \in \mathbb{N}$, $v > 0$, there exist $\epsilon = \epsilon(n, v) > 0, \delta = \delta(n, v) > 0$ so that the following holds. Let (M_i, p_i) be a sequence of pointed Kähler manifolds converging to (X, p) in the Gromov-Hausdorff sense. Assume the bisectional curvature of M_i is bounded from below by $-\epsilon^3$ and $vol(B(p_i, \frac{1}{\epsilon})) \geq \frac{v}{\epsilon^{2n}}$. Assume that $B(p, \frac{1}{\epsilon})$ is ϵ-Gromov-Hausdorff close to $B_V(o, \frac{1}{\epsilon})$, where (V, o) is a metric cone isometric to $\mathbb{R}^{2k} \times W$. Then there exists an open set $B(p, \delta) \subset U \subset B(p, 1)$ so that U is biholomorphic to a product $\mathbb{D}^k \times Z$ where Z is an irreducible normal complex analytic space.*

Proof. The proof is almost the same as in proposition 5.2. We skip the argument here. □

The following is a local stability result.

Proposition 5.7. *Under the same assumptions of proposition 5.6, there exists $\gamma = \gamma(n, v) > 0$ so that we can find a homeomorphism Φ_i from an open set $B(p_i, \gamma) \subset U_i \subset B(p_i, 1)$ to $B(p, \gamma) \subset U \subset B(p, 1)$ respecting the holomorphic factor $\mathbb{D}^k$. Also Φ_i is a $\Phi(\epsilon)$-Hausdorff approximation of subsets in $\mathbb{C}^N$, if we consider the embeddings as in lemma 3.2 (the conical structure of V in lemma 3.2 is not essential). Thus X is a topological manifold.*

Proof. We use reverse induction. If $k = n$, then the conclusion follows from proposition 5.3. Assume the proposition is proved for $k \geq j+1$. We need to prove it for $k \geq j$. Let ϵ_{j+1}, δ_{j+1} be the constants in proposition 5.7 and proposition 5.6 corresponding to $k \geq j+1$.

Just assume $k = j$. By Gromov compactness, we can find small $a = a(n, v) > 0$ so that $B(p_i, a)$ and $B(p, a)$ are all embedded in $\mathbb{C}^N$. Let H_i, z_s^i, z_s be defined as in proposition 5.3 (recall z_s^i, z_s are defined in the beginning. H_i is defined right above claim 5.3). Since X is not necessarily a metric cone, z_s are not necessarily homogeneous. However, by a compactness argument, if ϵ is small, we may assume that they are almost homogeneous in the sense that $|Z(z_s) - \alpha_s z_s| < \rho$. Here ρ is an arbitrarily prescribed small number, Z is the limit of the holomorphic vector field Z_i.

Let Z_j^i be zeros of $z_1^i, ..., z_j^i$, Σ be the zeros of $z_1, .., z_j$. Let us assume H_i converges to a holomorphic vector field H on Σ. Let E be the Euclidean distance function on $\mathbb{C}^N$. We may assume that $(z_1, .., z_N)_i^{-1}(B_E(0, \lambda)) \subset\subset B(p_i, \frac{a}{2})$, where λ depends only on n, v. Let $S_i = \partial B_E(p_i, \lambda) \cap Z_j^i$, $S = \partial B_E(p, \lambda) \cap \Sigma$. Now we fix the value $\lambda = \lambda(n, v)$.

