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BACKGROUND VELOCITY ESTIMATION BY CROSS CORRELATION OF AMBIENT NOISE SIGNALS IN THE RADIATIVE TRANSPORT REGIME*

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Abstract. The cross correlation of the wave signals emitted by ambient noise sources can be used to estimate the Green's function of the wave equation in an inhomogeneous medium. In this paper we clarify the role of random scattering in the Green's function estimation in the radiative transport regime and we show how this insight can be used to estimate the velocity of propagation of a smooth background medium.

Key words. Green's function estimation, passive sensor imaging, noise sources, random media.

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1. Introduction

Recently, it has been shown that the Green's function of the wave equation in an inhomogeneous medium can be estimated from the cross correlation of the wave fields measured at different points in the medium, in the case in which the signals are generated by ambient noise sources. This technique allows for travel time estimation and background velocity estimation and it has been successfully applied in geophysics [8, 18, 22].

From the theoretical point of view, the relation between the Green's function and the cross correlation of ambient noise signals was first established in configurations where the noise sources surround the region of interest [16, 21]. This relation also holds when the field is equipartitioned, for instance in an ergodic cavity [6, 7]. When the enegy flux coming from the sources does not have enough directional diversity, the Green's function estimation is not always possible [10].

Scattering by random fluctuations of the medium has been shown to help Green's function and travel time estimation because scatterers play the role of secondary sources. This was analyzed in a simple single-scattering regime in [10] and in the parabolic approximation in [14]. In this paper we would like to clarify the role of random scattering in the radiative transport regime. This is an interesting situation because it allows us to consider regimes close to geometric optics (where random scattering is negligible), regimes where single scattering is dominant, and regimes where multiple scattering is so strong that diffusion approximation is valid and equipartition of waves in direction can be reached.

In the applications to surface wave tomography one has a situation where the directionality of the energy flow is clearly noticeable [19, 22], which means that there is no equipartition of waves, but there are scattered contributions in the ambient noise data as shown in [20]. Therefore this is a situation that is well modeled by the radiative transfer regime discussed in this paper. In other applications where one uses coda waves associated with earthquakes, i.e., waves that have been scattered, one may

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have a situation that is well modeled by the diffusion approximation [13], which is a regime that we also discuss here. We stress that in our framework the clutter and multiple scattering serves to enhance the estimation resolution since it enhances the phase space diversity of the wave field. This is in particular illustrated in Section 7 where we discuss how the presence of a scattering layer may enable estimation in a surrounding region without clutter.

The paper is organized as follows. In Section 2 we introduce the wave equation with noise sources. The empirical wave field cross correlations are introduced in Section 3. In Section 4, we recapitulate briefly the transport equations. In Section 5 we show how the cross correlations relate to the transport equations and how the Green's function and the velocity of the background medium can be estimated using the cross correlations. Next we recapitulate briefly the probabilistic interpretation of the transport equations in Section 6 and discuss how this can be used to analyze Green's function estimation in various scattering regimes. We show, in Section 7, that the presence of a heterogeneous layer may affect the estimation of the Green's function in the homogeneous part of the medium.

2. The wave equation with noise sources

We consider scalar waves propagating in a three-dimensional random medium. The governing equation is

$$\frac{1}{c^\varepsilon(\boldsymbol{x})^2}\frac{\partial^2 u^\varepsilon}{\partial t^2} - \Delta u^\varepsilon = f^\varepsilon, \tag{2.1}$$

where $c^\varepsilon(\boldsymbol{x})$ is the inhomogeneous propagation speed in the medium and $\boldsymbol{x}\in\mathbb{R}^3$ is the space coordinate. The source term f^ε models a distribution of ambient noise sources emitting stationary random signals.

The velocity profile in the medium is assumed to be of the form

$$\frac{1}{c^\varepsilon(\boldsymbol{x})^2} = \frac{1}{c_0^2(\boldsymbol{x})}\left[1+\varepsilon^{1/2}\nu\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\varepsilon}\right)\right]. \tag{2.2}$$

The slowly varying background velocity $c_0(\boldsymbol{x})$ is assumed to be smooth and bounded (as well as its first two derivatives). The rapid random fluctuations of the medium are described by the process $\nu(\boldsymbol{x},\boldsymbol{x}')$. For any fixed $\boldsymbol{x}$, the process $\boldsymbol{x}'\mapsto\nu(\boldsymbol{x},\boldsymbol{x}')$ is stationary and zero mean and it has strong mixing properties. The $\boldsymbol{x}$-dependence of $\nu(\boldsymbol{x},\boldsymbol{x}')$ models a slow spatial variation in the statistical properties of the medium. The covariance function R and local power spectral density $\hat{R}$ of the fluctuations are defined by

$$R(\boldsymbol{x},\boldsymbol{y}) = \mathbb{E}[\nu(\boldsymbol{x},\boldsymbol{y}')\nu(\boldsymbol{x},\boldsymbol{y}'+\boldsymbol{y})], \tag{2.3}$$

$$\hat{R}(\boldsymbol{x},\boldsymbol{k}) = \int R(\boldsymbol{x},\boldsymbol{y})e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y}, \tag{2.4}$$

where $\mathbb{E}[\cdot]$ stands for the expectation with respect to the distribution of the random medium. We assume that the propagation speed has fluctuations only in a bounded region, that is, we assume that $c_0(\boldsymbol{x})=\bar{c}$ (where $\bar{c}$ is a constant) and $\hat{R}(\boldsymbol{x},\boldsymbol{k})=0$ for $\boldsymbol{x}$ outside a ball that surrounds the inhomogeneous region.

The ambient noise sources emit signals that are random and stationary in time with short coherence time. Moreover they have a spatially limited support and short-range spatial correlation. The covariance function of the source function f^ε is

$$\left\langle f^\varepsilon(t,\boldsymbol{x})f^\varepsilon(t',\boldsymbol{x}')\right\rangle = \gamma\left(\frac{t-t'}{\varepsilon}\right)\Gamma\left(\frac{\boldsymbol{x}+\boldsymbol{x}'}{2},\frac{\boldsymbol{x}-\boldsymbol{x}'}{\varepsilon}\right), \tag{2.5}$$

where $\langle\cdot\rangle$ stands for the expectation with respect to the distribution of the noise sources. The function $t\mapsto\gamma(t)$ is the time correlation function of the sources. It is even and its Fourier transform

$$\hat{\gamma}(\omega)=\int\gamma(t)e^{i\omega t}\mathrm{d}t \tag{2.6}$$

is the power spectral density of the sources, and is a nonnegative integrable function. For any fixed $\boldsymbol{x}$, the function $\boldsymbol{y}\mapsto\Gamma(\boldsymbol{x},\boldsymbol{y})$ is the local spatial covariance function. The function $\boldsymbol{x}\mapsto\Gamma(\boldsymbol{x},\boldsymbol{0})$ describes the spatial support of the source distribution. We assume that the spatial support is limited and that $\int\Gamma(\boldsymbol{x},\boldsymbol{0})\mathrm{d}\boldsymbol{x}<\infty$. The function $\boldsymbol{y}\to\Gamma(\boldsymbol{x},\boldsymbol{y})$ is even and its Fourier transform

$$\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})=\int\Gamma(\boldsymbol{x},\boldsymbol{y})e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y} \tag{2.7}$$

is a nonnegative integrable function.

3. The wave field cross correlation

The wave fields recorded at the observation points $\boldsymbol{x}$ and $\boldsymbol{x}'$ are incoherent and we convert these to the primary imaging data by forming the cross correlation. The empirical cross correlation is

$$C_T(\tau,\boldsymbol{x},\boldsymbol{x}')=\frac{1}{T}\int_0^T u^\varepsilon(t,\boldsymbol{x})u^\varepsilon(t+\tau,\boldsymbol{x}')\mathrm{d}t. \tag{3.1}$$

The following proposition shows that the empirical cross correlation is statistically stable with respect to the distribution of the noise sources.

PROPOSITION 3.1. *The empirical cross correlation is a self-averaging quantity with respect to the distribution of the noise sources:*

$$C_T(\tau,\boldsymbol{x}',\boldsymbol{x}')\overset{T\to\infty}{\longrightarrow}C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}'), \tag{3.2}$$

where the statistical cross correlation is defined by

$$C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')=\langle u^\varepsilon(0,\boldsymbol{x})u^\varepsilon(\tau,\boldsymbol{x}')\rangle\,. \tag{3.3}$$

The proof of this proposition is given in [10].

From now on we will assume that the integration time T is large enough so that the empirical cross correlation C_T can be considered as a good approximation of the statistical cross correlation $C^{(1)}$. Our objective is thus now to use the primary data $C^{(1)}$ to solve various imaging problems. The fundamental tool for describing propagation in the scaling regime at hand is the radiative transport theory. In the next section we quickly review the radiative transport equation as it will be fundamental for the representation of the cross correlation. The next lemma gives a particular representation for the statistical cross correlation $C^{(1)}$ based on Duhamel's formula. This representation is useful to connect to radiative transport theory in standard form, in which the wave propagation problem has no source term but a non-zero initial condition.

Lemma 3.2. *Let $v^\varepsilon(s,\boldsymbol{x})$ be the solution of*

$$\frac{1}{c^\varepsilon(\boldsymbol{x})^2}\frac{\partial^2 v^\varepsilon}{\partial s^2}-\Delta v^\varepsilon=0, \tag{3.4}$$

starting from $v^\varepsilon(s=0,\boldsymbol{x})=0$, $\partial_s v^\varepsilon(s=0,\boldsymbol{x})=c^\varepsilon(\boldsymbol{x})^2F^\varepsilon(\boldsymbol{x})$.
Here $F^\varepsilon(\boldsymbol{x})=f^\varepsilon(0,\boldsymbol{x})/\sqrt{\gamma(0)}$ is a zero-mean random process with covariance

$$\langle F^\varepsilon(\boldsymbol{x})F^\varepsilon(\boldsymbol{x}')\rangle=\Gamma\left(\frac{\boldsymbol{x}+\boldsymbol{x}'}{2},\frac{\boldsymbol{x}-\boldsymbol{x}'}{\varepsilon}\right).$$

Then the statistical cross correlation (3.3) can be expressed as

$$C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')=\varepsilon\int\gamma(s)\tilde{C}^{(1)}(\tau-\varepsilon s,\boldsymbol{x},\boldsymbol{x}')\mathrm{d}s, \tag{3.5}$$

where

$$\tilde{C}^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')=\begin{cases}\displaystyle\int_0^\infty\langle v^\varepsilon(s,\boldsymbol{x})v^\varepsilon(s+\tau,\boldsymbol{x}')\rangle\mathrm{d}s & \textit{if } \tau\geq 0,\\ \displaystyle\int_0^\infty\langle v^\varepsilon(s-\tau,\boldsymbol{x})v^\varepsilon(s,\boldsymbol{x}')\rangle\mathrm{d}s & \textit{if } \tau<0.\end{cases} \tag{3.6}$$

We prove this lemma in Appendix A. Thus, in the limit case when the sources are delta-correlated in time, $\gamma(t)=\delta(t)$, then $C^{(1)}$ is proportional to $\tilde{C}^{(1)}$. Moreover, in the general case when the sources are correlated in time, then $C^{(1)}$ is obtained from $\tilde{C}^{(1)}$ through a convolution with the correlation function $\gamma(\cdot/\varepsilon)$.

4. Transport equations for wave energy

In the description of the wave field cross correlations we shall use the radiative transport equations that we recall here. Let us consider the solution v^ε of (3.4). We use the results of [17, Section 3] in our context in which the wave equation (3.4) can be written in the general hyperbolic form

$$\frac{\partial\boldsymbol{w}^\varepsilon}{\partial s}=-\nabla v^\varepsilon,\qquad \frac{1}{c^\varepsilon(\boldsymbol{x})^2}\frac{\partial v^\varepsilon}{\partial s}=-\nabla\cdot\boldsymbol{w}^\varepsilon, \tag{4.1}$$

starting from

$$v^\varepsilon(s=0,\boldsymbol{x})=0,\qquad \boldsymbol{w}^\varepsilon(s=0,\boldsymbol{x})=\nabla\phi^\varepsilon(\boldsymbol{x}), \tag{4.2}$$

with the potential ϕ^ε satisfying the Poisson equation $\Delta\phi^\varepsilon(\boldsymbol{x})=-F^\varepsilon(\boldsymbol{x})$. The eigenvalues ω^+ and ω^- and the eigenvectors $\boldsymbol{b}^+$ and $\boldsymbol{b}^-$ (representing the acoustic waves $(\boldsymbol{w}^\varepsilon,v^\varepsilon)$) of the hyperbolic system (4.1) are

$$\omega^\pm(\boldsymbol{x},\boldsymbol{k})=\pm c_0(\boldsymbol{x})|\boldsymbol{k}|,\qquad \boldsymbol{b}^\pm(\boldsymbol{x},\boldsymbol{k})=\frac{1}{\sqrt{2}}\begin{pmatrix}\frac{\boldsymbol{k}}{|\boldsymbol{k}|}\\ \pm c_0(\boldsymbol{x})\end{pmatrix}.$$

We introduce the matrix Wigner transform

$$\mathbf{W}^\varepsilon(s,\boldsymbol{x},\boldsymbol{k})=\int\mathbb{E}\Big[\begin{pmatrix}\boldsymbol{w}^\varepsilon\\ v^\varepsilon\end{pmatrix}\Big(s,\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\Big)\begin{pmatrix}\boldsymbol{w}^\varepsilon\\ v^\varepsilon\end{pmatrix}^T\Big(s,\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\Big)\Big]e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y}, \tag{4.3}$$

where the expectation is with respect to the distribution of the medium and the noise sources. The matrix Wigner transform weakly converges as $\varepsilon \to 0$ [17]:

$$\mathbf{W}^\varepsilon(s,\boldsymbol{x},\boldsymbol{k}) \xrightarrow{\varepsilon\to 0} a^+(s,\boldsymbol{x},\boldsymbol{k})\boldsymbol{b}^+(\boldsymbol{x},\boldsymbol{k})\boldsymbol{b}^+(\boldsymbol{x},\boldsymbol{k})^T + a^-(s,\boldsymbol{x},\boldsymbol{k})\boldsymbol{b}^-(\boldsymbol{x},\boldsymbol{k})\boldsymbol{b}^-(\boldsymbol{x},\boldsymbol{k})^T. \tag{4.4}$$

