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REGULARITY CRITERIA FOR THE 2D BOUSSINESQ EQUATIONS WITH SUPERCRITICAL DISSIPATION*

JINGNA LI[†], HAIFENG SHANG[‡], JIAHONG WU[§], XIAOJING XU[¶], AND ZHUAN YE[‖]

Abstract. This paper focuses on the 2D incompressible Boussinesq equations with fractional dissipation, given by $\Lambda^\alpha u$ in the velocity equation and by $\Lambda^\beta\theta$ in the temperature equation, where $\Lambda=\sqrt{-\Delta}$ denotes the Zygmund operator. Due to the vortex stretching and the lack of sufficient dissipation, the global regularity problem for the supercritical regime $\alpha+\beta<1$ remains an outstanding problem. This paper presents several regularity criteria for the supercritical Boussinesq equations. These criteria are sharp and reflect the level of difficulty of the supercritical Boussinesq problem. In addition, these criteria are important tools in understanding some crucial properties of Boussinesq solutions such as the eventual regularity.

Key words. Boussinesq equations, fractional dissipation, global well-posedness.

AMS subject classifications. 35Q35, 35B65, 76B03.

1. Introduction

The Boussinesq equations model large scale atmospheric and oceanic flows and play an important role in the study of Rayleigh–Bénard convection, one of the most commonly studied convection phenomena (see, e.g., [12,18,19,30,35,40]). The 2D Boussinesq equations have recently attracted considerable attention in the community of mathematical fluid mechanics due to their mathematical significance. Mathematically the 2D Boussinesq equations serve as a lower-dimensional model of the 3D hydrodynamics equations. In fact, the 2D Boussinesq equations retain some key features of the 3D Euler and Navier–Stokes equations such as the vortex stretching mechanism. The inviscid 2D Boussinesq equations are identical to the Euler equations for the 3D axisymmetric swirling flows (away from the symmetry axis) (see, e.g., [31]).

One of the fundamental problems concerning the Boussinesq system is whether or not its solutions remain smooth for all time or they blow up in a finite time. We briefly explain why this problem could be extremely difficult and how much the dissipation and the thermal diffusion can help. When there is no dissipation or thermal diffusion,

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[†]Department of Mathematics, Jinan University, Guangzhou, 510632, P.R. China (jingna8005@hotmail.com).
[‡]School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454000, P.R. China (hfshang@hpu.edu.cn, hfshang@163.com).
[§]Department of Mathematics, Oklahoma State University, Stillwater, OK 74078, USA (jiahong.wu@okstate.edu).
[¶]School of Mathematical Sciences, Beijing Normal University, and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, P.R. China (xjxu@bnu.edu.cn).
[‖]School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, P.R. China (yezhuan815@126.com).

the 2D Boussinesq equation is given by

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p + \theta\mathbf{e}_2, \quad x\in\mathbb{R}^2, t>0, \\ \partial_t\theta + (u\cdot\nabla)\theta = 0, \\ \nabla\cdot u = 0, \end{cases} \tag{1.1}$$

where u represents the 2D velocity field, p the pressure, $\mathbf{e}_2$ the unit vector in the vertical direction and θ the temperature. A standard approach to the global regularity problem is to first obtain the local existence and regularity and then extend the local solution to a global one by establishing global *a priori* bounds for the solution. Due to the divergence-free condition $\nabla\cdot u = 0$, any solution (u,θ) of Equation (1.1) with sufficiently smooth data admits global L^2-bound for u and global L^q-bound for θ ($q\in[1,\infty]$). However, it appears impossible to obtain global bounds for any derivative of u or θ. The main obstacle is the "vortex stretching" term in the equation of the vorticity $\omega = \nabla\times u$ and $\nabla^\perp\theta$, where $\nabla^\perp = (-\partial_{x_2},\partial_{x_1})$. More precisely, $(\omega,\nabla^\perp\theta)$ satisfies

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = \partial_{x_1}\theta, \\ \partial_t\nabla^\perp\theta + (u\cdot\nabla)\nabla^\perp\theta = (\nabla^\perp\theta\cdot\nabla)u, \end{cases} \tag{1.2}$$

which resembles the 3D Euler vorticity equation

$$\partial_t\omega^E + (u^E\cdot\nabla)\omega^E = (\omega^E\cdot\nabla)u^E,$$

where u^E and ω^E denote the 3D Euler velocity and the corresponding vorticity, respectively. The global regularity problem for the 3D Euler equations appear to be out of reach due to the term $(\omega^E\cdot\nabla)u^E$. Potential finite time singularities have been explored from different perspectives including boundary effects and 1D models (see [10,11,29,37]).

Dissipation helps control the derivatives and thus regularizes solutions. When Δu and $\Delta\theta$ are added to the velocity equation and the equation of θ in Equation (1.1), respectively, the global regularity can then be established following a similar proof as that for the 2D Navier–Stokes equations. The issue that arises naturally is how much dissipation is really needed for the global regularity. This problem has attracted considerable interests recently and important progress has been made (see, e.g., [1–3, 6, 9, 13, 17, 20–22, 25–28, 33, 34, 38, 42–48]). We briefly describe some of the relevant work to provide a background for the results we will present.

There have been exciting developments on the 2D Boussinesq equations with fractional dissipation,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nu\Lambda^\alpha u = -\nabla p + \theta\mathbf{e}_2, \quad x\in\mathbb{R}^2, t>0, \\ \partial_t\theta + (u\cdot\nabla)\theta + \kappa\Lambda^\beta\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.3}$$

where $\nu\geq 0$, $\kappa\geq 0$ and $\alpha,\beta\in(0,2)$ are parameters, and $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator defined via the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha\widehat{f}(\xi).$$

This generalization allows us to study a family of equations simultaneously and may be physically relevant. In fact, there are geophysical circumstances in which the Boussinesq equations with fractional Laplacian may arise. Flows in the middle atmosphere traveling upward undergo changes due to the changes in atmospheric properties, although the incompressibility and Boussinesq approximations are applicable. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian (see [7, 19]).

Recent efforts are devoted to the global regularity of Equation (1.3) with the smallest possible $\alpha \in (0,2)$ and $\beta \in (0,2)$. As pointed out in [23], it is useful to classify α and β into three categories: the subcritical case when $\alpha+\beta>1$, the critical case when $\alpha+\beta=1$, and the supercritical case when $\alpha+\beta<1$. This classification gives us a sense of the level of difficulty for different parameter ranges. The subcritical case is relatively easy and the global regularity for several parameter ranges have been established. We note that not all subcritical cases have been resolved. For instance, we do not know the global regularity for the case when α and β are close to $\frac{1}{2}$ and $\alpha+\beta>1$. The critical case is more difficult. Two special critical cases, $\alpha=1, \kappa=0$ and $\beta=1, \nu=0$, were studied and resolved by [20, 21]. General critical cases with $\alpha+\beta=1$ and $\alpha\in(0,1)$ were dealt with by Jiu, Miao, Wu and Zhang [23], which established the global regularity for Equation (1.3) with $\alpha+\beta=1$ and $1>\alpha>\alpha_0 \equiv \frac{23-\sqrt{145}}{12} \approx 0.9132$. Very recently Stefanov and Wu improved the result of Jiu, Miao, Wu, and Zhang by further enlarging the range of α with $\alpha+\beta=1$ and $1>\alpha>\frac{\sqrt{1777}-23}{24}\approx 0.7981$ [38].

The global regularity problem for the supercritical regime $\alpha+\beta<1$ appears to be out of reach at this moment. Very few results are currently available. To help understand this difficult problem, we examine the regularization effects of the fractional dissipation. It appears reasonable to conjecture that solutions of Equation (1.3) with any $\alpha>0$ and $\beta>0$ will become regular eventually, namely for $t>T$ for some $T>0$. Previous work in this direction includes an eventual regularity result of Jiu, Wu, and Yang for $\alpha+\beta<1$ and $\alpha>\frac{23-\sqrt{145}}{12}\approx 0.9132$ [24]. The approach there converts the supercritical 2D Boussinesq equations into a generalized supercritical surface quasi-geostrophic equation. We intend to employ a more direct approach in order to establish the eventual regularity for larger ranges of α and β in the supercritical regime.

This paper presents several regularity criteria for Equation (1.3). These criteria are important first steps toward the eventual resolution of the global regularity issue on the supercritical Boussinesq equations. They specify the regularity window in which any possible finite singularity scenario can occur. The first regularity criterion is for the case when the velocity dissipation dominates, namely $\alpha>\beta$. Roughly speaking, it states that the solution can be extended globally if $(1-\alpha)$-derivative of θ remains bounded. In other words, any finite-time singular solution (u,θ) must have θ blow up in the regularity window between $H^{\frac{\beta}{2}}$ and $B^{1-\alpha}_{\infty,1}$.

THEOREM 1.1. *Consider Equation (1.3) with $\nu>0$, $\kappa>0$, $0<\alpha<1$, and $0\le\beta<\alpha$. Assume $(u_0,\theta_0)\in H^s(\mathbb{R}^2)$ with $s>2$ and $\nabla\cdot u_0=0$. Let (u,θ) be the corresponding local solution of Equation (1.3) on $[0,T_0)$. If, for some $T\ge T_0$, θ satisfies*

$$\theta\in L^1([0,T];B^{1-\alpha}_{\infty,1}(\mathbb{R}^2)), \tag{1.4}$$

then the local solution can be extended to $[0,T]$. Especially, $\theta\in L^1([0,T];C^\gamma(\mathbb{R}^2))$ with $\gamma\in(1-\alpha,1)$ implies the extension to $[0,T]$.

We remark that Theorem 1.1 holds for $\kappa=0$, when there is no thermal diffusion. The proof of Theorem 1.1 is not trivial. Direct energy estimates would not work. To overcome the difficulty, we work with a combined quantity of ω and θ and apply the Besov space techniques. In addition, this paper also involves an effective approach to handle the difficulty caused by terms generated by working with a combined quantity. The details can be found in Section 2.

The second regularity criterion is for the case when the fractional thermal diffusion dominates, namely $\alpha \le \beta$.

THEOREM 1.2. *Consider Equation (1.3) with $\nu > 0$, $\kappa > 0$, $0 \le \alpha < 1$ and $\alpha \le \beta < 1$. Assume $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$ and $\nabla \cdot u_0 = 0$. Let (u,θ) be the corresponding local solution of Equation (1.3) on $[0,T_0)$. If, for some $T \ge T_0$, θ satisfies*

$$\theta \in L^1([0,T]; B^{1-\beta}_{\infty,1}(\mathbb{R}^2)), \tag{1.5}$$

then the local solution can be extended to $[0,T]$. Especially, $\theta \in L^1([0,T]; C^\gamma(\mathbb{R}^2))$ with $\gamma \in (1-\beta, 1)$ implies the regularity of the solution on $[0,T]$.

Especially, Theorem 1.2 holds for $\nu = 0$, when there is no velocity dissipation. The proof of Theorem 1.2 relies on an alternative regularity criterion in terms of u. This criterion in terms of u is valid for any $0 \le \alpha < 1$ and $0 < \beta < 1$.