Pick a point $q \in S$. Consider points $S_i \ni q_i \to q$. If ϵ is sufficiently small, we can find δ_j^1, δ_j^2 depending only on n, v so that for some $\delta_j^2 < \delta_j^0 < \delta_j^1$, $B(q, \frac{\delta_j^0}{\epsilon_{j+1}})$ and $B(q_i, \frac{\delta_j^0}{\epsilon_{j+1}})$ are $\epsilon_{j+1}\delta_j^0$-Gromov-Hausdorff close to a geodesic ball in a metric cone which splits off $\mathbb{R}^{2j+2}$. By proposition 5.1 and proposition 5.6, we find some open set $B(q_i, \delta_{j+1}\delta_j^0) \subset U_i \subset B(q_i, \delta_j^0)$ $(B(q, \delta_{j+1}\delta_j^0) \subset U \subset B(q, \delta_j^0))$ so that U_i (U) is biholomorphic to $\mathbb{D}^{j+1} \times \hat{Z}_i$ $(\hat{Z})$. Say the coordinate on $\mathbb{D}^{j+1}$ is given by $(z_1^i, .., z_j^i, w^i)$ $((z_1, .., z_j, w))$. Furthermore, $z_1^i(q_i) = \cdots = z_j^i(q_i) = w^i(q_i) = 0$ $(z_1(q) = \cdots = z_j(q) = w(q) = 0)$, where w^i (w) come from the splitting along gradient of distance to p_i (p), roughly speaking. We may assume that $z_s^i \to z_s$, $w^i \to w$.

By the induction hypothesis, there exists a homeomorphism Φ_i from U_i to U respecting the holomorphic factor $\mathbb{D}^{j+1}$. Write the coordinate function $w^i = x_i + \sqrt{-1}y_i$ $(w = x + \sqrt{-1}y)$. By a compactness argument, we find some $c = c(n, v) > 0$ so that if ϵ is small enough depending only on n, v,

$$-ReH_i(d_E(\cdot, 0)) < -c < 0, -ReH_i(x_i) < -c < 0 \tag{5.5}$$

on S_i for all large i. Let G_i be the subset of U_i which is given by the common zeros of $z_1^i, .., z_j^i, x_i$. We can project G_i and $\Phi_i(G_i)$ to S_i and S by the flow generated by ReH_i and ReH. From (5.5), we see that the projections are all local homeomorphisms. Therefore, we have a homeomorphism from local parts of S_i to S. This implies that S is a topological manifold. Moreover, by simple ode argument, one can verify that this is a Hausdorff approximation. Furthermore, if we have two points which are close to each other on S_i, then we can connect them by the shortest geodesic on M_i with small length. Therefore, the diameter is small. We project the curve to S_i by using the flow generated by H_i. By ode argument, we see that the projected curve still has small diameter. This implies that there exists a function ξ as in definition 2.2 so that S_i are all ξ-connected for all sufficiently large i. Thus S is also ξ-connected.

By applying the gluing theorem (proposition 2.3), we have a homeomorphism from S_i to S which is also a Hausdorff approximation. By using the flow generated by $-ReH_i$ and $-ReH$, we can extend this as a homeomorphism for domains bounded by S_i and S on Z^i_j and Σ. This is still a Hausdorff approximation. The product with $\mathbb{D}^j$ becomes a homeomorphism. Recall holomorphic vector field W_m in claim 5.1 is used to construct the biholomorphism in proposition 5.1. Since these holomorphic vector fields have a convergence subsequence, so this is a Hausdorff approximation. This completes the induction. □

Applying proposition 2.3 again, we can glue local homeomorphisms to a global homeomorphism if X is compact. The proof of theorem 1.2 is complete. □

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References

[1] A. Chau and L.-F. Tam, *On the complex structure of Kähler manifolds with nonnegative curvature*, J. Diff. Geom, 73(2006), 491–530.

[2] A. Chau and L.-F. Tam, *Nonnegatively curved Kähler manifolds with average quadratic curvature decay*, Comm. Anal. Geom, 15(2007), 121–146.

[3] A. Chau and L.-F. Tam, *On the Steiness of a class of Kähler manifolds*, J. Diff. Geom, 79(2008), 167–183.

[4] J. Cheeger and T. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. of Math. (2) 144 (1996), no. 1, 189–237.