There are no vortical mode terms in (4.4) because of the initial conditions of the form (4.2). As remarked in [17] the amplitudes a^+ and a^- of the modes are such that $a^-(s,\boldsymbol{x},\boldsymbol{k}) = a^+(s,\boldsymbol{x},-\boldsymbol{k})$ with $a^+(s,\boldsymbol{x},\boldsymbol{k}) = a(s,\boldsymbol{x},\boldsymbol{k})$ the solution of the transport equation

$$\frac{\partial a}{\partial s} + \nabla_{\boldsymbol{k}}\omega\cdot\nabla_{\boldsymbol{x}} a - \nabla_{\boldsymbol{x}}\omega\cdot\nabla_{\boldsymbol{k}} a = \int \sigma(\boldsymbol{x},\boldsymbol{k},\boldsymbol{k}')\big(a(s,\boldsymbol{x},\boldsymbol{k}') - a(s,\boldsymbol{x},\boldsymbol{k})\big)\mathrm{d}\boldsymbol{k}', \tag{4.5}$$

with the initial condition

$$a(s=0,\boldsymbol{x},\boldsymbol{k}) = \frac{1}{2}\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})\frac{c_0^2(\boldsymbol{x})}{\omega^2(\boldsymbol{x},\boldsymbol{k})}. \tag{4.6}$$

Here

$$\omega(\boldsymbol{x},\boldsymbol{k}) = c_0(\boldsymbol{x})|\boldsymbol{k}| \tag{4.7}$$

is the dispersion relation (the frequency at $\boldsymbol{x}$ of the wave with wavevector $\boldsymbol{k}$) and the scattering cross section is

$$\sigma(\boldsymbol{x},\boldsymbol{k},\boldsymbol{k}') = \frac{\pi c_0^2(\boldsymbol{x})|\boldsymbol{k}|^2}{2(2\pi)^3}\hat{R}(\boldsymbol{x},\boldsymbol{k}-\boldsymbol{k}')\delta\big(c_0(\boldsymbol{x})|\boldsymbol{k}| - c_0(\boldsymbol{x})|\boldsymbol{k}'|\big). \tag{4.8}$$

5. Transport representation of wave field cross correlations

5.1. The fundamental representation formula. The following proposition is the fundamental result for the analysis of imaging by cross correlations. It shows that the local cross correlation function has a τ-dependence that has a simple form in terms of the background velocity and the limit Wigner transform.

PROPOSITION 5.1. *We have*

$$\begin{aligned}\frac{1}{\varepsilon}\int \frac{\partial}{\partial\tau}C^{(1)}\Big(\varepsilon\tau,\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2},\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\Big)e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y} &\xrightarrow{\varepsilon\to 0} \frac{ic_0(\boldsymbol{x})}{2|\boldsymbol{k}|}A(\boldsymbol{x},\boldsymbol{k})\gamma(\tau)*_\tau\Big[e^{ic_0(\boldsymbol{x})|\boldsymbol{k}|\tau}\Big]\\ &\quad - \frac{ic_0(\boldsymbol{x})}{2|\boldsymbol{k}|}A(\boldsymbol{x},-\boldsymbol{k})\gamma(\tau)*_\tau\Big[e^{-ic_0(\boldsymbol{x})|\boldsymbol{k}|\tau}\Big],\end{aligned} \tag{5.1}$$

where the amplitude factor is

$$A(\boldsymbol{x},\boldsymbol{k}) = \int_0^\infty W(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}s. \tag{5.2}$$

Here $W(s,\boldsymbol{x},\boldsymbol{k})$ is the solution of the transport equation (4.5) with the initial condition

$$W(s=0,\boldsymbol{x},\boldsymbol{k}) = \frac{1}{2}\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})c_0^2(\boldsymbol{x}). \tag{5.3}$$

We remark that the amplitude $A(\boldsymbol{x},\boldsymbol{k})$ is a quantity that depends on the background velocity, the statistics of the random fluctuations of the medium, and the spatial support and correlation of the sources.

The proof of the proposition is given in Appendices A-B. In Appendix A we convert the problem with source conditions in (2.1) into a problem with initial conditions to facilitate the transport analysis. Then we use this representation and the Wigner transform to relate the wave field cross correlations to solutions of transport equations in Appendix B. The convergence is proved for the expectation of the wave field cross correlation, where the expectation is taken with respect to the distribution of the random fluctuations of the medium. In a practical context such an expectation is usually not available since we deal with only one realization of the random medium. The question of the statistical stability of the quantity of interest is here important. A statistically stable quantity is a random quantity which has small fluctuations around its expected value, and we anticipate that the cross correlation or equivalently the Wigner transform has such a property when $\varepsilon \to 0$. Although the proof of this assertion is not yet available, there is theoretical and numerical evidence that this should be the case [3, 4, 5]. Anyway it is likely that smoothing in the $(\boldsymbol{x},\boldsymbol{k})$-space is necessary to get stability when ε is small but not zero, which is a manifestation of the usual trade-off between signal-to-noise ratio and resolution.

5.2. Emergence of the Green's function in the equipartition regime. The following proposition shows that, if waves are coming to $\boldsymbol{x}$ in an equipartitioned way in direction and in wavenumber, then the time-derivative of the cross correlation is proportional to the symmetrized local Green's function convolved with the time correlation function of the sources.

PROPOSITION 5.2. *If the amplitude factor $A(\boldsymbol{x},\boldsymbol{k})$ is independent of $\boldsymbol{k}$,*

$$A(\boldsymbol{x},\boldsymbol{k}) = \mathcal{A}(\boldsymbol{x}),$$

then

$$\frac{1}{\varepsilon}\frac{\partial}{\partial \tau} C^{(1)}\Big(\varepsilon\tau, \boldsymbol{x} - \frac{\varepsilon \boldsymbol{y}}{2}, \boldsymbol{x} + \frac{\varepsilon \boldsymbol{y}}{2}\Big) \overset{\varepsilon\to 0}{\longrightarrow} -\mathcal{A}(\boldsymbol{x})\gamma(\tau) *_\tau \Big[\mathrm{sgn}(\tau) G_{c_0(\boldsymbol{x})}(|\tau|, \boldsymbol{y})\Big], \tag{5.4}$$

where G_{c_0} is the Green's function of the homogeneous wave equation with background velocity c_0:

$$G_{c_0}(\tau, \boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{y}|}\delta\Big(\tau - \frac{|\boldsymbol{y}|}{c_0}\Big). \tag{5.5}$$

The convergence (5.4) holds weakly in $\boldsymbol{y}$. If the sources are delta-correlated in time, then the $(\tau,\boldsymbol{y})$-dependence of the cross correlation function is proportional to the local Green's function. If the sources are correlated, then the cross correlation function is a smoothed version of the Green's function, where the smoothing is a convolution in time with the time correlation function of the sources.

Proof. For any τ we have

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\tau}\tilde{C}^{(1)}\Big(\varepsilon\tau, \boldsymbol{x} - \frac{\varepsilon\boldsymbol{y}}{2}, \boldsymbol{x} + \frac{\varepsilon\boldsymbol{y}}{2}\Big) = -\frac{\tau\mathcal{A}(\boldsymbol{x})c_0^2(\boldsymbol{x})}{(2\pi)^3}\int \mathrm{sinc}\big(c_0(\boldsymbol{x})|\boldsymbol{k}|\tau\big)e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{k},$$

and

$$\mathrm{sinc}\big(c_0(\boldsymbol{x})|\boldsymbol{k}|\tau\big) = \frac{1}{4\pi}\int_{\partial B(\boldsymbol{0},1)} e^{ic_0(\boldsymbol{x})\tau\boldsymbol{k}\cdot\boldsymbol{z}}\mathrm{d}\boldsymbol{z}, \tag{5.6}$$

which gives, for any test function $\phi(\boldsymbol{y})$,

$$\lim_{\varepsilon\to 0}\int \frac{\partial}{\partial\tau}\tilde{C}^{(1)}\Big(\varepsilon\tau,\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2},\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\Big)\phi(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = -\frac{\mathcal{A}(\boldsymbol{x})c_0^2(\boldsymbol{x})\tau}{4\pi}\int_{\partial B(\mathbf{0},1)}\phi\big(c_0(\boldsymbol{x})\tau\boldsymbol{z}\big)\mathrm{d}\boldsymbol{z}.$$

Finally, comparison with

$$\int G_{c_0(\boldsymbol{x})}(|\tau|,\boldsymbol{y})\phi(\boldsymbol{y})\mathrm{d}\boldsymbol{y} = \frac{c_0^2(\boldsymbol{x})|\tau|}{4\pi}\int_{\partial B(\mathbf{0},1)}\phi\big(c_0(\boldsymbol{x})\tau\boldsymbol{z}\big)\mathrm{d}\boldsymbol{z}$$

gives the result. ∎

5.3. Emergence of the Green's function in the diffusive regime. The following proposition shows that if the waves are arriving at $\boldsymbol{x}$ in an equipartitioned way in direction, but not necessarily in wavenumber, then we can observe a smoothed version of the local Green's function. We will see in the next section that this configuration happens when scattering is strong enough and diffusion approximation for the radiative transport equation holds.

PROPOSITION 5.3. *If the amplitude factor $A(\boldsymbol{x},\boldsymbol{k})$ is independent of $\boldsymbol{k}/|\boldsymbol{k}|$,*

$$A(\boldsymbol{x},\boldsymbol{k}) = \mathcal{A}(\boldsymbol{x},|\boldsymbol{k}|),$$

then

$$\begin{aligned}&\frac{1}{\varepsilon}\frac{\partial}{\partial\tau}C^{(1)}\Big(\varepsilon\tau,\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2},\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\Big)\\ &\overset{\varepsilon\to 0}{\longrightarrow} -\iint K_{\boldsymbol{x}}(\tau-\tau',\boldsymbol{y}-\boldsymbol{y}')\big[\mathrm{sgn}(\tau')G_{c_0(\boldsymbol{x})}(|\tau'|,\boldsymbol{y}')\big]\mathrm{d}\boldsymbol{y}'\mathrm{d}\tau',\end{aligned} \tag{5.7}$$

where

$$K_{\boldsymbol{x}}(\tau,\boldsymbol{y}) = \frac{\gamma(\tau)}{2\pi^2}\int_0^\infty \mathcal{A}(\boldsymbol{x},k)\mathrm{sinc}(k|\boldsymbol{y}|)k^2\mathrm{d}k. \tag{5.8}$$

Proof. For any τ we have, by (5.1),

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\tau}\tilde{C}^{(1)}\Big(\varepsilon\tau,\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2},\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\Big) = -\frac{c_0^2(\boldsymbol{x})\tau}{(2\pi)^3}\int \mathcal{A}(\boldsymbol{x},|\boldsymbol{k}|)\mathrm{sinc}\big(c_0(\boldsymbol{x})|\boldsymbol{k}|\tau\big)e^{-i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{k}.$$

Using (5.6) this also reads

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\partial}{\partial\tau}\tilde{C}^{(1)}\big(\varepsilon\tau,\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2},\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\big)\\ &= -\frac{c_0^2(\boldsymbol{x})\tau}{32\pi^4}\int_0^\infty \mathrm{d}k\int_{\partial B(\mathbf{0},1)}\mathrm{d}\hat{\boldsymbol{k}}\int_{\partial B(\mathbf{0},1)}\mathrm{d}\boldsymbol{z}\mathcal{A}(\boldsymbol{x},k)k^2e^{ic_0(\boldsymbol{x})\tau k\hat{\boldsymbol{k}}\cdot\boldsymbol{z}-ik\hat{\boldsymbol{k}}\cdot\boldsymbol{y}}\\ &= -\frac{c_0^2(\boldsymbol{x})\tau}{(2\pi)^3}\int_0^\infty \mathrm{d}k\,\mathcal{A}(\boldsymbol{x},k)k^2\int_{\partial B(\mathbf{0},1)}\mathrm{d}\boldsymbol{z}\,\mathrm{sinc}\big(k|c_0(\boldsymbol{x})\tau\boldsymbol{z}-\boldsymbol{y}|\big).\end{aligned}$$

Comparison with

$$\begin{aligned}&\int \mathrm{d}\boldsymbol{y}'\frac{1}{2\pi^2}\int_0^\infty \mathrm{d}k\mathcal{A}(\boldsymbol{x},k)k^2\mathrm{sinc}\big(k|\boldsymbol{y}-\boldsymbol{y}'|\big)G_{c_0(\boldsymbol{x})}(\tau,\boldsymbol{y}')\\ &= \frac{c_0(\boldsymbol{x})}{8\pi^3}\int_0^\infty \mathrm{d}k\mathcal{A}(\boldsymbol{x},k)k^2\int_0^\infty \mathrm{d}y'y'\int_{\partial B(\mathbf{0},1)}\mathrm{d}\boldsymbol{z}\,\mathrm{sinc}\big(k|y'\boldsymbol{z}-\boldsymbol{y}|\big)\delta\big(y'-c_0(\boldsymbol{x})\tau\big)\\ &= \frac{c_0^2(\boldsymbol{x})\tau}{(2\pi)^3}\int_0^\infty \mathrm{d}k\mathcal{A}(\boldsymbol{x},k)k^2\int_{\partial B(\mathbf{0},1)}\mathrm{d}\boldsymbol{z}\,\mathrm{sinc}\big(k|c_0(\boldsymbol{x})\tau\boldsymbol{z}-\boldsymbol{y}|\big)\end{aligned}$$

gives the result. ∎

We remark that, for instance, if

$$\mathcal{A}(\boldsymbol{x},k)=\mathcal{A}_0(\boldsymbol{x})e^{-k^2 r_c^2},$$

then

$$K_{\boldsymbol{x}}(\tau,\boldsymbol{y})=\mathcal{A}_0(\boldsymbol{x})\frac{\gamma(\tau)}{(2\sqrt{\pi}r_c)^3}e^{-\frac{|\boldsymbol{y}|^2}{4r_c^2}}.$$

Thus, the cross correlation function is a smoothed version of the Green's function, where the smoothing is a convolution in time with the time correlation function of the sources and a convolution in space with a function that depends on the spatial covariance function of the sources and the effective transport of energy from the source region to the observation region.