THEOREM 1.3. *Consider Equation (1.3) with $\nu > 0$, $\kappa > 0$, $0 \le \alpha < 1$, and $0 < \beta < 1$. Assume $(u_0, \theta_0) \in H^s(\mathbb{R}^2)$ with $s > 2$ and $\nabla \cdot u_0 = 0$. Let (u,θ) be the corresponding local solution on $[0,T_0)$. If, for some $\gamma > 1 - \beta$ and $T \ge T_0$,*

$$u \in L^\infty([0,T]; C^\gamma(\mathbb{R}^2)), \tag{1.6}$$

then the local classical solution can be extended to the time interval $[0,T]$.

The proofs of these results rely on Besov space techniques and are given in the next two sections. Section 2 proves Theorem 1.1. In order to prove this theorem, we state and prove three lemmas in this section. Section 3 proves Theorem 1.2 and Theorem 1.3. An appendix containing the Littlewood–Paley decomposition and the definition of Besov spaces is also given for the convenience of the readers.

2. The case when the velocity dissipation dominates

This section is devoted to the proof of Theorem 1.1, which provides a regularity criterion for the case when the velocity dissipation dominates, namely $\alpha > \beta$ in Equation (1.3). Without loss of generality, we set $\nu = \kappa = 1$ in this section.

The proof of Theorem 1.1 involves working with a combined quantity and applying the Besov space techniques. In addition, as we shall see in the proof of Theorem 1.1, the dissipative term $\Lambda^\beta \theta$ generates a term that hinders our approach. To facilitate the proof of Theorem 1.1, we state and prove three lemmas. The first lemma (Lemma 2.1) provides an easy-to-use upper bound for the L^p type estimates of the localized nonlinear term. The second lemma (Lemma 2.2) gives an upper bound for a commutator. The third lemma (Lemma 2.3) presents an estimate to deal with the aforementioned difficulty due to the dissipative term $\Lambda^\beta \theta$.

The rest of this section starts with the proof of Theorem 1.1, followed by the statements and proofs of the lemmas.

Proof. (**Proof of Theorem 1.1).** The aim is to show that the local solution (u,θ) can be extended to $[0,T]$ under the condition (1.4). More precisely, we show $(u,\theta) \in H^s$ with $s > 2$ for any $t \in [0,T]$. As is well-known, if u satisfies

$$\|\nabla u\|_{L^1(0,T; L^\infty(\mathbb{R}^2))} < \infty,$$

then (u,θ) can be extended to $[0,T]$. Due to the simple inequality

$$\|\nabla u\|_{L^\infty} \le C(\|u\|_{L^2} + \|\omega\|_{B^0_{\infty,1}}),$$

it suffices to show that

$$\|\omega\|_{L^1(0,T;B^0_{\infty,1}(\mathbb{R}^2))} < \infty. \tag{2.1}$$

Direct energy estimates on the vorticity equation

$$\partial_t \omega + u\cdot\nabla\omega + \Lambda^\alpha \omega = \partial_{x_1}\theta \tag{2.2}$$

do not appear to allow us to verify condition (2.1). The idea is to eliminate the "vortex stretching" term $\partial_{x_1}\theta$ by considering a combined quantity (see, e.g., [20, 23, 33]). To this end, we apply $\mathcal{R}_\alpha \equiv \Lambda^{-\alpha}\partial_{x_1}$ to the equation of θ to obtain

$$\partial_t \mathcal{R}_\alpha\theta + u\cdot\nabla\mathcal{R}_\alpha\theta + \Lambda^\beta\mathcal{R}_\alpha\theta = -[\mathcal{R}_\alpha, u\cdot\nabla]\theta. \tag{2.3}$$

Taking the difference of Equations (2.2) and (2.3) yields that

$$G = \omega - \mathcal{R}_\alpha\theta$$

satisfies

$$\partial_t G + u\cdot\nabla G + \Lambda^\alpha G = \Lambda^\beta\mathcal{R}_\alpha\theta + [\mathcal{R}_\alpha, u\cdot\nabla]\theta. \tag{2.4}$$

Our first step is to show that, for any $q\in[2,\infty)$ and for any $T>0$ and $0<t\le T$,

$$\|\Lambda^{1-\alpha}\theta(t)\|_{L^q} \le C(T,u_0,\theta_0), \qquad \|G(t)\|_{L^q} \le C(T,u_0,\theta_0). \tag{2.5}$$

Multiplying Equation (2.4) by $G|G|^{q-2}$ and integrating in space, we obtain, after integration by parts,

$$\frac{1}{q}\frac{d}{dt}\|G\|_{L^q}^q + \int G|G|^{q-2}\Lambda^\alpha G = \int \Lambda^\beta\mathcal{R}_\alpha\theta\, G|G|^{q-2} + \int [\mathcal{R}_\alpha, u\cdot\nabla]\theta\, G|G|^{q-2}.$$

The dissipative term is nonnegative (see, e.g., [15])

$$\int G|G|^{q-2}\Lambda^\alpha G \ge 0.$$

Applying Hölder's inequality to the terms on the right-hand side yields

$$\frac{d}{dt}\|G\|_{L^q} \le \|\Lambda^\beta\mathcal{R}_\alpha\theta\|_{L^q} + \|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q},$$

or, after integration in time,

$$\|G(t)\|_{L^q} \le \|G(0)\|_{L^q} + \|\Lambda^\beta\mathcal{R}_\alpha\theta\|_{L^1_tL^q} + \|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^1_tL^q}.$$

$\|\Lambda^\beta\mathcal{R}_\alpha\theta\|_{L^1_tL^q}$ can not be directly controlled in terms of (1.4). It is necessary to return to the equation of θ to obtain a suitable bound. This is done in Lemma 2.3 below. By Lemma 2.3,

$$\|\Lambda^\beta\mathcal{R}_\alpha\theta\|_{L^1_tL^q} \le \sum_{j=-1}^{\infty} \|\Delta_j\Lambda^\beta\mathcal{R}_\alpha\theta\|_{L^1_tL^q}$$

$$\begin{aligned}&\leq Ct\|\theta_0\|_{L^q}+\sum_{j=0}^{\infty}2^{\beta j}\|\Delta_j\Lambda^{1-\alpha}\theta\|_{L^1_tL^q}\\&\leq C(T,\theta_0)+C\int_0^t\|\nabla u(\tau)\|_{L^q}\|\theta(\tau)\|_{B^{1-\alpha}_{\infty,1}}d\tau.\end{aligned}\tag{2.6}$$

By Lemma 2.2 with $q_1=q$ and $q_2=\infty$,

$$\|[\mathcal{R}_\alpha,u\cdot\nabla]\theta\|_{L^q}\leq C\|\nabla u\|_{L^q}\|\theta\|_{B^{1-\alpha}_{\infty,1}}.\tag{2.7}$$

Therefore,

$$\|G(t)\|_{L^q}\leq C(T,u_0,\theta_0)+C\int_0^t\|\nabla u(\tau)\|_{L^q}\|\theta(\tau)\|_{B^{1-\alpha}_{\infty,1}}d\tau.\tag{2.8}$$

Applying $\Lambda^{1-\alpha}$ to the equation of θ and then dotting it by $\Lambda^{1-\alpha}\theta|\Lambda^{1-\alpha}\theta|^{q-2}$, we have

$$\frac{1}{q}\frac{d}{dt}\|\Lambda^{1-\alpha}\theta\|^q_{L^q}=-\int\Lambda^{1-\alpha}\theta|\Lambda^{1-\alpha}\theta|^{q-2}[\Lambda^{1-\alpha},u\cdot\nabla]\theta.$$

By Hölder's inequality,

$$\frac{d}{dt}\|\Lambda^{1-\alpha}\theta\|_{L^q}\leq C\|[\Lambda^{1-\alpha},u\cdot\nabla]\theta\|_{L^q}.$$

Applying Lemma 2.2 and proceeding as in Equation(2.7), we have

$$\frac{d}{dt}\|\Lambda^{1-\alpha}\theta\|_{L^q}\leq C\|\nabla u\|_{L^q}\|\theta\|_{B^{1-\alpha}_{\infty,1}},$$

or

$$\|\Lambda^{1-\alpha}\theta\|_{L^q}\leq\|\Lambda^{1-\alpha}\theta_0\|_{L^q}+C\int_0^t\|\nabla u(\tau)\|_{L^q}\|\theta(\tau)\|_{B^{1-\alpha}_{\infty,1}}d\tau.\tag{2.9}$$

Adding Equations (2.8) and (2.9) and noticing that

$$\|\nabla u\|_{L^q}\leq C\|\omega\|_{L^q}\leq C(\|G\|_{L^q}+\|\Lambda^{1-\alpha}\theta\|_{L^q}),$$

we conclude Equation (2.5) by Gronwall's inequality and Equation (1.4). Consequently,

$$\|\omega(t)\|_{L^q}\leq C(T,u_0,\theta_0).$$

Next we show that

$$\sup_{j\geq 0}2^{\alpha j}\|\Delta_jG\|_{L^1_tL^q}\leq C(T,u_0,\theta_0).$$

Applying Δ_j with $j\geq 0$ to Equation (2.4) and then dotting with $\Delta_jG|\Delta_jG|^{q-2}$, we obtain

$$\frac{1}{q}\frac{d}{dt}\|\Delta_jG\|^q_{L^q}+\int\Delta_jG|\Delta_jG|^{q-2}\Lambda^\alpha\Delta_jG=K_1+K_2+K_3,\tag{2.10}$$

where K_1, K_2, and K_3 are given by

$$K_1=-\int\Delta_jG|\Delta_jG|^{q-2}\Delta_j(u\cdot\nabla G),$$

$$K_2 = \int \Delta_j G|\Delta_j G|^{q-2}\Delta_j \Lambda^\beta \mathcal{R}_\alpha \theta,$$
$$K_3 = \int \Delta_j G|\Delta_j G|^{q-2}\Delta_j [\mathcal{R}_\alpha, u\cdot\nabla]\theta.$$

The dissipative terms admits the lower bound (see [8])

$$\int \Delta_j G|\Delta_j G|^{q-2}\Lambda^\alpha \Delta_j G \geq C 2^{\alpha j}\|\Delta_j G\|_{L^q}^q.$$

For $q\in[2,\infty)$, we choose $q_1, q_2 \in [2,\infty)$ satisfying $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$. By Lemma 2.1,

$$\begin{aligned}|K_1| \leq{}& C\|\Delta_j G\|_{L^q}^{q-1}\|\nabla u\|_{L^{q_1}}\|\Delta_j G\|_{L^{q_2}}\\ &+ C\|\Delta_j G\|_{L^q}^{q-1}\|\nabla u\|_{L^{q_1}}\sum_{k\leq j-1} 2^{k-j}\|\Delta_k G\|_{L^{q_2}}\\ &+ C\|\Delta_j G\|_{L^q}^{q-1}\|\nabla u\|_{L^{q_1}}\sum_{k\geq j-1} 2^{j-k}\|\Delta_k G\|_{L^{q_2}}.\end{aligned}$$

The bounds in Equation (2.5) allow us to conclude that

$$|K_1| \leq C\|\Delta_j G\|_{L^q}^{q-1}.$$

By Hölder's inequality,

$$|K_2| \leq C\|\Delta_j G\|_{L^q}^{q-1}\|\Delta_j \Lambda^\beta \mathcal{R}_\alpha \theta\|_{L^q}$$

