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GLOBAL EXISTENCE AND POINTWISE ESTIMATES OF SOLUTIONS FOR THE GENERALIZED SIXTH-ORDER BOUSSINESQ EQUATION*

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Abstract. This paper studied the Cauchy problem for the generalized sixth-order Boussinesq equation in multi-dimension ($n\geq 3$), which was derived in the shallow fluid layers and nonlinear atomic chains. Firstly the global classical solution for the problem is obtained by means of long wave-short wave decomposition, energy method and the Green's function. Secondly and what's more, the pointwise estimates of the solutions are derived by virtue of the Fourier analysis and Green's function, which concludes that $|D_x^\alpha u(x,t)|\leq C(1+t)^{-\frac{n+|\alpha|-1}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-N}$ for $N>\left[\frac{n}{2}\right]+1$.

Keywords. global existence; pointwise estimates; generalized sixth-order Boussinesq equation; Green's function.

AMS subject classifications. 35G25; 35B40; 35Q35.

1. Introduction

In this paper, we consider the global existence and pointwise estimates for the following Cauchy problem of generalized sixth-order Boussinesq equation in $(x,t)\in\mathbb{R}^n\times[0,+\infty)$:

$$\begin{cases} u_{tt}-\varepsilon\Delta u_t-\Delta u+\mu\Delta^2 u-\nu\Delta^3 u=\Delta f(u),\\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \end{cases} \tag{1.1}$$

where $n\geq 3$, $\varepsilon>0$ is a small enough constant. The coefficients $\mu>0$ and $\nu>0$. Δu_t is a damping term, $\Delta^3 u$ is a dispersive term. $f(u)=O(u^\gamma),\gamma\geq 2$ is the nonlinear term. $u_0(x)$ and $u_1(x)$ are given two initial value functions.

For the equation (1.1) in the space dimension $n=1$ and without the damping term, it reduces as

$$u_{tt}-u_{xx}+\mu u_{xxxx}-\nu u_{xxxxxx}=(f(u))_{xx}. \tag{1.2}$$

When $f(u)=u^2$, this equation (1.2) was derived in the shallow fluid layers and nonlinear atomic chains, and described the bi-directional propagation of small amplitude and long capillary-gravity waves on the surface of shallow water for bond number (surface tension parameter) less than but very close to 1/3 [1,2]. Neglecting the sixth-order term, one obtains the classical Boussinesq equation [3]

$$u_{tt}-u_{xx}+\mu u_{xxxx}=(u^2)_{xx}, \tag{1.3}$$

which arises as mathematical model for describing the nonlinear motions of long waves in shallow water. Thus equation (1.1) was called generalized sixth-order Boussinesq equation.

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For the sixth-order Boussinesq equation (1.2), G. Maugin also proposed in modeling the nonlinear lattice dynamics in elastic crystals [4]. B. F. Feng et al. studied the solitary waves and their interactions [5]. C. I. Christov et al. [6] investigated the stationary propagating localized solutions, a numerical simulation of the collision of two solitary waves was conducted and the impact of Galilean invariance on phase shift was discussed in [7]. A. Esfahani and his coworkers also have done lots of work about the sixth-order Boussinesq equation (1.2) under different conditions in one dimension space, such as local well-posedness [8], global well-posedness [9,10], stability of solitary waves [11], and blow-up [12]. For some other research results about the sixth-order Boussinesq equation (1.2), see [13,14] and reference therein.

All works above studied the problem in one dimension space. For the space dimension $n=1,2,3$, we particularly mention V. Varlamov [15–18], who did lots of work, such as local well-posedness and long-time decay for the generalized Boussinesq-type equation

$$u_{tt}-2b\Delta u_t-\Delta u+\alpha\Delta^2 u=\beta\Delta(u^2),\quad x\in\mathbb{R}^n(n=1,2,3),t>0. \tag{1.4}$$

For $n\geq 3$, some global existence of the weak solutions for the Boussinesq-type equation without the sixth-order term are also studied by various researchers. we refer readers to [19–21], and reference therein. We particularly refer M. Liu and W. K. Wang here, who obtained the existence of classical solutions of the Cauchy problem for multidimensional Boussinesq-type equations with the help of the method of long wave-short wave decomposition, energy method and the Green's function [22]. However, up to now, there are few works on the problem for the multidimensional sixth-order Boussinesq equation. Thus in this paper, we are going try to study the Cauchy problem for multidimensional generalized sixth-order Boussinesq equation (1.1), first about the existence of classical solutions and the pointwise estimates for the solution.

Notice that the generalized sixth-order Boussinesq equation (1.1) contains higher order derivatives. Thus it is difficult to only use energy estimates for the Cauchy problem, since the lower-order derivatives of $u(x,t)$ and $u(x,t)$ itself cannot be estimated by the higher-order derivatives in the case of losing compactness. Thus we employ another powerful methodologies, which are long wave-short wave decomposition, energy estimates and the Green's function to handle our problem. The method of Green's function provides extremely powerful tools for studying pointwise estimates for various nonlinear evolution equations, and its point is to make use of Fourier transform to build the Green's function for the linearized equation and thereby obtain the representation formulas entailing the Green's functions for the nonlinear problem by Duhamel's principle [23]. The method has been successfully employed to study the pointwise estimates for kinds of nonlinear evolution equations, such as Navier–Stokes systems [24], the Euler equations with damping [25], linear thermoelastic system with second sound [26], dissipative wave equation [27,28], etc.

The rest of paper is organized as follows. In Section 2, we briefly give some notations and preliminaries. In Section 3, we make use of the Fourier transform to build the Green's function for the linearized equation of the generalized sixth-order Boussinesq equation (1.1). In Section 4, we study the unique global classical solution for the problem (1.1), by dividing $u(x,t)$ into two parts, long wave part and short wave part. For the short wave part, we employ energy estimate to obtain the exponential decay by virtue of the Poincaré-like inequality. For the long wave part, we make use of the Green's function to construct the estimates on the L^2-norm and L^∞-norm. In Section 5, the pointwise estimates of the solutions are obtained by applying the Green's function, where we use the cut-off functions to divide the frequency space into low frequency part, middle frequency part and high frequency part, and we make estimates for each part. In the last Section 6, we make some conclusions.

2. Notations and preliminaries

We shall use the following conventional notations throughout the paper. Without any ambiguity, we denote a generic positive constant by C which may vary from line to line. $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\cdots,\beta_n)$ are multi-indexes. We denote $D_{x_i}=\frac{1}{\sqrt{-1}}\frac{\partial}{\partial_{x_i}}, i=1,2,\cdots,n$. $W^{k,p}(\mathbb{R}^n)$ is the usual k-th order Sobolev space with its norm

$$\|u\|_{W^{k,p}(\mathbb{R}^n)}=\sum_{|\alpha|\le k}\|\partial_x^\alpha u(x)\|_{L^p(\mathbb{R}^n)},$$

and when $p=2$, we write $\|\cdot\|_{H^k}=\|\cdot\|_{W^{k,2}(\mathbb{R}^n)}$. And we denote

$$\|u\|_{\dot{H}^m}=\sum_{|\alpha|=k}\|\partial_x^\alpha u(x)\|_{L^2(\mathbb{R}^n)}.$$

Meanwhile, we define the Fourier transform and inverse Fourier transform as

$$\widehat{u}(\xi,t)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u(x,t)dx,\quad u(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\widehat{u}(\xi,t)d\xi.$$

where the notation i denotes the imaginary unit satisfying $i^2=-1$.

In what follows, we will frequently use following inequalities and lemmas.

LEMMA 2.1 (Gagliardo-Nirenberg inequality [29]). *Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let u be any function in $W^{m,r}(\Omega)\cap L^q(\Omega)$, $1\le q,r\le\infty$. For any integer j, $0\le j<m$, and for any number a in the interval $j/m\le a\le 1$, set*

$$\frac{1}{p}=\frac{j}{n}+a\left(\frac{1}{r}-\frac{m}{n}\right)+(1-a)\frac{1}{q}.$$

If $m-j-n/r$ is not a nonnegative integer, then

$$\|D^j u\|_{L^p}\le C\|u\|_{W^{m,r}}^a\|u\|_{L^q}^{1-a}. \tag{2.1}$$

If $m-j-n/r$ is a nonnegative integer, then inequality (2.1) holds for $a=j/m$. The constant C depends only on Ω,r,q,j,a.

LEMMA 2.2. *Let s and $\gamma\ge 2$ be two positive integers, and let $\delta>0,p,q,r\in[1,\infty]$ be numbers such that $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. Suppose that $F(u)=u^\gamma$ is a function of class $\mathcal{C}^s$. If $u,v\in W^{s,q}\cap L^p\cap L^\infty$ and $\|u\|_{L^\infty}\le\delta,\|v\|_{L^\infty}\le\delta$, then*

$$\|F(u)\|_{W^{s,r}}\le C\|u\|_{W^{s,q}}\|u\|_{L^p}\|u\|_{L^\infty}^{\gamma-2}, \tag{2.2}$$

$$\|\partial_x^\alpha F(u)\|_{L^r}\le C\|\partial_x^\alpha u\|_{L^q}\|u\|_{L^p}\|u\|_{L^\infty}^{\gamma-2},\quad |\alpha|\le s, \tag{2.3}$$

and

$$\begin{aligned}\|F(u)-F(v)\|_{W^{s,r}}\le{}& C(\|u\|_{L^\infty}+\|v\|_{L^\infty})^{\gamma-2}\\ &\cdot(\|u-v\|_{W^{s,q}}(\|u\|_{L^p}+\|v\|_{L^p})+\|u-v\|_{L^p}(\|u\|_{W^{s,q}}+\|v\|_{W^{s,q}})).\end{aligned} \tag{2.4}$$

Proof. The proof of the estimates (2.2) and (2.3) can be found in [28]. And the estimates of (2.4) can be also obtained after a direct computation. □

LEMMA 2.3. *For any nonnegative integers α,β and real number $b>0$, there holds*

$$D_\xi^\beta(\xi^\alpha e^{-b\xi^2 t}) = \begin{cases} O(1)\xi^{\alpha-\beta}(1+\xi^2 t)^\beta e^{-b\xi^2 t}, & \alpha\geq\beta, \\ O(1)t^{\frac{\beta-\alpha}{2}}(1+\xi^2 t)^{\frac{\alpha+\beta}{2}} e^{-b\xi^2 t}, & \alpha<\beta, \end{cases} \tag{2.5}$$

and

$$\left|D_\xi^\beta(\xi^\alpha e^{-b\xi^2 t})\right| \leq C\left|\xi^{\alpha-\beta}(1+\xi^2 t)^\beta e^{-b\xi^2 t}\right|, \tag{2.6}$$

where C is a positive constant.

Proof. First the estimates (2.5) can be obtained by the induction. Furthermore, for $\alpha<\beta$, one gets

$$\begin{aligned} t^{\frac{\beta-\alpha}{2}}(1+\xi^2 t)^{\frac{\alpha+\beta}{2}} e^{-b\xi^2 t} &= \xi^{\alpha-\beta}(\xi^2 t)^{\frac{\beta-\alpha}{2}}(1+\xi^2 t)^{\frac{\alpha+\beta}{2}} e^{-b\xi^2 t} \\ &\leq \xi^{\alpha-\beta}(1+\xi^2 t)^{\frac{\beta-\alpha}{2}}(1+\xi^2 t)^{\frac{\alpha+\beta}{2}} e^{-b\xi^2 t} \\ &= \xi^{\alpha-\beta}(1+\xi^2 t)^\beta e^{-b\xi^2 t}. \end{aligned}$$

That is to say, the estimates (2.5) can be written in the same form (2.6). □

LEMMA 2.4. *Suppose that $\widehat{f}(\xi,t)$ is the Fourier transform of $f(x,t)$. For any multi-indexes α and positive integer N, if there exists a constant $b>0$, such that $\widehat{f}(\xi,t)$ satisfies*

$$\left|D_\xi^\beta(\xi^\alpha \widehat{f}(\xi,t))\right| \leq C(|\xi|^{(|\alpha|+k-|\beta|)_+} + |\xi|^{|\alpha|+k} t^{|\beta|/2})(1+|\xi|^2 t)^m e^{-b|\xi|^2 t}, \tag{2.7}$$

for any fixed integers k and m and any multi-indexes β with $|\beta|\leq 2N$, where $(a)_+ = \max\{a,0\}$. Then we have the estimates

$$|D_x^\alpha f(x,t)| \leq C_N t^{-\frac{n+|\alpha|+k}{2}} B_N(|x|,t), \tag{2.8}$$

where C_N is a positive constant and $B_N(|x|,t) = \left(1+\frac{|x|^2}{1+t}\right)^{-N}$.

Furthermore, if supp$\{\widehat{f}(\xi,t)\}\subset\{(\xi,t)||\xi|<r_0, t>0\}$, $0<r_0<+\infty$, we also have the following estimates

$$|D_x^\alpha f(x,t)| \leq C_N (1+t)^{-\frac{n+|\alpha|+k}{2}} B_N(|x|,t). \tag{2.9}$$

Proof.

