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ON THE GLOBAL ATTRACTOR OF THE DAMPED ROSENAU EQUATION ON THE WHOLE LINE*

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Abstract. We consider the asymptotic behaviour of the solution for the damped Rosenau equation on $\mathbb{R}^1$. By applying the $I-$ method and a variant form of Riesz-Rellich criteria, we prove that this damped Rosenau equation possesses a global attractor in $H^s(\mathbb{R})$ for any $s\in(\frac{1}{2},2)$. Moreover, the global attractor $\mathcal{A}_s$ is contained in $H^2(\mathbb{R})$ for any $s\in(\frac{1}{2},2)$. Our results establish the lower regularity of the global attractor for the damped Rosenau equation in fractional order Sobolev space and give a partial answer to the open problem in [D. Zhou and C. Mu, Appl. Anal., 1–10, 2016].

Keywords. Rosenau equation; global solution; global attractor.

AMS subject classifications. Primary: 35B40; Secondary: 35B41; 35Q53.

1. Introduction

In this paper, we consider the damped Rosenau equation

$$\partial_t u+\partial_t\partial_x^4 u+\gamma\partial_x^4 u+\mu u+\partial_x u+u\partial_x u=g(x),\quad x\in\mathbb{R},\quad t>0, \tag{1.1}$$

where γ and μ are positive constants, and the time independent source term $g(x)$ is given. The Rosenau equation was proposed by P. Rosenau in [24, 25] when describing the higher order effects of dense discrete systems in late 1980s. And the function $u(x,t)$ has different physical meanings in different contexts. Especially, in the context of the vibration of a single 1-D lattice, $u=u(x,t)$ represents the displacement of the particle x at the time t from its equilibrium. While in the context of RLC circuit, $u=u(x,t)$ represents the charge at the location x and the time t(see [24, 25] for more physical meanings). The Rosenau equation is regarded as a higher-order generalization of the regularized long wave equation (BBM equation)

$$\partial_t u-\partial_t\partial_x^2 u+\partial_x u+u\partial_x u=0,\qquad \text{(see [5,23])}$$

or dispersive perturbation of the Burgers equations

$$\partial_t u+\partial_x u+u\partial_x u+D^\alpha\partial_t u=0,\quad \alpha\in\mathbb{R}^+,\qquad \text{(see [16])}.$$

The existence, uniqueness, decay rates as well as long time behaviour of the solutions for the damped BBM equation has been extensively researched (see [1, 2, 7, 8, 12–14, 17, 20, 28, 29, 32–37, 39–42], and the references therein). Especially, Wang in [34] has proved that the damped BBM equation has a global attractor in $H^k(\mathbb{R})$ for every integer $k\geq 2$. And this result was improved to $H^1(\mathbb{R})$ by Littlewood–Paley decomposition and the

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Riesz-Rellich criteria (see [40] for detail). Recently, Wang [37] improved this result to $L^2(\mathbb{R})$ by $I-$ method and high-low frequency decomposition method. Yet, there are a few analytical results on the Rosenau equation except [10, 21, 22, 43, 44]. Up to our knowledge, the long time behaviour of the solution to the Rosenau equation has only been proved as $u_0 \in H^s(\mathbb{R})$ with $s \geq 2$. More exactly,

THEOREM 1.1. *(see [44]) Given $s \geq 2$. For the damped Rosenau equation* (1.1) *with initial data $u_0 \in H^s(\mathbb{R})$ and the time independent source term $g \in H^{s-3}(\mathbb{R})$, there exists a global attractor $\mathcal{A}_s$ in $H^s(\mathbb{R})$. Moreover, for any $\epsilon > 0$ and $g \in H^{s-4+k}(\mathbb{R})(k=1,2)$, the global attractor $\mathcal{A}_s \subset H^{s+\frac{k}{2}-\epsilon}(\mathbb{R})$.*

However, as $s \in [0,2)$, whether the damped Rosenau equation (1.1) with $(u_0, g) \in H^s(\mathbb{R}) \times H^{s-3}(\mathbb{R})$ possesses a global attractor is still a unsolved problem. As analyzed and mentioned in [44], the a priori estimates

$$\sup_{t \in [0,\infty)} \|u\|_{H^s(\mathbb{R})} \leq C(\|u_0\|_{H^s(\mathbb{R})}, \|g\|_{H^{s-3}(\mathbb{R})}), \qquad s \in [0,2)$$

is not directly obtained by the classical energy method. In the present paper, we overcome this difficulty by constructing an I_s operator (see Section 2) for any $s \in [0,2)$, then coming the property of I_s operator (see (2.2)) and under the assumption on time independent source term $\|g\|_{H^{-1}(\mathbb{R})} \leq \sqrt{\frac{|\mu-\gamma|\min\{\mu,\gamma\}}{2}} \|u_0\|_{H^s(\mathbb{R})}$, we immediately get the a priori estimates in the form

$$\sup_{t \in [0,\infty)} \|u\|_{H^s(\mathbb{R})} \leq C(\|u_0\|_{H^s(\mathbb{R})}), \qquad s \in [0,2). \tag{1.2}$$

With this a priori estimates in hand, applying Riesz-Rellich criteria, which involves the Littlewood–Paley projection operators, as well as the two properties of I_s opertaor (see (2.2), (2.3)), we obtain the following results.

THEOREM 1.2. *Given $s \in (\frac{1}{2}, 2)$. For the damped Rosenau equation* (1.1) *with the coefficients γ, $\mu \in \mathbb{R}^+$, $\mu \neq \gamma$, the initial data $u_0 \in H^s(\mathbb{R})$, the source term $g \in H^{-1}(\mathbb{R})$ and*

$$\|g\|_{H^{-1}(\mathbb{R})} \leq \sqrt{\frac{|\mu-\gamma|\min\{\mu,\gamma\}}{2}} \|u_0\|_{H^s(\mathbb{R})}, \tag{1.3}$$

there exists a global attractor $\mathcal{A}_s$ in $H^s(\mathbb{R})$. Moreover, for any $\epsilon > 0$, the global attractor $\mathcal{A}_s \subset H^{s+\frac{1}{2}-\epsilon}(\mathbb{R})$.

REMARK 1.1. We extend the results on existences of the global attractor for BBM equation in [32,40] to the damped Rosenau equation while requiring slightly less smoothness for the source term space. More exactly, we only require the source term

$$g(x) \in \begin{cases} H^{s-3}(\mathbb{R}), & s \geq 2, \\ H^{-1}(\mathbb{R}), & \frac{1}{2} < s < 2. \end{cases}$$

REMARK 1.2. We leave an open problem on the existence of the global attractor for the damped Rosenau equation (1.1) as $(u_0, g) \in H^s(\mathbb{R}) \times H^{-1}(\mathbb{R})$ with $s \in [0, \frac{1}{2}]$ and $\gamma = \mu$ in this paper. The main difficulties to deal with the two cases are that inequalities (4.18) and (3.4) do not hold for $s \in [0, \frac{1}{2}]$ and $\gamma = \mu \in \mathbb{R}^+$, separately.

As a byproduct of Theorem 1.2, we have the following Corollary.

COROLLARY 1.1. *The constructed global attractor $\mathcal{A}_s$ in Theorem 1.2 does not depend on s. More exactly, given $u_0 \in H^s(\mathbb{R})$, $g \in H^{-1}(\mathbb{R})$ satisfying (1.3), for the constructed global attractor $\mathcal{A}_s$, we have*

$$\mathcal{A}_s = \mathcal{A}_2, \qquad s \in (\frac{1}{2}, 2).$$

REMARK 1.3. Combing this Corollary for $s \in (\frac{1}{2}, 2)$ and Corollary 1.4 in [44] for $s \in [2,\infty)$, we immediately get $\mathcal{A}_s = \mathcal{A}_2 = H^2(\mathbb{R}), s \in (\frac{1}{2}, \infty)$. And then one may naturally ask whether $\mathcal{A}_s = \mathcal{A}_2$ for $s \in [0, \frac{1}{2}]$. In fact, as $s = 0$, through the criteria of the global attractor on the Kuratowski measure of non-compactness [15,27], after using the similar arguments as that of the work by Wang [37,38], one can get $\mathcal{A}_0 = H^2(\mathbb{R})$ (we omit the exact proof for this claim). However, the cost of such expected result is that the time independent source term $g \in L^2(\mathbb{R})$, which is not our original aim in this paper.

The paper is organized as follows. In Section 2, we mainly recall some basic results on the Littlewood–Palay projection operator, present a variant form of the Riesz-Rellich theorem, and prove several properties of the I_s operator. In Section 3, we prove that the damped Rosenau equation has a unique global solution by contraction mapping principle and a priori estimates. In section 4, we prove Theorem 1.2. And in the last section, we prove Corollary 1.1.

