COMMUN. MATH. SCI.
Vol. 15, No. 8, pp. 2219–2278

UNIFORM REGULARITY AND VANISHING VISCOSITY LIMIT FOR THE COMPRESSIBLE NEMATIC LIQUID CRYSTAL FLOWS*

JINCHENG GAO†, BOLING GUO‡, AND YAQING LIU§

Abstract. In this paper, we study the uniform regularity and vanishing viscosity limit for the compressible nematic liquid crystal flows in three-dimensional bounded domains. One establishes the uniform estimates for the solutions in a conormal Sobolev space and obtains the uniform estimates for the density and velocity in $W^{1,\infty}$. Then, it is shown that there exists a unique strong solution for the compressible nematic liquid crystal flows in a finite time interval which is independent of the viscosity coefficient. Based on the uniform estimates, we also obtain the convergence rate of the viscous solutions to the inviscid ones with a rate of convergence.

Keywords. Nematic liquid crystal flows; vanishing viscosity limit; conormal Sobolev space; convergence rate.

AMS subject classifications. 35Q35; 35B65; 76N10.

1. Introduction

Nematic liquid crystals contain a large number of elongated, rod-like molecules and possess the same orientational order. The continuum theory of liquid crystals due to Ericksen [1] and Leslie [2] was developed around the 1960s, see also [3]. Since then, numerous researchers have obtained some important developments for liquid crystals not only in theory but also in the application. The Ericksen–Leslie system is a macroscopic description of the time evolution of the materials under the influence of both the flow velocity field and the macroscopic description of the microscopic orientation configuration of rod-like liquid crystals. In this paper, we investigate the motion of compressible nematic liquid crystal flows, which are governed by the following simplified version of the Ericksen–Leslie equations as follows

$$\rho_t^\varepsilon + \mathrm{div}(\rho^\varepsilon u^\varepsilon) = 0, \tag{1.1}$$

$$\rho^\varepsilon u_t^\varepsilon + \rho^\varepsilon u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon = \mu\varepsilon\Delta u^\varepsilon + (\mu+\lambda)\varepsilon\nabla \mathrm{div} u^\varepsilon - \nabla d^\varepsilon \cdot \Delta d^\varepsilon, \tag{1.2}$$

$$d_t^\varepsilon + u^\varepsilon \cdot \nabla d^\varepsilon = \Delta d^\varepsilon + |\nabla d^\varepsilon|^2 d^\varepsilon, \tag{1.3}$$

where $(x,t) \in \Omega \times (0,T)$. Here $0 < T \leq +\infty$ and Ω is a bounded domain of $\mathbb{R}^3$. The unknown functions $\rho^\varepsilon(x,t)$, $u^\varepsilon(x,t) = (u_1^\varepsilon(x,t), u_2^\varepsilon(x,t), u_3^\varepsilon(x,t),)$ and $d^\varepsilon(x,t) = (d_1^\varepsilon(x,t), d_2^\varepsilon(x,t), d_3^\varepsilon(x,t))$ represent the density, velocity field of the fluid and the macroscopic average of the nematic liquid crystal orientation field, respectively. The scalar function $p^\varepsilon = p(\rho^\varepsilon)$ is the pressure function and satisfies the $\gamma-$law

$$p(\rho) = \rho^\gamma \quad \text{with} \quad \gamma > 1.$$

The viscous coefficients μ and λ satisfy the physical restrictions

$$\mu > 0, \quad 2\mu + 3\lambda \geq 0.$$

*Received: November 29, 2016; accepted (in revised form): August 23, 2017. Communicated by Chun Liu.

†School of Mathematics, Sun Yat-sen University, 510275, Guangzhou, P.R. China (gaojc1998@163.com).

‡Institute of Applied Physics and Computational Mathematics, 100088, Beijing, P.R. China (gbl@iapcm.ac.cn).

§School of Applied Science, Beijing Information Science and Technology University, 100192, Beijing, P.R. China (liuyaqing1981@163.com).

Here the parameter $\varepsilon>0$ is the inverse of the Reynolds number.

Corresponding to the system (1.1)-(1.3), one imposes the following Navier-slip type and Neumann boundary conditions:

$$u^\varepsilon\cdot n=0,\quad ((Su^\varepsilon)n)_\tau=-(Au^\varepsilon)_\tau,\quad \text{and}\quad \frac{\partial d^\varepsilon}{\partial n}=0,\quad \text{on } \partial\Omega, \tag{1.4}$$

where n is the unit outward vector normal to $\partial\Omega$, A is a given smooth symmetric matrix(see [4]), $(Au^\varepsilon)_\tau$ represents the tangential component of Au^ε. The strain tensor Su^ε is defined by

$$Su^\varepsilon\triangleq\frac{1}{2}\left((\nabla u^\varepsilon)+(\nabla u^\varepsilon)^t\right).$$

For any smooth solutions v, it is easy to check that

$$(2S(v)n-(\nabla\times v)\times n)_\tau=-(2S(n)v)_\tau,$$

see [5] for detail. Define $B\triangleq 2(A-S(n))$, then the boundary condition (1.4) can be written in the form of the vorticity as

$$u^\varepsilon\cdot n=0,\quad n\times(\nabla\times u^\varepsilon)=[Bu^\varepsilon]_\tau,\quad \text{and}\quad \frac{\partial d^\varepsilon}{\partial n}=0,\quad \text{on } \partial\Omega. \tag{1.5}$$

Actually, it turns out that the form (1.5) will be more convenient than the form (1.4) in the energy estimates, see [6,7].

In this paper, we are interested in the existence of strong solution of (1.1)-(1.3) with uniform bounds on an interval of time independent of viscosity coefficient $\varepsilon\in(0,1]$ and the vanishing viscosity limit to the corresponding inviscid nematic liquid crystal flows as viscosity coefficient ε vanishes, i.e.

$$\rho_t+\operatorname{div}(\rho u)=0,\qquad (x,t)\in\Omega\times(0,T), \tag{1.6}$$

$$\rho u_t+\rho u\cdot\nabla u+\nabla p=-\nabla d\cdot\Delta d,\qquad (x,t)\in\Omega\times(0,T), \tag{1.7}$$

$$d_t+u\cdot\nabla d=\Delta d+|\nabla d|^2 d,\qquad (x,t)\in\Omega\times(0,T), \tag{1.8}$$

with the boundary conditions

$$u\cdot n=0,\quad \text{and}\quad \frac{\partial d}{\partial n}=0,\quad \text{on } \partial\Omega. \tag{1.9}$$

If the parameter $\varepsilon>0$ is fixed, Jiang et al. [8] established the global existence of weak solutions to the initial-boundary problem with large initial energy and without any smallness condition on the initial density and velocity if some component of the initial direction field is small. Recently, Lin et al. [9] established the existence of global weak solutions in three-dimensional space, provided the initial orientational director field lies in the hemisphere S^2_+. Local existence of a unique strong solution was proved when the initial data was sufficiently regular and satisfied a natural compatibility condition in a recent work [10]. On the other hand, Hu and Wu [11] obtained the existence and uniqueness of global strong solution in critical Besov spaces provided that the initial data were close to an equilibrium state $(1,0,w_0)$ with a constant vector $w_0\in S^2$. Recently, Gao et al. [12] established the global-in-time existence for the compressible nematic liquid crystal flows in three dimensional whole space, and time decay rates for the high-order spatial derivatives of density, velocity and director were obtained by using the

Green function, energy estimates and Fourier splitting method, see also other models in [13]. For more results about the compressible Ericksen–Leslie system (1.1)-(1.3), the readers can refer to [14–19] and references therein.

When the density and director field are a constant scalar function and constant vector field, respectively, the systems (1.1)-(1.3) and (1.6)-(1.8) are the well-known incompressible Navier–Stokes equations and incompressible Euler equations, respectively. There is lots of literature on the uniform bounds and the vanishing viscosity limit for the Navier–Stokes equations without boundaries, refer to [20–23]. The time of existence T^ε always depend on the viscosity coefficient when the boundary appears. It is difficult to prove that the existence of time stays bounded away from zero. However, for the domain with some special types of Navier-slip boundary conditions, some uniform H^3 (or $W^{2,p}$, with p large enough) estimates and a uniform time of existence have recently been established in [6, 24, 25]. These uniform estimates in some limited regularity Sobolev space can be obtained because the main parts of boundary layer vanish by virtue of the special boundary conditions. For the three dimensional domains with smooth boundary, Masmoudi and Rousset [26] recently obtained conormal uniform estimates for the incompressible Navier–Stokes equations with Naiver-slip type boundary condition. Furthermore, they also applied the compact argument to establish the convergence of the viscous solutions to the inviscid ones. Due to the uniform estimates in [4], better convergence with rates have been studied in [4] and [27]. In particular, Xiao and Xin [27] have proved the convergence in $L^\infty(0,T;H^1)$ with a rate of convergence. Motivated by the work of [26] and [27], Gao et al. [28] investigated the vanishing viscosity limit of incompressible nematic liquid crystal flows. More precisely, we proved that there exists a unique strong solution for the incompressible nematic liquid crystal flows in a finite time interval which is independent of the viscosity coefficient and obtained the convergence rate of the viscous solutions to the inviscid ones.

For the compressible Navier–Stokes equations, Paddick [29] obtained uniform estimates in three dimensionl half-space with a Navier boundary condition, which was improved by Wang et al. [7] to generalized bounded domain. Specially, Wang et al. [7] shown that the boundary layers for the density must be weaker than the one for the velocity and established the convergence of the viscous solutions to the inviscid ones. For more results about the inviscid limit for the compressible Navier–Stokes equations, the readers can refer to [30,31] and the references therein. Motivated by the work of [28] and [7], we hope to investigate the vanishing viscosity limit for the compressible nematic liquid crystal flows (1.1)-(1.3).

Before stating the main results, we first explain the notations and conventions used throughout this paper. Similar to [7, 26], we suppose that the bounded domain $\Omega \subset \mathbb{R}^3$ has a covering such that

$$\Omega \subset \Omega_0 \cup_{k=1}^n \Omega_k,$$

where $\overline{\Omega}_0 \subset \Omega$, and in each $\Omega_k (k=1,...,n)$ there exists a function ψ_k such that

$$\begin{aligned} &\Omega \cap \Omega_k = \{x=(x_1,x_2,x_3) | x_3 > \psi_k(x_1,x_2)\} \cap \Omega_k, \\ &\partial\Omega \cap \Omega_k = \{x_3 = \psi_k(x_1,x_2)\} \cap \Omega_k. \end{aligned}$$

Here, Ω is said to be $\mathcal{C}^m$ if every function $\psi_k (k=1,...,n)$ is $\mathcal{C}^m$-function. In order to define the conormal Sobolev space, we suppose $(Z_k)_{1\le k\le 3}$ is a finite set of generators of vector fields that are tangential to the domain boundary $\partial\Omega$, and set

$$H^m_{co} = \{f \in L^2(\Omega) | Z^I f \in L^2(\Omega), \text{ for } |I| \le m\},$$

where $I=(k_1,k_2,k_3)$, and $Z^I f=\{Z^{k_1}Z^{k_2}Z^{k_3}f,\ \text{for}\ k_1+k_2+k_3=|I|\}$. We also define

$$\|u\|_m^2=\|u\|_{H^m_{co}}^2=\sum_{j=1}^3\sum_{|I|\le m}\|Z^I u_j\|_{L^2}^2,$$
$$\|u\|_{m,\infty}^2=\sum_{|I|\le m}\|Z^I u\|_{L^\infty}^2,\quad \|\nabla Z^m u\|^2=\sum_{|I|=m}\|\nabla Z^I u\|_{L^2}^2.$$

Noting that by using the covering of Ω, we also suppose that each vector field $(p^\varepsilon,u^\varepsilon,d^\varepsilon)$ is supported in one of the Ω_i, and moreover, in Ω_0 the norm $\|\cdot\|_m$ yields control of the standard H^m norm, whereas if $\Omega_i\cap\partial\Omega\neq\emptyset$, there is no control of the normal derivatives.

Since $\partial\Omega$ is given locally by $x_3=\psi(x_1,x_2)$ (we omit the subscript j of notational convenience), it is convenient to use the coordinates

$$\Psi:\ (y,z)\mapsto(y,\psi(y)+z)=x.$$

Hence, a basis can be given by the vector fields (e_{y^1},e_{y^2},e_z), where $e_{y^1}=(1,0,\partial_1\psi)^t$, $e_{y^2}=(0,1,\partial_2\psi)^t$, and $e_z=(0,0,-1)^t$. On the boundary, e_{y^1} and e_{y^2} are tangent to $\partial\Omega$, and in general, e_z is not a normal vector field. By using this parametrization, one can take as suitable vector fields compactly supported in Ω_j in the definition of the $\|\cdot\|_m$ norms

$$Z_i=\partial_{y^i}=\partial_i+\partial_i\psi\partial_z,\ i=1,2,\quad Z_3=\varphi(z)\partial_z,$$

where $\varphi(z)=\frac{z}{1+z}$ is smooth, supported in $\mathbb{R}_+$ and enjoys the properties $\varphi(0)=0,\varphi'(0)>0,\varphi(z)>0$ for $z>0$. Furthermore, it is easy to check that

$$Z_kZ_j=Z_jZ_k,\quad j,k=1,2,3, \tag{1.10}$$

and

$$\partial_zZ_i=Z_i\partial_z,\ i=1,2;\quad \partial_zZ_3\neq Z_3\partial_z.$$

We shall still denote by ∂_j, $j=1,2,3$, or ∇ the derivatives in the physical space. The coordinates of a vector field u in the basis (e_{y^1},e_{y^2},e_z) will be denoted by u^i, and thus

$$u=u^1e_{y^1}+u^2e_{y^2}+u^3e_z.$$

We shall denote by u_j the coordinates in the standard basis of $\mathbb{R}^3$, i.e, $u=u_1e_1+u_2e_2+u_3e_3$. Denote by n the unit outward normal in the physical space which is given locally by

$$n(x)\equiv n(\Psi(y,z))=\frac{1}{\sqrt{1+|\nabla\psi(y)|^2}}\begin{pmatrix}\partial_1\psi(y)\\ \partial_2\psi(y)\\ -1\end{pmatrix}\triangleq\frac{-N(y)}{\sqrt{1+|\nabla\psi(y)|^2}}, \tag{1.11}$$

and by Π the orthogonal projection

$$\Pi u\equiv\Pi(\Psi(y,z))u=u-[u\cdot n(\Psi(y,z))]n(\Psi(y,z)), \tag{1.12}$$

which gives the orthogonal projection on to the tangent space of the boundary. Note that n and Π are defined in the whole Ω_k and do not depend on z. For later use and notational convenience, set

$$\mathcal{Z}^\alpha=\partial_t^{\alpha_0}Z^{\alpha_1}=\partial_t^{\alpha_0}Z_1^{\alpha_{11}}Z_2^{\alpha_{12}}Z_3^{\alpha_{13}},$$

where α,α_0 and α_1 are the differential multi-indices with $\alpha=(\alpha_0,\alpha_1),\alpha_1=(\alpha_{11},\alpha_{12},\alpha_{13})$ and we also use the notation

$$\|f(t)\|_{\mathcal{H}^m}^2=\|f(t)\|_{\mathcal{H}^m}^2=\sum_{|\alpha|\le m}\|\mathcal{Z}^\alpha f(t)\|_{L_x^2}^2, \tag{1.13}$$

and

$$\|f(t)\|_{\mathcal{H}^{k,\infty}}=\sum_{|\alpha|\le k}\|\mathcal{Z}^\alpha f(t)\|_{L_x^\infty} \tag{1.14}$$

for a smooth space-time function $f(x,t)$. Throughout this paper, C denotes the positive generic constants independent of the viscosity coefficient ε. Similarly, we also denote by C_k a positive constant independent of $\varepsilon\in(0,1]$ which depends only on the $\mathcal{C}^k$-norm of the functions $\psi_j,\ j=1,...,n$. Here, $\|\cdot\|_{L^2}$ denotes the standard $L^2(\Omega;dx)$ norm, and $\|\cdot\|_{H^m}(m=1,2,3,...)$ denotes the Sobolev $H^m(\Omega,dx)$ norm. The notation $|\cdot|_{H^m}$ will be used for the standard Sobolev norm of functions defined on $\partial\Omega$, which only involves the tangential derivatives. $P(\cdot)$ denotes a polynomial function.

To obtain the uniform estimates for solutions to the nematic liquid crystal flows with Navier-slip and Neumann boundary conditions, we need to find a suitable functional space because the boundary layers may appear in the presence of physical boundaries. In the spirit of Wang et al. [7], we also investigate the vanishing viscosity limit for the nematic liquid crystal flows in conormal Sobolev space. Hence, the functional space should include some information for the direction field d. On the other hand, due to the nonlinear higher order derivatives term $\nabla d\cdot\Delta d$, one should control this term by using the dissipative term Δd on the right hand side of equation (1.2) which involves the time derivatives term d_t. Thus, one also includes some information involving the time derivatives in the functional space. Therefore, we define the functional space $X_m^\varepsilon(T)$ for a pair of functions $(u,p,d)(x,t)$ as follows

$$X_m^\varepsilon(T)=\{(p,u,d)\in L^\infty([0,T],L^2);\ \operatorname*{esssup}_{0\le t\le T}\|(p,u,d)(t)\|_{X_m^\varepsilon}<+\infty\}, \tag{1.15}$$

where the norm $\|(\cdot,\cdot)\|_{X_m^\varepsilon}$ is given by

$$\begin{aligned}\|(p,u,d)(t)\|_{X_m^\varepsilon}\triangleq&\|(u,p)(t)\|_{\mathcal{H}^m}^2+\|d(t)\|_{L^2}^2+\|\nabla d(t)\|_{\mathcal{H}^m}^2+\|(\nabla u,\Delta d)(t)\|_{\mathcal{H}^{m-1}}^2\\&+\|\nabla u(t)\|_{\mathcal{H}^{1,\infty}}^2+\sum_{k=0}^{m-2}\|\partial_t^k\nabla p(t)\|_{m-1-k}^2+\varepsilon\|\nabla\partial_t^{m-1}p(t)\|_{L^2}^2\\&+\|\Delta p(t)\|_{\mathcal{H}^1}^2+\varepsilon\|\Delta p(t)\|_{\mathcal{H}^2}^2.\end{aligned} \tag{1.16}$$

In the present paper, we supplement the nematic liquid crystal flows system (1.1)-(1.3) with initial data

$$(p^\varepsilon,u^\varepsilon,d^\varepsilon)(x,0)=(p_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)(x), \tag{1.17}$$

such that

$$0<\frac{1}{\hat{C}_0}\le\rho_0^\varepsilon\le\hat{C}_0<\infty,$$

and

$$\sup_{0<\varepsilon\le1}\|(p_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)\|_{X_m^\varepsilon}=\sup_{0<\varepsilon\le1}\{\|(u_0^\varepsilon,p_0^\varepsilon)\|_{\mathcal{H}^m}^2+\|d_0^\varepsilon\|_{L^2}^2+\|\nabla d_0^\varepsilon\|_{\mathcal{H}^m}^2+\|\nabla u_0^\varepsilon\|_{\mathcal{H}^{m-1}}^2$$

$$+\sum_{k=0}^{m-2}\|\partial_t^k\nabla p_0^\varepsilon\|_{m-1-k}^2+\varepsilon\|\nabla\partial_t^{m-1}p_0^\varepsilon\|_{L^2}^2+\|\Delta p_0^\varepsilon\|_{\mathcal{H}^1}^2$$
$$+\varepsilon\|\Delta p_0^\varepsilon\|_{\mathcal{H}^2}^2+\|\Delta d_0^\varepsilon\|_{\mathcal{H}^{m-1}}^2+\|\nabla u_0^\varepsilon\|_{\mathcal{H}^{1,\infty}}^2\}\le\widetilde{C}_0, \quad (1.18)$$

where $\widetilde{C}_0$ is a positive constant independent of $\varepsilon\in(0,1]$, and the time derivatives of initial data are defined through equation (1.1)-(1.3). Thus, the initial data $(\rho_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)$ is assumed to have a higher space regularity and compatibilities. Notice that the a priori estimates in Theorem 3.1 below are obtained in the case that the approximate solution is sufficiently smooth up to the boundary, and therefore, in order to obtain a selfconstained result, one needs to assume the approximated initial data satisfies the boundary compatibilities condition (1.5). For the initial data $(\rho_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)$ satisfying equation (1.17), it is not clear if there exists an approximate sequences $(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta},d_0^{\varepsilon,\delta})$ (δ being a regularization parameter) which satisfy the boundary compatibilities and $\|(p_0^{\varepsilon,\delta}-p_0^\varepsilon,u_0^{\varepsilon,\delta}-u_0^\varepsilon,d_0^{\varepsilon,\delta}-d_0^\varepsilon)\|_{X_m^\varepsilon}\to 0$ as $\delta\to 0$. Therefore, we set

$$X_{n,m}^{\varepsilon,ap}=\{(p,u,d)\in H^{4m}\times H^{4m}\times H^{4m+1}\,|\,\partial_t^k p,\partial_t^k u,\partial_t^k d,k=1,\dots,m \text{ are defined through the Equations } (1.1)-(1.3) \text{ and } \partial_t^k u,\ \partial_t^k\nabla d,k=0,\dots,m-1,\text{satisfy the boundary compatibility condition}\}.$$

and

$$X_{n,m}^\varepsilon=\text{ the closure of } X_{n,m}^{\varepsilon,ap} \text{ in the norm } \|(\cdot,\cdot)\|_{X_m^\varepsilon}.$$

Now, we state the first results concerning the uniform regularity for the nematic liquid crystal flows (1.1)-(1.3), (1.5) and (1.17) as follows.