(1) If $|\beta|<|\alpha|+k|$, then by the direct calculation, we have

$$\begin{aligned} \left|x^\beta D^\alpha f(x,t)\right| &= \left|\frac{i^\beta}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} D^\beta \xi^\alpha \widehat{f}(\xi,t)d\xi\right| \\ &\leq C\left|\int_{\mathbb{R}^n}(|\xi|^{|\alpha|+k-|\beta|} + |\xi|^{|\alpha|+k} t^{|\beta|/2})(1+|\xi|^2 t)^m e^{-b|\xi|^2 t} d\xi\right| \\ &\leq C t^{-\frac{n+|\alpha|+k-|\beta|}{2}} \\ &\leq C(1+t)^{\frac{|\beta|}{2}} t^{-\frac{n+|\alpha|+k}{2}}, \end{aligned} \tag{2.10}$$

where we use the known results

$$\left|\int_{\mathbb{R}^n} x^m e^{-bx^2}dx\right| \le Cb^{-\frac{m+n}{2}}, \text{ for } m\in N^+ \text{and } b>0.$$

(2) If $|\beta|\ge|\alpha|+k|$, by the direct calculation, we also have

$$\begin{aligned}
\left|x^\beta D^\alpha f(x,t)\right| &= \left|\frac{i^\beta}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}D^\beta\xi^\alpha\widehat{f}(\xi,t)d\xi\right| \\
&\le C\left|\int_{\mathbb{R}^n}(1+|\xi|^{|\alpha|+k}t^{|\beta|/2})(1+|\xi|^2t)^m e^{-b|\xi|^2t}d\xi\right| \\
&\le C(1+t^{-\frac{|\alpha|+k-|\beta|}{2}})t^{-\frac{n}{2}} \\
&\le C(1+t)^{-\frac{|\alpha|+k-|\beta|}{2}}t^{-\frac{n}{2}} \\
&= C(1+t)^{\frac{|\beta|}{2}}t^{-\frac{n+|\alpha|+k}{2}}\left(\frac{t}{1+t}\right)^{\frac{|\alpha|+k}{2}} \\
&\le C(1+t)^{\frac{|\beta|}{2}}t^{-\frac{n+|\alpha|+k}{2}}. \qquad (2.11)
\end{aligned}$$

Taking $\beta=0$ when $|x|^2\le 1+t$ and $|\beta|=2N$ when $|x|^2>1+t$, from inequalities (2.10) and (2.11), we obtain

$$\left|x^\beta D^\alpha f(x,t)\right| \le Ct^{-\frac{n+|\alpha|+k}{2}}\min\left(1,\left(\frac{1+t}{|x|^2}\right)^N\right). \qquad (2.12)$$

Noticing the fact

$$1+\frac{|x|^2}{1+t}\le\begin{cases}2, & |x|^2\le 1+t,\\ \frac{2|x|^2}{1+t}, & |x|^2>1+t,\end{cases}$$

we obtain

$$\min\left(1,\left(\frac{1+t}{|x|^2}\right)^N\right)\le\frac{2^N}{\left(1+\left(\frac{|x|^2}{1+t}\right)\right)^N}=2^N B_N(|x|,t). \qquad (2.13)$$

Thus combining inequalities (2.12) and (2.13) yields estimate (2.8).

Furthermore, if $\widehat{f}(\xi,t)$ satisfies the condition $supp\{\widehat{f}(\xi,t)\}\subset\{(\xi,t)||\xi|<r_0,t>0\}$, $0<r_0<+\infty$. Then

(1) If $|\beta|<|\alpha|+k|$ and $t\ge 1$, from (2.10) we can compute that

$$\begin{aligned}
\left|x^\beta D^\alpha f(x,t)\right| &\le C(1+t)^{\frac{|\beta|}{2}}t^{-\frac{n+|\alpha|+k}{2}} \\
&\le C(1+t)^{\frac{|\beta|}{2}}(1+t)^{-\frac{n+|\alpha|+k}{2}}\left(\frac{1+t}{t}\right)^{\frac{n+|\alpha|+k}{2}} \\
&\le C(1+t)^{\frac{|\beta|}{2}}(1+t)^{-\frac{n+|\alpha|+k}{2}}\cdot 2^{\frac{n+|\alpha|+k}{2}} \\
&\le C(1+t)^{\frac{|\beta|}{2}}(1+t)^{-\frac{n+|\alpha|+k}{2}}. \qquad (2.14)
\end{aligned}$$

While $|\beta|<|\alpha|+k|$ and $0<t<1$, there also holds

$$\left|x^\beta D^\alpha f(x,t)\right| = \left|\frac{i^\beta}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}D^\beta\xi^\alpha\widehat{f}(\xi,t)d\xi\right|$$

$$
\begin{aligned}
&\leq C\left|\int_{|\xi|<r_0}(|\xi|^{|\alpha|+k-|\beta|}+|\xi|^{|\alpha|+k}t^{|\beta|/2})(1+|\xi|^2t)^m e^{-b|\xi|^2t}d\xi\right|\\
&\leq C\left|\int_{|\xi|<r_0}(r_0^{|\alpha|+k-|\beta|}+r_0^{|\alpha|+k})(1+r_0^2)^m d\xi\right|\\
&\leq C\\
&\leq C(1+t)^{\frac{|\beta|}{2}}(1+t)^{-\frac{|\beta|}{2}}\\
&\leq C(1+t)^{\frac{|\beta|}{2}}(1+t)^{-\frac{n+|\alpha|+k}{2}}(1+t)^{\frac{n+|\alpha|+k}{2}-\frac{|\beta|}{2}}\\
&\leq C(1+t)^{\frac{|\beta|}{2}}(1+t)^{-\frac{n+|\alpha|+k}{2}}\cdot 2^{\frac{n+|\alpha|+k-|\beta|}{2}}\\
&\leq C(1+t)^{\frac{|\beta|}{2}}(1+t)^{-\frac{n+|\alpha|+k}{2}}.
\end{aligned}
\tag{2.15}
$$

Combining inequalities (2.14) and (2.15) together, we have

$$
\left|x^\beta D^\alpha f(x,t)\right|\leq C(1+t)^{\frac{|\beta|}{2}}(1+t)^{-\frac{n+|\alpha|+k}{2}}. \tag{2.16}
$$

For the case $|\beta|\geq|\alpha|+k|$, we can obtain the similar estimates. Together with the property of $B_N(|x|,t)$, one can prove estimate (2.9). Thus the proof of Lemma 2.4 is completed.

□

LEMMA 2.5. *Suppose $f(x,t)$ and $g(x,t)$ are two given bounded functions and $\widehat{f}(\xi,t)=e^{ic\xi t}\widehat{g}(\xi,t)$, where $c\in(-\infty,\infty)$ is some real constant. Then if $|g(x,t)|\leq z(x,t)$, we have*

$$
|f(x,t)|\leq z(x+ct,t). \tag{2.17}
$$

Proof. It is easily obtained from the following induction

$$
\begin{aligned}
|f(x,t)| &= \left|\tfrac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\widehat{f}(\xi,t)d\xi\right|\\
&= \left|\tfrac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}e^{ic\xi t}\widehat{g}(\xi,t)d\xi\right|\\
&= \left|\tfrac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i(x+ct)\cdot\xi}\widehat{g}(\xi,t)d\xi\right|\\
&= |g(x+ct,t)|\\
&\leq z(x+ct,t).
\end{aligned}
$$

□

LEMMA 2.6. *When $n_1,n_2>\frac{n}{2}$, and $n_3=\min(n_1,n_2)$, we have that*

$$
\int_{\mathbb{R}^n}\left(1+\frac{|x-y|^2}{1+t}\right)^{-n_1}(1+|y|^2)^{-n_2}dy\leq C\left(1+\frac{|x|^2}{1+t}\right)^{-n_3}. \tag{2.18}
$$

Proof. The proof of the Lemma 2.6 can be found in [25]. □

LEMMA 2.7. *For the function $B_N(|x|,t)=\left(1+\frac{|x|^2}{1+t}\right)^{-N}$, we have the following estimates*

$$
\int_{\mathbb{R}^n}B_N(|x|,t)dx\leq C(1+t)^{\frac{N}{2}}, \tag{2.19}
$$

and

$$
\left|\left(\int_0^t(1+t-s)^{-\frac{n+|\alpha|+1}{2}}B_N(|x|,t-s)\right)*\left((1+s)^{-\frac{n+|\alpha|-1}{2}}B_N(|x|,s)\right)ds\right|
$$

$$\leq C(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t), \tag{2.20}$$

where C are positive constants.

Proof. The estimates (2.19) can be obtained by direct induction. we mainly prove the estimate (2.20), and denote the left-hand of (2.20) as I. First when $|x|^2 \leq t$, we have

$$1 \leq 2^N \left(1+\frac{|x|^2}{1+t}\right)^{-N}, \tag{2.21}$$

and thus

$$\begin{aligned}
I \leq & \left|\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}B_N(|x-y|,t-s)B_N(|y|,s)dyds\right| \\
& +\left|\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}B_N(|x-y|,t-s)B_N(|y|,s)dyds\right| \\
\leq & C\left|\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}\int_{\mathbb{R}^n}B_N(|y|,s)dyds\right| \\
& +C\left|\int_{\frac{t}{2}}^t(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}\int_{\mathbb{R}^n}B_N(|x-y|,t-s)dyds\right| \\
\leq & C\left|\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1-N}{2}}ds\right| \\
& +C\left|\int_{\frac{t}{2}}^t(1+t-s)^{-\frac{n+|\alpha|+1-N}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}ds\right| \\
\leq & C(1+t)^{-\frac{n+|\alpha|+1}{2}}+(1+t)^{-\frac{n+|\alpha|-1}{2}} \\
\leq & C(1+t)^{-\frac{n+|\alpha|-1}{2}}\times 1 \\
\leq & C(1+t)^{-\frac{n+|\alpha|-1}{2}}B_N(|x|,t).
\end{aligned} \tag{2.22}$$

When $|x|^2 \geq t$, there holds the estimates for $\tau \in [0,t]$

$$\left(1+\frac{|x|^2}{1+\tau}\right)^{-N} \leq 3^N\left(\frac{1+\tau}{1+t}\right)^N\left(1+\frac{|x|^2}{1+t}\right)^{-N}, \tag{2.23}$$

and

$$\begin{aligned}
I \leq & \left|\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}B_N(|x-y|,t-s)B_N(|y|,s)dyds\right| \\
& +\left|\int_{\frac{t}{2}}^t\int_{\mathbb{R}^n}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}B_N(|x-y|,t-s)B_N(|y|,s)dyds\right| \\
\leq & C\left|\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}\right. \\
& \left.\cdot\int_{\mathbb{R}^n}\left(1+\frac{|x-y|^2}{1+t-s}\right)^{-N}\left(\frac{1+s}{1+t}\right)^N\left(1+\frac{|x|^2}{1+t}\right)^{-N}dyds\right|
\end{aligned}$$

$$
\begin{aligned}
&+C\left|\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{n+|\alpha|+1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}}\right.\\
&\quad\left.\cdot\int_{\mathbb{R}^n}\left(\frac{1+t-s}{1+t}\right)^{N}\left(1+\frac{|x|^2}{1+t}\right)^{-N}\left(1+\frac{|y|^2}{1+s}\right)^{-N}dyds\right|\\
\le& C\frac{1}{(1+t)^N}\left(1+\frac{|x|^2}{1+t}\right)^{-N}\left|\int_0^{\frac{t}{2}}(1+t-s)^{-\frac{n+|\alpha|+1-N}{2}}(1+s)^{-\frac{n+|\alpha|-1-2N}{2}}ds\right|\\
&+C\frac{1}{(1+t)^N}\left(1+\frac{|x|^2}{1+t}\right)^{-N}\left|\int_{\frac{t}{2}}^{t}(1+t-s)^{-\frac{n+|\alpha|+1-2N}{2}}(1+s)^{-\frac{n+|\alpha|-1-N}{2}}ds\right|\\
\le& C\frac{1}{(1+t)^N}\left(1+\frac{|x|^2}{1+t}\right)^{-N}(1+t)^{-\frac{n+|\alpha|+1-N}{2}}\\
&+C\frac{1}{(1+t)^N}\left(1+\frac{|x|^2}{1+t}\right)^{-N}(1+t)^{-\frac{n+|\alpha|-1-N}{2}}\\
\le& C(1+t)^{-\frac{n+|\alpha|-1}{2}}B_N(|x|,t). \qquad (2.24)
\end{aligned}
$$

Combining estimates (2.22) and (2.24) yields (2.20). Thus the proof of Lemma 2.7 is completed.

□

3. The Green's function

In this section, we use the Fourier transform to build the Green's function $G(x,t)$ for the linearized equation of the generalized sixth-order Boussinesq problem (1.1). Thus the Green's function $G(x,t)$ satisfies

$$
\begin{cases}
G_{tt}-\varepsilon\Delta G_t-\Delta G+\mu\Delta^2 G-\nu\Delta^3 G=0,\\
G(x,0)=0,\quad G_t(x,0)=\delta(x),
\end{cases} \qquad (3.1)
$$

where $\delta(x)$ is the Dirac function. By the Fourier transform, we obtain that

$$
\begin{cases}
\widehat{G}_{tt}+\varepsilon|\xi|^2\widehat{G}_t+(|\xi|^2+\mu|\xi|^4+\nu|\xi|^6)\widehat{G}=0,\\
\widehat{G}(\xi,0)=0,\quad \widehat{G}_t(\xi,0)=1,
\end{cases} \qquad (3.2)
$$

where $\xi\in\mathbb{R}^n$. The symbol of the operator for the equation (3.2) is

$$
\sigma=\lambda^2+\varepsilon|\xi|^2\lambda+|\xi|^2+\mu|\xi|^4+\nu|\xi|^6, \qquad (3.3)
$$

where λ and ξ correspond to $\frac{\partial}{\partial t}$, and $D_{x_i},(i=1,2,\cdots,n)$. We can compute the eigenvalues of equation (3.3) are

$$
\lambda_{\pm}(\xi)=-\frac{1}{2}\varepsilon|\xi|^2\pm\frac{1}{2}\theta_0(\xi), \qquad (3.4)
$$

where $\theta_0(\xi)=2i|\xi|\sqrt{1+(\mu-\frac{1}{4}\varepsilon^2)|\xi|^2+\nu|\xi|^4}$. By direct calculation, we have

$$
\widehat{G}(\xi,t)=\frac{1}{\theta_0(\xi)}e^{\lambda_+(\xi)t}-\frac{1}{\theta_0(\xi)}e^{\lambda_-(\xi)t}=\widehat{G}^+(\xi,t)+\widehat{G}^-(\xi,t), \qquad (3.5)
$$

where $\widehat{G}^+(\xi,t)=\frac{1}{\theta_0(\xi)}e^{\lambda_+(\xi)t}$ and $\widehat{G}^-(\xi,t)=-\frac{1}{\theta_0(\xi)}e^{\lambda_-(\xi)t}$.

Let

$$\chi_1(\xi)=\begin{cases} 1, & |\xi|<r,\\ 0, & |\xi|>2r,\end{cases} \qquad \chi_3(\xi)=\begin{cases} 1, & |\xi|>R+1,\\ 0, & |\xi|<R,\end{cases} \tag{3.6}$$

be two smooth cut-off functions, where r and R are any fixed positive numbers satisfying $2r<1<R$. Moreover we set

$$\chi_2(\xi)=1-\chi_1(\xi)-\chi_3(\xi). \tag{3.7}$$

Meanwhile, we define the long-short wave decomposition $(u_L(x),u_S(x))$ for a function $u(x)$ based on the Fourier transform

$$\widehat{u_L}(\xi)=(1-\chi_3(\xi))\widehat{u}(\xi), \quad \widehat{u_S}(\xi)=\chi_3(\xi)\widehat{u}(\xi),$$

which imply $u_L(x)=(1-\chi_3(D))u(x)$, and $u_S(x)=\chi_3(D)u(x)$. Here $\chi_3(D)$ is the operator with the symbol $\chi_3(\xi)$. Then we have

$$u(x,t)=u_L(x,t)+u_S(x,t), \tag{3.8}$$

which implies that the function $u(x,t)$ can be divided into two parts: long wave part $u_L(x,t)$ and short wave part $u_S(x,t)$.

