

GLOBAL WEAK SOLUTIONS TO A STRONGLY DEGENERATE HAPTOTAXIS MODEL*

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Abstract. We consider a one-dimensional version of a model obtained in [C. Engwer, A. Hunt, and C. Surulescu, *IMA J. Math. Med. Biol.*, 33(4):435–459, 2016] and describing the anisotropic spread of tumor cells in a tissue network. The model consists of a reaction-diffusion-taxis equation for the density of tumor cells coupled with an ODE for the density of tissue fibers and allows for strong degeneracy both in the diffusion and the haptotaxis terms. In this setting we prove the global existence of weak solutions to an associated no-flux initial-boundary value problem. Numerical simulations are performed in order to illustrate the model behavior.

Keywords. haptotaxis; degenerate diffusion; global existence.

AMS subject classifications. 35K65, 35K51, 35K57 (primary); 35D30, 35K55, 92C17, 35Q30, 35Q92 (secondary).

1. Introduction

Models with degenerate diffusion in the context of taxis equations have received increased interest during the last decade. They describe the dynamics of a cell population in response to a chemoattractant [7, 14, 20], moving up the gradient of an insoluble signal (haptotaxis) [24], or performing both chemo- and haptotaxis [15, 19, 22].

In this work we consider a reaction-diffusion-transport-haptotaxis model which is inspired by the effective equations obtained in [9] via parabolic scaling upon starting from a multiscale model for glioma invasion in the anisotropic brain tissue and relying on the setting introduced in [8]. More precisely, the following PDE-ODE model was considered for the density function $p(t, x, v, y)$ of glioma cells depending on time t , position $x \in \mathbb{R}^n$, velocity $v \in V := s\mathbb{S}^{n-1}$, and density $y \in Y := (0, R_0)$ of cell surface receptors¹ bound to tissue fibers, and for the subcellular dynamics simplified to mass action kinetics of the mentioned receptor binding:

$$\partial_t p + \nabla_x \cdot (vp) + \nabla_y \cdot (G(y, w)p) = \mathcal{L}[\lambda]p + \mathcal{P}(p) \tag{1.1}$$

$$\dot{y} = G(y, w). \tag{1.2}$$

Thereby, $w(x)$ represents the (macroscopic) volume fraction of tissue, the turning operator $\mathcal{L}[\lambda]p := -\lambda(y)p + \int_V \lambda(y)K(x, v, v')p(v')dv'$ describes the reorientation of cells due to contact guidance by tissue, and the term $\mathcal{P}(p) := \mu(x, \bar{p}, v) \int_Y \chi(x, y, y')p(t, x, v, y')w(x)dy'$ models proliferation subsequent to cell-tissue interactions. The function $\lambda(y)$ denotes the cell reorientation rate, $K(x, v, v')$ is the turning kernel depending on the directional distribution $q(x, v)$ of tissue fibers (obtained from diffusion tensor imaging data), μ represents the proliferation rate depending on the macroscopic cell density $\bar{p} = \int_V \int_Y p(t, x, v, y)dydv$, and χ is a kernel characterizing the transition from the state y to the state y' during a proliferative action.

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¹ R_0 denotes the total amount of receptors, assumed to be constant

An appropriate parabolic scaling led to the macroscopic equation for (an approximation of) the tumor cell density:

$$\partial_t u - \nabla \nabla : (\mathbb{D}_T u) + \nabla \cdot (a(w) \mathbb{D}_T \nabla w u) = w \mu(x, u) u, \quad (1.3)$$

where $a(w)$ is a function containing both macroscopic and subcellular level information, $\mathbb{D}_T = \text{const} \int_V q v \otimes v dv$ is the tumor diffusion tensor encrypting the medical data about the structure of brain tissue, and

$$\nabla \nabla : (\mathbb{D}_T u) = \nabla \cdot (\mathbb{D}_T(x) \nabla u) + \nabla \cdot (\zeta(x) u) \quad (1.4)$$

with the drift velocity $\zeta(x) = \text{const} \int_V v \otimes v \nabla q dv$. For more details and the precise definitions we refer to [9].

Equation (1.3) is of the reaction-diffusion-transport-(hapto)taxis type and characterizes the evolution of the tumor cell density for a known underlying structure of brain tissue; in practice, the functions q and w are assessed at a certain time point t from medical data. This facilitates both its mathematical analysis and efficient numerical handling, however in fact the tumor evolution in a patient also induces dynamical changes in the tissue such as e.g. depletion or remodeling, which play an essential role in the disease development, see e.g. [2, 17] and the references therein. Therefore, a further equation is needed to describe these tissue modifications under the influence of tumor cells. Although in practice it is not feasible from the viewpoint of medical imaging to assess the tissue structure dynamically, by way of model-based predictions relying on such PDE-ODE coupled systems it is possible to use a sequence of just a few images in order to obtain via numerical simulations a good approximation of the dynamics over the whole timespan of interest.

Another issue is related to possible (local) degeneracies of the tumor diffusion tensor $\mathbb{D}_T(x)$, which is particularly relevant e.g. when modeling resected or irradiated regions of the tumor, where the tissue has been depleted as well. In the respective domains, this indeed reduces the otherwise diffusion-dominated PDE (1.3) to a hyperbolic transport equation with nonlinear source term.

The mathematically quite delicate features of such strongly degenerate systems become manifest already in the case when any taxis or source terms are absent, that is, when $a \equiv 0$ and $\mu \equiv 0$ in equation (1.3). Indeed, in [11] the linear scalar parabolic equation

$$\partial_t u = (d_1(y)u)_{xx} + (d_2(y)u)_{yy}, \quad (x, y) \in \Omega = (0, L_x) \times (0, L_y), \quad t > 0, \quad (1.5)$$

has been studied, motivated among others by a monoscale model for anisotropic glioma spread in [16], and it was shown there that if the functions d_1 and d_2 are smooth and nonnegative and such that d_1 is strictly positive but d_2 vanishes precisely in some subinterval $[a, b]$ of $(0, L_y)$, then solutions to an associated no-flux initial-boundary value problem asymptotically approach a singular state reflecting concentration of mass within the degeneracy region $[0, L_x] \times [a, b]$ and extinction outside.

In this paper we intend to provide a first step toward a mathematical understanding of corresponding systems when beyond such strongly degenerate diffusion processes, further crucial mechanisms and especially nonlinear haptotaxis are involved. In order to concentrate on essential aspects of such types of interplay within the framework of a model that captures the essential properties but beyond that remains as simple as possible, we may restrict to the spatially one-dimensional case, in which the tumor diffusion tensor \mathbb{D}_T in equation (1.3) actually reduces to a scalar function. In the

context of a simple evolution law for the haptotactic attractant, particularly neglecting remodeling mechanisms, this leads to coupled parabolic-ODE systems of the form

$$\begin{cases} u_t = (d(x)u)_{xx} - (d(x)u\psi(v)v_x)_x, \\ v_t = -uh(v), \end{cases} \tag{1.6}$$

with given nonnegative functions d, ψ and h .

Although in our current 1D setting (1.6) the model in [9] loses most of its anisotropy relevance, some of it is retained in the space-dependent diffusion and haptotactic sensitivity coefficients. Likewise, the multiscale dynamics considered in [9] and leading to a haptotactic coefficient depending on the subcellular dynamics can still be partially retained in this model, in spite of the modified transport term, in which the drift velocity has now a simpler form, yet depending on $d(x)$. The very presence of the haptotaxis term is a consequence of taking the receptor binding dynamics into account when describing the evolution of the cell density function on the mesoscopic level and scaling up to the macroscopic one. Hence, essential features of the model obtained in [9] are preserved even in this simplified, dimension-reduced setting.

Other related models featuring degenerate diffusion in the context of haptotaxis were proposed and investigated in [23, 24]. The kind of degeneracy considered there is, however, different from the one in this and previous models, as it affects both the diffusion and the haptotaxis coefficients, thereby allowing the diffusion to degenerate due to one or both solution components (tumor cell density and tissue density). The model in [23] involves two subpopulations of tumor cells, differentiating between moving and proliferating² ones, but allowing mutual transitions. Unlike the present model, in [23, 24] there is (apart from the taxis) no other transport term.

2. Problem setup and main result

In order to make the essential mathematical aspects of the system (1.6) more transparent, let us write the system (1.6) in a form involving a constant haptotactic sensitivity, which according to the simple ODE structure of the second equation therein can readily be achieved on substituting $w = \Psi(v)$ with $\Psi(v) := \int_0^v \psi(\sigma) d\sigma, v \geq 0$. Accordingly, in an open bounded interval $\Omega \subset \mathbb{R}$ we will henceforth consider the initial-boundary value problem

$$\begin{cases} u_t = (d(x)u)_{xx} - (d(x)uw_x)_x + uf(x, u, w), & x \in \Omega, t > 0, \\ w_t = -ug(w), & x \in \Omega, t > 0, \\ (d(x)u)_x - d(x)uw_x = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad w(x, 0) = w_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

with given parameter functions $d: \bar{\Omega} \rightarrow [0, \infty), f: \bar{\Omega} \times [0, \infty)^2 \rightarrow \mathbb{R}$ and $g: [0, \infty) \rightarrow [0, \infty)$ satisfying

$$\sqrt{d} \in W^{1, \infty}(\Omega), \quad f \in C^1(\bar{\Omega} \times [0, \infty)^2) \quad \text{and} \quad g \in C^1([0, \infty)), \tag{2.2}$$

and with prescribed initial data u_0 and w_0 which are such that

$$\begin{cases} 0 \leq u_0 \in C^0(\bar{\Omega}) \text{ satisfies } u_0 \not\equiv 0 \quad \text{and} \\ 0 \leq w_0 \in W^{1, 2}(\Omega) \text{ has the property that } \int_{\Omega} \frac{w_0^2}{g(w_0)} < \infty. \end{cases} \tag{2.3}$$

²whence in virtue of the go-or-grow dichotomy assumed to be non-motile

As for the parameter functions in the system (2.1), throughout our analysis we shall furthermore assume that

$$f(x, u, w) \leq \rho(w) \text{ for all } (x, u, w) \in \bar{\Omega} \times [0, \infty)^2 \text{ with a nondecreasing } \rho: [0, \infty) \rightarrow [0, \infty), \tag{2.4}$$

and that there exists $\delta > 0$ such that writing

$$M := \|w_0\|_{L^\infty(\Omega)} + \delta, \tag{2.5}$$

we have

$$g(0) = 0, \quad g(w) > 0 \text{ for all } w \in (0, M] \quad \text{and} \quad g'(w) > 0 \text{ for all } w \in [0, M] \tag{2.6}$$

as well as

$$\liminf_{w \searrow 0} \frac{g'(w)}{g(w)} > 0, \tag{2.7}$$

whence in particular there exist $\Gamma > 0$ and $\gamma > 0$ fulfilling

$$g(w) \leq \Gamma w \text{ for all } w \in [0, M] \tag{2.8}$$

and

$$\frac{g'(w)}{g(w)} \geq \gamma \text{ for all } w \in (0, M]. \tag{2.9}$$

Beyond the analytically simplest case obtained on letting

$$g(w) = w, \quad w \geq 0,$$

this inter alia includes more general choices such as

$$g(w) = w(1 - w), \quad w \geq 0,$$

upon which via the substitution $w = \frac{v}{1+v}$, on the set of solutions fulfilling $v < 1$ the system (2.1) becomes formally equivalent to a corresponding initial-boundary value problem for the special version

$$\begin{cases} u_t = (d(x)u)_{xx} - \left(\frac{d(x)u}{(1+v)^2}v_x\right)_x, \\ v_t = -uv, \end{cases} \tag{2.10}$$

of the system (1.6), as proposed in [24] for modeling tumor invasion in a tissue network, thereby paying increased attention to the form of the haptotaxis coefficient. Specifically, the latter accounts for microscopic cell-tissue interactions, which—besides having a haptotaxis term at all—retains a supplementary trace of multiscality in our macroscopic model, although in a rather indirect way, as we do not explicitly couple some ODE for receptor binding kinetics to the two PDEs for u and v . The presence of $d(x)$ in both diffusion/transport and haptotaxis coefficients is motivated by the deduction in [9].

The main results of our analysis indicate that even in this general setting, thus allowing for virtually arbitrary strength of degeneracies in diffusion, haptotactic cross-diffusion does not result in a finite-time collapse of solutions into e.g. persistent Dirac-type singularities. More precisely, let us introduce the following solution concept to

pursued below, in which we use the abbreviation $\{d > 0\} := \{x \in \bar{\Omega} \mid d(x) > 0\}$ which along with a corresponding definition of $\{d = 0\}$ will frequently be used throughout the sequel.

DEFINITION 2.1. *A pair (u, w) of nonnegative functions*

$$\begin{cases} u \in L^1_{loc}(\bar{\Omega} \times [0, \infty)), \\ w \in L^\infty_{loc}(\bar{\Omega} \times [0, \infty)) \cap L^1_{loc}([0, \infty); W^{1,1}(\{d > 0\})) \end{cases} \tag{2.11}$$

satisfying

$$uf(\cdot, u, w) \in L^1_{loc}(\bar{\Omega} \times [0, \infty)) \quad \text{and} \quad ug(w) \in L^1_{loc}(\bar{\Omega} \times [0, \infty)) \tag{2.12}$$

as well as

$$duw_x \in L^1_{loc}([0, \infty); L^1(\{d > 0\})) \tag{2.13}$$

will be called a global weak solution of the system (2.1) if

$$\begin{aligned} - \int_0^\infty \int_\Omega u \varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) &= \int_0^\infty \int_{\{d > 0\}} du \varphi_{xx} + \int_0^\infty \int_{\{d > 0\}} duw_x \varphi_x \\ &\quad + \int_0^\infty \int_\Omega uf(\cdot, u, w) \varphi \end{aligned} \tag{2.14}$$

for all $\varphi \in C^\infty_0(\bar{\Omega} \times [0, \infty))$ such that $\varphi_x = 0$ on $\partial\Omega \times (0, \infty)$ and

$$\int_0^\infty \int_\Omega w \varphi_t + \int_\Omega w_0 \varphi(\cdot, 0) = \int_0^\infty \int_\Omega ug(w) \varphi \tag{2.15}$$

for all $\varphi \in C^\infty_0(\Omega \times [0, \infty))$.

Within this framework, a global solution of the system the system (2.1) can always be constructed:

THEOREM 2.1. *Suppose that $\Omega \subset \mathbb{R}$ is a bounded interval, and that u_0, w_0, d, f and g satisfy the conditions (2.3), (2.2), (2.4) and (2.6). Then (2.1) possesses at least one global weak solution in the sense specified in Definition 2.1 below.*

This paper is organized as follows: In Section 3 we introduce a regularized version of the degenerate problem, for which some useful properties are obtained. Section 4 is concerned with studying an entropy functional which allows to deduce a quasi-dissipative property of the regularized system, inter alia asserting global existence of its solution. Some precompactness and regularity properties of terms involved in that system follow in Sections 5 and 6, respectively, succeeded in Section 7 by regularity features of corresponding time derivatives. Sections 8 and 9 provide convergence properties of the approximate solution in the region with no degeneracy; further properties of the respective limits are obtained in Section 10. Finally, Section 11 concludes the existence proof for the strongly degenerate problem (2.1).