2. Preliminaries

In this section, we aim to present some notations, definitions and lemmas which will be used several times in the following sections.

Denote by $\mathcal{S}$ the Schwartz class. Then $\forall f \in \mathcal{S}$, the Fourier transform of f is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i <x,\xi>} dx.$$

Fixed an even function $\psi \in C_0^\infty(\mathbb{R}^n)$, so that supp$\psi \subset [\frac{1}{2}, 2]$ and $\sum_{k=-\infty}^{\infty} \psi(2^{-k}x) = 1$ for all $x \neq 0$. Define the operator P_δ via $\widehat{P_\delta f}(\xi) = \psi(\frac{\xi}{\delta})\widehat{f}(\xi)$. Observe that P_δ essentially restricts the Fourier support of the function f to the annulus $\frac{\delta}{2} \le |\xi| \le 2\delta$ and $\sum_{\delta \text{ dyadic}} P_\delta = 1$. And we call P_δ the **Littlewood–Paley projection operator** at frequency δ.

Let $n = 1$, since supp$\widehat{P_\delta f} \subseteq 2^\delta \Upsilon$, where Υ denotes the annulus $\{x | \frac{1}{2} \le |x| \le 2\}$, from Bernstein's inequality(see [18], Lemma 1.1) and the fact that $\|P_\delta f\|_{L^2(\mathbb{R})} \le \|f\|_{L^2(\mathbb{R})}$, we immediately have

$$\|P_{>\delta} f\|_{L^2(\mathbb{R})} \le \frac{C^{k+1}}{2^{\delta k}} \|\partial_x^k f\|_{L^2(\mathbb{R})} \le \frac{C^{k+1}}{\delta^k} \|\partial_x^k f\|_{L^2(\mathbb{R})}, \qquad k \in \mathbb{Z}^+. \tag{2.1}$$

Note that $P_\delta f = \delta^n \widehat{\psi}(\delta \cdot) * f$ and

$$\|\delta^n \widehat{\psi(\delta \cdot)}\|_{L^1(\mathbb{R})} \lesssim 1,$$

we get $P_\delta : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ for all $p \in [1,\infty]$.

And let $S(t)f_n$ solves the evolution system (see [19])

$$\frac{d}{dt} u(t) = F(u(t)), \qquad u(0) = f_n.$$

We say that there is a global attractor $\mathcal{A}$ for the dynamical system $\{S(t)\}_{t\geq 0}$ on X, if $\mathcal{A}$ is compact, invariant ($S(t)\mathcal{A}=\mathcal{A}, t\geq 0$) and $\text{dist}(S(t)(B),\mathcal{A})\to 0$ as $t\to\infty$ for any bounded set $B\subset X$ (see [4,15,31] for general theorems that provide the existence of the global attractor for a dynamic system).

Suppose that $\sup_n\|f_n\|_{H^s(\mathbb{R})}<B$. Let $t_n\to\infty$ and denote $u_n(\cdot,t_n)=S(t_n)f_n$. We have the following version for asymptotic compactness.

Lemma 2.1 (A variant form of Riesz-Rellich criteria). *Given $s\geq 0$. Assume*
*(c1)**(uniformally boundedness)** $\sup_n\|u_n(\cdot,t_n)\|_{H^s(\mathbb{R})}\leq C(B)$;*
*(c2)**(uniformally small outside a ball)** $\limsup_n\|u_n(\cdot,t_n)\|_{H^s(|x|>N)}\to 0$ as $N\to\infty$;*
*(c3)**(unifomally continuous)** $\limsup_n\|P_{>N}u_n(\cdot,t_n)\|_{H^s(\mathbb{R})}\to 0$ as $N\to\infty$; Then the sequence $\{u_n(\cdot,t_n)\}$ is precompact in $H^s(\mathbb{R})$.*

See [28, Proposition 3, p.827],for more details of asymptotic compactness criterion and see [26, p. 248 Theorem XIII] on Riesz-Rellich criteria.

Lemma 2.2 (Bilinear Estimates). *Let $u,v\in H^s(\mathbb{R})$, $s\geq 0$. Then*

$$\|\varphi(D_x)(uv)\|_{H^s(\mathbb{R})}\leq C\|u\|_{H^s(\mathbb{R})}\|v\|_{H^s(\mathbb{R})},$$

where $\varphi(\xi)=\frac{\xi}{1+\xi^4}$ and $\varphi(D_x)$ is the Fourier multiplier defined by $\widehat{\varphi(D_x)u}=\varphi(\xi)\widehat{u}(\xi)$.

This lemma can be easily proved following the method and techniques in [7, Lemma 1].

Let $s\in[0,2)$, $A\gg 1$. We define the I_s-operator

$$I_s: H^s(\mathbb{R})\to H^2(\mathbb{R}),\quad \phi\mapsto I_s\phi,$$

where $\widehat{I_s\phi}(\xi)=m(\xi)\widehat{\phi}(\xi)$, $m(\xi)$ is a symmetric, decreasing and smoothing function satisfying

$$m(\xi)=\begin{cases}1, & |\xi|\leq A,\\ (\frac{|\xi|}{A})^{s-2}, & |\xi|\geq 2A.\end{cases}$$

For any $f\in H^s(\mathbb{R})$, $s\in[0,2)$, one can check the following two assertions

$$\|f\|_{H^s(\mathbb{R})}\leq\|I_sf\|_{H^2(\mathbb{R})}\leq CA^{2-s}\|f\|_{H^s(\mathbb{R})},\quad s\in[0,2), \tag{2.2}$$

and

$$\|I_sf\|_{H^m(\mathbb{R})}\leq\|f\|_{H^m(\mathbb{R})},\quad m\in(-\infty,2). \tag{2.3}$$

In order to get the global solution of the Cauchy problem (1.1), we need to use the following two lemmas.

Lemma 2.3. *There exists a constant C independent of A such that*

$$\|(I+\partial_x^4)^{-1}I_s\partial_x(uv)\|_{H^2(\mathbb{R})}\leq C\|I_su\|_{H^2(\mathbb{R})}\|I_sv\|_{H^2(\mathbb{R})}. \tag{2.4}$$

Proof. If we have proved

$$\|(1+\partial_x^4)^{-1}I_s\partial_x(uv)\|_{H^2(\mathbb{R})}\leq C\|u\|_{H^s(\mathbb{R})}\|v\|_{H^s(\mathbb{R})},\qquad s\in[0,2), \tag{2.5}$$

we immediately get estimate (2.4) using the estimate (2.2).

In order to prove estimate (2.5), it suffices to prove that

$$\Gamma := \left|\int_{\mathbb{R}^2} \frac{i\xi <\xi>^2 m(\xi)\widehat{u}(\xi_1)\widehat{v}(\xi-\xi_1)\overline{\widehat{w}}(\xi)}{[1+(i\xi)^4]<\xi_1>^s<\xi-\xi_1>^s} d\xi_1 d\xi\right| \le C\|u\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})}\|w\|_{L^2(\mathbb{R})} \quad (2.6)$$

holds for any $\widehat{u}(\xi)\ge 0$, $\widehat{v}(\xi)\ge 0$, and $\widehat{w}(\xi)\ge 0$, $\forall \xi\in\mathbb{R}$.

Since $<\xi>^s \lesssim <\xi_1>^s<\xi-\xi_1>^s$, using $|m(\xi)|\le 1$, Young's inequality, Plancheral's theorem as well as $\frac{\xi<\xi>^{2-s}}{1+\xi^4}\in L^2(\mathbb{R})$ for all $s\ge 0$, we have

$$\begin{aligned}
\Gamma &\le \int_{\mathbb{R}^2} \frac{\xi<\xi>^{2-s}\widehat{u}(\xi_1)\widehat{v}(\xi-\xi_1)\overline{\widehat{w}}(\xi)}{(1+\xi^4)} d\xi_1 d\xi \\
&= (\overline{m}, \widehat{u}*\widehat{v}) \qquad (\text{ where } \overline{m}(\xi) = \frac{\xi<\xi>^{2-s}\overline{\widehat{w}}(\xi)}{1+\xi^4}) \\
&= (\overline{\widehat{v}}, u_1 * m) \qquad (\text{ where } u_1(\xi) = \widehat{u}(-\xi)) \\
&\le \|\overline{\widehat{v}}\|_{L^2(\mathbb{R})}\|u_1 * m\|_{L^2(\mathbb{R})} \\
&\le \|v\|_{L^2(\mathbb{R})}\|u_1\|_{L^2(\mathbb{R})}\|m\|_{L^1(\mathbb{R})} \\
&\le \|v\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}\|w\|_{L^2(\mathbb{R})}\left\|\frac{\xi<\xi>^{2-s}}{1+\xi^4}\right\|_{L^2(\mathbb{R})} \\
&\le C\|v\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})}\|w\|_{L^2(\mathbb{R})}.
\end{aligned}$$