On the other hand, we denote $\widehat{G}_i^+(\xi,t)=\chi_i(\xi)\widehat{G}^+(\xi,t)$, $\widehat{G}_i^-(\xi,t)=\chi_i(\xi)\widehat{G}^-(\xi,t)$, and $\widehat{G}_i^{\pm}(\xi,t)=\widehat{G}_i^+(\xi,t)+\widehat{G}_i^-(\xi,t)$ for $i=1,2,3$. Then $\widehat{G}(\xi,t)$ in (3.5) can be rewrote as

$$\begin{aligned}\widehat{G}(\xi,t)&=\widehat{G}^+(\xi,t)+\widehat{G}^-(\xi,t)\\ &=\widehat{G}_1^+(\xi,t)+\widehat{G}_2^+(\xi,t)+\widehat{G}_3^+(\xi,t)+\widehat{G}_1^-(\xi,t)+\widehat{G}_2^-(\xi,t)+\widehat{G}_3^-(\xi,t)\\ &=\widehat{G}_1^{\pm}(\xi,t)+\widehat{G}_2^{\pm}(\xi,t)+\widehat{G}_3^{\pm}(\xi,t).\end{aligned} \tag{3.9}$$

That is to say, the frequency space is divided into low frequency part $\widehat{G}_1^{\pm}(\xi,t)$, middle frequency part $\widehat{G}_2^{\pm}(\xi,t)$ and high frequency part $\widehat{G}_3^{\pm}(\xi,t)$.

4. Global existence for problem (1.1)

In this section, we will make use of the long-short wave decomposition (3.8) to study the global existence for the nonlinear problem (1.1). For a given integer $l\geq\left[\frac{n}{2}\right]+5$ and some constant $M>0$, we define a function space

$$X_{l,M}=\{u(x,t)|D_l(u)\leq M\}, \tag{4.1}$$

where $D_l(u)$ is defined as

$$D_l(u)=\sup_{t\geq 0}\left\{(1+t)^{\frac{n-1}{2}}\|u(x,t)\|_{W^{l-\left[\frac{n}{2}\right]-1,\infty}}+(1+t)^{\frac{n-2}{4}}\|u(x,t)\|_{H^l}\right\}. \tag{4.2}$$

Thus $(X_{l,M},D_l(u))$ is a Banach space, and we will consider the problem (1.1) in this space. Next we construct a convergent sequence $\{u^m(x,t)\}_{m=1}^{\infty}$ to get the global solution to the nonlinear problem. Here $u^m(x;t)$ is the solution of the following linear problem:

$$\begin{cases} u_{tt}^m-\varepsilon\Delta u_t^m-\Delta u^m+\mu\Delta^2u^m-\nu\Delta^3u^m=\Delta f(u^{m-1}),\\ u^m(x,0)=u_0(x), \quad u_t^m(x,0)=u_1(x),\end{cases} \tag{4.3}$$

for $m\geq 1$ and $u^0(x,t)=0$.

According the definition of short wave part of $u(x,t)$, we have the following lemma and proposition.

LEMMA 4.1 (Poincaré-like inequality). *Assume $u(x)\in H^m, m\geq 0$ is an integer. Then for short wave part $u_S(x,t)$, there exists constant C sunch that*

$$||u_S||_{L^2}\leq C||u_S||_{\dot{H}^s}$$

holds for any integer $s\in[0,m]$.

PROPOSITION 4.1. *Suppose that $u(x,t)$ is the solution of problem (1.1) and $u_S(x,t)$ is the short wave part of $u(x,t)$. Then for $|\alpha|\leq \min\{l-[\frac{n}{2}]-1,n\}$, where $l\geq [\frac{n}{2}]+5$, there exists a constant $C>0$ such that*

$$\begin{aligned}&\|D_x^\alpha u_S(x,t)\|_{H^3}^2+\|D_x^\alpha u_{St}(x,t)\|_{L^2}^2\\ \leq & C\int_0^t e^{-\frac{t-\tau}{C^*}}\|D_x^\alpha f(u(x,\tau))\|_{H^1}^2 d\tau + C\int_0^t e^{-\frac{t-\tau}{C^*}}\int_0^\tau \|D_x^\alpha f(u(x,s))\|_{H^1}^2 ds d\tau\\ &+C\int_0^t e^{-\frac{t-\tau}{C^*}}E_0^\alpha d\tau + Ce^{-\frac{t}{C^*}}(\|D_x^\alpha u_0\|_{H^3}^2+\|D_x^\alpha u_1\|_{L^2}^2), \end{aligned}\tag{4.4}$$

where $E_0^\alpha=\|D_x^\alpha u_{S1}\|^2+\|D_x^{\alpha+1}u_{S0}\|^2+\mu\|D_x^{\alpha+2}u_{S0}\|^2+\nu\|D_x^{\alpha+3}u_{S0}\|^2$ depending on the initial values.

Proof. Considering the short wave part $u_S(x,t)$ of the problem (1.1), there holds

$$\begin{cases} u_{Stt}-\varepsilon\Delta u_{St}-\Delta u_S+\mu\Delta^2 u_S-\nu\Delta^3 u_S=\Delta(\chi_3(D)f(u(x,t))),\\ u_S(x,0)=u_{S0}(x),\quad u_{St}(x,0)=u_{S1}(x).\end{cases}\tag{4.5}$$

Multiplying the above equation by $u_{St}(x,t)$ and integrating over $\mathbb{R}^n$, we have

$$\frac{d}{dt}\left(\|u_{St}\|^2+\|\nabla u_S\|^2+\mu\|\Delta u_S\|^2+\nu\|\Delta\nabla u_S\|^2\right)+\varepsilon\|\nabla u_{St}\|^2\frac{1}{\varepsilon}\|\nabla(\chi_3(D)f(u(x,t)))\|^2.\tag{4.6}$$

Integrating over the region $(0,t)$ to give that

$$\begin{aligned}\|u_{St}\|^2\leq & \frac{1}{\varepsilon}\int_0^t\|\nabla(\chi_3(D)f(u(x,s)))\|^2 ds+\|u_{S1}\|^2+\|\nabla u_{S0}\|^2+\mu\|\Delta u_{S0}\|^2+\nu\|\Delta\nabla u_{S0}\|^2\\ =&\frac{1}{\varepsilon}\int_0^t\|\nabla(\chi_3(D)f(u(x,s)))\|^2 ds+E_0^0,\end{aligned}\tag{4.7}$$

where $E_0^0=\|u_{S1}\|^2+\|\nabla u_{S0}\|^2+\mu\|\Delta u_{S0}\|^2+\nu\|\Delta\nabla u_{S0}\|^2\ \leq C(\|u_0\|_{H^3}^2+\|u_1\|_{L^2}^2)$ is a constant depending on the initial values.

Meanwhile, we multiply equation (4.5) by $u_S(x,t)$ and integrate over $\mathbb{R}^n$

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^n}\left(u_S u_{St}+\frac{\varepsilon}{2}|\nabla u_S|^2\right)dx+\frac{3}{4}\|\nabla u_S\|^2+\mu\|\Delta u_S\|^2+\nu\|\Delta\nabla u_S\|^2\\ \leq&\|\nabla(\chi_3(D)f(u(x,t)))\|^2+\|u_{St}\|^2.\end{aligned}\tag{4.8}$$

Adding inequalities (4.6) and (4.8) together, and noticing inequality (4.7), we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^n}\left(u_S u_{St}+|u_{St}|^2+\frac{\varepsilon+1}{2}|\nabla u_S|^2+\mu|\Delta u_S|^2+\nu|\Delta\nabla u_S|^2\right)dx$$

$$+\varepsilon\|\nabla u_{St}\|^2+\frac{3}{4}\|\nabla u_S\|^2+\mu\|\Delta u_S\|^2+\nu\|\Delta\nabla u_S\|^2$$
$$\leq\|\nabla(\chi_3(D)f(u(x,t)))\|^2+\frac{1}{\varepsilon}\int_0^t\|\nabla(\chi_3(D)f(u(x,s)))\|^2ds+E_0^0. \tag{4.9}$$

According the Poincaré-like inequality in Lemma 4.1, there exists some positive constants C_0 and C_1 such that

$$\|u_S\|^2\leq C_0\|\nabla u_S\|^2,\quad \|u_{St}\|^2\leq C_1\|\nabla u_{St}\|^2. \tag{4.10}$$

Then taking $C^*=\min\left\{1,\frac{2\varepsilon}{3C_1},\frac{3}{2(C_0+\varepsilon+1)}\right\}$, we have

$$\begin{aligned}
&\varepsilon\|\nabla u_{St}\|^2+\frac{3}{4}\|\nabla u_S\|^2+\mu\|\Delta u_S\|^2+\nu\|\Delta\nabla u_S\|^2\\
&\geq\frac{3C_1C^*}{2}\|\nabla u_{St}\|^2+\left(\frac{C_0}{2}+\frac{\varepsilon+1}{2}\right)C^*\|\nabla u_S\|^2+\mu C^*\|\Delta u_S\|^2+\nu C^*\|\Delta\nabla u_S\|^2\\
&=\frac{C_0C^*}{2}\|\nabla u_S\|^2+\frac{C_1C^*}{2}\|\nabla u_{St}\|^2+C_1C^*\|\nabla u_{St}\|^2+\frac{\varepsilon+1}{2}C^*\|\nabla u_S\|^2\\
&\quad+\mu C^*\|\Delta u_S\|^2+\nu C^*\|\Delta\nabla u_S\|^2\\
&\geq\frac{C^*}{2}\|u_S\|^2+\frac{C^*}{2}\|u_{St}\|^2+C^*\|u_{St}\|^2+\frac{\varepsilon+1}{2}C^*\|\nabla u_S\|^2+\mu C^*\|\Delta u_S\|^2+\nu C^*\|\Delta\nabla u_S\|^2\\
&\geq C^*\int_{\mathbb{R}^n}\left(u_Su_{St}+|u_{St}|^2+\frac{\varepsilon+1}{2}|\nabla u_S|^2+\mu|\Delta u_S|^2+\nu|\Delta\nabla u_S|^2\right)dx.
\end{aligned} \tag{4.11}$$

Then combining inequalities (4.9) and (4.11), we get

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\frac{t}{C^*}}\int_{\mathbb{R}^n}\left(u_Su_{St}+|u_{St}|^2+\frac{\varepsilon+1}{2}|\nabla u_S|^2+\mu|\Delta u_S|^2+\nu|\Delta\nabla u_S|^2\right)dx\right)\\
&\leq e^{\frac{t}{C^*}}\left(\|\nabla(\chi_3(D)f(u(x,t)))\|^2+\frac{1}{\varepsilon}\int_0^t\|\nabla(\chi_3(D)f(u(x,s)))\|^2ds+E_0^0\right).
\end{aligned} \tag{4.12}$$

Changing the variables (x,t) of the inequality above to (x,τ), integrating over the region $\tau\in(0,t)$ and employing Lemma 4.1 again to yields that

$$\begin{aligned}
&\|u_S(x,t)\|_{H^3}^2+\|u_{St}(x,t)\|_{L^2}^2\\
&\leq C\int_0^t e^{-\frac{t-\tau}{C^*}}\|f(u(x,\tau))\|_{H^1}^2d\tau+C\int_0^t e^{-\frac{t-\tau}{C^*}}\int_0^\tau\|f(u(x,s))\|_{H^1}^2dsd\tau.\\
&\quad+C\int_0^t e^{-\frac{t-\tau}{C^*}}E_0^0d\tau+Ce^{-\frac{t}{C^*}}(\|u_0\|_{H^3}^2+\|u_1\|_{L^2}^2).
\end{aligned} \tag{4.13}$$

For the estimates for higher-order derivatives, we can process the deduction in the similar way and obtain

$$\begin{aligned}
&\|D_x^\alpha u_S(x,t)\|_{H^3}^2+\|D_x^\alpha u_{St}(x,t)\|_{L^2}^2\\
&\leq C\int_0^t e^{-\frac{t-\tau}{C^*}}\|D_x^\alpha f(u(x,\tau))\|_{H^1}^2d\tau+C\int_0^t e^{-\frac{t-\tau}{C^*}}\int_0^\tau\|D_x^\alpha f(u(x,s))\|_{H^1}^2dsd\tau\\
&\quad+C\int_0^t e^{-\frac{t-\tau}{C^*}}E_0^\alpha d\tau+Ce^{-\frac{t}{C^*}}(\|D_x^\alpha u_0\|_{H^3}^2+\|D_x^\alpha u_1\|_{L^2}^2),
\end{aligned} \tag{4.14}$$

where $E_0^\alpha = \|D_x^\alpha u_{S1}\|^2 + \|D_x^{\alpha+1} u_{S0}\|^2 + \mu \|D_x^{\alpha+2} u_{S0}\|^2 + \nu \|D_x^{\alpha+3} u_{S0}\|^2 \le C(\|D_x^\alpha u_0\|_{H^3}^2 + \|D_x^\alpha u_1\|_{L^2}^2)$ is a constant. The proof of Proposition4.1 is completed. ∎

For the long wave part of $u(x,t)$, we have the following proposition.