3. Regularized problems and their basic properties

In order to prepare the construction of an appropriate family of non-degenerate approximations of the system (2.1), according to the nonnegativity of d and the inclusion

$\sqrt{d} \in W^{1,\infty}(\Omega)$ we may first choose $(d_\varepsilon)_{\varepsilon \in (0,1)} \subset C^3(\bar{\Omega})$ in such a way that $d_{\varepsilon x} = 0$ on $\partial\Omega$ and that with some $K_1 > 0$, for each $\varepsilon \in (0,1)$ we have

$$\sqrt{\varepsilon} \leq d_\varepsilon(x) \leq \|d\|_{L^\infty(\Omega)} + 1 \quad \text{for all } x \in \bar{\Omega}, \tag{3.1}$$

as well as

$$\frac{d_{\varepsilon xx}^2(x)}{d_\varepsilon(x)} \leq K_1 \quad \text{for all } x \in \bar{\Omega}, \tag{3.2}$$

and such that moreover

$$d_\varepsilon \rightarrow d \quad \text{in } L^\infty(\Omega) \quad \text{as } \varepsilon \searrow 0 \tag{3.3}$$

and

$$d_{\varepsilon x} \rightarrow d_x \quad \text{a.e. in } \Omega \quad \text{as } \varepsilon \searrow 0. \tag{3.4}$$

We next note that according to the properties (2.6) it is possible to fix $\varepsilon_0 \in (0,1)$ such that $g(M) > \varepsilon_0$, whereupon with δ as introduced in the course of the definition (2.5) of M , for each $\varepsilon \in (0,\varepsilon_0)$ we can choose $\delta_\varepsilon \in (0,\delta^2)$ such that

$$g(w) \geq \varepsilon \quad \text{for all } w \in [\delta_\varepsilon, M], \tag{3.5}$$

and such that moreover $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \searrow 0$. It is then easy to see that one can find $(\eta_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)} \subset (0,1)$ with the two properties that

$$\eta_\varepsilon \ln \frac{1}{\sqrt{\delta_\varepsilon}} \rightarrow +\infty \quad \text{as } \varepsilon \searrow 0, \tag{3.6}$$

and that

$$\eta_\varepsilon \rightarrow 0 \quad \text{as } \varepsilon \searrow 0; \tag{3.7}$$

indeed, it can readily be checked that this can be achieved on choosing

$$\eta_\varepsilon := \frac{\ln \ln \frac{A}{\sqrt{\delta_\varepsilon}}}{\ln \frac{A}{\sqrt{\delta_\varepsilon}}}, \quad \varepsilon \in (0,\varepsilon_0),$$

with some suitably large $A > 0$. For $\varepsilon \in (0,\varepsilon_0)$, we then let

$$w_{0\varepsilon}(x) := w_0(x) + \sqrt{\delta_\varepsilon}, \quad x \in \bar{\Omega}, \tag{3.8}$$

and consider the regularized variant of the system (2.1) given by

$$\begin{cases} u_{\varepsilon t} = (d_\varepsilon u_\varepsilon)_{xx} - \left(d_\varepsilon \frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} \right)_x + u_\varepsilon f(x, u_\varepsilon, w_\varepsilon), & x \in \Omega, \ t > 0, \\ w_{\varepsilon t} = \varepsilon \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \right)_x - \frac{u_\varepsilon}{1+\eta_\varepsilon u_\varepsilon} g(w_\varepsilon), & x \in \Omega, \ t > 0, \\ u_{\varepsilon x} = w_{\varepsilon x} = 0, & x \in \partial\Omega, \ t > 0, \\ u_\varepsilon(x, 0) = u_0(x), \quad w_\varepsilon(x, 0) = w_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{3.9}$$

Due to the additionally introduced artificial diffusion in the equation for w_ε each of these problems can be viewed as a variant of the well-studied Keller-Segel chemotaxis system;

in fact, as can be seen by straightforward adaptation of arguments well-established in the analysis of chemotaxis problems ([1, 12, 21]), all these problems allow for local-in-time classical solutions which enjoy a favorable extensibility criterion:

LEMMA 3.1. *For each $\varepsilon \in (0, \varepsilon_0)$, there exist $T_{max,\varepsilon} \in (0, \infty]$ and nonnegative functions*

$$\begin{cases} u_\varepsilon \in C^0(\bar{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \\ w_\varepsilon \in C^0([0, T_{max,\varepsilon}); W^{1,2}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max,\varepsilon})), \end{cases}$$

which solve the system (3.9) in the classical sense in $\Omega \times (0, T_{max,\varepsilon})$, and which are such that if $T_{max,\varepsilon} < \infty$, then

$$\limsup_{t \nearrow T_{max,\varepsilon}} \left(\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|w_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} + \left\| \frac{1}{g(w_\varepsilon(\cdot, t))} \right\|_{L^\infty(\Omega)} \right) = \infty. \tag{3.10}$$

Let us first collect some basic properties of these solutions. We first assert some useful pointwise upper and lower bounds for w_ε .

LEMMA 3.2. *Let $\varepsilon \in (0, \varepsilon_0)$. Then*

$$w_\varepsilon(x, t) \leq M \quad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max,\varepsilon}) \tag{3.11}$$

and

$$w_\varepsilon(x, t) \geq \sqrt{\delta_\varepsilon} e^{-\frac{\Gamma}{\eta_\varepsilon} t} \quad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max,\varepsilon}), \tag{3.12}$$

where $\Gamma > 0$ is as in the inequality (2.8).

Proof. Since according to our choices of δ_ε and M we have

$$w_\varepsilon(x, 0) = w_0(x) + \sqrt{\delta_\varepsilon} \leq \|w_0\|_{L^\infty(\Omega)} + \delta = M \quad \text{for all } x \in \Omega,$$

the inequality in (3.11) immediately results from the maximum principle applied to the second equation in (3.9). As a consequence thereof, in view of the properties (2.6) we know that $g'(w_\varepsilon) \geq 0$ in $\Omega \times (0, T_{max,\varepsilon})$, whence

$$\frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \leq \frac{1}{\eta_\varepsilon} g(w_\varepsilon) \quad \text{in } \Omega \times (0, T_{max,\varepsilon}),$$

so that using the system (3.9) and the inequality (2.8) we see that

$$w_{\varepsilon t} \geq \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \right)_x - \frac{\Gamma}{\eta_\varepsilon} w_\varepsilon \quad \text{in } \Omega \times (0, T_{max,\varepsilon}).$$

Since

$$\underline{w}(x, t) := \sqrt{\delta_\varepsilon} e^{-\frac{\Gamma}{\eta_\varepsilon} t}, \quad x \in \bar{\Omega}, \quad t \geq 0,$$

satisfies

$$\underline{w}_t - \left(\frac{\underline{w}_x}{\sqrt{g(\underline{w})}} \right)_x + \frac{\Gamma}{\eta_\varepsilon} \underline{w} = 0 \quad \text{in } \Omega \times (0, \infty)$$

and $\frac{\partial \underline{w}}{\partial \nu} = 0$ on $\partial\Omega \times (0, \infty)$ as well as

$$\underline{w}(x, 0) = \sqrt{\delta_\varepsilon} \leq w_\varepsilon(x, 0) \quad \text{for all } x \in \Omega$$

by the definition (3.8), the comparison principle therefore ensures that $w_\varepsilon \geq \underline{w}$ in $\Omega \times (0, T_{max,\varepsilon})$ and that thus also the inequality (3.12) is valid. \square

Using the latter along with the inequality (2.4), we easily obtain the following information on the evolution of $\int_\Omega u_\varepsilon$.

LEMMA 3.3. *With M as in definition (2.5) and ρ taken from the inequality (2.4), we have*

$$\frac{d}{dt} \int_\Omega u_\varepsilon \leq \rho(M) \int_\Omega u_\varepsilon \quad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{3.13}$$

and

$$\int_\Omega u_\varepsilon(\cdot, t) \leq \left\{ \int_\Omega u_0 \right\} \cdot e^{\rho(M)t} \quad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{3.14}$$

Proof. We integrate the first equation in (3.9) and use the inequality (2.4) together with the inequality (3.11) to find that

$$\frac{d}{dt} \int_\Omega u_\varepsilon = \int_\Omega u_\varepsilon f(x, u_\varepsilon, w_\varepsilon) \leq \int_\Omega u_\varepsilon \rho(w_\varepsilon) \leq \rho(M) \int_\Omega u_\varepsilon \quad \text{for all } t \in (0, T_{max,\varepsilon}),$$

and that hence the estimate (3.13) holds, from which in turn the estimate (3.14) results upon integration in time. \square

4. Implications of an entropy-like structure

Now the core of our approach consists in the detection of a favorable quasi-dissipative property of the system (3.9) which can be revealed by following the well-established strategy of considering the time evolution of a functional that combines a logarithmic entropy of the cell distribution with a properly chosen summand annihilating the correspondingly obtained cross-diffusive interaction integral. In order to clarify which precise form the latter takes in the context of the approximate problems (3.9), let us begin by separately tracking the logarithmic entropy.

LEMMA 4.1. *Let ρ, M and K_1 be as introduced in the expressions (2.4), (2.5) and (3.2). Then for each $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\begin{aligned} \frac{d}{dt} \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2} \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} &\leq \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}}{(1 + \eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} + \left(\rho(M) + \frac{K_1}{2} \right) \cdot \left\{ \int_\Omega u_0 \right\} \cdot e^{\rho(M)t} \\ &\quad + \int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x, u_\varepsilon, w_\varepsilon) \quad \text{for all } t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{4.1}$$

Proof. Since u_ε is positive in $\bar{\Omega} \times (0, T_{max,\varepsilon})$ by the strong maximum principle, we may multiply the first equation in (3.9) by $\ln u_\varepsilon$ and integrate by parts to see that

$$\begin{aligned} \frac{d}{dt} \int_\Omega u_\varepsilon \ln u_\varepsilon &= \int_\Omega u_{\varepsilon t} \ln u_\varepsilon + \frac{d}{dt} \int_\Omega u_\varepsilon \\ &= - \int_\Omega (d_\varepsilon u_\varepsilon)_x \cdot \frac{u_{\varepsilon x}}{u_\varepsilon} + \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}}{(1 + \eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} + \int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x, u_\varepsilon, w_\varepsilon) \\ &\quad + \frac{d}{dt} \int_\Omega u_\varepsilon \quad \text{for all } t \in (0, T_{max,\varepsilon}). \end{aligned} \tag{4.2}$$

Here by Lemma 3.3 we have

$$\frac{d}{dt} \int_{\Omega} u_{\varepsilon} \leq \rho(M) \int_{\Omega} u_{\varepsilon} \quad \text{for all } t \in (0, T_{max, \varepsilon}), \tag{4.3}$$

and using Young’s inequality and the inequality (3.2) we obtain

$$\begin{aligned} - \int_{\Omega} (d_{\varepsilon} u_{\varepsilon})_x \cdot \frac{u_{\varepsilon x}}{u_{\varepsilon}} &= - \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} - d_{\varepsilon x} u_{\varepsilon x} \\ &\leq - \frac{1}{2} \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} + \frac{1}{2} \int_{\Omega} \frac{d_{\varepsilon x}^2}{d_{\varepsilon}} u_{\varepsilon} \\ &\leq - \frac{1}{2} \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} + \frac{K_1}{2} \int_{\Omega} u_{\varepsilon} \quad \text{for all } t \in (0, T_{max, \varepsilon}). \end{aligned} \tag{4.4}$$

Since

$$\left(\rho(M) + \frac{K_1}{2} \right) \int_{\Omega} u_{\varepsilon} \leq \left(\rho(M) + \frac{K_1}{2} \right) \cdot \left\{ \int_{\Omega} u_0 \right\} \cdot e^{\rho(M)t} \quad \text{for all } t \in (0, T_{max, \varepsilon})$$

by Lemma 3.3, combining the results (4.2)-(4.4) thus yields the estimate (4.1). □

Thanks to a favorable exact relationship between the approximate signal absorption rate $0 \leq u \mapsto \frac{u}{1+\eta_{\varepsilon}u}$ and the tactic sensitivity $0 \leq \frac{1}{(1+\eta_{\varepsilon}u)^2}$ in (3.9), an exact compensation of the first summand on the right of the estimate (4.1) can be achieved on complementing the above by the following

LEMMA 4.2. *With K_1 as in the inequality (3.2), we have*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} d_{\varepsilon} \frac{w_{\varepsilon x}^2}{g(w_{\varepsilon})} + \frac{\varepsilon}{2} \int_{\Omega} d_{\varepsilon} \cdot \frac{1}{\sqrt{g(w_{\varepsilon})}} \cdot \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x^2 + \frac{1}{2} \int_{\Omega} d_{\varepsilon} \cdot \frac{u_{\varepsilon}}{1+\eta_{\varepsilon}u_{\varepsilon}} \cdot \frac{g'(w_{\varepsilon})}{g(w_{\varepsilon})} w_{\varepsilon x}^2 \\ &\leq - \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}}{(1+\eta_{\varepsilon}u_{\varepsilon})^2} w_{\varepsilon x} + \frac{\varepsilon K_1}{2} \int_{\Omega} \frac{w_{\varepsilon x}^2}{\sqrt{g(w_{\varepsilon})^3}} \quad \text{for all } t \in (0, T_{max, \varepsilon}) \end{aligned} \tag{4.5}$$

whenever $\varepsilon \in (0, \varepsilon_0)$.

Proof. Using that $g(w_{\varepsilon})$ is positive in $\bar{\Omega} \times [0, T_{max, \varepsilon})$ due to Lemma 3.2 and the properties (2.6), on the basis of the second equation in (3.9) we compute

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} d_{\varepsilon} \frac{w_{\varepsilon x}^2}{g(w_{\varepsilon})} &= 2 \int_{\Omega} d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} w_{\varepsilon x t} - \int_{\Omega} d_{\varepsilon} \frac{g'(w_{\varepsilon})}{g^2(w_{\varepsilon})} w_{\varepsilon x}^2 w_{\varepsilon t} \\ &= 2 \int_{\Omega} d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \cdot \left\{ \varepsilon \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_{xx} - \left(\frac{u_{\varepsilon}}{1+\eta_{\varepsilon}u_{\varepsilon}} g(w_{\varepsilon}) \right)_x \right\} \\ &\quad - \int_{\Omega} d_{\varepsilon} \frac{g'(w_{\varepsilon})}{g^2(w_{\varepsilon})} w_{\varepsilon x}^2 \cdot \left\{ \varepsilon \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x - \frac{u_{\varepsilon}}{1+\eta_{\varepsilon}u_{\varepsilon}} g(w_{\varepsilon}) \right\} \\ &= -2\varepsilon \int_{\Omega} \left(d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \right)_x \cdot \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x \\ &\quad - 2 \int_{\Omega} d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \cdot \left(\frac{u_{\varepsilon}}{1+\eta_{\varepsilon}u_{\varepsilon}} g(w_{\varepsilon}) \right)_x - \varepsilon \int_{\Omega} d_{\varepsilon} \frac{g'(w_{\varepsilon})}{g^2(w_{\varepsilon})} w_{\varepsilon x}^2 \cdot \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x \\ &\quad + \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon}}{1+\eta_{\varepsilon}u_{\varepsilon}} \frac{g'(w_{\varepsilon})}{g(w_{\varepsilon})} w_{\varepsilon x}^2 \quad \text{for all } t \in (0, T_{max, \varepsilon}). \end{aligned} \tag{4.6}$$

Here we expand

$$-2 \int_{\Omega} d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \cdot \left(\frac{u_{\varepsilon}}{1 + \eta_{\varepsilon} u_{\varepsilon}} g(w_{\varepsilon}) \right)_x = -2 \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}}{(1 + \eta_{\varepsilon} u_{\varepsilon})^2} w_{\varepsilon x} - 2 \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon}}{1 + \eta_{\varepsilon} u_{\varepsilon}} \frac{g'(w_{\varepsilon})}{g(w_{\varepsilon})} w_{\varepsilon x}^2$$

for all $t \in (0, T_{max, \varepsilon})$, so that

$$\begin{aligned} & -2 \int_{\Omega} d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \cdot \left(\frac{u_{\varepsilon}}{1 + \eta_{\varepsilon} u_{\varepsilon}} g(w_{\varepsilon}) \right)_x + \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon}}{1 + \eta_{\varepsilon} u_{\varepsilon}} \frac{g'(w_{\varepsilon})}{g(w_{\varepsilon})} w_{\varepsilon x}^2 \\ & = -2 \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}}{(1 + \eta_{\varepsilon} u_{\varepsilon})^2} w_{\varepsilon x} - \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon}}{1 + \eta_{\varepsilon} u_{\varepsilon}} \frac{g'(w_{\varepsilon})}{g(w_{\varepsilon})} w_{\varepsilon x}^2 \quad \text{for all } t \in (0, T_{max, \varepsilon}). \end{aligned} \tag{4.7}$$

We next use the identity

$$w_{\varepsilon x x} = \sqrt{g(w_{\varepsilon})} \cdot \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x + \frac{g'(w_{\varepsilon})}{2g(w_{\varepsilon})} w_{\varepsilon x}^2 \quad \text{in } \Omega \times (0, T_{max, \varepsilon}) \tag{4.8}$$

to rewrite

$$\begin{aligned} \left(d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \right)_x & = d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x x} - d_{\varepsilon} \frac{g'(w_{\varepsilon})}{g^2(w_{\varepsilon})} w_{\varepsilon x}^2 + d_{\varepsilon x} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \\ & = d_{\varepsilon} \frac{1}{\sqrt{g(w_{\varepsilon})}} \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x - \frac{1}{2} d_{\varepsilon} \frac{g'(w_{\varepsilon})}{g^2(w_{\varepsilon})} w_{\varepsilon x}^2 \\ & \quad + d_{\varepsilon x} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \quad \text{in } \Omega \times (0, T_{max, \varepsilon}), \end{aligned}$$

so that on the right-hand side of the equality (4.6) we can employ Young’s inequality to see that

$$\begin{aligned} & -2\varepsilon \int_{\Omega} \left(d_{\varepsilon} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \right)_x \cdot \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x - \varepsilon \int_{\Omega} d_{\varepsilon} \frac{g'(w_{\varepsilon})}{g^2(w_{\varepsilon})} w_{\varepsilon x}^2 \cdot \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x \\ & = -2\varepsilon \int_{\Omega} d_{\varepsilon} \frac{1}{\sqrt{g(w_{\varepsilon})}} \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x^2 - 2\varepsilon \int_{\Omega} d_{\varepsilon x} \frac{1}{g(w_{\varepsilon})} w_{\varepsilon x} \cdot \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x \\ & \leq -\varepsilon \int_{\Omega} d_{\varepsilon} \frac{1}{\sqrt{g(w_{\varepsilon})}} \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x^2 + \varepsilon \int_{\Omega} \frac{d_{\varepsilon x}^2}{d_{\varepsilon}} \frac{1}{\sqrt{g(w_{\varepsilon})}^3} w_{\varepsilon x}^2 \\ & \leq -\varepsilon \int_{\Omega} d_{\varepsilon} \frac{1}{\sqrt{g(w_{\varepsilon})}} \left(\frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \right)_x^2 + \varepsilon K_1 \int_{\Omega} \frac{1}{\sqrt{g(w_{\varepsilon})}^3} w_{\varepsilon x}^2 \quad \text{for all } t \in (0, T_{max, \varepsilon}), \end{aligned}$$

again due to the inequality (3.2). In conjunction with the equalities (4.7) and (4.6) this yields the estimate (4.5). □

In fact, combining the latter two lemmata yields a quasi-entropy inequality, the essential implications of which can be summarized as follows.