THEOREM 1.1 (Uniform Regularity). *Let m be an integer satisfying $m\ge 6$, Ω be a $\mathcal{C}^{m+2}$ domain, and $A\in C^{m+1}(\partial\Omega)$. Consider the initial data $(p_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)\in X_{n,m}^\varepsilon$ satisfying inequality (1.18) and $|d_0^\varepsilon|=1$ in $\overline{\Omega}$. Then, there exists a time $T_0>0$ and $\widetilde{C}_1>0$ independent of $\varepsilon\in(0,1]$, such that there exits a unique solution of system (1.1)-(1.3), (1.5) and (1.17) which is defined on $[0,T_0]$ and satisfies the estimates*

$$\begin{aligned}
&\sup_{0\le t\le T_0}(\|d^\varepsilon(t)\|_{L^2}^2+\|(u^\varepsilon,p^\varepsilon,\nabla d^\varepsilon)(t)\|_{\mathcal{H}^m}^2+\|(\nabla u^\varepsilon,\Delta d^\varepsilon)(t)\|_{\mathcal{H}^{m-1}}^2+\|\nabla u^\varepsilon(t)\|_{\mathcal{H}^{1,\infty}}^2)\\
&+\sup_{0\le t\le T_0}(\sum_{k=0}^{m-2}\|\partial_t^k\nabla p^\varepsilon(t)\|_{m-1-k}^2+\varepsilon\|\partial_t^{m-1}\nabla p^\varepsilon(t)\|_{L^2}^2+\|\Delta p^\varepsilon(t)\|_{\mathcal{H}^1}^2+\varepsilon\|\Delta p^\varepsilon(t)\|_{\mathcal{H}^2}^2)\\
&+\int_0^{T_0}(\|\nabla\partial_t^{m-1}p^\varepsilon(t)\|_{L^2}^2+\|\Delta p^\varepsilon(t)\|_{\mathcal{H}^2}^2)dt+\varepsilon\int_0^{T_0}\|\nabla u^\varepsilon(t)\|_{\mathcal{H}^m}^2dt\\
&+\varepsilon^2\int_0^{T_0}\|\nabla^2\partial_t^{m-1}u^\varepsilon(t)\|_{L^2}^2dt+\varepsilon\sum_{k=0}^{m-2}\int_0^{T_0}\|\nabla^2\partial_t^k u^\varepsilon(t)\|_{m-k-1}^2dt\\
&+\int_0^{T_0}\|\Delta d^\varepsilon\|_{\mathcal{H}^m}^2dt+\int_0^{T_0}\|\nabla\Delta d^\varepsilon\|_{\mathcal{H}^{m-1}}^2dt\le\widetilde{C}_1,
\end{aligned}\quad (1.19)$$

and

$$\frac{1}{2\widehat{C}_0}\le\rho^\varepsilon(t)\le 2\widehat{C}_0,\quad t\in[0,T_0],$$

where $\widetilde{C}_1$ *depends only on* $\widehat{C}_0, \widetilde{C}_0$ *and* C_{m+2}.

REMARK 1.1. For $(p_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon) \in X_{n,m}^\varepsilon$, it must hold that $u_0^\varepsilon \cdot n|_{\partial\Omega} = 0$, $((Su_0^\varepsilon)n)_\tau|_{\partial\Omega} = -(Au_0^\varepsilon)_\tau|_{\partial\Omega}$, and $n \cdot \nabla d_0^\varepsilon|_{\partial\Omega} = 0$ in the trace sense for every fixed $\varepsilon \in (0,1]$.

We state the main steps for the proof of Theorem 1.1 as follows. First, we obtained a conormal energy estimate for $(p^\varepsilon, u^\varepsilon, \nabla d^\epsilon)$ in $\mathcal{H}^m$-norm. The second step is to give the estimate for $\|\partial_n u^\varepsilon\|_{\mathcal{H}^{m-1}}$. In order to obtain this estimate by an energy method, $\partial_n u^\varepsilon$ is not a convenient quantity because it does not vanish on the boundary. Similar to Wang et al. [7], $\partial_n u^\varepsilon$ can be controlled by $\partial_n u^\varepsilon \cdot n$(or $\mathrm{div} u^\varepsilon$) and $(\partial_n u)_\tau$. In order to give the estimate for $(\partial_n u^\varepsilon)_\tau$, we choose the convenient quantity $\eta = w^\varepsilon \times n + (Bu^\varepsilon)_\tau$ with homogeneous Dirichlet boundary conditions. The third step is to give the estimates for Δd^ε and $\mathrm{div} u^\varepsilon$. Indeed, it is easy to obtain the estimate for the quantity Δd^ε since there exists a dissipative term Δd^ε on the right-hand side of equation (1.3). In the spirit of Wang et al. [7], we obtain a control of $\sum_{j=0}^{m-2} \|\partial_t^j (\mathrm{div} u^\varepsilon, \nabla p^\varepsilon)\|_{m-1-j}^2$ at the cost that the term $\int_0^t \|\nabla \mathcal{Z}^{m-2} \mathrm{div} u^\varepsilon\|_{L^2}^2 d\tau$ appears in the right-hand side of the inequality. Following the idea as Wang et al. [7], we can obtain the uniform estimates for $\int_0^t \|\partial_t^{m-1} \nabla p^\varepsilon\|_{L^2}^2 d\tau$ and get a control of $\|\partial_t^{m-1} \mathrm{div} u^\varepsilon\|_{L^2}^2$ in terms of $\sum_{j=0}^{m-2} \|\partial_t^j (\nabla u^\varepsilon, \nabla p^\varepsilon)\|_{m-1-j}^2$ and $\|(p^\varepsilon, u^\varepsilon)\|_{\mathcal{H}^m}^2$. The fourth step is to estimate $\|\Delta d^\varepsilon\|_{W^{1,\infty}}$. Indeed, this estimate is easy to obtain since there exists a dissipation term Δd^ε on the right-hand side of equation (1.3). The fifth step is to estimate $\|\nabla u^\varepsilon\|_{\mathcal{H}^{1,\infty}}$. In fact, it suffices to estimate $\|(\partial_n u^\varepsilon)_\tau\|_{\mathcal{H}^{1,\infty}}$ since the other terms can be estimated by the Sobolev embedding. We choose an equivalent quantity such that it satisfies a homogeneous Dirichlet condition and solves a convection-diffusion equation at the leading order. The last step is to obtain the uniform estimate of $\|\Delta p^\varepsilon\|_{\mathcal{H}^1}$, which gives a control of $\|\nabla p^\varepsilon\|_{\mathcal{H}^{1,\infty}}$ from Proposition 2.3. Then Theorem 1.1 can be proved by these a priori estimates and a classical iteration method.

Next, we hope to prove the vanishing viscosity limit with rates of convergence, which can be stated as follows.

THEOREM 1.2 (Inviscid Limit). *Let* $(\rho, u, d)(t) \in L^\infty(0, T_1; H^4 \times H^4 \times H^5)$ *be the smooth solution to equations* (1.6)-(1.8) *and boundary condition* (1.9) *with initial data* (ρ_0, u_0, d_0) *satisfying*

$$(\rho_0, u_0, d_0) \in (H^4 \times H^4 \times H^5) \cap X_{n,m}^\varepsilon \text{ with } m \geq 6. \tag{1.20}$$

Let $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)(t)$ *be the solution to the initial boundary value problem of the nematic liquid crystal flows* (1.1)-(1.4) *with initial data* (ρ_0, u_0, d_0) *satisfying property* (1.20). *Then, there exists* $T_2 = \min\{T_0, T_1\} > 0$, *which is independent of* $\varepsilon > 0$, *such that*

$$\|(\rho^\varepsilon - \rho, u^\varepsilon - u)(t)\|_{L^2}^2 + \|(d^\varepsilon - d)(t)\|_{H^1}^2 \leq C\varepsilon^{\frac{3}{2}}, \quad t \in [0, T_2], \tag{1.21}$$

$$\|(\rho^\varepsilon - \rho, u^\varepsilon - u)(t)\|_{H^1}^2 \leq C\varepsilon^{\frac{1}{6}}, \quad \|(d^\varepsilon - d)(t)\|_{H^2}^2 \leq C\varepsilon^{\frac{1}{2}}, \quad t \in [0, T_2], \tag{1.22}$$

and

$$\|(\rho^\varepsilon - \rho, u^\varepsilon - u)\|_{L^\infty(0,T_2;L^\infty(\Omega))} + \|(d^\varepsilon - d)\|_{L^\infty(0,T_2;W^{1,\infty}(\Omega))} \leq C\varepsilon^{\frac{3}{10}}, \tag{1.23}$$

which C *depends only on the norm* $\|(\rho_0, u_0)\|_{H^3}, \|d_0\|_{H^4}$ *and* $\|(p(\rho_0), u_0, d_0)\|_{X_{m,n}^\varepsilon}$.

The rest of the paper is organized as follows: In Section 2, we collect some inequalities that will be used later. In Section 3, the a priori estimates in Theorem 3.1 are proved. By using these a priori estimates, we give the proof for the Theorem 1.1 in

Section 4. Based on the uniform estimates obtained in Theorem 1.1, we establish the convergence rate for the solutions from equations (1.1)-(1.3) to equations (1.6)-(1.8) and complete the proof for Theorem 1.2.

2. Preliminaries

The following lemma (see [6, 32]) tells the basic fact that the $H^m(\Omega)$-norm of a vector valued function u can be controlled by its H^{m-1}-norm of $\nabla\times u$ and div$u$ and the $H^{m-\frac{1}{2}}(\partial\Omega)$ of $u\cdot n$.

PROPOSITION 2.1. *Let $m\in\mathbb{N}_+$ be an integer. Let $u\in H^m$ be a vector-valued function. Then, there exists a constant $C>0$ independent of u, such that*

$$\|u\|_{H^m}\le C(\|\nabla\times u\|_{H^{m-1}}+\|\mathrm{div}u\|_{H^{m-1}}+\|u\|_{H^{m-1}}+|u\cdot n|_{H^{m-\frac{1}{2}}(\partial\Omega)}), \tag{2.1}$$

and

$$\|u\|_{H^m}\le C(\|\nabla\times u\|_{H^{m-1}}+\|\mathrm{div}u\|_{H^{m-1}}+\|u\|_{H^{m-1}}+|n\times u|_{H^{m-\frac{1}{2}}(\partial\Omega)}). \tag{2.2}$$

In this paper, we repeatedly use the Gagliardo–Nirenberg–Morser type inequality, whose proof can be found in [33]. One defines

$$W^m(\Omega\times[0,T])=\{f(x,t)\in L^2(\Omega\times[0,T])|\mathcal{Z}^\alpha f\in L^2(\Omega\times[0,T]),|\alpha|\le m\}. \tag{2.3}$$

Hence, we have the following Gagliardo–Nirenberg–Morser type inequality:

PROPOSITION 2.2. *For $u,v\in L^\infty(\Omega\times[0,T])\cap\mathcal{W}^m(\Omega\times[0,T])$ with $m\in\mathbb{N}_+$ an integer, it holds that*

$$\int_0^t\|(\mathcal{Z}^\beta u\mathcal{Z}^\gamma v)(\tau)\|_{L^2}^2d\tau\lesssim\|u\|_{L^\infty_{t,x}}^2\int_0^t\|v(\tau)\|_{\mathcal{H}^m}^2d\tau+\|v\|_{L^\infty_{t,x}}^2\int_0^t\|u(\tau)\|_{\mathcal{H}^m}^2d\tau, \tag{2.4}$$

where $|\beta|+|\gamma|=m$.

Finally, we need the following anisotropic Sobolev embedding and trace theorems, refer to [7].

PROPOSITION 2.3. *Let $m_1\ge 0,m_2\ge 0$ be integers and $f\in H_{co}^{m_1}(\Omega)\cap H_{co}^{m_2}(\Omega)$ and $\nabla f\in H_{co}^{m_2}(\Omega)$.*

(1) *The following anisotropic Sobolev embedding holds:*

$$\|f\|_{L^\infty}^2\le C(\|\nabla f\|_{H_{co}^{m_2}}+\|f\|_{H_{co}^{m_2}})\cdot\|f\|_{H_{co}^{m_1}}, \tag{2.5}$$

provided $m_1+m_2\ge 3$.

(2) *The following trace estimate holds:*

$$|f|_{H^s(\partial\Omega)}^2\le C(\|\nabla f\|_{H_{co}^{m_2}}+\|f\|_{H_{co}^{m_2}})\cdot\|f\|_{H_{co}^{m_1}}, \tag{2.6}$$

provided $m_1+m_2\ge 2s\ge 0$.

3. A priori estimates

The aim of this section is to prove the following a priori estimates, which are crucial to prove Theorem 1.1. For notational convenience, we drop the superscript ε throughout this section.

Theorem 3.1 (a priori estimates). *Let m be an integer satisfying $m\geq 6$, Ω be a C^{m+2} domain, and $A\in\mathcal{C}^{m+1}(\partial\Omega)$. For sufficiently smooth solutions defined on $[0,T]$, of the problem (1.1)-(1.4), it holds that*

$$|\rho(x,0)|\exp\left(-\int_0^t\|\mathrm{div}u(\tau)\|_{L^\infty}d\tau\right)\leq\rho(x,t)\leq|\rho(x,0)|\exp\left(\int_0^t\|\mathrm{div}u(\tau)\|_{L^\infty}d\tau\right),\quad(3.1)$$

for $(x,t)\in\Omega\times[0,T]$. In addition, if

$$0<c_0\leq\rho(x,t)\leq\frac{1}{c_0}<\infty,\quad(x,t)\in\Omega\times[0,T],\quad(3.2)$$

where c_0 is any given small positive constant, then the following a priori estimates hold

$$\begin{aligned}&N_m(t)+\int_0^t(\|\nabla\partial_t^{m-1}p(\tau)\|_{L^2}^2+\|\Delta p(\tau)\|_{\mathcal{H}^2}^2)d\tau+\varepsilon\int_0^t\|\nabla u(\tau)\|_{\mathcal{H}^m}^2d\tau\\&\quad+\varepsilon\sum_{k=0}^{m-2}\int_0^t\|\nabla^2\partial_t^k u(\tau)\|_{m-1-k}^2d\tau+\varepsilon^2\int_0^t\|\nabla^2\partial_t^{m-1}u(\tau)\|_{L^2}^2d\tau\\&\quad+\int_0^t\|\Delta d(\tau)\|_{\mathcal{H}^m}^2d\tau+\int_0^t\|\nabla\Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2d\tau\\&\leq\widetilde{C}_2C_{m+2}\left\{P(N_m(0))+P(N_m(t))\int_0^tP(N_m(\tau))d\tau\right\},\quad\forall t\in[0,T],\end{aligned}\quad(3.3)$$

where $\widetilde{C}_2$ depends only on $\frac{1}{c_0}$, $P(\cdot)$ is a polynomial, and

$$\begin{aligned}N_m(t)\triangleq\sup_{0\leq\tau\leq t}\Big\{&1+\|(p,u)(\tau)\|_{\mathcal{H}^m}^2+\|d(\tau)\|_{L^2}^2+\|\nabla d(\tau)\|_{\mathcal{H}^m}^2+\|\nabla u(\tau)\|_{\mathcal{H}^{m-1}}^2\\&+\|\Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2+\sum_{k=0}^{m-2}\|\partial_t^k\nabla p(\tau)\|_{m-1-k}^2+\varepsilon\|\nabla\partial_t^{m-1}p(\tau)\|_{L^2}^2\\&+\|\Delta p(\tau)\|_{\mathcal{H}^1}^2+\varepsilon\|\Delta p(\tau)\|_{\mathcal{H}^2}^2+\|\nabla u(\tau)\|_{\mathcal{H}^{1,\infty}}^2\Big\}.\end{aligned}\quad(3.4)$$

Throughout this section, we shall work on the interval of time $[0,T]$ such that $c_0\leq\rho(x,t)\leq\frac{1}{c_0}$. Furthermore, we point out that the generic constant C may depend on $\frac{1}{c_0}$ in this section. Since the proof of Theorem 3.1 is quite lengthy and involved, we divide the proof into the following several subsections.

3.1. Conormal energy estimates for ρ,u and ∇d. For any smooth function f, notice that

$$\Delta f=\nabla\mathrm{div}f-\nabla\times(\nabla\times f),$$

and then equation (1.2) can be written as

$$\rho u_t+\rho u\cdot\nabla u+\nabla p=-\mu\varepsilon\nabla\times(\nabla\times u)+(2\mu+\lambda)\varepsilon\nabla\mathrm{div}u-\nabla d\cdot\Delta d.\quad(3.5)$$

In this subsection, we first give the basic a priori L^2-estimate which holds for equations (1.1)-(1.3) and (1.5).

Lemma 3.1. *For a smooth solution to equations (1.1)-(1.3) and (1.5), it holds that for* $\varepsilon\in(0,1]$

$$
\begin{aligned}
&\int(\frac{1}{2}\rho|u|^2+\frac{\gamma}{\gamma-1}\rho^\gamma+\frac{1}{2}|\nabla d|^2)dx+c_1\varepsilon\int_0^t\|\nabla u\|_{L^2}^2d\tau+\int_0^t\|\Delta d\|_{L^2}^2d\tau\\
&\le\int(\frac{1}{2}\rho_0|u_0|^2+\frac{\gamma}{\gamma-1}\rho_0^\gamma+\frac{1}{2}|\nabla d_0|^2)dx+\|\nabla d\|_{L^\infty}^2\int_0^t\|\nabla d\|_{L^2}^2d\tau+C_2\int_0^t\|u\|_{L^2}^2d\tau. \quad (3.6)
\end{aligned}
$$

Proof. Multiplying equation (1.3) by d and integrating over Ω, one arrives at

$$\frac{d}{dt}\frac{1}{2}\int(|d|^2-1)dx+\int u\cdot\nabla(|d|^2-1)dx=\int\Delta d\cdot d\ dx+\int|\nabla d|^2|d|^2dx,$$

which, integrating by parts and applying the boundary condition (1.5), yields that

$$\frac{d}{dt}\int(|d|^2-1)dx+2\int(|d|^2-1)(|\nabla d|^2-\mathrm{div}u)dx=0. \quad (3.7)$$

In view of the Grönwall inequality, one deduces from the identity (3.7) that

$$|d(x,t)|=1,\quad (x,t)\in\overline{\Omega}\times[0,T].$$

Multiplying equation (3.5) by u, integrating by parts and applying the boundary condition (1.5), we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\rho|u|^2dx+\int\nabla p\cdot u\ dx+\mu\varepsilon\int\nabla\times(\nabla\times u)\cdot u\ dx\\
=&(2\mu+\lambda)\varepsilon\int\nabla\mathrm{div}u\cdot u\ dx-\int(u\cdot\nabla)d\cdot\Delta d\ dx. \quad (3.8)
\end{aligned}
$$

By virtue of equation (1.1), one deduces that

$$\int\nabla p\cdot u\ dx=\frac{\gamma}{\gamma-1}\int\nabla(\rho^{\gamma-1})\cdot\rho u dx=\frac{\gamma}{\gamma-1}\int\rho^{\gamma-1}\rho_t dx=\frac{d}{dt}\frac{\gamma}{\gamma-1}\int\rho^\gamma dx. \quad (3.9)$$

Integrating by parts and applying the boundary condition (1.5), we get

$$
\begin{aligned}
\int\nabla\times(\nabla\times u)u\ dx&=\int_{\partial\Omega}n\times(\nabla\times u)\cdot u\ d\sigma+\int|\nabla\times u|^2dx\\
&=\int_{\partial\Omega}[Bu]_\tau\cdot u_\tau\ d\sigma+\int|\nabla\times u|^2dx,
\end{aligned}
$$

and

$$\int\nabla\mathrm{div}u\cdot u\ dx=\int_{\partial\Omega}(\mathrm{div}u)u\cdot n\ d\sigma-\int|\mathrm{div}u|^2dx=-\int|\mathrm{div}u|^2dx.$$

which, together with equation (3.8), gives directly

$$
\begin{aligned}
&\frac{d}{dt}\int(\frac{1}{2}\rho|u|^2+\frac{\gamma}{\gamma-1}\rho^\gamma)dx+\mu\varepsilon\int|\nabla\times u|^2dx+(2\mu+\lambda)\varepsilon\int|\mathrm{div}u|^2dx\\
=&-\varepsilon\int_{\partial\Omega}[Bu]_\tau\cdot u_\tau\ d\sigma-\int(u\cdot\nabla)d\cdot\Delta d\ dx. \quad (3.10)
\end{aligned}
$$

Multiplying equation (1.3) by Δd and integrating over Ω, one arrives at

$$\int (d_t + u\cdot\nabla d)\cdot\Delta d\, dx = \int |\Delta d|^2 dx + \int |\nabla d|^2 d\cdot\Delta d\, dx. \tag{3.11}$$

Integration by parts and application of boundary condition (1.5) yield directly

$$\int d_t\cdot\Delta d\, dx = \int_{\partial\Omega} d_t\cdot\frac{\partial d}{\partial n}d\sigma - \frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx = -\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx. \tag{3.12}$$

By virtue of the basic fact $|d|=1$, we find $\Delta d\cdot d = -|\nabla d|^2$. Then, the combination of equations (3.11) and (3.12) gives

$$\frac{1}{2}\frac{d}{dt}\int |\nabla d|^2 dx + \int |\Delta d|^2 dx = \int (u\cdot\nabla)d\cdot\Delta d\, dx + \int |\nabla d|^4 dx,$$

which, together with equation (3.10), yields directly

$$\begin{aligned}
&\frac{d}{dt}\int (\frac{1}{2}\rho|u|^2 + \frac{\gamma}{\gamma-1}\rho^\gamma + \frac{1}{2}|\nabla d|^2)dx + \int |\Delta d|^2 dx \\
&\quad + \mu\varepsilon\int |\nabla\times u|^2 dx + (2\mu+\lambda)\varepsilon\int |\mathrm{div} u|^2 dx \\
&= -\varepsilon\int_{\partial\Omega}[Bu]_\tau\cdot u_\tau\, d\sigma + \int |\nabla d|^4 dx.
\end{aligned} \tag{3.13}$$

It follows from the trace theorem in Proposition 2.3 that

$$|u|^2_{L^2(\partial\Omega)} \le \delta\|\nabla u\|^2_{L^2} + C_\delta\|u\|^2_{L^2}. \tag{3.14}$$