PROPOSITION 4.2. *For $n \ge 3$ and $f \in L^1(\mathbb{R}^n)$, then there exists some constant $C>0$ such that for $h=0,1$ and $\alpha \ge 0$*

$$\|D_x^\alpha \partial_t^h G_L(x,t) * f(x)\|_{L^2} \le C(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|+h}{2}} \|f\|_{L^1}, \tag{4.15}$$

$$\|D_x^\alpha \partial_t^h G_L(x,t) * f(x)\|_{L^\infty} \le C(1+t)^{-\frac{n-1}{2}-\frac{|\alpha|+h}{2}} \|f\|_{L^1}. \tag{4.16}$$

Proof. For $h=0$, we first know that

$$|\theta_0(\xi)| = \left| 2i|\xi| \sqrt{1+(\mu - \frac{1}{4}\varepsilon^2)|\xi|^2 + \nu |\xi|^4} \right| \ge 2|\xi|.$$

And from equation (3.5), we have

$$|\widehat{G}_L(\xi,t)| \le C|\xi|^{-1} e^{-\frac{1}{2}\varepsilon|\xi|^2 t}. \tag{4.17}$$

Applying the Plancherel's identity and Hausdorff-Young inequality, we obtain

$$\begin{aligned}
\|D_x^\alpha G_L(x,t) * f(x)\|_{L^2} &= \left(\int_{\mathbb{R}^n} |(i\xi)^\alpha \widehat{G}_L(\xi,t) \widehat{f}(\xi)|^2 dx \right)^{\frac{1}{2}} \\
&\le C \left(\int_{|\xi|<R+1} ||\xi|^{\alpha-1} e^{-\frac{1}{2}\varepsilon|\xi|^2 t} \widehat{f}(\xi)|^2 dx \right)^{\frac{1}{2}} \\
&\le C \|\widehat{f}\|_{L^\infty} \left(\int_{|\xi|<R+1} ||\xi|^{\alpha-1} e^{-\frac{1}{2}\varepsilon|\xi|^2 t} dx \right)^{\frac{1}{2}} \\
&\le C \|f\|_{L^1} (1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}},
\end{aligned} \tag{4.18}$$

where we use the following results

$$\begin{aligned}
\int_{|\xi|<R+1} ||\xi|^{|\alpha|-1} e^{-\frac{1}{2}\varepsilon|\xi|^2 t} dx &\le C t^{-\frac{n}{2}+1-|\alpha|} \int_{|\xi|<R+1} |\eta|^{2|\alpha|-2} e^{-\frac{1}{2}\varepsilon|\eta|^2} d\eta \\
&\le C t^{-\frac{n}{2}+1-|\alpha|} \int_0^\infty Y^{2|\alpha|-2} e^{-\frac{1}{2}\varepsilon Y^2} Y^{n-1} dY \\
&\le C(1+t)^{-\frac{n-2}{2}-|\alpha|} (1+\frac{1}{t})^{\frac{n-2}{2}+|\alpha|} \int_0^\infty e^{-\frac{1}{2}\varepsilon Y^2} Y^{n+2|\alpha|-3} dY \\
&\le C(1+t)^{-\frac{n-2}{2}-|\alpha|}.
\end{aligned} \tag{4.19}$$

In the same way, we can compute

$$\begin{aligned}
\|D_x^\alpha G_L(x,t) * f(x)\|_{L^\infty} &\le C\|f\|_{L^1} \|D_x^\alpha G_L(x,t)\|_{L^\infty} \\
&\le C\|f\|_{L^1} \| \int_{|\xi|<R+1} (i\xi)^\alpha \widehat{G}_L(\xi,t) e^{ix\xi} d\xi \|_{L^\infty} \\
&\le C\|f\|_{L^1} \| \int_{|\xi|<R+1} |\xi|^{\alpha-1} e^{-\frac{1}{2}\varepsilon|\xi|^2 t} d\xi
\end{aligned}$$

$$\leq C\|f\|_{L^1}(1+t)^{-\frac{n-1}{2}-\frac{|\alpha|}{2}}. \tag{4.20}$$

Finally, we can complete the proof for $h=1$ in the same way. Thus the proof of Proposition 4.2 is completed. ∎

Based on the results of Proposition 4.1 and Proposition 4.2, we have the following theorem for the global well-posedness.

THEOREM 4.1. *Let* $l\geq\left[\frac{n}{2}\right]+5$, $|\alpha|\leq\min\left\{l-\left[\frac{n}{2}\right]-1,n\right\}$, *and set*

$$E_0=\|u_0\|_{H^l\cap L^1}+\|u_1\|_{H^{l-3}\cap L^1}.$$

If E_0 *is suitably small and* $E_0\leq C(1+t)^{-\frac{n-1}{2}}$ *for some positive constant* C, *and* M *is also suitably small. Then the sequence* $\{u^m(x,t)\}_{m=1}^{\infty}\in X_{l,M}$ *is a Cauchy sequence, which means the problem (1.1) has a unique global classical solution* $u(x,t)\in X_{l,M}$.

Proof. We will prove the theorem by induction on m for $\{u^m(x,t)\}_{m=1}^{\infty}\in X_{l,M}$. Firstly we consider $m=1$. Since $H^k(\mathbb{R}^n)\hookrightarrow L^\infty(\mathbb{R}^n)$ for $k>\frac{n}{2}$, then we have the estimates for the short wave part

$$\begin{aligned}\|u_S^m(x,t)\|^2_{W^{l-[\frac{n}{2}]-1,\infty}}&\leq\sum_{|\beta|\leq l-[\frac{n}{2}]-1}\|D_x^\beta u_S^m(x,t)\|^2_{L^\infty}\\&\leq C\sum_{|\beta|\leq l-\left[\frac{n}{2}\right]-1}\|D_x^\beta u_S^m(x,t)\|^2_{H^{\left[\frac{n}{2}\right]+1}}\\&\leq C\|u_S^m(x,t)\|^2_{H^l}.\end{aligned}\tag{4.21}$$

According to the estimates in Proposition 4.1, we have

$$\begin{aligned}\|u_S^1(x,t)\|_{H^l}&\leq C\int_0^t e^{-\frac{t-\tau}{C^*}}(\|u_0\|^2_{H^l}+\|u_1\|^2_{H^{l-3}})d\tau+Ce^{-\frac{t}{C^*}}(\|u_0\|^2_{H^l}+\|u_1\|^2_{H^{l-3}})\\&\leq CE_0\int_0^t e^{-\frac{t-\tau}{C^*}}(1+\tau)^{-\frac{n-1}{2}}d\tau+CE_0e^{-\frac{t}{C^*}}\\&\leq CE_0(1+t)^{-\frac{n-1}{2}}\\&\leq CE_0(1+t)^{-\frac{n-2}{4}}.\end{aligned}\tag{4.22}$$

Then from inequality (4.21), there holds

$$\|u_S^1(x,t)\|_{W^{l-[\frac{n}{2}]-1,\infty}}\leq C\|u_S^m(x,t)\|_{H^l}\leq CE_0(1+t)^{-\frac{n-1}{2}}. \tag{4.23}$$

Thus we have

$$(1+t)^{\frac{n-1}{2}}\|u_S^1(x,t)\|_{W^{l-[\frac{n}{2}]-1,\infty}}+(1+t)^{-\frac{n-2}{4}}\|u_S^1(x,t)\|_{H^l}\leq CE_0. \tag{4.24}$$

On the other hand, for the long wave part, based on the Duhamel's principle, we have

$$\begin{aligned}D_x^\alpha u_L^m(x,t)=&D_x^\alpha\partial_t G_L(x,t)*u_0+D_x^\alpha G_L(x,t)*(u_1-\varepsilon\Delta u_0)\\&+\int_0^t D_x^\alpha\Delta G_L(x,t-s)*f(u^{m-1}(x,s))ds.\end{aligned}\tag{4.25}$$

Then from Proposition 4.2, we have

$$\|D_x^\alpha u_L^1(x,t)\|_{L^2}\leq\|D_x^\alpha\partial_t G_L(x,t)*u_0\|_{L^2}+\|D_x^\alpha G_L(x,t)*(u_1-\varepsilon\Delta u_0)\|_{L^2}$$

$$
\begin{aligned}
&\le C(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|+1}{2}}\|u_0\|_{L^1}+C(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}\|u_1-\varepsilon\Delta u_0\|_{L^1}\\
&\le CE_0(1+t)^{-\frac{n-2}{4}},
\end{aligned}\tag{4.26}
$$

and

$$
\begin{aligned}
\|D_x^\alpha u_L^1(x,t)\|_{L^\infty}&\le\|D_x^\alpha\partial_t G_L(x,t)*u_0\|_{L^\infty}+\|D_x^\alpha G_L(x,t)*(u_1-\varepsilon\Delta u_0)\|_{L^\infty}\\
&\le C(1+t)^{-\frac{n-1}{2}-\frac{|\alpha|+1}{2}}\|u_0\|_{L^1}+C(1+t)^{-\frac{n-1}{2}-\frac{|\alpha|}{2}}\|u_1-\varepsilon\Delta u_0\|_{L^1}\\
&\le CE_0(1+t)^{-\frac{n-1}{2}}.
\end{aligned}\tag{4.27}
$$

Thus from inequalities (4.26) and (4.27), there holds

$$
(1+t)^{\frac{n-1}{2}}\|D_x^\alpha u_L^1(x,t)\|_{L^\infty}+(1+t)^{-\frac{n-2}{4}}\|D_x^\alpha u_L^1(x,t)\|_{L^2}\le CE_0. \tag{4.28}
$$

Combining inequalities (4.24) (4.28) and taking E_0 suitably small, one can prove $u^1(x,t)\in X_{l,M}$.

Next we assume that $\{u^j(x,t)\}_{j=1}^\infty\in X_{l,M}$ is valid for $j\le m-1(m>1)$, and one needs to get that $u^j(x,t)\in X_{l,M}(j=m)$ holds. Firstly from Lemma 2.2 , we have

$$
\begin{aligned}
\|f(u^{m-1}(x,t))\|_{H^{l-2}}&\le C\|u^{m-1}(x,t)\|_{H^{l-2}}\|u^{m-1}(x,t)\|_{L^\infty}\|u^{m-1}(x,t)\|_{L^\infty}^{\gamma-2}\\
&\le CM^\gamma(1+t)^{-\frac{n-2}{4}}(1+t)^{-\frac{n-1}{2}(\gamma-1)}\\
&\le CM^\gamma(1+t)^{-\frac{n-2}{4}},
\end{aligned}\tag{4.29}
$$

and

$$
\begin{aligned}
\|f(u^{m-1}(x,t))\|_{L^1}&\le C\|u^{m-1}(x,t)\|_{L^2}\|u^{m-1}(x,t)\|_{L^\infty}\|u^{m-1}(x,t)\|_{L^\infty}^{\gamma-2}\\
&\le CM^\gamma(1+t)^{-\frac{n-2}{4}}(1+t)^{-\frac{n-1}{2}(\gamma-1)}.
\end{aligned}\tag{4.30}
$$

By the estimates in Proposition 4.1, one can obtain

$$
\begin{aligned}
\|u_S^m(x,t)\|_{H^l}^2\le{}& C\int_0^t e^{-\frac{t-\tau}{C^*}}\|f(u^{m-1}(x,\tau))\|_{H^{l-2}}^2d\tau\\
&+C\int_0^t e^{-\frac{t-\tau}{C^*}}\int_0^\tau\|f(u^{m-1}(x,s))\|_{H^{l-2}}^2dsd\tau\\
&+C\int_0^t e^{-\frac{t-\tau}{C^*}}(\|u_0\|_{H^l}^2+\|u_1\|_{H^{l-3}}^2)d\tau+Ce^{-\frac{t}{C^*}}(\|u_0\|_{H^l}^2+\|u_1\|_{H^{l-3}}^2)\\
\le{}& CM^{2\gamma}\int_0^t e^{-\frac{t-\tau}{C^*}}(1+\tau)^{-\frac{n-2}{2}}(1+\tau)^{-(n-1)(\gamma-1)}d\tau\\
&+CM^{2\gamma}\int_0^t e^{-\frac{t-\tau}{C^*}}\int_0^\tau(1+s)^{-\frac{n-2}{2}}(1+s)^{-(n-1)(\gamma-1)}dsd\tau\\
&+CE_0\int_0^t e^{-\frac{t-\tau}{C^*}}(1+\tau)^{-\frac{n-1}{2}}d\tau+CE_0e^{-\frac{t}{C^*}}\\
\le{}& CM^{2\gamma}(1+t)^{-\frac{n-2}{2}}+CM^{2\gamma}\int_0^t e^{-\frac{t-\tau}{C^*}}(1+\tau)^{-\frac{n-2}{2}}d\tau\\
&+CE_0(1+t)^{-\frac{n-2}{2}}+CE_0(1+t)^{-\frac{n-2}{2}}\\
\le{}&(CM^{2\gamma}+CE_0)(1+t)^{-\frac{n-2}{2}},
\end{aligned}\tag{4.31}
$$

which implies

$$(1+t)^{\frac{n-2}{4}}\|u_S^m(x,t)\|_{H^l}\leq (CM^{2\gamma}+CE_0)^{\frac{1}{2}}. \tag{4.32}$$

In the same way, we can compute

$$(1+t)^{\frac{n-1}{2}}\|u_S^m(x,t)\|_{W^{l-[\frac{n}{2}]-1,\infty}}\leq (CM^{2\gamma}+CE_0)^{\frac{1}{2}}. \tag{4.33}$$

And therefore

$$(1+t)^{\frac{n-1}{2}}\|u_S^m(x,t)\|_{W^{l-[\frac{n}{2}]-1,\infty}}+(1+t)^{\frac{n-2}{4}}\|u_S^m(x,t)\|_{H^l}\leq (CM^{2\gamma}+CE_0)^{\frac{1}{2}}. \tag{4.34}$$

On the other hand, from equation (4.25) and inequality (4.30), we compute

$$\begin{aligned}\|D_x^\alpha u_L^m(x,t)\|_{L^2}\leq &\|D_x^\alpha \partial_t G_L(x,t)*u_0\|_{L^2}+\|D_x^\alpha G_L(x,t)*(u_1-\varepsilon\Delta u_0)\|_{L^2}\\ &+\int_0^t\|D_x^\alpha \Delta G_L(x,t-s)*f(u^{m-1})(x,s)\|_{L^2}ds.\\ \leq& C(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|+1}{2}}\|u_0\|_{L^1}+C(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}\|u_1-\varepsilon\Delta u_0\|_{L^1}\\ &+C(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|+2}{2}}\|f(u^{m-1}(x,t))\|_{L^1}\\ \leq& C(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|}{2}}(\|u_0\|_{H^2}+\|u_1\|_{L^1})\\ &+CM^\gamma(1+t)^{-\frac{n-2}{4}-\frac{|\alpha|+2}{2}}(1+t)^{-\frac{n-2}{4}}(1+t)^{-\frac{n-1}{2}(\gamma-1)}\\ \leq& C(E_0+M^\gamma)(1+t)^{-\frac{n-2}{4}},\end{aligned} \tag{4.35}$$

and similarly to obtain that

$$\|D_x^\alpha u_L^m(x,t)\|_{L^\infty}\leq C(E_0+M^\gamma)(1+t)^{-\frac{n-1}{2}}. \tag{4.36}$$

And thus there holds

$$(1+t)^{\frac{n-1}{2}}\|D_x^\alpha u_L^m(x,t)\|_{L^\infty}+(1+t)^{\frac{n-2}{4}}\|D_x^\alpha u_L^m(x,t)\|_{L^2}\leq C(E_0+M^\gamma). \tag{4.37}$$

Combining inequalities (4.34) and (4.37), taking E_0 and M suitably small, we can conclude that $\{u^m(x,t)\}_{m=1}^\infty\in X_{l,M}$.