5.4. Background velocity estimation without equipartition. In the previous subsection we have shown that it is possible to estimate the local Green's function in favorable configurations (equipartition). Note that the local Green's function gives the background velocity. In this section we analyze configurations in which wave equipartition is not satisfied, which does not mean, however, that background velocity estimation is impossible. In the next proposition we give the conditions under which it is possible to estimate the background velocity given the observation of the cross correlation around $\boldsymbol{x}$.

PROPOSITION 5.4. *If for some $\boldsymbol{k}$ the amplitude factor $A(\boldsymbol{x},\boldsymbol{k})$ is not zero and the power spectral density of the noise sources $\hat{\gamma}(\omega)$ is not zero at $\omega=c_0(\boldsymbol{x})|\boldsymbol{k}|$, then it is possible to extract the background velocity at $\boldsymbol{x}$ from the cross correlation $C^{(1)}$ around $\boldsymbol{x}$.*

The background velocity $c_0(\boldsymbol{x})$ can be obtained by looking at the (time) Fourier transform of $\partial_\tau C^{(1)}$ that is concentrated at $\pm c_0(\boldsymbol{x})|\boldsymbol{k}|$:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\iint\frac{\partial}{\partial\tau}C^{(1)}\Big(\varepsilon\tau,\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2},\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\Big)e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y}e^{i\omega\tau}\mathrm{d}\tau$$
$$=\frac{i\pi c_0(\boldsymbol{x})}{|\boldsymbol{k}|}\hat{\gamma}(c_0(\boldsymbol{x})|\boldsymbol{k}|)\Big[A(\boldsymbol{x},\boldsymbol{k})\delta\big(\omega+c_0(\boldsymbol{x})|\boldsymbol{k}|\big)-A(\boldsymbol{x},-\boldsymbol{k})\delta\big(\omega-c_0(\boldsymbol{x})|\boldsymbol{k}|\big)\Big].$$

Proof. Consider (5.1) and compute the Fourier transform in time using the fact that γ is even so that $\hat{\gamma}(-\omega_0)=\hat{\gamma}(\omega_0)$. ∎

The problem is now to identify the configurations for which the amplitude factor $A(\boldsymbol{x},\boldsymbol{k})$ is not zero. This can be done by considering the probabilistic representation of the solution of the transport Equation (4.5-4.6). In the next section we distinguish different types of configurations depending on the scattering properties of the medium.

6. Transport regimes for wave field cross correlations

6.1. Geometrical optics connection. First, we briefly recall the ray equations of the geometric optics approximation of the scalar wave equation with speed of propagation $c_0(\boldsymbol{x})$. The ray starting from $\boldsymbol{x}$ with the direction $\boldsymbol{k}$ is the solution of the Hamilton's equations (with the Hamiltonian $\omega(\boldsymbol{x},\boldsymbol{k})=c_0(\boldsymbol{x})|\boldsymbol{k}|$)

$$\frac{\mathrm{d}\mathbf{X}_s}{\mathrm{d}s}=c_0(\mathbf{X}_s)\frac{\mathbf{K}_s}{|\mathbf{K}_s|}, \tag{6.1}$$

$$\frac{\mathrm{d}\mathbf{K}_s}{\mathrm{d}s}=-\nabla c_0(\mathbf{X}_s)|\mathbf{K}_s|, \tag{6.2}$$

starting from $\mathbf{X}_0(\boldsymbol{x},\boldsymbol{k})=\boldsymbol{x}$ and $\mathbf{K}_0(\boldsymbol{x},\boldsymbol{k})=\boldsymbol{k}$. From now on we assume that the inhomogeneous region and the noise source region are within the ball $B(\mathbf{0},R)$ and that the ray equations have unique solutions within the ball $B(\mathbf{0},R)$. A sufficient condition is that $\|c_0-\bar{c}\|_{\mathcal{C}^2}$ is small enough ($\bar{c}$ is a constant reference background velocity). We next make the connection to the transport equations (4.5). Note first that the normalized function

$$\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{k})=\frac{c_0^2(\boldsymbol{x})\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})}{\hat{\Gamma}_{\rm tot}}, \qquad \hat{\Gamma}_{\rm tot}=\iint c_0^2(\boldsymbol{x}')\hat{\Gamma}(\boldsymbol{x}',\boldsymbol{k}')\mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{k}'$$

is a probability density function and let us assume that $(\mathbf{X}_0,\mathbf{K}_0)$ has the distribution with density $\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{k})$, and moreover that $(\mathbf{X}_s,\mathbf{K}_s)$ follows the ray equations (6.1-6.2). Then the solution $W_d(s,\boldsymbol{x},\boldsymbol{k})$ of the transport equation

$$\frac{\partial W_d}{\partial s}+\nabla_{\boldsymbol{k}}\omega\cdot\nabla_{\boldsymbol{x}}W_d-\nabla_{\boldsymbol{x}}\omega\cdot\nabla_{\boldsymbol{k}}W_d=0, \tag{6.3}$$

with the initial condition (5.3) satisfies, for any test function ϕ,

$$\iint \phi(\boldsymbol{x},\boldsymbol{k})W_d(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k}=\hat{\Gamma}_{\rm tot}\mathbf{E}\big[\phi(\mathbf{X}_s,\mathbf{K}_s)\big].$$

Here the expectation $\mathbf{E}$ is taken with respect to the initial distribution of $(\mathbf{X}_0,\mathbf{K}_0)$:

$$\mathbf{E}\big[\phi(\mathbf{X}_0,\mathbf{K}_0)\big]=\iint \phi(\boldsymbol{x},\boldsymbol{k})\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k}.$$

6.2. Smooth medium. The next proposition shows that, in the absence of random scattering, it is not always possible to extract the background velocity from the cross correlation. The flux of energy of the waves given by the rays issued from the sources must reach the region in which the cross correlations are evaluated. This important result was already mentioned in [10] using a geometric optics approach.

PROPOSITION 6.1. *If there is no scattering $\hat{R}=0$, then the amplitude factor $A(\boldsymbol{x},\boldsymbol{k})$ is not zero at $(\boldsymbol{x},\boldsymbol{k})$ if and only if the ray starting from $(\boldsymbol{x},-\boldsymbol{k})$ reaches the interior of the support of $\hat{\Gamma}$ (i.e. the source region).*

Proof. In the absence of scattering, the solution W of the radiative transport Equation (4.5) is equal to the solution W_d of (6.3), and we have

$$\iint \phi(\boldsymbol{x},\boldsymbol{k})W_d(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k}=\iint \phi\big((\mathbf{X}_s,\mathbf{K}_s)(\boldsymbol{x},\boldsymbol{k})\big)c_0^2(\boldsymbol{x})\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k},$$

where $\big((\mathbf{X}_s,\mathbf{K}_s)(\boldsymbol{x},\boldsymbol{k})\big)_{s\geq 0}$ is the solution of the ray equations (6.1-6.2) starting from $(\mathbf{X}_0,\mathbf{K}_0)=(\boldsymbol{x},\boldsymbol{k})$. In the right-hand side we make the change of variables $(\boldsymbol{x},\boldsymbol{k})\mapsto(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}}):=(\mathbf{X}_s,\mathbf{K}_s)(\boldsymbol{x},\boldsymbol{k})$. The inverse transform is $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}})\mapsto(\boldsymbol{x},\boldsymbol{k})=(\mathbf{X}_s,-\mathbf{K}_s)(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})$. We find

$$\begin{aligned}&\iint \phi(\boldsymbol{x},\boldsymbol{k})W_d(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k}\\ &=\iint \phi(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}})c_0^2(\mathbf{X}_s(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}))\hat{\Gamma}\big((\mathbf{X}_s,\mathbf{K}_s)(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})\big)J^{-1}(s,\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}})\mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}\tilde{\boldsymbol{k}},\end{aligned}$$

where $J(s,\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}})$ is the Jacobian of the transform $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}})\mapsto(\mathbf{X}_s,-\mathbf{K}_s)(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})$ and we have used the fact that $\hat{\Gamma}(\boldsymbol{x},-\boldsymbol{k})=\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})$. Since this holds true for any test function, we have

$$W_d(s,\boldsymbol{x},\boldsymbol{k})=c_0^2(\mathbf{X}_s(\boldsymbol{x},-\boldsymbol{k}))\hat{\Gamma}\big((\mathbf{X}_s,\mathbf{K}_s)(\boldsymbol{x},-\boldsymbol{k})\big)J^{-1}(s,\boldsymbol{x},\boldsymbol{k}),$$

and

$$A(\boldsymbol{x},\boldsymbol{k})=\int_0^\infty c_0^2(\mathbf{X}_s(\boldsymbol{x},-\boldsymbol{k}))\hat{\Gamma}\big((\mathbf{X}_s,\mathbf{K}_s)(\boldsymbol{x},-\boldsymbol{k})\big)J^{-1}(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}s.$$

Therefore $A(\boldsymbol{x},\boldsymbol{k})$ is positive if and only if the ray $\big((\mathbf{X}_s,\mathbf{K}_s)(\boldsymbol{x},-\boldsymbol{k})\big)_{s\geq 0}$ "spends" a positive time in the support of $\hat{\Gamma}$. ∎

6.3. Probabilistic representation of the transport equations. Here we will give a probabilistic representation of the radiative transport Equation (4.5). Let us assume that $(\mathbf{X}_0,\mathbf{K}_0)$ has the distribution with density $\tilde{\Gamma}(\boldsymbol{x},\boldsymbol{k})$ and $(\mathbf{X}_s,\mathbf{K}_s)$ follows the ray equations (6.1-6.2) until a random time T_1. The distribution of this random time is

$$\mathbf{P}(T_1>t|\mathbf{X}_0,\mathbf{K}_0)=\exp\Big(-\int_0^t\Sigma(\mathbf{X}_s,\mathbf{K}_s)\mathrm{d}s\Big),$$

where $\Sigma(\boldsymbol{x},\boldsymbol{k})$ is the total cross section

$$\Sigma(\boldsymbol{x},\boldsymbol{k})=\int\sigma(\boldsymbol{x},\boldsymbol{k},\boldsymbol{k}')\mathrm{d}\boldsymbol{k}'. \tag{6.4}$$

At the time T_1 the wavevector jumps. The conditional distribution of $(\mathbf{X}_{T_1},\mathbf{K}_{T_1})$ given $(\mathbf{X}_{T_1-},\mathbf{K}_{T_1-})$ is given by

$$\mathbf{E}\big[\phi(\mathbf{X}_{T_1},\mathbf{K}_{T_1})\big|\mathbf{X}_{T_1-},\mathbf{K}_{T_1-}\big]=\frac{1}{\Sigma(\mathbf{X}_{T_1-},\mathbf{K}_{T_1-})}\int\phi(\mathbf{X}_{T_1-},\boldsymbol{k}')\sigma(\mathbf{X}_{T_1-},\mathbf{K}_{T_1-},\boldsymbol{k}')\mathrm{d}\boldsymbol{k}',$$

so that $\mathbf{X}_{T_1}=\mathbf{X}_{T_1-}$. After the jump at time T_1, $(\mathbf{X}_s,\mathbf{K}_s)$ follows the ray equations (6.1-6.2) until it jumps at time T_1+T_2, with the conditional distribution of T_2 being

$$\mathbf{P}(T_2>t|\mathbf{X}_{T_1},\mathbf{K}_{T_1})=\exp\Big(-\int_0^t\Sigma(\mathbf{X}_{T_1+s},\mathbf{K}_{T_1+s})\mathrm{d}s\Big),$$

and the conditional distribution for the wavevector jump being analogous to the one above, and so on.

The solution $W(s,\boldsymbol{x},\boldsymbol{k})$ of (4.5) with the initial condition (5.3) satisfies, for any test function ϕ [15],

$$\iint\phi(\boldsymbol{x},\boldsymbol{k})W(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k}=\hat{\Gamma}_{\mathrm{tot}}\mathbf{E}\big[\phi(\mathbf{X}_s,\mathbf{K}_s)\big]. \tag{6.5}$$

Here the expectation $\mathbf{E}$ is taken with respect to the initial distribution of $(\mathbf{X}_0,\mathbf{K}_0)$ and the distribution of the jumps.

6.4. Weakly scattering medium. If scattering is weak, then $W(s,\boldsymbol{x},\boldsymbol{k})$ can be approximated by the sum

$$W(s,\boldsymbol{x},\boldsymbol{k}) = W_0(s,\boldsymbol{x},\boldsymbol{k}) + W_1(s,\boldsymbol{x},\boldsymbol{k}),$$

where W_0 is the contribution of the direct waves and W_1 is the contribution of the single-scattered waves. The same approximation holds for the amplitude factor

$$A(\boldsymbol{x},\boldsymbol{k}) = A_0(\boldsymbol{x},\boldsymbol{k}) + A_1(\boldsymbol{x},\boldsymbol{k}), \qquad A_j(\boldsymbol{x},\boldsymbol{k}) = \int_0^\infty W_j(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}s, \quad j = 0,1.$$

The contribution of the direct waves is the contribution in (6.5) of the paths with no jump. It is given by

$$\iint \phi(\boldsymbol{x},\boldsymbol{k}) W_0(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k} = \hat{\Gamma}_{\mathrm{tot}} \mathbf{E}\big[\phi(\mathbf{X}_s,\mathbf{K}_s)\mathbf{1}_{T_1>s}\big],$$

where $(\mathbf{X}_s,\mathbf{K}_s)_{s\geq 0}$ follows the random dynamics (with deterministic transport and random jumps) described in Subsection 6.3. Therefore we have

$$\iint \phi(\boldsymbol{x},\boldsymbol{k}) W_0(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k} = \iint \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k} c_0^2(\boldsymbol{x})\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})\phi\big((\mathbf{X}_s,\mathbf{K}_s)(\boldsymbol{x},\boldsymbol{k})\big) \times e^{-\int_0^s \Sigma\big((\mathbf{X}_t,\mathbf{K}_t)(\boldsymbol{x},\boldsymbol{k})\big)\mathrm{d}t},$$

where $\big((\mathbf{X}_t,\mathbf{K}_t)(\boldsymbol{x},\boldsymbol{k})\big)_{t\geq 0}$ is a classical ray starting form $(\boldsymbol{x},\boldsymbol{k})$ and moving according to (6.1-6.2). Note that $W_0(s,\boldsymbol{x},\boldsymbol{k})$ is positive if and only if $W_d(s,\boldsymbol{x},\boldsymbol{k})$ is positive. In the configurations in which the amplitude factor is zero in the absence of scattering, the amplitude factor A_0 is also zero in the presence of scattering. This is expected, since A_0 is the contribution of the direct waves.