The application of Proposition 2.1 gives immediately

$$\begin{aligned}
&\mu\|\nabla\times u\|^2_{L^2} + (2\mu+\lambda)\|\mathrm{div} u\|^2_{L^2} \\
&\ge \min\{\mu, 2\mu+\lambda\}(\|\nabla\times u\|^2_{L^2} + \|\mathrm{div} u\|^2_{L^2}) \\
&\ge 2c_1\|\nabla u\|^2_{L^2} - C\|u\|^2_{L^2}.
\end{aligned} \tag{3.15}$$

Substituting inequalities (3.14) and (3.15) into equation (3.13) and choosing δ small enough, one arrives at

$$\begin{aligned}
&\frac{d}{dt}\int (\frac{1}{2}\rho|u|^2 + \frac{\gamma}{\gamma-1}\rho^\gamma + \frac{1}{2}|\nabla d|^2)dx + c_1\varepsilon\int |\nabla u|^2 dx + \int |\Delta d|^2 dx \\
&\le \int |\nabla d|^4 dx + C_2\int |u|^2 dx,
\end{aligned}$$

which, integrating over $[0,t]$, yields

$$\begin{aligned}
&\int (\frac{1}{2}\rho|u|^2 + \frac{\gamma}{\gamma-1}\rho^\gamma + \frac{1}{2}|\nabla d|^2)dx + c_1\varepsilon\int_0^t\int |\nabla u|^2 dxd\tau + \int_0^t\int |\Delta d|^2 dxd\tau \\
&\le \int (\frac{1}{2}\rho_0|u_0|^2 + \frac{\gamma}{\gamma-1}\rho_0^\gamma + \frac{1}{2}|\nabla d_0|^2)dx + \|\nabla d\|^2_{L^\infty}\int_0^t\int |\nabla d|^2 dxd\tau + C_2\int_0^t\int |u|^2 dxd\tau.
\end{aligned}$$

Therefore, we complete the proof of lemma. □

However, the above basic energy estimation is insufficient to get the vanishing viscosity limit. Some conormal derivative estimates are needed. Let

$$Q(t)\triangleq \sup_{0\le\tau\le t}\{\|(\nabla p,\nabla u)\|^2_{\mathcal{H}^{1,\infty}}+\|(p,u,p_t,u_t)\|^2_{L^\infty} \\ +\|d_t\|^2_{W^{1,\infty}}+\|\nabla d\|^2_{W^{1,\infty}}+\|\nabla\Delta d\|^2_{L^\infty}\} \tag{3.16}$$

and

$$\Lambda_m(t)\triangleq \|(p,u,\nabla d)(t)\|^2_{\mathcal{H}^m}+\|(\nabla u,\Delta d)(t)\|^2_{\mathcal{H}^{m-1}}+\|\nabla u(\tau)\|^2_{\mathcal{H}^{1,\infty}} \\ +\sum_{k=0}^{m-2}\|\nabla\partial_t^k p(t)\|^2_{m-1-k}+\varepsilon\|\nabla\partial_t^{m-1}p(t)\|^2_{L^2}. \tag{3.17}$$

Lemma 3.2. *For $m\in\mathbb{N}^+$ and a smooth solution to equations (1.1)-(1.3) and (1.5), it holds that for $\varepsilon\in(0,1]$,*

$$\sup_{0\le\tau\le t}\|(u,p,\nabla d)(\tau)\|^2_{\mathcal{H}^m}+C\varepsilon\int_0^t\|\nabla u(\tau)\|^2_{\mathcal{H}^m}d\tau+\int_0^t\|\Delta d(\tau)\|^2_{\mathcal{H}^m}d\tau \\ \le C_{m+2}\Big\{\|(u_0,p_0,\nabla d_0)\|^2_{\mathcal{H}^m}+\delta\int_0^t\|\nabla\Delta d(\tau)\|^2_{\mathcal{H}^{m-1}}d\tau+\delta\varepsilon^2\int_0^t\|\nabla^2u(\tau)\|^2_{\mathcal{H}^{m-1}}d\tau \\ +\delta\int_0^t\|\nabla\partial_t^{m-1}p(\tau)\|^2_{L^2}d\tau+C_\delta[1+P(Q(t))]\int_0^t\Lambda_m(\tau)d\tau\Big\}, \tag{3.18}$$

where δ is a small constant which will be chosen late, C_δ is a polynomial function of $\frac{1}{\delta}$, and the generic positive constant $C>0$ depends on μ and λ.

Proof. The case for $m=0$ is already proved in Lemma 3.1. Assume that inequality (3.18) has been proved for $k=m-1$. We shall prove that holds for $k=m\ge1$. Applying the operator $\mathcal{Z}^\alpha(|\alpha_0|+|\alpha_1|=m)$ to equation (3.5), we find

$$\rho\mathcal{Z}^\alpha u_t+\rho u\cdot\nabla\mathcal{Z}^\alpha u+\mathcal{Z}^\alpha\nabla p \\ =-\mu\varepsilon\mathcal{Z}^\alpha\nabla\times(\nabla\times u)+(2\mu+\lambda)\varepsilon\mathcal{Z}^\alpha\nabla\mathrm{div}u-\mathcal{Z}^\alpha(\nabla d\cdot\Delta d)+\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha, \tag{3.19}$$

where

$$\mathcal{C}_1^\alpha=-[\mathcal{Z}^\alpha,\rho]u_t,\quad \mathcal{C}_2^\alpha=-[\mathcal{Z}^\alpha,\rho u\cdot\nabla]u.$$

Multiplying equation (3.19) by $\mathcal{Z}^\alpha u$ and integrating over Ω, we obtain

$$\frac12\frac{d}{dt}\int\rho|\mathcal{Z}^\alpha u|^2dx+\int\mathcal{Z}^\alpha\nabla p\cdot\mathcal{Z}^\alpha u\,dx \\ =-\mu\varepsilon\int\mathcal{Z}^\alpha\nabla\times(\nabla\times u)\cdot\mathcal{Z}^\alpha u\,dx-(2\mu+\lambda)\varepsilon\int\mathcal{Z}^\alpha\nabla\mathrm{div}u\cdot\mathcal{Z}^\alpha u\,dx \\ -\int\mathcal{Z}^\alpha(\nabla d\cdot\Delta d)\cdot\mathcal{Z}^\alpha u\,dx+\int\mathcal{C}_1^\alpha\cdot\mathcal{Z}^\alpha u\,dx+\int\mathcal{C}_2^\alpha\cdot\mathcal{Z}^\alpha u\,dx. \tag{3.20}$$

Using the same arguments as Lemma 3.4 in [7], one can obtain the following estimates

$$-\varepsilon\int\mathcal{Z}^\alpha\nabla\times(\nabla\times u)\cdot\mathcal{Z}^\alpha u\,dx$$

$$\leq -\frac{3\varepsilon}{4}\|\nabla\times\mathcal{Z}^\alpha u\|_{L^2}^2+\delta\varepsilon^2\|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2+C_\delta C_{m+2}(\|\nabla u\|_{\mathcal{H}^{m-1}}^2+\|u\|_{\mathcal{H}^m}^2) \tag{3.21}$$

and

$$\begin{aligned}&\varepsilon\int \mathcal{Z}^\alpha\nabla \mathrm{div} u\cdot\mathcal{Z}^\alpha u\ dx\\ \leq& -\frac{3\varepsilon}{4}\|\mathrm{div}\mathcal{Z}^\alpha u\|_{L^2}^2+\delta\varepsilon^2\|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2+C_\delta C_{m+2}(\|\nabla u\|_{\mathcal{H}^{m-1}}^2+\|u\|_{\mathcal{H}^m}^2).\end{aligned} \tag{3.22}$$

On the other hand, it follows from Proposition 2.1 that

$$\begin{aligned}&2c_1\|\nabla\mathcal{Z}^\alpha u\|_{L^2}^2\\ \leq&(\mu\|\nabla\times\mathcal{Z}^\alpha u\|_{L^2}^2+(2\mu+\lambda)\|\mathrm{div}\mathcal{Z}^\alpha u\|_{L^2}^2+\|\mathcal{Z}^\alpha u\|_{L^2}^2+|\mathcal{Z}^\alpha u\cdot n|_{H^{\frac12}(\partial\Omega)})\\ \leq&(\mu\|\nabla\times\mathcal{Z}^\alpha u\|_{L^2}^2+(2\mu+\lambda)\|\mathrm{div}\mathcal{Z}^\alpha u\|_{L^2}^2)+C_{m+2}(\|u\|_{\mathcal{H}^m}^2+\|\nabla u\|_{\mathcal{H}^{m-1}}^2),\end{aligned} \tag{3.23}$$

where we have using the fact

$$|\mathcal{Z}^\alpha u\cdot n|_{H^{\frac12}(\partial\Omega)}\leq C_{m+2}(\|u\|_{\mathcal{H}^m}^2+\|\nabla u\|_{\mathcal{H}^{m-1}}^2).$$

Substituting inequalities (3.21)-(3.23) into equation (3.20) and integrating the resulting inequality over $[0,t]$, we find

$$\begin{aligned}&\frac12\int\rho|\mathcal{Z}^\alpha u(t)|^2dx+\frac{3c_1\varepsilon}{2}\int_0^t\int|\nabla\mathcal{Z}^\alpha u|^2dxd\tau+\int_0^t\int\mathcal{Z}^\alpha\nabla p\cdot\mathcal{Z}^\alpha u\ dxd\tau\\ \leq&\frac12\int\rho_0|\mathcal{Z}^\alpha u_0|^2dx+C\delta_1\varepsilon\int_0^t\|\nabla u\|_{\mathcal{H}^m}^2d\tau+C\delta\varepsilon^2\int_0^t\|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2d\tau\\ &+C_\delta C_{m+2}\int_0^t(\|\nabla u\|_{\mathcal{H}^{m-1}}^2+\|u\|_{\mathcal{H}^m}^2)d\tau+\int_0^t\int\mathcal{C}_1^\alpha\cdot\mathcal{Z}^\alpha u\ dxd\tau\\ &+\int_0^t\int\mathcal{C}_2^\alpha\cdot\mathcal{Z}^\alpha u\ dxd\tau-\int_0^t\int\mathcal{Z}^\alpha(\nabla d\cdot\Delta d)\cdot\mathcal{Z}^\alpha u\ dxd\tau.\end{aligned} \tag{3.24}$$

Applying the transport equation (1.1), we follow the same argument as Lemma 3.4 of [7] to obtain

$$\begin{aligned}&-\int\mathcal{Z}^\alpha\nabla p\cdot\mathcal{Z}^\alpha u\ dx\\ \leq&-\int\frac{1}{2\gamma p}|\mathcal{Z}^\alpha p|^2dx+\int\frac{1}{2\gamma p_0}|\mathcal{Z}^\alpha p_0|^2dx+C\delta\int_0^t\|\nabla p\|_{\mathcal{H}^{m-1}}^2dx\\ &+C_\delta[1+P(Q(t))]\int(\|(p,u)\|_{\mathcal{H}^m}^2+\|\nabla u\|_{\mathcal{H}^{m-1}}^2)d\tau.\end{aligned} \tag{3.25}$$

In view of the Proposition 2.2, we obtain

$$\int_0^t\|\mathcal{Z}^\alpha(\nabla d\cdot\Delta d)\|_{L^2}^2d\tau\leq C\|\nabla d\|_{L^\infty_{x,t}}^2\int_0^t\|\Delta d\|_{\mathcal{H}^m}^2d\tau+C\|\Delta d\|_{L^\infty_{x,t}}^2\int_0^t\|\nabla d\|_{\mathcal{H}^m}^2d\tau$$

which, by using the Cauchy inequality, yields directly

$$\left|-\int_0^t\int\mathcal{Z}^\alpha(\nabla d\cdot\Delta d)\cdot\mathcal{Z}^\alpha u\ dxd\tau\right|$$

$$\leq \delta_1 \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1}(\|\nabla d\|_{L^\infty_{x,t}}^2 + \|\Delta d\|_{L^\infty_{x,t}}^2)\int_0^t (\|u\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2) d\tau. \tag{3.26}$$

Similarly, it is easy to deduce that (or see Lemma 3.4 of [7])

$$\int_0^t \int \mathcal{C}_1^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau \leq C[1+P(Q(t))]\int_0^t \|(p,u)\|_{\mathcal{H}^m}^2 d\tau \tag{3.27}$$

and

$$\int_0^t \int \mathcal{C}_2^\alpha \cdot \mathcal{Z}^\alpha u \, dx d\tau \leq C[1+P(Q(t))]\int_0^t (\|(p,u)\|_{\mathcal{H}^m}^2 + \|\nabla u\|_{\mathcal{H}^{m-1}}^2) d\tau. \tag{3.28}$$

Substituting inequalities (3.25)-(3.28) into inequality (3.24), one attains

$$\begin{aligned}
&\frac{1}{2}\int \rho |\mathcal{Z}^\alpha u|^2 dx + \int \frac{1}{2\gamma p}|\mathcal{Z}^\alpha p|^2 dx + \frac{3c_1\varepsilon}{2}\int_0^t \int |\nabla \mathcal{Z}^\alpha u|^2 dx d\tau \\
\leq & \frac{1}{2}\int \rho_0 |\mathcal{Z}^\alpha u_0|^2 dx + \int \frac{1}{2\gamma p_0}|\mathcal{Z}^\alpha p_0|^2 dx + C\delta_1 \varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + C\delta\varepsilon^2 \int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&+ C_\delta C_{m+2}[1+P(Q(t))]\int_0^t (\|\nabla u\|_{\mathcal{H}^{m-1}}^2 + \|(p,u,\nabla d)\|_{\mathcal{H}^m}^2) d\tau.
\end{aligned} \tag{3.29}$$

Applying the operator $\mathcal{Z}^\alpha \nabla (|\alpha_0|+|\alpha_1| = m)$ to equation (1.3), we find

$$\mathcal{Z}^\alpha \nabla d_t - \mathcal{Z}^\alpha \nabla \Delta d = -\mathcal{Z}^\alpha \nabla (u \cdot \nabla d) + \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d). \tag{3.30}$$

Multiplying equation (3.30) by $\mathcal{Z}^\alpha \nabla d$ and integrating over Ω, it is easy to deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\mathcal{Z}^\alpha \nabla d|^2 dx - \int \mathcal{Z}^\alpha \nabla \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\
=& -\int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx + \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \mathcal{Z}^\alpha \nabla d \, dx.
\end{aligned} \tag{3.31}$$

The integration by parts yields directly

$$\begin{aligned}
&-\int \mathcal{Z}^\alpha \nabla \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\
=& -\int \nabla \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx - \int [\mathcal{Z}^\alpha, \nabla]\Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\
=& -\int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma + \int \mathcal{Z}^\alpha \Delta d \cdot \mathrm{div}(\mathcal{Z}^\alpha \nabla d) \, dx - \int [\mathcal{Z}^\alpha, \nabla]\Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx \\
=& -\int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n \, d\sigma + \int |\mathrm{div}(\mathcal{Z}^\alpha \nabla d)|^2 dx + \int [\mathcal{Z}^\alpha, \mathrm{div}]\nabla d \cdot \mathrm{div}(\mathcal{Z}^\alpha \nabla d) \, dx \\
&- \int [\mathcal{Z}^\alpha, \nabla]\Delta d \cdot \mathcal{Z}^\alpha \nabla d \, dx.
\end{aligned}$$

This, together with equation (3.31), reads

$$\begin{aligned}
&\frac{1}{2}\int |\mathcal{Z}^\alpha \nabla d(t)|^2 dx + \int_0^t \int |\mathrm{div}(\mathcal{Z}^\alpha \nabla d)|^2 dx \\
=& \frac{1}{2}\int |\mathcal{Z}^\alpha \nabla d_0|^2 dx - \int_0^t \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \mathcal{Z}^\alpha \nabla d \, dx d\tau
\end{aligned}$$

$$+\int_0^t\int \mathcal{Z}^\alpha\nabla(|\nabla d|^2 d)\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau-\int_0^t\int[\mathcal{Z}^\alpha,\mathrm{div}]\nabla d\cdot\mathrm{div}(\mathcal{Z}^\alpha\nabla d)\, dxd\tau$$
$$+\int_0^t\int[\mathcal{Z}^\alpha,\nabla]\Delta d\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau+\int_0^t\int_{\partial\Omega}\mathcal{Z}^\alpha\Delta d\cdot\mathcal{Z}^\alpha\nabla d\cdot n\, d\sigma d\tau$$
$$\triangleq I_1+I_2+I_3+I_4+I_5+I_6. \tag{3.32}$$

Deal with the term I_2. Integrating by parts, one arrives at

$$\begin{aligned} I_2=&-\int_0^t\int\nabla\mathcal{Z}^\alpha(u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau-\int_0^t\int[\mathcal{Z}^\alpha,\nabla](u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau\\ =&-\int_0^t\int_{\partial\Omega}\mathcal{Z}^\alpha(u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\cdot n\, d\sigma d\tau-\int_0^t\int\mathcal{Z}^\alpha(u\cdot\nabla d)\cdot\mathrm{div}(\mathcal{Z}^\alpha\nabla d)\, dxd\tau\\ &-\int_0^t\int[\mathcal{Z}^\alpha,\nabla](u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau. \end{aligned}\tag{3.33}$$

To estimate the boundary term on the right hand side of equation (3.33), if $|\alpha_0|=m$, we apply the boundary condition (1.5) to deduce that

$$-\int_0^t\int_{\partial\Omega}\mathcal{Z}^\alpha(u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\cdot n\, d\sigma d\tau=0.$$

If $|\alpha_{13}|\neq 0$, the proposition of (1.10) implies $\mathcal{Z}^\alpha\nabla d=0$ on the boundary. Then, one arrives at

$$-\int_0^t\int_{\partial\Omega}\mathcal{Z}^\alpha(u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\cdot n\, d\sigma d\tau=0.$$

Hence, we deal with the case of $|\alpha_{13}|=0$ and $|\alpha_0|\le m-1$. For $|\beta|=m-1-\alpha_0(|\alpha_0|\le m-1)$, one integrates by parts along the boundary to deduce that

$$-\int_0^t\int_{\partial\Omega}\mathcal{Z}^\alpha(u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\cdot n\, d\sigma d\tau\le\int_0^t|\partial_t^{\alpha_0}Z_y^\beta(u\cdot\nabla d)|_{L^2(\partial\Omega)}|\mathcal{Z}^\alpha\nabla d\cdot n|_{H^1(\partial\Omega)}d\tau. \tag{3.34}$$

Applying the trace theorem in Propositions 2.3 and 2.2, one arrives at

$$\begin{aligned} &\int_0^t|\partial_t^{\alpha_0}Z_y^\beta(u\cdot\nabla d)|_{L^2(\partial\Omega)}^2d\tau\\ \le& C\int_0^t(\|\nabla\partial_t^{\alpha_0}(u\cdot\nabla d)\|_{m-1-\alpha_0}^2+\|\partial_t^{\alpha_0}(u\cdot\nabla d)\|_{m-1-\alpha_0}^2)d\tau\\ \le& C\int_0^t(\|\nabla(u\cdot\nabla d)\|_{\mathcal{H}^{m-1}}^2+\|u\cdot\nabla d\|_{\mathcal{H}^{m-1}}^2)d\tau\\ \le& CQ(t)\int_0^t(\|(u,\nabla d)\|_{\mathcal{H}^{m-1}}^2+\|\nabla(u,\nabla d)\|_{\mathcal{H}^{m-1}}^2)d\tau. \end{aligned}\tag{3.35}$$

By virtue of boundary condition (1.5) and trace theorem in Proposition 2.3, we find

$$\int_0^t|\mathcal{Z}^\alpha\nabla d\cdot n|_{H^1(\partial\Omega)}^2d\tau\le C_{m+2}\int_0^t(\|\nabla^2 d\|_{\mathcal{H}^m}^2+\|\nabla d\|_{\mathcal{H}^m}^2)d\tau. \tag{3.36}$$

The combination of inequalities (3.34)-(3.36) and Cauchy's inequality gives directly

$$\begin{aligned}&-\int_0^t\int_{\partial\Omega}\mathcal{Z}^\alpha(u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\cdot n\, d\sigma d\tau\\ &\le C_{\delta_1}C_{m+2}(1+Q(t))\int_0^t(\|(u,\nabla d)\|_{\mathcal{H}^m}^2+\|\nabla(u,\nabla d)\|_{\mathcal{H}^{m-1}}^2)d\tau+\delta_1\int_0^t\|\nabla^2 d\|_{\mathcal{H}^m}^2d\tau.\end{aligned}\tag{3.37}$$

Applying the Young inequality and the Proposition 2.2, one attains immediately

$$\begin{aligned}&-\int_0^t\int\mathcal{Z}^\alpha(u\cdot\nabla d)\cdot\mathrm{div}(\mathcal{Z}^\alpha\nabla d)\, dxd\tau\\ &\le\delta_1\int_0^t\|\mathrm{div}(\mathcal{Z}^\alpha\nabla d)\|_{L^2}^2d\tau+C_{\delta_1}\|u\|_{L^\infty_{x,t}}^2\int_0^t\|\nabla d\|_{\mathcal{H}^m}^2d\tau+C_{\delta_1}\|\nabla d\|_{L^\infty_{x,t}}^2\int_0^t\|u\|_{\mathcal{H}^m}^2d\tau\\ &\le\delta_1\int_0^t\|\mathrm{div}(\mathcal{Z}^\alpha\nabla d)\|_{L^2}^2d\tau+C_{\delta_1}C_1Q(t)\int_0^t(\|u\|_{\mathcal{H}^m}^2+\|\nabla d\|_{\mathcal{H}^m}^2)d\tau\end{aligned}\tag{3.38}$$

and

$$\begin{aligned}&-\int_0^t\int[\mathcal{Z}^\alpha,\nabla](u\cdot\nabla d)\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau\\ &\le\sum_{|\beta|\le m-1}\int_0^t\|\mathcal{Z}^\beta(\nabla u\cdot\nabla d+u\cdot\nabla^2 d)\|_{L^2}\|\mathcal{Z}^\alpha\nabla d\|_{L^2}dx\\ &\le C\int_0^t\|\nabla d\|_{\mathcal{H}^m}^2d\tau+C(\|\nabla u\|_{L^\infty_{x,t}}^2+\|\nabla d\|_{L^\infty_{x,t}}^2)\int_0^t(\|\nabla u\|_{\mathcal{H}^{m-1}}^2+\|\nabla d\|_{\mathcal{H}^{m-1}}^2)d\tau\\ &\quad+C(\|u\|_{L^\infty_{x,t}}^2+\|\nabla^2 d\|_{L^\infty_{x,t}}^2)\int_0^t(\|u\|_{\mathcal{H}^{m-1}}^2+\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2)d\tau\\ &\le C(1+Q(t))\int_0^t(\|u\|_{\mathcal{H}^{m-1}}^2+\|\nabla u\|_{\mathcal{H}^{m-1}}^2+\|\nabla d\|_{\mathcal{H}^m}^2+\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2)d\tau.\end{aligned}\tag{3.39}$$