In what follows, we will prove that $\{u^m(x,t)\}_{m=1}^\infty\in X_{l,M}$ is a Cauchy sequence. Set $v^m=u^m-u^{m-1}$ and $v^{m-1}=u^{m-1}-u^{m-2}$ for $m\geq 2$ ($u^0(x,t)=0$), then $v^m(x,t)$ satisfies the following system

$$\begin{cases} v_{tt}^m-\varepsilon\Delta v_t^m-\Delta v^m+\mu\Delta^2 v^m-\nu\Delta^3 v^m=\Delta f(u^{m-1})-\Delta f(u^{m-2}),\\ v^m(x,0)=v_t^m(x,0)=0.\end{cases} \tag{4.38}$$

Then the Duhamel's principle shows

$$D_x^\alpha v^m(x,t)=\int_0^t D_x^\alpha\Delta G(x,t-s)*(f(u^{m-1}(x,s))-f(u^{m-2}(x,s)))ds. \tag{4.39}$$

First from Lemma 2.2, we have

$$\begin{aligned}&\|f(u^{m-1})-f(u^{m-2})\|_{H^{l-2}}\\ &\leq C(\|u^{m-1}\|_{L^\infty}+\|u^{m-2}\|_{L^\infty})^{\gamma-2}\end{aligned}$$

$$
\begin{aligned}
&\cdot\big[\|u^{m-1}-u^{m-2}\|_{H^{l-2}}(\|u^{m-1}\|_{L^\infty}+\|u^{m-2}\|_{L^\infty})\\
&\quad+\|u^{m-1}-u^{m-2}\|_{L^\infty}(\|u^{m-1}\|_{H^{l-2}}+\|u^{m-2}\|_{H^{l-2}})\big]\\
\le& CM^{\gamma-1}(1+t)^{-\frac{n-1}{2}(\gamma-2)}\\
&\cdot\Big(\|u^{m-1}-u^{m-2}\|_{H^{l-2}}(1+t)^{-\frac{n-1}{2}}+\|u^{m-1}-u^{m-2}\|_{L^\infty}(1+t)^{-\frac{n-2}{4}}\Big)\\
\le& CM^{\gamma-1}(1+t)^{-\frac{n-1}{2}(\gamma-2)}(1+t)^{-\frac{n-1}{2}}\|v^{m-1}\|_{H^{l-2}}\\
&+CM^{\gamma-1}(1+t)^{-\frac{n-2}{4}(\gamma-2)}(1+t)^{-\frac{n-2}{4}}\|v^{m-1}\|_{L^\infty}\\
\le& CM^{\gamma-1}(1+t)^{-\frac{n-1}{2}(\gamma-2)}(1+t)^{-\frac{n-1}{2}}(1+t)^{-\frac{n-2}{4}}D_l(v^{m-1})\\
&+CM^{\gamma-1}(1+t)^{-\frac{n-2}{4}(\gamma-2)}(1+t)^{-\frac{n-2}{4}}(1+t)^{-\frac{n-1}{2}}D_l(v^{m-1})\\
\le& CM^{\gamma-1}(1+t)^{-\frac{3n-4}{4}-\frac{n-2}{4}(\gamma-2)}D_l(v^{m-1}),
\end{aligned}\tag{4.40}
$$

and

$$
\begin{aligned}
\|f(u^{m-1})-f(u^{m-2})\|_{L^1}\le& C(\|u^{m-1}\|_{L^\infty}+\|u^{m-2}\|_{L^\infty})^{\gamma-2}\\
&\cdot(\|u^{m-1}\|_{L^2}+\|u^{m-2}\|_{L^2})\|u^{m-1}-u^{m-2}\|_{L^2}\\
\le& CM^{\gamma-1}(1+t)^{-\frac{n-1}{2}(\gamma-2)}(1+t)^{-\frac{n-2}{4}}\|v^{m-1}\|_{L^2}\\
\le& CM^{\gamma-1}(1+t)^{-\frac{n-1}{2}(\gamma-2)}(1+t)^{-\frac{n-2}{4}}(1+t)^{-\frac{n-2}{4}}D_l(v^{m-1})\\
\le& CM^{\gamma-1}(1+t)^{-\frac{n-2}{2}-\frac{n-1}{2}(\gamma-2)}D_l(v^{m-1}).
\end{aligned}\tag{4.41}
$$

Similar to obtain the estimate in Proposition 4.1, we can obtain the estimates

$$
\begin{aligned}
\|v_S^m(x,t)\|_{H^l}^2\le& C\int_0^t e^{-\frac{t-\tau}{C}}\|f(u^{m-1})(x,\tau)-f(u^{m-2})(x,\tau)\|_{H^{l-2}}^2d\tau\\
&+C\int_0^t e^{-\frac{t-\tau}{C^*}}\int_0^\tau\|f(u^{m-1}(x,s))-f(u^{m-2}(x,s))\|_{H^{l-2}}^2dsd\tau\\
&+C\int_0^t e^{-\frac{t-\tau}{C^*}}(\|v^m(x,0)\|_{H^l}^2+\|\partial_t v(x,0)\|_{H^{l-3}}^2)d\tau\\
&+Ce^{-\frac{t}{C^*}}(\|v^m(x,0)\|_{H^l}^2+\|\partial_t v(x,0)\|_{H^{l-3}}^2)\\
\le& CM^{2(\gamma-1)}(D_l(v^{m-1}))^2\int_0^t e^{-\frac{t-\tau}{C}}(1+\tau)^{-\frac{3n-4}{2}-\frac{n-2}{2}(\gamma-2)}d\tau\\
&+CM^{2(\gamma-1)}(D_l(v^{m-1}))^2\int_0^t e^{-\frac{t-\tau}{C^*}}\int_0^\tau(1+s)^{-\frac{3n-4}{2}-\frac{n-2}{2}(\gamma-2)}dsd\tau\\
\le& CM^{2(\gamma-1)}(D_l(v^{m-1}))^2(1+t)^{-(n-1)}\\
\le& CM^{2(\gamma-1)}(D_l(v^{m-1}))^2(1+t)^{-\frac{n-2}{2}},
\end{aligned}\tag{4.42}
$$

which also means

$$
(1+t)^{-\frac{n-2}{4}}\|v_S^m(x,t)\|_{H^l}\le CM^{(\gamma-1)}D_l(v^{m-1}).\tag{4.43}
$$

Furthermore, there holds

$$
\|v_S^m(x,t)\|_{W^{l-[\frac{n}{2}]-1,\infty}}\le C\|v_S^m(x,t)\|_{H^l}\le CM^{(\gamma-1)}D_l(v^{m-1})(1+t)^{-\frac{n-1}{2}}.\tag{4.44}
$$

Combining inequalities (4.43)(4.44) together implies

$$
D_l(v_S^m)\le CM^{(\gamma-1)}D_l(v^{m-1}).\tag{4.45}
$$

On the other hand, from Proposition 4.2, estimates (4.39) and (4.41), we get

$$\begin{aligned}\|D_x^\alpha v_L^m\|_{L^2} \le& \int_0^t \|D_x^\alpha \Delta G(x,t-s)*(f(u^{m-1}(x,s))-f(u^{m-2}(x,s)))\|_{L^2}ds\\ \le& C\int_0^t (1+t-s)^{-\frac{n-2}{4}-\frac{|\alpha|+2}{2}}\|f(u^{m-1}(x,s))-f(u^{m-2}(x,s))\|_{L^1}ds\\ \le& CM^{\gamma-1}D_l(v^{m-1})\int_0^t (1+t-s)^{-\frac{n-2}{4}-\frac{|\alpha|+2}{2}}(1+s)^{-\frac{n-2}{2}-\frac{n-1}{2}(\gamma-2)}ds\\ \le& CM^{\gamma-1}(1+t)^{-\frac{n-2}{4}}D_l(v^{m-1}),\end{aligned} \tag{4.46}$$

and also similarly

$$\|D_x^\alpha v_L^m\|_{L^\infty} \le CM^{\gamma-1}(1+t)^{-\frac{n-1}{2}}D_l(v^{m-1}). \tag{4.47}$$

From inequalities (4.46)(4.47), one gets

$$D_l(v_L^m) \le CM^{(\gamma-1)}D_l(v^{m-1}). \tag{4.48}$$

Finally using inequalities (4.45)(4.48), setting $\theta = CM^{(\gamma-1)}$ and taking M suitably small yields

$$D_l(v^m) \le \theta D_l(v^{m-1}), \tag{4.49}$$

where $0<\theta<1$. Thus we can conclude that $\{u^m(x,t)\}_{m=1}^\infty \in X_{l,M}$ is a Cauchy sequence, and there exists $u(x,t)\in X_{l,M}$, which is the limit of $\{u^m(x,t)\}_{m=1}^\infty$ since $X_{l,M}$ is a Banach space. Thus, the proof of Theorem 4.1 is completed. □

5. Pointwise estimates for problem (1.1)

In this section, we will employ the Green's function to study the pointwise estimates for the solutions of the problem (1.1). As is well known, the decay of the solution is mainly related to the properties of $\widehat{G}(\xi,t)$ near $\xi=0$ in the frequency space. According to the value of ξ, we make use of equation (3.9) to divide the frequency space into low frequency part $\widehat{G}_1^\pm(\xi,t)$, middle frequency part $\widehat{G}_2^\pm(\xi,t)$ and high frequency part $\widehat{G}_3^\pm(\xi,t)$. And we study the pointwise estimates for each part respectively.

Firstly we have the following proposition about the estimates on $G_1(x,t)$.

Proposition 5.1. *Suppose $G_1(x,t)$ is the inverse Fourier transform of $\widehat{G}_1(\xi,t)$, and $|\alpha|\ge 0$. Then for sufficient small $r>0$ and $h=0,1$, we have the following estimate*

$$\left|\partial_t^h D_x^\alpha G_1(x,t)\right| \le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}B_N(|x|,t), \tag{5.1}$$

where C is a positive constant and $B_N(|x|,t)=\left(1+\frac{|x|^2}{1+t}\right)^{-N}$.

Proof. For $|\xi|$ being sufficiently small, by applying the Taylor expansion, we can obtain

$$\lambda_+(\xi) = -\frac{1}{2}\varepsilon|\xi|^2+\frac{1}{2}\theta_0(\xi) = -\frac{1}{2}\varepsilon|\xi|^2+i\xi+O(\xi^3), \tag{5.2}$$

$$\lambda_-(\xi) = -\frac{1}{2}\varepsilon|\xi|^2-\frac{1}{2}\theta_0(\xi) = -\frac{1}{2}\varepsilon|\xi|^2-i\xi+O(\xi^3), \tag{5.3}$$

$$\frac{1}{\theta_0(\xi)}=\frac{1}{\sqrt{\varepsilon^2|\xi|^4-4(|\xi|^2+\mu|\xi|^4+\nu|\xi|^6)}}=-\frac{i}{2\xi}+\frac{4\mu-\varepsilon^2}{16}i\xi+O(\xi^2). \tag{5.4}$$

Then there holds that

$$e^{\lambda_+(\xi)t}=e^{\left(-\frac{1}{2}\varepsilon|\xi|^2+i\xi\right)t}(1+O(\xi^3)t),\quad e^{\lambda_-(\xi)t}=e^{\left(-\frac{1}{2}\varepsilon|\xi|^2-i\xi\right)t}(1+O(\xi^3)t),$$

and

$$\begin{aligned}\left|\partial_t^h\widehat{G}^+(\xi,t)\right|&=\left|\frac{(\lambda_+(\xi))^h}{\theta_0(\xi)}e^{\lambda_+(\xi)t}\right|\\&=\left|\left(-\frac{1}{2}\varepsilon|\xi|^2+i\xi+O(\xi^3)\right)^h\left(-\frac{i}{2\xi}+\frac{4\mu-\varepsilon^2}{16}i\xi+O(\xi^2)\right)\right.\\&\quad\left.\cdot e^{\left(-\frac{1}{2}\varepsilon|\xi|^2+i\xi\right)t}(1+O(\xi^3)t)\right|\\&\le C\left(|\xi|^{h-1}+|\xi|^{h+1}+|\xi|^{h+2}+|\xi|^{h+2}t\right)e^{-\frac{1}{2}\varepsilon|\xi|^2t}\\&\le C\left(|\xi|^{h-1}+|\xi|^{h+1}+|\xi|^{h+2}t\right)e^{-\frac{1}{2}\varepsilon|\xi|^2t}.\end{aligned} \tag{5.5}$$

Firstly for $t\ge 1$ and from the properties of the Fourier transform, we have

$$\begin{aligned}\left|D_x^\alpha\partial_t^hG_1^+(x,t)\right|&=\left|\frac{i^\alpha}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\xi^\alpha\partial_t^h\widehat{G_1^+}(\xi,t)d\xi\right|\\&=\left|\frac{i^\alpha}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\xi^\alpha\chi_1(\xi)\partial_t^h\widehat{G^+}(\xi,t)d\xi\right|\\&\le C\left|\int_{0<|\xi|<2r}e^{ix\cdot\xi}\left(|\xi|^{|\alpha|+h-1}+|\xi|^{|\alpha|+h+1}+|\xi|^{|\alpha|+h+2}t\right)e^{-\frac{1}{2}\varepsilon|\xi|^2t}d\xi\right|\\&\le C\left|\int_{\mathbb{R}^n}e^{ix\cdot\xi}|\xi|^{|\alpha|+h-1}e^{-\frac{1}{2}\varepsilon|\xi|^2t}d\xi\right|\\&\quad+C\left|\int_{0<|\xi|<2r}e^{ix\cdot\xi}\left((2r)^2|\xi|^{|\alpha|+h-1}+2r|\xi|^{|\alpha|+h+1}t\right)e^{-\frac{1}{2}\varepsilon|\xi|^2t}d\xi\right|\\&\le Ct^{-\frac{n+|\alpha|+h-1}{2}}+Ct^{-\frac{n+|\alpha|+h-1}{2}}+Ct\cdot t^{-\frac{n+|\alpha|+h+1}{2}}\\&\le Ct^{-\frac{n+|\alpha|+h-1}{2}}\\&\le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}\left(\frac{1+t}{t}\right)^{\frac{n+|\alpha|+h-1}{2}}\\&\le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}.\end{aligned} \tag{5.6}$$