□

LEMMA 2.4. *There exists a constant C independent of A such that*

$$|(I_s\partial_x(u^2), I_s u)| \le CA^{-4}\|I_s u\|^3_{H^2(\mathbb{R})}. \quad (2.7)$$

Proof. Since $(\mathcal{F}f, g) = (f, \mathcal{F}g)$ and $\mathcal{F}^2[f(-x)] = f(x)$, we get $(\widehat{f}, \widehat{g}(-\cdot)) = (f, g(\cdot))$. Then

$$\begin{aligned}
&(I_s\partial_x(u^2), I_s u) \\
&= \int_{\xi_1} \widehat{I_s\partial_x(u^2)}\widehat{I_s u(-\cdot)} d\xi_1 \\
&= \int_{\xi_1}\int_{\xi_2} i\xi_1 m^2(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_1-\xi_2)\widehat{u}(-\xi_1) d\xi_2 d\xi_1 \\
&= \int_{\xi_1}\int_{\xi_2} (-i\xi_1)m^2(-\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_1-\xi_2)\widehat{u}(-\xi_1) d\xi_2 d(-\xi_1) \qquad (\text{where } m(\xi) = m(-\xi)) \\
&= \int_{\eta_1+\eta_2+\eta_3=0} i\eta_1 m^2(\eta_1)\widehat{u}(\eta_1)\widehat{u}(\eta_2)\widehat{u}(\eta_3) d\eta_1 d\eta_2.
\end{aligned}$$

Similarly,

$$(I_s\partial_x(u^2), I_s u) = \int_{\eta_1+\eta_2+\eta_3=0} i\eta_2 m^2(\eta_2)\widehat{u}(\eta_1)\widehat{u}(\eta_2)\widehat{u}(\eta_3) d\eta_1 d\eta_2$$

and

$$(I_s\partial_x(u^2), I_s u) = \int_{\eta_1+\eta_2+\eta_3=0} i\eta_3 m^2(\eta_3)\widehat{u}(\eta_1)\widehat{u}(\eta_2)\widehat{u}(\eta_3) d\eta_1 d\eta_2.$$

Then

$$(I_s\partial_x(u^2),I_su)=\int_{\xi_1+\xi_2+\xi_3=0}\frac{i(\xi_1m^2(\xi_1)+\xi_2m^2(\xi_2)+\xi_3m^2(\xi_3))}{3}\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)d\xi_1d\xi_2.$$

In order to prove estimate (2.7), it suffices to prove

$$\begin{aligned}\Xi=&|\int_{\xi_1+\xi_2+\xi_3=0}\frac{i(\xi_1m^2(\xi_1)+\xi_2m^2(\xi_2)+\xi_3m^2(\xi_3))\widehat{u}(\xi_1)\widehat{u}(\xi_2)\widehat{u}(\xi_3)}{3<\xi_1>^2<\xi_2>^2<\xi_3>^2m(\xi)m(\xi_2)m(\xi_3)}d\xi_1d\xi_2|\\ &\le CA^{-4}\|u\|^3_{L^2(\mathbb{R})}.\end{aligned}\tag{2.8}$$

By the symmetry of the integral, we assume that $|\xi_3|\le|\xi_2|\le|\xi_1|$ without loss of generality.

Case 1: as $|\xi_1|\le A$, we have $m(\xi_1)=m(\xi_2)=m(\xi_3)=1$. Since $\xi_1+\xi_2+\xi_3=0$, we immediately get that $\Xi=0$.

Case 2: as $|\xi_1|\gtrsim A$, we have $|\xi_2|\ge A$. Otherwise $|\xi_2|<A$, then $|\xi_1|=-|\xi_2+\xi_3|\le|\xi_3|+|\xi_2|<2A$, which contradicts our assumption. In this case, one can check that (see [9] Lemma 4.3)

$$|\xi_1m^2(\xi_1)+\xi_2m^2(\xi_2)+\xi_3m^2(\xi_3)|\lesssim\xi_3m^2(\xi_3).$$

Moreover, $<\xi_i>^2m(\xi_i)\ge<\xi_i>^2\frac{|\xi_i|^{s-2}}{A^{s-2}}\ge\frac{|\xi_i|^2|\xi_i|^{s-2}}{A^{s-2}}\ge A^2$, as $i=1,2$. Applying Young's inequality, Plancheral's theorem, $|m(\xi_3)|\le1$ for $\xi_3\le A$ as well as $<\xi>^{-1}\in L^2(\mathbb{R})$, we have the following estimates

$$\begin{aligned}\Xi&\le\int_{|\xi_1|\ge A,|\xi_2|\ge A}\frac{m(\xi_3)|\widehat{u}(\xi_1)||\widehat{u}(\xi_2)||\widehat{u}(\xi_3)|}{<\xi_1>^2<\xi_2>^2<\xi_3>m(\xi_1)m(\xi_2)}d\xi_1d\xi_2\\ &\le A^{-4}\int_{\mathbb{R}^2}\frac{|\widehat{u}(\xi_1)||\widehat{u}(\xi_2)||\widehat{u}(\xi_3)|}{<\xi_3>}d\xi_1d\xi_2\\ &\le A^{-4}\|\widehat{u}\|_{L^2(\mathbb{R})}\|\hat{u}*\hat{g}\|_{L^2(\mathbb{R})}\qquad(\widehat{g}(-\xi_3)=\frac{\widehat{u}(-\xi_3)}{<-\xi_3>})\\ &\le A^{-4}\|\widehat{u}\|_{L^2(\mathbb{R})}\|\widehat{u}\|_{L^2(\mathbb{R})}\|\widehat{g}\|_{L^1(\mathbb{R})}\\ &\le A^{-4}\|\widehat{u}\|^3_{L^2(\mathbb{R})}\|<-\xi_3>^{-1}\|_{L^2(\mathbb{R})},\end{aligned}$$

which implies inequality (2.8). □

Here and after, by $f\lesssim g$, we mean that there is a constant C such that $f\le Cg$. Throughout this paper, we denote by $\|\cdot\|$ the norm of $L^2(\mathbb{R})$, $C(*,*,\cdots,*)$ a constant which may be different from line to line but only depend on the quantities appearing in the parentheses.

3. Global solution

In this section we aim to prove the existence of the global solution to the Cauchy problem (1.1) by the properties of I_s operator constructed in last section.

PROPOSITION 3.1 (Global well-posedness). *Given $0<T<\infty$ and $s\in[0,2)$. For the damped Rosenau equation (1.1) with $\mu,\nu\in\mathbb{R}^+$, $\nu\ne\mu$, the initial data $u_0\in H^s(\mathbb{R})$, the source term $g\in H^{-1}(\mathbb{R})$ and*

$$\|g\|_{H^{-1}(\mathbb{R})}\le\sqrt{\frac{|\mu-\gamma|\min\{\mu,\gamma\}}{2}}\|u_0\|_{H^s(\mathbb{R})},\tag{3.1}$$

it has a unique global weak solution

$$u\in C([0,T];H^s(\mathbb{R})).$$

REMARK 3.1. By using the higher-lower order frequency decomposition on the initial data, we can also get the global weak solution to problem (1.1) with $u_0\in H^s(\mathbb{R})$ and $g\in H^{s-2}(\mathbb{R})(s\geq 0)$, but without the restriction conditions (3.1) or $\nu\neq\mu$. However, it is not obvious to get the long time behaviour of the solution using this higher-lower order frequency decomposition method in our case.

Proof. Since problem (1.1) can be rewritten as

$$\partial_t u+\gamma u=(I+\partial_x^4)^{-1}[g-\partial_x u-u\partial_x u-(\mu-\gamma)u],$$

we define the nonlinear mapping

$$G(u)=e^{-\gamma t}u_0+\int_0^t e^{-\gamma(t-\tau)}(1+\partial_x^4)^{-1}[g-\partial_x u-u\partial_x u-(\mu-\gamma)u]dx.$$

First, we aim to prove that problem (1.1) has a unique local solution by contraction mapping principle, i.e. we can find some positive constants r and T_s such that

$$I_sG:B(r,T_s)\to B(r,T_s),$$

is a contraction mapping.