Plugging the estimates (3.37)-(3.39) into the identity (3.33), we obtain

$$\begin{aligned}|I_2|\le& C_{\delta_1}C_1(1+Q(t))\int_0^t(\|(u,\nabla d)\|_{\mathcal{H}^m}^2+\|\nabla(u,\nabla d)\|_{\mathcal{H}^{m-1}}^2)d\tau\\ &+\delta_1\int_0^t\|\nabla^2 d\|_{\mathcal{H}^m}^2d\tau.\end{aligned}\tag{3.40}$$

Deal with the term I_3. Indeed, by integrating by parts, one arrives at

$$\begin{aligned}I_3=&\int_0^t\int\nabla\mathcal{Z}^\alpha(|\nabla d|^2 d)\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau+\int_0^t\int[\mathcal{Z}^\alpha,\nabla](|\nabla d|^2 d)\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau\\ =&-\int_0^t\int\mathcal{Z}^\alpha(|\nabla d|^2 d)\cdot\mathrm{div}(\mathcal{Z}^\alpha\nabla d)\, dxd\tau+\int_0^t\int[\mathcal{Z}^\alpha,\nabla](|\nabla d|^2 d)\cdot\mathcal{Z}^\alpha\nabla d\, dxd\tau\\ &+\int_0^t\int_{\partial\Omega}\mathcal{Z}^\alpha(|\nabla d|^2 d)\cdot\mathcal{Z}^\alpha\nabla d\cdot n\, d\sigma d\tau.\end{aligned}\tag{3.41}$$

It is easy to deduce that

$$-\int_0^t\int\mathcal{Z}^\alpha(|\nabla d|^2 d)\cdot\mathrm{div}(\mathcal{Z}^\alpha\nabla d)\, dxd\tau$$

$$= -\sum_{|\beta|\geq 1}\int_0^t\int \mathcal{Z}^\gamma(|\nabla d|^2)\mathcal{Z}^\beta d\cdot \mathrm{div}(\mathcal{Z}^\alpha\nabla d)\ dxd\tau$$
$$-\int_0^t\int \mathcal{Z}^\alpha(|\nabla d|^2)d\cdot \mathrm{div}(\mathcal{Z}^\alpha\nabla d)\ dxd\tau. \tag{3.42}$$

By virtue of the Proposition 2.2, we obtain

$$\begin{aligned}
&\sum_{|\beta|\geq 1}\int_0^t \|\mathcal{Z}^\gamma(|\nabla d|^2)\mathcal{Z}^\beta d\|_{L^2}^2 d\tau\\
&\leq \|\mathcal{Z}d\|_{L^\infty_{x,t}}^2\int_0^t \||\nabla d|^2\|_{\mathcal{H}^{m-1}}^2 d\tau + \||\nabla d|^2\|_{L^\infty_{x,t}}^2\int_0^t \|\mathcal{Z}d\|_{\mathcal{H}^{m-1}}^2 d\tau\\
&\leq \|\mathcal{Z}d\|_{L^\infty}^2\|\nabla d\|_{L^\infty}^2\int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + \|\nabla d\|_{L^\infty}^4\int_0^t \|\mathcal{Z}d\|_{\mathcal{H}^{m-1}}^2 d\tau\\
&\leq \|\nabla d\|_{L^\infty}^4\int_0^t \|\partial_t d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_1(1+P(Q(t)))\int_0^t \Lambda_m(\tau)d\tau.
\end{aligned} \tag{3.43}$$

In view of equation (1.3), one attains directly

$$\begin{aligned}
\int_0^t \|\partial_t d\|_{\mathcal{H}^{m-1}}^2 d\tau \leq& \|u\|_{L^\infty}^2\int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + \|\nabla d\|_{L^\infty}^2\int_0^t \|u\|_{\mathcal{H}^{m-1}}^2 d\tau\\
&+\int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \int_0^t \||\nabla d|^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau.
\end{aligned} \tag{3.44}$$

By virtue of the Proposition 2.2, we find

$$\begin{aligned}
&\int_0^t \||\nabla d|^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau\\
&\leq \sum_{|\gamma|\geq 1, |\beta|+|\gamma|\leq m-1}\int_0^t \|\mathcal{Z}^\beta(|\nabla d|^2)\mathcal{Z}^\gamma d\|_{L^2}^2 d\tau + \int_0^t \||\nabla d|^2\|_{\mathcal{H}^{m-1}}^2 d\tau\\
&\lesssim \|\mathcal{Z}d\|_{L^\infty}^2\|\nabla d\|_{L^\infty}^2\int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau + \|\nabla d\|_{L^\infty}^4\int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2 d\tau\\
&\quad + \|\nabla d\|_{L^\infty}^4\int_0^t \|\partial_t d\|_{\mathcal{H}^{m-2}}^2 d\tau + \|\nabla d\|_{L^\infty}^2\int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau.
\end{aligned} \tag{3.45}$$

Substituting inequality (3.45) into inequality (3.44), one arrives at immediately

$$\begin{aligned}
\int_0^t \|d_t\|_{\mathcal{H}^{m-1}}^2 d\tau \lesssim& \|\nabla d\|_{L^\infty}^4\int_0^t \|\partial_t d\|_{\mathcal{H}^{m-2}}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau\\
&+C_1(1+P(Q(t)))\int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2)d\tau.
\end{aligned} \tag{3.46}$$

On the other hand, it is easy to check that

$$\int_0^t \|d_t\|_{L^2}^2 d\tau \lesssim \int_0^t \|\Delta d\|_{L^2}^2 d\tau + (1+\|\nabla d\|_{L^\infty}^2)\int_0^t (\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2)d\tau. \tag{3.47}$$

The combination of inequalities (3.46) and (3.47) yields directly

$$\int_0^t \|d_t\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C_1(1+P(Q(t)))\int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau$$

$$+C_1(1+P(Q(t)))\int_0^t (\|u\|_{\mathcal{H}^{m-1}}^2+\|\nabla d\|_{\mathcal{H}^{m-1}}^2)d\tau, \tag{3.48}$$

which, together with inequality (3.43), gives immediately

$$\begin{aligned}&\left|-\sum_{|\beta|\ge 1}\int_0^t\int \mathcal{Z}^\gamma(|\nabla d|^2)\mathcal{Z}^\beta d\cdot \operatorname{div}(\mathcal{Z}^\alpha\nabla d)\ dxd\tau\right|\\ \le&\delta_1\int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha\nabla d)\|_{L^2}d\tau+C_{\delta_1}C_1(1+P(Q(t)))\int_0^t \Lambda_m(\tau)d\tau.\end{aligned} \tag{3.49}$$

In view of the Proposition 2.2 and Cauchy's inequality, we obtain

$$\begin{aligned}&\left|-\int_0^t\int \mathcal{Z}^\alpha(|\nabla d|^2)d\cdot \operatorname{div}(\mathcal{Z}^\alpha\nabla d)\ dxd\tau\right|\\ \le&\delta_1\int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha\nabla d)\|_{L^2}d\tau+C_{\delta_1}\|\nabla d\|_{L^\infty}^2\int_0^t \|\nabla d\|_{\mathcal{H}^m}^2d\tau.\end{aligned} \tag{3.50}$$

Then, the combination of inequalities (3.49) and (3.50) yields immediately

$$\begin{aligned}&-\int_0^t\int \mathcal{Z}^\alpha(|\nabla d|^2 d)\cdot \operatorname{div}(\mathcal{Z}^\alpha\nabla d)\ dxd\tau\\ \le&\delta_1\int_0^t \|\operatorname{div}(\mathcal{Z}^\alpha\nabla d)\|_{L^2}d\tau+C_1C_{\delta_1}(1+P(Q(t)))\int_0^t \Lambda_m(\tau)d\tau.\end{aligned} \tag{3.51}$$

On the other hand, it is easy to check that

$$\begin{aligned}&\int_0^t\int [\mathcal{Z}^\alpha,\nabla](|\nabla d|^2 d)\cdot \mathcal{Z}^\alpha\nabla d\ dxd\tau\\ =&\sum_{|\beta|\le m-1}\int_0^t\int d\cdot \mathcal{Z}^\beta(\nabla d\cdot\nabla^2 d)\cdot \mathcal{Z}^\alpha\nabla d\ dxd\tau\\ &+\sum_{|\beta|+|\gamma|\le m-1}^{|\beta|\ge 1|}\int_0^t\int \mathcal{Z}^\beta d\cdot \mathcal{Z}^\gamma(\nabla d\cdot\nabla^2 d)\cdot \mathcal{Z}^\alpha\nabla d\ dxd\tau\\ &+\sum_{|\beta|+|\gamma|\le m-1}\int_0^t\int \mathcal{Z}^\beta(|\nabla d|^2)\mathcal{Z}^\gamma\nabla d\cdot \mathcal{Z}^\alpha\nabla d\ dxd\tau\\ =&II_1+II_2+II_3.\end{aligned} \tag{3.52}$$

In view of the Proposition 2.2, we find

$$II_1\lesssim \|\nabla d\|_{L^\infty_{x,t}}^2\int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2d\tau+\|\nabla^2 d\|_{L^\infty_{x,t}}^2\int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2d\tau+\int_0^t \|\nabla d\|_{\mathcal{H}^m}^2d\tau. \tag{3.53}$$

and

$$\begin{aligned}II_2\lesssim& \|\nabla d\|_{W^{1,\infty}_{x,t}}^4\int_0^t \|\mathcal{Z}d\|_{\mathcal{H}^{m-2}}^2d\tau+\|\mathcal{Z}d\|_{L^\infty_{x,t}}^2\|\nabla d\|_{L^\infty}^2\int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-2}}^2d\tau\\ &+\|\mathcal{Z}d\|_{L^\infty_{x,t}}^2\|\nabla^2 d\|_{L^\infty}^2\int_0^t \|\nabla d\|_{\mathcal{H}^{m-2}}^2d\tau+\int_0^t \|\nabla d\|_{\mathcal{H}^m}^2d\tau\end{aligned}$$

$$\leq C_1(1+P(Q(t)))\int_0^t \Lambda_m(\tau)d\tau, \tag{3.54}$$

where we have used the estimate (3.48). Similarly, it is easy to deduce that

$$II_3 \leq C\|\nabla d\|_{L^\infty}^4 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + C\int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \tag{3.55}$$

Plugging the estimates (3.53)-(3.55) into equation (3.52), one arrives at

$$\int_0^t \int [\mathcal{Z}^\alpha, \nabla](|\nabla d|^2 d)\cdot \mathcal{Z}^\alpha \nabla d \; dxd\tau \leq C_1(1+P(Q(t)))\int_0^t \Lambda_m(\tau)d\tau. \tag{3.56}$$

Deal with the boundary term on the right-hand side of equation (3.41). If $|\alpha_0|=m$ or $|\alpha_{13}|\geq 1$, we obtain

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(|\nabla d|^2 d)\cdot \mathcal{Z}^\alpha \nabla d\cdot n \; d\sigma d\tau = 0. \tag{3.57}$$

On one hand, it is easy to deduce that for $|\beta|=m-1-\alpha_0$

$$\begin{aligned}&\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(|\nabla d|^2 d)\cdot \mathcal{Z}^\alpha \nabla d\cdot n \; d\sigma d\tau\\ \leq &\int_0^t |\partial_t^{\alpha_0} Z_y^\beta(|\nabla d|^2 d)|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \nabla d\cdot n|_{H^1(\partial\Omega)} d\tau.\end{aligned} \tag{3.58}$$

By virtue of the trace theorem in Proposition 2.3, we find for $|\beta|=m-1-\alpha_0$

$$\begin{aligned}&|\partial_t^{\alpha_0} Z_y^\beta(|\nabla d|^2 d)|_{L^2(\partial\Omega)}^2\\ \leq &C\|\nabla \partial_t^{\alpha_0}(|\nabla d|^2 d)\|_{m-1-\alpha_0}\|\partial_t^{\alpha_0}(|\nabla d|^2 d)\|_{m-1-\alpha_0} + C\|\partial_t^{\alpha_0}(|\nabla d|^2 d)\|_{m-1-\alpha_0}^2\\ \leq &C\|\nabla(|\nabla d|^2 d)\|_{\mathcal{H}^{m-1}}\||\nabla d|^2 d\|_{\mathcal{H}^{m-1}} + C\||\nabla d|^2 d\|_{\mathcal{H}^{m-1}}^2,\end{aligned} \tag{3.59}$$

and

$$|\mathcal{Z}^\alpha \nabla d\cdot n|_{H^1(\partial\Omega)}^2 \leq C_{m+2}(\|\nabla^2 d\|_{\mathcal{H}^m}^2 + \|\nabla d\|_{\mathcal{H}^m}^2). \tag{3.60}$$

On the other hand, we obtain, just following the idea behind equations (3.42) and (3.52), that

$$\int_0^t \||\nabla d|^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C(1+P(Q(t)))\int_0^t \Lambda_m(\tau)d\tau \tag{3.61}$$

and

$$\int_0^t \|\nabla(|\nabla d|^2 d)\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C(1+P(Q(t)))\int_0^t \Lambda_m(\tau)d\tau. \tag{3.62}$$

The combination of inequalities (3.58)-(3.62) gives directly

$$\begin{aligned}&\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha(|\nabla d|^2 d)\cdot \mathcal{Z}^\alpha \nabla d\cdot n \; d\sigma d\tau\\ \leq &\delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + C_{\delta_1} C_{m+2}(1+P(Q(t)))\int_0^t \Lambda_m(\tau)d\tau.\end{aligned} \tag{3.63}$$

Substituting the estimates (3.51), (3.56) and (3.63) into equation (3.41), one attains

$$|I_3| \le \delta_1 \int_0^t \|\mathrm{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2}^2 d\tau + \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau \\ + C_{\delta_1} C_{m+2}(1+P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \tag{3.64}$$

Deal with the term I_4 and I_5. In view of the Cauchy inequality, one arrives at

$$|I_4| \le \delta_1 \int_0^t \|\mathrm{div}(\mathcal{Z}^\alpha \nabla d)\|_{L^2}^2 d\tau + C_{\delta_1} \int_0^t \|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau, \tag{3.65}$$

and

$$|I_5| \le \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \int_0^t \|\nabla d\|_{\mathcal{H}^m}^2 d\tau. \tag{3.66}$$

Deal with the term I_6. If $|\alpha_0| = m$ or $|\alpha_{13}| \ge 1$, it is easy to check that

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n d\sigma d\tau = 0. \tag{3.67}$$

For the case of $|\alpha_0| \le m-1$ or $|\alpha_{13}| = 0$, integrating by parts along the boundary, we have for $|\beta| = m-1-\alpha_0$

$$\int_0^t \int_{\partial\Omega} \mathcal{Z}^\alpha \Delta d \cdot \mathcal{Z}^\alpha \nabla d \cdot n d\sigma d\tau \le C \int_0^t |\partial_t^{\alpha_0} Z_y^\beta \Delta d|_{L^2(\partial\Omega)} |\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} d\tau. \tag{3.68}$$

By virtue of the trace theorem in Proposition 2.3, one arrives at

$$|\partial_t^{\alpha_0} Z_y^\beta \Delta d|_{L^2(\partial\Omega)} \le C(\|\nabla \partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac12} + \|\partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac12}) \|\partial_t^{\alpha_0} Z_y^\beta \Delta d\|^{\frac12} \\ \le C(\|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^{\frac12} + \|\Delta d\|_{\mathcal{H}^{m-1}}^{\frac12}) \|\Delta d\|_{\mathcal{H}^{m-1}}^{\frac12}. \tag{3.69}$$

Similarly, in view of boundary condition (1.5) and trace theorem in Proposition 2.3, one attains

$$|\mathcal{Z}^\alpha \nabla d \cdot n|_{H^1(\partial\Omega)} \le C_{m+2}(\|\nabla^2 d\|_{\mathcal{H}^m}^{\frac12} + \|\nabla d\|_{\mathcal{H}^m}^{\frac12}) \|\nabla d\|_{\mathcal{H}^m}^{\frac12}. \tag{3.70}$$

Substituting inequalities (3.69) and (3.70) into inequality (3.68) and applying the Cauchy inequality, we find

$$|I_6| \le \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta_1 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau \\ + C_{\delta,\delta_1} C_{m+2} \int_0^t (\|\nabla d\|_{\mathcal{H}^m}^2 + \|\Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau. \tag{3.71}$$

Substituting inequalities (3.40), (3.64)-(3.66) and (3.71) into equation (3.32) and choosing δ_1 small enough, we obtain

$$\frac12 \int |\mathcal{Z}^\alpha \nabla d(t)|^2 dx + \frac34 \int_0^t \int |\mathcal{Z}^\alpha \Delta d|^2 dx d\tau$$

$$\leq \frac{1}{2}\int |\mathcal{Z}^\alpha \nabla d_0|^2 dx + \delta_2 \int_0^t \|\nabla^2 d\|_{\mathcal{H}^m}^2 d\tau + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau$$
$$+ C_{\delta,\delta_2} C_{m+2}(1+P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \tag{3.72}$$

In view of the standard elliptic regularity results with Neumann boundary condition, we get that

$$\begin{aligned}\|\nabla^2 d\|_{\mathcal{H}^m}^2 &= \|\nabla^2 \partial_t^{\alpha_0} d\|_{m-\alpha_0}^2\\ &\leq C_{m+2}(\|\nabla \partial_t^{\alpha_0} d\|_{L^2}^2 + \|\Delta \partial_t^{\alpha_0} d\|_{m-\alpha_0}^2)\\ &\leq C_{m+2}(\|\nabla d\|_{\mathcal{H}^m}^2 + \|\Delta d\|_{\mathcal{H}^m}^2).\end{aligned} \tag{3.73}$$

The combination of inequalities (3.29), (3.72) and (3.73) yields directly

$$\sup_{0\leq\tau\leq t} \|(u,p,\nabla d)\|_{\mathcal{H}^m}^2 + C\varepsilon \int_0^t \|\nabla u\|_{\mathcal{H}^m}^2 d\tau + \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau$$
$$\leq C_{m+2}\Big\{ \|(u_0,p_0,\nabla d_0)\|_{\mathcal{H}^m}^2 + \delta \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + \delta\varepsilon^2 \int \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau$$
$$+ \delta \int_0^t \|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 d\tau + C_\delta[1+P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau \Big\}.$$

Therefore, we complete the proof of Lemma 3.2. □

3.2. Normal derivatives estimates. In order to estimate $\|\nabla u\|_{\mathcal{H}^{m-1}}$, it remains to estimate $\|\chi_j \partial_n u\|_{\mathcal{H}^{m-1}}$, where χ_j is supported compactly in one of the Ω_j and with value one in a neighborhood of the boundary. Indeed, it follows from the definition of the norm that $\|\chi \partial_{y_i} u\|_{\mathcal{H}^{m-1}} \leq C\|u\|_{\mathcal{H}^m}, i=1,2$. Then, it suffices to estimate $\|\chi \partial_n u\|_{\mathcal{H}^{m-1}}$.

Note that

$$\mathrm{div} u = \partial_n u \cdot n + (\Pi \partial_{y_1} u)_1 + (\Pi \partial_{y_1} u)_2 \tag{3.74}$$

and

$$\partial_n u = (\partial_n u \cdot n) n + \Pi(\partial_n u). \tag{3.75}$$

Then, it follows from equations (3.74) and (3.75) that

$$\begin{aligned}\|\chi \partial_n u\|_{\mathcal{H}^{m-1}} &\leq \|\chi \partial_n u \cdot n\|_{\mathcal{H}^{m-1}} + \|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}}\\ &\leq C_m\{\|\chi \mathrm{div} u\|_{\mathcal{H}^{m-1}} + \|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}} + \|u\|_{\mathcal{H}^m}\}.\end{aligned}$$

Thus, it suffices to estimate $\|\chi \Pi(\partial_n u)\|_{\mathcal{H}^{m-1}}$ and $\|\chi \mathrm{div} u\|_{\mathcal{H}^{m-1}}$, since $\|u\|_{\mathcal{H}^m}$ has been estimated before(see Lemma 3.2). We extend the smooth symmetric matrix A to be $A(y,z) = A(y)$. Define

$$\eta \triangleq \chi(w \times n + \Pi(Bu)) = \chi(\Pi(w \times n) + \Pi(Bu)). \tag{3.76}$$

In view of the boundary condition (1.5), η satisfies

$$\eta|_{\partial\Omega} = 0. \tag{3.77}$$

Since $w\times n=(\nabla u-(\nabla u)^t)\cdot n$, then η can be rewritten as

$$\eta=\chi\left\{\Pi(\partial_n u)-\Pi(\nabla(u\cdot n))+\Pi((\nabla n)^t\cdot u)+\Pi(Bu)\right\},$$

which, yields immediately that

$$\|\chi\Pi(\partial_n u)\|_{\mathcal{H}^{m-1}}\le C_{m+1}(\|\eta\|_{\mathcal{H}^{m-1}}+\|u\|_{\mathcal{H}^m}). \tag{3.78}$$

Hence, it remains to estimate $\|\eta\|_{\mathcal{H}^{m-1}}$.