When $0<t<1$, we obtain the estimates as

$$\begin{aligned}&\left|D_x^\alpha\partial_t^hG_1^+(x,t)\right|\\&=\left|\frac{i^\alpha}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\xi^\alpha\partial_t^h\widehat{G_1^+}(\xi,t)d\xi\right|\\&=\left|\frac{i^\alpha}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\cdot\xi}\xi^\alpha\chi_1(\xi)\partial_t^h\widehat{G^+}(\xi,t)d\xi\right|\\&\le C\left|\int_{0<|\xi|<2r}e^{ix\cdot\xi}\left(|\xi|^{|\alpha|+h-1}+|\xi|^{|\alpha|+h+1}+|\xi|^{|\alpha|+h+2}t\right)e^{-\frac{1}{2}\varepsilon|\xi|^2t}d\xi\right|\end{aligned}$$

$$\begin{aligned}
&\le C\left|\int_{0<|\xi|<2r}\left((2r)^{|\alpha|+h-1}+(2r)^{|\alpha|+h+1}+(2r)^{|\alpha|+h+2}\right)d\xi\right|\\
&\le C\\
&\le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}(1+t)^{\frac{n+|\alpha|+h-1}{2}}\\
&\le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}.
\end{aligned}\tag{5.7}$$

Combining inequalities (5.6) and (5.7) to yield that

$$\left|D_x^\alpha\partial_t^h G_1^+(x,t)\right|\le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}.\tag{5.8}$$

Secondly since $\widehat{G}^+(\xi,t)$ is a smooth function to the variable ξ near $\xi=0$, and from Lemma 2.3, we have

$$\begin{aligned}
\left|D_\xi^\beta\left(\xi^\alpha\partial_t^h\widehat{G}^+(\xi,t)\right)\right|&\le C\left|D_\xi^\beta\left(\left(|\xi|^{|\alpha|+h-1}+|\xi|^{|\alpha|+h+1}+|\xi|^{|\alpha|+h+2}t\right)e^{-\frac{1}{2}\varepsilon|\xi|^2t}\right)\right|\\
&\le C\left(|\xi|^{(|\alpha|+h-1-|\beta|)_+}+|\xi|^{|\alpha|+h-1}t^{\frac{|\beta|}{2}}\right)\cdot\left(1+|\xi|^2t\right)^{\frac{|\beta|}{2}+1}e^{-\frac{1}{2}\varepsilon|\xi|^2t}.
\end{aligned}\tag{5.9}$$

for $1\le\beta<2N$. Then there holds

$$\begin{aligned}
\left|D_\xi^\beta\left(\xi^\alpha\partial_t^h\widehat{G}_1^+(\xi,t)\right)\right|&=\left|D_\xi^\beta\left(\xi^\alpha\partial_t^h\left(\chi_1(\xi)\widehat{G}^+(\xi,t)\right)\right)\right|\\
&\le\left|\sum_{l+m=\beta}\frac{\beta!}{l!m!}D_\xi^l(\chi_1(\xi))D_\xi^m(\xi^\alpha\partial_t^h\widehat{G}^+(\xi,t))\right|\\
&\le C(|\xi|^{(|\alpha|+h-1-|m|)_+}+|\xi|^{|\alpha|+h-1}t^{\frac{|m|}{2}})(1+|\xi|^2t)^{\frac{|m|}{2}+1}e^{-\frac{1}{2}\varepsilon|\xi|^2t}\\
&\le C(|\xi|^{(|\alpha|+h-1-|\beta|)_+}+|\xi|^{|\alpha|+h-1}t^{\frac{|\beta|}{2}})(1+|\xi|^2t)^{\frac{|\beta|}{2}+1}e^{-\frac{1}{2}\varepsilon|\xi|^2t}.
\end{aligned}\tag{5.10}$$

Since $\widehat{G}_1^+(\xi,t)=\chi_1(\xi)\widehat{G}^+(\xi,t)$, thus from inequality (2.9) in Lemma 2.4, we have

$$\left|\partial_t^h D_x^\alpha G_1^+(x,t)\right|\le C_N(1+t)^{-\frac{n+|\alpha|+h-1}{2}}B_N(|x|,t),\tag{5.11}$$

where C_N is a positive constant. Combining inequalities (5.8) and (5.11), we obtain

$$\left|\partial_t^h D_x^\alpha G_1^+(x,t)\right|\le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}B_N(|x|,t).\tag{5.12}$$

Similarly, we can obtain the estimate for $G_1^-(x,t)$ in the same way, which concludes

$$\left|\partial_t^h D_x^\alpha G_1^-(x,t)\right|\le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}B_N(|x|,t).\tag{5.13}$$

Combining inequalities (5.12) and (5.13) together, we have that

$$\left|\partial_t^h D_x^\alpha G_1(x,t)\right|\le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}B_N(|x|,t).\tag{5.14}$$

Thus the proof of Proposition 5.1 is completed. □

Secondly we have the following proposition for the estimate on $G_2(x,t)$.

Proposition 5.2. *Suppose $G_2(x,t)$ is the inverse Fourier transform of $\widehat{G}_2(\xi,t)$, and $|\alpha|\geq 0$. Then for any fixed r and R, and $h=0,1$, we have the following estimate*

$$\left|\partial_t^h D_x^\alpha G_2(x,t)\right|\leq Ce^{-b_1 t}B_N(|x|,t), \tag{5.15}$$

where C and b_1 are some positive constants.

Proof. For any fixed r and R, when ξ lies in the bounded interval $[r,R+1]$, we know that the real part of the eigenvalues of λ in (3.4) are negative. And

$$\left|\frac{1}{\theta_0(\xi)}\right| = \left|\frac{1}{2i|\xi|\sqrt{1+(\mu-\frac{1}{4}\varepsilon^2)|\xi|^2+\nu|\xi|^4}}\right| \leq \frac{1}{2r\sqrt{1+(\mu-\frac{1}{4}\varepsilon^2)r^2+\nu r^4}}\leq C. \tag{5.16}$$

Thus we can choose some positive constant $b_0\left(0<b_0\leq\frac{1}{2}\varepsilon r^2\right)$ such that

$$\left|\widehat{G}(\xi,t)\right|=\left|\frac{1}{\theta_0(\xi)}e^{\lambda_+(\xi)t}-\frac{1}{\theta_0(\xi)}e^{\lambda_-(\xi)t}\right|\leq Ce^{-\frac{1}{2}\varepsilon|\xi|^2t}\leq Ce^{-b_0t}, \tag{5.17}$$

and

$$\left|\partial_t^h\widehat{G}(\xi,t)\right|=\left|\frac{(\lambda_+(\xi))^h}{\theta_0(\xi)}e^{\lambda_+(\xi)t}-\frac{(\lambda_-(\xi))^h}{\theta_0(\xi)}e^{\lambda_-(\xi)t}\right|\leq Ce^{-\frac{1}{2}\varepsilon|\xi|^2t}\leq Ce^{-b_0t}. \tag{5.18}$$

Then we have

$$\left|\partial_t^h\widehat{G}_2(\xi,t)\right|\leq C\left|\partial_t^h\left(\chi_2(\xi)\widehat{G}(\xi,t)\right)\right|\leq Ce^{-b_0t}, \tag{5.19}$$

and furthermore

$$\begin{aligned}\left|\partial_t^h D_x^\alpha G_2(x,t)\right| &= \left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{\partial_t^h D_x^\alpha G_2}(\xi,t)d\xi\right|\\ &\leq C\left|\int_{r\leq|\xi|<R+1} e^{ix\xi}\xi^\alpha\partial_t^h\widehat{G}_2(\xi,t)d\xi\right|\\ &\leq Ce^{-b_0t}\int_{r\leq|\xi|<R+1}|\xi|^\alpha d\xi\\ &\leq Ce^{-b_0t}.\end{aligned} \tag{5.20}$$

Next, we will prove the following estimate by induction on β

$$\left|D_\xi^\beta\partial_t^h\widehat{G}_2(\xi,t)\right|\leq C(1+t)^\beta e^{-b_0t}. \tag{5.21}$$

When $\beta=0$, the estimate holds obviously from inequality (5.19). Now we suppose that it holds for $\beta\leq l-1$, which implies

$$\left|D_\xi^{l-1}\partial_t^h\widehat{G}_2(\xi,t)\right|\leq C(1+t)^{l-1}e^{-b_0t}, \tag{5.22}$$

we will prove that it is true for $\beta=l$. First from equation (3.2), we can get

$$\begin{cases}(\partial_{tt}^2+\varepsilon|\xi|^2\partial_t+|\xi|^2+\mu|\xi|^4+\nu|\xi|^6)D_\xi^l\partial_t^h\widehat{G}(\xi,t)\\ =-\sum\limits_{\beta_1+\beta_2=l,\beta_1\neq 0}\frac{l!}{\beta_1!\beta_2!}\Big(\varepsilon D_\xi^{\beta_1}(|\xi|^2)D_\xi^{\beta_2}(\partial_t^h\widehat{G}(\xi,t))\\ \qquad+D_\xi^{\beta_1}(|\xi|^2+\mu|\xi|^4+\nu|\xi|^6)D_\xi^{\beta_2}(\partial_t^h\widehat{G}(\xi,t))\Big),\\ D_\xi^l(\partial_t^h\widehat{G})(\xi,0)=a_0,\quad \partial_t\left(D_\xi^l(\partial_t^h\widehat{G})\right)(\xi,0)=a_1,\end{cases} \tag{5.23}$$

where a_0 and a_1 are polynomials of $|\xi|$. Multiplying (5.23), whose variables are now changed to $(\xi;s)$ by $\widehat{G}(\xi;t-s)$ and integrating over the region $s\in(0,t)$, then we can have that

$$\begin{aligned}\left|D_\xi^l\partial_t^h\widehat{G}(\xi,t)\right| &\le C(|a_0|+|a_1|)|\widehat{G}(\xi,t)| \\ &\quad + \left|\sum_{\beta_1+\beta_2=l,\,\beta_1\neq 0}\frac{l!}{\beta_1!\beta_2!}\int_0^t\Big(\varepsilon D_\xi^{\beta_1}(|\xi|^2)D_\xi^{\beta_2}(\partial_t^h\widehat{G}(\xi,s))\right. \\ &\quad \left. + D_\xi^{\beta_1}(|\xi|^2+\mu|\xi|^4+\nu|\xi|^6)D_\xi^{\beta_2}(\partial_t^h\widehat{G}(\xi,s))\Big)\widehat{G}(\xi;t-s)ds\right| \\ &\le Ce^{-b_0t}+C\int_0^t(1+s)^{l-1}e^{-b_0s}e^{-b_0(t-s)}ds \\ &\le C(1+t)^le^{-b_0t}. \end{aligned} \tag{5.24}$$

Thus the assertion of inequality (5.22) is completed. In all, for $1\le\beta\le l$ and taking $0<b_1<b_0$, there holds

$$\begin{aligned}\left|x^\beta\partial_t^hD_x^\alpha G_2(x,t)\right| &= \left|\frac{i^\beta}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\xi}D_\xi^\beta(\widehat{\partial_t^hD_x^\alpha G_2}(\xi,t))d\xi\right| \\ &= \left|\frac{i^{\alpha+\beta}}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\xi}D_\xi^\beta(\xi^\alpha\partial_t^h\widehat{G_2}(\xi,t))d\xi\right| \\ &= \left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{ix\xi}D_\xi^\beta(\xi^\alpha\partial_t^h(\chi_2(\xi)\widehat{G}(\xi,t))d\xi\right| \\ &\le C\left|\int_{r\le|\xi|<R+1}\sum_{\beta_l+\beta_2=\beta}\frac{\beta!}{\beta_1!\beta_2!}D_\xi^{\beta_1}(\xi^\alpha\chi_2(\xi))D_\xi^{\beta_2}(\partial_t^h\widehat{G}^+(\xi,t))d\xi\right| \\ &\le Ce^{-b_0t}(1+t)^\beta\int_{r\le|\xi|<R+1}d\xi \\ &\le Ce^{-b_0t}(1+t)^\beta \\ &\le Ce^{-b_1t}e^{-(b_0-b_1)t}(1+t)^\beta \\ &\le Ce^{-b_1t}(1+t)^{\frac{\beta}{2}}, \end{aligned} \tag{5.25}$$

which implies

$$\left|\partial_t^hD_x^\alpha G_2(x,t)\right|\le Ce^{-b_1t}\left(\frac{1+t}{x^2}\right)^{\frac{\beta}{2}}. \tag{5.26}$$

Since $0<b_1<b_0$, thus from (5.20), there holds

$$\left|\partial_t^hD_x^\alpha G_2(x,t)\right|\le Ce^{-b_0t}\le Ce^{-b_1t}. \tag{5.27}$$

By using (5.26) when $|x|^2\le 1+t$ and (5.27) when $|x|^2>1+t$, we have

$$\left|\partial_t^hD_x^\alpha G_2(x,t)\right|\le Ce^{-b_1t}\min\left(1,\left(\frac{1+t}{x^2}\right)^N\right). \tag{5.28}$$

Noticing the fact that

$$1+\frac{|x|^2}{1+t}\le\begin{cases}2, & |x|^2\le 1+t,\\ \frac{2|x|^2}{1+t}, & |x|^2>1+t,\end{cases}$$

we get that

$$\left|\partial_t^h D_x^\alpha G_2(x,t)\right| \le Ce^{-b_1 t} B_N(|x|,t), \tag{5.29}$$

which completes the proof of Proposition 5.2. ∎

Finally about the estimate on $G_3(x,t)$, we have the following proposition.