Now, for any $I_su,I_sv\in C([0,T_s];H^2(\mathbb{R}))$, using estimate (2.2) and Lemma 2.3, we have

$$\begin{aligned}
&\|I_sG(u)\|_{C([0,T_s];H^2(\mathbb{R}))}\\
=&\|e^{-\gamma t}I_su_0+\int_0^t e^{-\gamma(t-\tau)}(1+\partial_x^4)^{-1}I_s[g-\partial_x u-u\partial_x u-(\mu-\gamma)u]dx\|_{C([0,T_s];H^2(\mathbb{R}))}\\
\leq&\|I_su_0\|_{H^2(\mathbb{R})}+\int_0^{T_s}\Big(CA^2\|(1+\partial_x^4)^{-1}g\|+C\|I_su\|^2_{H^2(\mathbb{R})}+(1+|\mu-\gamma|)\|I_su\|_{H^2(\mathbb{R})}\Big)dt\\
\leq&\|I_su_0\|_{H^2(\mathbb{R})}+T_sCA^2\|g\|_{H^{-4}(\mathbb{R})}+T_sC\left(\|I_su\|^2_{C([0,T_s];H^2(\mathbb{R}))}+\|I_su\|_{C([0,T_s];H^2(\mathbb{R}))}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\|I_sG(u)-I_sG(v)\|_{C([0,T_s];H^2(\mathbb{R}))}\\
=&\|\int_0^t e^{-\gamma(t-\tau)}(1+\partial_x^4)^{-1}I_s[-\frac{\partial_x}{2}(u^2-v^2)-\partial_x(u-v)-(\mu-\gamma)(u-v)]dx\|_{C([0,T_s];H^2(\mathbb{R}))}\\
\leq&C\int_0^{T_s}\left(\|I_su-I_sv\|_{H^2(\mathbb{R})}\|I_su+I_sv\|_{H^2(\mathbb{R})}+(1+|\mu-\gamma|)\|I_su-I_sv\|_{H^2(\mathbb{R})}\right)dt\\
\leq&T_sC\left(\|I_su+I_sv\|_{C([0,T_s];H^2(\mathbb{R}))}+1\right)\|I_su-I_sv\|_{C([0,T_s];H^2(\mathbb{R}))}.
\end{aligned}$$

Therefore, if we choose

$$r=4\|I_su_0\|_{H^2(\mathbb{R})},$$

and

$$T_s\leq\min\{\frac{r}{4CA^2\|g\|_{H^{-2}(\mathbb{R})}},\frac{1}{2C(8\|I_su_0\|_{H^2(\mathbb{R})}+1)}\}$$

we immediately get that

$$I_sG: B(r,T_s) \to B(r,T_s) \text{ is a contraction mapping.}$$

Consequently, due to estimate (2.2), $\forall u_0 \in H^s(\mathbb{R})$ with $s \in [0,2)$ and $g \in H^{-2}(\mathbb{R})$, we claim that $G(\cdot)$ has a unique fixed point $u \in C([0,T_s];H^s(\mathbb{R}))$, which is a local solution to problem (1.1).

Next, we aim to show that the local solution $u \in C([0,T_s];H^s(\mathbb{R}))$ can be extended to the global solution $u \in C([0,\infty);H^s(\mathbb{R}))$ after a priori estimates.

Now, acting the I_s operator on both sides of equation (1.1) and taking the L^2 inner product with I_su, we arrive at

$$\partial_t\left(\|I_su\|^2 + \|\partial_x^2 I_su\|^2\right) + 2\gamma\|\partial_x^2 I_su\|^2 + 2\mu\|I_su\|^2 = -(I_s\partial_x(u^2), I_su) + 2(I_sg, I_su). \quad (3.2)$$

In the following, we will discuss (3.2) according to $\mu < \gamma$ and $\mu > \gamma$.

Case 1: As $\mu < \gamma$, we rewrite (3.2) as

$$\begin{aligned} &\partial_t\left(\|I_su\|^2 + \|\partial_x^2 I_su\|^2\right) + 2\mu\left(\|I_su\|^2 + \|\partial_x^2 I_su\|^2\right) \\ =& -(I_s\partial_x(u^2), I_su) + 2(I_sg, I_su) + 2(\mu-\gamma)\|\partial_x^2 I_su\|^2. \end{aligned} \quad (3.3)$$

Hence, applying Lemma 2.4, Cauchy-Schwartz inequality as well as $\|I_sg\|_{H^{-2}(\mathbb{R})} \leq \|g\|_{H^{-2}(\mathbb{R})}$, we obtain

$$\begin{aligned} &\left(\|I_su\|^2 + \|\partial_x^2 I_su\|^2\right)(t) \\ =& e^{-2\mu t}\left(\|I_su_0\|^2 + \|\partial_x^2 I_su_0\|^2\right) + \int_0^t e^{-2\mu(t-\tau)}[-(I_s\partial_x(u^2), I_su) + 2(I_sg, I_su)]d\tau \\ &+ 2\int_0^t e^{-2\mu(t-\tau)}(\mu-\gamma)\|\partial_x^2 I_su\|^2 d\tau \\ \leq& e^{-2\mu t}\|I_su_0\|^2_{H^2(\mathbb{R})} + \frac{1-e^{-2\mu t}}{2\epsilon\mu}\|g\|^2_{H^{-2}(\mathbb{R})} \\ &+ \int_0^t e^{-2\mu(t-\tau)}(\epsilon + CA^{-4}\|I_su\|_{H^2(\mathbb{R})} - 2(\gamma-\mu))\left(\|I_su\|^2 + \|\partial_x^2 I_su\|^2\right)d\tau, \end{aligned}$$

for all $t \in [0,\infty)$.

Since $\forall \tau \in [0,T_s]$, $\|I_su\|_{H^2(\mathbb{R})}(\tau) \leq r = 4\|I_su_0\|_{H^2(\mathbb{R})}$, choosing $\epsilon = \frac{\gamma-\mu}{4}$ and sufficiently large A such that $4CA^{-4}\|I_su_0\|_{H^2(\mathbb{R})} < \frac{\gamma-\mu}{4}$, then

$$\epsilon + \frac{C\|I_su\|_{H^2(R)}}{A^4} - (\gamma-\mu) < 0. \quad (3.4)$$

And then

$$\begin{aligned} &\left(\|I_su\|^2 + \|\partial_x^2 I_su\|^2\right)(t) \\ \leq& e^{-2\mu t}\|I_su_0\|^2_{H^2(\mathbb{R})} + \frac{2(1-e^{-2\mu t})\|g\|^2_{H^{-2}(\mathbb{R})}}{(\gamma-\mu)\mu}, \quad \forall t \in [0,T_s]. \end{aligned} \quad (3.5)$$

Recalling the given condition (3.1) and the fact $\|u_0\|_{H^s(\mathbb{R})} \leq \|I_su_0\|_{H^2(\mathbb{R})}$, choosing $t = T_s$ in inequality (3.5), it follows

$$\left(\|I_su\|^2 + \|\partial_x^2 I_su\|^2\right)(T_s) \leq \|I_su_0\|^2_{H^2(\mathbb{R})}.$$

Thanks to the estimates, we can take $u(x,T_s)$ as a new initial data to obtain a unique solution on $[T_s,2T_s]$. Moreover, recalling estimate (3.5), the solution satisfies

$$\begin{aligned}\left(\|I_su\|^2+\|\partial_x^2I_su\|^2\right)(2T_s) &\le e^{-\mu T_s}\|I_su\|^2_{H^2(\mathbb{R})}(T_s)+\tfrac{2(1-e^{-2\mu t})\|g\|^2_{H^{-2}(\mathbb{R})}}{(\gamma-\mu)\mu}\\ &\le \|I_su_0\|^2_{H^2(\mathbb{R})}.\end{aligned}$$

Repeating the same process, we obtain the solution on $[kT_s,(k+1)T_s]$, $k=0,1,2,\cdots$. Moreover, estimate (3.5) holds for all $t\in[kT_s,(k+1)T_s], k=0,1,2,\cdots$. Which implies that equation (1.1) has a unique global weak solution $u\in C([0,\infty);H^s(\mathbb{R}))$ with the bound

$$\|I_su\|_{C([0,\infty);H^2(\mathbb{R}))}\le \|I_su_0\|_{H^2(\mathbb{R})}+\frac{2\|g\|^2_{H^{-2}(\mathbb{R})}}{(\gamma-\mu)\mu}. \tag{3.6}$$

Case 2: As $\gamma<\mu$, we rewrite equation (3.2) as

$$\begin{aligned}&\partial_t\left(\|I_su\|^2+\|\partial_x^2I_su\|^2\right)+2\gamma\left(\|I_su\|^2+\|\partial_x^2I_su\|^2\right)\\ &=-(I_s\partial_x(u^2),I_su)+2(I_sg,I_su)+2(\gamma-\mu)\|I_su\|^2.\end{aligned}$$