LEMMA 4.3. *Let $T > 0$. Then there exist $\varepsilon_{\star}(T) \in (0, \varepsilon_0)$ and $C(T) > 0$ such that for any choice of $\varepsilon \in (0, \varepsilon_{\star}(T))$, the solution of the system (3.9) satisfies*

$$\int_{\{u_{\varepsilon}(\cdot, t) \geq 1\}} u_{\varepsilon}(\cdot, t) \ln u_{\varepsilon}(\cdot, t) \leq C(T) \quad \text{for all } t \in (0, \widehat{T}_{\varepsilon}) \tag{4.9}$$

and

$$\int_{\Omega} d_{\varepsilon} \frac{w_{\varepsilon x}^2(\cdot, t)}{g(w_{\varepsilon}(\cdot, t))} \leq C(T) \quad \text{for all } t \in (0, \widehat{T}_{\varepsilon}), \tag{4.10}$$

and such that moreover

$$\int_0^{\widehat{T}_\varepsilon} \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \leq C(T) \tag{4.11}$$

and

$$\int_0^{\widehat{T}_\varepsilon} \int_{\{u_\varepsilon(\cdot, t) \geq 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(\cdot, u_\varepsilon, w_\varepsilon) \leq C(T) \tag{4.12}$$

as well as

$$\int_0^{\widehat{T}_\varepsilon} \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon} \frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \leq C(T), \tag{4.13}$$

where with $T_{max, \varepsilon}$ as in Lemma 3.1 we have set $\widehat{T}_\varepsilon := \min\{T, T_{max, \varepsilon}\}$.

Proof. We add the inequalities provided by Lemma 4.1 and Lemma 4.2 to see on dropping a nonnegative summand on the right that for all $\varepsilon \in (0, \varepsilon_0)$ we have

$$\begin{aligned} & \frac{d}{dt} \left\{ \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{1}{2} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \right\} + \frac{1}{2} \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} + \frac{1}{2} \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon} \frac{g'(w_\varepsilon)}{g(w_\varepsilon)} w_{\varepsilon x}^2 \\ & \leq c_1 + \int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x, u_\varepsilon, w_\varepsilon) + \frac{\varepsilon K_1}{2} \int_\Omega \frac{w_{\varepsilon x}^2}{\sqrt{g(w_\varepsilon)}^3} \quad \text{for all } t \in (0, T_{max, \varepsilon}) \end{aligned} \tag{4.14}$$

with $c_1 \equiv c_1(T) := \left(\rho(M) + \frac{K_1}{2}\right) \cdot \left\{ \int_\Omega u_0 \right\} \cdot e^{\rho(M)T}$. Here we split $f = f_+ - f_-$ and

$$\begin{aligned} \int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x, u_\varepsilon, w_\varepsilon) &= \int_{\{u_\varepsilon < 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_+(x, u_\varepsilon, w_\varepsilon) - \int_{\{u_\varepsilon < 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(x, u_\varepsilon, w_\varepsilon) \\ & \quad + \int_{\{u_\varepsilon \geq 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_+(x, u_\varepsilon, w_\varepsilon) \\ & \quad - \int_{\{u_\varepsilon \geq 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(x, u_\varepsilon, w_\varepsilon) \end{aligned} \tag{4.15}$$

for $t \in (0, T_{max, \varepsilon})$, where clearly

$$\int_{\{u_\varepsilon < 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_+(x, u_\varepsilon, w_\varepsilon) \leq 0 \quad \text{for all } t \in (0, T_{max, \varepsilon}), \tag{4.16}$$

and where using that

$$\xi \ln \xi \geq -\frac{1}{e} \quad \text{for all } \xi > 0, \tag{4.17}$$

we see that

$$- \int_{\{u_\varepsilon < 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(x, u_\varepsilon, w_\varepsilon) \leq \frac{|\Omega|}{e} c_2 \quad \text{for all } t \in (0, T_{max, \varepsilon}) \tag{4.18}$$

with

$$c_2 := \max_{(x, u, w) \in \Omega \times [0, 1] \times [0, M]} f_-(x, u, w)$$

being finite by continuity of f . Since

$$f_+(\cdot, u_\varepsilon, w_\varepsilon) \leq \rho(M) \quad \text{in } \Omega \times (0, T_{max, \varepsilon}) \tag{4.19}$$

by the inequality (2.4) and Lemma 3.2, again relying on the inequality (4.17) we see that

$$\begin{aligned} \int_{\{u_\varepsilon \geq 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_+(x, u_\varepsilon, w_\varepsilon) &\leq \rho(M) \int_{\{u_\varepsilon \geq 1\}} u_\varepsilon \ln u_\varepsilon \\ &= \rho(M) \cdot \left\{ \int_\Omega u_\varepsilon \ln u_\varepsilon - \int_{\{u_\varepsilon < 1\}} u_\varepsilon \ln u_\varepsilon \right\} \\ &\leq \rho(M) \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{\rho(M)|\Omega|}{e} \quad \text{for all } t \in (0, T_{max, \varepsilon}), \end{aligned}$$

so that from the results (4.15), (4.16) and (4.18) we infer that

$$\int_\Omega u_\varepsilon \ln u_\varepsilon \cdot f(x, u_\varepsilon, w_\varepsilon) \leq \rho(M) \int_\Omega u_\varepsilon \ln u_\varepsilon - \int_{\{u_\varepsilon \geq 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(x, u_\varepsilon, w_\varepsilon) + c_3, \tag{4.20}$$

for all $t \in (0, T_{max, \varepsilon})$ and with $c_3 := \frac{|\Omega|}{e} \cdot (c_2 + \rho(M))$.

Next, the rightmost summand in the inequality (4.14) can be estimated using Lemma 3.2 along with the defining properties of $(d_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ and $(\eta_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$: Indeed, given $T > 0$ we may use the property (3.6) to fix $\varepsilon_\star = \varepsilon_\star(T) \in (0, \varepsilon_0)$ small enough such that with Γ as in the inequality (2.8) we have

$$T \leq \frac{\eta_\varepsilon}{\Gamma} \cdot \ln \frac{1}{\sqrt{\delta_\varepsilon}} \quad \text{for all } \varepsilon \in (0, \varepsilon_\star),$$

which implies that for any such ε ,

$$e^{-\frac{T}{\eta_\varepsilon} t} \geq e^{-\ln \frac{1}{\sqrt{\delta_\varepsilon}}} = \sqrt{\delta_\varepsilon} \quad \text{for all } t \in (0, T)$$

and hence, by Lemma 3.2,

$$M \geq w_\varepsilon(x, t) \geq \sqrt{\delta_\varepsilon} \cdot e^{-\frac{T}{\eta_\varepsilon} t} \geq \delta_\varepsilon \quad \text{for all } x \in \Omega \text{ and } t \in (0, \widehat{T}_\varepsilon).$$

Therefore, the inequality (3.5) applies so as to ensure that

$$g(w_\varepsilon) \geq \varepsilon \quad \text{for all } x \in \Omega \text{ and } t \in (0, \widehat{T}_\varepsilon),$$

so that the term in question satisfies

$$\begin{aligned} \frac{\varepsilon K_1}{2} \int_\Omega \frac{w_{\varepsilon x}^2}{\sqrt{g(w_\varepsilon)}^3} &\leq \frac{\sqrt{\varepsilon} K_1}{2} \int_\Omega \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \\ &\leq \frac{K_1}{2} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \quad \text{for all } t \in (0, \widehat{T}_\varepsilon), \end{aligned} \tag{4.21}$$

because $d_\varepsilon \geq \sqrt{\varepsilon}$ in Ω by the inequality (3.1). Together with the inequalities (4.20) and (4.14), this shows that

$$y_\varepsilon(t) := \int_\Omega u_\varepsilon(\cdot, t) \ln u_\varepsilon(\cdot, t) + \frac{1}{2} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2(\cdot, t)}{g(w_\varepsilon(\cdot, t))}, \quad t \in [0, T_{max, \varepsilon}),$$

and

$$\begin{aligned}
 h_\varepsilon(t) := & \frac{1}{2} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2(\cdot, t)}{u_\varepsilon(\cdot, t)} + \frac{1}{2} \int_\Omega d_\varepsilon \frac{u_\varepsilon(\cdot, t)}{1 + \eta_\varepsilon u_\varepsilon(\cdot, t)} \frac{g'(w_\varepsilon(\cdot, t))}{g(w_\varepsilon(\cdot, t))} w_{\varepsilon x}^2(\cdot, t) \\
 & + \int_{\{u_\varepsilon(\cdot, t) \geq 1\}} u_\varepsilon(\cdot, t) \ln u_\varepsilon(\cdot, t) \cdot f_-(\cdot, u_\varepsilon(\cdot, t), w_\varepsilon(\cdot, t)), \quad t \in (0, T_{max, \varepsilon}),
 \end{aligned}$$

have the property that

$$\begin{aligned}
 y'_\varepsilon(t) + h_\varepsilon(t) & \leq c_1 + c_3 + \rho(M) \int_\Omega u_\varepsilon \ln u_\varepsilon + \frac{K_1}{2} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \\
 & = c_1 + c_3 + \rho(M) \cdot \left\{ y_\varepsilon(t) - \frac{1}{2} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \right\} + K_1 \cdot \left\{ y_\varepsilon(t) - \int_\Omega u_\varepsilon \ln u_\varepsilon \right\} \\
 & \leq c_4 + c_5 y_\varepsilon(t) \quad \text{for all } t \in (0, \widehat{T}_\varepsilon)
 \end{aligned} \tag{4.22}$$

with $c_4 := c_1 + c_3 + \frac{K_1|\Omega|}{e}$ and $c_5 := \rho(M) + K_1$, where we again have used the inequality (4.17).

Now by nonnegativity of h_ε and the inequality (3.1), an integration of the inequality (4.22) firstly yields

$$\begin{aligned}
 y_\varepsilon(t) & \leq y_\varepsilon(0) e^{c_5 t} + \frac{c_4}{c_5} e^{c_5 t} (1 - e^{-c_5 t}) \\
 & \leq c_6 := \left\{ \int_\Omega u_0 \ln u_0 + \frac{1}{2} (\|d\|_{L^\infty(\Omega)} + 1) \cdot \sup_{\varepsilon \in (0, \varepsilon_0)} \int_\Omega \frac{w_{0x}^2}{g(w_0 + \delta_\varepsilon)} + \frac{c_4}{c_5} \right\} \cdot e^{c_5 T}
 \end{aligned} \tag{4.23}$$

for all $t \in [0, \widehat{T}_\varepsilon]$ and $\varepsilon \in (0, \varepsilon_*)$, where we note that c_6 is finite according to the properties (2.3), because $\delta_\varepsilon \rightarrow 0$ as $\varepsilon \searrow 0$, and because due to the fact that $g' \geq 0$ on $[0, M]$, as asserted by the properties (2.6), Beppo Levi's theorem warrants that as $k \rightarrow \infty$ we have

$$\int_\Omega \frac{w_{0x}^2}{g(w_0 + \frac{1}{k})} \nearrow \int_\Omega \frac{w_{0x}^2}{g(w_0)} < \infty.$$

Once more in view of the inequality (4.17), this entails both inequalities (4.9) and (4.10) with some suitably large $C(T) > 0$, whereas another integration of the inequality (4.22), this time making use of the inequality (4.23), shows that

$$\begin{aligned}
 \int_0^{\widehat{T}_\varepsilon} h_\varepsilon(t) dt & \leq y_\varepsilon(0) - y_\varepsilon(\widehat{T}_\varepsilon) + c_4 \widehat{T}_\varepsilon + c_5 \int_0^{\widehat{T}_\varepsilon} y_\varepsilon(t) dt \\
 & \leq c_7 := c_6 + \frac{|\Omega|}{e} + c_4 T + c_5 c_6 T,
 \end{aligned} \tag{4.24}$$

and hence that the estimates (4.10)-(4.13) hold with some possibly enlarged $C(T)$. \square

Due to the boundedness property (3.11) of w_ε , and the properties (2.6) and (2.9), from the estimates (4.10) and (4.13) we particularly obtain corresponding estimates for integrals no longer containing $\frac{1}{g(w_\varepsilon)}$ and $\frac{g'(w_\varepsilon)}{g(w_\varepsilon)}$.

COROLLARY 4.1. *Suppose that $T > 0$, and let $\varepsilon_*(T) \in (0, \varepsilon_0)$ be as given by Lemma 4.3. Then there exists $C(T) > 0$ with the property that for all $\varepsilon \in (0, \varepsilon_*(T))$,*

$$\int_\Omega d_\varepsilon w_{\varepsilon x}^2(\cdot, t) \leq C(T) \quad \text{for all } t \in (0, \widehat{T}_\varepsilon) \tag{4.25}$$

and

$$\int_0^{\widehat{T}_\varepsilon} \int_\Omega d_\varepsilon \frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon} w_{\varepsilon x}^2 \leq C(T), \tag{4.26}$$

where again $\widehat{T}_\varepsilon := \min\{T, T_{max,\varepsilon}\}$.

Proof. Since $g(w_\varepsilon) \leq g(M)$ in $\Omega \times (0, T_{max,\varepsilon})$ by Lemma 3.2 and the properties (2.6), we immediately obtain the inequality (4.25) from the inequality (4.10). As furthermore the inequality (2.9) warrants that

$$\frac{g'(w_\varepsilon)}{g(w_\varepsilon)} \geq \gamma > 0 \quad \text{in } \Omega \times (0, T_{max,\varepsilon})$$

by Lemma 3.2, we also infer from the inequality (4.13) that the inequality (4.26) is valid with some adequately large $C(T) > 0$. \square

As one consequence of (4.25) when combined with the boundedness of $\frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2}$ and the uniform positivity of d_ε , we can infer that in fact our approximate solutions cannot blow up in finite time:

LEMMA 4.4. *For each $\varepsilon \in (0, \varepsilon_0)$, the solution of the system (3.9) is global in time; that is, in Lemma 3.1 we have $T_{max,\varepsilon} = \infty$.*

Proof. Assuming on the contrary that $T_{max,\varepsilon}$ be finite, combining the bound (3.12) with the properties (2.6) we first obtain that then there would exist $c_1 > 0$ such that

$$\frac{1}{g(w_\varepsilon)} \leq c_1 \quad \text{in } \Omega \times (0, T_{max,\varepsilon}). \tag{4.27}$$

Moreover, as $d_\varepsilon > 0$ in $\bar{\Omega}$ by the inequality (3.1), Corollary 4.1 and Lemma 3.2 would yield $c_2 > 0$ fulfilling

$$\|w_\varepsilon(\cdot, t)\|_{W^{1,2}(\Omega)} \leq c_2 \quad \text{for all } t \in (0, T_{max,\varepsilon}). \tag{4.28}$$

In particular, the latter along with the inequality (3.1) and the fact that $\frac{\xi}{(1 + \eta_\varepsilon \xi)^2} \leq \frac{1}{4\eta_\varepsilon}$ for all $\xi \geq 0$ ensures that the cross-diffusive flux in the first equation in (3.9) satisfies

$$\left\| d_\varepsilon \frac{u_\varepsilon(\cdot, t)}{(1 + \eta_\varepsilon u_\varepsilon(\cdot, t))^2} w_{\varepsilon x}(\cdot, t) \right\|_{L^2(\Omega)} \leq \left(\|d\|_{L^\infty(\Omega)} + 1 \right) \cdot \frac{1}{4\eta_\varepsilon} \cdot c_2 \quad \text{for all } t \in (0, T_{max,\varepsilon}).$$

Since furthermore, by (2.4) and again Lemma 3.2,

$$f(\cdot, u_\varepsilon, w_\varepsilon) \leq \rho(M) \quad \text{in } \Omega \times (0, T_{max,\varepsilon}),$$

a standard argument based on the smoothing properties of the non-degenerate linear semigroup $(e^{t(d_\varepsilon \cdot)_{xx}})_{t \geq 0}$ (cf. e.g. the reasoning in [3, Lemma 3.2]) provides $c_3 > 0$ such that

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq c_3 \quad \text{for all } t \in (0, T_{max,\varepsilon}).$$

In view of the extensibility criterion (3.10), together with the inequalities (4.27) and (4.28) this shows that our assumption $T_{max,\varepsilon} < \infty$ was absurd. \square

5. Weak precompactness properties of $u_\varepsilon f(x, u_\varepsilon, w_\varepsilon)$ and $u_\varepsilon g(w_\varepsilon)$ in L^1

In appropriately passing to the limit in the zero-order integrals appearing in the respective weak formulations of the system (3.9), we shall make essential use of two compactness properties of the solutions thereof which appear to go beyond trivial implications of the bounds provided by Lemma 4.3.

As a preparation for our arguments in this respect, let us state the following observation on a lower bound for all possible values of $u \geq 0$ at which $u \cdot f_-(x, u, w)$ may become large for some $x \in \bar{\Omega}$ and $w \in [0, M]$. This will be used in Lemma 5.2 to assert that for arbitrarily large $\kappa > 0$ one can pick $N > 0$ in such a way that whenever $u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \geq N$, we know that $u_\varepsilon \geq \kappa$.