By Hölder's inequality, Lemma 2.2, and Equation (2.5),

$$\begin{aligned}|K_3| &\leq \|\Delta_j G\|_{L^q}^{q-1}\|\Delta_j[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q}\\ &\leq C\|\Delta_j G\|_{L^q}^{q-1}\|\nabla u\|_{L^q}\|\theta\|_{B^{1-\alpha}_{\infty,1}}\\ &\leq C\|\Delta_j G\|_{L^q}^{q-1}\|\theta\|_{B^{1-\alpha}_{\infty,1}}.\end{aligned}$$

Inserting the bounds above in Equation (2.10), integrating in time and using Lemma 2.3 or Equation (2.6), we obtain

$$\|\Delta_j G(t)\|_{L^q} \leq e^{-C2^{\alpha j}t}\|\Delta_j G(0)\|_{L^q} + C\int_0^t e^{-C2^{\alpha j}(t-\tau)}(1+\|\theta\|_{B^{1-\alpha}_{\infty,1}})\,d\tau.$$

Taking L^1-norm in time yields

$$2^{\alpha j}\|\Delta_j G\|_{L^1_t L^q} \leq C\|\Delta_j G(0)\|_{L^q} + C\|\theta\|_{L^1_t B^{1-\alpha}_{\infty,1}}.$$

or

$$\sup_{j\geq 0} 2^{\alpha j}\|\Delta_j G\|_{L^1_t L^q} \leq C(T, u_0, \theta_0) + C\|\theta\|_{L^1_t B^{1-\alpha}_{\infty,1}}.$$

A special consequence of this global bound is that, for $\frac{2}{q} - \alpha < 0$,

$$\|G(t)\|_{L^1_t B^0_{\infty,1}} = \sum_{j\geq -1}\|\Delta_j G\|_{L^1_t L^\infty} \leq \sum_{j\geq -1} 2^{\frac{2}{q}j}\|\Delta_j G\|_{L^1_t L^q}$$

$$=\sum_{j\geq -1} 2^{(\frac{2}{q}-\alpha)j}2^{\alpha j}\|\Delta_j G\|_{L^1_t L^q}\leq C(T,u_0,\theta_0).$$

Therefore,

$$\|\omega\|_{L^1_t B^0_{\infty,1}}\leq \|G(t)\|_{L^1_t B^0_{\infty,1}}+\|\theta\|_{L^1_t B^{1-\alpha}_{\infty,1}}\leq C(T,u_0,\theta_0).$$

This global bound allows us to conclude the desired regularity. This completes the proof of Theorem 1.1. ∎

In the proof of Theorem 1.1, we invoked three lemmas. We now provide the precise statements and proofs of these lemmas. The first lemma provides an *a priori* bound for the term generated by the nonlinearity when we perform the L^q-estimate. In this lemma Δ_j denotes the Fourier localization operator. Its precise definition and other notation can be found in the appendix.

Lemma 2.1. *Let $j\geq 0$ be an integer. Assume $q\in[2,\infty)$ and $q_1,q_2\in[2,\infty]$ satisfy $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$ (Note that q_1 and q_2 are allowed to be ∞). Assume $\nabla\cdot u=0$. Then,*

$$\begin{aligned}\left|\int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(u\cdot\nabla G)\right| &\leq C\|\nabla u\|_{L^{q_1}}\|\Delta_j G\|_{L^q}^{q-1}\|\Delta_j G\|_{L^{q_2}}\\ &\quad + C\|\nabla u\|_{L^{q_1}}\|\Delta_j G\|_{L^q}^{q-1}\sum_{k\leq j-1}2^{k-j}\|\Delta_k G\|_{L^{q_2}}\\ &\quad + C\|\nabla u\|_{L^{q_1}}\|\Delta_j G\|_{L^q}^{q-1}\sum_{k\geq j-1}2^{j-k}\|\Delta_k G\|_{L^{q_2}}.\end{aligned}$$

where C's are constants.

Proof. Following the notion of paraproducts, we write

$$I\equiv\int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(u\cdot\nabla G)=I_1+I_2+I_3+I_4+I_5, \tag{2.11}$$

where

$$\begin{aligned} I_1&=\sum_{|j-k|\leq 2}\int \Delta_j G|\Delta_j G|^{q-2}[\Delta_j,S_{k-1}u\cdot\nabla]\Delta_k G,\\ I_2&=\sum_{|j-k|\leq 2}\int \Delta_j G|\Delta_j G|^{q-2}(S_{k-1}u-S_j u)\cdot\nabla\Delta_j\Delta_k G,\\ I_3&=\int \Delta_j G|\Delta_j G|^{q-2}S_j u\cdot\nabla\Delta_j G,\\ I_4&=\sum_{|j-k|\leq 2}\int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}G),\\ I_5&=\sum_{k\geq j-1}\int \Delta_j G|\Delta_j G|^{q-2}\Delta_j(\widetilde{\Delta}_k u\cdot\nabla\Delta_k G)\end{aligned}$$

with $\widetilde{\Delta}_k=\Delta_{k-1}+\Delta_k+\Delta_{k+1}$. We remark that the decomposition (2.11) follows from the paraproduct decomposition of $\Delta_j(u\cdot\nabla G)$,

$$\Delta_j(u\cdot\nabla G)=\sum_{|j-k|\leq 2}\Delta_j(S_{k-1}u\cdot\nabla\Delta_k G)+\sum_{|j-k|\leq 2}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}G)$$

$$+\sum_{k\geq j-1}\Delta_j(\widetilde{\Delta}_k u\cdot\nabla\Delta_k G)$$

and a further splitting of the first term,

$$\begin{aligned}\sum_{|j-k|\leq 2}\Delta_j(S_{k-1}u\cdot\nabla\Delta_k G)=&\sum_{|j-k|\leq 2}[\Delta_j,S_{k-1}u\cdot\nabla]\Delta_k G\\&+\sum_{|j-k|\leq 2}(S_{k-1}u-S_j u)\cdot\nabla\Delta_j\Delta_k G+S_j u\cdot\nabla\Delta_j G.\end{aligned}$$

$\nabla\cdot u=0$ implies $I_3=0$. For fixed j, the summation over $|j-k|\leq 2$ involves only a finite number of k's. For the sake of brevity, we shall replace the summations by their representative term with $k=j$ in I_1, I_2, and I_4. This practice does not change the estimates. To estimate I_1, we make use of the commutator structure to write

$$\begin{aligned}&[\Delta_j,S_{j-1}u\cdot\nabla]\Delta_j G\\&=\int\Phi_j(x-y)(S_{j-1}u(y)-S_{j-1}u(x))\cdot\nabla\Delta_j G(y)dy\\&=\int\Phi_j(x-y)\int_0^1(y-x)\cdot(\nabla S_{j-1}u)(\rho y+(1-\rho)x)\,d\rho\cdot\nabla\Delta_j G(y)dy\\&=\int_0^1\int\Phi_j\left(\frac{z}{\rho}\right)\frac{z}{\rho^3}\cdot(\nabla S_{j-1}u)(x-z)\cdot\nabla\Delta_j G\left(x-\frac{z}{\rho}\right)dz\,d\rho,\end{aligned}$$

where Φ_j is the kernel function corresponding to the operator Δ_j (see the appendix). By Minkowski's inequality and Hölder's inequality,

$$\begin{aligned}\|[\Delta_j,S_{j-1}u\cdot\nabla]\Delta_j G\|_{L^q}&\leq\int_0^1\int\left|\Phi_j\left(\frac{z}{\rho}\right)\frac{z}{\rho^3}\right|\|\nabla S_{j-1}u\|_{L^{q_1}_x}\|\nabla\Delta_j G\|_{L^{q_2}_x}\,dz\,d\rho\\&\leq\|\nabla S_{j-1}u\|_{L^{q_1}}\|\nabla\Delta_j G\|_{L^{q_2}}\int_0^1\int\left|\Phi_j\left(\frac{z}{\rho}\right)\frac{z}{\rho^3}\right|dz\,d\rho\\&=\|\nabla S_{j-1}u\|_{L^{q_1}}\|\nabla\Delta_j G\|_{L^{q_2}}\|x\Phi_j(x)\|_{L^1_x}.\end{aligned}\tag{2.12}$$

By Bernstein's inequality (see the appendix),

$$\|\nabla\Delta_j G\|_{L^{q_2}}\leq C2^j\|\Delta_j G\|_{L^{q_2}}$$

for a constant C independent of j. Furthermore, according to the definition of Φ_j,

$$\|x\Phi_j(x)\|_{L^1}=2^{-j}\|x\Phi_0(x)\|_{L^1}=C2^{-j}.$$

Therefore,

$$\|[\Delta_j,S_{j-1}u\cdot\nabla]\Delta_j G\|_{L^q}\leq C\|\nabla S_{j-1}u\|_{L^{q_1}}\|\Delta_j G\|_{L^{q_2}}.$$

Consequently, by Hölder's inequality,

$$\begin{aligned}|I_1|&\leq C\|\nabla S_{j-1}u\|_{L^{q_1}}\|\Delta_j G\|_{L^{q_2}}\|\Delta_j G\|_{L^q}^{q-1}\\&\leq C\|\nabla u\|_{L^{q_1}}\|\Delta_j G\|_{L^{q_2}}\|\Delta_j G\|_{L^q}^{q-1}.\end{aligned}$$

By Hölder's inequality and Bernstein's inequality,

$$|I_2|\leq C2^j\|\Delta_j u\|_{L^{q_1}}\|\Delta_j G\|_{L^{q_2}}\|\Delta_j G\|_{L^q}^{q-1}$$

$$\begin{aligned}&\le C\,\|\nabla\Delta_j u\|_{L^{q_1}}\,\|\Delta_j G\|_{L^{q_2}}\,\|\Delta_j G\|_{L^q}^{q-1}\\&\le C\,\|\nabla u\|_{L^{q_1}}\,\|\Delta_j G\|_{L^{q_2}}\,\|\Delta_j G\|_{L^q}^{q-1}.\end{aligned}$$

Invoking the remark we made previously and applying Hölder's and Bernstein's inequalities, we have

$$\begin{aligned}|I_4|&\le C\,\|\Delta_j(\Delta_j u\cdot\nabla S_{j-1}G)\|_{L^q}\,\|\Delta_j G\|_{L^q}^{q-1}\\&\le C\,\|\Delta_j u\|_{L^{q_1}}\,\|\nabla S_{j-1}G\|_{L^{q_2}}\,\|\Delta_j G\|_{L^q}^{q-1}\\&\le C\,2^{-j}\|\nabla\Delta_j u\|_{L^{q_1}}\sum_{k\le j-1}2^k\|\Delta_k G\|_{L^{q_2}}\,\|\Delta_j G\|_{L^q}^{q-1}\\&\le C\,\|\nabla u\|_{L^{q_1}}\,\|\Delta_j G\|_{L^q}^{q-1}\sum_{k\le j-1}2^{k-j}\|\Delta_k G\|_{L^{q_2}}.\end{aligned}$$

To estimate I_5, we apply $\nabla\cdot u=0$ and Hölder's inequality to obtain

$$|I_5|\le\|\Delta_j G\|_{L^q}^{q-1}2^j\sum_{k\ge j-1}\|\widetilde{\Delta}_k u\|_{L^{q_1}}\,\|\Delta_k G\|_{L^{q_2}}.$$

An application of Bernstein's inequality yields

$$\begin{aligned}|I_5|&\le C\,\|\Delta_j G\|_{L^q}^{q-1}\sum_{k\ge j-1}2^{j-k}\|\nabla\widetilde{\Delta}_k u\|_{L^{q_1}}\,\|\Delta_k G\|_{L^{q_2}}\\&\le C\,\|\nabla u\|_{L^{q_1}}\,\|\Delta_j G\|_{L^q}^{q-1}\sum_{k\ge j-1}2^{j-k}\|\Delta_k G\|_{L^{q_2}}.\end{aligned}$$

The desired bound of this lemma then follows from combining the bounds for I_1 through I_5. This completes the proof of Lemma 2.1. □

Recall that $\mathcal{R}_\alpha=\Lambda^{-\alpha}\partial_{x_1}$. The second lemma provides an *a priori* bound for the L^q-norm of the commutator $\Delta_j[\mathcal{R}_\alpha,u\cdot\nabla]\theta\equiv\Delta_j\mathcal{R}_\alpha(u\cdot\nabla\theta)-\Delta_j(u\cdot\mathcal{R}_\alpha\nabla\theta)$.