Repeating the similar process in step 1, using (3.1) again, we immediately get that equation (1.1) has a unique global weak solution $u\in C([0,\infty);H^s(\mathbb{R}))$ with the estimates

$$\|I_su\|_{C([0,\infty);H^2(\mathbb{R}))}\le \|I_su_0\|_{H^2(\mathbb{R})}+\frac{2\|g\|^2_{H^{-2}(\mathbb{R})}}{(\mu-\gamma)\gamma}. \tag{3.7}$$

∎

COROLLARY 3.1. *Under the same conditions in Proposition 3.1, we can find two constants M_i $(i=1,2)$ which only depend on $\|u_0\|_{H^s(\mathbb{R})},\gamma$ and μ such that*

$$\|u\|_{C([0,\infty);H^s(\mathbb{R}))}\le M_1(\|u_0\|_{H^s(\mathbb{R})},\gamma,\mu), \tag{3.8}$$

and

$$\|\partial_tu\|_{C([0,\infty);H^s(\mathbb{R}))}\le M_2(\|u_0\|_{H^s(\mathbb{R})},\gamma,\mu). \tag{3.9}$$

Proof. Estimate (3.8) is immediately get from inequalities (2.2), (3.6) and (3.7). And we only need to prove estimate (3.9). Since

$$\partial_tu=(1+\partial_x^4)^{-1}g-\gamma u+(\mu-\gamma)(1+\partial_x^4)^{-1}u-(1+\partial_x^4)^{-1}\partial_xu-\frac{1}{2}(1+\partial_x^4)^{-1}\partial_xu^2,$$

it follows that

$$\begin{aligned}&\|\partial_tu\|_{H^s(\mathbb{R})}(t)\\ &=\|(1+\partial_x^4)^{-1}g-\gamma u+(\mu-\gamma)(1+\partial_x^4)^{-1}u-(1+\partial_x^4)^{-1}\partial_xu-\tfrac{1}{2}(1+\partial_x^4)^{-1}\partial_xu^2\|_{H^s(\mathbb{R})}(t)\\ &\le \|g\|_{H^{s-4}(\mathbb{R})}+C(\gamma,\mu)\|u\|_{H^s(\mathbb{R})}(t)+\|u^2\|_{H^{s-3}(\mathbb{R})}(t)\\ &\le \|g\|_{H^{-2}(\mathbb{R})}+C(\gamma,\mu)\|u\|_{H^s(\mathbb{R})}(t)+\|u^2\|_{H^{-1}(\mathbb{R})}(t)\\ &\le \|g\|_{H^{-2}(\mathbb{R})}+C(\gamma,\mu)\|u\|_{H^s(\mathbb{R})}(t)+\|u\|^2(t),\qquad \forall t\in[0,\infty),\quad s\in[0,2),\end{aligned}$$

here in the last estimate we have used the fact (see [6])

$$\|f_1f_2\|_{H^s(\mathbb{R})}\le C\|f_1\|_{L^2(\mathbb{R})}\|f_2\|_{L^2(\mathbb{R})},\quad \forall s\in[-1,-\frac{1}{2}).$$

Recalling estimates (3.1) and (3.8),we immediately get estimate (3.9). ∎

4. Proof of Theorem 1.2

In this section, we main to prove that the solution sequence $\{u_n(\cdot,t_n)\}_{n=1}^{\infty}$ to the damped Rosenau equation

$$\begin{cases}\partial_t u_n+\partial_t\partial_x^4 u_n+\gamma\partial_x^4 u_n+\mu u_n+\partial_x u_n+u_n\partial_x u_n=g(x), & x\in\mathbb{R}, \quad t>0,\\ u_n(x,0)=u_{0,n}(x), & x\in\mathbb{R},\end{cases} \tag{4.1}$$

is precompact in $H^s(\mathbb{R})$ for any $s\in[0,2)$ and any initial data sequence

$$\{u_{0,n}\}_{n=1}^{\infty}\subset B_s(0,r)=\{f:\|f\|_{H^s(\mathbb{R})}\le r\}.$$

And we mainly use Lemma 2.1 to reach this goal.

The condition $(c1)$:

$$\sup_n\|u_n(\cdot,t_n)\|_{H^s(\mathbb{R})}\le C(B), \tag{4.2}$$

with $s\in[0,2)$ is implied from (3.8).

And we only need to prove
condition $(c2)$:

$$\lim\sup_n\|u_n(\cdot,t_n)\|_{H^s(|x|>N)}\to 0, \qquad N\to\infty; \tag{4.3}$$

condition $(c3)$:

$$\lim\sup_n\|P_{>N}u_n(\cdot,t_n)\|_{H^s(\mathbb{R})}\to 0, \qquad N\to\infty; \tag{4.4}$$

From Corollary 3.1, there exists a constant M_3 only depends on s, r, γ and μ, such that

$$\|u_n\|_{C([0,\infty);H^s(\mathbb{R}))}+\|\partial_t u_n\|_{C([0,\infty);H^s(\mathbb{R}))}\le M_3, \qquad \forall n\in\mathbb{Z}^+, s\in[0,2). \tag{4.5}$$

Let

$$v_{N,n}(x,t)=u_n(x,t)(1-\phi(\frac{x}{N})),$$

where $\phi(x)$ is a nonincreasing and smooth cut-off function defined as

$$\phi(x)=\begin{cases}1, & |x|<1,\\ 0, & |x|>2.\end{cases}$$

Operating multiplier $(1-\phi(\frac{x}{N}))$ to the first equation in (4.1), it yields

$$\partial_t v_{N,n}+\partial_t\partial_x^4 v_{N,n}+\gamma\partial_x^4 v_{N,n}+\mu v_{N,n}+\partial_x v_{N,n}+v_{N,n}\partial_x u_n=G, \tag{4.6}$$

where

$$G(x,t)=g(x)(1-\phi(\frac{x}{N}))+\frac{1}{N}\phi^{'}(\frac{x}{N})u_n+\sum_{k=0}^{3}C_4^k\partial_t\partial_x^k u_n\partial_x^{4-k}(1-\phi(\frac{x}{N}))$$
$$+\gamma\sum_{k=0}^{3}C_4^k\partial_x^k u_n\partial_x^{4-k}(1-\phi(\frac{x}{N})).$$

Operating I_s operator on both sides of equation (4.6), multiplying it by $I_s v_{N,n}$ and integrating over $\mathbb{R}$, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|I_s v_{N,n}\|^2+\|I_s\partial_x^2 v_{N,n}\|^2\right)+\gamma\|I_s\partial_x^2 v_{N,n}\|^2+\mu\|I_s v_{N,n}\|^2\\ =&-(I_s(v_{N,n}\partial_x u_n),I_s v_{N,n})+(I_s G,I_s v_{N,n}).\end{aligned} \tag{4.7}$$

In order to get the uniform estimates of $\|I_s v_{N,n}\|_{C([0,\infty);H^2(\mathbb{R}))}$, we need to estimate the two terms on the right hand sides of equation (4.7).

For $-(I_s(v_{N,n}\partial_x u_n),I_s v_{N,n})$**:**
Using integrating by parts formula for several times, Young's inequality, Holder's inequality, $H^1(\mathbb{R})\hookrightarrow C(\mathbb{R})$ and $\|I_s f\|_{H^p(\mathbb{R})}\le\|f\|_{H^p(\mathbb{R})}$, $\forall p\in\mathbb{R}, s\in[0,2)$, it follows that

$$\begin{aligned}&-(I_s(v_{N,n}\partial_x u_n),I_s v_{N,n})\\ =&-\frac{1}{2}(I_s(\partial_x u_n^2(1-\phi(\frac{x}{N}))),I_s v_{N,n})\\ =&-\frac{1}{2N}(I_s(u_n^2\phi'(\frac{x}{N})),I_s v_{N,n})+\frac{1}{2}(I_s(u_n^2(1-\phi(\frac{x}{N}))),\partial_x I_s v_{N,n})\\ \le&\frac{C}{N}\|I_s(u_n^2\phi'(\frac{x}{N}))\|_{H^{-1}(\mathbb{R})}\|I_s v_{N,n}\|_{H^1(\mathbb{R})}+C\|I_s\partial_x v_{N,n}\|_{H^1(\mathbb{R})}\|I_s(u_n^2(1-\phi(\frac{x}{N})))\|_{H^{-1}(\mathbb{R})}\\ \le&\epsilon\|I_s v_{N,n}\|^2_{H^1(\mathbb{R})}+\frac{C(\epsilon)}{N}\|I_s(u_n^2\phi'(\frac{x}{N}))\|^2_{H^{-1}(\mathbb{R})}\\ &+\epsilon\|I_s\partial_x v_{N,n}\|^2_{H^1(\mathbb{R})}+C(\epsilon)\|I_s(u_n^2(1-\phi(\frac{x}{N})))\|^2_{H^{-1}(\mathbb{R})}\\ \le&4\epsilon\left(\|I_s v_{N,n}\|^2+\|I_s\partial_x^2 v_{N,n}\|^2\right)+\frac{C(\epsilon)}{N}\|u_n^2\phi'(\frac{x}{N})\|^2_{H^{-1}}+C(\epsilon)\|u_n^2(1-\phi(\frac{x}{N}))\|^2_{H^{-1}}.\end{aligned}$$