LEMMA 5.1. *With $M > 0$ as in the definition (2.5), let*

$$\mathcal{S}(N) := \left\{ u \geq 0 \mid u \cdot f_-(x, u, w) \geq N \text{ for some } x \in \bar{\Omega} \text{ and } w \in [0, M] \right\} \tag{5.1}$$

and

$$\kappa(N) := \begin{cases} \inf \mathcal{S}(N) & \text{if } \mathcal{S}(N) \neq \emptyset, \\ +\infty & \text{else} \end{cases} \tag{5.2}$$

for $N \in \mathbb{N}$. Then

$$\limsup_{N \rightarrow \infty} \kappa(N) = +\infty. \tag{5.3}$$

Proof. If the equality (5.3) were false, then there would exist $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we would have $\mathcal{S}(N) \neq \emptyset$ and $\kappa(N) < c_1$ with some $c_1 > 0$. By definition of $\mathcal{S}(N)$ and $\kappa(N)$, this would mean that we could find $(x_N)_{N \geq N_0} \subset \bar{\Omega}$, $(u_N)_{N \geq N_0} \subset [0, \infty)$ and $(w_N)_{N \geq N_0} \subset [0, M]$ fulfilling

$$u_N \cdot f_-(x_N, u_N, w_N) \geq N \quad \text{for all } N \geq N_0 \tag{5.4}$$

and

$$u_N \leq c_1 \quad \text{for all } N \geq N_0,$$

where passing to a subsequence if necessary we may assume that as $N \rightarrow \infty$ we have $x_N \rightarrow x_\infty$, $u_N \rightarrow u_\infty$ and $w_N \rightarrow w_\infty$ with some $x_\infty \in \bar{\Omega}$, $u_\infty \in [0, c_1]$ and $w_\infty \in [0, M]$. By continuity of f_- , however, this would imply that

$$u_N \cdot f_-(x_N, u_N, w_N) \rightarrow u_\infty \cdot f_-(x_\infty, u_\infty, w_\infty) \quad \text{as } N \rightarrow \infty$$

and thereby contradict the property (5.4). □

Making use of the latter, by means of the Dunford–Pettis theorem we can now establish suitable compactness properties of the rightmost summands in the first two equations in (3.9).

LEMMA 5.2. *Let $T > 0$. Then with $\varepsilon_*(T) \in (0, \varepsilon_0)$ as in Lemma 4.3,*

$$\begin{aligned} & \left(u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon) \right)_{\varepsilon \in (0, \varepsilon_*(T))} \quad \text{is relatively compact} \\ & \text{with respect to the weak topology in } L^1(\Omega \times (0, T)), \end{aligned} \tag{5.5}$$

and moreover

$$\begin{aligned} & \left(u_\varepsilon g(w_\varepsilon) \right)_{\varepsilon \in (0, \varepsilon_*(T))} \text{ is relatively compact} \\ & \text{with respect to the weak topology in } L^1(\Omega \times (0, T)). \end{aligned} \tag{5.6}$$

Proof. According to Lemma 4.3, we can fix positive constants c_1 and c_2 such that

$$\int_{\{u_\varepsilon(\cdot, t) \geq 1\}} u_\varepsilon(\cdot, t) \ln u_\varepsilon(\cdot, t) \leq c_1 \quad \text{for all } t \in (0, T) \tag{5.7}$$

and

$$\int_0^T \int_{\{u_\varepsilon(\cdot, t) \geq 1\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(\cdot, u_\varepsilon, w_\varepsilon) \leq c_2 \tag{5.8}$$

whenever $\varepsilon \in (0, \varepsilon_*(T))$. Aiming at an application of the Dunford-Pettis theorem, given $\mu > 0$ we first fix an integer $N \geq 1$ suitably large such that

$$\frac{c_1 \rho(M) T}{\ln N} < \frac{\mu}{4} \tag{5.9}$$

and

$$\frac{c_1 g(M) T}{\ln N} < \frac{\mu}{2}, \tag{5.10}$$

and such that with $\kappa(N)$ as defined in Lemma 5.1 we have $\kappa(N) > 1$ and

$$\frac{c_2}{\ln \kappa(N)} < \frac{\mu}{4}, \tag{5.11}$$

where the latter is possible due to the outcome of Lemma 5.1. Thereafter, we choose $\iota > 0$ small enough fulfilling

$$\rho(M) N \iota < \frac{\mu}{4} \tag{5.12}$$

and

$$N \iota < \frac{\mu}{4} \tag{5.13}$$

as well as

$$g(M) N \iota < \frac{\mu}{2}, \tag{5.14}$$

and fix an arbitrary measurable set $E \subset \Omega \times (0, T)$ satisfying $|E| < \iota$. Then decomposing

$$\int \int_E \left| u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon) \right| = \int \int_E u_\varepsilon f_+(\cdot, u_\varepsilon, w_\varepsilon) + \int \int_I u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon), \tag{5.15}$$

by combining the bound (5.7) with Lemma 3.2 and the properties (2.4) and (5.12), we can estimate

$$\int_E u_\varepsilon f_+(\cdot, u_\varepsilon, w_\varepsilon) = \int \int_{E \cap \{u_\varepsilon < N\}} u_\varepsilon f_+(\cdot, u_\varepsilon, w_\varepsilon) + \int_{E \cap \{u_\varepsilon \geq N\}} u_\varepsilon f_+(\cdot, u_\varepsilon, w_\varepsilon)$$

$$\begin{aligned}
 &\leq \rho(M) \int_{E \cap \{u_\varepsilon < N\}} u_\varepsilon + \frac{\rho(M)}{\ln N} \int \int_{E \cap \{u_\varepsilon \geq N\}} u_\varepsilon \ln u_\varepsilon \\
 &\leq \rho(M) \cdot N|E| + \frac{\rho(M)}{\ln N} \int_0^T \int_{\{u_\varepsilon(\cdot, t) \geq 1\}} u_\varepsilon \ln u_\varepsilon \\
 &\leq \rho(M)Nt + \frac{\rho(M)}{\ln N} \cdot c_1 T \\
 &< \frac{\mu}{4} + \frac{\mu}{4} = \frac{\mu}{2} \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)).
 \end{aligned} \tag{5.16}$$

Likewise, relying on the property (5.13) we see that

$$\begin{aligned}
 &\int \int_E u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
 &= \int \int_{E \cap \{u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) < N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) + \int \int_{E \cap \{u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \geq N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
 &\leq N|E| + \int \int_{E \cap \{u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \geq N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
 &< \frac{\mu}{4} + \int \int_{E \cap \{u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \geq N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)),
 \end{aligned} \tag{5.17}$$

and in order to appropriately control the last summand herein we recall the definition (5.2) of $\kappa(N)$ to observe that whenever $u_\varepsilon(x, t)f_-(x, u_\varepsilon(x, t), w_\varepsilon(x, t)) \geq N$ for some $\varepsilon \in (0, \varepsilon_0)$, $x \in \bar{\Omega}$ and $t \geq 0$, we necessarily must have $u_\varepsilon(x, t) \geq \kappa(N)$.

Consequently, $E \cap \{u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \geq N\} \subset E \cap \{u_\varepsilon \geq \kappa(N)\}$, so that the conditions (5.8) and (5.11) become applicable so as to guarantee that

$$\begin{aligned}
 \int \int_{E \cap \{u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \geq N\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) &\leq \int \int_{E \cap \{u_\varepsilon \geq \kappa(N)\}} u_\varepsilon f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
 &\leq \frac{1}{\ln \kappa(N)} \int \int_{E \cap \{u_\varepsilon \geq \kappa(N)\}} u_\varepsilon \ln u_\varepsilon \cdot f_-(\cdot, u_\varepsilon, w_\varepsilon) \\
 &\leq \frac{1}{\ln \kappa(N)} \cdot c_2 \\
 &< \frac{\mu}{4} \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)),
 \end{aligned}$$

which along with the results (5.15), (5.16) and (5.17) shows that for any such E we have

$$\int \int_E \left| u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon) \right| < \mu \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)). \tag{5.18}$$

Similarly, using Lemma 3.2 together with the properties (2.6) we obtain

$$\begin{aligned}
 \int \int_E \left| u_\varepsilon g(w_\varepsilon) \right| &= \int \int_{E \cap \{u_\varepsilon < N\}} u_\varepsilon g(w_\varepsilon) + \int \int_{E \cap \{u_\varepsilon \geq N\}} u_\varepsilon g(w_\varepsilon) \\
 &\leq g(M) \int \int_{E \cap \{u_\varepsilon < N\}} u_\varepsilon + g(M) \int \int_{E \cap \{u_\varepsilon \geq N\}} u_\varepsilon \\
 &\leq g(M) \cdot N|E| + \frac{g(M)}{\ln N} \int_0^T \int_{\{u_\varepsilon(\cdot, t) \geq 1\}} u_\varepsilon
 \end{aligned}$$

$$\begin{aligned} &\leq g(M)N\iota + \frac{g(M)}{\ln N} \cdot c_1 T \\ &< \frac{\mu}{2} + \frac{\mu}{2} = \mu \quad \text{for all } \varepsilon \in (0, \varepsilon_*(T)), \end{aligned}$$

because of the inequalities (5.10) and (5.14). By means of the Dunford–Pettis compactness criterion, from this we infer that the property (5.6) holds, and that the property (5.5) is a consequence of the inequality (5.18). \square

6. Regularity properties of $\sqrt{d_\varepsilon}u_\varepsilon$

In order to further prepare our limit procedure, especially with regard to pointwise convergence of u_ε and of convergence in the cross-diffusive flux term $d_\varepsilon \frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x}$ in (3.9), we next plan to combine the weak compactness feature of the part $\sqrt{d_\varepsilon}w_{\varepsilon x}$ thereof, as naturally implied by Corollary 4.1, by an appropriate result on convergence in the complementary factor $\sqrt{d_\varepsilon} \frac{u_\varepsilon}{(1+\eta_\varepsilon u_\varepsilon)^2}$ in a strong L^2 topology. To achieve this in Lemma 9.1 by using underway an argument based on the Aubin–Lions lemma, let us suitably interpolate between the inequalities in (3.14) and (4.11) to derive the following spatio-temporal estimates for the quantity $\sqrt{d_\varepsilon}u_\varepsilon$ forming the core of the factor in question.

LEMMA 6.1. *Let $T > 0$ and $\varepsilon_*(T) \in (0, \varepsilon_0)$ be as in Lemma 4.3. Then there exists $C(T) > 0$ such that for all $\varepsilon \in (0, \varepsilon_*(T))$,*

$$\int_0^T \left\| (\sqrt{d_\varepsilon}u_\varepsilon(\cdot, t))_x \right\|_{L^1(\Omega)}^2 dt \leq C(T) \tag{6.1}$$

and

$$\int_0^T \left\| \sqrt{d_\varepsilon}u_\varepsilon(\cdot, t) \right\|_{L^\infty(\Omega)}^2 dt \leq C(T) \tag{6.2}$$

as well as

$$\int_0^T \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \leq C(T). \tag{6.3}$$

Proof. According to Lemma 3.3 and Lemma 4.3, there exist $c_1 = c_1(T) > 0$ and $c_2 = c_2(T) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ we have

$$\int_\Omega u_\varepsilon \leq c_1 \quad \text{for all } t \in (0, T), \tag{6.4}$$

and that

$$\int_0^T \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \leq c_2 \tag{6.5}$$

whenever $\varepsilon \in (0, \varepsilon_*(T))$. Since

$$\begin{aligned} \left| (\sqrt{d_\varepsilon}u_\varepsilon)_x \right| &= \left| \sqrt{d_\varepsilon}u_{\varepsilon x} + \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}}u_\varepsilon \right| \\ &\leq \sqrt{d_\varepsilon}|u_{\varepsilon x}| + \frac{\sqrt{K_1}}{2}u_\varepsilon \quad \text{in } \Omega \times (0, \infty) \end{aligned}$$

due to the bound (3.2), by the Cauchy–Schwarz inequality these estimates imply that

$$\begin{aligned} \int_0^T \left\| (\sqrt{d_\varepsilon} u_\varepsilon(\cdot, t))_x \right\|_{L^1(\Omega)}^2 dt &\leq \int_0^T \left\{ \int_\Omega \sqrt{d_\varepsilon} |u_{\varepsilon x}| \right\}^2 + \frac{\sqrt{K_1}}{2} \int_0^T \left\{ \int_\Omega u_\varepsilon \right\}^2 \\ &\leq 2 \int_0^T \left\{ \int_\Omega d_\varepsilon \frac{u_{\varepsilon x}^2}{u_\varepsilon} \right\} \cdot \left\{ \int_\Omega u_\varepsilon \right\} + \frac{K_1}{2} \int_0^T \left\{ \int_\Omega u_\varepsilon \right\}^2 \\ &\leq c_3 \equiv c_3(T) := 2c_1 c_2 + \frac{K_1}{2} c_1^2 T \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)) \end{aligned}$$

and thereby proves the inequality (6.1), whereupon the inequality (6.2) follows from Lemma 3.3 and the fact that $W^{1,1}(\Omega) \hookrightarrow L^\infty(\Omega)$. As the Gagliardo-Nirenberg inequality provides $c_4 > 0$ such that

$$\|\varphi\|_{L^3(\Omega)}^3 \leq c_4 \|\varphi_x\|_{L^1(\Omega)}^2 \|\varphi\|_{L^1(\Omega)} + c_4 \|\varphi\|_{L^1(\Omega)}^3 \quad \text{for all } \varphi \in W^{1,1}(\Omega),$$

in view of the inequality (3.1) this furthermore entails that with $c_5 := \sqrt{\|d\|_{L^\infty(\Omega)} + 1}$ we have

$$\begin{aligned} \int_0^T \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 &= \int_0^T \left\| \sqrt{d_\varepsilon} u_\varepsilon(\cdot, t) \right\|_{L^3(\Omega)}^3 dt \\ &\leq c_4 \int_0^T \left\| (\sqrt{d_\varepsilon} u_\varepsilon(\cdot, t))_x \right\|_{L^1(\Omega)}^2 \cdot \left\| \sqrt{d_\varepsilon} u_\varepsilon(\cdot, t) \right\|_{L^1(\Omega)} dt + c_4 \int_0^T \left\| \sqrt{d_\varepsilon} u_\varepsilon(\cdot, t) \right\|_{L^1(\Omega)}^3 dt \\ &\leq c_4 c_5 \|u_\varepsilon\|_{L^\infty((0,T);L^1(\Omega))} \int_0^T \left\| (\sqrt{d_\varepsilon} u_\varepsilon(\cdot, t))_x \right\|_{L^1(\Omega)}^2 dt + c_4 c_5^3 \int_0^T \left\| u_\varepsilon(\cdot, t) \right\|_{L^1(\Omega)}^3 dt \\ &\leq c_4 c_5 c_1 c_3 + c_4 c_5^3 c_1^3 T \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)), \end{aligned}$$

and that thus also the inequality (6.3) holds. □

7. Regularity in time

As a final preparation for our first subsequence extraction procedure, we combine our previously gained estimates to obtain some regularity properties involving time derivatives of the solution components u_ε and w_ε .