LEMMA 2.2. *Let $j\ge 0$ be an integer. Let $\alpha\in(0,2)$. Assume $q\in[2,\infty)$ and $q_1,q_2\in[2,\infty]$ satisfy $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$. Assume $\nabla\cdot u=0$. Then*

$$\begin{aligned}\|\Delta_j[\mathcal{R}_\alpha,u\cdot\nabla]\theta\|_{L^q}\le{}&C\,2^{(1-\alpha)j}\,\|\nabla u\|_{L^{q_1}}\,\|\Delta_j\theta\|_{L^{q_2}}\\&+C\,\|\nabla u\|_{L^{q_1}}\sum_{k\le j-1}2^{k-j}2^{(1-\alpha)k}\|\Delta_k\theta\|_{L^{q_2}}\\&+C\,\|\nabla u\|_{L^{q_1}}\sum_{k\ge j-1}2^{(2-\alpha)(j-k)}2^{(1-\alpha)k}\|\Delta_k\theta\|_{L^{q_2}}\\&+C\,\|\nabla u\|_{L^{q_1}}\sum_{k\ge j-1}2^{j-k}\,2^{(1-\alpha)k}\|\Delta_k\theta\|_{L^{q_2}},\end{aligned}\tag{2.13}$$

where C's are constants. In addition, Equation (2.13) *still holds if $\mathcal{R}_\alpha$ is replaced by $\Lambda^{1-\alpha}$. A special consequence of Equation* (2.13) *is the following bound,*

$$\|[\mathcal{R}_\alpha,u\cdot\nabla]\theta\|_{L^q}\le C\,\|\nabla u\|_{L^{q_1}}\,\|\theta\|_{B^{1-\alpha}_{q_2,1}}.\tag{2.14}$$

Similarly,

$$\|[\Lambda^{1-\alpha},u\cdot\nabla]\theta\|_{L^q}\le C\,\|\nabla u\|_{L^{q_1}}\,\|\theta\|_{B^{1-\alpha}_{q_2,1}}.$$

Proof. We write

$$\Delta_j[\mathcal{R}_\alpha, u\cdot\nabla]\theta = L_1 + L_2 + L_3,$$

where

$$\begin{aligned}
L_1 &= \sum_{|j-k|\le 2} [\Delta_j \mathcal{R}_\alpha(S_{k-1}u\cdot\nabla\Delta_k\theta) - \Delta_j(S_{k-1}u\cdot\nabla\mathcal{R}_\alpha\Delta_k\theta)],\\
L_2 &= \sum_{|j-k|\le 2} [\Delta_j \mathcal{R}_\alpha(\Delta_k u\cdot\nabla S_{k-1}\theta) - \Delta_j(\Delta_k u\cdot\nabla\mathcal{R}_\alpha S_{k-1}\theta)],\\
L_3 &= \sum_{k\ge j-1} \Big[\Delta_j \mathcal{R}_\alpha(\widetilde{\Delta}_k u\cdot\nabla\Delta_k\theta) - \Delta_j(\widetilde{\Delta}_k u\cdot\nabla\mathcal{R}_\alpha\Delta_k\theta)\Big].
\end{aligned}$$

To estimate L_1, we make use of the commutator structure. For the sake of clarity, we further divide L_1 into two parts

$$L_1 = L_{11} + L_{12},$$

where

$$\begin{aligned}
L_{11} &= \sum_{|j-k|\le 2} [\Delta_j \mathcal{R}_\alpha(S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla\Delta_j\mathcal{R}_\alpha\Delta_k\theta],\\
L_{12} &= \sum_{|j-k|\le 2} [S_{k-1}u\cdot\nabla\Delta_j\mathcal{R}_\alpha\Delta_k\theta - \Delta_j(S_{k-1}u\cdot\nabla\mathcal{R}_\alpha\Delta_k\theta)].
\end{aligned}$$

We denote by $h_j(x)$ the kernel function for the operator $\Delta_j\mathcal{R}_\alpha$, namely

$$\Delta_j\mathcal{R}_\alpha f = h_j * f \qquad \text{or} \qquad \widehat{h}_j(\xi) = \Phi_j(\xi)\, i\xi_1\, |\xi|^{-\alpha},$$

where Φ_j is the kernel function corresponding to Δ_j. It is not difficult to check that

$$h_j(x) = 2^{(1-\alpha)j}\, 2^{2j} h_0(2^j x), \qquad \widehat{h}_0(\xi) = \widehat{\Phi}_0(\xi) i\xi_1 |\xi|^{-\alpha}.$$

As in the estimate of Equation(2.12), we have

$$\begin{aligned}
\|L_{11}\|_{L^q} &\le C\, \|x h_j(x)\|_{L^1}\, \|\nabla S_{j-1}u\|_{L^{q_1}}\, \|\nabla\Delta_j\theta\|_{L^{q_2}}\\
&\le C\, 2^{-\alpha j} \|\nabla u\|_{L^{q_1}}\, 2^j\, \|\Delta_j\theta\|_{L^{q_2}}\\
&\le C\, 2^{(1-\alpha)j} \|\nabla u\|_{L^{q_1}}\, \|\Delta_j\theta\|_{L^{q_2}}.
\end{aligned}$$

Making use of the commutator structure again yields

$$\begin{aligned}
\|L_{12}\|_{L^q} &\le C\, \|x\Phi_j(x)\|_{L^1}\, \|\nabla u\|_{L^{q_1}}\, \|\mathcal{R}_\alpha\nabla\Delta_j\theta\|_{L^{q_2}}\\
&\le C\, 2^{-j}\, \|\nabla u\|_{L^{q_1}}\, 2^{(1-\alpha)j} 2^j \|\Delta_j\theta\|_{L^{q_2}}\\
&= C\, 2^{(1-\alpha)j} \|\nabla u\|_{L^{q_1}}\, \|\Delta_j\theta\|_{L^{q_2}}.
\end{aligned}$$

The estimates for L_2 are similar and we have

$$\|L_2\|_{L^q} \le C\, 2^{-j} \|\nabla u\|_{L^{q_1}}\, \|\nabla\mathcal{R}_\alpha S_{j-1}\theta\|_{L^{q_2}}$$

$$\leq C\,\|\nabla u\|_{L^{q_1}} \sum_{k\leq j-1} 2^{k-j} 2^{(1-\alpha)k} \|\Delta_k \theta\|_{L^{q_2}}.$$

We remark that the divergence-free condition $\nabla\cdot u=0$ in not used in the estimates of L_1 and L_2. The commutator structure is not needed to bound L_3. Due to $\nabla\cdot u=0$,

$$\begin{aligned}
\|L_3\|_{L^q} &\leq C \sum_{k\geq j-1} \Big[\|\Delta_j \mathcal{R}_\alpha(\nabla\cdot(\widetilde{\Delta}_k u \Delta_k\theta))\|_{L^q} + \|\Delta_j \nabla\cdot(\widetilde{\Delta}_k u \mathcal{R}_\alpha \Delta_k \theta)\|_{L^q}\Big] \\
&\leq C \sum_{k\geq j-1} \Big[2^{(2-\alpha)j} \|\widetilde{\Delta}_k u\|_{L^{q_1}}\, \|\Delta_k\theta\|_{L^{q_2}} + 2^j \|\widetilde{\Delta}_k u\|_{L^{q_1}}\, 2^{(1-\alpha)k} \|\Delta_k \theta\|_{L^{q_2}}\Big] \\
&\leq C \sum_{k\geq j-1} 2^{(2-\alpha)j} 2^{-k} \|\nabla \widetilde{\Delta}_k u\|_{L^{q_1}}\, \|\Delta_k\theta\|_{L^{q_2}} \\
&\quad + C \sum_{k\geq j-1} 2^{j-k} \|\nabla\widetilde{\Delta}_k u\|_{L^{q_1}}\, 2^{(1-\alpha)k} \|\Delta_k\theta\|_{L^{q_2}} \\
&\leq C\,\|\nabla u\|_{L^{q_1}} \sum_{k\geq j-1} \Big[2^{(2-\alpha)(j-k)}\, 2^{(1-\alpha)k} \|\Delta_k\theta\|_{L^{q_2}} + 2^{j-k}\, 2^{(1-\alpha)k} \|\Delta_k\theta\|_{L^{q_2}}\Big].
\end{aligned}$$

Combining the estimates above yields the desired bound in Equation (2.13). We finally remark that the estimates above still holds when $\mathcal{R}_\alpha$ is replaced by $\Lambda^{1-\alpha}$. The main reason is that $\Lambda^{1-\alpha}$ differs from $\mathcal{R}_\alpha$ by a Riesz transform and Riesz transforms are bounded on functions of the form $\Delta_j f$ for any $j\geq 0$. Therefore Equation (2.13) holds when $\mathcal{R}_\alpha$ is replaced by $\Lambda^{1-\alpha}$. The inequality (2.14) is an easy consequence of Equation (2.13). In fact,

$$\begin{aligned}
\|[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q} &\leq \sum_{j\geq -1} \|\Delta_j[\mathcal{R}_\alpha, u\cdot\nabla]\theta\|_{L^q} \\
&\leq C\,\|\nabla u\|_{L^{q_1}} \sum_{j\geq -1} 2^{(1-\alpha)j}\, \|\Delta_j\theta\|_{L^{q_2}} \\
&\quad + C\|\nabla u\|_{L^{q_1}} \sum_{j\geq -1}\sum_{k\leq j-1} 2^{k-j} 2^{(1-\alpha)k}\, \|\Delta_k\theta\|_{L^{q_2}} \\
&\quad + C\,\|\nabla u\|_{L^{q_1}} \sum_{j\geq -1}\sum_{k\geq j-1} 2^{(2-\alpha)(j-k)} 2^{(1-\alpha)k} \|\Delta_k\theta\|_{L^{q_2}} \\
&\quad + C\,\|\nabla u\|_{L^{q_1}} \sum_{j\geq -1}\sum_{k\geq j-1} 2^{j-k}\, 2^{(1-\alpha)k} \|\Delta_k\theta\|_{L^{q_2}} \\
&\leq C\,\|\nabla u\|_{L^{q_1}}\, \|\theta\|_{B^{1-\alpha}_{q_2,1}},
\end{aligned}$$

where we have used the Young's inequality for series convolution. This completes the proof of Lemma 2.2. ∎

The last lemma of this section presents an estimate for the term $\Lambda^\beta \mathcal{R}_\alpha \theta$. This estimate has been used in the proof of Theorem 1.1.