In order to continue to estimate the above inequality, we denote

$$M_4=\max\{\sup_{x\in\mathbb{R}}|\phi(x)|,\sup_{x\in\mathbb{R}}|\phi'(x)|,\sup_{x\in\mathbb{R}}|\phi''(x)|,\sup_{x\in\mathbb{R}}|\phi'''(x)|\}.$$

Recalling estimate (4.5) and using estimate (3.4) again, we immediately get

$$\|u_n^2\phi'(\tfrac{x}{N})\|_{H^{-1}(\mathbb{R})}\le\|u_n\|\|u_n\phi'(\tfrac{x}{N})\|\le CM_4\|u_n\|^2\le CM_4M_3^2;$$

Similarly,

$$\|u_n^2(1-\phi(\tfrac{x}{N}))\|_{H^{-1}(\mathbb{R})}\le CM_4M_3^2.$$

Substituting the above two estimates into the estimate of $-(I_s(v_{N,n}\partial_x u_n),I_s v_{N,n})$, we arrive at

$$-(I_s(v_{N,n}\partial_x u_n),I_s v_{N,n})\le 4\epsilon\|I_s v_{N,n}\|^2_{H^2(\mathbb{R})}+\tfrac{C(\epsilon)M_4^2M_3^4}{N}+C(\epsilon)M_4^2M_3^4. \tag{4.8}$$

For the term $(I_s G,I_s v_{N,n})$**:**
Since $(I_s G,I_s v_{N,n})=\sum_{j=1}^4 B_j$, where

$$\begin{aligned}B_1=&\left(I_s(g(x)(1-\phi(\frac{x}{N}))),I_s v_{N,n}\right),\\ B_2=&\frac{1}{N}\left(I_s\phi'(\frac{x}{N})u_n,I_s v_{N,n}\right),\end{aligned}$$

$$B_3 = \gamma \sum_{k=0}^{3} C_4^k \left(I_s \partial_x^k u_n \partial_x^{4-k} (1-\phi(\frac{x}{N})), I_s v_{N,n} \right),$$
$$B_4 = \sum_{k=0}^{3} C_4^k \left(I_s \partial_t \partial_x^k u_n \partial_x^{4-k} (1-\phi(\frac{x}{N})), I_s v_{N,n} \right).$$

For B_1:

Applying Cauchy-Schwartz inequality, Young's inequality and inequality (2.2), we get that

$$\begin{aligned} B_1 &\le \|I_s(g(x)(1-\phi(\frac{x}{N})))\| \|I_s v_{N,n}\| \\ &\le \epsilon \|I_s v_{N,n}\|^2 + C(\epsilon)\|I_s(g(x)(1-\phi(\frac{x}{N})))\|^2 \\ &\le \epsilon \|I_s v_{N,n}\|^2 + C(\epsilon)\|g(x)(1-\phi(\frac{x}{N}))\|^2 \\ &\le \epsilon \|I_s v_{N,n}\|^2 + C(\epsilon)\|g\|^2_{L^2(|x|>N)}; \end{aligned} \tag{4.9}$$

For B_2: Similar to the estimate of B_1, we also get

$$B_2 \le \epsilon \|I_s v_{N,n}\|^2 + \frac{C(\epsilon) M_3^2 M_4^2}{N}. \tag{4.10}$$

For B_3:

$$\begin{aligned} B_3 &= \gamma \sum_{k=0}^{3} C_4^k \left(I_s \partial_x^k u_n \partial_x^{4-k} (1-\phi(\frac{x}{N})), I_s v_{N,n} \right) \\ &\le C \sum_{k=0}^{3} \|I_s \partial_x^k u_n \partial_x^{4-k} \phi(\frac{x}{N})\|_{H^{-2}(\mathbb{R})} \|I_s v_{N,n}\|_{H^2(\mathbb{R})} \\ &\le \epsilon \left(\|I_s v_{N,n}\|^2 + \|I_s \partial_x^2 v_{N,n}\|^2 \right) + C(\epsilon) \sum_{k=0}^{3} \|I_s \partial_x^k u_n \partial_x^{4-k} (1-\phi(\frac{x}{N}))\|^2_{H^{-2}(\mathbb{R})}. \end{aligned}$$

In order to get the upper bound of B_3, we need to find the upper bound of $\|I_s \partial_x^k u_n \partial_x^{4-k} (1-\phi(\frac{x}{N}))\|_{H^{-2}(\mathbb{R})}, k=0,1,2,3$.

By direct computation, one can check that

$$\begin{aligned} &\partial_x u_n \partial_x^3 \phi(\frac{x}{N}) = \partial_x (u_n \partial_x^3 \phi(\frac{x}{N})) - u_n \partial_x^4 \phi(\frac{x}{N}); \\ &\partial_x^2 u_n \partial_x^2 \phi(\frac{x}{N}) = \partial_x^2 (u_n \partial_x^2 \phi(\frac{x}{N})) - 2\partial_x u_n \partial_x^3 \phi(\frac{x}{N}) - u_n \partial_x^4 \phi(\frac{x}{N}); \\ &\partial_x^3 u_n \partial_x \phi(\frac{x}{N}) = \partial_x^3 (u_n \partial_x \phi(\frac{x}{N})) - 3\partial_x^2 (u_n \partial_x^2 \phi(\frac{x}{N})) + 3\partial_x (u_n \partial_x^3 \phi(\frac{x}{N})) - u_n \partial_x^4 \phi(\frac{x}{N}). \end{aligned}$$

Then for $\|I_s \partial_x^k u_n \partial_x^{4-k} \phi(\frac{x}{N})\|_{H^{-2}(\mathbb{R})} (k=0,1,2,3)$, one has
as $k=0$,

$$\|I_s u_n \partial_x^4 \phi(\frac{x}{N})\|_{H^{-2}(\mathbb{R})} \le \|u_n \partial_x^4 \phi(\frac{x}{N})\| \le \frac{M_3 M_4}{N};$$

as $k=1$,

$$\begin{aligned} \|I_s \partial_x u_n \partial_x^3 \phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})} &\le \|\partial_x u_n \partial_x^3 \phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})} \\ &\le \|\partial_x (u_n \partial_x^3 \phi(\tfrac{x}{N})) - u_n \partial_x^4 \phi(\tfrac{x}{N})\|_{H^{-1}(\mathbb{R})} \\ &\le \|u_n \partial_x^3 \phi(\tfrac{x}{N})\| + \|u_n \partial_x^4 \phi(\tfrac{x}{N})\| \\ &\le \tfrac{2M_3 M_4}{N}; \end{aligned}$$

as $k=2$,

$$\begin{aligned}\|I_s\partial_x^2u\partial_x^2\phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})} &\le \|\partial_x^2u\partial_x^2\phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})}\\ &=\|\partial_x^2(u_n\partial_x^2\phi(\tfrac{x}{N}))-2\partial_xu_n\partial_x^3\phi(\tfrac{x}{N})-u_n\partial_x^4\phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})}\\ &\le \|u_n\partial_x^2\phi(\tfrac{x}{N})\|+2\|\partial_xu_n\partial_x^3\phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})}+\|u_n\partial_x^4\phi(\tfrac{x}{N})\|\\ &\le \tfrac{6M_3M_4}{N};\end{aligned}$$

as $k=3$, we claim that

$$\|I_s\partial_x^3u_n\partial_x\phi(\frac{x}{N})\|_{H^{-2}(\mathbb{R})}\le\begin{cases}\frac{C(1+A)M_3M_4+CA^{2-s}M_3M_4}{N}, & s\in[0,1),\\ \frac{9M_3M_4}{N}, & s\in[1,2).\end{cases}$$