The contribution of the single-scattered waves is the contribution in (6.5) of the paths with a single jump:

$$\iint \phi(\boldsymbol{x},\boldsymbol{k}) W_1(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k} = \hat{\Gamma}_{\mathrm{tot}} \mathbf{E}\big[\phi(\mathbf{X}_s,\mathbf{K}_s)\mathbf{1}_{T_1\leq s<T_1+T_2}\big], \tag{6.6}$$

where $(\mathbf{X}_s,\mathbf{K}_s)_{s\geq 0}$ follows the random dynamics described in Subsection 6.3. This gives

$$\begin{aligned}\iint \phi(\boldsymbol{x},\boldsymbol{k}) W_1(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k} = &\iint \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k} c_0^2(\boldsymbol{x})\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k}) \int \mathrm{d}\boldsymbol{k}' \int_0^s \mathrm{d}t \\ &\times e^{-\int_0^t \Sigma\big((\mathbf{X}_{t'},\mathbf{K}_{t'})(\boldsymbol{x},\boldsymbol{k})\big)\mathrm{d}t'} \sigma(\mathbf{X}_t(\boldsymbol{x},\boldsymbol{k}),\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k}),\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})+\boldsymbol{k}') \\ &\times e^{-\int_t^s \Sigma\big((\mathbf{X}_{t'-t},\mathbf{K}_{t'-t})(\mathbf{X}_t(\boldsymbol{x},\boldsymbol{k}),\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})+\boldsymbol{k}')\big)\mathrm{d}t'} \\ &\times \phi\big((\mathbf{X}_{s-t},\mathbf{K}_{s-t})(\mathbf{X}_t(\boldsymbol{x},\boldsymbol{k}),\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})+\boldsymbol{k}')\big),\end{aligned} \tag{6.7}$$

where $\big((\mathbf{X}_t,\mathbf{K}_t)(\boldsymbol{x},\boldsymbol{k})\big)_{t\geq 0}$ is a classical ray starting form $(\boldsymbol{x},\boldsymbol{k})$ and moving according to (6.1-6.2). In the expression (6.7), $(\boldsymbol{x},\boldsymbol{k})$ is the starting point of the broken ray, t is the jump time, $\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})$ is the angle just before the jump, and $\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})+\boldsymbol{k}'$ is the new angle just after the jump. The position of the particle at time s (after t) is $\mathbf{X}_{s-t}(\mathbf{X}_t(\boldsymbol{x},\boldsymbol{k}),\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})+\boldsymbol{k}')$ and its direction is $\mathbf{K}_{s-t}(\mathbf{X}_t(\boldsymbol{x},\boldsymbol{k}),\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})+\boldsymbol{k}')$. From this description the following intuitive result is derived explicitly in Appendix C:

PROPOSITION 6.2. *If $\hat{R}$ is non-vanishing, then the amplitude factor $A_1(\boldsymbol{x},\boldsymbol{k})$ due to the single-scattered waves is not zero at $(\boldsymbol{x},\boldsymbol{k})$ if and only if the ray starting from $(\boldsymbol{x},-\boldsymbol{k})$ can reach the interior of the support of $\hat{\Gamma}$ after one scattering event (i.e. a change in direction at some point).*

This explicit geometric condition explains at which locations we can estimate the background velocity. The scatterers act as secondary noise sources, they "illuminate" a region that is determined by the scattering cross section and the wavevectors associated with rays coming in from the primary sources. Thus, the region in which we can estimate the background velocity can be significantly enlarged by the presence of scatterers.

6.5. Strongly scattering medium. If the scattering is strong, then $W(s,\boldsymbol{x},\boldsymbol{k})$ can be approximated by a quantity independent of the direction $\boldsymbol{k}/|\boldsymbol{k}|$ that satisfies a diffusion equation, according to the diffusion-approximation theory [17]. Proposition 5.3 then shows that it is possible to get the local Green's function from the cross correlation up to a smoothing kernel that depends on the correlation properties of the noise sources and on the transport properties of the random medium. The next proposition states this result quantitatively.

PROPOSITION 6.3.

1. When the distance L between the sources and the observation area around $\boldsymbol{x}$ is much larger than the transport mean free path, then the amplitude factor $A(\boldsymbol{x},\boldsymbol{k})$ becomes independent of the direction $\boldsymbol{k}/|\boldsymbol{k}|$ and we have

$$\frac{1}{\varepsilon}\frac{\partial}{\partial\tau}C^{(1)}\Big(\varepsilon\tau,\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2},\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\Big) \overset{\varepsilon\to 0}{\longrightarrow} -\iint K_{\boldsymbol{x}}(\tau-\tau',\boldsymbol{y}-\boldsymbol{y}')\big[\mathrm{sgn}(\tau')G_{c_0(\boldsymbol{x})}(|\tau'|,\boldsymbol{y}')\big]\mathrm{d}\boldsymbol{y}'\mathrm{d}\tau'. \tag{6.8}$$

The smoothing kernel $K_{\boldsymbol{x}}(\tau,\boldsymbol{y})$ is given in terms of the amplitude factor $\mathcal{A}(\boldsymbol{x},k)$ by (5.8).

2. If, additionally, $\hat{R}(\boldsymbol{x},\boldsymbol{k})=\hat{r}(|\boldsymbol{k}|)$ is spherically symmetric and the background velocity has small variations around the constant value $\bar{c}$, then the transport mean free path $l^*(k)$ is given by

$$\frac{1}{l^*(k)}=\frac{\pi^2k^4}{(2\pi)^3}\int_{-1}^{1}\hat{r}\big(k\sqrt{2(1-\mu)}\big)(1-\mu)\mathrm{d}\mu, \tag{6.9}$$

and the amplitude factor is given by

$$\mathcal{A}(\boldsymbol{x},k)=\frac{3}{4\pi\bar{c}L}\frac{b_0(k)}{l^*(k)},\qquad b_0(k)=\frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\partial B(\boldsymbol{0},1)}c_0^2(\boldsymbol{z})\hat{\Gamma}(\boldsymbol{z},k\boldsymbol{n})\mathrm{d}\boldsymbol{n}\mathrm{d}\boldsymbol{z}. \tag{6.10}$$

The resolution of the Green's function estimation is determined by

1) the amplitude factor $\mathcal{A}(\boldsymbol{x},k)$, which itself depends on the transport mean free path $l^*(k)$ and the spatial covariance of the sources through the term $b_0(k)$, and

2) the coherence time of the noise sources.

Therefore the resolution depends both on the (spatial and time) coherence of the noise sources and on the statistics of the random fluctuations of the medium.

Proof. When the propagation distance is large, the diffusion approximation can be used and $W(s,\boldsymbol{x},\boldsymbol{k})$ can be approximated by a quantity independent of the direction $\boldsymbol{k}/|\boldsymbol{k}|$ that satisfies a diffusion equation. Proposition 5.3 then gives the first part of the proposition. Here we give a simple proof of the form (6.10) of the amplitude factor when the fluctuations of the medium are statistically homogeneous and the background velocity is constant (in the region of interest). The solution of the transport Equation (4.5) is of the form [17]

$$W(s,\boldsymbol{x},\boldsymbol{k}) \simeq w(s,\boldsymbol{x},|\boldsymbol{k}|),$$

where $w(s,\boldsymbol{x},k)$ satisfies the diffusion equation

$$\frac{\partial w}{\partial s} = \frac{\bar{c}l^*(k)}{3}\Delta_{\boldsymbol{x}} w$$

with the initial condition

$$w(s=0,\boldsymbol{x},k) = \frac{\bar{c}^2}{8\pi}\int_{\partial B(\boldsymbol{0},1)} \hat{\Gamma}(\boldsymbol{x},k\boldsymbol{n})\mathrm{d}\boldsymbol{n}.$$

The transport mean free path $l^*(k)$ is given by (6.9). Since the region of the sources is spatially limited and the observation area is far from it, we can center the reference frame to the center of the source distribution and we can approximate the initial condition by a δ-distribution in $\boldsymbol{x}$:

$$w(s=0,\boldsymbol{x},k) = \frac{1}{8\pi}\int_{\mathbb{R}^3}\int_{\partial B(\boldsymbol{0},1)} \bar{c}^2\hat{\Gamma}(\boldsymbol{z},k\boldsymbol{n})\mathrm{d}\boldsymbol{n}\mathrm{d}\boldsymbol{z}\,\delta(\boldsymbol{x}) = b_0(k)\delta(\boldsymbol{x}).$$

This allows us to obtain a closed-form expression for the function w:

$$w(s,\boldsymbol{x},k) = \left(\frac{3}{4\pi\bar{c}l^*(k)s}\right)^{3/2}\exp\left(-\frac{3|\boldsymbol{x}|^2}{4\bar{c}l^*(k)s}\right)b_0(k)$$

and

$$\mathcal{A}(\boldsymbol{x},k) = \int_0^\infty w(s,\boldsymbol{x},k)\mathrm{d}s = \frac{3}{4\pi\bar{c}|\boldsymbol{x}|l^*(k)}b_0(k),$$

which gives the desired result. □

7. Noise mixing transport

In this section we consider the situation with a randomly scattering medium located in the halfspace $x_3<0$ while the halfspace $x_3>0$ is not scattering and has a slowly and smoothly varying background velocity (using the notation $\boldsymbol{x}=(\hat{\boldsymbol{x}},x_3)$). The random sources are located in the heterogeneous halfspace $x_3<0$. We observe the wave field at two locations $\boldsymbol{x}$ and $\boldsymbol{x}'$ in the smooth halfspace $x_3>0$ (see Figure 7.1). Then we can use transport theory to identify the spectrum of the wavefield at the boundary of the random medium at $x_3=0$ in order to get an expression for the cross correlation of the field at the two observation points. The following result expresses the result for the cross correlation in terms of the transport solution and the high-frequency approximation of the background Green's function.

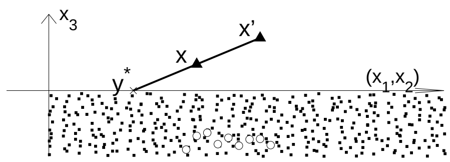

FIG. 7.1. *Configuration considered in Section 7. The open circles are the random sources, the bottom half-space is randomly heterogeneous, the observation points $\boldsymbol{x}$ and $\boldsymbol{x}'$ are in the homogeneous upper half-space, and $\boldsymbol{y}^*$ is the intersection of the ray going through $\boldsymbol{x}$ and $\boldsymbol{x}'$ with the plane $x_3 = 0$.*

PROPOSITION 7.1. *Let us assume that $\hat{R}(\boldsymbol{x},\boldsymbol{k})$ in (2.4) and $\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})$ in (2.7) are supported in the halfspace $x_3 < 0$. Let us consider two observation points $\boldsymbol{x} = (\hat{\boldsymbol{x}}, x_3)$ and $\boldsymbol{x}' = (\hat{\boldsymbol{x}}', x_3')$ with $x_3' > x_3 > 0$. Then the wave field cross correlation is concentrated around the time lag $\mathcal{T}(\boldsymbol{x},\boldsymbol{x}')$ and it can be expressed as*

$$\frac{1}{\varepsilon^2}\frac{\partial}{\partial s}C^{(1)}\big(\mathcal{T}(\boldsymbol{x},\boldsymbol{x}') + \varepsilon s, \boldsymbol{x}, \boldsymbol{x}'\big) \tag{7.1}$$
$$\stackrel{\varepsilon\to 0}{\longrightarrow} \frac{-2\alpha(\boldsymbol{y}^*,\boldsymbol{x})\alpha(\boldsymbol{y}^*,\boldsymbol{x}')}{\sqrt{\det \mathbf{H}^*}}\Big|\frac{\partial}{\partial y_3}\mathcal{T}(\boldsymbol{y}^*,\boldsymbol{x})\Big|\int A\big(\boldsymbol{y}^*, -|\omega|\nabla_{\boldsymbol{y}}\mathcal{T}(\boldsymbol{y}^*,\boldsymbol{x}')\big)\hat{\gamma}(\omega)e^{-i\omega s}\mathrm{d}\omega,$$

where $\boldsymbol{y}^* = (\hat{\boldsymbol{y}}^*, 0)$ is the intersection of the ray going through $\boldsymbol{x}$ and $\boldsymbol{x}'$ with the plane $x_3 = 0$, the amplitude $\alpha(\boldsymbol{y},\boldsymbol{x})$ and the travel time $\mathcal{T}(\boldsymbol{y},\boldsymbol{x})$ are obtained from the high-frequency approximation of the background Green's function solution of $\Delta_{\boldsymbol{x}}\hat{G}(\omega,\boldsymbol{x},\boldsymbol{y}) + \omega^2 c_0^{-2}(\boldsymbol{x})\hat{G}(\omega,\boldsymbol{x},\boldsymbol{y}) = -\delta(\boldsymbol{y}-\boldsymbol{x})$,

$$\hat{G}(\omega,\boldsymbol{x},\boldsymbol{y}) \simeq \alpha(\boldsymbol{x},\boldsymbol{y})\exp\big(i\omega\mathcal{T}(\boldsymbol{x},\boldsymbol{y})\big),$$

and $\mathbf{H}^*$ is the 2×2 matrix

$$\mathbf{H}^* = \nabla_{\hat{\boldsymbol{y}}}\otimes\nabla_{\hat{\boldsymbol{y}}}\big[\mathcal{T}\big((\hat{\boldsymbol{y}}^*,0),\boldsymbol{x}\big) - \mathcal{T}\big((\hat{\boldsymbol{y}}^*,0),\boldsymbol{x}'\big)\big].$$

Here non-degenerate ray paths have been assumed.