LEMMA 2.3. *Assume that θ solves*

$$\partial_t\theta + u\cdot\nabla\theta + \Lambda^\beta\theta = 0. \tag{2.15}$$

Then, for any $t>0$ and any $1<q<\infty$,

$$\sum_{j=0}^{\infty} 2^{\beta j} \|\Delta_j \Lambda^{1-\alpha}\theta\|_{L^1_t L^q} \leq C\,\|\theta_0\|_{B^{1-\alpha}_{q,1}} + C\int_0^t \|\nabla u(\tau)\|_{L^q}\, \|\theta(\tau)\|_{B^{1-\alpha}_{\infty,1}}\, d\tau,$$

where C's are constants.

Proof. Applying $\Lambda^{1-\alpha}$ to Equation (2.15) leads to the equation of $\varphi \equiv \Lambda^{1-\alpha}\theta$,

$$\partial_t \varphi + u\cdot\nabla\varphi + \Lambda^\beta \varphi = -[\Lambda^{1-\alpha}, u\cdot\nabla]\theta. \tag{2.16}$$

Applying Δ_j to Equation (2.16), multiplying by $\Delta_j\varphi|\Delta_j\varphi|^{q-2}$ and integrating over $\mathbb{R}^2$, we obtain

$$\frac{1}{q}\frac{d}{dt}\|\Delta_j\varphi\|_{L^q}^q + \int (\Lambda^\beta \Delta_j\varphi)\,\Delta_j\varphi|\Delta_j\varphi|^{q-2} = L_1 + L_2,$$

where

$$L_1 = -\int \Delta_j(u\cdot\nabla\varphi)\,\Delta_j\varphi|\Delta_j\varphi|^{q-2},$$
$$L_2 = -\int \Delta_j[\Lambda^{1-\alpha}, u\cdot\nabla]\theta\,\Delta_j\varphi|\Delta_j\varphi|^{q-2}.$$

The second term above admits the following lower bound (see [8]),

$$\int (\Lambda^\beta \Delta_j\varphi)\,\Delta_j\varphi|\Delta_j\varphi|^{q-2} \geq C_0\, 2^{\beta j}\, \|\Delta_j\varphi\|_{L^q}^q$$

for some constant $C_0 > 0$. By Lemma 2.1,

$$|L_1| \leq C\,\|\Delta_j\varphi\|_{L^q}^{q-1}\,\|\nabla u\|_{L^q}\, A_1,$$

where

$$A_1 = \|\Delta_j\varphi\|_{L^\infty} + \sum_{k\leq j-1} 2^{k-j}\|\Delta_k\varphi\|_{L^\infty} + \sum_{k\geq j-1} 2^{j-k}\|\Delta_k\varphi\|_{L^\infty}.$$

By Lemma 2.2,

$$|L_2| \leq C\,\|\Delta_j\varphi\|_{L^q}^{q-1}\,\|\nabla u\|_{L^q}\, A_2,$$

where

$$A_2 = 2^{(1-\alpha)j}\,\|\Delta_j\theta\|_{L^\infty} + \sum_{k\leq j-1} 2^{k-j} 2^{(1-\alpha)k}\,\|\Delta_k\theta\|_{L^\infty}$$
$$+ \sum_{k\geq j-1} 2^{(2-\alpha)(j-k)} 2^{(1-\alpha)k}\|\Delta_k\theta\|_{L^\infty} + \sum_{k\geq j-1} 2^{j-k}\, 2^{(1-\alpha)k}\|\Delta_k\theta\|_{L^\infty}.$$

These bounds allow us to obtain

$$\frac{d}{dt}\|\Delta_j\varphi\|_{L^q} + C_0\, 2^{\beta j}\|\Delta_j\varphi\|_{L^q} \leq C\,\|\nabla u\|_{L^q}\,(A_1 + A_2).$$

Integrating w.r.t. time yields

$$\|\Delta_j\varphi(t)\|_{L^q} \leq e^{-C_0\, 2^{\beta j} t}\,\|\Delta_j\varphi(0)\|_{L^q} + C\int_0^t e^{-C_0\, 2^{\beta j}(t-\tau)}\,\|\nabla u(\tau)\|_{L^q}\,(A_1+A_2)(\tau)\,d\tau.$$

Taking L^1-norm in time and applying Young's inequality to the last term, we have

$$\|\Delta_j\varphi\|_{L_t^1 L^q} \leq C\, 2^{-\beta j}\,\|\Delta_j\varphi(0)\|_{L^q} + C\, 2^{-\beta j}\int_0^t \|\nabla u(\tau)\|_{L^q}\,(A_1+A_2)(\tau)\,d\tau.$$

Multiplying by $2^{\beta j}$ and summing over $j\geq 0$, we find

$$\sum_{j=0}^{\infty}2^{\beta j}\|\Delta_j\varphi\|_{L^1_tL^q}\leq C\sum_{j=0}^{\infty}\|\Delta_j\varphi(0)\|_{L^q}+C\int_0^t\|\nabla u(\tau)\|_{L^q}\sum_{j=0}^{\infty}(A_1+A_2)(\tau)\,d\tau.$$

Recalling $\varphi=\Lambda^{1-\alpha}\theta$ and applying Young's inequality for series convolution, we have

$$\sum_{j=0}^{\infty}(A_1+A_2)\leq C\|\theta\|_{B^{1-\alpha}_{\infty,1}}.$$

Therefore,

$$\sum_{j=0}^{\infty}2^{\beta j}\|\Delta_j\Lambda^{1-\alpha}\theta\|_{L^1_tL^q}\leq C\|\theta_0\|_{B^{1-\alpha}_{q,1}}+C\int_0^t\|\nabla u(\tau)\|_{L^q}\|\theta(\tau)\|_{B^{1-\alpha}_{\infty,1}}\,d\tau.$$

This completes the proof of Lemma 2.3. ∎

3. The case when the thermal diffusion dominates

This section proves Theorem 1.2 and Theorem 1.3. Theorem 1.2 deals with the case when $\alpha\leq\beta$. Since we need Theorem 1.3 to prove Theorem 1.2, we shall prove Theorem 1.3 first. Without loss of generality, we set $\nu=\kappa=1$.

As a preparation for the proof of Theorem 1.3. We state and prove a lemma first. This lemma helps with the estimates of the nonlinear term when we bound the L^q-norm of the localized equation. This lemma involves Besov spaces $B^s_{p,q}$. Its precise definition is given in the appendix. In particular, $B^\gamma_{\infty,\infty}$ is equivalent to the standard Hölder space C^γ when $\gamma\in(0,1)$.

Lemma 3.1. *Let $j\geq 0$ be an integer. Let $q\in[2,\infty)$ and $\gamma\in(0,1)$. Assume $\nabla\cdot u=0$. Then,*

$$\begin{aligned}\left|\int\Delta_j\theta|\Delta_j\theta|^{q-2}\Delta_j(u\cdot\nabla\theta)\right|\leq{}& C2^{(1-\gamma)j}\|u\|_{B^\gamma_{\infty,\infty}}\|\Delta_j\theta\|^q_{L^q}\\&+C2^{-\gamma j}\|u\|_{B^\gamma_{\infty,\infty}}\|\Delta_j\theta\|^{q-1}_{L^q}\sum_{k\leq j-1}2^k\|\Delta_k\theta\|_{L^q}\\&+C2^{j}\|u\|_{B^\gamma_{\infty,\infty}}\sum_{k\geq j-1}2^{-\gamma k}\|\Delta_k\theta\|_{L^q},\end{aligned}$$

where C's are constants.

Proof. The proof of this lemma bears some similarity to that of Lemma 2.1. For reader's convenience, we provide the details. Following the notion of paraproducts, we write

$$\widetilde{I}\equiv\int\Delta_j\theta|\Delta_j\theta|^{q-2}\Delta_j(u\cdot\nabla\theta)=\widetilde{I}_1+\widetilde{I}_2+\widetilde{I}_3+\widetilde{I}_4+\widetilde{I}_5,$$

where

$$\widetilde{I}_1=\sum_{|j-k|\leq 2}\int\Delta_j\theta|\Delta_j\theta|^{q-2}[\Delta_j,S_{k-1}u\cdot\nabla]\Delta_k\theta,$$

$$\widetilde{I}_2=\sum_{|j-k|\leq 2}\int\Delta_j\theta|\Delta_j\theta|^{q-2}(S_{k-1}u-S_ju)\cdot\nabla\Delta_j\Delta_k\theta,$$

$$\widetilde{I}_3 = \int \Delta_j\theta|\Delta_j\theta|^{q-2} S_j u\cdot\nabla\Delta_j\theta,$$
$$\widetilde{I}_4 = \sum_{|j-k|\le 2}\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta),$$
$$\widetilde{I}_5 = \sum_{k\ge j-1}\int \Delta_j\theta|\Delta_j\theta|^{q-2}\Delta_j(\widetilde{\Delta}_k u\cdot\nabla\Delta_k\theta)$$

Again, $\nabla\cdot u=0$ implies $\widetilde{I}_3=0$. For fixed j, the summation over $|j-k|\le 2$ involves only a finite number of k's. For the sake of brevity, we shall replace the summations by their representative term with $k=j$ in $\widetilde{I}_1$, $\widetilde{I}_2$, and $\widetilde{I}_4$. This practice does not change the estimates. To estimate $\widetilde{I}_1$, we make use of the commutator structure to write

$$\begin{aligned}
&[\Delta_j, S_{j-1}u\cdot\nabla]\Delta_j\theta\\
&\quad= \int \Phi_j(x-y)(S_{j-1}u(y)-S_{j-1}u(x))\cdot\nabla\Delta_j\theta(y)dy\\
&\quad\le C\|S_{j-1}u\|_{C^{\gamma}_{\infty,\infty}}\int |\Phi_j(x-y)|\,|x-y|^{\gamma}|\nabla\Delta_j\theta(y)|dy.
\end{aligned}$$

By Young's inequality,

$$\|[\Delta_j, S_{j-1}u\cdot\nabla]\Delta_j\theta\|_{L^q}\le C\|S_{j-1}u\|_{C^{\gamma}_{\infty,\infty}}\|\nabla\Delta_j\theta\|_{L^q}\||x|^{\gamma}\Phi_j(x)\|_{L^1_x}. \tag{3.1}$$

By Bernstein's inequality,

$$\|\nabla\Delta_j\theta\|_{L^{q_2}}\le C2^j\|\Delta_j\theta\|_{L^{q_2}}$$

for a constant C independent of j. Furthermore, according to the definition of Φ_j,

$$\||x|^{\gamma}\Phi_j(x)\|_{L^1} = 2^{-\gamma j}\||x|^{\gamma}\Phi_0(x)\|_{L^1} = C2^{-\gamma j}.$$