LEMMA 7.1. *Let $T > 0$ and $\varepsilon_\star(T) \in (0, \varepsilon_0)$ be as in Lemma 4.3. Then there exists $C(T) > 0$ such that*

$$\int_0^T \left\| \partial_t \sqrt{d_\varepsilon (u_\varepsilon(\cdot, t) + 1)} \right\|_{(W^{1,3}(\Omega))^\star} dt \leq C(T) \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)). \tag{7.1}$$

Proof. For fixed $t > 0$ and $\psi \in C^1(\bar{\Omega})$, from the first equation in (3.9) we obtain that

$$\begin{aligned} &\int_\Omega \partial_t \sqrt{d_\varepsilon (u_\varepsilon(\cdot, t) + 1)} \psi \\ &= -\frac{1}{2} \int_\Omega \left(\sqrt{d_\varepsilon} \frac{1}{\sqrt{u_\varepsilon + 1}} \psi \right)_x \cdot (d_\varepsilon u_\varepsilon)_x + \frac{1}{2} \int_\Omega \left(\sqrt{d_\varepsilon} \frac{1}{\sqrt{u_\varepsilon + 1}} \psi \right)_x \cdot d_\varepsilon \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} \\ &\quad + \frac{1}{2} \int_\Omega \sqrt{d_\varepsilon} \frac{1}{\sqrt{u_\varepsilon + 1}} u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon) \psi \\ &= \frac{1}{4} \int_\Omega \sqrt{d_\varepsilon}^3 \frac{1}{\sqrt{u_\varepsilon + 1}^3} u_{\varepsilon x}^2 \psi + \frac{1}{4} \int_\Omega \sqrt{d_\varepsilon} d_{\varepsilon x} \frac{u_\varepsilon}{\sqrt{u_\varepsilon + 1}^3} u_{\varepsilon x} \psi \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{4} \int_{\Omega} \sqrt{d_{\varepsilon}} d_{\varepsilon x} \frac{1}{\sqrt{u_{\varepsilon}+1}} u_{\varepsilon x} \psi - \frac{1}{4} \int_{\Omega} \frac{d_{\varepsilon x}^2}{\sqrt{d_{\varepsilon}}} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon}+1}} \psi \\
 & -\frac{1}{2} \int_{\Omega} \sqrt{d_{\varepsilon}}^3 \frac{1}{\sqrt{u_{\varepsilon}+1}} u_{\varepsilon x} \psi_x - \frac{1}{2} \int_{\Omega} \sqrt{d_{\varepsilon}} d_{\varepsilon x} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon}+1}} \psi_x \\
 & -\frac{1}{4} \int_{\Omega} \sqrt{d_{\varepsilon}}^3 \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon}+1}^3 (1+\eta_{\varepsilon} u_{\varepsilon})^2} u_{\varepsilon x} w_{\varepsilon x} \psi + \frac{1}{2} \int_{\Omega} \sqrt{d_{\varepsilon}}^3 \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon}+1} (1+\eta_{\varepsilon} u_{\varepsilon})^2} w_{\varepsilon x} \psi_x \\
 & + \frac{1}{2} \int_{\Omega} \sqrt{d_{\varepsilon}} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon}+1}} f(\cdot, u_{\varepsilon}, w_{\varepsilon}) \psi \quad \text{for all } \varepsilon \in (0, \varepsilon_0).
 \end{aligned} \tag{7.2}$$

Here using the inequality (3.1) and Young’s inequality, we see that

$$\begin{aligned}
 \left| \frac{1}{4} \int_{\Omega} \sqrt{d_{\varepsilon}}^3 \frac{1}{\sqrt{u_{\varepsilon}+1}^3} u_{\varepsilon x}^2 \psi \right| & \leq \frac{\sqrt{c_1}}{4} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \|\psi\|_{L^{\infty}(\Omega)} \\
 & \leq \frac{\sqrt{c_1}}{8} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} + 1 \right\} \|\psi\|_{L^{\infty}(\Omega)}
 \end{aligned} \tag{7.3}$$

with $c_1 := \|d\|_{L^{\infty}(\Omega)} + 1$, and similarly,

$$\begin{aligned}
 \left| \frac{1}{4} \int_{\Omega} \sqrt{d_{\varepsilon}} d_{\varepsilon x} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon}+1}^3} u_{\varepsilon x} \psi \right| & \leq \frac{1}{4} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} d_{\varepsilon x}^2 \frac{u_{\varepsilon}^3}{(u_{\varepsilon}+1)^3} \psi^2 \right\}^{\frac{1}{2}} \\
 & \leq \frac{\sqrt{K_1 c_1}}{4} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)} \\
 & \leq \frac{\sqrt{K_1 c_1}}{8} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} + 1 \right\} \|\psi\|_{L^2(\Omega)},
 \end{aligned} \tag{7.4}$$

because

$$d_{\varepsilon x}^2 \leq K_1 d_{\varepsilon} \leq K_1 c_1 \quad \text{in } \Omega \tag{7.5}$$

thanks to the inequalities (3.2) and (3.1). Next, by the Hölder inequality and again due to the inequalities (7.5), (3.2), (3.1), and Young’s inequality,

$$\begin{aligned}
 \left| -\frac{1}{4} \int_{\Omega} \sqrt{d_{\varepsilon}} d_{\varepsilon x} \frac{1}{\sqrt{u_{\varepsilon}+1}} u_{\varepsilon x} \psi \right| & \leq \frac{1}{4} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} d_{\varepsilon x}^2 \frac{u_{\varepsilon}}{u_{\varepsilon}+1} \psi^2 \right\}^{\frac{1}{2}} \\
 & \leq \frac{\sqrt{K_1 c_1}}{4} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)} \\
 & \leq \frac{\sqrt{K_1 c_1}}{8} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} + 1 \right\} \|\psi\|_{L^2(\Omega)}
 \end{aligned} \tag{7.6}$$

and

$$\begin{aligned}
 \left| -\frac{1}{4} \int_{\Omega} \frac{d_{\varepsilon x}^2}{\sqrt{d_{\varepsilon}}} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon}+1}} \psi \right| & \leq \frac{1}{4} \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}}^3 u_{\varepsilon}^3 \right\}^{\frac{1}{6}} \cdot \left\{ \int_{\Omega} d_{\varepsilon}^{-\frac{9}{10}} |d_{\varepsilon x}|^{\frac{12}{5}} \left(\frac{u_{\varepsilon}}{u_{\varepsilon}+1} \right)^{\frac{3}{5}} |\psi|^{\frac{6}{5}} \right\}^{\frac{5}{6}} \\
 & \leq \frac{K_1}{4} \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}}^3 u_{\varepsilon}^3 \right\}^{\frac{1}{6}} \cdot \left\{ \int_{\Omega} d_{\varepsilon}^{\frac{3}{10}} |\psi|^{\frac{6}{5}} \right\}^{\frac{5}{6}} \\
 & \leq \frac{K_1 c_1^{\frac{1}{4}}}{4} \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}}^3 u_{\varepsilon}^3 \right\}^{\frac{1}{6}} \|\psi\|_{L^{\frac{6}{5}}(\Omega)}
 \end{aligned}$$

$$\leq \frac{K_1 c_1^{\frac{3}{4}}}{24} \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}^{-3}} u_{\varepsilon}^3 + 1 \right\} \|\psi\|_{L^{\frac{6}{5}}(\Omega)} \tag{7.7}$$

as well as

$$\begin{aligned} \left| -\frac{1}{2} \int_{\Omega} \sqrt{d_{\varepsilon}^{-3}} \frac{1}{\sqrt{u_{\varepsilon} + 1}} u_{\varepsilon x} \psi_x \right| &\leq \frac{1}{2} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} d_{\varepsilon}^2 \frac{u_{\varepsilon}}{u_{\varepsilon} + 1} \psi_x^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{c_1}{2} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \|\psi_x\|_{L^2(\Omega)} \\ &\leq \frac{c_1}{4} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} + 1 \right\} \|\psi_x\|_{L^2(\Omega)} \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} \left| -\frac{1}{2} \int_{\Omega} \sqrt{d_{\varepsilon}} d_{\varepsilon x} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon} + 1}} \psi_x \right| &\leq \frac{1}{2} \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}^{-3}} u_{\varepsilon}^3 \right\}^{\frac{1}{6}} \cdot \left\{ d_{\varepsilon}^{\frac{3}{10}} |d_{\varepsilon x}|^{\frac{6}{5}} \left(\frac{u_{\varepsilon}}{u_{\varepsilon} + 1} \right)^{\frac{3}{5}} |\psi_x|^{\frac{6}{5}} \right\}^{\frac{5}{6}} \\ &\leq \frac{\sqrt{K_1}}{2} \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}^{-3}} u_{\varepsilon}^3 \right\}^{\frac{1}{6}} \cdot \left\{ d_{\varepsilon}^{\frac{9}{10}} |\psi_x|^{\frac{6}{5}} \right\}^{\frac{5}{6}} \\ &\leq \frac{\sqrt{K_1} c_1^{\frac{3}{4}}}{2} \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}^{-3}} u_{\varepsilon}^3 \right\}^{\frac{1}{6}} \|\psi_x\|_{L^{\frac{6}{5}}(\Omega)} \\ &\leq \frac{\sqrt{K_1} c_1^{\frac{3}{4}}}{2} \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}^{-3}} u_{\varepsilon}^3 + 1 \right\} \|\psi_x\|_{L^{\frac{6}{5}}(\Omega)}. \end{aligned} \tag{7.9}$$

Likewise, the integrals in the equality (7.2) stemming from the cross-diffusive interaction can be estimated according to

$$\begin{aligned} &\left| -\frac{1}{4} \int_{\Omega} \sqrt{d_{\varepsilon}^{-3}} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon} + 1}^3 (1 + \eta_{\varepsilon} u_{\varepsilon})^2} u_{\varepsilon x} w_{\varepsilon x} \psi \right| \\ &\leq \frac{1}{4} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} d_{\varepsilon}^2 \frac{u_{\varepsilon}^3}{(u_{\varepsilon} + 1)^3} w_{\varepsilon x}^2 \right\}^{\frac{1}{2}} \|\psi\|_{L^{\infty}(\Omega)} \\ &\leq \frac{\sqrt{c_1}}{4} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 \right\}^{\frac{1}{2}} \|\psi\|_{L^{\infty}(\Omega)} \\ &\leq \frac{\sqrt{c_1}}{8} \left\{ \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon}} + \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 \right\} \|\psi\|_{L^{\infty}(\Omega)} \end{aligned} \tag{7.10}$$

and

$$\begin{aligned} \left| \frac{1}{4} \int_{\Omega} \sqrt{d_{\varepsilon}} d_{\varepsilon x} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon} + 1} (1 + \eta_{\varepsilon} u_{\varepsilon})^2} w_{\varepsilon x} \psi \right| &\leq \frac{1}{4} \left\{ \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} d_{\varepsilon}^2 \frac{u_{\varepsilon}^2}{u_{\varepsilon} + 1} \psi^2 \right\}^{\frac{1}{2}} \\ &\leq \frac{\sqrt{K_1} c_1}{4} \left\{ \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} u_{\varepsilon} \right\}^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)} \\ &\leq \frac{\sqrt{K_1} c_1}{8} \left\{ \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 + \int_{\Omega} u_{\varepsilon} \right\} \|\psi\|_{L^2(\Omega)} \end{aligned} \tag{7.11}$$

as well as

$$\left| \frac{1}{2} \int_{\Omega} \sqrt{d_{\varepsilon}^{-3}} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon} + 1} (1 + \eta_{\varepsilon} u_{\varepsilon})^2} w_{\varepsilon x} \psi_x \right|$$

$$\begin{aligned}
 &\leq \frac{1}{2} \left\{ \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} d_{\varepsilon}^2 \frac{u_{\varepsilon}^2}{u_{\varepsilon} + 1} \psi_x^2 \right\}^{\frac{1}{2}} \\
 &\leq \frac{c_1^{\frac{3}{4}}}{2} \left\{ \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}} u_{\varepsilon} \psi_x^2 \right\}^{\frac{1}{2}} \\
 &\leq \frac{c_1^{\frac{3}{4}}}{2} \left\{ \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}}^3 u_{\varepsilon}^3 \right\}^{\frac{1}{6}} \|\psi_x\|_{L^3(\Omega)} \\
 &\leq \frac{c_1^{\frac{3}{4}}}{4} \left\{ \int_{\Omega} d_{\varepsilon} w_{\varepsilon x}^2 + \int_{\Omega} \sqrt{d_{\varepsilon}}^3 u_{\varepsilon}^3 + 1 \right\} \|\psi_x\|_{L^3(\Omega)}. \tag{7.12}
 \end{aligned}$$

Since finally

$$\left| \frac{1}{2} \int_{\Omega} \sqrt{d_{\varepsilon}} \frac{u_{\varepsilon}}{\sqrt{u_{\varepsilon} + 1}} f(\cdot, u_{\varepsilon}, w_{\varepsilon}) \psi \right| \leq \frac{c_1}{2} \left\{ \int_{\Omega} u_{\varepsilon} |f(\cdot, u_{\varepsilon}, w_{\varepsilon})| \right\} \|\psi\|_{L^{\infty}(\Omega)}, \tag{7.13}$$

and since in the present one-dimensional setting we have $W^{1,3}(\Omega) \subset W^{1,\frac{6}{5}}(\Omega) \hookrightarrow L^{\infty}(\Omega) \subset L^3(\Omega) \subset L^2(\Omega) \subset L^{\frac{6}{5}}(\Omega)$, in view of the estimates implied by Lemma 4.3, Corollary 4.1, Lemma 6.1, Lemma 3.3 and Lemma 5.2 we only need to collect the inequalities (7.3), (7.4) and (7.6)-(7.13) to derive (7.1) from (7.2). \square

It may be not surprising that our derivation of a corresponding property of w_{ε} is much less involved:

LEMMA 7.2. *Let $T > 0$, and let $\varepsilon_{\star}(T) \in (0, \varepsilon_0)$ be as in Lemma 4.3. Then one can find $C(T) > 0$ such that*

$$\int_0^T \left\| \partial_t \left(\sqrt{d_{\varepsilon}} w_{\varepsilon}(\cdot, t) \right) \right\|_{(W^{1,2}(\Omega))^{\star}}^3 dt \leq C(T) \quad \text{for all } \varepsilon \in (0, \varepsilon_{\star}(T)). \tag{7.14}$$

Proof. For arbitrary $\psi \in C^1(\bar{\Omega})$, the second equation in (3.9) shows that

$$\begin{aligned}
 \int_{\Omega} \partial_t \left(\sqrt{d_{\varepsilon}} w_{\varepsilon}(\cdot, t) \right) \psi &= -\varepsilon \int_{\Omega} \frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} (\sqrt{d_{\varepsilon}} \psi)_x - \int_{\Omega} \sqrt{d_{\varepsilon}} \frac{u_{\varepsilon}}{1 + \eta_{\varepsilon} u_{\varepsilon}} g(w_{\varepsilon}) \psi \\
 &= -\frac{\varepsilon}{2} \int_{\Omega} \frac{d_{\varepsilon x}}{\sqrt{d_{\varepsilon}}} \frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \psi - \varepsilon \int_{\Omega} \sqrt{d_{\varepsilon}} \frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \psi_x \\
 &\quad - \int_{\Omega} \sqrt{d_{\varepsilon}} \frac{u_{\varepsilon}}{1 + \eta_{\varepsilon} u_{\varepsilon}} g(w_{\varepsilon}) \psi \tag{7.15}
 \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$, where by the Cauchy–Schwarz inequality and the inequality (3.2),

$$\begin{aligned}
 \left| -\frac{\varepsilon}{2} \int_{\Omega} \frac{d_{\varepsilon x}}{\sqrt{d_{\varepsilon}}} \frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \psi \right| &\leq \frac{\varepsilon}{2} \left\{ \int_{\Omega} d_{\varepsilon} \frac{w_{\varepsilon x}^2}{g(w_{\varepsilon})} \right\}^{\frac{1}{2}} \cdot \left\{ \int_{\Omega} \frac{d_{\varepsilon x}^2}{d_{\varepsilon}^2} \psi^2 \right\}^{\frac{1}{2}} \\
 &\leq \frac{\sqrt{K_1} \varepsilon^{\frac{3}{4}}}{2} \left\{ \int_{\Omega} d_{\varepsilon} \frac{w_{\varepsilon x}^2}{g(w_{\varepsilon})} \right\}^{\frac{1}{2}} \|\psi\|_{L^2(\Omega)}, \tag{7.16}
 \end{aligned}$$

because $\frac{1}{d_{\varepsilon}} \leq \frac{1}{\sqrt{\varepsilon}}$ in Ω according to the inequality (3.1). Furthermore,

$$\left| -\varepsilon \int_{\Omega} \sqrt{d_{\varepsilon}} \frac{w_{\varepsilon x}}{\sqrt{g(w_{\varepsilon})}} \psi_x \right| \leq \varepsilon \left\{ \int_{\Omega} d_{\varepsilon} \frac{w_{\varepsilon x}^2}{g(w_{\varepsilon})} \right\}^{\frac{1}{2}} \|\psi_x\|_{L^2(\Omega)}, \tag{7.17}$$

whereas again invoking Lemma 3.2 along with the properties (2.6) we see that

$$\left| - \int_{\Omega} \sqrt{d_{\varepsilon}} \frac{u_{\varepsilon}}{1 + \eta_{\varepsilon} u_{\varepsilon}} g(w_{\varepsilon}) \psi \right| \leq g(M) \int_{\Omega} \sqrt{d_{\varepsilon}} u_{\varepsilon} \psi \leq g(M) \left\{ \int_{\Omega} \sqrt{d_{\varepsilon}}^3 u_{\varepsilon}^3 \right\}^{\frac{1}{3}} \|\psi\|_{L^{\frac{3}{2}}(\Omega)} \tag{7.18}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Thus, since $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega) \subset L^{\frac{3}{2}}(\Omega)$, and due to Lemma 4.3 and Lemma 6.1, we obtain that for any $T > 0$,

$$\begin{aligned} & \sup_{\varepsilon \in (0, \varepsilon_*(T))} \int_0^T \left\{ \int_{\Omega} d_{\varepsilon}(x) \frac{w_{\varepsilon x}^2(x, t)}{g(w_{\varepsilon}(x, t))} dx \right\}^{\frac{3}{2}} dt \\ & \leq T \cdot \left\{ \sup_{\varepsilon \in (0, \varepsilon_*(T))} \sup_{t \in (0, T)} \int_{\Omega} d_{\varepsilon}(x) \frac{w_{\varepsilon x}^2(x, t)}{g(w_{\varepsilon}(x, t))} dx \right\}^{\frac{3}{2}} < \infty \end{aligned}$$

and

$$\sup_{\varepsilon \in (0, \varepsilon_*(T))} \int_0^T \int_{\Omega} \sqrt{d_{\varepsilon}(x)}^3 u_{\varepsilon}^3(x, t) dx dt < \infty,$$

it follows from the inequalities (7.16), (7.17), and (7.18) that the equation (7.15) entails the inequality (7.14). □

8. Construction of limit functions in $\{d > 0\}$

We are now prepared for the construction of a limit function inside the positivity set of d through a straightforward extraction process based on straightforward compactness arguments. We remark that at this stage, besides the weighted functions $\sqrt{d_{\varepsilon}} w_{\varepsilon}$, our reasoning yet involves the quantities $\sqrt{d_{\varepsilon}(u_{\varepsilon} + 1)}$, rather than those addressed in Lemma 6.1.