LEMMA 3.3. *For $m\ge 1$, it holds that*

$$\begin{aligned}&\sup_{0\le\tau\le t}\|\eta(\tau)\|_{L^2}^2+\varepsilon\int_0^t\|\nabla\eta(\tau)\|_{L^2}^2d\tau\\ &\le CC_3\left\{\|u_0\|_{H^1}^2+\delta\varepsilon^2\int_0^t\|\nabla^2u(\tau)\|_{L^2}^2d\tau+\delta\int_0^t\|\nabla\Delta d\|_{L^2}^2d\tau\right\}\\ &\quad+C_3C_\delta[1+P(Q(t))]\int_0^t\Lambda_m(\tau)d\tau.\end{aligned} \tag{3.79}$$

Proof. Notice that

$$\nabla\times((u\cdot\nabla)u)=(u\cdot\nabla)w-(w\cdot\nabla)u+w\mathrm{div}u,$$

so w satisfies the following equation

$$\rho w_t+\rho(u\cdot\nabla)w=\mu\varepsilon\Delta w+F_1, \tag{3.80}$$

where

$$F_1\triangleq-\nabla\rho\times u_t-\nabla\rho\times(u\cdot\nabla)u+\rho(w\cdot\nabla)u-\rho w\mathrm{div}u-\nabla\times(\nabla d\cdot\Delta d).$$

Consequently, the system for η is

$$\begin{aligned}&\rho\eta_t+\rho u_1\partial_{y_1}\eta+\rho u_2\partial_{y_2}\eta+\rho u\cdot N\partial_z\eta-\mu\varepsilon\Delta\eta\\ &=\chi[F_1\times n+\Pi(BF_2)]+\chi F_3+F_4-\mu\varepsilon\chi\Delta(\Pi B)\cdot u,\end{aligned} \tag{3.81}$$

where

$$\begin{aligned}F_2&=(\mu+\lambda)\varepsilon\nabla\mathrm{div}u-\nabla p-\nabla d\cdot\Delta d,\\ F_3&=-2\mu\varepsilon\sum_{i=1}^2\partial_j w\times\partial_j n-\mu\varepsilon w\times\Delta n+\sum_{j=1}^2\rho u_i w\times\partial_i n\\ &\quad+\sum_{i=1}^2\rho u_i\partial_i(\Pi B)u-2\mu\varepsilon\sum_{i=1}^2\partial_i(\Pi B)\partial_i u,\\ F_4&=\sum_{i=1}^2\rho u_i\partial_{y_i}\chi\cdot(w\times n+\Pi(Bu))+\rho u\cdot N\partial_z\chi\cdot(w\times n+\Pi(Bu))\\ &\quad-2\mu\varepsilon\sum_{i=1}^3\partial_i\chi\partial_i(w\times n+\Pi(Bu))-\mu\varepsilon\Delta\chi\cdot(w\times n+\Pi(Bu)).\end{aligned}$$

Multiplying equation (3.81) by η and integrating over Ω, one arrives at

$$\frac{1}{2}\frac{d}{dt}\int|\eta|^2dx+\varepsilon\int|\nabla\eta|^2dx=\int F\cdot\eta dx-\mu\varepsilon\int\chi\Delta(\Pi B)\cdot u\cdot\eta dx, \tag{3.82}$$

where $F\triangleq\chi[F_1\times n+\Pi(BF_2)]+\chi F_3+F_4$. It is easy to deduce that

$$\|\chi F_1\times n\|_{L^2}\le C_2\{[1+P(Q(t))](\|\nabla u\|_{L^2}+\|\nabla p\|_{L^2})+\|\nabla d\|_{L^\infty}\|\nabla\Delta d\|_{L^2}\}, \tag{3.83}$$
$$\|\chi\Pi(BF_2)\|_{L^2}\le C_2(\varepsilon\|\nabla^2u\|_{L^2}+\|\nabla d\|_{L^\infty}\|\Delta d\|_{L^2}+\|\nabla p\|_{L^2}), \tag{3.84}$$
$$\|\chi F_3\|_{L^2}\le\varepsilon\|\nabla^2u\|_{L^2}+C_3(1+\|u\|_{L^\infty})(\|u\|_{L^2}+\|\nabla u\|_{L^2}). \tag{3.85}$$

Notice that the term F_4 are supported away from the boundary, we can control all the derivatives by the $\|\cdot\|_{\mathcal{H}^m}$. Hence, we find

$$\|F_4\|_{L^2}\le\varepsilon\|\nabla^2u\|_{L^2}+C_3(1+\|u\|_{L^\infty})\|u\|_{\mathcal{H}^1}. \tag{3.86}$$

Integrating by parts, it is easy to deduce that

$$-\mu\varepsilon\int\chi\Delta(\Pi B)\cdot u\cdot\eta dx\le\delta\varepsilon\int|\nabla\eta|^2dx+C_\delta C_3(\|\nabla u\|_{L^2}^2+\|u\|_{\mathcal{H}^1}^2). \tag{3.87}$$

Substituting inequalities (3.83)-(3.87) into equation (3.82) and integrating the resultant inequality over $[0,t]$, we have

$$\begin{aligned}&\frac{1}{2}\int|\eta|^2(t)dx+\varepsilon\int_0^t\int|\nabla\eta|^2dxd\tau\\&\le\frac{1}{2}\int|\eta_0|^2dx+\delta\varepsilon^2\int_0^t\|\nabla^2u\|_{L^2}^2d\tau+\delta\int_0^t\|\nabla\Delta d\|_{L^2}^2d\tau+C[1+P(Q(t))]\int_0^t\Lambda_1(\tau)d\tau.\end{aligned}$$

Therefore, we complete the proof of Lemma 3.3. ∎

Lemma 3.4. *For $m\ge1$, it holds that*

$$\begin{aligned}&\sup_{0\le\tau\le t}\|\eta(\tau)\|_{\mathcal{H}^{m-1}}^2+\mu\varepsilon\int_0^t\|\nabla\eta(\tau)\|_{\mathcal{H}^{m-1}}^2d\tau\\&\le CC_{m+2}\left\{\|(u_0,\nabla u_0)\|_{\mathcal{H}^{m-1}}^2+\delta\int_0^t\|\nabla\partial_t^{m-1}p\|_{L^2}^2d\tau+\delta\varepsilon^2\int_0^t\|\nabla^2u(\tau)\|_{\mathcal{H}^{m-1}}^2d\tau\right\}\\&\quad+CC_{m+2}\left\{\delta\int_0^t\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2d\tau+C_\delta[1+P(Q(t))]\int_0^t\Lambda_m(\tau)d\tau\right\}.\end{aligned} \tag{3.88}$$

Proof. The case for $m=1$ has been proved in Lemma 3.3. Assume that inequality (3.88) is proved for $k=m-2$. We shall prove that holds for $k=m-1\ge1$. For $|\alpha|=m-1$, applying the operator $\mathcal{Z}^\alpha$ to equation (3.81) , we find

$$\rho\mathcal{Z}^\alpha\eta_t+\rho(u\cdot\nabla)\mathcal{Z}^\alpha\eta-\mu\varepsilon\mathcal{Z}^\alpha\Delta\eta=\mathcal{Z}^\alpha F-\mathcal{Z}^\alpha[\mu\varepsilon\chi(\Delta(\Pi B)\cdot u)]+\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha, \tag{3.89}$$

where

$$\mathcal{C}_3^\alpha=-\sum_{|\beta|\ge1,\beta+\gamma=\alpha}C_{\alpha,\beta}\mathcal{Z}^\beta\rho\mathcal{Z}^\gamma\eta_t,$$

$$\begin{aligned}\mathcal{C}_4^\alpha = & -\sum_{|\beta|\geq 1,\beta+\gamma=\alpha}\sum_{i=1}^{2} C_{\alpha,\beta}\mathcal{Z}^\beta(\rho u_i)\mathcal{Z}^\gamma \partial_{y_i}\eta, \\ & -\sum_{|\beta|\geq 1,\beta+\gamma=\alpha} C_{\alpha,\beta}\mathcal{Z}^\beta(\rho u\cdot N)\mathcal{Z}^\gamma\partial_z\eta, \\ & -\rho(u\cdot N)\sum_{|\beta|\geq m-2} C(\alpha,\beta,z)\partial_z\mathcal{Z}^\beta\eta,\end{aligned}$$

where $C(\alpha,\beta,z)$ is smooth function depending on α,β and $\varphi(z)$. Multiplying equation (3.89) by $\mathcal{Z}^\alpha\eta$ and integrating over $[0,t]\times\Omega$, it is easy to deduce that

$$\begin{aligned}&\frac{1}{2}\int \rho|\mathcal{Z}^\alpha\eta(t)|^2dx - \frac{1}{2}\int\rho_0|\mathcal{Z}^\alpha\eta_0|^2dx \\ =&\mu\varepsilon\int_0^t\int \mathcal{Z}^\alpha\Delta\eta\cdot\mathcal{Z}^\alpha\eta\ dxd\tau + \int_0^t\int \mathcal{Z}^\alpha F\cdot\mathcal{Z}^\alpha\eta\ dxd\tau \\ & -\mu\varepsilon\int_0^t\int\mathcal{Z}^\alpha(\chi\Delta(\Pi B)\cdot u)\cdot\mathcal{Z}^\alpha\eta\ dxd\tau + \int_0^t\int(\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha)\cdot\mathcal{Z}^\alpha\eta\ dxd\tau. \qquad (3.90)\end{aligned}$$

In the local basis, it holds that

$$\partial_j = \beta_j^1\partial_{y_1}+\beta_j^2\partial_{y_2}+\beta_j^3\partial_z,\quad j=1,2,3,$$

for harmless functions β_j^i, $i,j=1,2,3$ depending on the boundary regularity and weighted function $\varphi(z)$. Then, the following commutation expansion holds:

$$\mathcal{Z}^\alpha\Delta\eta = \Delta\mathcal{Z}^\alpha\eta + \sum_{|\beta|\leq m-2} C_{1\beta}\partial_{zz}\mathcal{Z}^\beta\eta + \sum_{|\beta|\leq m-1}(C_{2\beta}\partial_z\mathcal{Z}^\beta\eta + C_{3\beta}Z_y\mathcal{Z}^\beta\eta).$$

Then integrating by parts and applying the Cauchy inequality, we obtain

$$\begin{aligned}&\mu\varepsilon\int_0^t\int\mathcal{Z}^\alpha\Delta\eta\cdot\mathcal{Z}^\alpha\eta dxd\tau \\ =&\mu\varepsilon\int_0^t\int\Delta\mathcal{Z}^\alpha\eta\cdot\mathcal{Z}^\alpha\eta dxd\tau + \sum_{|\beta|\leq m-2}\mu\varepsilon\int_0^t\int C_{1\beta}\partial_{zz}\mathcal{Z}^\beta\eta\cdot\mathcal{Z}^\alpha\eta dxd\tau \\ & + \sum_{|\beta|\leq m-1}\mu\varepsilon\int_0^t\int(C_{2\beta}\partial_z\mathcal{Z}^\beta\eta + C_{3\beta}Z_y\mathcal{Z}^\beta\eta)\cdot\mathcal{Z}^\alpha\eta dxd\tau \\ \leq & -\frac{3}{4}\mu\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha\eta\|_{L^2}^2d\tau + C\mu\varepsilon\int_0^t\|\nabla\eta\|_{\mathcal{H}^{m-2}}^2d\tau + C_{m+2}\mu\varepsilon\int_0^t\|\eta\|_{\mathcal{H}^{m-1}}^2d\tau. \qquad (3.91)\end{aligned}$$

Note that there is no boundary term in the integration by parts since $\mathcal{Z}^\alpha\eta$ vanishes on the boundary. Substituting the estimate (3.91) into (3.90), we find

$$\begin{aligned}&\frac{1}{2}\int|\mathcal{Z}^\alpha\eta(t)|^2dx + \frac{3}{4}\mu\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha\eta\|_{L^2}^2d\tau \\ \leq & \frac{1}{2}\int|\mathcal{Z}^\alpha\eta_0|^2dx + C\mu\varepsilon\int_0^t\|\nabla\eta\|_{\mathcal{H}^{m-2}}^2d\tau + C_{m+2}\varepsilon\int_0^t\|\eta\|_{\mathcal{H}^{m-1}}^2d\tau \\ & + \int_0^t\int\mathcal{Z}^\alpha F\cdot\mathcal{Z}^\alpha\eta dxd\tau - \mu\varepsilon\int_0^t\int\mathcal{Z}^\alpha(\chi\Delta(\Pi B)\cdot u)\cdot\mathcal{Z}^\alpha\eta\ dxd\tau\end{aligned}$$

$$+\int_0^t\int(\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha)\cdot\mathcal{Z}^\alpha\eta\ dxd\tau. \tag{3.92}$$

Similar to inequalities (3.83)-(3.86), we apply the Proposition 2.2 to deduce that

$$\begin{aligned}&\int_0^t\int\mathcal{Z}^\alpha(\chi F_1\times n)\cdot\mathcal{Z}^\alpha\eta dxd\tau\\ \le&C_m\delta\int_0^t\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2d\tau+C_mC_\delta(1+P(Q(t)))\int_0^t\Lambda_m(\tau)d\tau,\end{aligned} \tag{3.93}$$

$$\begin{aligned}&\int_0^t\int\mathcal{Z}^\alpha(\chi\Pi(BF_2))\cdot\mathcal{Z}^\alpha\eta dxd\tau\\ \le&C_{m+1}\delta\int_0^t\|\nabla\partial_t^{m-1}p\|_{L^2}^2d\tau+C_{m+1}\delta\varepsilon^2\int_0^t\|\nabla^2u\|_{\mathcal{H}^{m-1}}^2d\tau\\ &+C_{m+1}(1+P(Q(t)))\int_0^t\Lambda_m(\tau)d\tau,\end{aligned} \tag{3.94}$$

$$\begin{aligned}&\int_0^t\int\mathcal{Z}^\alpha(\chi F_3)\cdot\mathcal{Z}^\alpha\eta dxd\tau\\ \le&C_{m+2}\delta\varepsilon^2\int_0^t\|\nabla^2u\|_{\mathcal{H}^{m-1}}^2d\tau+C_{m+2}C_\delta(1+P(Q(t)))\int_0^t\Lambda_m(\tau)d\tau,\end{aligned} \tag{3.95}$$

and

$$\begin{aligned}&\int_0^t\int\mathcal{Z}^\alpha F_4\cdot\mathcal{Z}^\alpha\eta dxd\tau\\ \le&C_{m+1}\left\{\delta\varepsilon^2\int_0^t\|\nabla^2u\|_{\mathcal{H}^{m-1}}^2d\tau+C_\delta(1+P(Q(t)))\int_0^t\Lambda_m(\tau)d\tau\right\}.\end{aligned} \tag{3.96}$$

Then, the combination of inequalities (3.93)-(3.96) gives directly

$$\begin{aligned}&\int_0^t\int\mathcal{Z}^\alpha F\cdot\mathcal{Z}^\alpha\eta dxd\tau\\ \le&C_{m+2}\left\{\delta\int_0^t\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2d\tau+\delta\varepsilon^2\int\|\nabla^2u\|_{\mathcal{H}^{m-1}}^2d\tau\right\}\\ &+C_{m+2}\left\{\delta\int_0^t\|\nabla\partial_t^{m-1}p\|_{L^2}^2d\tau+C_\delta(1+P(Q(t)))\int_0^t\Lambda_m(\tau)d\tau\right\}.\end{aligned} \tag{3.97}$$

Integrating by parts and applying the Cauchy inequality, one arrives at directly

$$\begin{aligned}&\left|\mu\varepsilon\int_0^t\int\mathcal{Z}^\alpha(\chi\Delta(\Pi B)\cdot u)\cdot\mathcal{Z}^\alpha\eta\ dxd\tau\right|\\ \le&\delta\mu\varepsilon^2\int_0^t\|\nabla\eta\|_{\mathcal{H}^{m-1}}^2d\tau+C_\delta C_{m+2}\int_0^t\Lambda_m(\tau)d\tau.\end{aligned} \tag{3.98}$$

Using the same argument as Lemma 3.13 of [7], one can obtain the following estimate

$$\int_0^t\int(\mathcal{C}_3^\alpha+\mathcal{C}_4^\alpha)\cdot\mathcal{Z}^\alpha\eta dxd\tau\le C_m(1+P(Q(t)))\int_0^t\Lambda_m(\tau)d\tau. \tag{3.99}$$

Substituting the estimates (3.97), (3.98) and (3.99) into estimate (3.92), we find

$$\begin{aligned}
&\frac{1}{2}\int|\mathcal{Z}^\alpha\eta(t)|^2dx+\frac{3\mu\varepsilon}{4}\int_0^t\|\nabla\mathcal{Z}^\alpha\eta\|_{L^2}^2d\tau\\
\le& C_{m+2}\left\{\frac{1}{2}\int|\mathcal{Z}^\alpha\eta_0|^2dx+C\varepsilon\int_0^t\|\nabla\eta\|_{\mathcal{H}^{m-2}}^2d\tau\right\}\\
&+C_{m+2}\left\{\delta\int_0^t\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2d\tau+\delta\varepsilon^2\int_0^t\|\nabla^2u\|_{\mathcal{H}^{m-1}}^2d\tau\right\}.\\
&+C_{m+2}\left\{\delta\int_0^t\|\nabla\partial_t^{m-1}p\|_{L^2}^2d\tau+C_\delta(1+P(Q(t)))\int_0^t\Lambda_m(t)d\tau\right\}.
\end{aligned}$$

By the induction assumption, one can eliminate the term $\varepsilon\int_0^t\|\nabla\eta\|_{\mathcal{H}^{m-2}}^2d\tau$. Therefore, we complete the proof of Lemma 3.4. □

3.3. Estimates for the Δd, div$u$ and Δp. In this subsection, we shall get some uniform estimates for Δd, div$u$ and Δp in conormal Sobolev space.

LEMMA 3.5. *For a smooth solution to equations (1.1)-(1.3) and (1.5), it holds that for $\varepsilon\in(0,1]$*

$$\sup_{0\le\tau\le t}\|\Delta d(\tau)\|_{L^2}^2+\int_0^t\|\nabla\Delta d\|_{L^2}^2d\tau\le\|\Delta d_0\|_{L^2}^2+C(1+Q(t)^2)\int_0^t\Lambda_1(\tau)d\tau. \tag{3.100}$$

Proof. Taking ∇ operator to equation (1.3), one arrives at

$$\nabla d_t-\nabla\Delta d=-\nabla(u\cdot\nabla d)+\nabla(|\nabla d|^2d). \tag{3.101}$$

Multiplying equation (3.101) by $-\nabla\Delta d$ and integrating over Ω, we find

$$\begin{aligned}
&-\int\nabla d_t\cdot\nabla\Delta d\,dx+\int|\nabla\Delta d|^2dx\\
&=\int\nabla(u\cdot\nabla d)\cdot\nabla\Delta d\,dx-\int\nabla(|\nabla d|^2d)\cdot\nabla\Delta d\,dx.
\end{aligned}\tag{3.102}$$

By integrating by parts and applying the Neumann boundary condition (1.5), we get

$$-\int\nabla d_t\cdot\nabla\Delta d\,dx=-\int_{\partial\Omega}n\cdot\nabla d_t\cdot\Delta d\,d\sigma+\int\Delta d_t\cdot\Delta d\,dx=\frac{1}{2}\frac{d}{dt}\int|\Delta d|^2dx. \tag{3.103}$$

In view of the Cauchy inequality, we obtain

$$\int\nabla(u\cdot\nabla d)\cdot\nabla\Delta d\,dx\le\delta\|\nabla\Delta d\|_{L^2}^2+C_\delta\|u\|_{W^{1,\infty}}^2(\|\nabla d\|_{L^2}^2+\|\nabla^2d\|_{L^2}^2), \tag{3.104}$$

$$-\int\nabla(|\nabla d|^2d)\cdot\nabla\Delta d\,dx\le\delta\|\nabla\Delta d\|_{L^2}^2+\|\nabla d\|_{L^\infty}^4\|\nabla d\|_{L^2}^2+\|\nabla d\|_{L^2}^2\|\nabla^2d\|_{L^2}^2. \tag{3.105}$$

Substituting equation (3.103) and inequalities (3.104) and (3.105) into equation (3.102), choosing δ small enough and integrating over $[0,t]$, one attains directly

$$\frac{1}{2}\int|\Delta d|^2(t)dx+\frac{3}{4}\int|\nabla\Delta d|^2dx$$

$$\leq \int |\Delta d_0|^2 dx + C(\|u\|_{W^{1,\infty}}^2 + \|\nabla d\|_{L^\infty}^4) \int_0^t (\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2) d\tau.$$

Therefore, we complete the proof of Lemma 3.5. □

Next, we can establish the following conormal estimates for the quantity Δd.