Therefore,

$$\|[\Delta_j, S_{j-1}u\cdot\nabla]\Delta_j\theta\|_{L^q}\le C2^{(1-\gamma)j}\|S_{j-1}u\|_{C^{\gamma}_{\infty,\infty}}\|\Delta_j\theta\|_{L^q}.$$

Consequently, by Hölder's inequality,

$$|\widetilde{I}_1|\le C2^{(1-\gamma)j}\|u\|_{C^{\gamma}_{\infty,\infty}}\|\Delta_j\theta\|^q_{L^q}.$$

By Hölder's inequality and Bernstein's inequality,

$$\begin{aligned}
|\widetilde{I}_2| &\le C2^j\|\Delta_j u\|_{L^\infty}\|\Delta_j\theta\|^q_{L^q}\\
&\le C2^{(1-\gamma)j}2^{\gamma j}\|\Delta_j u\|_{L^\infty}\|\Delta_j\theta\|^q_{L^q}\\
&\le C2^{(1-\gamma)j}\|u\|_{C^{\gamma}_{\infty,\infty}}\|\Delta_j\theta\|^q_{L^q}.
\end{aligned}$$

Applying Hölder's and Bernstein's inequalities, we have

$$\begin{aligned}
|\widetilde{I}_4| &\le C\|\Delta_j(\Delta_j u\cdot\nabla S_{j-1}\theta)\|_{L^q}\|\|\Delta_j\theta\|^{q-1}_{L^q}\\
&\le C\|\Delta_j u\|_{L^\infty}\|\nabla S_{j-1}\theta\|_{L^q}\|\Delta_j\theta\|^{q-1}_{L^q}\\
&\le C2^{-\gamma j}\|u\|_{C^{\gamma}_{\infty,\infty}}\sum_{k\le j-1}2^k\|\Delta_k\theta\|_{L^q}\|\Delta_j\theta\|^{q-1}_{L^q}.
\end{aligned}$$

To estimate $\widetilde{I}_5$, we apply $\nabla\cdot u=0$ and Hölder's inequality to obtain

$$|\widetilde{I}_5|\le\|\Delta_j\theta\|_{L^q}^{q-1}2^j\sum_{k\ge j-1}\|\widetilde{\Delta}_k u\|_{L^\infty}\|\Delta_k\theta\|_{L^q}.$$

An application of Bernstein's inequality yields

$$|\widetilde{I}_5|\le C\|\Delta_j\theta\|_{L^q}^{q-1}\|u\|_{C^\gamma_{\infty,\infty}}\sum_{k\ge j-1}2^{-\gamma k}\|\Delta_k\theta\|_{L^q}.$$

The estimates above then yield the desired bound. This completes the proof of Lemma 3.1. ∎

We are now ready to prove Theorem 1.3. The proof follows the idea in [14].

Proof. (**Proof of Theorem 1.3.**) We show that Equation (1.6) implies the following bound for θ,

$$\sup_{t\in[0,T]}\|\theta(t)\|_{B^\delta_{q,\infty}}<\infty\quad\text{for any }\delta\in(1,\gamma+\beta)\text{ and for any }q\in[2,\infty).$$

As explained later, this bound then implies $\|\nabla\theta\|_{L^\infty}<\infty$, which, in turn, implies the regularity of the solution.

We start by applying Δ_j with $j\ge0$ to the θ equation

$$\partial_t(\Delta_j\theta)+\Delta_j\Lambda^\beta\theta=-\Delta_j(u\cdot\nabla\theta).\tag{3.2}$$

Dotting it with $\Delta_j\theta|\Delta_j\theta|^{q-2}$ and applying Lemma 3.1 yields

$$\begin{aligned}\frac{d}{dt}\|\Delta_j\theta\|_{L^q}^q+C_0 2^{\beta j}\|\Delta_j\theta\|_{L^q}\le{}& C2^{(1-\gamma)j}\|u\|_{B^\gamma_{\infty,\infty}}\|\Delta_j\theta\|_{L^q}\\&+C2^{-\gamma j}\|u\|_{B^\gamma_{\infty,\infty}}\sum_{k\le j-2}2^k\|\Delta_k\theta\|_{L^q}\\&+C2^j\|u\|_{B^\gamma_{\infty,\infty}}\sum_{k\ge j-1}2^{-\gamma k}\|\Delta_k\theta\|_{L^q}.\end{aligned}$$

Multiplying above inequality by $2^{\delta j}$, integrating in time and taking the supremum with respect to $j\ge0$, we get

$$\|\theta\|_{B^\delta_{q,\infty}}\le\|\theta_0\|_{L^q}+\|\theta_0\|_{B^\delta_{q,\infty}}+J_1+J_2+J_3,$$

where we have used the fact that

$$\|\theta\|_{B^\delta_{q,\infty}}\le\|\theta\|_{L^q}+\sup_{j\ge0}2^{\delta j}\|\Delta_j\theta\|_{L^q}$$

and J_1, J_2, and J_3 are given by

$$\begin{aligned}J_1&=C_1\sup_{j\ge0}2^{j(1-\gamma)}\int_0^t e^{-C_0(t-\tau)2^{\beta j}}2^{j\delta}\|u\|_{B^\gamma_{\infty,\infty}}\|\Delta_j\theta(\tau)\|_{L^q}d\tau,\\J_2&=C_2\sup_{j\ge0}2^{-j\gamma}\int_0^t e^{-C_0(t-\tau)2^{\beta j}}2^{j\delta}\|u\|_{B^\gamma_{\infty,\infty}}\sum_{k\le j-2}2^k\|\Delta_k\theta(\tau)\|_{L^q}d\tau,\end{aligned}$$

$$J_3 = C_3 \sup_{j\geq 0} 2^j \int_0^t e^{-C_0(t-\tau)2^{\beta j}} 2^{j\delta} \|u\|_{B^\gamma_{\infty,\infty}} \sum_{k\leq j-2} 2^{-\gamma k} \|\Delta_k \theta(\tau)\|_{L^q} d\tau.$$

We now estimate the terms above. For notational convenience, we write

$$M \equiv \sup_{t\in[0,T]} \|u\|_{B^\gamma_{\infty,\infty}} < \infty.$$

To bound J_1, we choose a sufficiently large integer j_1 such that

$$\frac{C_1 M}{C_0} 2^{(1-\gamma-\beta)j_1} \leq \frac{1}{16}.$$

We note that this can be done due to $\gamma > 1-\beta$. Then,

$$\begin{aligned}
J_1 \leq & C_1 M \sup_{j\geq j_1} 2^{j(1-\gamma)} \int_0^t e^{-C_0(t-\tau)2^{\beta j}} 2^{j\delta} \|\Delta_j \theta(\tau)\|_{L^q} d\tau \\
& + C_1 M \sup_{-1\leq j<j_1} 2^{j(1-\gamma)} \int_0^t e^{-C_0\kappa(t-\tau)2^{\beta j}} 2^{j\delta} \|\Delta_j \theta(\tau)\|_{L^q} d\tau \\
\leq & \sup_{\tau\in[0,t]} \|\theta\|_{B^\delta_{q,\infty}} \sup_{j\geq j_1} \frac{C_1 M}{C_0} 2^{(1-\gamma-\beta)j} \left(1 - e^{-C_0 2^{\beta j} t}\right) \\
& + \frac{C_1 M}{C_0} \|\theta_0\|_{L^q} \sup_{j\leq j_1} 2^{(1-\gamma-\beta+\delta)j} \left(1 - e^{-C_0 2^{\beta j} t}\right) \\
\leq & \frac{1}{16} \sup_{\tau\in[0,t]} \|\theta\|_{B^\delta_{q,\infty}} + C(M, j_1, \|\theta_0\|_{L^q}),
\end{aligned}$$

where $C(M,j_1,\|\theta_0\|_{L^q})$ is a constant depending on the quantities inside the parenthesis. We also note that j_1 depends only on C_0, C_1, and M. To bound J_2, we note that, due to $1<\delta<\gamma+\beta$, there exists a positive integer j_2 such that

$$\frac{C_2 M}{C_0} 2^{(\delta-\gamma-\beta)j_2} \leq \frac{1}{16}.$$

Then J_2 can be bounded by

$$\begin{aligned}
J_2 = & C_2 M \sup_{j\geq -1} 2^{(\delta-\gamma)j} \int_0^t e^{-C_0(t-\tau)2^{\beta j}} \sum_{k\leq j-2} 2^{(1-\delta)k} 2^{\delta k} \|\Delta_k \theta(\tau)\|_{L^q} d\tau \\
\leq & \frac{C_2 M}{C_0} \sup_{j\geq j_2} 2^{(\delta-\gamma-\beta)j} \sup_{\tau\in[0,t]} \|\theta\|_{B^\delta_{q,\infty}} \\
& + C_2 M \sup_{-1\leq j\leq j_2} 2^{(\delta-\gamma)j} \int_0^t e^{-C_0(t-\tau)2^{\beta j}} \sum_{k\leq j-2} 2^{(1-\delta)k} 2^{\delta k} \|\Delta_k \theta(\tau)\|_{L^q} d\tau \\
\leq & \frac{C_2 M}{C_0} \sup_{j\geq j_2} 2^{(\delta-\gamma-\beta)j} \sup_{\tau\in[0,t]} \|\theta\|_{B^\delta_{q,\infty}} + C(C_0, C_2, M, j_2, \|\theta_0\|_{L^q}) \\
\leq & \frac{1}{16} \sup_{\tau\in[0,t]} \|\theta\|_{B^\delta_{q,\infty}} + C(C_0, C_2, M, j_2, \|\theta_0\|_{L^q}),
\end{aligned}$$

where we have used the fact that the summation of $2^{(1-\delta)k}$ for $\delta>1$ and $-1\leq k\leq j-2$ is finite. To bound J_3, we first write

$$J_3 = C_3 M \sup_{j\geq -1} 2^j \int_0^t e^{-C_0(t-\tau)2^{\beta j}} 2^{j\delta} \sum_{k\leq j-2} 2^{-\gamma k} \|\Delta_k \theta(\tau)\|_{L^q} d\tau$$

$$=C_3 M \sup_{j\geq -1} 2^{(1-\gamma)j} \int_0^t e^{-C_0(t-\tau)2^{\beta j}} \sum_{k\leq j-2} 2^{-(\gamma+\delta)(k-j)} 2^{\delta k} \|\Delta_k \theta(\tau)\|_{L^q} d\tau.$$

Similarly, we choose a positive integer j_3 such that

$$\frac{C_3 M}{C_0} 2^{(1-\beta-\gamma)j_3} \leq \frac{1}{16}.$$

Then, as in J_2,

$$J_3 \leq \frac{1}{16} \sup_{\tau\in[0,t]} \|\theta\|_{B^\delta_{q,\infty}} + C(C_0, C_2, M, j_3, \|\theta_0\|_{L^q}).$$

Putting together the estimates above yields

$$\|\theta(t)\|_{B^\delta_{q,\infty}} \leq \|\theta_0\|_{B^\delta_{q,\infty}} + \frac{3}{16} \sup_{\tau\in[0,t]} \|\theta(\tau)\|_{B^\delta_{q,\infty}} + C(M, \|\theta_0\|_{L^q}).$$

Therefore, for $\delta \in (1, \beta+\gamma)$,

$$\sup_{\tau\in[0,t]} \|\theta(\tau)\|_{B^\delta_{q,\infty}} \leq \|\theta_0\|_{B^\delta_{q,\infty}} + C(M, \|\theta_0\|_{L^q}) < \infty.$$

As a special consequence, we have, for $t\in[0,T]$,

$$\|\nabla\theta(t)\|_{L^\infty} < \infty.$$

In fact, if we choose $q\in[2,\infty)$ large enough such that $1+\frac{2}{q}-\delta<0$, then

$$\begin{aligned} \|\nabla\theta\|_{L^\infty} &\leq \sum_{j\geq -1} \|\nabla\Delta_j\theta\|_{L^\infty} \leq \sum_{j\geq -1} 2^{\left(1+\frac{2}{q}\right)j} \|\Delta_j\theta\|_{L^q} \\ &= \sum_{j\geq -1} 2^{\left(1+\frac{2}{q}-\delta\right)j} 2^{\delta j} \|\Delta_j\theta\|_{L^q} \leq C\|\theta\|_{B^\delta_{q,\infty}} < \infty. \end{aligned} \tag{3.3}$$

The regularity of our solution on $[0,T]$ then follows from the regularity criteria that

$$\int_0^T \|\nabla\theta(\tau)\|_{L^\infty} d\tau < \infty$$

implies the regularity on $[0,T]$. This completes the proof of Theorem 1.3. □

We now turn to the proof of Theorem 1.2.