LEMMA 8.1. *There exist a sequence $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \varepsilon_0)$ and nonnegative functions \tilde{u} and \tilde{w} defined in $\{d > 0\} \times (0, \infty)$ such that $\varepsilon_k \searrow 0$ as $k \rightarrow \infty$ and*

$$u_{\varepsilon} \rightarrow \tilde{u} \quad \text{a.e. in } \{d > 0\} \times (0, \infty), \tag{8.1}$$

$$u_{\varepsilon} \rightarrow \tilde{u} \quad \text{in } L^1_{loc}([0, \infty); L^1(\{d > 0\})), \tag{8.2}$$

$$\sqrt{d_{\varepsilon}(u_{\varepsilon} + 1)} \rightarrow \sqrt{d(\tilde{u} + 1)} \quad \text{in } L^2_{loc}([0, \infty); L^2(\{d > 0\})), \tag{8.3}$$

$$w_{\varepsilon} \rightarrow \tilde{w} \quad \text{a.e. in } \{d > 0\} \times (0, \infty) \quad \text{and} \tag{8.4}$$

$$\sqrt{d_{\varepsilon}} w_{\varepsilon} \rightarrow \sqrt{d} \tilde{w} \quad \text{in } L^2_{loc}([0, \infty); W^{1,2}(\{d > 0\})) \tag{8.5}$$

as $\varepsilon = \varepsilon_k \searrow 0$.

Proof. Since given $T > 0$ we can use the inequality (3.2) to estimate

$$\begin{aligned} \int_0^T \int_{\Omega} \left| (\sqrt{d_{\varepsilon}(u_{\varepsilon} + 1)})_x \right|^2 &= \frac{1}{4} \int_0^T \int_{\Omega} \left| \sqrt{d_{\varepsilon}} \frac{u_{\varepsilon x}}{\sqrt{u_{\varepsilon} + 1}} + \frac{d_{\varepsilon x}}{\sqrt{d_{\varepsilon}}} \sqrt{u_{\varepsilon} + 1} \right|^2 \\ &\leq \frac{1}{2} \int_0^T \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon} + 1} + \frac{1}{2} \int_0^T \int_{\Omega} \frac{d_{\varepsilon x}^2}{d_{\varepsilon}} (u_{\varepsilon} + 1) \\ &\leq \frac{1}{2} \int_0^T \int_{\Omega} d_{\varepsilon} \frac{u_{\varepsilon x}^2}{u_{\varepsilon} + 1} + \frac{K_1}{2} \int_0^T \int_{\Omega} (u_{\varepsilon} + 1) \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$, it follows from Lemma 4.3 and Lemma 3.3 that with $\varepsilon_*(T)$ as introduced there,

$$\left(\sqrt{d_{\varepsilon}(u_{\varepsilon} + 1)} \right)_{\varepsilon \in (0, \varepsilon_*(T))} \text{ is bounded in } L^2((0, T); W^{1,2}(\Omega)).$$

Therefore, in view of Lemma 7.1 the Aubin–Lions lemma asserts that for any such T ,

$$\left(\sqrt{d_\varepsilon(u_\varepsilon + 1)}\right)_{\varepsilon \in (0, \varepsilon_*(T))} \text{ is relatively compact in } L^2((0, T); L^2(\Omega)),$$

from which it follows by a standard argument that for a suitable sequence $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \varepsilon_0)$ and some $z \in L^2_{loc}([0, \infty); L^2(\Omega))$ we have $\varepsilon_k \searrow 0$ as $k \rightarrow \infty$ and

$$\sqrt{d_\varepsilon(u_\varepsilon + 1)} \rightarrow z \quad \text{in } L^2_{loc}([0, \infty); L^2(\Omega)) \tag{8.6}$$

and

$$\sqrt{d_\varepsilon(u_\varepsilon + 1)} \rightarrow z \quad \text{a.e. in } \Omega \times (0, \infty) \tag{8.7}$$

as $\varepsilon = \varepsilon_k \searrow 0$. Since $d_\varepsilon \rightarrow d$ a.e. in Ω as $\varepsilon \searrow 0$ by the property (3.3), this means that if we let $\tilde{u}(x, t) := \frac{z^2(x, t)}{d(x)} - 1$ for $x \in \{d > 0\}$ and $t > 0$, then the convergence results (8.7) and (8.6) imply the results (8.1) and (8.3), whereupon (8.1) a posteriori also shows that \tilde{u} must be nonnegative.

We next make use of the estimate (4.9) from Lemma 4.3 to infer that for $T > 0$ and $\varepsilon_*(T)$ as above,

$$\left(u_\varepsilon \ln u_\varepsilon\right)_{\varepsilon \in (0, \varepsilon_*(T))} \text{ is bounded in } L^1(\Omega \times (0, T)),$$

so that the Dunford–Pettis theorem guarantees that $(u_\varepsilon)_{\varepsilon \in (0, \varepsilon_*(T))}$ is relatively compact with respect to the weak topology in $L^1(\Omega \times (0, T))$, and that hence the convergence (8.2) can be achieved on extracting a subsequence of $(\varepsilon_k)_{k \in \mathbb{N}}$ if necessary.

As for the second solution component, we first use the inequality (3.2) to see that

$$\begin{aligned} \int_\Omega \left|(\sqrt{d_\varepsilon} w_\varepsilon)_x\right|^2 &= \int_\Omega \left|\sqrt{d_\varepsilon} w_{\varepsilon x} + \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} w_\varepsilon\right|^2 \\ &\leq 2 \int_\Omega d_\varepsilon w_{\varepsilon x}^2 + \frac{1}{2} \int_\Omega \frac{d_{\varepsilon x}^2}{d_\varepsilon} w_\varepsilon^2 \\ &\leq 2 \int_\Omega d_\varepsilon w_{\varepsilon x}^2 + \frac{K_1}{2} \int_\Omega w_\varepsilon^2 \quad \text{for all } t > 0, \end{aligned}$$

so that for $T > 0$ and $\varepsilon_*(T)$ as before, Corollary 4.1 and Lemma 3.2 warrant that

$$\left(\sqrt{d_\varepsilon} w_\varepsilon\right)_{\varepsilon \in (0, \varepsilon_*(T))} \text{ is bounded in } L^\infty((0, T); W^{1,2}(\Omega)). \tag{8.8}$$

Thus,

$$\left(\sqrt{d_\varepsilon} w_\varepsilon\right)_{\varepsilon \in (0, \varepsilon_*(T))} \text{ is relatively compact with respect to the weak topology in } L^2((0, T); W^{1,2}(\Omega)), \tag{8.9}$$

whereas the property (8.8) in conjunction with Lemma 7.2 and the Aubin–Lions lemma ensures that

$$\left(\sqrt{d_\varepsilon} w_\varepsilon\right)_{\varepsilon \in (0, \varepsilon_*(T))} \text{ is relatively compact in } L^2(\Omega \times (0, T)).$$

Consequently, arguing as above we conclude upon passing to a further subsequence if necessary that both the convergence results (8.4) and (8.5) hold with some $\tilde{w} : \{d > 0\} \times (0, \infty) \rightarrow [0, \infty)$. □

In dealing with the taxis term in the system (3.9), the following consequence of the convergence (8.5) will turn out to be more convenient.

COROLLARY 8.1. *With $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \varepsilon_0)$ and \tilde{w} as in Lemma 8.1, we have*

$$\sqrt{d_\varepsilon} w_{\varepsilon x} \rightharpoonup \sqrt{d} \tilde{w}_x \quad \text{in } L^2_{loc}([0, \infty); L^2(\{d > 0\})) \tag{8.10}$$

as $\varepsilon = \varepsilon_k \searrow 0$.

Proof. We rewrite

$$\sqrt{d_\varepsilon} w_{\varepsilon x} = (\sqrt{d_\varepsilon} w_\varepsilon)_x - \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} w_\varepsilon \tag{8.11}$$

and note that in view of the dominated convergence theorem, combining the convergence results (3.4), (3.3) and (8.4) with the inequality (3.2) and Lemma 3.2 shows that for any $T > 0$ and $\varphi \in L^2(\{d > 0\} \times (0, T))$ we obtain

$$\int_0^T \int_\Omega \int_{\{d > 0\}} \frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} w_\varepsilon \varphi \rightarrow \int_0^T \int_{\{d > 0\}} \frac{d_x}{2\sqrt{d}} \tilde{w} \varphi$$

and hence

$$\frac{d_{\varepsilon x}}{2\sqrt{d_\varepsilon}} w_\varepsilon \rightharpoonup \frac{d_x}{2\sqrt{d}} \tilde{w} \quad \text{in } L^2_{loc}([0, \infty); L^2(\{d > 0\}))$$

as $\varepsilon = \varepsilon_k \searrow 0$. Therefore, the convergence (8.10) results from the equality (8.11) on using the convergence (8.5). □

9. Further convergence and integrability properties

Let us now make use of the pointwise convergence property (4.9) from Lemma 4.3 to accomplish our previously formulated goal concerning strong L^2 convergence of $\sqrt{d_\varepsilon} \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2}$. Indeed, through an argument based on Egorov’s theorem this will result from the fact that Lemma 6.1 implies bounds for this quantity in Lebesgue spaces involving superquadratic integrability.

LEMMA 9.1. *Let $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \varepsilon_0)$ be as provided by Lemma 8.1. Then*

$$\sqrt{d_\varepsilon} \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2} \rightarrow \sqrt{d} \tilde{u} \quad \text{in } L^2_{loc}([0, \infty); L^2(\{d > 0\})) \tag{9.1}$$

as $\varepsilon = \varepsilon_k \searrow 0$.

Proof. As a consequence of Lemma 6.1, given $T > 0$ we can find $c_1 = c_1(T) > 0$ such that with $\varepsilon_\star(T)$ as in Lemma 4.3,

$$\int_0^T \int_\Omega \sqrt{d_\varepsilon}^3 u_\varepsilon^3 \leq c_1 \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)). \tag{9.2}$$

Since $\eta_\varepsilon > 0$ for all $\varepsilon \in (0, \varepsilon_0)$, this implies that for

$$z_\varepsilon := \sqrt{d_\varepsilon} \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2}, \quad \varepsilon \in (0, \varepsilon_\star(T)),$$

we have

$$\int_0^T \int_\Omega z_\varepsilon^3 \leq c_1 \quad \text{for all } \varepsilon \in (0, \varepsilon_\star(T)). \tag{9.3}$$

Since $\eta_\varepsilon \rightarrow 0$ as $\varepsilon \searrow 0$ by the property (3.7), from Lemma 8.1 we moreover know that

$$z_\varepsilon \rightarrow \sqrt{d\tilde{u}} \quad \text{a.e. in } \{d > 0\} \times (0, \infty) \quad \text{as } \varepsilon = \varepsilon_k \searrow 0. \tag{9.4}$$

Therefore, according to a standard argument involving Egorov’s theorem it particularly follows from the inequality (9.3) that

$$z_\varepsilon \rightharpoonup \sqrt{d\tilde{u}} \quad \text{in } L^2(\{d > 0\} \times (0, T)) \quad \text{as } \varepsilon = \varepsilon_k \searrow 0,$$

so that it remains to show that

$$\limsup_{\varepsilon = \varepsilon_k \searrow 0} \int_0^T \int_{\{d > 0\}} z_\varepsilon^2 \leq \int_0^T \int_{\{d > 0\}} d\tilde{u}^2. \tag{9.5}$$

To this end, supposing on the contrary that for some $c_2 > \int_0^T \int_{\{d > 0\}} du^2$ and some subsequence $(\varepsilon_{k_j})_{j \in \mathbb{N}}$ of $(\varepsilon_k)_{k \in \mathbb{N}}$ we had

$$\int_0^T \int_{\{d > 0\}} z_\varepsilon^2 \rightarrow c_2 \quad \text{as } \varepsilon = \varepsilon_{k_j} \searrow 0, \tag{9.6}$$

once more by means of the inequality (9.3) we could extract a further subsequence, again denoted by $(\varepsilon_{k_j})_{j \in \mathbb{N}}$ here fore convenience, along which for some $\widehat{z} \in L^{\frac{3}{2}}(\{d > 0\} \times (0, T))$ we would have

$$z_\varepsilon^2 \rightharpoonup \widehat{z} \quad \text{in } L^{\frac{3}{2}}(\{d > 0\} \times (0, T)) \quad \text{as } \varepsilon = \varepsilon_{k_j} \searrow 0.$$

Since the property (9.4) warrants that $z_\varepsilon^2 \rightarrow du^2$ a.e. in $\{d > 0\} \times (0, \infty)$ as $\varepsilon = \varepsilon_k \searrow 0$, again by Egorov’s theorem this would imply that actually

$$z_\varepsilon^2 \rightharpoonup d\tilde{u}^2 \quad \text{in } L^{\frac{3}{2}}(\{d > 0\} \times (0, T)) \quad \text{as } \varepsilon = \varepsilon_{k_j} \searrow 0,$$

so that, since the boundedness of $\{d > 0\} \times (0, T)$ allows for choosing nontrivial constants as test functions here, we would conclude that

$$\int_0^T \int_{\{d > 0\}} z_\varepsilon^2 \rightarrow \int_0^T \int_{\{d > 0\}} du^2 \quad \text{as } \varepsilon = \varepsilon_{k_j} \searrow 0.$$

This contradiction to the convergence assumption (9.6) shows that in fact the estimate (9.5) must hold, whence the proof becomes complete. \square

A further property of the limit couple (\tilde{u}, \tilde{w}) , quite plausible in view of Corollary 4.1, can also be justified on the basis of Egorov’s theorem.

LEMMA 9.2. *Suppose that \tilde{u} and \tilde{w} are as constructed in Lemma 8.1. Then for all $T > 0$,*

$$\int_0^T \int_{\{d > 0\}} d\tilde{u}\tilde{w}_x^2 < \infty. \tag{9.7}$$

Proof. According to Lemma 8.1, the property (3.7), and Corollary 8.1, with $(\varepsilon_k)_{k \in \mathbb{N}} \subset (0, \varepsilon_0)$ as in Lemma 8.1 we have

$$\sqrt{\frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon}} \rightarrow \sqrt{\tilde{u}} \quad \text{a.e. in } \{d > 0\} \times (0, T) \tag{9.8}$$

and

$$\sqrt{d_\varepsilon} w_{\varepsilon x} \rightharpoonup \sqrt{d} \tilde{w}_x \quad \text{in } L^2(\{d > 0\} \times (0, T)) \tag{9.9}$$

as $\varepsilon = \varepsilon_k \searrow 0$. Next, Corollary 4.1 entails that with $\varepsilon_*(T) \in (0, \varepsilon_0)$ taken from Lemma 4.3, the family $\left(\sqrt{d_\varepsilon \frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon}} w_{\varepsilon x}\right)_{\varepsilon \in (0, \varepsilon_*(T))}$ is bounded in $L^2(\{d > 0\} \times (0, T))$, so that we can find

$$z \in L^2(\{d > 0\} \times (0, T)) \tag{9.10}$$

and a subsequence $(\varepsilon_{k_j})_{j \in \mathbb{N}}$ of $(\varepsilon_k)_{k \in \mathbb{N}}$ in such a way that

$$\sqrt{\frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon}} \cdot \left(\sqrt{d_\varepsilon} w_{\varepsilon x}\right) \equiv \sqrt{d_\varepsilon \frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon}} w_{\varepsilon x} \rightharpoonup z \quad \text{in } L^2(\{d > 0\} \times (0, T))$$

as $\varepsilon = \varepsilon_{k_j} \searrow 0$. Here a known consequence of Egorov’s theorem ([24, Lemma A.1]) asserts that due to the convergence results (9.8) and (9.9) we may identify

$$z = \sqrt{\tilde{u}} \cdot \left(\sqrt{d} \tilde{w}_x\right) \equiv \sqrt{d \tilde{u}} \tilde{w}_x \quad \text{a.e. in } \{d > 0\} \times (0, T),$$

so that the property (9.7) results from the inclusion (9.10). □

10. Solution properties of \tilde{u} and \tilde{w}

We are now ready to make sure that (\tilde{u}, \tilde{w}) indeed solves the system (2.1) when restricted to $\{d > 0\}$ in the following sense.