In fact, for any $s\in[1,2)$, recalling estimate (3.8), and the definition of M_3 and M_4, one has

$$\begin{aligned}&\|I_s\partial_x^3u_n\partial_x\phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})}\\ \le\ &\|\partial_x^3u_n\partial_x\phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})}\\ =\ &\|\partial_x^3(u_n\partial_x\phi(\tfrac{x}{N}))-3\partial_x^2(u_n\partial_x^2\phi(\tfrac{x}{N}))+3\partial_x(u_n\partial_x^3\phi(\tfrac{x}{N}))-u_n\partial_x^4\phi(\tfrac{x}{N})\|_{H^{-2}(\mathbb{R})}\\ \le\ &\|\partial_x(u_n\partial_x\phi(\tfrac{x}{N}))\|+3\|u_n\partial_x^2\phi(\tfrac{x}{N})\|+3\|u_n\partial_x^3\phi(\tfrac{x}{N})\|+\|u_n\partial_x^4\phi(\tfrac{x}{N})\|\\ \le\ &\tfrac{9M_3M_4}{N}.\end{aligned}$$

For any $s\in[0,1)$: since the symmetric, decreasing and smoothing function

$$m(\xi)=\begin{cases}1, & |\xi|\le A,\\ (\frac{|\xi|}{A})^{s-2}, & |\xi|\ge 2A,\end{cases}$$

one can check that

$$\frac{m^2(\xi)(1+\xi^6)}{<\xi>^4}\le(1+A)^2,\qquad \text{as } |\xi|\le 2A;$$

$$\frac{m^2(\xi)(1+\xi^6)}{<\xi>^4}\le A^{4-2s},\qquad \text{as } s\in[0,1) \quad\text{and}\quad |\xi|\ge 2A;$$

then $\frac{m^2(\xi)(1+\xi^6)}{<\xi>^4}\lesssim(1+A+A^{2-s})^2$. Therefore,

$$\begin{aligned}&\|I_s\partial_x^3u_n\partial_x\phi(\frac{x}{N})\|^2_{H^{-2}(\mathbb{R})}\\ =&\int_{\mathbb{R}}\frac{m^2(\xi)}{<\xi>^4}|\mathcal{F}(\partial_x^3u_n\partial_x\phi(\frac{x}{N}))|^2d\xi\\ =&\int_{\mathbb{R}}\frac{m^2(\xi)}{<\xi>^4}|\mathcal{F}(\partial_x^3(u_n\partial_x\phi(\frac{x}{N}))-3\partial_x^2(u_n\partial_x^2\phi(\frac{x}{N}))+3\partial_x(u_n\partial_x^3\phi(\frac{x}{N}))-u_n\partial_x^4\phi(\frac{x}{N}))|^2d\xi\\ \lesssim&\int_{\mathbb{R}}\frac{m^2(\xi)(1+\xi^6)}{<\xi>^4}\\ &\quad\times\Big(|\mathcal{F}(u_n\partial_x\phi(\frac{x}{N})|^2+|\mathcal{F}(u_n\partial_x^2\phi(\frac{x}{N}))|^2+|\mathcal{F}(u_n\partial_x^3\phi(\frac{x}{N}))|^2+|\mathcal{F}(u_n\partial_x^4\phi(\frac{x}{N}))|^2\Big)d\xi\\ \lesssim&(1+A+A^{2-s})^2\Big(\|u_n\partial_x\phi(\frac{x}{N})\|^2+\|u_n\partial_x^2\phi(\frac{x}{N})\|^2+\|u_n\partial_x^3\phi(\frac{x}{N})\|^2+\|u_n\partial_x^4\phi(\frac{x}{N})\|^2\Big)\end{aligned}$$

$$\lesssim \left(\frac{(1+A+A^{2-s})M_3M_4}{N}\right)^2.$$

Consequently,

$$B_3 \le \epsilon\left(\|I_sv_{N,n}\|^2 + \|I_s\partial_x^2 v_{N,n}\|^2\right) + \frac{C(\epsilon,A,M_3,M_4)}{N}. \tag{4.11}$$

For B_4:

Repeating the similar argument as the estimate of B_3 and using estimate (3.9), we also get

$$B_4 \le \epsilon\left(\|I_sv_{N,n}\|^2 + \|I_s\partial_x^2 v_{N,n}\|^2\right) + \frac{C(\epsilon,A,M_3,M_4)}{N}. \tag{4.12}$$

Therefore, combing estimates (4.9), (4.10), (4.11) and (4.12), we arrive at

$$(I_sG, I_sv_{N,n}) \le 4\epsilon\left(\|I_sv_{N,n}\|^2 + \|I_s\partial_x^2 v_{N,n}\|^2\right) + C(\epsilon)\|g\|^2_{L^2(|x|>N)} + \frac{C(\epsilon,A,M_3,M_4)}{N}. \tag{4.13}$$

Substituting inequalities (4.8) and (4.13) into equation (4.7), it follows

$$\begin{aligned}&\tfrac{d}{2dt}\left(\|I_sv_{N,n}\|^2 + \|I_s\partial_x^2 v_{N,n}\|^2\right) + \gamma\|I_s\partial_x^2 v_{N,n}\|^2 + \mu\|I_sv_{N,n}\|^2\\ &\le 8\epsilon\left(\|I_sv_{N,n}\|^2 + \|I_s\partial_x^2 v_{N,n}\|^2\right) + C(\epsilon)\|g\|^2_{L^2(|x|>N)} + \tfrac{C(\epsilon,A,M_3,M_4)}{N} + C(\epsilon,M_3,M_4).\end{aligned} \tag{4.14}$$

Choosing $a = \min\{\mu,\gamma\}$, and $\epsilon = \frac{a}{16}$ in the above estimate, then

$$\frac{d}{dt}\left(\|I_sv_{N,n}\|^2 + \|I_s\partial_x^2 v_{N,n}\|^2\right) + a\left(\|I_s\partial_x^2 v_{N,n}\|^2 + \|I_sv_{N,n}\|^2\right) \le C\|g\|^2_{L^2(|x|>N)} + C, \tag{4.15}$$

where the constant C only depends on A, M_3 and M_4.

In order to obtain the unform estimate of $\|v_{N,n}(x,0)\|_{H^s(\mathbb{R})}$, we introduce a linear function

$$L(v) = v\left(1-\phi\left(\frac{x}{N}\right)\right), \quad v \in H^s(\mathbb{R}).$$

One can check that $L(\cdot)$ fulfil the three conditions of the Tartar's linear interpolation theorem [30], then

$$L: H^s(\mathbb{R}) \to H^s(\mathbb{R}), \quad s \in [0,2).$$

Moreover,

$$\|L(v)\|_{H^s(\mathbb{R})} \le C\|v\|_{H^s(\mathbb{R})},$$

where C only depends on M_4. Thanks to

$$v_{N,n}(x,t) = u_n(x,t)\left(1-\phi\left(\frac{x}{N}\right)\right)$$

and the initial data sequence $\{u_{0,n}\}_{n=1}^{\infty} \subset B_s(0,r) = \{f : \|f\|_{H^s(\mathbb{R})} \le r, s \in [0,2)\}$, we immediately get that

$$\|v_{N,n}\|_{H^s(\mathbb{R})}(0) = \|u_{0,n}(1-\phi(\frac{x}{N}))\|_{H^s(\mathbb{R})} \le Cr. \tag{4.16}$$

With estimate (4.16) in hand, applying Gronwall's lemma to inequality (4.15), the condition $g\in H^{-2}(\mathbb{R})$ and the fact inequality (2.2) holds, we finally arrive at

$$\begin{aligned}
&\lim_{N\to\infty}\lim_n\sup\|u_n\|^2_{H^s(|x|\geq N)}(t_n)\\
&=\lim_{N\to\infty}\lim_n\sup\|u_n(1-\phi(\tfrac{x}{N}))\|^2_{H^s(\mathbb{R})}(t_n)\\
&\leq\lim_{N\to\infty}\lim_n\sup\|I_s v_{N,n}\|^2_{H^2(\mathbb{R})}(t_n)\\
&\leq\lim_{N\to\infty}\lim_n\sup\Big(e^{-at_n}\|I_s v_{N,n}\|^2_{H^2(\mathbb{R})}(0)+e^{-at_n}t_n(C+\|g\|^2_{H^{-2}(|x|\geq N)})\Big)\\
&=0.
\end{aligned}$$

Recalling the definition of the Littlewood–Palay projection operator P_δ, operating $I_sP_{>N}$ on both sides of the damped Rosenau equation in (4.1), multiplying it by $I_sP_{>N}u_n$, and then integrating over $\mathbb{R}$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|I_sP_{>N}u_n\|^2+\|\partial_x^2I_sP_{>N}u_n\|^2\right)+\gamma\|\partial_x^2I_sP_{>N}u_n\|^2+\mu\|I_sP_{>N}u_n\|^2\\
&=-\frac{1}{2}\int_{\mathbb{R}}I_sP_{>N}\partial_xu_n^2I_sP_{>N}u_ndx+\int_{\mathbb{R}}I_sP_{>N}gI_sP_{>N}u_ndx.
\end{aligned}\tag{4.17}$$