LEMMA 3.6. *For $m \geq 1$ and a smooth solution to equations (1.1)-(1.3) and (1.5), it holds that for $\varepsilon \in (0,1]$*

$$\begin{aligned}
&\sup_{0\leq\tau\leq t} \|\Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2 + \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\
&\leq C_m \left\{ \|\Delta d_0\|_{\mathcal{H}^{m-1}}^2 + \delta \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C_\delta (1 + P(Q(t))) \int_0^t \Lambda_m(t) d\tau \right\}.
\end{aligned} \tag{3.106}$$

Proof. The case for $m=1$ is already proved in Lemma 3.5. Assume that inequality (3.106) is proved for $k=m-2$. We shall prove that is holds for $k=m-1\geq 1$. For $|\alpha|=m-1$, multiplying equation (3.101) by $-\nabla \mathcal{Z}^\alpha \Delta d$ and integrating over Ω, we find

$$\begin{aligned}
&-\int \mathcal{Z}^\alpha \nabla d_t \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx + \int \mathcal{Z}^\alpha \nabla \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx \\
&= \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx + \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx.
\end{aligned} \tag{3.107}$$

The integration by parts yields directly

$$\begin{aligned}
&-\int \mathcal{Z}^\alpha \nabla d_t \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx \\
&= -\int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma + \int \nabla \cdot (\mathcal{Z}^\alpha \nabla d_t) \cdot \mathcal{Z}^\alpha \Delta d \, dx \\
&= -\int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma + \frac{1}{2} \frac{d}{dt} \int |\mathcal{Z}^\alpha \Delta d|^2 dx - \int [\mathcal{Z}^\alpha, \nabla \cdot] \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, dx.
\end{aligned} \tag{3.108}$$

It is easy to check that

$$\int \mathcal{Z}^\alpha \nabla \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx = \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx + \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx. \tag{3.109}$$

Substituting equations (3.108) and (3.109) into equation (3.107) and integrating over $[0,t]$, we find

$$\begin{aligned}
&\frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d(t)|^2 dx + \int_0^t \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx d\tau \\
&= \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d_0|^2 dx + \int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, d\sigma d\tau \\
&\quad + \int_0^t \int [\mathcal{Z}^\alpha, \nabla \cdot] \nabla d_t \cdot \mathcal{Z}^\alpha \Delta d \, dx d\tau - \int_0^t \int [\mathcal{Z}^\alpha, \nabla] \Delta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&\quad + \int_0^t \int \mathcal{Z}^\alpha \nabla (u \cdot \nabla d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau + \int_0^t \int \mathcal{Z}^\alpha \nabla (|\nabla d|^2 d) \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\
&:= III_1 + III_2 + III_3 + III_4 + III_5 + III_6.
\end{aligned} \tag{3.110}$$

To deal with the boundary term on the right-hand side of equation (3.110). If $|\alpha_0|=m-1$, then we have

$$\int_0^t\int_{\partial\Omega} n\cdot\mathcal{Z}^\alpha\nabla d_t\cdot\mathcal{Z}^\alpha\Delta d\ d\sigma d\tau=0. \tag{3.111}$$

On the other hand, it is easy to deduce that for $|\alpha_0|\le m-2$

$$\int_0^t\int_{\partial\Omega} n\cdot\mathcal{Z}^\alpha\nabla d_t\cdot\mathcal{Z}^\alpha\Delta d\ d\sigma d\tau\le\int_0^t |n\cdot\mathcal{Z}^\alpha\nabla d_t|_{L^2(\partial\Omega)}|\mathcal{Z}^\alpha\Delta d|_{L^2(\partial\Omega)}d\tau. \tag{3.112}$$

The application of trace inequality in Proposition 2.3 and the boundary condition (1.5) implies

$$\begin{aligned}
&|\mathcal{Z}^\alpha\Delta d|_{L^2(\partial\Omega)}=|\partial_t^{\alpha_0}\Delta d|_{H^{m-1-|\alpha_0|}(\partial\Omega)}\\
\le& C\|\nabla\partial_t^{\alpha_0}\Delta d\|_{m-1-|\alpha_0|}+C\|\partial_t^{\alpha_0}\Delta d\|_{m-|\alpha_0|}\\
\le& C\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}+C\|\Delta d\|_{\mathcal{H}^m},
\end{aligned} \tag{3.113}$$

and

$$\begin{aligned}
&|n\cdot\mathcal{Z}^\alpha\nabla d_t|_{L^2(\partial\Omega)}\le C_m|\partial_t^{\alpha_0}\nabla d_t|_{H^{m-2-|\alpha_0|}(\partial\Omega)}\\
\le& C_m\|\partial_t^{\alpha_0}\nabla^2 d_t\|_{m-2-|\alpha_0|}+C_m\|\partial_t^{\alpha_0}\nabla d_t\|_{m-1-|\alpha_0|}\\
\le& C_m\|\nabla^2 d\|_{\mathcal{H}^{m-1}}+C_m\|\nabla d\|_{\mathcal{H}^m}.
\end{aligned} \tag{3.114}$$

Substituting inequalities (3.113) and (3.114) into inequality (3.112) and applying the Cauchy inequality, one attains

$$\begin{aligned}
III_2\le&\delta_1\int_0^t\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau+\delta\int_0^t\|\Delta d\|_{\mathcal{H}^m}^2 d\tau\\
&+C_m\left\{C_{\delta,\delta_1}\int_0^t\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau+C_{\delta,\delta_1}\int_0^t\|\nabla d\|_{\mathcal{H}^m}^2 d\tau\right\}.
\end{aligned} \tag{3.115}$$

By virtue of the Cauchy inequality, one arrives at

$$III_3\le C\int_0^t\|\Delta d_t\|_{\mathcal{H}^{m-2}}\|\Delta d\|_{\mathcal{H}^{m-1}}d\tau\le C\int_0^t\|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau, \tag{3.116}$$

$$III_4\le\delta_1\int_0^t\|\nabla\mathcal{Z}^\alpha\Delta d\|_{L^2}^2 d\tau+C_{\delta_1}\int_0^t\|\nabla\Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau. \tag{3.117}$$

The application of Proposition 2.2 yields directly

$$\begin{aligned}
III_5=&\int_0^t\int\mathcal{Z}^\alpha(\nabla u\cdot\nabla d)\cdot\nabla\mathcal{Z}^\alpha\Delta d\ dxd\tau+\int_0^t\int\mathcal{Z}^\alpha(u\cdot\nabla^2 d)\cdot\nabla\mathcal{Z}^\alpha\Delta d\ dxd\tau\\
\le&\delta_1\int_0^t\|\nabla\mathcal{Z}^\alpha\Delta d\|_{L^2}^2 d\tau+C_\delta\|\nabla u\|_{L^\infty_{x,t}}^2\int_0^t\|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau\\
&+C_\delta\|\nabla d\|_{L^\infty_{x,t}}^2\int_0^t\|\nabla u\|_{\mathcal{H}^{m-1}}^2 d\tau+C_\delta\|u\|_{L^\infty_{x,t}}^2\int_0^t\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 d\tau\\
&+C_\delta\|\nabla^2 d\|_{L^\infty_{x,t}}^2\int_0^t\|u\|_{\mathcal{H}^{m-1}}^2 d\tau
\end{aligned}$$

$$\leq \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_{\delta_1}(1+P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \tag{3.118}$$

It is easy to deduce that

$$\begin{aligned} III_6 = & \sum_{|\beta|\geq 1} \int_0^t \int \mathcal{Z}^\gamma(\nabla d \cdot \nabla^2 d) \cdot \mathcal{Z}^\beta d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\ & + \int_0^t \int \mathcal{Z}^\alpha (\nabla d \cdot \nabla^2 d) \cdot d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\ & + \sum_{|\beta|+|\gamma|=m-1} \int_0^t \int \mathcal{Z}^\gamma (|\nabla d|^2) \mathcal{Z}^\beta \nabla d \cdot \nabla \mathcal{Z}^\alpha \Delta d \, dx d\tau \\ = & III_{61} + III_{62} + III_{63}. \end{aligned} \tag{3.119}$$

By virtue of Proposition 2.2 and Cauchy inequality, one arrives at

$$\begin{aligned} III_{61} \leq & \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_\delta \|\mathcal{Z} d\|_{L^\infty_{x,t}}^2 \int_0^t \|\nabla d \cdot \nabla^2 d\|_{\mathcal{H}^{m-2}}^2 d\tau \\ & + C_\delta \|\nabla d \cdot \nabla^2 d\|_{L^\infty_{x,t}}^2 \int_0^t \|\mathcal{Z} d\|_{\mathcal{H}^{m-2}}^2 d\tau \\ \leq & \delta_1 \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_1 C_{\delta_1}(1+P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \end{aligned} \tag{3.120}$$

Similarly, it is easy to deduce that

$$III_{62} \leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_\delta \|\nabla d\|_{W^{1,\infty}_{x,t}}^2 \int_0^t (\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau, \tag{3.121}$$

$$III_{63} \leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_\delta \|\nabla d\|_{L^\infty_{x,t}}^4 \int_0^t (\|\nabla^2 d\|_{\mathcal{H}^{m-1}}^2 + \|\nabla d\|_{\mathcal{H}^{m-1}}^2) d\tau. \tag{3.122}$$

Substituting inequalities (3.120), (3.121) and (3.122) into equation (3.119), we obtain

$$III_6 \leq \delta \int_0^t \|\nabla \mathcal{Z}^\alpha \Delta d\|_{L^2}^2 d\tau + C_1 C_\delta (1+P(Q(t))) \int_0^t \Lambda_m(\tau) d\tau. \tag{3.123}$$

Plugging inequalities (3.115)-(3.118) and (3.123) into equation (3.110) and choosing δ small enough, we find

$$\begin{aligned} & \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d(t)|^2 dx + \int_0^t \int |\nabla \mathcal{Z}^\alpha \Delta d|^2 dx d\tau \\ \leq & C_m \left\{ \frac{1}{2} \int |\mathcal{Z}^\alpha \Delta d_0|^2 dx + \delta \int_0^t \|\Delta d\|_{\mathcal{H}^m}^2 d\tau + C \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau \right\} \\ & + C_m C_\delta [1+P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \end{aligned}$$

By the induction assumption, one can eliminate the term $\int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-2}}^2 d\tau$. Therefore, we complete the proof of Lemma 3.6. □

Next, we derive the following lower order estimates on $\|(\mathrm{div} u, p)\|_{L^2}^2$.

LEMMA 3.7. *For every $m\in\mathbb{N}_+$, it holds that*

$$\sup_{0\le\tau\le t}\int\left(\frac{1}{2}\rho|\mathrm{div}u|^2+\frac{1}{2\gamma p}|\nabla p(\tau)|^2\right)dx+\varepsilon\int_0^t\|\nabla\mathrm{div}u(\tau)\|_{L^2}^2d\tau$$
$$\le\frac{1}{2}\int\rho_0|\mathrm{div}u_0|^2dx+\int\frac{1}{2\gamma p_0}|\nabla p_0|^2dx$$
$$+C_3[1+P(Q(t))]\int_0^t(\Lambda_m(\tau)+\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2)d\tau. \tag{3.124}$$

Proof. Multiplying equation (3.5) by $\nabla\mathrm{div}u$ and integrating over $[0,t]\times\Omega$, we find

$$\int_0^t\int(\rho u_t+\rho u\cdot\nabla u)\cdot\nabla\mathrm{div}u\ dxd\tau+\int_0^t\int\nabla p\cdot\nabla\mathrm{div}u\ dxd\tau$$
$$=-\mu\varepsilon\int_0^t\int\nabla\times w\cdot\nabla\mathrm{div}u\ dxd\tau+(2\mu+\lambda)\varepsilon\int_0^t\int|\nabla\mathrm{div}u|^2dxd\tau$$
$$-\int_0^t\int(\nabla d\cdot\Delta d)\cdot\nabla\mathrm{div}u\ dxd\tau=IV_1+IV_2+IV_3+IV_4. \tag{3.125}$$

Using the same argument as Lemma 3.5 of [7], one can obtain the following estimates

$$IV_1\le-\frac{1}{2}\int\rho|\mathrm{div}u|^2dx+\frac{1}{2}\int\rho_0|\mathrm{div}u_0|^2dx+C_2[1+P(Q(t))]\int_0^t\Lambda_m(\tau)d\tau, \tag{3.126}$$
$$IV_2\le-\int\frac{1}{2\gamma p}|\nabla p|^2dx+\int\frac{1}{2\gamma p_0}|\nabla p_0|^2dx+C_2[1+P(Q(t))]\int_0^t\|\nabla p\|_{L^2}^2d\tau, \tag{3.127}$$

and

$$IV_3\le\frac{\varepsilon}{4}\int_0^t\|\nabla\mathrm{div}u\|_{L^2}^2d\tau+C_3\varepsilon\int_0^t(\|u(\tau)\|_{L^2}^2+\|\nabla u(\tau)\|_{L^2}^2)d\tau. \tag{3.128}$$

Integrating by parts and applying the boundary condition (1.5), we get

$$IV_4=-\int_0^t\int_{\partial\Omega}n\cdot(\nabla d\cdot\Delta d)\mathrm{div}u\ d\sigma d\tau+\int_0^t\int\nabla(\nabla d\cdot\Delta d)\mathrm{div}u\ dxd\tau$$
$$\le C(\|\nabla d\|_{L^\infty}^2+\|\Delta d\|_{L^\infty})\int_0^t(\|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2)d\tau. \tag{3.129}$$

Substituting the estimates (3.126)-(3.129) into equation (3.125), one attains directly

$$\int\left(\frac{1}{2}\rho|\mathrm{div}u|^2+\frac{1}{2\gamma p}|\nabla p|^2\right)dx+\varepsilon\int_0^t\|\nabla\mathrm{div}u(\tau)\|_{L^2}^2d\tau$$
$$\le\int\left(\frac{1}{2}\rho_0|\mathrm{div}u_0|^2+\frac{1}{2\gamma p_0}|\nabla p_0|^2\right)dx+C_3[1+P(Q(t))]\int_0^t(\Lambda_m(\tau)+\|\nabla\Delta d\|_{L^2}^2)d\tau.$$

Therefore, we complete the proof of Lemma 3.7. ∎

LEMMA 3.8. *For $m\ge1$ and $|\alpha|\le m-1$ with $|\alpha_0|\le m-2$, it holds that*

$$\sup_{0\le\tau\le t}\int\left(\rho|\mathcal{Z}^\alpha\mathrm{div}u(\tau)|^2+\frac{1}{\gamma p}|\mathcal{Z}^\alpha\nabla p(\tau)|^2\right)dx+\varepsilon\int_0^t\|\nabla\mathcal{Z}^\alpha\mathrm{div}u\|_{L^2}^2d\tau$$

$$
\begin{aligned}
&\le C\int\left(\rho_0|\mathcal{Z}^\alpha \mathrm{div} u_0|^2+\frac{1}{\gamma p_0}|\mathcal{Z}^\alpha\nabla p_0|^2\right)dx+CC_{m+2}\delta\int_0^t\|\nabla\partial_t^{m-1}p(\tau)\|_{L^2}^2d\tau\\
&\quad+CC_{m+2}\left\{\varepsilon\int_0^t\|\nabla^2 u\|_{\mathcal{H}^{m-2}}^2d\tau+(\delta+\varepsilon)\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div} u\|_{L^2}^2d\tau\right\}\\
&\quad+C_\delta C_{m+2}[1+P(Q(t))]\int_0^t(\Lambda_m+\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2)d\tau. \qquad (3.130)
\end{aligned}
$$

Proof. The case for $|\alpha|=0$ is already proved in Lemma 3.7. Assuming it is proved for $|\alpha|\le m-2$, one needs to prove it for $|\alpha|=m-1$ with $|\alpha_0|\le m-2$. Multiplying equation (3.19) by $\nabla\mathcal{Z}^\alpha\mathrm{div} u$ and integrating over $[0,t]\times\Omega$, one attains directly

$$
\begin{aligned}
&\underbrace{\int_0^t\int(\rho\mathcal{Z}^\alpha u_t+\rho u\cdot\nabla\mathcal{Z}^\alpha u)\cdot\nabla\mathcal{Z}^\alpha\mathrm{div} u\,dxd\tau}_{V_1}+\underbrace{\int_0^t\int\mathcal{Z}^\alpha\nabla p\cdot\nabla\mathcal{Z}^\alpha\mathrm{div} u\,dxd\tau}_{V_2}\\
&=\underbrace{-\mu\varepsilon\int_0^t\int\mathcal{Z}^\alpha\nabla\times w\cdot\nabla\mathcal{Z}^\alpha\mathrm{div} u\,dxd\tau}_{V_3}+\underbrace{\int_0^t\int(\mathcal{C}_1^\alpha+\mathcal{C}_2^\alpha)\cdot\nabla\mathcal{Z}^\alpha\mathrm{div} u\,dx\tau}_{V_6}\\
&\quad\underbrace{+(2\mu+\lambda)\varepsilon\int_0^t\int\mathcal{Z}^\alpha\nabla\mathrm{div} u\cdot\nabla\mathcal{Z}^\alpha\mathrm{div} u\,dxd\tau}_{V_4}\\
&\quad\underbrace{-\int_0^t\int\mathcal{Z}^\alpha(\nabla d\cdot\Delta d)\cdot\nabla\mathcal{Z}^\alpha\mathrm{div} u\,dxd\tau}_{V_5}. \qquad (3.131)
\end{aligned}
$$

Using the same argument as Lemma 3.6 of [7], one can obtain the following estimates

$$
\begin{aligned}
V_1\le&-\int\frac{\rho}{2}|\mathcal{Z}^\alpha\mathrm{div} u|^2dx+\int\frac{\rho_0}{2}|\mathcal{Z}^\alpha\mathrm{div} u_0|^2dx\\
&+\delta\int_0^t\|\nabla\mathcal{Z}^{\alpha-2}\mathrm{div} u\|_{L^2}^2d\tau+C_\delta[1+P(Q(t))]\int_0^t\Lambda_m(\tau)d\tau, \qquad (3.132)\\
V_4\ge&\frac{3(2\mu+\lambda)\varepsilon}{4}\int_0^t\|\nabla\mathcal{Z}^\alpha\mathrm{div} u\|_{L^2}^2d\tau-C\varepsilon\int_0^t\Lambda_m(\tau)d\tau\\
&-C\varepsilon\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div} u\|_{L^2}^2d\tau, \qquad (3.133)\\
V_3\ge&-\frac{(2\mu+\lambda)\varepsilon}{4}\int_0^t\|\nabla\mathcal{Z}^\alpha\mathrm{div} u\|_{L^2}^2d\tau-C\varepsilon\int_0^t\|\nabla^2 u\|_{\mathcal{H}^{m-2}}^2d\tau\\
&-C_{m+2}\int_0^t P(\Lambda_m(\tau))d\tau, \qquad (3.134)\\
V_2\le&-\int\frac{1}{2\gamma p}|\mathcal{Z}^\alpha\nabla p|^2dx+\int\frac{1}{2\gamma p_0}|\mathcal{Z}^\alpha\nabla p_0|^2dx\\
&+C\delta\int_0^t\|\nabla\mathcal{Z}^{m-2}\mathrm{div} u\|_{L^2}^2d\tau+C\delta\int_0^t\|\nabla\partial_t^{m-1}p\|_{L^2}^2d\tau\\
&+C_\delta C_{m+1}(1+P(Q(t)))\int_0^t\Lambda_m(\tau)d\tau, \qquad (3.135)
\end{aligned}
$$

and

$$V_6 \leq \delta \int_0^t \|\nabla \mathcal{Z}^{m-2}\mathrm{div}u\|_{L^2}^2 d\tau + C_\delta [1+P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \tag{3.136}$$

On the other hand, the integration by parts yields directly

$$\begin{aligned} V_5 = & -\int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha(\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha \mathrm{div}u \, d\sigma d\tau \\ & + \int_0^t \int \mathrm{div}\mathcal{Z}^\alpha(\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha \mathrm{div}u \, dx d\tau. \end{aligned} \tag{3.137}$$

In view of the trace inequality in Proposition 2.3, we find

$$\begin{aligned} & |Z_y^{m-2-\alpha_0} Z_t^{\alpha_0} \mathrm{div}u|_{L^2(\partial\Omega)}^2 \\ \leq & C \|\nabla Z_y^{m-2-\alpha_0} Z_t^{\alpha_0} \mathrm{div}u\|_{L^2}^2 + C \|Z_y^{m-2-\alpha_0} Z_t^{\alpha_0} \mathrm{div}u\|_{L^2}^2, \end{aligned} \tag{3.138}$$

and

$$\begin{aligned} & |Z_y(n \cdot Z_y^{m-1-\alpha_0} Z_t^{\alpha_0})(\nabla d \cdot \Delta d)|_{L^2(\partial\Omega)}^2 \\ \leq & C \|\nabla Z_t^{\alpha_0}(\nabla d \cdot \Delta d)\|_{H_{co}^{|\alpha_1|}}^2 + C \|Z_t^{\alpha_0}(\nabla d \cdot \Delta d)\|_{H_{co}^{|\alpha_1|}}^2. \end{aligned} \tag{3.139}$$

Integrating by parts along the boundary and applying the estimates (3.138) and (3.139), one attains

$$\begin{aligned} & -\int_0^t \int_{\partial\Omega} n \cdot \mathcal{Z}^\alpha(\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha \mathrm{div}u \, d\sigma d\tau \\ \leq & \int_0^t |Z_y(n \cdot Z_y^{m-1-\alpha_0} Z_t^{\alpha_0})(\nabla d \cdot \Delta d)|_{L^2(\partial\Omega)} |Z_y^{m-2-\alpha_0} Z_t^{\alpha_0} \mathrm{div}u|_{L^2(\partial\Omega)} d\tau \\ \leq & \delta \int_0^t \|\nabla \mathcal{Z}^{m-2}\mathrm{div}u\|_{L^2}^2 d\tau + C_\delta \int_0^t \|\mathcal{Z}^{m-2}\mathrm{div}u\|_{L^2}^2 d\tau \\ & + C_\delta \|\nabla^2 d\|_{L^\infty}^2 \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \|\nabla d\|_{L^\infty}^2 \int_0^t \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & + C_\delta \|\nabla \Delta d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau + C_\delta \|\nabla d\|_{L^\infty}^2 \int_0^t \|\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ & + C_\delta \|\Delta d\|_{L^\infty}^2 \int_0^t \|\nabla d\|_{\mathcal{H}^{m-1}}^2 d\tau \\ \leq & \delta \int_0^t \|\nabla \mathcal{Z}^{m-2}\mathrm{div}u\|_{L^2}^2 d\tau + C_\delta [1+P(Q(t))] \int_0^t (\Lambda_m + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau. \end{aligned} \tag{3.140}$$

Applying the Proposition 2.2, it is easy to check that

$$\begin{aligned} & \int_0^t \int \mathrm{div}\mathcal{Z}^\alpha(\nabla d \cdot \Delta d) \cdot \mathcal{Z}^\alpha \mathrm{div}u \, dx d\tau \\ \leq & C[1+P(Q(t))] \int_0^t (\Lambda_m + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau. \end{aligned} \tag{3.141}$$

Substituting inequalities (3.140) and (3.141) into equation (3.137), we obtain

$$V_5 \leq \delta \int_0^t \|\nabla \mathcal{Z}^{m-2}\mathrm{div}u\|_{L^2}^2 d\tau + C_\delta [1+P(Q(t))] \int_0^t (\Lambda_m + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) d\tau. \tag{3.142}$$

Substituting inequalities (3.132)-(3.136) and (3.142) into equation (3.131), we complete the proof of Lemma 3.8. ∎