In the case with a constant background, $c_0(\boldsymbol{x}) \equiv c_0$, we have

$$\mathcal{T}(\boldsymbol{x},\boldsymbol{x}') = \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c_0}, \qquad \boldsymbol{y}^* = \frac{x_3'\boldsymbol{x} - x_3\boldsymbol{x}'}{x_3' - x_3},$$

and

$$\frac{1}{\varepsilon^2}\frac{\partial}{\partial s}C^{(1)}\Big(\frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c_0} + \varepsilon s, \boldsymbol{x}, \boldsymbol{x}'\Big) \stackrel{\varepsilon\to 0}{\longrightarrow} \frac{-1}{8\pi^2|\boldsymbol{x}-\boldsymbol{x}'|}\int A\Big(\boldsymbol{y}^*, \frac{|\omega|}{c_0}\frac{\boldsymbol{x}'-\boldsymbol{x}}{|\boldsymbol{x}-\boldsymbol{x}'|}\Big)\hat{\gamma}(\omega)e^{-i\omega s}\mathrm{d}\omega.$$

This shows that the cross correlation between $\boldsymbol{x}$ and $\boldsymbol{x}'$ depends only the energy flux oriented in the direction from $\boldsymbol{x}$ to $\boldsymbol{x}'$ and is concentrated around the travel time from $\boldsymbol{x}$ to $\boldsymbol{x}'$.

This result generalizes in that the random medium could be made up of different connected subsections. Then the cross correlation would be non-zero only if the ray going through the two points intersects a subsection that is exposed to the random sources, or in the more special case that this ray intersects the source region. This result shows that we can estimate travel times for well separated points of the medium exploiting the enhanced phase space diversity in the ambient noise that has been created by the random medium. Thus, this result differs from the above ones in that it describes a situation in which one can estimate travel times at the macroscopic scale, but it is similar in that the presence of clutter significantly enhances the estimation process. We remark that the above result can now be generalized to the situation with imaging of reflectors in the slowly varying part of the medium via cross correlations. In this case one may assume measurements on an array and then compute the cross correlations between the measurements on the array. These cross correlation traces can then be migrated in order to carry out an imaging task [10, 11, 12].

8. Conclusion

We have studied the role of random scattering for Green's function estimation from cross correlations of ambient noise signals. We have considered the radiative transport regime and we have shown that random scattering improves the Green's function estimation by enhancing the directional diversity of the waves recorded by the sensors. Here the sensors are supposed to be close to each other (that is, closer than the mean free path), but the observation region can be far from the source region. In particular, in the diffusion-approximation regime, that is, when the propagation distance from the sources to the observation region is much larger than the mean free path, we have shown that the local Green's function can be estimated up to a convolution in time and space that depends on the time and space correlation of the noise sources and on the frequency dependence of the mean free path. We have also shown how a scattering layer can serve to enhance the angular spread of a noise field and therefore also correlation imaging resolution.

As mentioned in Section 5.1, the main limitation of the radiative transport model is statistical stability, since it describes the behavior of averaged quantities (averaged over the possible realizations of the random medium), while the data set corresponds to one particular realization of the random medium. However quadratic quantities such as the cross correlation or equivalently the Wigner transform that we study here become asymptotically statistically stable, i.e., independent of the realization of the random medium. This stability property is supported by theoretical and numerical evidence [3, 4, 5].

The results presented in this paper serve to explain some of the recent progress made in tomography imaging based on wave field cross correlations. In particular they provide a bridge between the situation with a partly directional wave field and a fully equipartitioned situation. The results give concrete information about imaging functional construction and resolution by explicitly providing the connection between the local Green's function and wave field cross correlations. The analysis provides a uniform framework for identifying how clutter and therefore enhanced phase space diversity may enhance correlation based Green's function estimation in regimes ranging from geometrical optics and only weak scattering to the diffusive regime with strong scattering. The main applications of this technique to surface wave tomography have so far been in a regime in between these that is well captured by the general radiative transfer model that we consider.

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Appendix A. Proof of Lemma 3.2. Using Duhamel's formula, the solution of the wave equation with source (2.1) can be expressed as

$$u^\varepsilon(t,\boldsymbol{x}) = \int_{-\infty}^{t} \tilde{u}^\varepsilon(t,\boldsymbol{x};s)\mathrm{d}s,$$

where $\tilde{u}^\varepsilon$ is the solution of

$$\frac{1}{c^\varepsilon(\boldsymbol{x})^2}\frac{\partial^2 \tilde{u}^\varepsilon}{\partial t^2} - \Delta \tilde{u}^\varepsilon = 0,$$

starting from $\tilde{u}^\varepsilon(t=s,\boldsymbol{x};s)=0$, $\partial_t \tilde{u}^\varepsilon(t=s,\boldsymbol{x};s) = c^\varepsilon(\boldsymbol{x})^2 f^\varepsilon(s,\boldsymbol{x})$. Let us denote by $\tilde{G}^\varepsilon$ the Green's function of the wave equation with local velocity $c^\varepsilon(\boldsymbol{x})$:

$$\frac{1}{c^\varepsilon(\boldsymbol{x})^2}\frac{\partial^2 \tilde{G}^\varepsilon}{\partial t^2} - \Delta_{\boldsymbol{x}} \tilde{G}^\varepsilon = 0, \quad \tilde{G}^\varepsilon(t=0,\boldsymbol{x},\boldsymbol{y}) = 0, \quad \partial_t \tilde{G}^\varepsilon(t=0,\boldsymbol{x},\boldsymbol{y}) = c^\varepsilon(\boldsymbol{y})^2 \delta(\boldsymbol{x}-\boldsymbol{y}). \tag{A.1}$$

We can then express the solution $\tilde{u}^\varepsilon$ as

$$\tilde{u}^\varepsilon(t,\boldsymbol{x};s) = \int \tilde{G}^\varepsilon(t-s,\boldsymbol{x},\boldsymbol{y}) f^\varepsilon(s,\boldsymbol{y})\mathrm{d}\boldsymbol{y},$$

and the solution v^ε of (3.4) can be expressed as

$$v^\varepsilon(s,\boldsymbol{x}) = \int \tilde{G}^\varepsilon(s,\boldsymbol{x},\boldsymbol{y}) F^\varepsilon(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

Using these integral representations the cross correlation of $\tilde{u}^\varepsilon$ can be expressed in the form

$$\begin{aligned}\langle \tilde{u}^\varepsilon(0,\boldsymbol{x};s)\tilde{u}^\varepsilon(\tau,\boldsymbol{x}';s')\rangle = \iint & \tilde{G}^\varepsilon(-s,\boldsymbol{x},\boldsymbol{y})\tilde{G}^\varepsilon(\tau-s',\boldsymbol{x}',\boldsymbol{y}') \\ & \times \Gamma\left(\frac{\boldsymbol{y}+\boldsymbol{y}'}{2}, \frac{\boldsymbol{y}-\boldsymbol{y}'}{\varepsilon}\right)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{y}'\gamma\left(\frac{s-s'}{\varepsilon}\right),\end{aligned}$$

and the cross correlation of v^ε can be written as

$$\langle v^\varepsilon(s,\boldsymbol{x}) v^\varepsilon(s',\boldsymbol{x}')\rangle = \iint \tilde{G}^\varepsilon(s,\boldsymbol{x},\boldsymbol{y})\tilde{G}^\varepsilon(s',\boldsymbol{x}',\boldsymbol{y}')\Gamma\left(\frac{\boldsymbol{y}+\boldsymbol{y}'}{2}, \frac{\boldsymbol{y}-\boldsymbol{y}'}{\varepsilon}\right)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{y}'.$$

Using the above two results we then find that, for any τ,

$$\begin{aligned} C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}') &= \langle u^\varepsilon(0,\boldsymbol{x}) u^\varepsilon(\tau,\boldsymbol{x}')\rangle \\ &= \int_{-\infty}^{0}\mathrm{d}\tilde{s}\int_{-\infty}^{\tau}\mathrm{d}\tilde{s}'\,\langle \tilde{u}^\varepsilon(0,\boldsymbol{x};\tilde{s})\tilde{u}^\varepsilon(\tau,\boldsymbol{x}';\tilde{s}')\rangle \\ &= \varepsilon\iint \mathrm{d}s\mathrm{d}s'\gamma(s')\iint \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{y}'\tilde{G}^\varepsilon(s,\boldsymbol{x},\boldsymbol{y})\tilde{G}^\varepsilon(\tau-\varepsilon s'+s,\boldsymbol{x}',\boldsymbol{y}')\Gamma\left(\frac{\boldsymbol{y}+\boldsymbol{y}'}{2}, \frac{\boldsymbol{y}-\boldsymbol{y}'}{\varepsilon}\right) \\ &= \varepsilon\iint \mathrm{d}s\mathrm{d}s'\gamma(s')\,\langle v^\varepsilon(s,\boldsymbol{x}) v^\varepsilon(s+\tau-\varepsilon s',\boldsymbol{x}')\rangle \\ &= \varepsilon\int \mathrm{d}s'\gamma(s')\tilde{C}^{(1)}(\tau-\varepsilon s',\boldsymbol{x},\boldsymbol{x}'),\end{aligned}$$

with

$$\tilde{C}^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}') = \int \mathrm{d}s \, \langle v^\varepsilon(s,\boldsymbol{x}) v^\varepsilon(s+\tau,\boldsymbol{x}')\rangle \,.$$

Since $v^\varepsilon(s,\boldsymbol{x})$ is zero for $s<0$ we get the desired result.

Appendix B. Proof of the fundamental representation formula (5.1). Let v^ε and $\tilde{C}^{(1)}$ be defined as in Lemma 3.2. We know that the matrix Wigner transform (4.3) weakly converges as described in Section 4. This implies the weak convergence of the Wigner transform

$$V^\varepsilon(s,\tau,\boldsymbol{x},\boldsymbol{k}) = \int \big\langle v^\varepsilon\big(s,\boldsymbol{x}\big) v^\varepsilon\big(s+\varepsilon\tau,\boldsymbol{x}+\varepsilon\boldsymbol{y}\big)\big\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{y}} \mathrm{d}\boldsymbol{y}$$

for $\tau=0$. Let us fix $\tau>0$. Using the Kirchhoff representation formula and the fact that c^ε has only small variations for propagation distances of order ε we have, to leading order in ε,

$$\begin{aligned} v^\varepsilon(s+\varepsilon\tau,\boldsymbol{x}) = \frac{1}{4\pi}\int_{\partial B(\boldsymbol{0},1)} & v\big(s,\boldsymbol{x}+\varepsilon c_0(\boldsymbol{x})\tau\boldsymbol{z}\big) \\ & + \varepsilon c_0(\boldsymbol{x})\tau\boldsymbol{z}\cdot\nabla v\big(s,\boldsymbol{x}+\varepsilon c_0(\boldsymbol{x})\tau\boldsymbol{z}\big) + \varepsilon\tau\partial_s v\big(s,\boldsymbol{x}+\varepsilon c_0(\boldsymbol{x})\tau\boldsymbol{z}\big)\mathrm{d}\boldsymbol{z}. \end{aligned} \tag{B.1}$$

The following asymptotic relations are obtained from (4.3-4.4) and the fact that $a^-(-\boldsymbol{k}) = a^+(\boldsymbol{k}) \equiv a(\boldsymbol{k})$:

$$\begin{aligned} \lim_{\varepsilon\to 0}\langle v^\varepsilon(s,\boldsymbol{x}) v^\varepsilon(s,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle &= \frac{c_0^2(\boldsymbol{x})}{(2\pi)^3}\int a(s,\boldsymbol{x},\boldsymbol{k})\cos(\boldsymbol{k}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{k}, \\ \lim_{\varepsilon\to 0}\varepsilon\langle v^\varepsilon(s,\boldsymbol{x})\partial_s v^\varepsilon(s,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle &= -\lim_{\varepsilon\to 0} c_0^2(\boldsymbol{x})\varepsilon\langle v^\varepsilon(s,\boldsymbol{x})\nabla_{\boldsymbol{x}}\cdot\boldsymbol{w}^\varepsilon(s,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle \\ &= -\lim_{\varepsilon\to 0} c_0^2(\boldsymbol{x})\nabla_{\boldsymbol{y}}\cdot\langle v^\varepsilon(s,\boldsymbol{x})\boldsymbol{w}^\varepsilon(s,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle \\ &= \frac{c_0^3(\boldsymbol{x})}{(2\pi)^3}\int |\boldsymbol{k}| a(s,\boldsymbol{x},\boldsymbol{k})\sin(\boldsymbol{k}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{k}, \\ \lim_{\varepsilon\to 0}\varepsilon\langle v^\varepsilon(s,\boldsymbol{x})\nabla v^\varepsilon(s,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle &= \lim_{\varepsilon\to 0}\nabla_{\boldsymbol{y}}\langle v^\varepsilon(s,\boldsymbol{x}) v^\varepsilon(s,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle \\ &= -\frac{c_0^2(\boldsymbol{x})}{(2\pi)^3}\int \boldsymbol{k} a(s,\boldsymbol{x},\boldsymbol{k})\sin(\boldsymbol{k}\cdot\boldsymbol{y})\mathrm{d}\boldsymbol{k}. \end{aligned}$$