PROPOSITION 5.3. *Let $G_3(x,t)$ is the inverse Fourier transform of $\widehat{G}_3(\xi,t)$, and $|\alpha|\ge 0$. Then for $h=0,1$, there exists some positive constants b_4 and C such that*

$$\left|\partial_t^h D_x^\alpha G_3(x,t) - O(1)e^{-\frac{1}{2}\varepsilon R^2 t} F(x,t)\right| \le Ce^{-b_4 t} B_N(|x|,t). \tag{5.30}$$

where $F(x,t)$ is defined as

$$F(x,t) = \sum_{k=0}^{(\alpha+3h-3)_+} D_x^k \delta(x). \tag{5.31}$$

Proof. When ξ is sufficient large, we can obtain the approximate Tayloy expansion of the eigenvalues near $\xi=\infty$, which are

$$\lambda_+(\xi) = -\frac{1}{2}\varepsilon|\xi|^2 + i\sqrt{v}\xi^3 - i\sqrt{v}a\xi + i\sqrt{v}d\frac{1}{\xi} + O(1)\frac{1}{\xi^3}, \tag{5.32}$$

$$\lambda_-(\xi) = -\frac{1}{2}\varepsilon|\xi|^2 - i\sqrt{v}\xi^3 + i\sqrt{v}a\xi - i\sqrt{v}d\frac{1}{\xi} + O(1)\frac{1}{\xi^3}, \tag{5.33}$$

and

$$\frac{1}{\theta_0(\xi)} = \frac{1}{\sqrt{\varepsilon^2|\xi|^4 - 4(|\xi|^2 + \mu|\xi|^4 + \nu|\xi|^6)}} = -\frac{i}{2\sqrt{v}}\frac{1}{\xi^3} + \frac{ic}{\sqrt{v}}\frac{1}{\xi^5} + O(1)\frac{1}{\xi^7}, \tag{5.34}$$

where $a = \frac{\varepsilon^2 - 4\mu}{8v}, d = \frac{64v - \varepsilon^4 + 8\varepsilon^2\mu - 16\mu^2}{128v^2}$ and $c = \frac{4\mu - \varepsilon^2}{16v}$ are constants. Next by direct computation, we have

$$\begin{aligned}
\left|D_x^\alpha \partial_t^h G_3^+(x,t)\right| &= \left|\frac{i^\alpha}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\xi^\alpha \widehat{\partial_t^h G_3^+}(\xi,t)d\xi\right| \\
&= \left|\frac{i^\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}^n} e^{ix\xi}\xi^\alpha \chi_3(\xi)\left(\partial_t^h \widehat{G}^+(\xi,t)\right)d\xi\right| \\
&= \left|\frac{i^\alpha}{\sqrt{2\pi}}\int_{\mathbb{R}^n} e^{ix\xi}\xi^\alpha \chi_3(\xi)\left(\frac{(\lambda_+(\xi))^h}{\theta_0(\xi)}e^{\lambda_+(\xi)t}\right)d\xi\right| \\
&\le C\left|\int_{|\xi|>R} e^{ix\xi}\xi^\alpha\left(i\sqrt{v}\xi^3 - \frac{1}{2}\varepsilon|\xi|^2 - i\sqrt{v}a\xi + i\sqrt{v}d\frac{1}{\xi} + O(1)\frac{1}{\xi^3}\right)^h\right. \\
&\quad \cdot\left(-\frac{i}{2\sqrt{v}}\frac{1}{\xi^3} + \frac{ic}{\sqrt{v}}\frac{1}{\xi^5} + O(1)\frac{1}{\xi^7}\right)e^{\left(i\sqrt{v}\xi^3 - \frac{1}{2}\varepsilon|\xi|^2 - i\sqrt{v}a\xi\right)t} \\
&\quad \left.\cdot\left(1 + i\sqrt{v}d\frac{1}{\xi}t + O(1)\frac{1}{\xi^3}t\right)d\xi\right| \\
&\le C\left|\int_{|\xi|>R} e^{i(x\xi + (\sqrt{v}\xi^3 - \sqrt{v}a\xi)t)}e^{-\frac{1}{2}\varepsilon|\xi|^2 t}|\xi|^{\alpha+3h}\right.
\end{aligned}$$

$$\cdot\left(-\frac{i}{2\sqrt{v}}\frac{1}{\xi^3}+\frac{ic}{\sqrt{v}}\frac{1}{\xi^5}+O(1)\frac{1}{\xi^7}\right)\left(1+i\sqrt{v}d\frac{1}{\xi}t+O(1)\frac{1}{\xi^3}t\right)d\xi\Bigg|$$

$$\leq C\left|\int_{|\xi|>R}e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}e^{-\frac{1}{2}\varepsilon|\xi|^2t}|\xi|^{\alpha+3h}\right.$$

$$\left.\cdot\left(\frac{1}{|\xi|^3}+\frac{1}{|\xi|^5}+\frac{1}{|\xi|^4}t+\frac{1}{|\xi|^6}t\right)d\xi\right|. \tag{5.35}$$

In what follows, we will discuss the estimates under different conditions. They are: (1) When $\alpha=0$ and $h=0$, we have

$$\begin{aligned}&\left|D_x^\alpha\partial_t^hG_3^+(x,t)\right|\\&\leq C\left|\int_{|\xi|>R}e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}e^{-\frac{1}{2}\varepsilon|\xi|^2t}\left(\frac{1}{|\xi|^3}+\frac{1}{|\xi|^5}+\frac{1}{|\xi|^4}t+\frac{1}{|\xi|^6}t\right)d\xi\right|\\&\leq Ce^{-\frac{1}{2}\varepsilon R^2t}(1+t)\int_{|\xi|>R}\frac{1}{|\xi|^3}d\xi\\&\leq Ce^{-\frac{1}{2}\varepsilon R^2t}(1+t).\end{aligned}\tag{5.36}$$

(2) When $\alpha=0$ and $h=1$, we get

$$\begin{aligned}&\left|D_x^\alpha\partial_t^hG_3^+(x,t)\right|\\&\leq C\left|\int_{|\xi|>R}e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}e^{-\frac{1}{2}\varepsilon|\xi|^2t}|\xi|^3\left(\frac{1}{|\xi|^3}+\frac{1}{|\xi|^5}+\frac{1}{|\xi|^4}t+\frac{1}{|\xi|^6}t\right)d\xi\right|\\&\leq Ce^{-\frac{1}{2}\varepsilon R^2t}\left|\int_{|\xi|>R}e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}d\xi\right|\\&\quad+Ce^{-\frac{1}{2}\varepsilon R^2t}\left|\int_{|\xi|>R}\frac{1}{\xi^2}d\xi\right|+Ce^{-\frac{1}{2}\varepsilon R^2t}t\left|\int_{|\xi|>R}\frac{1}{\xi}e^{ix\xi}d\xi\right|\\&\leq Ce^{-\frac{1}{2}\varepsilon R^2t}(1+t).\end{aligned}\tag{5.37}$$

(3) When $\alpha=1$ and $h=0$, there holds

$$\begin{aligned}&\left|D_x^\alpha\partial_t^hG_3^+(x,t)\right|\\&\leq C\left|\int_{|\xi|>R}e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}e^{-\frac{1}{2}\varepsilon|\xi|^2t}|\xi|\left(\frac{1}{|\xi|^3}+\frac{1}{|\xi|^5}+\frac{1}{|\xi|^4}t+\frac{1}{|\xi|^6}t\right)d\xi\right|\\&\leq Ce^{-\frac{1}{2}\varepsilon R^2t}(1+t)\int_{|\xi|>R}\frac{1}{|\xi|^2}d\xi\\&\leq Ce^{-\frac{1}{2}\varepsilon R^2t}(1+t).\end{aligned}\tag{5.38}$$

(4) When $\alpha=1$ and $h=1$, there yields

$$\begin{aligned}&\left|D_x^\alpha\partial_t^hG_3^+(x,t)\right|\\&\leq C\left|\int_{|\xi|>R}e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}e^{-\frac{1}{2}\varepsilon|\xi|^2t}|\xi|^4\left(\frac{1}{|\xi|^3}+\frac{1}{|\xi|^5}+\frac{1}{|\xi|^4}t+\frac{1}{|\xi|^6}t\right)d\xi\right|\\&\leq Ce^{-\frac{1}{2}\varepsilon R^2t}\int_{|\xi|>R}e^{ix\xi}|\xi|d\xi+Ce^{-\frac{1}{2}\varepsilon R^2t}\left|\int_{|\xi|>R}\frac{1}{\xi}e^{ix\xi}d\xi\right|\end{aligned}$$

$$
\begin{aligned}
&+Ce^{-\frac{1}{2}\varepsilon R^2 t}t\left|\int_{|\xi|>R} e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}d\xi\right|+Ce^{-\frac{1}{2}\varepsilon R^2 t}t\int_{|\xi|>R}\frac{1}{|\xi|^2}d\xi\\
\le& Ce^{-\frac{1}{2}\varepsilon R^2 t}\int_{|\xi|>R} e^{ix\xi}|\xi|d\xi+Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t). \qquad (5.39)
\end{aligned}
$$

(5) When $\alpha=2$ and $h=0$, we obtain

$$
\begin{aligned}
&\left|D_x^\alpha\partial_t^h G_3^+(x,t)\right|\\
\le& C\left|\int_{|\xi|>R} e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}e^{-\frac{1}{2}\varepsilon|\xi|^2 t}|\xi|^2\left(\frac{1}{|\xi|^3}+\frac{1}{|\xi|^5}+\frac{1}{|\xi|^4}t+\frac{1}{|\xi|^6}t\right)d\xi\right|\\
\le& Ce^{-\frac{1}{2}\varepsilon R^2 t}\int_{|\xi|>R} e^{ix\xi}\frac{1}{|\xi|}d\xi+Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{|\xi|>R}\frac{1}{|\xi|^2}e^{ix\xi}d\xi\right|\\
&+Ce^{-\frac{1}{2}\varepsilon R^2 t}t\int_{|\xi|>R}\frac{1}{|\xi|^2}d\xi\\
\le& Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t). \qquad (5.40)
\end{aligned}
$$

(6) Finally, for $\alpha+h\ge 3$, we have the estimate

$$
\begin{aligned}
\left|D_x^\alpha\partial_t^h G_3^+(x,t)\right|\le& C\left|\int_{|\xi|>R} e^{i(x\xi+(\sqrt{v}\xi^3-\sqrt{v}a\xi)t)}e^{-\frac{1}{2}\varepsilon|\xi|^2 t}|\xi|^{\alpha+3h}\right.\\
&\left.\cdot\left(\frac{1}{|\xi|^3}+\frac{1}{|\xi|^5}+\frac{1}{|\xi|^4}t+\frac{1}{|\xi|^6}t\right)d\xi\right|\\
\le& Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{|\xi|>R} e^{i\xi x}\sum_{k=0}^{(\alpha+3h-3)_+}\xi^k d\xi\right|\\
&+Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t)\left|\int_{|\xi|>R} e^{i\xi x}\frac{1}{|\xi|}d\xi\right|\\
&+Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t)\left|\int_{|\xi|>R}\frac{1}{|\xi|^2}d\xi\right|\\
\le& Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{|\xi|>R} e^{i\xi x}\sum_{k=0}^{(\alpha+3h-3)_+}\xi^k d\xi\right|+Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t). \qquad (5.41)
\end{aligned}
$$

where $(\alpha+3h-3)_+=\max(\alpha+3h-3,0)$. In sum, based on the results from estimates (5.36)-(5.41), the estimate (5.35) can be computed as the uniform form

$$
\begin{aligned}
\left|D_x^\alpha\partial_t^h G_3^+(x,t)\right|\le& Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{|\xi|>R} e^{i\xi x}\sum_{k=0}^{(\alpha+3h-3)_+}\xi^k d\xi\right|+Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t)\\
=& I_1+Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t). \qquad (5.42)
\end{aligned}
$$

From Lemma 2.5 and the inverse Fourier transform, we have

$$
I_1=Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{|\xi|>R} e^{i\xi x}\sum_{k=0}^{(\alpha+3h-3)_+}\xi^k d\xi\right|
$$

$$\leq Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{\mathbb{R}^n} e^{i\xi x}\sum_{k=0}^{(\alpha+3h-3)_+}\xi^k d\xi\right|$$

$$= O(1)e^{-\frac{1}{2}\varepsilon R^2 t}\sum_{k=0}^{(\alpha+3h-3)_+} D_x^k\delta(x) = O(1)e^{-\frac{1}{2}\varepsilon R^2 t}F(x,t), \tag{5.43}$$

where $F(x,t)$ is defined in equation (5.31). Combining estimates (5.42) and (5.43), and taking some suitable number $0 < b_2 < \frac{1}{4}\varepsilon R^2$, we can obtain

$$\begin{aligned}\left|D_x^\alpha\partial_t^h G_3^+(x,t) - O(1)e^{-\frac{1}{2}\varepsilon R^2 t}F(x,t)\right| &\leq Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t)\\ &\leq Ce^{-\frac{1}{2}\varepsilon R^2 t}e^{\frac{1}{4}\varepsilon R^2 t}\\ &\leq Ce^{-b_2 t}.\end{aligned} \tag{5.44}$$

Meanwhile, we also can compute $x^\beta D_x^\alpha\partial_t^h G_3^+(x,t)$ in the same way

$$\begin{aligned}\left|x^\beta D_x^\alpha\partial_t^h G_3^+(x,t)\right| &= \left|\frac{i^{\alpha+\beta}}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}D_\xi^\beta\left(\xi^\alpha(\partial_t^h\widehat{G}_3^+(\xi,t))\right)\right|\\ &\leq C\left|\int_{|\xi|>R} e^{ix\xi}D_\xi^\beta\left(\xi^\alpha\chi_3(\xi)(\partial_t^h\widehat{G}^+(\xi,t))\right)\right|\\ &\leq C\left|\int_{|\xi|>R} e^{ix\xi}D_\xi^\beta\left(\xi^\alpha\chi_3(\xi)\left(\frac{(\lambda_+(\xi))^h}{\theta_0(\xi)}e^{\lambda_+(\xi)t}\right)\right)\right|\\ &\leq Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{|\xi|>R} e^{i\xi x}\sum_{k=0}^{(\alpha+3h-3-\beta)_+}\xi^k d\xi\right|\\ &\quad + Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t)\left|\int_{|\xi|>R}\frac{1}{|\xi|}e^{ix\xi}d\xi\right|\\ &\quad + Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t)\left|\int_{|\xi|>R}\frac{1}{|\xi|^2}d\xi\right|\\ &= I_1' + Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t),\end{aligned} \tag{5.45}$$

where

$$\begin{aligned}I_1' &= Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{|\xi|>R} e^{i\xi x}\sum_{k=0}^{(\alpha+3h-3-\beta)_+}\xi^k d\xi\right|\\ &\leq Ce^{-\frac{1}{2}\varepsilon R^2 t}\left|\int_{\mathbb{R}^n} e^{i\xi x}\sum_{k=0}^{(\alpha+3h-3-\beta)_+}\xi^k d\xi\right|\\ &\leq Ce^{-\frac{1}{2}\varepsilon R^2 t}\sum_{k=0}^{(\alpha+3h-3-\beta)_+} D_x^k\delta(x).\end{aligned} \tag{5.46}$$

With the help of the following relationship, there holds

$$x^\beta\sum_{k=0}^{(\alpha+3h-3)_+} D_x^k\delta(x) = \frac{i^\beta}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}D_\xi^\beta\left(\sum_{k=0}^{(\alpha+3h-3)_+}\xi^k\right)d\xi$$

$$= \frac{i^\beta}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} \sum_{k=0}^{(\alpha+3h-3-\beta)_+} \xi^k d\xi$$

$$= \sum_{k=0}^{(\alpha+3h-3-\beta)_+} D_x^k\delta(x), \tag{5.47}$$

which also implies that

$$I_1' \le Ce^{-\frac{1}{2}\varepsilon R^2 t} \sum_{k=0}^{(\alpha+3h-3-\beta)_+} D_x^k\delta(x) \le Ce^{-\frac{1}{2}\varepsilon R^2 t}\cdot x^\beta \cdot F(x,t). \tag{5.48}$$

Combining inequalities (5.45) and (5.48), and taking some suitable $b_3>0$, we have

$$\begin{aligned}\left|x^\beta D_x^\alpha \partial_t^h G_3^+(x,t) - O(1)x^\beta e^{-\frac{1}{2}\varepsilon R^2 t}F(x,t)\right| &\le Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+t)\\ &\le Ce^{-b_3 t}(1+t)^{\frac{\beta}{2}},\end{aligned} \tag{5.49}$$

which concludes that

$$\left|D_x^\alpha \partial_t^h G_3^+(x,t) - O(1)e^{-\frac{1}{2}\varepsilon R^2 t}F(x,t)\right| \le Ce^{-b_3 t}\left(\frac{1+t}{x^2}\right)^{\frac{\beta}{2}}. \tag{5.50}$$

Combining inequalities (5.44) and (5.50) together, taking $b_4=\min(b_2,b_3)$ and noticing the result $\min\left(1,\left(\frac{1+t}{x^2}\right)^N\right)\le 2^N B_N(|x|,t)$ again, we can obtain

$$\left|D_x^\alpha \partial_t^h G_3^+(x,t) - O(1)e^{-\frac{1}{2}\varepsilon R^2 t}F(x,t)\right| \le Ce^{-b_4 t}B_N(|x|,t). \tag{5.51}$$

Similarly, we can obtain the estimate for $D_x^\alpha \partial_t^h G_3^-(x,t)$, which implies

$$\left|D_x^\alpha \partial_t^h G_3^-(x,t) - O(1)e^{-\frac{1}{2}\varepsilon R^2 t}F(x,t)\right| \le Ce^{-b_4 t}B_N(|x|,t). \tag{5.52}$$

Thus combining inequalities (5.51) and (5.52), we complete the proof of Proposition 5.3. □

In summary, we have the following theorem about the Green's function $G(x,t)$.