LEMMA 10.1. *Let \tilde{u} and \tilde{w} be as obtained in Lemma 8.1.*

i) *If $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$ is such that $\varphi_x = 0$ on $\partial\Omega \times (0, \infty)$ and additionally*

$$\text{supp } \varphi \subset \{d > 0\} \times [0, \infty), \tag{10.1}$$

then

$$\begin{aligned} - \int_0^\infty \int_{\{d > 0\}} \tilde{u} \varphi_t - \int_{\{d > 0\}} u_0 \varphi(\cdot, 0) &= \int_0^\infty \int_{\{d > 0\}} d \tilde{u} \varphi_{xx} + \int_0^\infty \int_{\{d > 0\}} d \tilde{u} \tilde{w}_x \varphi_x \\ &\quad + \int_0^\infty \int_{\{d > 0\}} \tilde{u} f(\cdot, \tilde{u}, \tilde{w}) \varphi. \end{aligned} \tag{10.2}$$

ii) *For all $\varphi \in C_0^\infty(\Omega \times [0, \infty))$ fulfilling the condition (10.1), we have*

$$\int_0^\infty \int_{\{d > 0\}} \tilde{w} \varphi_t + \int_{\{d > 0\}} w_0 \varphi(\cdot, 0) = \int_0^\infty \tilde{u} g(\tilde{w}) \varphi. \tag{10.3}$$

Proof. On testing the first equation in (3.9) by φ we see that

$$\begin{aligned} - \int_0^\infty \int_\Omega u_\varepsilon \varphi_t - \int_\Omega u_0 \varphi(\cdot, 0) &= \int_0^\infty \int_\Omega d_\varepsilon u_\varepsilon \varphi_{xx} + \int_0^\infty \int_\Omega d_\varepsilon \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} \varphi_x \\ &\quad + \int_0^\infty \int_\Omega u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon) \varphi \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \end{aligned} \tag{10.4}$$

where since $u_\varepsilon \rightharpoonup \tilde{u}$ in $L^1_{loc}([0, \infty); L^1(\{d > 0\}))$ as $\varepsilon = \varepsilon_k \searrow 0$ by Lemma 8.1, according to the condition (10.1) we have

$$- \int_0^\infty \int_\Omega u_\varepsilon \varphi_t \rightarrow - \int_0^\infty \int_{\{d > 0\}} \tilde{u} \varphi_t$$

and

$$\int_0^\infty \int_\Omega d_\varepsilon u_\varepsilon \varphi_{xx} \rightarrow \int_0^\infty \int_{\{d>0\}} d\tilde{u} \varphi_{xx}$$

as $\varepsilon = \varepsilon_k \searrow 0$, because $d_\varepsilon \rightarrow d$ in $L^\infty(\Omega)$ as $\varepsilon \searrow 0$ due to the property (3.3).

Next, since Lemma 8.1 warrants that also $u_\varepsilon \rightarrow \tilde{u}$ and $w_\varepsilon \rightarrow \tilde{w}$ a.e. in $\{d > 0\} \times (0, \infty)$ as $\varepsilon = \varepsilon_k \searrow 0$, it follows from Lemma 5.2 and a standard argument, again involving Egorov’s theorem, that

$$u_\varepsilon f(\cdot, u_\varepsilon, w_\varepsilon) \rightharpoonup \tilde{u} f(\cdot, \tilde{u}, \tilde{w}) \quad \text{in } L^1_{loc}([0, \infty); L^1(\{d > 0\})),$$

and hence that

$$\int_0^\infty \int_\Omega u_\varepsilon f(x, u_\varepsilon, w_\varepsilon) \rightarrow \int_0^\infty \int_{\{d>0\}} \tilde{u} f(x, \tilde{u}, \tilde{w})$$

as $\varepsilon = \varepsilon_k \searrow 0$.

Finally, from Corollary 8.1 we know that

$$\sqrt{d_\varepsilon} w_{\varepsilon x} \rightharpoonup \sqrt{d} \tilde{w}_x \quad \text{in } L^2_{loc}([0, \infty); L^2(\{d > 0\}))$$

as $\varepsilon = \varepsilon_k \searrow 0$, which combined with the strong convergence property of $\sqrt{d_\varepsilon} \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2}$ in $L^2_{loc}([0, \infty); L^2(\{d > 0\}))$ asserted by Lemma 9.1 ensures that

$$\int_0^\infty \int_\Omega d_\varepsilon \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2} w_{\varepsilon x} \varphi_x = \int_0^\infty \int_\Omega \left(\sqrt{d_\varepsilon} \frac{u_\varepsilon}{(1 + \eta_\varepsilon u_\varepsilon)^2} \right) \cdot \left(\sqrt{d_\varepsilon} w_{\varepsilon x} \right) \varphi_x \rightarrow \int_0^\infty \int_{\{d>0\}} d\tilde{u} \tilde{w}_x \varphi_x$$

as $\varepsilon = \varepsilon_k \searrow 0$. Therefore, the equality (10.2) is a consequence of the equality (10.4).

To verify the equality (10.3), given $\varphi \in C^\infty_0(\Omega \times [0, \infty))$ fulfilling the condition (10.1) we obtain from the system (3.9) that

$$\int_0^\infty \int_\Omega w_\varepsilon \varphi_t + \int_\Omega w_{0\varepsilon} \varphi(\cdot, 0) = -\varepsilon \int_0^\infty \int_\Omega \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \varphi_x - \int_0^\infty \int_\Omega \frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \varphi \quad (10.5)$$

for all $\varepsilon \in (0, \varepsilon_0)$. Here by Lemma 8.1, Lemma 3.2 and the dominated convergence theorem,

$$\int_0^\infty \int_\Omega w_\varepsilon \varphi_t \rightarrow \int_0^\infty \int_{\{d>0\}} \tilde{w} \varphi_t \quad (10.6)$$

as $\varepsilon = \varepsilon_k \searrow 0$, whereas the definition (3.8) trivially ensures that

$$\int_\Omega w_{0\varepsilon} \varphi(\cdot, 0) \rightarrow \int_{\{d>0\}} w_0 \varphi(\cdot, 0) \quad (10.7)$$

as $\varepsilon = \varepsilon_k \searrow 0$. Moreover, combining Lemma 5.2 with the pointwise convergence properties in (8.1) and (8.4) we easily infer that

$$u_\varepsilon g(w_\varepsilon) \rightharpoonup \tilde{u} g(\tilde{w}) \quad \text{in } L^1_{loc}([0, \infty); L^1(\{d > 0\}))$$

and thus, also relying on the property (3.7) and again on Lemma 8.1, we obtain

$$\int_0^\infty \int_\Omega \frac{u_\varepsilon}{1 + \eta_\varepsilon u_\varepsilon} g(w_\varepsilon) \varphi \rightarrow \int_0^\infty \int_{\{d>0\}} \tilde{u} g(\tilde{w}) \varphi \quad (10.8)$$

as $\varepsilon = \varepsilon_k \searrow 0$. Finally, once more relying on the fact that $d_\varepsilon \geq \sqrt{\varepsilon}$ by the inequality (3.1), we see by using the Cauchy–Schwarz inequality that if $T > 0$ is large enough such that $\varphi \equiv 0$ in $\Omega \times (T, \infty)$, then

$$\begin{aligned} \left| \varepsilon \int_0^\infty \int_\Omega \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \varphi_x \right| &\leq \varepsilon \int_0^\infty \left\{ \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2}{g(w_\varepsilon)} \right\}^{\frac{1}{2}} \cdot \left\{ \int_\Omega \frac{\varphi_x^2(\cdot, t)}{d_\varepsilon} \right\}^{\frac{1}{2}} dt \\ &\leq \varepsilon^{\frac{3}{4}} \sup_{t \in (0, T)} \left\{ \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2(\cdot, t)}{g(w_\varepsilon(\cdot, t))} \right\}^{\frac{1}{2}} \cdot \int_0^\infty \left\{ \int_\Omega \varphi_x^2(\cdot, t) \right\}^{\frac{1}{2}} dt \end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Therefore, since with $\varepsilon_*(T) \in (0, \varepsilon_0)$ as given by Lemma 4.3 we know from (4.10) that

$$\sup_{\varepsilon \in (0, \varepsilon_*(T))} \sup_{t \in (0, T)} \int_\Omega d_\varepsilon \frac{w_{\varepsilon x}^2(\cdot, t)}{g(w_\varepsilon(\cdot, t))} < \infty,$$

it follows that

$$\varepsilon \int_0^\infty \int_\Omega \frac{w_{\varepsilon x}}{\sqrt{g(w_\varepsilon)}} \varphi_x \rightarrow 0$$

as $\varepsilon = \varepsilon_k \searrow 0$. In combination with the convergence results (10.6)–(10.8), this shows that the equality (10.5) implies the equality (10.3). □

11. Proof of Theorem 2.1

We can finally extend the above spatially local solution in an evident manner so as to become a global weak solution in the flavor of Definition 2.1. In the verification of the desired solution property near the boundary of $\{d > 0\}$ we shall make use of the following consequence of the inclusion $\sqrt{d} \in W^{1, \infty}(\Omega)$.

LEMMA 11.1. *Let $x \in \bar{\Omega}$. Then*

$$d(x) \leq \frac{K_1}{4} \left\{ \text{dist}(x, \{d = 0\}) \right\}^2. \tag{11.1}$$

Proof. We only need to consider the case when $d(x) > 0$, in which by the closedness of $\{d = 0\}$ we can pick $x_0 \in \bar{\Omega}$ such that $d(x_0) = 0$ and $|x - x_0| = \text{dist}(x, \{d = 0\}) > 0$. Thanks to the inequality (3.2), we then have

$$\sqrt{d(x)} = \int_{x_0}^x (\sqrt{d})_y dy = \frac{1}{2} \int_{x_0}^x \frac{d_x(y)}{\sqrt{d(y)}} dy \leq \frac{\sqrt{K_1}}{2} |x - x_0| = \frac{\sqrt{K_1}}{2} \cdot \text{dist}(x, \{d = 0\}),$$

from which the inequality (11.1) follows. □

By means of an appropriate cut-off procedure we can thereby proceed to show that the natural extension of (\tilde{u}, \tilde{w}) , consisting of a solution to the ODE system formally associated with the system (2.1) in $\{d = 0\}$ indeed solves the system (2.1) in the desired sense.

Proof. (Proof of Theorem 2.1.) We let \tilde{u} and \tilde{w} denote the functions defined on $\{d > 0\} \times (0, \infty)$ in Lemma 8.1, and for fixed $x \in \{d = 0\}$ we let $(\hat{u}(x, \cdot), \hat{w}(x, \cdot)) \in$

$(C^1([0, \infty)))^2$ be the solution of the initial-value problem

$$\begin{cases} \widehat{u}_t = \widehat{u}f(x, \widehat{u}, \widehat{w}), & t > 0, \\ \widehat{w}_t = -\widehat{u}g(\widehat{w}), & t > 0, \\ \widehat{u}(x, 0) = u_0(x), \quad \widehat{w}(x, 0) = w_0(x). \end{cases} \tag{11.2}$$

Indeed, it follows from the properties (2.2), (2.4) and (2.6) that for any such x this ODE problem possesses a globally defined solution fulfilling

$$0 \leq \widehat{w}(x, t) \leq M \quad \text{for all } t > 0 \tag{11.3}$$

and

$$0 \leq \widehat{u}(x, t) \leq u_0(x)e^{\rho(M)t} \quad \text{for all } t > 0, \tag{11.4}$$

and since u_0 and w_0 are continuous in $\bar{\Omega}$ by the property (2.3), standard ODE theory warrants that both \widehat{u} and \widehat{w} are continuous in $\{d=0\} \times [0, \infty)$. Therefore,

$$(u, w)(x, t) := \begin{cases} (\widetilde{u}, \widetilde{w})(x, t), & x \in \{d > 0\}, t > 0, \\ (\widehat{u}, \widehat{w})(x, t), & x \in \{d = 0\}, t > 0, \end{cases} \tag{11.5}$$

defines a pair of nonnegative measurable functions on all of $\Omega \times (0, \infty)$ which, thanks to Lemma 8.1, Lemma 5.2, Lemma 3.2, the properties (2.6) and the inequalities (11.3) and (11.4), satisfy (2.11) and (2.12), and for which Lemma 9.2 in particular entails that also the condition (2.13) holds.

In order to verify the equality (2.14), we first make use of the fact that by continuity of d the set $\{d > 0\}$ is relatively open in $\bar{\Omega}$, and hence consists of countably many connected components; that is, there exist an index set $I \subset \mathbb{N}$ and intervals $P_i \subset \bar{\Omega}$, $i \in I$, such that $\{d > 0\} = \bigcup_{i \in I} P_i$ and $P_i \cap P_j = \emptyset$ for $i, j \in I$ with $i \neq j$. Now for each $i \in I$, there exist $a_i \in \bar{\Omega}$ and $b_i \in \bar{\Omega}$ such that $(a_i, b_i) \subset P_i \subset [a_i, b_i]$, where $a_i \in P_i$ if and only if $a_i \in \partial\Omega$ and $b_i \in P_i$ if and only if $b_i \in \partial\Omega$. For fixed $\delta \in (0, 1)$, defining $\delta_i := 2^{-i}\delta$, $i \in I$, it is then possible to pick $(\zeta_\delta^{(i)})_{i \in I} \subset C^\infty(\bar{\Omega})$ such that for all $i \in I$ we have $0 \leq \zeta_\delta^{(i)} \leq 1$ in $\bar{\Omega}$, $\zeta_\delta^{(i)}(x) = 1$ whenever $x \in P_i$ is such that $\text{dist}(x, \partial P_i) \geq \delta_i$, $\zeta_\delta^{(i)} \equiv 0$ in $\bar{\Omega} \setminus P_i$, and

$$|\zeta_{\delta x}^{(i)}| \leq \frac{2}{\delta_i} \quad \text{in } \bar{\Omega} \tag{11.6}$$

as well as

$$|\zeta_{\delta xx}^{(i)}| \leq \frac{16}{\delta_i^2} \quad \text{in } \bar{\Omega}, \tag{11.7}$$

where in the exceptional case $a_i \in \partial\Omega$ we can additionally achieve that $\zeta_\delta^{(i)} \equiv 1$ holds even throughout $[a_i, b_i - \delta_i]$, and where, similarly, in the case $b_i \in \partial\Omega$ we require that $\zeta_\delta^{(i)} \equiv 1$ in $[a_i + \delta_i, b_i]$.

Now given $\varphi \in C_0^\infty(\bar{\Omega} \times [0, \infty))$ satisfying $\varphi_x = 0$ on $\partial\Omega \times (0, \infty)$, from the system (11.2) and the inequality (11.4) we immediately see that

$$-\int_0^\infty \int_{\{d=0\}} u \varphi_t - \int_{\{d=0\}} u_0 \varphi(\cdot, 0) = \int_0^\infty \int_{\{d=0\}} u f(x, u, w) \varphi. \tag{11.8}$$

Moreover, Lemma 10.1 guarantees that if we let

$$\zeta_\delta := \sum_{i \in I} \zeta_\delta^{(i)}, \quad \delta \in (0, 1),$$

then since $\text{supp}(\zeta_\delta \cdot \varphi) \subset \{d > 0\} \times [0, \infty)$, we have

$$\begin{aligned} & - \int_0^\infty \int_{\{d>0\}} \zeta_\delta u \varphi_t - \int_{\{d>0\}} \zeta_\delta u_0 \varphi(\cdot, 0) \\ &= \int_0^\infty \int_{\{d>0\}} du \cdot (\zeta_\delta \varphi)_{xx} + \int_0^\infty \int_{\{d>0\}} du w_x \cdot (\zeta_\delta \varphi)_x + \int_0^\infty \int_{\{d>0\}} \zeta_\delta u f(\cdot, u, w) \varphi \\ &= \int_0^\infty \int_{\{d>0\}} \zeta_\delta du \varphi_{xx} + 2 \int_0^\infty \int_{\{d>0\}} \zeta_{\delta x} du \varphi_x + \int_0^\infty \int_{\{d>0\}} \zeta_{\delta xx} du \varphi \\ &+ \int_0^\infty \int_{\{d>0\}} \zeta_\delta du w_x \varphi_x + \int_0^\infty \int_{\{d>0\}} \zeta_{\delta x} du w_x \varphi \\ &+ \int_0^\infty \int_{\{d>0\}} \zeta_\delta u f(\cdot, u, w) \varphi \quad \text{for all } \delta \in (0, 1). \end{aligned} \tag{11.9}$$

Here we may use that $0 \leq \zeta_\delta \leq 1$ and that as $\delta \searrow 0$ we have $\zeta_\delta \rightarrow 1$ a.e. in $\{d > 0\}$ to infer from the dominated convergence theorem that

$$- \int_0^\infty \int_{\{d>0\}} \zeta_\delta u \varphi_t \rightarrow - \int_0^\infty \int_{\{d>0\}} u \varphi_t \tag{11.10}$$

and

$$- \int_{\{d>0\}} \zeta_\delta u_0 \varphi(\cdot, 0) \rightarrow - \int_{\{d>0\}} u_0 \varphi(\cdot, 0) \tag{11.11}$$

as well as

$$\int_0^\infty \int_{\{d>0\}} \zeta_\delta du \varphi_{xx} \rightarrow \int_0^\infty \int_{\{d>0\}} du \varphi_{xx} \tag{11.12}$$

and

$$\int_0^\infty \int_{\{d>0\}} \zeta_\delta du w_x \varphi_x \rightarrow \int_0^\infty \int_{\{d>0\}} du w_x \varphi_x \tag{11.13}$$

and

$$\int_0^\infty \int_{\{d>0\}} \zeta_\delta u f(\cdot, u, w) \varphi \rightarrow \int_0^\infty \int_{\{d>0\}} u f(\cdot, u, w) \varphi \tag{11.14}$$

as $\delta \searrow 0$. In order to estimate the integrals on the right of the inequality (11.9) which contain derivatives of ζ_δ , let us first observe that as a consequence of the inequalities (11.6) and (11.7) and Lemma 11.1 we know that whenever $x \in \bar{\Omega}$ is such that $\zeta_{\delta x}(x) \neq 0$, for some $i \in I$ we have $x \in P_i$ and $\text{dist}(x, \{d = 0\}) \leq \delta_i$ and hence

$$d(x) \zeta_{\delta x}^2(x) \leq \frac{K_1 \delta_i^2}{4} \cdot \left(\frac{2}{\delta_i}\right)^2 = K_1 \tag{11.15}$$

as well as

$$d(x) \cdot |\zeta_{\delta xx}(x)| \leq \frac{K_1 \delta_i^2}{4} \cdot \frac{16}{\delta_i^2} = 4K_1. \tag{11.16}$$

Furthermore, again by mutual disjointness of the P_i ,

$$\left| \text{supp} \zeta_{\delta x} \right| \leq \sum_{i \in I} 2 \cdot \delta_i = \sum_{i \in I} 2 \cdot (2^{-i} \delta) \leq 2\delta \quad \text{for all } \delta \in (0, 1),$$

so that since we know from Lemma 6.1, Lemma 9.2 and Fatou's lemma that with $T > 0$ taken large enough fulfilling $\varphi \equiv 0$ in $\Omega \times (T, \infty)$ we have

$$\sqrt{d}^3 u^3 \in L^1(\{d > 0\} \times (0, T))$$

and

$$duw_x^2 \in L^1(\{d > 0\} \times (0, T)),$$

from the dominated convergence theorem it follows that

$$\int_0^T \int_{\text{supp} \zeta_{\delta x}} \sqrt{d}^3 u^3 \rightarrow 0 \tag{11.17}$$

and

$$\int_0^T \int_{\text{supp} \zeta_{\delta x}} duw_x^2 \rightarrow 0 \tag{11.18}$$

as $\delta \searrow 0$, whereas combining the inequality (11.16) with the dominated convergence theorem shows that also

$$\int_{\text{supp} \zeta_{\delta x}} |\zeta_{\delta xx}| \cdot d \rightarrow 0 \tag{11.19}$$

as $\delta \searrow 0$.