Substituting into the Kirchhoff representation formula (B.1) we find that, in the limit $\varepsilon\to 0$,

$$\begin{aligned} \langle v^\varepsilon(s,\boldsymbol{x}) v(s+\varepsilon\tau,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle = {} & \frac{c_0^2(\boldsymbol{x})}{(2\pi)^3}\frac{1}{4\pi}\int\int_{\partial B(\boldsymbol{0},1)} a(s,\boldsymbol{x},\boldsymbol{k})\cos\big[\boldsymbol{k}\cdot(\boldsymbol{y}+c_0(\boldsymbol{x})\tau\boldsymbol{z})\big]\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{k} \\ & -\frac{c_0^2(\boldsymbol{x})}{(2\pi)^3}\frac{1}{4\pi}\int\int_{\partial B(\boldsymbol{0},1)} c_0(\boldsymbol{x})\tau\boldsymbol{k}\cdot\boldsymbol{z} a(s,\boldsymbol{x},\boldsymbol{k})\sin\big[\boldsymbol{k}\cdot(\boldsymbol{y}+c_0(\boldsymbol{x})\tau\boldsymbol{z})\big]\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{k} \\ & +\frac{c_0^2(\boldsymbol{x})}{(2\pi)^3}\frac{1}{4\pi}\int\int_{\partial B(\boldsymbol{0},1)} c_0(\boldsymbol{x})\tau|\boldsymbol{k}| a(s,\boldsymbol{x},\boldsymbol{k})\sin\big[\boldsymbol{k}\cdot(\boldsymbol{y}+c_0(\boldsymbol{x})\tau\boldsymbol{z})\big]\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{k}. \end{aligned}$$

Using the integration formulas

$$\frac{1}{4\pi}\int_{\partial B(\boldsymbol{0},1)} e^{i\boldsymbol{k}\cdot\boldsymbol{z}c_0\tau}\mathrm{d}\boldsymbol{z} = \mathrm{sinc}\big(|\boldsymbol{k}|c_0\tau\big)$$

and

$$\frac{1}{4\pi}\int_{\partial B(\mathbf{0},1)} c_0\tau \boldsymbol{k}\cdot\boldsymbol{z} e^{i\boldsymbol{k}\cdot\boldsymbol{z}c_0\tau}\mathrm{d}\boldsymbol{z} = i\,\mathrm{sinc}\big(|\boldsymbol{k}|c_0\tau\big) - i\cos\big(|\boldsymbol{k}|c_0\tau\big),$$

we obtain

$$\langle v^\varepsilon(s,\boldsymbol{x})v(s+\varepsilon\tau,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle = \frac{c_0^2(\boldsymbol{x})}{(2\pi)^3}\int a(s,\boldsymbol{x},\boldsymbol{k})\cos\big(|\boldsymbol{k}|c_0(\boldsymbol{x})\tau - \boldsymbol{k}\cdot\boldsymbol{y}\big)\mathrm{d}\boldsymbol{k},$$

which also reads

$$\begin{aligned}&\int \langle v^\varepsilon(s,\boldsymbol{x})v(s+\varepsilon\tau,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y}\\ &= \frac{c_0^2(\boldsymbol{x})}{2}\big[a(s,\boldsymbol{x},\boldsymbol{k})e^{i|\boldsymbol{k}|c_0(\boldsymbol{x})\tau} + a(s,\boldsymbol{x},-\boldsymbol{k})e^{-i|\boldsymbol{k}|c_0(\boldsymbol{x})\tau}\big].\end{aligned}$$

We need the following elementary lemma:

LEMMA B.1. *If $a(s,\boldsymbol{x},\boldsymbol{k})$ is the solution of (4.5) with the initial condition (4.6), then $W(s,\boldsymbol{x},\boldsymbol{k}) := a(s,\boldsymbol{x},\boldsymbol{k})\omega(\boldsymbol{x},\boldsymbol{k})^2$ is the solution of (4.5) with the initial condition (5.3).*

Proof. We have, for any smooth function ϕ,

$$\big[\nabla_{\boldsymbol{k}}\omega\cdot\nabla_{\boldsymbol{x}} - \nabla_{\boldsymbol{x}}\omega\cdot\nabla_{\boldsymbol{k}}\big]\phi\big(\omega(\boldsymbol{x},\boldsymbol{k})\big) = 0,$$

and for $\boldsymbol{k}'$ such that $\sigma(\boldsymbol{x},\boldsymbol{k},\boldsymbol{k}')\neq 0$, we have $|\boldsymbol{k}| = |\boldsymbol{k}'|$ and therefore $\phi(\omega(\boldsymbol{x},\boldsymbol{k})) = \phi(\omega(\boldsymbol{x},\boldsymbol{k}'))$. As a consequence, if $a(s,\boldsymbol{x},\boldsymbol{k})$ is the solution of (4.5), then the function $a(s,\boldsymbol{x},\boldsymbol{k})\phi(\omega(\boldsymbol{x},\boldsymbol{k}))$ is the solution of (4.5) with the initial condition $a(s=0,\boldsymbol{x},\boldsymbol{k})\phi(\omega(\boldsymbol{x},\boldsymbol{k}))$. ∎

Using (3.6) and the previous lemma we find

$$\begin{aligned}\int \tilde{C}^{(1)}(\varepsilon\tau,\boldsymbol{x},\boldsymbol{x}+\varepsilon\boldsymbol{y})e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y} &= \int \mathrm{d}\boldsymbol{y}\int_0^\infty \mathrm{d}s\,\langle v^\varepsilon(s,\boldsymbol{x})v^\varepsilon(s+\varepsilon\tau,\boldsymbol{x}+\varepsilon\boldsymbol{y})\rangle e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\\ &= \frac{c_0^2(\boldsymbol{x})}{2\omega^2(\boldsymbol{x},\boldsymbol{k})}\big[A(\boldsymbol{x},\boldsymbol{k})e^{i\omega(\boldsymbol{x},\boldsymbol{k})\tau} + A(\boldsymbol{x},-\boldsymbol{k})e^{-i\omega(\boldsymbol{x},\boldsymbol{k})\tau}\big],\end{aligned}$$

where A is defined in Proposition 5.1. By (3.5) we obtain finally

$$\begin{aligned}\int C^{(1)}\big(\varepsilon\tau,\boldsymbol{x},\boldsymbol{x}+\varepsilon\boldsymbol{y}\big)e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\mathrm{d}\boldsymbol{y} &= \varepsilon\int\mathrm{d}\boldsymbol{y}\int \mathrm{d}s\gamma(s)\tilde{C}^{(1)}(\varepsilon\tau-\varepsilon s,\boldsymbol{x},\boldsymbol{x}+\varepsilon\boldsymbol{y})e^{i\boldsymbol{k}\cdot\boldsymbol{y}}\\ &= \frac{\varepsilon c_0^2(\boldsymbol{x})}{2\omega^2(\boldsymbol{x},\boldsymbol{k})}\gamma *_\tau \Big[A(\boldsymbol{x},\boldsymbol{k})e^{i\omega(\boldsymbol{x},\boldsymbol{k})\tau} + A(\boldsymbol{x},-\boldsymbol{k})e^{-i\omega(\boldsymbol{x},\boldsymbol{k})\tau}\Big],\end{aligned}$$

which gives the desired result.

Appendix C. Support of single scattering transport kernel. We prove in this appendix Proposition 6.2. Let us consider the expression (6.7). For a given pair $(t,\boldsymbol{k}')$, which characterizes the jump (time of the jump and wavevector jump), we make the change of variables

$$(\boldsymbol{x},\boldsymbol{k})\mapsto(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}}) = (\mathbf{X}_{s-t},\mathbf{K}_{s-t})\big(\mathbf{X}_t(\boldsymbol{x},\boldsymbol{k}),\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})+\boldsymbol{k}'\big).$$

Here $(\boldsymbol{x},\boldsymbol{k})$ is the starting point of the broken ray, and $(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}})$ is the end point. The inverse transform is

$$(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}}) \mapsto (\boldsymbol{x},\boldsymbol{k}) = (\mathbf{X}_t,-\mathbf{K}_t)\big(\mathbf{X}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}),\mathbf{K}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})+\boldsymbol{k}'\big). \tag{C.1}$$

Moreover we have

$$(\mathbf{X}_t,\mathbf{K}_t)(\boldsymbol{x},\boldsymbol{k}) = \big(\mathbf{X}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}),-\mathbf{K}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})-\boldsymbol{k}'\big),$$

for any $t'\in(0,t)$ we have

$$(\mathbf{X}_{t'},\mathbf{K}_{t'})(\mathbf{x},\mathbf{k}) = (\mathbf{X}_{t-t'},-\mathbf{K}_{t-t'})\big(\mathbf{X}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}),\mathbf{K}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})+\boldsymbol{k}'\big),$$

and for any $t'\in(t,s)$ we have

$$(\mathbf{X}_{t'-t},\mathbf{K}_{t'-t})(\mathbf{X}_t(\boldsymbol{x},\boldsymbol{k}),\mathbf{K}_t(\boldsymbol{x},\boldsymbol{k})+\boldsymbol{k}') = (\mathbf{X}_{s-t'},-\mathbf{K}_{s-t'})(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}).$$

Substituting into (6.7) we obtain

$$\begin{aligned}
&\iint \phi(\boldsymbol{x},\boldsymbol{k})W_1(s,\boldsymbol{x},\boldsymbol{k})\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{k} = \int_0^s \mathrm{d}t\int \mathrm{d}\boldsymbol{k}'\iint \mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}\tilde{\boldsymbol{k}}\phi(\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}})J^{-1}(s,t,\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}},\boldsymbol{k}')\\
&\quad\times c_0^2\Big(\mathbf{X}_t\big(\mathbf{X}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}),\mathbf{K}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})+\boldsymbol{k}'\big)\Big)\\
&\quad\times\hat{\Gamma}\Big((\mathbf{X}_t,-\mathbf{K}_t)\big(\mathbf{X}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}),\mathbf{K}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})+\boldsymbol{k}'\big)\Big)\\
&\quad\times e^{-\int_0^t \Sigma\big((\mathbf{X}_{t-t'},-\mathbf{K}_{t-t'})(\mathbf{X}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}),\mathbf{K}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})+\boldsymbol{k}')\big)\mathrm{d}t'}\\
&\quad\times\sigma\big(\mathbf{X}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}}),-\mathbf{K}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})-\boldsymbol{k}',-\mathbf{K}_{s-t}(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})\big)\\
&\quad\times e^{-\int_t^s \Sigma\big((\mathbf{X}_{s-t'},-\mathbf{K}_{s-t'})(\tilde{\boldsymbol{x}},-\tilde{\boldsymbol{k}})\big)\mathrm{d}t'},
\end{aligned}$$

where $J(s,t,\tilde{\boldsymbol{x}},\tilde{\boldsymbol{k}},\boldsymbol{k}')$ (for the given pair $(t,\boldsymbol{k}')$) is the Jacobian of the transform (C.1). Using the fact that $\sigma(\boldsymbol{x},-\boldsymbol{k},-\boldsymbol{k}')=\sigma(\boldsymbol{x},\boldsymbol{k}',\boldsymbol{k})$, $\Sigma(\boldsymbol{x},-\boldsymbol{k})=\Sigma(\boldsymbol{x},\boldsymbol{k})$, and $\hat{\Gamma}(\boldsymbol{x},-\boldsymbol{k})=\hat{\Gamma}(\boldsymbol{x},\boldsymbol{k})$, this shows that

$$\begin{aligned}
W_1(s,\boldsymbol{x},\boldsymbol{k}) &= \int_0^s \mathrm{d}t\int \mathrm{d}\boldsymbol{k}' J^{-1}(s,t,\boldsymbol{x},\boldsymbol{k},\boldsymbol{k}')\\
&\quad\times c_0^2\Big(\mathbf{X}_t\big(\mathbf{X}_{s-t}(\boldsymbol{x},-\boldsymbol{k}),\mathbf{K}_{s-t}(\boldsymbol{x},-\boldsymbol{k})+\boldsymbol{k}'\big)\Big)\\
&\quad\times\hat{\Gamma}\Big((\mathbf{X}_t,\mathbf{K}_t)\big(\mathbf{X}_{s-t}(\boldsymbol{x},-\boldsymbol{k}),\mathbf{K}_{s-t}(\boldsymbol{x},-\boldsymbol{k})+\boldsymbol{k}'\big)\Big)\\
&\quad\times e^{-\int_0^t \Sigma\big((\mathbf{X}_{t-t'},\mathbf{K}_{t-t'})(\mathbf{X}_{s-t}(\boldsymbol{x},-\boldsymbol{k}),\mathbf{K}_{s-t}(\boldsymbol{x},-\boldsymbol{k})+\boldsymbol{k}')\big)\mathrm{d}t'}\\
&\quad\times\sigma\big(\mathbf{X}_{s-t}(\boldsymbol{x},-\boldsymbol{k}),\mathbf{K}_{s-t}(\boldsymbol{x},-\boldsymbol{k}),\mathbf{K}_{s-t}(\boldsymbol{x},-\boldsymbol{k})+\boldsymbol{k}'\big)\\
&\quad\times e^{-\int_0^{s-t} \Sigma\big((\mathbf{X}_{t'},\mathbf{K}_{t'})(\boldsymbol{x},-\boldsymbol{k})\big)\mathrm{d}t'}.
\end{aligned} \tag{C.2}$$

Let us consider a broken ray starting from $(\boldsymbol{x},-\boldsymbol{k})$, jumping at time $s-t$ with the deviation $\boldsymbol{k}'$ and reaching the interior of the support of $\hat{\Gamma}$ at time s. It is possible to find an open interval I centered at s, an open time interval I' centered at $s-t$, and an open domain Ω centered at $\boldsymbol{k}'$ such that a ray starting from $(\boldsymbol{x},-\boldsymbol{k})$, jumping at some time in I with the new deviation in Ω still reaches the interior of $\hat{\Gamma}$ for any s in I'. These rays give a positive contribution to the integral (C.2) for any s in I'. This proves Proposition 6.2.