[5] J. Cheeger and T. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. 46 (1997), no. 3, 406–480.

[6] J. Cheeger and T. Colding, *On the structure of spaces with Ricci curvature bounded below. II*, J. Differential Geom. 54 (2000), no. 1, 13–35.

[7] J. Cheeger and T. Colding, *On the structure of spaces with Ricci curvature bounded below. III*, J. Differential Geom. 54 (2000), no. 1, 37–74.

[8] J. Cheeger, T. Colding and G. Tian, *On the singularities of spaces with bounded Ricci curvature*, Geom. Funct. Anal. 12(2002), 873–914.

[9] B.-L, Chen, X.-Y. Fu, Y. Le, X.-P. Zhu, *Sharp dimension estimates for holomorphic function and rigidity*, Trans. Amer. Math. Soc. 358(2006), no. 4, 1435–1454.

[10] B. L. Chen, S. H. Tang and X. P. Zhu, *A uniformization theorem of complete noncompact Kähler surfaces with positive bisectional curvature*, J. Diff. Geom. 67(2004), 519–570.

[11] B. L. Chen and X.-P. Zhu, *On complete noncompact Kähler manifolds with positive bisectional curvature*, Math. Ann. 327(2003), 1–23.

[12] S. Y. Cheng and S. T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. 28 (1975), 333–354.

[13] K. Cielieback and Y. Eliashberg, *From Stein to Weinstein and back: Symplectic geometry and affine complex manifolds*, Colloquium publications, volume 59.

[14] J. P. Demailly, *Analytic methods in algebraic geometry*, Surveys of modern mathematics, Volume 1.

[15] S. K. Donaldson and S. Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math., 213 (2014), 63–106.

[16] S. K. Donaldson and S. Sun, *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II*, To appear in J. Diff. Geom.

[17] H. Grauert, Th. Peternell, R. Remmert, *Several complex variables VII, Sheaf-Theoretical Methods in Complex Analysis*, Springer-Verlag, 1994.

[18] M. Gromov, *Metric structures for riemannian and non-riemannian spaces*, Progress in Mathematics, 152, Birkhäuser Boston, Inc., Boston, MA, xx+585 pp, 1999.

[19] K. Grove and P. Petersen, *Bounding homotopy types by geometry*, Ann. Math, 128(1988), 195–206.

[20] K. Grove, P. Petersen and J. Wu, *Geometric finiteness theorem via controlled topology*, Invent. Math. 99(1990), 205–213.

[21] R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, INC, 1965.

[22] R. Hartshone, *Algebraic geometry*, Graduate texts in mathematics, 52.

[23] F. He, *Existence and applications of Ricci flow via pseudolocality*, arxiv. 1610.01735.

[24] L. Hormander, *An introduction to complex analysis in several variables*, 3rd edition, North Holland, 1990.

[25] S. C. Huang and L.-F. Tam, *Kähler-Ricci flow with unbounded curvature*, arxiv: 1506.00322.

[26] J. C. Joo, K. T. Kim and G. Schmalz, *On the generalization of Forelli's theorem*, Math. Ann., (2016)365, 1187–1200.

[27] V. Kapovitch, *Perelman's stability theorem*, Surveys in Differential geometry, Metric and Comparison geometry, vol XI, International press, (2007), 103–136. Available at http://www.math.toronto.edu/vtk/.

[28] N. Kuhlmann, *uber holomorphe Abbildungen komplexer Résume*. Arch. Math. (Basel) 15 (1964), 81–90.

[29] M. C. Lee and L. F. Tam, *On existence and curvature estimates of Ricci flow*, arxiv. 1702.02667v2.

[30] M. C. Lee and L. F. Tam, *Chern-Ricci flows on noncompact complex manifolds*, arxiv. 1708.00141.

[31] P. Li and R. Schoen, *L^p and mean value inequalities for subharmonic functions on Riemannian manifolds*, Acta. Math., 153(1984), 279–303.