THEOREM 5.1. *For $h=0,1$, and any multi-indexes α, we have*

$$\left|\partial_t^h D_x^\alpha G(x,t) - K(x,t)\right| \le C(1+t)^{-\frac{n+|\alpha|+h-1}{2}} B_N(|x|,t). \tag{5.53}$$

where

$$K(x,t) = O(1)e^{-\frac{1}{2}\varepsilon R^2 t}F(x,t). \tag{5.54}$$

Proof. From the estimates of Proposition 5.1, Proposition 5.2 and Proposition 5.3, one can find that

$$\begin{aligned}&\left|\partial_t^h D_x^\alpha G(x,t) - K(x,t)\right|\\ &= \left|\partial_t^h D_x^\alpha G_1(x,t) + \partial_t^h D_x^\alpha G_2(x,t) + \partial_t^h D_x^\alpha G_3(x,t) - K(x,t)\right|\\ &\le \left|\partial_t^h D_x^\alpha G_1(x,t)\right| + \left|\partial_t^h D_x^\alpha G_2(x,t)\right| + \left|\partial_t^h D_x^\alpha G_3(x,t) - K(x,t)\right|.\end{aligned}$$

$$
\begin{aligned}
&\leq C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}B_N(|x|,t)+Ce^{-b_1t}B_N(|x|,t)+Ce^{-b_4t}B_N(|x|,t)\\
&\leq C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}B_N(|x|,t)+Ce^{-bt}B_N(|x|,t)\\
&\leq C(1+t)^{-\frac{n+|\alpha|+h-1}{2}}B_N(|x|,t),
\end{aligned}
\tag{5.55}
$$

which completes the proof of Theorem 5.1. □

Based on the pointwise estimate of the Green's function $G(x,t)$, we have our main theorem about the pointwise estimate for problem (1.1) as follows

THEOREM 5.2. *For the space dimension $n\geq 3$, let $l\geq\left[\frac{n}{2}\right]+5$ and the initial values satisfy*

$$
\|u_0\|_{H^l\cap L^1}+\|u_1\|_{H^{l-3}\cap L^1}\leq\epsilon,
\tag{5.56}
$$

where $\epsilon\ll 1$ is a constant. Furthermore assume that $u_0(x)$ and $u_1(x)$ have compact support. Then for $|\alpha|\leq min\{l-\left[\frac{n}{2}\right]-1,n\}$, the solutions of the Cauchy problem for the generalized sixth-order Boussinesq equation (1.1) possess the following estimate

$$
|D_x^\alpha u(x,t)|\leq C(1+t)^{-\frac{n+|\alpha|-1}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-N},
\tag{5.57}
$$

where $N>\left[\frac{n}{2}\right]+1$ and C is a positive constant.

Proof. First by applying the Duhamel's principle, we can obtain that the solution of the problem (1.1) can be expressed as

$$
u(x,t)=G_t(x,t)*u_0+G(x,t)*(u_1-\varepsilon\Delta u_0)+\int_0^t G(x,t-s)*\Delta(f(u(x,s)))ds.
\tag{5.58}
$$

Applying D_x^α to the solution yields that

$$
\begin{aligned}
D_x^\alpha u(x,t)&=D_x^\alpha\partial_t G(x,t)*u_0+D_x^\alpha G(x,t)*(u_1-\varepsilon\Delta u_0)\\
&\quad+\int_0^t D_x^\alpha G(x,t-s)*\Delta(f(u(x,s)))ds \qquad (5.59)\\
&=J_1+J_2+J_3. \qquad (5.60)
\end{aligned}
$$

In what follows, we will obtain the estimates for J_1,J_2 and J_3 respectively. First we notice that, since $u_0(x)$ has compact support, we have

$$
\left|D^k u_0(x)\right|\leq C(1+x^2)^{-s},\ k=0,1,2,\cdots,
\tag{5.61}
$$

for any positive numbers s. If we choose $s\geq N$, then

$$
\begin{aligned}
|K(x,t)*u_0|&=\left|O(1)e^{-\frac{1}{2}\varepsilon R^2 t}F(x,t)*u_0\right|\\
&=\left|O(1)e^{-\frac{1}{2}\varepsilon R^2 t}\sum_{k=0}^{(\alpha+3h-3)_+}D_x^k\delta(x)*u_0\right|\\
&=\left|O(1)e^{-\frac{1}{2}\varepsilon R^2 t}\sum_{k=0}^{(\alpha+3h-3)_+}u_0^{(k)}(x)\right|
\end{aligned}
$$

$$\begin{aligned}
&\le Ce^{-\frac{1}{2}\varepsilon R^2 t}(1+x^2)^{-s} \\
&\le Ce^{-\frac{1}{2}\varepsilon R^2 t}\left(1+\frac{x^2}{1+t^2}\right)^{-s}.
\end{aligned} \tag{5.62}$$

Then according to Lemma 2.6 and (5.62), we have

$$\begin{aligned}
|J_1| &= |D_x^\alpha \partial_t G(x,t)*u_0| \\
&= |(D_x^\alpha \partial_t G(x,t) - K(x,t) + K(x,t))*u_0| \\
&\le C(1+t)^{-(n+|\alpha|)/2}\left|\int_{\mathbb{R}^n} B_N(|x-y|,t)u_0(y)dy\right| + |K(x,t)*u_0| \\
&\le C(1+t)^{-(n+|\alpha|)/2}\left|\int_{\mathbb{R}^n} B_N(|x-y|,t)(1+|y|^2)^{-s}dy\right| + Ce^{-\frac{1}{2}\varepsilon R^2 t}\left(1+\frac{x^2}{1+t^2}\right)^{-s} \\
&\le C(1+t)^{-\frac{n+|\alpha|}{2}} B_N(|x|,t) + C(1+t)^{-\frac{n+|\alpha|-1}{2}}\left(1+\frac{x^2}{1+t^2}\right)^{-s} \\
&\le C(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t).
\end{aligned} \tag{5.63}$$

In the same way, we can also compute that

$$\begin{aligned}
|J_2| &= |D_x^\alpha G(x,t)*(u_1-\varepsilon\Delta u_0)| \\
&= |(D_x^\alpha G(x,t) - K(x,t) + K(x,t))*(u_1-\varepsilon\Delta u_0)| \\
&\le |(D_x^\alpha G(x,t) - K(x,t))*(u_1-\varepsilon\Delta u_0)| + |K(x,t)*(u_1-\varepsilon\Delta u_0)| \\
&\le C(1+t)^{-\frac{n+|\alpha|-1}{2}}\left|\int_{\mathbb{R}^n} B_N(|x-y|,t)(u_1(y)-\varepsilon\Delta u_0(y))dy\right| \\
&\quad + \left|O(1)e^{-\frac{1}{2}\varepsilon R^2 t}\sum_{k=0}^{(\alpha+3h-3)_+}\left(u_1^{(k)}(x) - \varepsilon u_0^{(k+2)}(x)\right)\right| \\
&\le C(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t) + C(1+t)^{-\frac{n+|\alpha|-1}{2}}\left(1+\frac{x^2}{1+t^2}\right)^{-s} \\
&\le C(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t).
\end{aligned} \tag{5.64}$$

In order to obtain the estimate for J_3, we set

$$\phi(x,t) = (1+t)^{\frac{n+|\alpha|-1}{2}}(B_N(|x|,t))^{-1}, \tag{5.65}$$

and

$$\Phi(t) = \sup_{0\le s\le t,\, x\in\mathbb{R}^n} |D_x^\alpha u(x,s)|\phi(x,s). \tag{5.66}$$

Then we have

$$|D_x^\alpha u(x,t)| \le \Phi(t)(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t). \tag{5.67}$$

And since $f(u) = O(u^\gamma)$, after some direct computation, there holds

$$|D_x^\alpha f(u(x,t))| \le \Phi^\gamma(t)(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t). \tag{5.68}$$

From inequality (2.24) in Lemma 2.7, we have

$$\begin{aligned}
&\left|\int_0^t (D_x^\alpha \Delta G(x,t-s)-K(x,t-s))*(f(u(x,s)))ds\right| \\
\le& C\left|\int_0^t (1+t-s)^{-\frac{n+|\alpha|+1}{2}} B_N(|x|,t-s)\right. \\
&\left. * \left(\Phi^\gamma(s)(1+s)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,s)\right) ds\right| \\
\le& C\Phi^\gamma(t)\left|\int_0^t (1+t-s)^{-\frac{n+|\alpha|+1}{2}} (1+s)^{-(n+|\alpha|-1)/2}\right. \\
&\left. \cdot \int_{\mathbb{R}^n} B_N(|x-y|,t-s) B_N(|y|,s) dy ds\right| \\
\le& C\Phi^\gamma(t)(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t).
\end{aligned} \tag{5.69}$$

Meanwhile according the definition of $K(x,t)$, there holds

$$\begin{aligned}
&\left|\int_0^t K(x,t-s)*f(u(x,s))ds\right| \\
=&\left|\int_0^t O(1)e^{-\frac{1}{2}\varepsilon R^2(t-s)} F(x,t-s)*f(u(x,s))ds\right| \\
=&\left|\int_0^t O(1)e^{-\frac{1}{2}\varepsilon R^2(t-s)} \sum_{k=0}^{(\alpha+3h-3)_+} D_x^k \delta(x)*f(u(x,s))ds\right| \\
=&\left|\int_0^t O(1)e^{-\frac{1}{2}\varepsilon R^2(t-s)} \sum_{k=0}^{(\alpha+3h-3)_+} f^{(k)}(u(x,s))ds\right| \\
\le& C\left|\int_0^t (1+t-s)^{-\frac{1}{2}} \Phi^\gamma(s)(1+s)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,s)ds\right| \\
\le& C\Phi^\gamma(t) B_N(|x|,t)\left|\int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\frac{n+|\alpha|-1}{2}} ds\right| \\
\le& C\Phi^\gamma(t)(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t).
\end{aligned} \tag{5.70}$$

Thus combining inequalities (5.69) and (5.70) together, we have

$$\begin{aligned}
|J_3| =& \left|\int_0^t D_x^\alpha G(x,t-s)*\Delta(f(u(x,s)))ds\right| \\
=& \left|\int_0^t (D_x^\alpha \Delta G(x,t-s)-K(x,t-s))*(f(u(x,s)))ds\right| \\
&+\left|\int_0^t K(x,t-s)*f(u(x,s))ds\right| \\
\le& C\Phi^\gamma(t)(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t).
\end{aligned} \tag{5.71}$$

In the end, from inequalities (5.59) (5.63) (5.64) and (5.71), there holds

$$|D_x^\alpha u(x,t)| \le C(1+\Phi^\gamma(t))(1+t)^{-\frac{n+|\alpha|-1}{2}} B_N(|x|,t). \tag{5.72}$$

By virtue of equation (5.66), we can conclude that

$$\Phi(t)\le C(1+\Phi^{\gamma}(t)). \tag{5.73}$$

Since we choose $s\ge N$, there holds

$$\Phi(0)=\sup_{0\le s\le t,x\in\mathbb{R}^n}|D_x^\alpha u(x,0)|(1+|x|^2)^N\le C(1+x^2)^{N-s}\le 1. \tag{5.74}$$

Thus using(5.73) and the continuity of $\Phi(t)$, we assert $\Phi(t)\le 1$. Then from estimate (5.74), we can complete that

$$|D_x^\alpha u(x,t)|\le C(1+t)^{-\frac{n+|\alpha|-1}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-N}. \tag{5.75}$$

Thus the proof of the Theorem 5.2 is completed. □

6. Conclusions

Since the sixth-order Boussinesq equation was derived in the shallow fluid layers and nonlinear atomic chains and was also proposed in modeling the nonlinear lattice dynamics in elastic crystals, it has significant physical backgrounds and practical meaning. Thus it is an interesting topic for physicists, mathematicians and other researchers. From the view point of mathematics, some research have been done for the equation in one dimension space. However, there are few works on problem for the sixth-order Boussinesq equation in multidimension. Thus in this paper, we studied the multidimensional generalized sixth-order Boussinesq equation mathematically. We applied the long wave-short wave decomposition method, energy method and the Green's function to obtain that the Cauchy problem for the generalized sixth-order Boussinesq equation (1.1) has a unique global classical solution $u(x,t)\in X_{l,M}$. And what's more, we made use of the Fourier analysis and the method of Green's function to derive the pointwise estimates of the solutions for problem (1.1), which concludes that $|D_x^\alpha u(x,t)|\le C(1+t)^{-\frac{n+|\alpha|-1}{2}}\left(1+\frac{|x|^2}{1+t}\right)^{-N}$ for $N>\left[\frac{n}{2}\right]+1$.

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