Appendix D. Derivation of Lemma 7.1. For $\boldsymbol{z}\in\mathbb{R}^3$ let $\hat{G}^\varepsilon(\omega,\boldsymbol{y},\boldsymbol{z})$ be the fundamental solution of

$$\frac{\omega^2}{c_\varepsilon(\boldsymbol{y})^2}\hat{G}^\varepsilon(\omega,\boldsymbol{y},\boldsymbol{z})+\Delta_{\boldsymbol{y}}\hat{G}^\varepsilon(\omega,\boldsymbol{y},\boldsymbol{z})=-\delta(\boldsymbol{y}-\boldsymbol{z})$$

with the Sommerfeld radiation condition. For $\boldsymbol{x}=(\hat{\boldsymbol{x}},x_3)$ in the upper halfspace $x_3>0$, we denote by $\check{G}(\omega,\boldsymbol{y},\boldsymbol{x})$ the solution of

$$\frac{\omega^2}{c_0^2((\hat{\boldsymbol{y}},|y_3|))}\check{G}(\omega,\boldsymbol{y},\boldsymbol{x})+\Delta_{\boldsymbol{y}}\check{G}(\omega,\boldsymbol{y},\boldsymbol{x})=-\big(\delta(\boldsymbol{y}-\boldsymbol{x})-\delta(\boldsymbol{y}-\boldsymbol{x}^-)\big),$$

where $\boldsymbol{x}^-=(\hat{\boldsymbol{x}},-x_3)$. Since $c^\varepsilon(\boldsymbol{y})=c_0(\boldsymbol{y})$ in the upper halfspace, we find by Green's second identity that, for any point $\boldsymbol{z}$ in the lower halfspace and $\boldsymbol{x}$ in the upper halfspace,

$$\begin{aligned}
\hat{G}^\varepsilon(\omega,\boldsymbol{x},\boldsymbol{z})&=\int_{\mathbb{R}^2\times(0,\infty)}\hat{G}^\varepsilon(\omega,\boldsymbol{y},\boldsymbol{z})\delta(\boldsymbol{y}-\boldsymbol{x})\mathrm{d}\boldsymbol{y}\\
&=\int_{\mathbb{R}^2\times(0,\infty)}\hat{G}^\varepsilon(\omega,\boldsymbol{y},\boldsymbol{z})\Big[-\frac{\omega^2}{c_0(\boldsymbol{x})^2}\check{G}(\omega,\boldsymbol{y},\boldsymbol{x})-\Delta_{\boldsymbol{y}}\check{G}(\omega,\boldsymbol{y},\boldsymbol{x})\Big]\mathrm{d}\boldsymbol{y}\\
&=\int_{\mathbb{R}^2\times(0,\infty)}\check{G}(\omega,\boldsymbol{y},\boldsymbol{x})\Delta_{\boldsymbol{y}}\hat{G}^\varepsilon(\omega,\boldsymbol{y},\boldsymbol{z})-\hat{G}^\varepsilon(\omega,\boldsymbol{y},\boldsymbol{z})\Delta_{\boldsymbol{y}}\check{G}(\omega,\boldsymbol{y},\boldsymbol{x})\mathrm{d}\boldsymbol{y}\\
&=\int_{\mathbb{R}^2}\check{G}\big(\omega,(\hat{\boldsymbol{y}},0),\boldsymbol{x}\big)\partial_{y_3}\hat{G}^\varepsilon\big(\omega,(\hat{\boldsymbol{y}},0),\boldsymbol{z}\big)-\hat{G}^\varepsilon\big(\omega,(\hat{\boldsymbol{y}},0),\boldsymbol{z}\big)\partial_{y_3}\check{G}\big(\omega,(\hat{\boldsymbol{y}},0),\boldsymbol{x}\big)\mathrm{d}\hat{\boldsymbol{y}}\\
&=-\int_{\mathbb{R}^2}\hat{G}^\varepsilon(\omega,(\hat{\boldsymbol{y}},0),\boldsymbol{z})\partial_{y_3}\check{G}(\omega,(\hat{\boldsymbol{y}},0),\boldsymbol{x})\mathrm{d}\hat{\boldsymbol{y}},
\end{aligned}$$

since $\check{G}\big(\omega,(\hat{\boldsymbol{y}},0),\boldsymbol{x}\big)=0$. Since the noise sources $f^\varepsilon(t,\boldsymbol{z})$ in Equation (2.1) are localized in the lower halfspace, we obtain

$$\begin{aligned}
\hat{u}^\varepsilon(\omega,\boldsymbol{x})&=\int_{\mathbb{R}^3}\hat{G}^\varepsilon(\omega,\boldsymbol{x},\boldsymbol{z})\hat{f}^\varepsilon(\omega,\boldsymbol{z})\mathrm{d}\boldsymbol{z}\\
&=-\int_{\mathbb{R}^2}\hat{u}^\varepsilon(\omega,(\hat{\boldsymbol{y}},0))\check{G}'(\omega,\hat{\boldsymbol{y}},\boldsymbol{x})\mathrm{d}\hat{\boldsymbol{y}},
\end{aligned}\tag{D.1}$$

using the notation

$$\check{G}'(\omega,\hat{\boldsymbol{y}},\boldsymbol{x})=\partial_{y_3}\check{G}(\omega,(\hat{\boldsymbol{y}},0),\boldsymbol{x}).$$

Equation (D.1) gives a representation of the field in the upper halfspace in terms of the field at the surface. Then we find for $C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')=\langle u^\varepsilon(t,\boldsymbol{x})u^\varepsilon(t+\tau,\boldsymbol{x}')\rangle$ the expression

$$\begin{aligned}
C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')&=\frac{1}{2\pi}\iint\hat{C}^{(1)}(\omega,\hat{\boldsymbol{y}}_1,\hat{\boldsymbol{y}}_2)\overline{\check{G}'(\omega,\hat{\boldsymbol{y}}_1,\boldsymbol{x})}\check{G}'(\omega,\hat{\boldsymbol{y}}_2,\boldsymbol{x}')e^{-i\omega\tau}\mathrm{d}\hat{\boldsymbol{y}}_1\mathrm{d}\hat{\boldsymbol{y}}_2\mathrm{d}\omega\\
&=\frac{\varepsilon^2}{2\pi}\iint C^{(1)}\left(\varepsilon s,\hat{\boldsymbol{y}}-\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\hat{\boldsymbol{y}}+\frac{\varepsilon\hat{\boldsymbol{y}}'}{2}\right)\overline{\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}-\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}\right)}\\
&\quad\times\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}+\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}'\right)e^{-i\frac{\omega}{\varepsilon}\tau+i\omega s}\mathrm{d}\hat{\boldsymbol{y}}\mathrm{d}\hat{\boldsymbol{y}}'\mathrm{d}s\mathrm{d}\omega,
\end{aligned}\tag{D.2}$$

using the notation

$$C^{(1)}(t,\hat{\boldsymbol{y}}_1,\hat{\boldsymbol{y}}_2) \equiv C^{(1)}(t,(\hat{\boldsymbol{y}}_1,0),(\hat{\boldsymbol{y}}_2,0))$$

for the cross correlation of the field at the surface $x_3=0$. Thus

$$\frac{\partial}{\partial\tau}C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}') = \frac{\varepsilon}{2\pi}\iint \frac{\partial}{\partial s}C^{(1)}\left(\varepsilon s,\hat{\boldsymbol{y}}-\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\hat{\boldsymbol{y}}+\frac{\varepsilon\hat{\boldsymbol{y}}'}{2}\right) \times\overline{\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}-\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}\right)}\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}+\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}'\right)e^{-i\frac{\omega}{\varepsilon}\tau+i\omega s}\mathrm{d}\hat{\boldsymbol{y}}\mathrm{d}\hat{\boldsymbol{y}}'\mathrm{d}s\mathrm{d}\omega.$$

Then, using the weak limit in Proposition 5.1, we find

$$\frac{\partial}{\partial\tau}C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}') = \frac{\varepsilon^2}{2(2\pi)^4}\iint \frac{c_0(\hat{\boldsymbol{y}})\gamma(s-\tilde{s})}{|\boldsymbol{k}|}\left[iA(\hat{\boldsymbol{y}},\boldsymbol{k})e^{ic_0(\hat{\boldsymbol{y}})|\boldsymbol{k}|\tilde{s}}-iA(\hat{\boldsymbol{y}},-\boldsymbol{k})e^{-ic_0(\hat{\boldsymbol{y}})|\boldsymbol{k}|\tilde{s}}\right] \times e^{-i\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{y}}'}\overline{\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}-\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}\right)}\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}+\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}'\right)e^{-i\frac{\omega}{\varepsilon}\tau+i\omega s}\mathrm{d}\tilde{s}\mathrm{d}\boldsymbol{k}\mathrm{d}\hat{\boldsymbol{y}}\mathrm{d}\hat{\boldsymbol{y}}'\mathrm{d}s\mathrm{d}\omega, \quad \text{(D.3)}$$

using the notation $c_0(\hat{\boldsymbol{y}})\equiv c_0((\hat{\boldsymbol{y}},0))$ and $\boldsymbol{k}=(\hat{\boldsymbol{k}},k_3)$. Note that the above substitution holds in a weak form. We comment on the measurement configuration that justifies this in Appendix E. By integrating in s we obtain

$$\frac{\partial}{\partial\tau}C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}) = \frac{\varepsilon^2}{2(2\pi)^4}\iint \frac{c_0(\hat{\boldsymbol{y}})\hat{\gamma}(\omega)}{|\boldsymbol{k}|}\left[iA(\hat{\boldsymbol{y}},\boldsymbol{k})e^{i(\omega+c_0(\hat{\boldsymbol{y}})|\boldsymbol{k}|)\tilde{s}}-iA(\hat{\boldsymbol{y}},-\boldsymbol{k})e^{i(\omega-c_0(\hat{\boldsymbol{y}})|\boldsymbol{k}|)\tilde{s}}\right] \times e^{-i\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{y}}'}\overline{\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}-\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}\right)}\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}+\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}'\right)e^{-i\frac{\omega}{\varepsilon}\tau}\mathrm{d}\tilde{s}\mathrm{d}\boldsymbol{k}\mathrm{d}\hat{\boldsymbol{y}}\mathrm{d}\hat{\boldsymbol{y}}'\mathrm{d}\omega.$$

The integral in $\tilde{s}$ can be evaluated:

$$\int e^{i(\omega+c_0(\hat{\boldsymbol{y}})|\boldsymbol{k}|)\tilde{s}}\mathrm{d}\tilde{s} = \frac{2\pi}{c_0(\hat{\boldsymbol{y}})}\delta\Big(|\boldsymbol{k}|-\frac{|\omega|}{c_0(\hat{\boldsymbol{y}})}\Big)\mathbf{1}_{(-\infty,0)}(\omega),$$

$$\int e^{i(\omega-c_0(\hat{\boldsymbol{y}})|\boldsymbol{k}|)\tilde{s}}\mathrm{d}\tilde{s} = \frac{2\pi}{c_0(\hat{\boldsymbol{y}})}\delta\Big(|\boldsymbol{k}|-\frac{\omega}{c_0(\hat{\boldsymbol{y}})}\Big)\mathbf{1}_{(0,\infty)}(\omega),$$

so that

$$\int \left[iA(\hat{\boldsymbol{y}},\boldsymbol{k})e^{i(\omega+c_0(\hat{\boldsymbol{y}})|\boldsymbol{k}|)\tilde{s}}-iA(\hat{\boldsymbol{y}},-\boldsymbol{k})e^{i(\omega-c_0(\hat{\boldsymbol{y}})|\boldsymbol{k}|)\tilde{s}}\right]\mathrm{d}\tilde{s} = -\frac{2\pi i\,\mathrm{sgn}(\omega)}{c_0(\hat{\boldsymbol{y}})}\delta\Big(|\boldsymbol{k}|-\frac{|\omega|}{c_0(\hat{\boldsymbol{y}})}\Big)A(\hat{\boldsymbol{y}},-\mathrm{sgn}(\omega)\boldsymbol{k}).$$

We can also apply the high-frequency approximation

$$\check{G}\left(\frac{\omega}{\varepsilon},\boldsymbol{y},\boldsymbol{x}\right)\simeq \alpha(\boldsymbol{y},\boldsymbol{x})e^{i\frac{\omega}{\varepsilon}\mathcal{T}(\boldsymbol{y},\boldsymbol{x})}-\alpha(\boldsymbol{y},\boldsymbol{x}^-)e^{i\frac{\omega}{\varepsilon}\mathcal{T}(\boldsymbol{y},\boldsymbol{x}^-)}.$$

In the homogeneous case we have simply

$$\alpha(\boldsymbol{y},\boldsymbol{x})=\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}, \qquad \mathcal{T}(\boldsymbol{y},\boldsymbol{x})=\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c_0}.$$

With the high-frequency approximation of the background Green's function we find

$$\frac{\partial}{\partial\tau}C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')=-\frac{1}{4\pi^3}\iint\frac{i\mathrm{sgn}(\omega)\omega^2}{|\boldsymbol{k}|}A\big(\hat{\boldsymbol{y}},-\mathrm{sgn}(\omega)\boldsymbol{k}\big)\hat{\gamma}(\omega)\delta\Big(|\boldsymbol{k}|-\frac{|\omega|}{c_0(\hat{\boldsymbol{y}})}\Big)e^{-i\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{y}}'}$$
$$\times\alpha(\hat{\boldsymbol{y}},\boldsymbol{x})\alpha(\hat{\boldsymbol{y}},\boldsymbol{x}')\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x})\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x}')e^{-i\frac{\omega}{\varepsilon}(\mathcal{T}(\hat{\boldsymbol{y}}-\varepsilon\hat{\boldsymbol{y}}'/2,\boldsymbol{x})-\mathcal{T}(\hat{\boldsymbol{y}}+\varepsilon\hat{\boldsymbol{y}}'/2,\boldsymbol{x}')+\tau)}\mathrm{d}\omega\mathrm{d}\boldsymbol{k}\mathrm{d}\hat{\boldsymbol{y}}\mathrm{d}\hat{\boldsymbol{y}}',$$

using here the notation

$$\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x})=\partial_{y_3}\mathcal{T}\big((\hat{\boldsymbol{y}},0),\boldsymbol{x}\big),$$

and the symmetry relations $\alpha(\hat{\boldsymbol{y}},\boldsymbol{x}^-)=\alpha(\hat{\boldsymbol{y}},\boldsymbol{x})$, $\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x}^-)=\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x})$, and $\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x}^-)=-\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x})$. Thus, we have

$$\frac{\partial}{\partial\tau}C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')=-\frac{1}{4\pi^3}\iint A\big(\hat{\boldsymbol{y}},-\mathrm{sgn}(\omega)\boldsymbol{k}\big)\frac{i\mathrm{sgn}(\omega)\omega^2}{|\boldsymbol{k}|}\hat{\gamma}(\omega)\delta\Big(|\boldsymbol{k}|-\frac{|\omega|}{c_0(\hat{\boldsymbol{y}})}\Big)$$
$$\times\alpha(\hat{\boldsymbol{y}},\boldsymbol{x})\alpha(\hat{\boldsymbol{y}},\boldsymbol{x}')\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x})\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x}')$$
$$\times e^{-i\frac{\omega}{\varepsilon}(\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x})-\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x}')+\tau)}e^{i\frac{\omega}{2}\hat{\boldsymbol{y}}'\cdot\nabla_{\hat{\boldsymbol{y}}}(\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x})+\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x}'))}e^{-i\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{y}}'}\mathrm{d}\omega\mathrm{d}\boldsymbol{k}\mathrm{d}\hat{\boldsymbol{y}}\mathrm{d}\hat{\boldsymbol{y}}'.$$