[32] P. Li and J. Wang, Comparison theorem for Kähler manifolds and positivity of spectrum, J. Diff. Geom. 69 (2005), 43–74.

[33] P. Li and S. T. Yau, *On the parabolic kernel of the Schr̈dinger operator*, Acta Math. 156(1986), 139–168.

[34] G. Liu, *Three circle theorem and dimension estimate for holomorphic functions on Kähler manifolds*, To appear in Duke Math Journal.

[35] G. Liu, *On the volume growth of Kähler manifolds with nonnegative bisectional curvature*, To appear in J. Diff. Geom.

[36] G. Liu, *Gromov-Hausdorff limits of Kähler manifolds and the finite generation conjecture*, To appear in Ann. Math.

[37] G. Liu, *Gromov-Hausdorff limits of Kähler manifolds with bisectional curvature lower bound I*, arxiv 1505.07521.

[38] G. Liu, *On the tangent cone of Kähler manifolds with Ricci curvature lower bound*, To appear in Math. Annalen.

[39] M. Mclean, *Reeb orbits and the minimal discrepancy of an isolated singularity*, Invent. Math. 204(2016), 505–594.

[40] N. Mok, *An embedding theorem of complete Kähler manifolds with positive bisectional curvature onto affine algebraic varieties*, Bull. Soc. Math. France, 112(1984), 197–258.

[41] N. Mok, Y. T. Siu and S. T. Yau, *The Poincare-Lelong equation on complete Kähler manifolds*, Compositio. Math. 44(1981), 183–281.

[42] S. Mori, *Projective manifolds with ample tangent bundle*, Ann. of Math.(2) 110(1979), 593–606.

[43] D. Mumford. *The topology of normal singularities of an algebraic surface and a criterion for simplicity.* Inst. Hautes Etudes Sci. Publ. Math., (9):5–22, 1961

[44] L. Ni, *Ancient solutions to Kähler-Ricci flow*, Math. Res. Lett. 12(2005), 633–654.

[45] L. Ni, *A monotonicity formula on complete Kähler manifolds with nonnegative bisectional curvature* , Jour of AMS, Vol 17, No. 4, 909–946.

[46] L. Ni and L.- F. Tam, *Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature*, J. Diff. Geom. 64(2003), 457–624.

[47] L. Ni and L.- F. Tam, *Poincare-Lelong equation via the Hodge-Laplace heat equation*, Compos. Math. 149 (2013), no. 11, 1856–1870.

[48] G. Perelman, *Alexandrov spaces with curvatures bounded from below, II*, preprint, 1991.

[49] J. P. Rosay and W. Rudin, *Holomorphic maps from* C^n *to* C^n, Trans. Amer. Math. Soc. 310 (1988), no. 1, 47–86.

[50] W. X. Shi, *Ricci deformation of the metric on complete noncompact Kähler manifolds*, PhD thesis, Harvard University, 1990.

[51] W. X. Shi, *Ricci flow and the uniformization on complete noncompact Kähler manifolds*, J. Diff. Geom. 45(1997), 94–220.

[52] V. Shokurov, *Letters of a Bi-Rationalist IV: Geometry of log flips*. 2002, arxiv.org/abs/math/0206004.

[53] Y. T. Siu and S. T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. 59(1980), 189–204.

[54] L. F. Tam and C. Yu, *Some comparison theorems for Kähler manifolds*, Manuscripta Math. 137 (2012), no. 3–4, 483–495.

[55] G. Tian, *Partial* C^0*-estimate for Kähler-Einstein metrics*, Commun. Math. Stat. 1 (2013), no. 2, 105–113.

[56] S. T. Yau, *Nonlinear analysis and geometry*, L'Enseignement Mathematique. 33(1987), 109–158.

[57] S. T. Yau, *Open problems in geometry*, Lectures on Differential Geometry, by Schoen and Yau 1 (1994), 365–404.

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