Proof. (**Proof of Theorem 1.2.**) The main effort is devoted to showing that (1.5) implies, for any $t\in[0,T]$ and any $q\in[2,\infty)$,

$$\|\omega(t)\|_{L^q} \leq C, \qquad \|\Lambda^{1-\beta}\theta\|_{L^q} \leq C, \tag{3.4}$$

where C's are constants depending on T and the initial data. In particular, (3.4) implies that the velocity u obeys, for any $0<\gamma<1$ and $q\in[2,\infty)$ satisfying $\gamma+\frac{2}{q}-1<0$,

$$\|u\|_{C^\gamma} = \sup_{j\geq -1} 2^{\gamma j} \|\Delta_j u\|_{L^\infty} \leq C\|u\|_{L^2} + \sup_{j\geq 0} 2^{\gamma j} \|\Delta_j u\|_{L^\infty}$$

$$
\begin{aligned}
&\le C\|u\|_{L^2}+\sup_{j\ge 0}2^{(\gamma+\frac{2}{q})j}\|\Delta_j u\|_{L^q}\\
&\le C\|u\|_{L^2}+C\sup_{j\ge 0}2^{(\gamma+\frac{2}{q}-1)j}\|\nabla\Delta_j u\|_{L^q}\\
&\le C\|u\|_{L^2}+C\,\|\omega\|_{L^q}\sup_{j\ge 0}2^{(\gamma+\frac{2}{q}-1)j}\\
&\le C(\|u\|_{L^2}+\|\omega\|_{L^q})<\infty.
\end{aligned}
$$

Thus, there holds the assumption in Theorem 1.3, namely

$$u\in L^\infty([0,T];C^\gamma)\quad \text{for some } \gamma>1-\beta.$$

Theorem 1.3 then yields Theorem 1.2. It then suffices to show Equation (3.4). To do so, we form the equation for

$$Q=\omega+\mathcal{R}_\beta\theta,\quad \mathcal{R}_\beta=\Lambda^{-\beta}\partial_{x_1}.$$

Combining the equation for ω and $\mathcal{R}_\beta\theta$, we have

$$\partial_t Q+u\cdot\nabla Q+\Lambda^\alpha Q=\Lambda^\alpha\mathcal{R}_\beta\theta-[\mathcal{R}_\beta,u\cdot\nabla]\theta.$$

For $q\in[2,\infty)$, performing the L^q-estimate on Q and using $\nabla\cdot u=0$, we have

$$\frac{d}{dt}\|Q\|_{L^q}\le\|\Lambda^\alpha\mathcal{R}_\beta\theta\|_{L^q}+\|[\mathcal{R}_\beta,u\cdot\nabla]\theta\|_{L^q},$$

or

$$\|Q(t)\|_{L^q}\le\|Q(0)\|_{L^q}+\|\Lambda^\alpha\mathcal{R}_\beta\theta\|_{L^1_tL^q}+\|[\mathcal{R}_\beta,u\cdot\nabla]\theta\|_{L^1_tL^q}.$$

As in the proof of Theorem 1.1, we apply Lemma 2.3 to bound $\|\Lambda^\alpha\mathcal{R}_\beta\theta\|_{L^q}$. In fact, by Lemma 2.3

$$\sum_{j=0}^{\infty}2^{\beta j}\|\Delta_j\Lambda^{1-\beta}\theta\|_{L^1_tL^q}\le C(\theta_0)+C\int_0^t\|\nabla u(\tau)\|_{L^q}\,\|\theta(\tau)\|_{B^{1-\beta}_{\infty,1}}\,d\tau.$$

Consequently, for $\beta\ge\alpha$,

$$
\begin{aligned}
\|\Lambda^\alpha\mathcal{R}_\beta\theta\|_{L^1_tL^q}&\le\|\mathcal{R}_\beta\theta\|_{L^1_tL^q}+\|\Lambda^\beta\mathcal{R}_\beta\theta\|_{L^1_tL^q}\\
&\le\|\Lambda^{1-\beta}\theta\|_{L^1_tL^q}+\sum_{j=-1}^{\infty}\|\Delta_j\Lambda^\beta\mathcal{R}_\beta\theta\|_{L^1_tL^q}\\
&\le\|\Lambda^{1-\beta}\theta\|_{L^1_tL^q}+Ct\|\theta_0\|_{L^q}+\sum_{j=0}^{\infty}2^{\beta j}\|\Delta_j\Lambda^{1-\beta}\theta\|_{L^1_tL^q}\\
&\le\|\Lambda^{1-\beta}\theta\|_{L^1_tL^q}+C(T,\theta_0)+C\int_0^t\|\nabla u(\tau)\|_{L^q}\,\|\theta(\tau)\|_{B^{1-\beta}_{\infty,1}}\,d\tau.
\end{aligned}
$$

By Equation (2.14) in Lemma 2.2,

$$\|[\mathcal{R}_\beta,u\cdot\nabla]\theta\|_{L^q}\le C\,\|\nabla u\|_{L^q}\,\|\theta\|_{B^{1-\beta}_{\infty,1}}.$$

Therefore,

$$\|Q(t)\|_{L^q}\le C(T,u_0,\theta_0)+\|\Lambda^{1-\beta}\theta\|_{L^1_tL^q}+C\int_0^t\|\nabla u(\tau)\|_{L^q}\,\|\theta(\tau)\|_{B^{1-\beta}_{\infty,1}}\,d\tau. \tag{3.5}$$

Now we consider the equation for $\Lambda^{1-\beta}\theta$,

$$\partial_t \Lambda^{1-\beta}\theta + u\cdot\nabla\Lambda^{1-\beta}\theta + \Lambda^{\beta}\Lambda^{1-\beta}\theta = -[\Lambda^{1-\beta}, u\cdot\nabla]\theta.$$

Performing the L^q-estimate on $\Lambda^{1-\beta}\theta$ and using $\nabla\cdot u = 0$, we have

$$\frac{d}{dt}\|\Lambda^{1-\beta}\theta\|_{L^q} \le C\,\|[\Lambda^{1-\beta}, u\cdot\nabla]\theta\|_{L^q}.$$

Applying Lemma 2.2 again yields

$$\frac{d}{dt}\|\Lambda^{1-\beta}\theta\|_{L^q} \le C\,\|\nabla u\|_{L^q}\,\|\theta\|_{B^{1-\beta}_{\infty,1}}.$$

Noticing that, for $q\in(1,\infty)$,

$$\|\nabla u\|_{L^q} \le C\,\|\omega\|_{L^q} \le C\,(\|Q\|_{L^q} + \|\Lambda^{1-\beta}\theta\|_{L^q}),$$

we have

$$\|\Lambda^{1-\beta}\theta(t)\|_{L^q} \le \|\Lambda^{1-\beta}\theta_0\|_{L^q} + C\int_0^t (\|Q\|_{L^q} + \|\Lambda^{1-\beta}\theta\|_{L^q})\,\|\theta(\tau)\|_{B^{1-\beta}_{\infty,1}}\,d\tau. \tag{3.6}$$

Adding Equations (3.5) and (3.6) and applying Gronwall's inequality yield, for any $t\le T$,

$$\|Q(t)\|_{L^q} \le C, \qquad \|\Lambda^{1-\beta}\theta(t)\|_{L^q} \le C,$$

where C's depend on T, u_0, and θ_0. Therefore,

$$\|\omega(t)\|_{L^q} \le C.$$

This completes the proof of Theorem 1.2. ∎

Appendix A. Functional spaces. This appendix provides the definition of the Littlewood–Paley decomposition and the definition of Besov spaces. Some related facts used in the previous sections are also included. The material presented in this appendix can be found in several books and many papers (see, e.g., [4, 5, 32, 36, 39]).

We start with several notational conventions. $\mathcal{S}$ denotes the usual Schwarz class and $\mathcal{S}'$ its dual, the space of tempered distributions. To introduce the Littlewood–Paley decomposition, we write for each $j\in\mathbb{Z}$

$$A_j = \left\{\xi\in\mathbb{R}^d : 2^{j-1} \le |\xi| < 2^{j+1}\right\}.$$

The Littlewood–Paley decomposition asserts the existence of a sequence of functions $\{\Phi_j\}_{j\in\mathbb{Z}}\in\mathcal{S}$ such that

$$\mathrm{supp}\widehat{\Phi}_j \subset A_j, \qquad \widehat{\Phi}_j(\xi) = \widehat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jd}\Phi_0(2^j x),$$

and

$$\sum_{j=-\infty}^{\infty}\widehat{\Phi}_j(\xi) = \begin{cases} 1\,, & \text{if } \xi\in\mathbb{R}^d\setminus\{0\},\\ 0\,, & \text{if } \xi = 0.\end{cases}$$

Therefore, for a general function $\psi \in \mathcal{S}$, we have

$$\sum_{j=-\infty}^{\infty} \widehat{\Phi}_j(\xi)\widehat{\psi}(\xi) = \widehat{\psi}(\xi) \quad \text{for } \xi \in \mathbb{R}^d \setminus \{0\}.$$

We now choose $\Psi \in \mathcal{S}$ such that

$$\widehat{\Psi}(\xi) = 1 - \sum_{j=0}^{\infty} \widehat{\Phi}_j(\xi), \quad \xi \in \mathbb{R}^d.$$

Then, for any $\psi \in \mathcal{S}$,

$$\Psi * \psi + \sum_{j=0}^{\infty} \Phi_j * \psi = \psi$$

and hence

$$\Psi * f + \sum_{j=0}^{\infty} \Phi_j * f = f \tag{A.1}$$

in $\mathcal{S}'$ for any $f \in \mathcal{S}'$. To define the inhomogeneous Besov space, we set

$$\Delta_j f = \begin{cases} 0, & \text{if } j \le -2, \\ \Psi * f, & \text{if } j = -1, \\ \Phi_j * f, & \text{if } j = 0, 1, 2, \cdots. \end{cases} \tag{A.2}$$

Besides the Fourier localization operators Δ_j, the partial sum S_j is also a useful notation. For an integer j,

$$S_j \equiv \sum_{k=-1}^{j-1} \Delta_k,$$

For any $f \in \mathcal{S}'$, the Fourier transform of $S_j f$ is supported on the ball of radius 2^j. It is clear from Equation (A.1) that $S_j \to \mathrm{Id}$ as $j \to \infty$ in the distributional sense. In addition, the notation $\widetilde{\Delta}_k$, defined by

$$\widetilde{\Delta}_k = \Delta_{k-1} + \Delta_k + \Delta_{k+1},$$

is also useful and has been used in the previous sections.