By integrating in $\hat{\boldsymbol{y}}'$, in $\hat{\boldsymbol{k}}$, and then in k_3, we get

$$\frac{\partial}{\partial\tau}C^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')=-\frac{1}{\pi}\iint\big[A(\hat{\boldsymbol{y}},|\omega|\boldsymbol{\kappa}(\hat{\boldsymbol{y}}))+A(\hat{\boldsymbol{y}},|\omega|\boldsymbol{\kappa}^-(\hat{\boldsymbol{y}}))\big]ic_0(\hat{\boldsymbol{y}})\omega\hat{\gamma}(\omega)$$
$$\times\alpha(\hat{\boldsymbol{y}},\boldsymbol{x})\alpha(\hat{\boldsymbol{y}},\boldsymbol{x}')\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x})\mathcal{T}'(\hat{\boldsymbol{y}},\boldsymbol{x}')e^{-i\frac{\omega}{\varepsilon}(\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x})-\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x}')+\tau)}\mathcal{H}(\hat{\boldsymbol{y}})^{-1}\mathrm{d}\omega\mathrm{d}\hat{\boldsymbol{y}},\quad\text{(D.4)}$$

for

$$\mathcal{H}(\hat{\boldsymbol{y}})=\sqrt{1-\frac{c_0(\hat{\boldsymbol{y}})^2|\nabla_{\hat{\boldsymbol{y}}}(\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x})+\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x}'))|^2}{4}},$$
$$\boldsymbol{\kappa}(\hat{\boldsymbol{y}})=\big(\hat{\boldsymbol{\kappa}}(\hat{\boldsymbol{y}}),\kappa_3(\hat{\boldsymbol{y}})\big),\qquad\hat{\boldsymbol{\kappa}}(\hat{\boldsymbol{y}})=-\frac{\nabla_{\hat{\boldsymbol{y}}}(\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x})+\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x}'))}{2},\qquad\kappa_3(\hat{\boldsymbol{y}})=\frac{\mathcal{H}(\hat{\boldsymbol{y}})}{c_0(\hat{\boldsymbol{y}})},$$

and $\boldsymbol{\kappa}^-(\hat{\boldsymbol{y}})=\big(\hat{\boldsymbol{\kappa}}(\hat{\boldsymbol{y}}),-\kappa_3(\hat{\boldsymbol{y}})\big)$. Next we make the assumption of no turning ray, so that there is no flux of energy going from the upper halfspace to the lower halfspace $A(\hat{\boldsymbol{y}},|\omega|\boldsymbol{\kappa}^-(\boldsymbol{y}))=0$.

In view of the fast phase term in (D.4) we now evaluate this expression by a stationary phase approximation. Consider the phase term $-\omega\Theta(\hat{\boldsymbol{y}},\boldsymbol{x},\boldsymbol{x}')$ with

$$\Theta(\hat{\boldsymbol{y}},\boldsymbol{x},\boldsymbol{x}')=\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x})-\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x}')+\tau.\qquad\text{(D.5)}$$

We have a stationary phase point satisfying $\nabla_{\hat{\boldsymbol{y}}}(\omega\Theta)=\boldsymbol{0}$ and $\partial_\omega(\omega\Theta)=0$ if and only if $\hat{\boldsymbol{y}}$ lies on the ray going through $\boldsymbol{x}$ and $\boldsymbol{x}'$ (and is outside the segment between $\boldsymbol{x}$ and $\boldsymbol{x}'$) and if $\tau=\mathcal{T}(\boldsymbol{x},\boldsymbol{x}')$ (here we use the fact that $x_3'>x_3$). We denote by $\hat{\boldsymbol{y}}^*(\boldsymbol{x},\boldsymbol{x}')$ the unique intersection of the ray going through $\boldsymbol{x}$ and $\boldsymbol{x}'$ with the surface $z=0$ and by $\mathbf{H}^*(\boldsymbol{x},\boldsymbol{x}')$ the Hessian

$$\mathbf{H}^*(\boldsymbol{x},\boldsymbol{x}')=\nabla_{\hat{\boldsymbol{y}}}\otimes\nabla_{\hat{\boldsymbol{y}}}\Theta(\hat{\boldsymbol{y}}^*,\boldsymbol{x},\boldsymbol{x}').\qquad\text{(D.6)}$$

We assume that $\mathbf{H}^*$ is positive definite. This holds in particular when the upper halfspace is homogeneous since $x_3'>x_3$, in which case

$$\hat{\boldsymbol{y}}^*=\frac{x_3'\hat{\boldsymbol{x}}-x_3\hat{\boldsymbol{x}}}{x_3'-x_3},\qquad\boldsymbol{k}(\hat{\boldsymbol{y}}^*)=\frac{1}{c_0}\frac{\boldsymbol{x}'-\boldsymbol{x}}{|\boldsymbol{x}'-\boldsymbol{x}|},\qquad\det\mathbf{H}^*=\frac{(x_3'-x_3)^6}{c_0^2x_3^2{x_3'}^2|\boldsymbol{x}-\boldsymbol{x}'|^4}.$$

More generally we have $\nabla_{\hat{\boldsymbol{y}}}\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x})=\nabla_{\hat{\boldsymbol{y}}}\mathcal{T}(\hat{\boldsymbol{y}},\boldsymbol{x}')$ at $\hat{\boldsymbol{y}}=\hat{\boldsymbol{y}}^*$. Using the eikonal equation

$$|\nabla_{\hat{\boldsymbol{y}}}\mathcal{T}((\hat{\boldsymbol{y}},0),\boldsymbol{x}')|^2+(\partial_{y_3}\mathcal{T}((\hat{\boldsymbol{y}},0),\boldsymbol{x}'))^2=c_0^{-2}(\hat{\boldsymbol{y}}),$$

we find that $\mathcal{H}(\hat{\boldsymbol{y}}^*)=-c_0(\hat{\boldsymbol{y}}^*)\mathcal{T}'(\hat{\boldsymbol{y}}^*,\boldsymbol{x}')$ and $\boldsymbol{\kappa}(\boldsymbol{y}^*)=-\nabla_{\boldsymbol{y}}\mathcal{T}((\boldsymbol{y}^*,0),\boldsymbol{x}')$. We then get, from the stationary phase evaluation,

$$\begin{aligned}&\frac{\partial}{\partial\tau}C^{(1)}\big(\mathcal{T}(\boldsymbol{x},\boldsymbol{x}')+\varepsilon s,\boldsymbol{x},\boldsymbol{x}'\big)\\&=-2\varepsilon\frac{c_0(\hat{\boldsymbol{y}}^*)\alpha(\hat{\boldsymbol{y}}^*,\boldsymbol{x})\alpha(\hat{\boldsymbol{y}}^*,\boldsymbol{x}')\mathcal{T}'(\hat{\boldsymbol{y}}^*,\boldsymbol{x})\mathcal{T}'(\hat{\boldsymbol{y}}^*,\boldsymbol{x}')}{\sqrt{|\det\mathbf{H}^*|}\mathcal{H}(\hat{\boldsymbol{y}}^*)}\int A\big(\hat{\boldsymbol{y}}^*,|\omega|\boldsymbol{\kappa}(\hat{\boldsymbol{y}}^*)\big)\hat{\gamma}(\omega)e^{-i\omega s}\mathrm{d}\omega\\&=-2\varepsilon\frac{\alpha(\hat{\boldsymbol{y}}^*,\boldsymbol{x})\alpha(\hat{\boldsymbol{y}}^*,\boldsymbol{x}')|\mathcal{T}'(\hat{\boldsymbol{y}}^*,\boldsymbol{x})|}{\sqrt{|\det\mathbf{H}^*|}}\int A\big(\hat{\boldsymbol{y}}^*,|\omega|\boldsymbol{\kappa}(\hat{\boldsymbol{y}}^*)\big)\hat{\gamma}(\omega)e^{-i\omega s}\mathrm{d}\omega.\end{aligned}\tag{D.7}$$

Appendix E. Measurement smoothing. In a practical setting one would have smoothed rather than point measurements. We examine here the role of such smoothing. This argument simultaneously justifies using the weak limit in (D.3). Assume that the measurements $u^{\varepsilon,\psi}$ are smoothed versions of the field:

$$\begin{aligned}u^{\varepsilon,\psi}(t,\boldsymbol{x})&=\varepsilon^{-4p}\iint u^\varepsilon(t-s,\boldsymbol{x}-\boldsymbol{z})\psi_t\left(\frac{s}{\varepsilon^p}\right)\psi_x\left(\frac{\boldsymbol{z}}{\varepsilon^p}\right)\mathrm{d}s\mathrm{d}\boldsymbol{z}\\&=\frac{1}{2\pi}\iint\hat{u}^\varepsilon(\omega,\boldsymbol{x}-\varepsilon^p\boldsymbol{z})\widehat{\psi_t}(\varepsilon^p\omega)\psi_x(\boldsymbol{z})e^{-i\omega t}\mathrm{d}\omega\mathrm{d}\boldsymbol{z},\end{aligned}\tag{E.1}$$

with $p>0$ and where $\psi_t(t)$ and $\psi_x(\boldsymbol{x})$ are smooth functions. The cross correlation $C_\psi^{(1)}$ of the smoothed version of the field is

$$\begin{aligned}&C_\psi^{(1)}(\tau,\boldsymbol{x},\boldsymbol{x}')\\&=\big\langle u_\psi^\varepsilon(0,\boldsymbol{x})u_\psi^\varepsilon(\tau,\boldsymbol{x}')\big\rangle\\&=\frac{1}{2\pi}\iint\hat{C}^{(1)}(\omega,\boldsymbol{x}-\varepsilon^p\boldsymbol{z},\boldsymbol{x}'-\varepsilon^p\boldsymbol{z}')\big|\widehat{\psi_t}(\varepsilon^p\omega)\big|^2\psi_x(\boldsymbol{z})\psi_x(\boldsymbol{z}')e^{-i\omega\tau}\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{z}'\mathrm{d}\omega\\&=\frac{\varepsilon^2}{2\pi}\iint C^{(1)}\left(\varepsilon s,\hat{\boldsymbol{y}}-\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\hat{\boldsymbol{y}}+\frac{\varepsilon\hat{\boldsymbol{y}}'}{2}\right)\overline{\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}-\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}-\varepsilon^p\boldsymbol{z}\right)}\check{G}'\left(\frac{\omega}{\varepsilon},\hat{\boldsymbol{y}}+\frac{\varepsilon\hat{\boldsymbol{y}}'}{2},\boldsymbol{x}-\varepsilon^p\boldsymbol{z}'\right)\\&\quad\times\big|\widehat{\psi_t}(\varepsilon^{p-1}\omega)\big|^2\psi_x(\boldsymbol{z})\psi_x(\boldsymbol{z}')e^{-i\frac{\omega}{\varepsilon}\tau+i\omega s}\mathrm{d}\hat{\boldsymbol{y}}\mathrm{d}\hat{\boldsymbol{y}}'\mathrm{d}\boldsymbol{z}\mathrm{d}\boldsymbol{z}'\mathrm{d}\omega.\end{aligned}$$

Comparing with the equivalent formula (D.2) in the absence of smoothing, it is now clear that if $p>1$ the smoothing will be a lower order effect, and if $p<1$ then the smoothing will, in the limit, remove the leading coherent travel time information in the cross correlations. The critical scale $p=1$ corresponds to the natural smoothing on the "effective" wave length of the ambient noise — that is, its coherence time scale. Thus, we subsequently assume $p=1$. Then by using the weak limit of $C^{(1)}$ and the stationary phase method as in the previous section, we find that the cross correlation is concentrated around the time lag $\mathcal{T}(\boldsymbol{x},\boldsymbol{x}')$ (assuming $x_3'>x_3$), and

$$\begin{aligned}&\frac{\partial}{\partial\tau}C^{(1)}\big(\mathcal{T}(\boldsymbol{x},\boldsymbol{x}')+\varepsilon s,\boldsymbol{x},\boldsymbol{x}'\big)\\&=\frac{-2\varepsilon}{\sqrt{\det\mathbf{H}^*}}\alpha(\boldsymbol{y}^*,\boldsymbol{x})\alpha(\boldsymbol{y}^*,\boldsymbol{x}')|\mathcal{T}'(\boldsymbol{y}^*,\boldsymbol{x})|\\&\quad\times\int A\big(\boldsymbol{y}^*,|\omega|\boldsymbol{\kappa}(\hat{\boldsymbol{y}}^*)\big)\widehat{\psi_x}\big(-\omega\nabla_{\boldsymbol{x}}\mathcal{T}(\boldsymbol{y}^*,\boldsymbol{x})\big)\widehat{\psi_x}\big(\omega\nabla_{\boldsymbol{x}'}\mathcal{T}(\boldsymbol{y}^*,\boldsymbol{x}')\big)\hat{\gamma}(\omega)\big|\widehat{\psi_t}(\omega)\big|^2e^{-i\omega s}\mathrm{d}\omega.\end{aligned}$$

In view of (D.7) we see that the smoothing involves a slight modification in the cross correlation traces via a convolution in the offset with the autocorrelation of the measurement time smoothing function. Moreover, there is a modulation of the phase space representation of the field spectrum with the Fourier transform of the correlation of the measurement space smoothing function.

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