Using the fact $\mathcal{F}(\partial_xI_sP_{>N}f)=\mathcal{F}(I_sP_{>N}\partial_xf)$, where $\mathcal{F}$ denotes the Fourier transformation with respect to the space variable, integration by parts formula, Holder inequality, and inequality (2.2), it follows

$$\begin{aligned}
-\frac{1}{2}\int_{\mathbb{R}}I_sP_{>N}\partial_xu_n^2I_sP_{>N}u_ndx&=-\frac{1}{2}\int_{\mathbb{R}}\partial_xI_sP_{>N}u_n^2I_sP_{>N}u_ndx\\
&=\frac{1}{2}\int_{\mathbb{R}}I_sP_{>N}u_n^2I_sP_{>N}\partial_xu_ndx\\
&\leq\|I_sP_{>N}\partial_xu_n\|\|I_sP_{>N}u_n^2\|\\
&\lesssim\frac{1}{N}\|\partial_x^2I_su_n\|\|P_{>N}u_n^2\|\\
&\lesssim\frac{1}{N}\|I_su_n\|_{H^2(\mathbb{R})}\|u_n^2\|.
\end{aligned}$$

Using the embedding $H^{\frac{1}{2}+\epsilon}(\mathbb{R})\hookrightarrow C(\mathbb{R})$, inequalities (3.6) and (3.7), we have

$$\begin{aligned}
&\frac{1}{N}\|I_su_n\|_{H^2(\mathbb{R})}\|u_n^2\|\\
&\lesssim\frac{1}{N}\|I_su_n\|_{H^2(\mathbb{R})}\|u_n\|\|u_n\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R})}\\
&\lesssim\frac{1}{N}\|I_su_n\|_{H^2(\mathbb{R})}\|I_{\frac{1}{2}+\epsilon}u_n\|^2_{H^2(\mathbb{R})}\\
&\lesssim\frac{1}{N}C(\|I_su_{0,n}\|_{H^2(\mathbb{R})},\|g\|_{H^{-2}(\mathbb{R})},\mu,\nu)C(\|I_{\frac{1}{2}+\epsilon}u_{0,n}\|_{H^2(\mathbb{R})},\|g\|_{H^2(\mathbb{R})},\mu,\nu)\\
&\lesssim\frac{1}{N}C(\|u_{0,n}\|_{H^s(\mathbb{R})},\|g\|_{H^{-2}(\mathbb{R})},\mu,\nu),\qquad\text{where}\quad s\geq\frac{1}{2}+\epsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{\mathbb{R}}I_sP_{>N}gI_sP_{>N}u_ndx&\leq\|I_sP_{>N}u_n\|_{H^1(\mathbb{R})}\|I_sP_{>N}g\|_{H^{-1}(\mathbb{R})}\\
&\lesssim\frac{1}{N}\|I_su_n\|_{H^2(\mathbb{R})}\|g\|_{H^{-1}(\mathbb{R})}
\end{aligned}$$

$$\lesssim \frac{1}{N} C(\|u_{0,n}\|_{H^s(\mathbb{R})}, \|g\|_{H^{-1}(\mathbb{R})}), \qquad \text{where} \quad s \in [0,2).$$

Substituting above two estimates into inequality (4.17), for any $s\in(\frac{1}{2},2)$, we arrive at

$$\frac{d}{dt}\left(\|I_sP_{>N}u_n\|^2+\|\partial_x^2 I_sP_{>N}u_n\|^2\right)+2\gamma\|\partial_x^2 I_sP_{>N}u_n\|^2+2\mu\|I_sP_{>N}u_n\|^2\le\frac{C_*}{N}, \quad (4.18)$$

where C_* is a constant depending only on s, γ and $\|g\|_{H^{-1}(\mathbb{R})}$. Hence, choosing $a=\min\{\mu,\gamma\}>0$, after applying Lemma 2.4 in [44] and inequality (2.2), we have

$$\limsup_n \|P_{>N}u_n(\cdot,t_n)\|_{H^s(\mathbb{R})} \le \limsup_n \|I_sP_{>N}u_n(\cdot,t_n)\|_{H^2(\mathbb{R})} \le \frac{C_*}{N}.$$

Consequently, for any $s\in(\frac{1}{2},2)$,

$$\limsup_n \|P_{>N}u_n(\cdot,t_n)\|_{H^s(\mathbb{R})} \to 0 \quad \text{as } N\to\infty.$$

Therefore, combing conditions (4.2), (4.3), (4.4) and Lemma 2.2, we can claim that problem (1.1) has a global attractor $\mathcal{A}_s$ for any $s\in(\frac{1}{2},2)$.

Finally, supposing that u is in the global attractor $\mathcal{A}_s$, we immediately get

$$\|P_{>N}u\|^2_{H^s(\mathbb{R})} = \sup_n \|P_{>N}u_n\|^2_{H^s(\mathbb{R})} \le \sup_n \|I_sP_{>N}u_n(\cdot,t_n)\|^2_{H^s(\mathbb{R})} \le \frac{C_*}{N}.$$

Therefore, $u\in H^{s+\frac{1}{2}-\epsilon}(\mathbb{R})$, for any $\epsilon>0$ and $s\in(\frac{1}{2},2)$.

5. Proof of Corollary 1.1

Proof. It is clear that

$$\mathcal{A}_2 \subseteq \mathcal{A}_s, \qquad \forall s\in(\frac{1}{2},2). \quad (5.1)$$

In order to prove $\mathcal{A}_2=\mathcal{A}_s$, we aim to prove

$$\mathcal{A}_s \subseteq \mathcal{A}_2, \qquad \forall s\in(\frac{1}{2},2). \quad (5.2)$$

Indeed, for any $\delta_1\in(0,\frac{1}{8}]$, $\delta_2=\frac{1}{4}$, let $s_1=\frac{1}{2}+\delta_1$ and

$$s_k = s_{k-1}+\frac{1}{2}-\delta_2 \qquad k=2,3,4,5,6.$$

One can easily check that $s_k\in(\frac{1}{2},2)$ for any $k=1,2,3,4,5,6$. From Theorem 1.2, the global attractor $\mathcal{A}_{s_1}\subseteq H^{s_2}(\mathbb{R})$. Since the set $\mathcal{A}_{s_1}$ is compact and then bounded in $H^{s_2}(\mathbb{R})$, from the definition of the global attractor $\mathcal{A}_{s_2}$, it follows

$$\lim_{t\to\infty} \operatorname{dist}(S(t)\mathcal{A}_{s_1}, \mathcal{A}_{s_2}) = 0,$$

i.e. $\mathcal{A}_{s_1}=\lim_{t\to\infty} S(t)\mathcal{A}_{s_1}\subseteq\mathcal{A}_{s_2}$. Repeating the similar process, we get that

$$\mathcal{A}_{s_1}\subseteq\mathcal{A}_{s_2}\subseteq\mathcal{A}_{s_3}\subseteq\mathcal{A}_{s_4}\subseteq\mathcal{A}_{s_5}\subseteq\mathcal{A}_{s_6}.$$

Using Theorem 1.2 again, Corollary 1.4 in [44], after choosing $\delta_3=\frac{1}{4}+\delta_1$, we have

$$\mathcal{A}_{s_6}\subseteq H^{s_6+\frac{1}{2}-\delta_3}(\mathbb{R})=\mathcal{A}_2.$$

Then

$$\mathcal{A}_{s_1} \subseteq \mathcal{A}_{s_2} \subseteq \mathcal{A}_{s_3} \subseteq \mathcal{A}_{s_4} \subseteq \mathcal{A}_{s_5} \subseteq \mathcal{A}_{s_6} \subseteq \mathcal{A}_2. \tag{5.3}$$

Since for any $s \in [\frac{5}{8}, 2)$, we can find some global attractor $\mathcal{A}_{s_k}$, $k \in \{1,2,3,4,5,6\}$, such that $\mathcal{A}_s \subseteq \mathcal{A}_{s_k}$, then combing (5.3) we arrive at

$$\mathcal{A}_s \subseteq \mathcal{A}_2, \qquad s \in [\frac{5}{8}, 2). \tag{5.4}$$

And for any $s \in (\frac{1}{2}, \frac{5}{8})$, since $s_1 = \frac{1}{2} + \delta_1$ with $\delta_1 \in (0, \frac{1}{8})$, (5.3) implies

$$\mathcal{A}_s \subseteq \mathcal{A}_2, \qquad s \in (\frac{1}{2}, \frac{5}{8}). \tag{5.5}$$

Now, combing (5.4) and (5.5), we immediately get (5.2).

Therefore, (5.1) and (5.2) imply $\mathcal{A}_s = \mathcal{A}_2$ for any $s \in (\frac{1}{2}, 2)$. □

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