Definition A.1. *The inhomogeneous Besov space $B^s_{p,q}$ with $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ consists of $f \in \mathcal{S}'$ satisfying*

$$\|f\|_{B^s_{p,q}} \equiv \|2^{js} \|\Delta_j f\|_{L^p}\|_{l^q} < \infty,$$

where $\Delta_j f$ is as defined in Equation (A.2)*.*

Many frequently used function spaces are special cases of Besov spaces. The following proposition lists some useful equivalence and embedding relations.

Proposition A.1. *For any $s \in \mathbb{R}$,*

$$H^s \sim B^s_{2,2}.$$

For any $s\in\mathbb{R}$ and $1<q<\infty$,

$$B^s_{q,\min\{q,2\}} \hookrightarrow W^s_q \hookrightarrow B^s_{q,\max\{q,2\}}.$$

For any non-integer $s>0$, the Hölder space C^s is equivalent to $B^s_{\infty,\infty}$.

Bernstein's inequalities are useful tools in dealing with Fourier localized functions. These inequalities trade integrability for derivatives. The following proposition provides Bernstein type inequalities for fractional derivatives. The upper bounds also hold when the fractional operators are replaced by partial derivatives.

Proposition A.2. *Let $\alpha\geq 0$. Let $1\leq p\leq q\leq\infty$.*

1) If f satisfies

$$\operatorname{supp}\widehat{f}\subset\{\xi\in\mathbb{R}^d : |\xi|\leq K2^j\},$$

for some integer j and a constant $K>0$, then

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_1\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)}.$$

2) If f satisfies

$$\operatorname{supp}\widehat{f}\subset\{\xi\in\mathbb{R}^d : K_1 2^j\leq|\xi|\leq K_2 2^j\}$$

for some integer j and constants $0<K_1\leq K_2$, then

$$C_1\, 2^{2\alpha j}\|f\|_{L^q(\mathbb{R}^d)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^d)} \leq C_2\, 2^{2\alpha j + jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^d)},$$

where C_1 and C_2 are constants depending on α, p, and q only.

REFERENCES

[1] D. Adhikari, C. Cao, and J. Wu, *The 2D Boussinesq equations with vertical viscosity and vertical diffusivity*, J. Diff. Eqs., 249, 1078–1088, 2010.

[2] D. Adhikari, C. Cao, and J. Wu, *Global regularity results for the 2D Boussinesq equations with vertical dissipation*, J. Diff. Eqs., 251, 1637–1655, 2011.

[3] D. Adhikari, C. Cao, J. Wu, and X. Xu, *Small global solutions to the damped two-dimensional Boussinesq equations*, J. Diff. Eqs., 256, 3594–3613, 2014.

[4] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer, 2011.

[5] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[6] C. Cao and J. Wu, *Global regularity for the 2D anisotropic Boussinesq equations with vertical dissipation*, Arch. Ration. Mech. Anal., 208, 985–1004, 2013.

[7] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II*, Geophy. J. R. Astr. Soc., 13, 529–539, 1967.

[8] Q. Chen, C. Miao, and Z. Zhang, *A new Bernstein's inequality and the 2D dissipative quasi-geostrophic equation*, Commun. Math. Phys., 271, 821–838, 2007.

[9] D. Chae, *Global regularity for the 2D Boussinesq equations with partial viscosity terms*, Adv. Math., 203, 497–513, 2006.

[10] K. Choi, T. Hou, A. Kiselev, G. Luo, V. Sverak, and Y. Yao, *On the finite-time blowup of a 1D model for the 3D axisymmetric Euler equations*, arXiv:1407.4776 [math.AP] 17 Jul. 2014.

[11] K. Choi, A. Kiselev, and Y. Yao, *Finite time blow up for a 1D model of 2D Boussinesq system*, Commun. Math. Phys., 334, 1667–1679, 2015.

[12] P. Constantin and C.R. Doering, *Infinite Prandtl number convection*, J. Stat. Phys., 94, 159–172, 1999.

[13] P. Constantin and V. Vicol, *Nonlinear maximum principles for dissipative linear nonlocal operators and applications*, Geom. Funct. Anal., 22, 1289–1321, 2012.

[14] P. Constantin and J. Wu, *Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25, 1103–1110, 2008.
[15] A. Córdoba and D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*, Commun. Math. Phys., 249, 511–528, 2004.
[16] X. Cui, C. Dou, and Q. Jiu, *Local well-posedness and blow up criterion for the inviscid Boussinesq system in Hölder spaces*, J. Part. Diff. Eqs., 25, 220–238, 2012.
[17] R. Danchin and M. Paicu, *Global existence results for the anisotropic Boussinesq system in dimension two*, Math. Models Meth. Appl. Sci., 21, 421–457, 2011.
[18] C. Doering and J. Gibbon, *Applied Analysis of the Navier–Stokes Equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1995.
[19] A.E. Gill, *Atmosphere-Ocean Dynamics*, Academic Press, London, 1982.
[20] T. Hmidi, S. Keraani, and F. Rousset, *Global well-posedness for a Boussinesq–Navier–Stokes system with critical dissipation*, J. Diff. Eqs., 249, 2147–2174, 2010.
[21] T. Hmidi, S. Keraani, and F. Rousset, *Global well-posedness for Euler–Boussinesq system with critical dissipation*, Commun. Part. Diff. Eqs., 36, 420–445, 2011.
[22] T. Hou and C. Li, *Global well-posedness of the viscous Boussinesq equations*, Discrete and Cont. Dyn. Syst., 12, 1–12, 2005.
[23] Q. Jiu, C. Miao, J. Wu, and Z. Zhang, *The 2D incompressible Boussinesq equations with general critical dissipation*, SIAM J. Math. Anal., 46, 3426–3454, 2014.
[24] Q. Jiu, J. Wu, and W. Yang, *Eventual regularity of the two-dimensional Boussinesq equations with supercritical dissipation*, J. Nonlinear Science, 25, 37–58, 2015.
[25] D. KC, D. Regmi, L. Tao, and J. Wu, *The 2D Euler–Boussinesq equations with a singular velocity*, J. Diff. Eqs., 257, 82–108, 2014.
[26] M. Lai, R. Pan, and K. Zhao, *Initial boundary value problem for two-dimensional viscous Boussinesq equations*, Arch. Ration. Mech. Anal., 199, 739–760, 2011.
[27] A. Larios, E. Lunasin, and E.S. Titi, *Global well-posedness for the 2D Boussinesq system with anisotropic viscosity and without heat diffusion*, J. Diff. Eqs., 255, 2636–2654, 2013.
[28] J. Li and E.S. Titi, *Global well-posedness of the 2D Boussinesq equations with vertical dissipation*, arXiv:1502.06180 [math.AP] 22 Feb. 2015.
[29] G. Luo and T. Hou, *Potentially singular solutions of the 3D incompressible Euler equations*, arXiv:1310.0497v2 [physics.flu-dyn] 8 Dec. 2013.
[30] A.J. Majda, *Introduction to PDEs and Waves for the Atmosphere and Ocean*, Courant Lecture Notes in Mathematics, 9, AMS/CIMS, 2003.
[31] A.J. Majda and A.L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, Cambridge, 2001.
[32] C. Miao, J. Wu, and Z. Zhang, *Littlewood–Paley Theory and its Applications in Partial Differential Equations of Fluid Dynamics*, Science Press, Beijing, China, 2012 (in Chinese).
[33] C. Miao and L. Xue, *On the global well-posedness of a class of Boussinesq–Navier–Stokes systems*, NoDEA Nonlinear Diff. Eqs. Appl., 18, 707–735, 2011.
[34] K. Ohkitani, *Comparison between the Boussinesq and coupled Euler equations in two dimensions*, Tosio Kato's method and principle for evolution equations in mathematical physics (Sapporo, 2001), Surikaisekikenkyusho Kokyuroku, 1234, 127–145, 2001.
[35] J. Pedlosky, *Geophysical Fluid Dyanmics*, Springer-Verlag, New York, 1987.
[36] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators and Nonlinear Partial Differential Equations*, Walter de Gruyter, Berlin, New York, 1996.
[37] A. Sarria and J. Wu, *Blowup in stagnation-point form solutions of the inviscid 2d Boussinesq equations*, J. Diff. Eqs., 259, 3559–3576, 2015.
[38] A. Stefanov and J. Wu, *A global regularity result for the 2D Boussinesq equations with critical dissipation*, arXiv: 1411.1362 [math.AP] 5 Nov. 2014.
[39] H. Triebel, *Theory of Function Spaces II*, Birkhauser Verlag, 1992.
[40] J. Whitehead and C. Doering, *Internal heating driven convection at infinite Prandtl number*, J. Math. Phys., 52, 093101, 11, 2011.
[41] J. Wu, *The 2D Boussinesq equations with partial or fractional dissipation*, Lectures on the analysis of nonlinear partial differential equations, Morningside Lectures in Mathematics, Edited by Fang-Hua Lin and Ping Zhang, International Press, Somerville, MA, 2014, in press.
[42] J. Wu and X. Xu, *Well-posedness and inviscid limits of the Boussinesq equations with fractional Laplacian dissipation*, Nonlinearity, 27, 2215–2232, 2014.
[43] J. Wu, X. Xu, and Z. Ye, *Global smooth solutions to the n-dimensional damped models of incompressible fluid mechanics with small initial datum*, J. Nonlinear Science, 25, 157–192, 2015.
[44] X. Xu and L. Xue, *Yudovich type solution for the 2D inviscid Boussinesq system with critical and supercritical dissipation*, J. Diff. Eqs., 256, 3179–3207, 2014.
[45] W. Yang, Q. Jiu, and J. Wu, *Global well-posedness for a class of 2D Boussinesq systems with*

fractional dissipation, J. Diff. Eqs., 257, 4188–4213, 2014.
[46] Z. Ye, *Blow-up criterion of smooth solutions for the Boussinesq equations*, Nonlinear Anal., 110, 97–103, 2014.
[47] Z. Ye, *On the regularity criterion for the 2D Boussinesq equations involving the temperature*, Applicable Anal., 95, 615-626, 2016.
[48] K. Zhao, *2D inviscid heat conductive Boussinesq equations on a bounded domain*, Michigan Math. J., 59, 329–352, 2010.