LEMMA 3.9. *For $m \geq 1$, it holds that*

$$\begin{aligned}&\sup_{0\leq\tau\leq t}\varepsilon\int\left(\rho|\partial_t^{m-1}\mathrm{div}u(\tau)|^2+\frac{1}{\gamma p}|\partial_t^{m-1}\nabla p(\tau)|^2\right)dx\\&\quad+\varepsilon^2\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u(\tau)\|_{L^2}^2d\tau\\&\leq C\varepsilon\int\left(\rho_0|\partial_t^{m-1}\mathrm{div}u_0|^2+\frac{1}{\gamma p_0}|\partial_t^{m-1}\nabla p_0|^2\right)dx\\&\quad+C_{m+1}[1+P(Q(t))]\int_0^t(\Lambda_m(\tau)+\|\nabla\Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2)d\tau.\end{aligned}\tag{3.143}$$

Proof. Applying ∂_t^{m-1} to equation (3.5), we find that

$$\begin{aligned}&\rho\partial_t^{m-1}u_t+\rho u\cdot\nabla\partial_t^{m-1}u+\mu\varepsilon\nabla\times\partial_t^{m-1}(\nabla\times u)+\partial_t^{m-1}\nabla p\\&=(2\mu+\lambda)\varepsilon\nabla\partial_t^{m-1}\mathrm{div}u-\partial_t^{m-1}(\nabla d\cdot\Delta d)+\mathcal{C}_1^{m-1}+\mathcal{C}_2^{m-1},\end{aligned}\tag{3.144}$$

where

$$\mathcal{C}_1^{m-1}\triangleq-[\partial_t^{m-1},\rho]u_t,\quad \mathcal{C}_2^{m-1}\triangleq-[\partial_t^{m-1},\rho u\cdot\nabla]u.$$

Multiplying equation (3.144) by $\varepsilon\nabla\mathrm{div}\partial_t^{m-1}u$ and integrating over $[0,t]\times\Omega$, one arrives at

$$\begin{aligned}&\underbrace{\varepsilon\int_0^t\int(\rho\partial_t^{m-1}u_t+\rho u\cdot\nabla\partial_t^{m-1}u)\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,dxd\tau}_{VI_1}\\&\quad+\underbrace{\varepsilon\int_0^t\int\partial_t^{m-1}\nabla p\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,dxd\tau}_{VI_2}\\&=\underbrace{-\mu\varepsilon^2\int_0^t\int\nabla\times\partial_t^{m-1}(\nabla\times u)\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,dxd\tau}_{VI_3}\\&\quad+\underbrace{(2\mu+\lambda)\varepsilon^2\int_0^t\int|\nabla\partial_t^{m-1}\mathrm{div}u|^2dxd\tau}_{VI_4}\\&\quad+\underbrace{\varepsilon\int_0^t\int(\mathcal{C}_1^{m-1}+\mathcal{C}_2^{m-1})\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,dxd\tau}_{VI_5}\\&\quad\underbrace{-\varepsilon\int_0^t\int\partial_t^{m-1}(\nabla d\cdot\Delta d)\cdot\nabla\mathrm{div}\partial_t^{m-1}u\,dxd\tau}_{VI_6}.\end{aligned}\tag{3.145}$$

Using the same argument as Lemma 3.8 of [7], one can obtain the following estimates

$$|VI_3|\leq\frac{2\mu+\lambda}{8}\varepsilon^4\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u\|_{L^2}^2d\tau+CC_3\int_0^t\Lambda_m(\tau)d\tau,\tag{3.146}$$

$$
\begin{aligned}
VI_1 \leq & -\varepsilon \int \rho |\partial_t^{m-1} \mathrm{div} u(t)|^2 dx + \varepsilon \int \rho_0 |\partial_t^{m-1} \mathrm{div} u_0|^2 dx \\
& + \frac{\varepsilon^2}{8} \int_0^t \|\nabla \partial_t^{m-1} \mathrm{div} u\|_{L^2}^2 d\tau + C[1+P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau, \qquad (3.147)
\end{aligned}
$$

$$
|VI_5| \leq \frac{2\mu+\lambda}{8} \varepsilon^2 \int_0^t \|\nabla \partial_t^{m-1} \mathrm{div} u\|_{L^2}^2 d\tau + C[1+P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau, \qquad (3.148)
$$

and

$$
\begin{aligned}
VI_2 \leq & -\varepsilon \int \frac{1}{2\gamma p} |\nabla \partial_t^{m-1} p|^2 dx + \varepsilon \int \frac{1}{2\gamma p_0} |\nabla \partial_t^{m-1} p_0|^2 dx \\
& + C_{m+1}[1+P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \qquad (3.149)
\end{aligned}
$$

Integrating by parts and applying the boundary condition (1.5), one attains

$$
\begin{aligned}
VI_6 = & -\varepsilon \int_0^t \int n \cdot \partial_t^{m-1} (\nabla d \cdot \Delta d) \cdot \mathrm{div} \partial_t^{m-1} u \, d\sigma d\tau \\
& + \varepsilon \int_0^t \int \mathrm{div} \partial_t^{m-1} (\nabla d \cdot \Delta d) \cdot \mathrm{div} \partial_t^{m-1} u \, dx d\tau \\
\leq & C[1+P(Q(t))] \int_0^t (\Lambda_m(\tau) + \|\nabla \Delta d\|_{\mathcal{H}^{m-1}}^2) \tau. \qquad (3.150)
\end{aligned}
$$

Substituting inequalities (3.146)-(3.150) into equation (3.145), we complete the proof of Lemma 3.9. ∎

Next, we recall an important estimate that has been proved by Wang et al. [7].

Lemma 3.10. *Define*

$$
\begin{aligned}
\Lambda_{1m}(t) \triangleq & \|(p,u,\nabla d)(t)\|_{\mathcal{H}^m}^2 + \|\Delta d(t)\|_{\mathcal{H}^{m-1}}^2 \\
& + \sum_{|\beta| \leq m-2} \|\mathcal{Z}^\beta \nabla p(t)\|_1^2 + \sum_{|\beta| \leq m-2} \|\mathcal{Z}^\beta \nabla u(t)\|_1^2. \qquad (3.151)
\end{aligned}
$$

Then, for every $m \geq 3$, it holds that

$$
\|\partial_t^{m-1} \mathrm{div} u(t)\|_{L^2}^2 \leq C_2 \{P(\Lambda_{1m}(t)) + P(Q(t))\}. \qquad (3.152)
$$

Lemma 3.11. *For every $m \geq 1$, it holds that*

$$
\begin{aligned}
& \int_0^t \|\nabla \partial_t^{m-1} p(\tau)\|_{L^2}^2 d\tau \\
\leq & C\varepsilon^2 \int_0^t \|\nabla^2 \partial_t^{m-1} u(\tau)\|_{L^2}^2 d\tau + C[1+P(Q(t))] \int_0^t \Lambda_m(\tau) d\tau. \qquad (3.153)
\end{aligned}
$$

Proof. Applying the operator ∂_t^{m-1} to equation (3.5), we find

$$
\partial_t^{m-1} \nabla p = \partial_t^{m-1} (-\rho u_t - \rho u \cdot \nabla u - \mu \varepsilon \nabla \times (\nabla \times u) + (2\mu+\lambda) \varepsilon \nabla \mathrm{div} u - \nabla d \cdot \Delta d).
$$

By using the Proposition 2.2, it is easy to deduce the estimate (3.153). Hence, we complete the proof of Lemma 3.11. ∎

Next, we recall an important estimate that has been proved by Wang et al. [7].

LEMMA 3.12. *For every $m \geq 1$, it holds that*

$$\begin{aligned}&\int_0^t \|\nabla \mathcal{Z}^{m-2}\mathrm{div}u(\tau)\|_{L^2}^2 d\tau\\ \leq &C\int_0^t \|\nabla \partial_t^{m-1}p(\tau)\|_{L^2}^2 d\tau + C_m[1+P(Q(t))]\int_0^t \Lambda_m(\tau)d\tau. \end{aligned} \tag{3.154}$$

Finally, we estimate the estimate for the quantity $\nabla\Delta d$.

LEMMA 3.13. *For every $m \geq 1$, it holds that*

$$\int_0^t \|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2 d\tau \leq C[1+P(Q(t))]\int_0^t \Lambda_m(\tau)d\tau. \tag{3.155}$$

Proof. Applying ∇ operator to equation (1.3), we find

$$\nabla\Delta d = \nabla d_t + \nabla(u\cdot\nabla d) - \nabla(|\nabla d|^2 d).$$

By using the Proposition 2.2, it is easy to deduce the estimate (3.155). Then, we complete the proof of Lemma 3.13. □

Substituting the estimates (3.153), (3.154) and (3.155) into estimate (3.130), it is easy to deduce that

$$\begin{aligned}&\sup_{0\leq\tau\leq t}\sum_{k=0}^{m-2}\|(\partial_t^k\nabla p, \partial_t^k \mathrm{div}u)(\tau)\|_{m-1-k}^2 + \varepsilon\int_0^t\sum_{k=0}^{m-2}\|\partial_t^k\nabla\mathrm{div}u(\tau)\|_{m-1-k}^2 d\tau\\ \leq &CC_{m+2}\left\{\Lambda_m(0) + C\delta\varepsilon^2\int_0^t\|\nabla^2\partial_t^{m-1}u(\tau)\|_{L^2}^2 d\tau + \varepsilon\int_0^t \|\nabla^2 u(\tau)\|_{\mathcal{H}^{m-2}}^2 d\tau\right\}\\ &\quad + C_\delta C_{m+2}[1+P(Q(t))]\int_0^t\Lambda_m(\tau)d\tau. \end{aligned} \tag{3.156}$$

Plugging the estimate (3.155) into estimate (3.143), we obtain

$$\begin{aligned}&\sup_{0\leq\tau\leq t}\varepsilon\left(\|\partial_t^{m-1}\mathrm{div}u(\tau)\|_{L^2}^2 + \|\partial_t^{m-1}\nabla p(\tau)\|_{L^2}^2\right) + \varepsilon^2\int_0^t \|\nabla\partial_t^{m-1}\mathrm{div}u(\tau)\|_{L^2}^2 d\tau\\ \leq &C\varepsilon\Lambda_m(0) + CC_{m+1}[1+P(Q(t))]\int_0^t \Lambda_m(\tau)d\tau. \end{aligned} \tag{3.157}$$

Then, the combination of estimates (3.156) and (3.157) yields directly

$$\begin{aligned}&\sup_{0\leq\tau\leq t}\sum_{k=0}^{m-2}\{\|(\partial_t^k\nabla p, \partial_t^k\mathrm{div}u)(\tau)\|_{m-1-k}^2 + \varepsilon\|(\partial_t^{m-1}\mathrm{div}u, \partial_t^{m-1}\nabla p)(\tau)\|_{L^2}^2\}\\ &\quad + \varepsilon\int_0^t\sum_{k=0}^{m-2}\|\partial_t^k\nabla\mathrm{div}u(\tau)\|_{m-1-k}^2 d\tau + \varepsilon^2\int_0^t\|\nabla\partial_t^{m-1}\mathrm{div}u(\tau)\|_{L^2}^2 d\tau\\ \leq &CC_{m+2}\left\{\Lambda_m(0) + C\delta\varepsilon^2\int_0^t\|\nabla^2\partial_t^{m-1}u(\tau)\|_{L^2}^2 d\tau + \varepsilon\int_0^t\|\nabla^2 u(\tau)\|_{\mathcal{H}^{m-2}}^2 d\tau\right\}\end{aligned}$$

$$+C_\delta C_{m+2}[1+P(Q(t))]\int_0^t \Lambda_m(\tau)d\tau. \tag{3.158}$$

On the other hand, it is easy to check that

$$\sum_{|\beta|\le m-2}\|\mathcal{Z}^\beta\nabla u\|_1^2\le C_{m+1}(\|u\|_{\mathcal{H}^m}^2+\|\eta\|_{\mathcal{H}^{m-1}}^2+\sum_{k=0}^{m-2}\|\partial_t^k \mathrm{div}u\|_{m-1-k}^2), \tag{3.159}$$

$$\int_0^t \|\nabla^2 u\|_{\mathcal{H}^{m-1}}^2 d\tau\le C_{m+2}\int_0^t(\|\nabla u\|_{\mathcal{H}^m}^2+\|\nabla\eta\|_{\mathcal{H}^{m-1}}^2+\|\nabla \mathrm{div}u\|_{\mathcal{H}^{m-1}}^2+\Lambda_m)d\tau, \tag{3.160}$$

$$\varepsilon\int_0^t \|\nabla^2\mathcal{Z}^{m-2}u\|_{L^2}^2 d\tau\le C_{m+1}\varepsilon\int_0^t\|\nabla\eta\|_{\mathcal{H}^{m-2}}^2 d\tau+C_{m+1}\int_0^t\Lambda_m(\tau)d\tau, \tag{3.161}$$

and

$$\begin{aligned}&\sum_{k=0}^{m-2}\int_0^t\|\nabla^2\partial_t^k u\|_{m-1-k}^2 d\tau\\ &\le C_{m+2}\int_0^t(\|\nabla u\|_{\mathcal{H}^m}^2+\|\nabla\eta\|_{\mathcal{H}^{m-1}}^2)d\tau+C_{m+2}\int_0^t\Lambda_m(\tau)d\tau\\ &\quad +C_{m+2}\sum_{k=0}^{m-2}\int_0^t\|\partial_t^k\nabla \mathrm{div}u\|_{m-1-k}^2 d\tau. \end{aligned}\tag{3.162}$$

The combination of estimates (3.158)-(3.162) yields immediately

$$\begin{aligned}&\sup_{0\le\tau\le t}\left\{\Lambda_{1m}(\tau)+\|(\eta,\Delta d)(\tau)\|_{\mathcal{H}^{m-1}}^2+\varepsilon\|(\partial_t^{m-1}\mathrm{div}u,\partial_t^{m-1}\nabla p)(\tau)\|_{L^2}^2\right\}\\ &\quad+\varepsilon\int_0^t(\|\nabla u\|_{\mathcal{H}^m}^2+\|\nabla\eta\|_{\mathcal{H}^{m-1}}^2)d\tau+\int_0^t(\|\Delta d\|_{\mathcal{H}^m}^2+\|\nabla\Delta d\|_{\mathcal{H}^{m-1}}^2)d\tau\\ &\quad+\varepsilon\int_0^t\sum_{k=0}^{m-2}\|\partial_t^k\nabla \mathrm{div}u\|_{m-1-k}^2 d\tau+\varepsilon^2\int_0^t\|\partial_t^{m-1}\nabla\mathrm{div}u\|_{L^2}^2 d\tau\\ &\quad+\varepsilon\int_0^t\sum_{k=0}^{m-2}\|\partial_t^k\nabla^2 u\|_{m-1-k}^2 d\tau+\varepsilon^2\int_0^t\|\partial_t^{m-1}\nabla^2 u\|_{L^2}^2 d\tau+\int_0^t\|\partial_t^{m-1}\nabla p\|_{L^2}^2 d\tau\\ &\le CC_{m+2}\left\{\Lambda_m(0)+[1+P(Q(t))]\int_0^t P(\Lambda_m(\tau))d\tau\right\}. \end{aligned}\tag{3.163}$$

3.4. *$L^\infty-$estimates.* In this subsection, we shall provide the L^∞-estimates of (p,u,d) which are needed to estimate on the right-hand side of the estimate (3.163).

LEMMA 3.14. *For a smooth solution (p,u,d) to equations (1.1)-(1.3) and (1.5), it holds that*

$$\|\mathcal{Z}^\alpha(\ln\rho,p,u)\|_{L^\infty}^2\le CP(\Lambda_{1m}(t)),\quad m\ge 2+|\alpha|, \tag{3.164}$$

$$\|\nabla(\ln\rho,p)\|_{\mathcal{H}^{1,\infty}}^2\le C_3\left(P(\|\Delta p\|_{\mathcal{H}^1}^2)+P(\Lambda_{1m}(t))\right),\quad m\ge 5, \tag{3.165}$$

$$\|\mathrm{div}u(t)\|_{\mathcal{H}^{1,\infty}}^2\le C_3\left(P(\|\Delta p\|_{\mathcal{H}^1}^2)+P(\Lambda_{1m}(t))\right),\quad m\ge 5, \tag{3.166}$$

$$\|\nabla \mathrm{div} u(t)\|_{L^\infty}^2 \le C_3 P(Q(t)), \tag{3.167}$$

$$\|\nabla \mathrm{div} u(t)\|_{\mathcal{H}^{1,\infty}}^2 \le C_4[1+P(Q(t))]\left(\delta\|\Delta p\|_{\mathcal{H}^2}^2 + C_\delta P(\Lambda_{1m})\right),\ m\ge 6, \tag{3.168}$$

$$\|d_t\|_{W^{1,\infty}}^2 + \|\nabla d\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla^2 d\|_{\mathcal{H}^{1,\infty}}^2 + \|\nabla \Delta d\|_{\mathcal{H}^{1,\infty}}^2 \le C_{m+2} P(\Lambda_m(t)), \quad m\ge 3. \tag{3.169}$$

Proof. The estimates (3.164)-(3.168) have been proven by Wang et al. [7] (see Lemma 3.14). Hence, we give the proof for the estimate (3.169). By virtue of the Sobolev inequality in Proposition 2.3, one arrives at

$$\|\nabla d\|_{L^\infty}^2 \le C(\|\nabla^2 d\|_1^2 + \|\nabla d\|_2^2). \tag{3.170}$$

In view of the standard elliptic regularity results with Neumann boundary condition, we get that

$$\|\nabla^2 d\|_m^2 \le C_{m+2}(\|\Delta d\|_m^2 + \|\nabla d\|_{L^2}^2). \tag{3.171}$$

Then, the combination of inequalities (3.170) and (3.171) yields directly

$$\|\nabla d\|_{L^\infty}^2 \le C_3(\|\Delta d\|_1^2 + \|\nabla d\|_2^2). \tag{3.172}$$

For $|\alpha|=1$, the application of Proposition 2.3 gives for $m\ge 3$

$$\|\mathcal{Z}^\alpha \nabla d\|_{L^\infty}^2 \le C(\|\nabla(\mathcal{Z}^\alpha \nabla d)\|_1 + \|\mathcal{Z}^\alpha \nabla d\|_1)\|\mathcal{Z}^\alpha \nabla d\|_2 \le C_{m+2} P(\Lambda_m(t)),$$

which, together with inequality (3.172), yields

$$\|\nabla d\|_{\mathcal{H}^{1,\infty}}^2 \le C_{m+2} P(\Lambda_m(t)), \quad \text{for } m\ge 3. \tag{3.173}$$

By virtue of equation (1.3), we find

$$\begin{aligned}\|d_t\|_{L^\infty}^2 &\le C(\|\nabla d_t\|_1^2 + \|d_t\|_2^2)\\ &\le C(\|\nabla d_t\|_1^2 + \|\Delta d\|_2^2 + \|u\cdot\nabla d\|_2^2 + \||\nabla d|^2 d\|_2^2).\end{aligned} \tag{3.174}$$

In view of Proposition 2.2, estimates (3.164) and (3.172), one attains

$$\|u\cdot\nabla d\|_2^2 \le C(\|u\|_{L^\infty}^2\|\nabla d\|_2^2 + \|\nabla d\|_{L^\infty}^2\|u\|_2^2) \le C_3 P(\Lambda_m(t)),\ \text{for } m\ge 2; \tag{3.175}$$

and

$$\begin{aligned}\||\nabla d|^2 d\|_2^2 &\le \sum_{|\gamma|\ge 1, |\beta|+|\gamma|\le 2} \int |Z^\beta(|\nabla d|^2) Z^\gamma d|^2 dx + \||\nabla d|^2\|_2^2\\ &\le \|Zd\|_{L^\infty}^2 \||\nabla d|^2\|_1^2 + \||\nabla d|^2\|_{L^\infty}^2 \|Zd\|_1^2 + \|\nabla d\|_{L^\infty}^2\|\nabla d\|_2^2\\ &\le C_3 \Lambda_m^3(t),\ \text{for } m\ge 3.\end{aligned} \tag{3.176}$$

Then the combination of inequalities (3.175) and (3.176) gives directly

$$\|d_t\|_{L^\infty}^2 \le C_3 P(\Lambda_m(t)),\ \text{for } m\ge 3. \tag{3.177}$$

By virtue of Proposition 2.3, we obtain for $m\geq 3$

$$\|\nabla d_t\|_{L^\infty}^2\leq C(\|\nabla^2 d_t\|_1^2+\|\nabla d_t\|_2^2)\leq C(\|\Delta d_t\|_1^2+\|\nabla d_t\|_2^2)\leq C(\|\Delta d\|_{\mathcal{H}^2}^2+\|\nabla d\|_{\mathcal{H}^3}^2),$$

which, together with inequality (3.177), yields immediately

$$\|d_t\|_{W^{1,\infty}}^2\leq C_3 P(\Lambda_m(t)),\ \text{for } m\geq 3. \tag{3.178}$$

On the other hand, it is easy to check that

$$\begin{aligned}
&\partial_{ii}=\partial_{y_i}^2-\partial_{y_i}(\partial_i\psi\partial_z)-\partial_i\psi\partial_z\partial_{y_i}+(\partial_i\psi)^2\partial_z^2,\quad i=1,2,\\
&\partial_1\partial_2=\partial_{y_1}\partial_{y_2}-\partial_{y_2}(\partial_1\psi\partial_z)-\partial_2\psi\partial_{y_1}\partial_z+\partial_2\psi\partial_1\psi\partial_z^2,\\
&\partial_i\partial_3=\partial_{y_i}\partial_z-\partial_i\psi\partial_z^2,\quad i=1,2.
\end{aligned}$$