Thus, using the Hölder inequality along with the inequality (11.15) and the convergence (11.17) we obtain that

$$\begin{aligned} & \left| 2 \int_0^\infty \int_{\{d > 0\}} \zeta_{\delta x} du \varphi_x \right| \\ & \leq 2 \|\varphi_x\|_{L^\infty(\Omega \times (0, \infty))} \left\{ \int_0^T \int_{\text{supp} \zeta_{\delta x}} \sqrt{d}^3 u^3 \right\}^{\frac{1}{3}} \cdot \left\{ \int_0^T \int_{\text{supp} \zeta_{\delta x}} |\zeta_{\delta x}|^{\frac{3}{2}} d^{\frac{3}{4}} \right\}^{\frac{2}{3}} \\ & \leq 2\sqrt{K_1} T^{\frac{2}{3}} \|\varphi_x\|_{L^\infty(\Omega \times (0, \infty))} \left\{ \int_0^T \int_{\text{supp} \zeta_{\delta x}} \sqrt{d}^3 u^3 \right\}^{\frac{1}{3}} \\ & \rightarrow 0 \end{aligned} \tag{11.20}$$

as $\delta \searrow 0$, while from the convergence (11.19) we infer that

$$\left| \int_0^\infty \int_{\{d > 0\}} \zeta_{\delta xx} du \varphi \right|$$

$$\begin{aligned}
 &\leq \|\varphi\|_{L^\infty(\Omega \times (0, \infty))} \int_0^T \left\| \sqrt{d}u(\cdot, t) \right\|_{L^\infty(\{d>0\})} \int_{\text{supp } \zeta_{\delta x}} |\zeta_{\delta x}| \cdot d \, dt \\
 &\leq \sqrt{T} \|\varphi\|_{L^\infty(\Omega \times (0, \infty))} \left\{ \int_0^T \left\| \sqrt{d}u(\cdot, t) \right\|_{L^\infty(\{d>0\})} dt \right\}^{\frac{1}{2}} \cdot \int_{\text{supp } \zeta_{\delta x}} |\zeta_{\delta x}| \cdot d \\
 &\rightarrow 0
 \end{aligned} \tag{11.21}$$

as $\delta \searrow 0$, because Lemma 6.1 together with Fatou’s lemma warrants that

$$\int_0^T \left\| \sqrt{d}u(\cdot, t) \right\|_{L^\infty(\{d>0\})}^2 dt \leq \liminf_{\varepsilon = \varepsilon_k \searrow 0} \int_0^T \left\| \sqrt{d_\varepsilon} u_\varepsilon(\cdot, t) \right\|_{L^\infty(\Omega)}^2 dt < \infty.$$

Since finally the convergence (11.18) along with the inequality (11.15) ensures that also

$$\begin{aligned}
 \left| \int_0^\infty \int_{\{d>0\}} \zeta_{\delta x} duw_x \varphi \right| &\leq \|\varphi\|_{L^\infty(\Omega \times (0, \infty))} \left\{ \int_0^T \int_{\text{supp } \zeta_{\delta x}} duw_x^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_0^T \int_\Omega \zeta_{\delta x}^2 d \right\}^{\frac{1}{2}} \\
 &\leq \sqrt{K_1 T} \|\varphi\|_{L^\infty(\Omega \times (0, \infty))} \left\{ \int_0^T \int_{\text{supp } \zeta_{\delta x}} duw_x^2 \right\}^{\frac{1}{2}} \\
 &\rightarrow 0
 \end{aligned}$$

as $\delta \searrow 0$, from the results (11.9)-(11.14), (11.20) and (11.21) we conclude that

$$\begin{aligned}
 - \int_0^\infty \int_{\{d>0\}} u \varphi_t - \int_{\{d>0\}} u_0 \varphi(\cdot, 0) &= \int_0^\infty \int_{\{d>0\}} du \varphi_{xx} + \int_0^\infty \int_{\{d>0\}} duw_x \varphi_x \\
 &\quad + \int_0^\infty \int_{\{d>0\}} u f(\cdot, u, w) \varphi,
 \end{aligned}$$

which in combination with the equality (11.8) shows that indeed the equality (2.14) is valid for any such φ .

The derivation of the equality (2.15) is much less involved: Given $\varphi \in C_0^\infty(\Omega \times [0, \infty))$, from the definition (11.5) and the system (11.2) we first obtain that

$$\int_0^\infty \int_{\{d=0\}} w \varphi_t + \int_{\{d=0\}} w_0 \varphi(\cdot, 0) = \int_0^\infty \int_{\{d=0\}} ug(w) \varphi, \tag{11.22}$$

whereas with $(\zeta_\delta)_{\delta \in (0,1)}$ as introduced above we obtain from Lemma 10.1 that

$$\int_0^\infty \int_\Omega \zeta_\delta w \varphi_t + \int_\Omega \zeta_\delta w_0 \varphi(\cdot, 0) = \int_0^\infty \int_\Omega \zeta_\delta ug(w) \varphi \tag{11.23}$$

for all $\delta \in (0, 1)$. Using that w and $ug(w)$ belong to $L^1_{loc}([0, \infty); L^1(\{d>0\}))$ by Lemma 3.2, Lemma 5.2 and Lemma 8.1, we may again employ the dominated convergence theorem here to see that in the limit $\delta \searrow 0$, the equality (11.23) implies that

$$\int_0^\infty \int_{\{d>0\}} w \varphi_t + \int_{\{d>0\}} w_0 \varphi(\cdot, 0) = \int_0^\infty \int_{\{d>0\}} ug(w) \varphi,$$

and thus that in view of the equality (11.22) also the equality (2.15) holds. □

12. Discussion

In this paper we considered a model for tumor cell migration in an anisotropic environment. More precisely, this is a 1D version of a model deduced in [9] to characterize glioma invasion. Here we explicitly allow for strongly degenerate diffusion and the setting also involves haptotaxis and a drift term coming from the non-Fickian transport obtained via parabolic scaling in [9]. We proved the global existence of weak solutions, the boundedness and the uniqueness issues remaining open.

In order to illustrate the solution behavior some numerical experiments have been performed upon using a finite volume method. It is well known that there are many serious numerical challenges related to handling chemotaxis and haptotaxis systems, even in the case with linear diffusion for the cell density, as in such settings the solutions can infer large aggregations and even exhibit blow-up. We are not aware of any well investigated numerical methods dealing with strongly degenerate diffusion and haptotaxis, thus had to rely on procedures which have been developed for other purposes in order to get a glimpse into the solution behavior in this framework. Figure 12.1 shows the densities of cancer cells and of tissue at several time points; the simulations confirm the expected qualitative behavior and, moreover, hint on possible blowups in finite time, which in this work we have neither been able to prove nor to rule out. Thereby, we considered for $x \in [0, 1]$ a diffusivity function

$$d(x) = (\sin(4\pi(x - 0.25)))^{2\alpha} \mathbb{1}_{[0.25, 0.5]} + (\sin(4\pi(x - 0.75)))^{2\alpha} \mathbb{1}_{[0.75, 1.0]}$$

with $\alpha > 0$ and started with initial conditions like shown in Figure 12.1a; they describe a tumor at time $t = 0$ situated in a tissue region which has already been degraded by the cancer cells, most of the latter still being located in the immediate neighbourhood of the tumor. Then the cells spread gradually into the available tissue, eventually aggregating at the sites with highest difference between diffusivities, see Figures 12.1b and 12.1c. Our results actually hint on the solution blowing up at those sites, see Figure 12.1c.

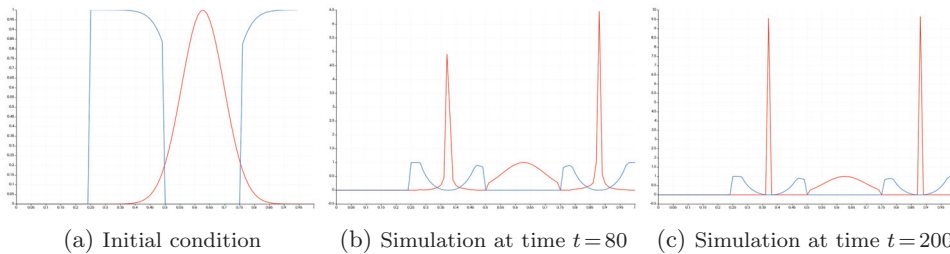


Fig. 12.1: *Simulation results. Red line: cancer cell density, blue line: tissue density; $\alpha = 0.14$.*

This behavior corresponds to previous observations, in particular in the context of glioma migration, which is heavily relying on the patient-specific brain tissue structure in which the tumor cells are invading. Indeed, glioma cells have been observed to accumulate along interfaces between white and gray matter [5]. This might be a consequence of the very different diffusivities in the two regions: the white matter is much softer and allows for cell motility (along white matter tracts) which is 5-25 times higher than in gray matter, see [4, 13, 18]. According to the discussion in [4], such behavior can be recovered, however, only if the equations involve myopic transport (i.e., the cells sense only locally their neighbourhood, which is described by a term of the form $\nabla \nabla : (D(x)u)$

instead of the classical Fickian diffusion, whence supplementing the diffusion with a transport term. The model we presented and analyzed here features this myopic transport, but moreover also accounts for haptotaxis, which is crucial for cells invading a tissue, [6] the more so for glioma cells following white matter tracts, [8, 10]. Most of the previous models addressing the issue of glioma invasion (including the already mentioned ones [4, 13, 18] and [16]) did not involve haptotaxis; as mentioned in Section 1 our macroscopic model is obtained from a micro-meso setting and the haptotaxis term is a consequence of the multiscality. Our numerical experiments suggest strong cell aggregation at sites with high diffusivity situated in the surroundings of regions with much lower diffusivity, whereby the latter may include areas with sparse tissue. Given the heterogeneity of brain tissue it is possible to have a more or less frequent alternation of such regions. This then leads to a rather fractal pattern for the tumor spread, featuring an alternation of large cancer cell accumulations with smaller tumor cell groups, possibly separated by hypocellular zones, occasionally exhibiting sharp transitions -as shown by our simulations. This behavior is in line with the highly infiltrative character of glioma spread.

Both the mathematical analysis and the numerical handling of a model featuring haptotaxis, degenerate diffusion, and myopic transport are much more challenging than for the prevalent systems with (nonlinear) diffusion and taxis. Not only are such models more realistic, but they also offer a research field for several interesting questions, both from the mathematical and the biological point of view.

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REFERENCES

- [1] H. Amann, *Dynamic theory of quasilinear parabolic systems III. Global existence*, Math. Z., 202:219–250, 1989.
- [2] A. Bellail, S.B. Hunter, D.J. Brat, C. Tan, and E.G. Van Meir, *Microregional extracellular matrix heterogeneity in brain modulates glioma cell invasion*, The Int. J. Biochem. Cell Biol., 36:1046–1069, 2004.
- [3] N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler, *Toward a Mathematical Theory of Keller-Segel Models of Pattern Formation in Biological Tissues*, Math. Mod. Meth. Appl. Sci., 25:1663–1763, 2015.
- [4] J. Belmonte-Beitia, T.E. Wooley, J.G. Scott, P.K. Maini, and E.A. Gaffney, *Modelling biological invasions: Individual to population scales at interfaces*, J. Theor. Biol., 334:1–12, 2013.
- [5] S.M. Burden-Gulley, M.Q. Quitash, K.E. Sullivant, et al., *Novel cryo-imaging of the glioma tumor microenvironment reveals migration and dispersal pathways in vivid three-dimensional detail*, Cancer Res., 71:5932–5940, 2011.
- [6] S.B. Carter, *Haptotaxis and the mechanism of cell motility*, Nature, 213:256–260, 1967.
- [7] H. Eberl, M. Efendiev, D. Wrzosek, and A. Zhigun, *Analysis of a degenerate biofilm model with a nutrient taxis term*, Discr. Cont. Dyn. Syst. A, 34:99–119, 2014.
- [8] C. Engwer, T. Hillen, M. Knappitsch, and C. Surulescu, *Glioma follow white matter tracts: a multiscale DTI-based model*, J. Math. Biol., 71:551–582, 2015.
- [9] C. Engwer, A. Hunt, and C. Surulescu, *Effective equations for anisotropic glioma spread with proliferation: a multiscale approach and comparisons with previous settings*, IMA J. Math. Med. Biol., 33(4):435–459, 2016.
- [10] A. Giese, L. Kluwe, H. Meissner, E. Michael, and M. Westphal, *Migration of human glioma cells on myelin*, Neurosurgery, 38:755–764, 1996.
- [11] T. Hillen, K. Painter, and M. Winkler, *Anisotropic diffusion in oriented environments can lead to singularity formation*, Eur. J. Appl. Math., 24:371–413, 2013.
- [12] D. Horstmann and M. Winkler, *Boundedness vs. blow-up in a chemotaxis system*, J. Diff. Eqs., 215:52–107, 2005.
- [13] E. Konukoglu, O. Clatz, B.H. Menze, B. Stieltjes, M.A. Weber, E. Mandonnet, H. Delingette, and

- N. Ayache, *Image guided personalization of reaction-diffusion type tumor growth models using modified anisotropic eikonal equations*, IEEE Trans. Med. Imaging, 29:77–95, 2010.
- [14] P. Laurençot and D. Wrzosek, *A chemotaxis model with threshold density and degenerate diffusion*, in Nonlinear Elliptic and Parabolic Problems, Progr. Nonlinear Differential Equations Appl., 64:273–290, 2005.
- [15] Y. Li and J. Lankeit, *Boundedness in a chemotaxis-haptotaxis model with nonlinear diffusion*, Nonlinearity, 29:1564–1595, 2016.
- [16] K.J. Painter and T. Hillen, *Mathematical modelling of glioma growth: the use of diffusion tensor imaging (DTI) data to predict the anisotropic pathways of cancer invasion*, J. Theoretical Biol., 323:25–39, 2013.
- [17] J. Rao, *Molecular mechanisms of glioma invasiveness: the role of proteases*, Nature Reviews Cancer, 3:489–501, 2003.
- [18] R. Rockne, E.C. Alvord, M. Szeto, S. Gu, G. Chakraborty, and K.R. Swanson, *Modeling diffusely invading brain tumors. an individualized approach to quantifying glioma evolution and response to therapy*, Modeling and Simulation in Science, Engineering and Technology, 1–15, August 2008.
- [19] Y. Tao and M. Winkler, *A Chemotaxis-Haptotaxis Model: The Roles of Nonlinear Diffusion and Logistic Source*, SIAM J. Math. Anal., 43:685–704, 2011.
- [20] Z.-A. Wang, M. Winkler, and D. Wrzosek, *Global regularity versus infinite-time singularity formation in a chemotaxis model with volume-filling effect and degenerate diffusion*, SIAM J. Math. Anal., 44:3502–3525, 2012.
- [21] M. Winkler, *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source*, Comm. Part. Diff. Eqs., 35:1516–1537, 2010.
- [22] P. Zheng, C. Mu, and X. Song, *On the boundedness and decay of solutions for a chemotaxis-haptotaxis system with nonlinear diffusion*, Discr. Cont. Dyn. Syst. A, 36:1737–1757, 2016.
- [23] A. Zhigun, C. Surulescu, and A. Hunt, *Global existence for a degenerate haptotaxis model of tumor invasion under the go-or-grow dichotomy hypothesis*, preprint, arXiv:1605.09226 (2016), submitted.
- [24] A. Zhigun, C. Surulescu, and A. Uatay, *Global existence for a degenerate haptotaxis model of cancer invasion*, Zeitsch. Angew. Math. Phys., 67:146, December 2016.