Then, we find that

$$\Delta=(1+|\nabla\psi|^2)\partial_z^2+\sum_{i=1,2}(\partial_{y_i}^2-\partial_{y_i}(\partial_i\psi\partial_z)-\partial_i\psi\partial_z\partial_{y_i}). \tag{3.179}$$

and

$$\begin{aligned}
\nabla^2=&[(1+|\nabla\psi|^2)+\partial_2\psi\partial_1\psi-\partial_1\psi-\partial_2\psi]\partial_z^2+\partial_{y_1}\partial_{y_2}\\
&+\sum_{i=1,2}(\partial_{y_i}^2-\partial_{y_i}(\partial_i\psi\partial_z)-\partial_i\psi\partial_z\partial_{y_i})-\partial_{y_2}(\partial_1\psi\partial_z)\\
&-\partial_2\psi\partial_{y_1}\partial_z+\partial_{y_1}\partial_z+\partial_{y_2}\partial_z.
\end{aligned}\tag{3.180}$$

The combination of equations (3.179)-(3.180) and Proposition 2.3 yield that

$$\begin{aligned}
\|\nabla^2 d\|_{L^\infty}^2&\leq C_1(\|\Delta d\|_{L^\infty}^2+\|\partial_z\partial_{y_i}d\|_{L^\infty}^2+\|\partial_{y_i}\partial_{y_j}d\|_{L^\infty}^2)\\
&\leq C_1(\|\nabla\Delta d\|_1^2+\|\Delta d\|_2^2+\|\nabla\partial_z\partial_{y_i}d\|_1^2+\|\partial_z\partial_{y_i}d\|_2^2)\\
&\quad+C(\|\nabla\partial_{y_i}\partial_{y_j}d\|_1^2+\|\partial_{y_i}\partial_{y_j}d\|_2^2)\\
&\leq C_1(\|\nabla\Delta d\|_1^2+\|\Delta d\|_2^2+\|\nabla^2 d\|_2^2+\|\nabla d\|_3^2)\\
&\leq C_4(\|\nabla\Delta d\|_1^2+\|\Delta d\|_2^2+\|\nabla d\|_3^2),
\end{aligned}\tag{3.181}$$

where we have used the estimate (3.171) in the last inequality. In order to deal with the first term on the right hand side of inequality (3.181), we apply equation (1.3) to obtain that

$$\begin{aligned}
\|\nabla\Delta d\|_1^2&\leq\|\nabla(d_t+u\cdot\nabla d-|\nabla d|^2 d)\|_1^2\\
&\leq\|\nabla d_t\|_1^2+\|\nabla u\cdot\nabla d\|_1^2+\|u\cdot\nabla^2 d\|_1^2+\|\nabla(|\nabla d|^2 d)\|_1^2.
\end{aligned}\tag{3.182}$$

It is easy to check that

$$\|\nabla d_t\|_1^2\leq\|\nabla d\|_{\mathcal{H}^2}^2\leq\Lambda_m(t),\ \text{for } m\geq 2, \tag{3.183}$$

$$\|\nabla u\cdot\nabla d\|_1^2\leq\|(\nabla u,\nabla d)\|_{L^\infty}^2\|(\nabla u,\nabla d)\|_1^2\leq C_3\Lambda_m^2(t),\ \text{for } m\geq 2, \tag{3.184}$$

and

$$\|u\cdot\nabla^2 d\|_1^2\leq\|u\|_{L^\infty}^2\|\nabla^2 d\|^2+\|Zu\|_{L^\infty}^2\|\nabla^2 d\|^2$$

$$+\|u\|_{L^\infty}^2\|\nabla^2 d\|_1^2$$
$$\leq C\Lambda_m^2(t), \text{ for } m\geq 3. \tag{3.185}$$

In view of the basic fact $|d|=1$, one arrives at

$$\|\nabla(|\nabla d|^2 d)\|_1^2\leq C\Lambda_m^3(t),\ m\geq 3. \tag{3.186}$$

Substituting inequalities (3.183)-(3.186) into inequality (3.182), we find

$$\|\nabla\Delta d\|_1^2\leq C_3\Lambda_m^3(t),\quad m\geq 3,$$

which, together with inequality (3.181), yields immediately

$$\|\nabla^2 d\|_{L^\infty}^2\leq C_4 P(\Lambda_m(t)),\quad m\geq 3. \tag{3.187}$$

Similarly, it is easy to check that for $|\alpha|=1$

$$\|\mathcal{Z}^\alpha\nabla^2 d\|_{L^\infty}^2\leq C_3 P(\Lambda_m(t)). \tag{3.188}$$

By virtue of equation (1.3) and inequalities (3.177) and (3.187), one attains for $m\geq 3$

$$\begin{aligned}\|\nabla\Delta d\|_{L^\infty}^2&\leq\|\nabla(d_t+u\cdot\nabla d-|\nabla d|^2 d)\|_{L^\infty}^2\\&\leq\|\nabla d_t\|_{L^\infty}^2+\|\nabla u\|_{L^\infty}^2\|\nabla d\|_{L^\infty}^2+\|u\|_{L^\infty}^2\|\nabla^2 d\|_{L^\infty}^2\\&\quad+\|\nabla d\|_{L^\infty}^2\|\nabla^2 d\|_{L^\infty}^2+\|\nabla d\|_{L^\infty}^4\|\nabla d\|_{L^\infty}^2\\&\leq C_4\Lambda_m^3(t).\end{aligned} \tag{3.189}$$

Similarly, it is easy to check that for $|\alpha|=1$

$$\|\mathcal{Z}^\alpha\nabla\Delta d\|_{L^\infty}^2\leq C_{m+2}P(\Lambda_m(t)),\quad m\geq 3. \tag{3.190}$$

The combination of inequalities (3.173), (3.178) and (3.187)-(3.190) yields the estimate (3.169). Therefore, we complete the proof of Lemma 3.14. ∎

To give the estimate for $\|\nabla u\|_{\mathcal{H}^{1,\infty}}$, we need the lemma as follows, refer to [7].

Lemma 3.15. *Let h be a smooth solution to*

$$a(t,y)[\partial_t h+b_1(t,y)\partial_{y_1}h+b_2(t,y)\partial_{y_2}h+zb_3(t,y)\partial_z h]-\varepsilon\partial_{zz}h=G,\ z>0, \tag{3.191}$$
$$h(t,y,0)=0, \tag{3.192}$$

for some smooth function $d(t,y)=\frac{1}{a(t,y)}$ and vector field $b=(b_1,b_2,b_3)^{tr}(t,y)$ satisfying equations (3.191)-(3.192). Assume that h and G are compactly supported in z. Then, it holds that

$$\begin{aligned}\|h\|_{\mathcal{H}^{1,\infty}}&\leq C\|h_0\|_{\mathcal{H}^{1,\infty}}+C\int_0^t\left\|\frac{1}{a}\right\|_{L^\infty}\|G\|_{\mathcal{H}^{1,\infty}}d\tau\\&\quad+C\int_0^t\left(1+\left\|\frac{1}{a}\right\|_{L^\infty}\right)\left(1+\|b\|_{L^\infty}^2+\sum_{i=0}^2\|Z_i(a,b)\|_{L^\infty}^2\right)\|h\|_{\mathcal{H}^{1,\infty}}d\tau.\end{aligned} \tag{3.193}$$

Finally, one gives the estimate for the quantity $\|\nabla u\|_{\mathcal{H}^{1,\infty}}$.

LEMMA 3.16. *For $m\geq 6$, we have the estimate*

$$\begin{aligned}\|\nabla u\|^2_{\mathcal{H}^{1,\infty}}\leq & CC_{m+2}\{\|(u_0,\nabla u_0)\|^2_{\mathcal{H}^{1,\infty}}+P(\Lambda_{1m}(t))+P(\|\Delta p(t)\|^2_{\mathcal{H}^1})\}\\ &+CC_{m+2}\ t\int_0^t(1+P(\Lambda_m(\tau))+P(Q(\tau)))(1+\varepsilon^2\|\Delta p\|^2_{\mathcal{H}^2})d\tau\\ &+CC_{m+2}\varepsilon^2 t\int_0^t\|\nabla^2 u\|^2_{\mathcal{H}^4}d\tau. \end{aligned}\tag{3.194}$$

Proof. Away from the boundary, we clearly have by the classical isotropic Sobolev embedding that

$$\|\chi\nabla u\|^2_{L^\infty}+\|\chi\mathcal{Z}^\alpha\nabla u\|^2_{L^\infty}\lesssim\|u\|^2_{\mathcal{H}^m},\quad m\geq 4,\quad |\alpha|=1,\tag{3.195}$$

where the support of χ is away from the boundary. Consequently, by using a partition of unity subordinated to the covering we only have to estimate $\|\chi_j\nabla u\|_{L^\infty}+\|\chi_j\mathcal{Z}^\alpha\nabla u\|_{L^\infty}$, $j\geq 1$, $|\alpha|=1$. For notational convenience, we shall denote χ_j by χ. Similar to [26] or [7], we use the local parametrization in the neighborhood of the boundary given by a normal geodesic system in which the Laplacian takes a convenient form. Denote

$$\Psi^n(y,z)=\begin{pmatrix} y\\ \psi(y)\end{pmatrix}-zn(y)=x,$$

where

$$n(y)=\frac{1}{\sqrt{1+|\nabla\psi(y)|^2}}\begin{pmatrix}\partial_1\psi(y)\\ \partial_2\psi(y)\\ -1\end{pmatrix}$$

is the unit outward normal. As before, one can extend n and Π in the interior by setting

$$n(\Psi^n(y,z))=n(y),\quad \Pi(\Psi^n(y,z))=\Pi(y)=I-n\otimes n,$$

where I is the unit matrix. Note that $n(y,z)$ and $\Pi(y,z)$ have different definitions from the ones used before. The advantages of this parametrization is that in the associated local basis (e_{y_1},e_{y_2},e_z) of $\mathbb{R}^3$, it holds that $\partial_z=\partial_n$ and

$$(e_{y_i})|_{\Psi^n(y,z)}\cdot(e_z)|_{\Psi^n(y,z)}=0,\quad i=1,2.$$

The scalar product on $\mathbb{R}^3$ induces in this coordinate system the Riemannian metric g with the norm

$$g(y,z)=\begin{pmatrix}\widetilde{g}(y,z) & 0\\ 0 & 1\end{pmatrix}.$$

Therefore, the Laplacian in this coordinate system has the form

$$\Delta f=\partial_{zz}f+\frac{1}{2}\partial_z(\ln|g|)\partial_z f+\Delta_{\widetilde{g}}f,\tag{3.196}$$

where $|g|$ denotes the determinant of the matrix g, and $\Delta_{\widetilde{g}}$ is defined by

$$\Delta_{\widetilde{g}}f=\frac{1}{\sqrt{|\widetilde{g}|}}\sum_{i,j=1,2}\partial_{y_i}(\widetilde{g}^{ij}|\widetilde{g}|^{\frac{1}{2}}\partial_{y_j}f),$$

which only involves the tangential derivatives and $\{\widetilde{g}^{ij}\}$ is the inverse matrix to g.

Next, thanks to equation (3.74) (in the coordinate system that we have just defined) and Lemma 3.14, we have for $m\geq 5$, $|\alpha|=1$

$$\begin{aligned}
&\|\chi\nabla u\|_{L^\infty}^2+\|\chi\mathcal{Z}^\alpha\nabla u\|_{L^\infty}^2\\
\leq & C_2(\|\chi\Pi(\partial_n u)\|_{L^\infty}^2+\|\chi\mathrm{div}u\|_{L^\infty}^2+\|\chi Z_y u\|_{L^\infty}^2)\\
&\quad +C_2(\|\chi\mathcal{Z}^\alpha\Pi(\partial_n u)\|_{L^\infty}^2+\|\chi\mathcal{Z}^\alpha\mathrm{div}u\|_{L^\infty}^2+\|\mathcal{Z}^\alpha(\chi Z_y u\|_{L^\infty}^2))\\
\leq & C_3\left\{\|\chi\Pi\partial_n u\|_{L^\infty}^2+\|\mathcal{Z}^\alpha(\chi\Pi\partial_n u)\|_{L^\infty}^2+P(\Lambda_{1m})+P(\|\Delta p\|_{\mathcal{H}^1}^2)\right\}.
\end{aligned} \tag{3.197}$$

Consequently, it suffices to estimate $\|\chi\Pi\partial_n u\|_{\mathcal{H}^{1,\infty}}$. To this end, it is useful to use the vorticity $w=\nabla\times u$, see [6,7,26]. Indeed, it is easy to deduce that

$$\Pi(w\times n)=\Pi((\nabla u-\nabla u^t)\cdot n)=\Pi(\partial_n u-\nabla(u\cdot n)+\nabla n^t\cdot u),$$

which implies

$$\|\chi\Pi\partial_n u\|_{\mathcal{H}^{1,\infty}}^2\leq C_3(\|\chi\Pi(w\times n)\|_{\mathcal{H}^{1,\infty}}^2+P(\Lambda_{1m}(t))), \tag{3.198}$$

where we have used the Lemma 3.14. In other words, we only need to estimate $\|\chi\Pi(w\times n)\|_{\mathcal{H}^{1,\infty}}$. It is easy to see that w solves the vorticity equation

$$\rho w_t+\rho(u\cdot\nabla)w=\mu\varepsilon\Delta w+F_1, \tag{3.199}$$

where

$$F_1\triangleq-\nabla\rho\times u_t-\nabla\rho\times(u\cdot\nabla)u+\rho(w\cdot\nabla)u-\rho w\mathrm{div}u-\nabla\times(\nabla d\cdot\Delta d).$$

In the support of χ, let

$$\widetilde{w}(y,z)=w(\Psi^n(y,z)),\quad (\widetilde{\rho},\widetilde{u},\widetilde{d})(y,z)=(\rho,u,d)(\Psi^n(y,z)),$$

The combination of equations (3.80) and (3.196) yields directly

$$\begin{aligned}
&\widetilde{\rho}\ \partial_t\widetilde{w}+\widetilde{\rho}\ \widetilde{u}^1\partial_{y_1}\widetilde{w}+\widetilde{\rho}\ \widetilde{u}^2\partial_{y_2}\widetilde{w}+\widetilde{\rho}\ \widetilde{u}\cdot n\partial_z\widetilde{w}\\
=&\mu\varepsilon(\partial_{zz}\widetilde{w}+\frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{w}+\Delta_{\widetilde{g}}\widetilde{w})+\widetilde{F}_1
\end{aligned} \tag{3.200}$$

and

$$\begin{aligned}
&\widetilde{\rho}\ \partial_t\widetilde{u}+\widetilde{\rho}\ \widetilde{u}^1\partial_{y_1}\widetilde{u}+\widetilde{\rho}\ \widetilde{u}^2\partial_{y_2}\widetilde{u}+\widetilde{\rho}\ \widetilde{u}\cdot n\partial_z\widetilde{u}\\
=&\mu\varepsilon(\partial_{zz}\widetilde{u}+\frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{u}+\Delta_{\widetilde{g}}\widetilde{u})+\widetilde{F}_2,
\end{aligned} \tag{3.201}$$

where $\widetilde{F}_2=F_2(\Psi^n(y,z))$ and $F_2=(\mu+\lambda)\varepsilon\nabla\mathrm{div}u-\nabla p-\nabla d\cdot\Delta d$. Similar to definition (3.76) , we define

$$\widetilde{\eta}=\chi(\widetilde{w}\times n+\Pi(B\widetilde{u})).$$

It is easy to deduce taht $\widetilde{\eta}$ satisfies

$$\widetilde{\eta}(y,0)=0.$$

and solves the equation

$$\begin{aligned}&\widetilde{\rho}\,\partial_t\widetilde{\eta}+\widetilde{\rho}\,\widetilde{u}^1\partial_{y_1}\widetilde{\eta}+\widetilde{\rho}\,\widetilde{u}^2\partial_{y_2}\widetilde{\eta}+\widetilde{\rho}\,\widetilde{u}\cdot n\partial_z\widetilde{\eta}\\&=\mu\varepsilon\left(\partial_{zz}\widetilde{\eta}+\frac{1}{2}\partial_z(\ln|g|)\partial_z\widetilde{\eta}\right)+\chi(\widetilde{F}_1\times n)+\chi\Pi(B\widetilde{F}_2)+F^\chi+\chi F^\kappa,\end{aligned}\tag{3.202}$$

where the source terms are given by

$$\begin{aligned}F^\chi=&\left[(\widetilde{\rho}\,\widetilde{u}^1\partial_{y_1}+\widetilde{\rho}\,\widetilde{u}^2\partial_{y_2}+\widetilde{\rho}\,\widetilde{u}\cdot n\partial_z)\chi\right](\widetilde{w}\times n+\Pi(B\widetilde{u}))\\&-\mu\varepsilon\left(\partial_{zz}\chi+2\partial_z\chi\partial_z+\frac{1}{2}\partial_z(\ln|g|)\partial_z\chi\right)(\widetilde{w}\times n+\Pi(B\widetilde{u})),\end{aligned}\tag{3.203}$$

and

$$\begin{aligned}F^\kappa=&(\widetilde{\rho}\,\widetilde{u}^1\partial_{y_1}\Pi+\widetilde{\rho}\,\widetilde{u}^2\partial_{y_2}\Pi)\cdot(B\widetilde{u})+w\times(\widetilde{\rho}\,\widetilde{u}^1\partial_{y_1}n+\widetilde{\rho}\,\widetilde{u}^2\partial_{y_2}n)\\&+\Pi\left[(\widetilde{\rho}\,\widetilde{u}^1\partial_{y_1}+\widetilde{\rho}\,\widetilde{u}^2\partial_{y_2}+\widetilde{\rho}\,\widetilde{u}\cdot n\partial_z)B\cdot\widetilde{u}\right]+\mu\varepsilon\Delta_{\widetilde{g}}\widetilde{w}\times n+\mu\varepsilon\Pi(B\Delta_{\widetilde{g}}\widetilde{u}).\end{aligned}\tag{3.204}$$

Note that in the derivation of the source terms above, in particular, F^κ, which contains all the commutators coming from the fact that n and Π are not constant, we have used the fact that in the coordinate system just defined, n and Π do not depend on the normal variable. Since $\Delta_{\widetilde{g}}$ involves only the tangential derivatives, and the derivatives of χ are compactly supported away from the boundary, the following estimates hold for $m\geq 6$

$$\begin{aligned}\|F^\chi\|^2_{\mathcal{H}^{1,\infty}}&\leq C_3(\|\rho u\|^2_{\mathcal{H}^{1,\infty}}\|u\|^2_{\mathcal{H}^{2,\infty}}+\varepsilon^2\|u\|^2_{\mathcal{H}^{3,\infty}})\\&\leq C_3\{P(Q(t))+P(\Lambda_{1m})\},\end{aligned}\tag{3.205}$$

$$\begin{aligned}\|\chi(\widetilde{F}^1\times n)\|^2_{\mathcal{H}^{1,\infty}}&\leq C_2(P(Q(t))+\|\nabla d\|^2_{\mathcal{H}^{1,\infty}}\|\nabla\Delta d\|^2_{\mathcal{H}^{1,\infty}})\\&\leq C_2(P(Q(t))+P(\Lambda_m)),\end{aligned}\tag{3.206}$$

$$\begin{aligned}\|\chi\Pi(B\widetilde{F}_2)\|^2_{\mathcal{H}^{1,\infty}}&\leq C_3(\varepsilon^2\|\nabla\mathrm{div}u\|^2_{\mathcal{H}^{1,\infty}}+\|\nabla p\|^2_{\mathcal{H}^{1,\infty}}+\|\nabla d\|^2_{\mathcal{H}^{1,\infty}}\|\Delta d\|^2_{\mathcal{H}^{1,\infty}})\\&\leq C_4\left\{P(Q(t))+P(\Lambda_m)+C\varepsilon^2[1+P(Q(t))]\|\Delta p\|^2_{\mathcal{H}^2}\right\},\end{aligned}\tag{3.207}$$

and

$$\begin{aligned}\|\chi F^\kappa\|^2_{\mathcal{H}^{1,\infty}}\leq&\, C_4\left\{\|u\|^8_{\mathcal{H}^{1,\infty}}+\|u\|^4_{\mathcal{H}^{1,\infty}}\|\nabla u\|^4_{\mathcal{H}^{1,\infty}}+\|\rho\|^2_{\mathcal{H}^{1,\infty}}\right.\\&\left.+\varepsilon^2(\|\nabla u\|^2_{\mathcal{H}^{3,\infty}}+\|u\|^2_{\mathcal{H}^{3,\infty}})\right\}\\\leq&\, C_4\left\{\varepsilon^2\|\nabla^2u\|^2_{\mathcal{H}^4}+P(\Lambda_{1m})+P(Q(t))\right\}.\end{aligned}\tag{3.208}$$

It follows from inequalities (3.205)-(3.208) that

$$\|F\|^2_{\mathcal{H}^{1,\infty}}\leq C_4\left\{\varepsilon^2\|\nabla^2u\|^2_{\mathcal{H}^4}+P(Q(t))+P(\Lambda_m)+\varepsilon^2[1+P(Q(t))]\|\Delta p\|^2_{\mathcal{H}^2}\right\},\tag{3.209}$$

where $\widetilde{F}=\chi(\widetilde{F}_1\times n)+\chi\Pi(B\widetilde{F}_2)+F^\chi+\chi F^\kappa$. In order to be able to use Lemma 3.15, we shall perform last change of unknown in order to eliminate the term $\partial_z(\ln|\widetilde{g}|)\partial_z\widetilde{\eta}$. We set

$$\widetilde{\eta}=\frac{1}{|g|^{\frac{1}{4}}}\overline{\eta}=\overline{\gamma}\;\eta.$$

Note that we have

$$\|\widetilde{\eta}\|_{\mathcal{H}^{1,\infty}}\leq C_3\|\overline{\eta}\|_{\mathcal{H}^{1,\infty}},\quad\|\overline{\eta}\|_{\mathcal{H}^{1,\infty}}\leq C_3\|\widetilde{\eta}\|_{\mathcal{H}^{1,\infty}}\tag{3.210}$$

and that, $\overline{\eta}$ solves the equation

$$\widetilde{\rho}\,\partial_t\overline{\eta}+\widetilde{\rho}\,\widetilde{u}^1\partial_{y_1}\overline{\eta}+\widetilde{\rho}\,\widetilde{u}^2\partial_{y_2}\overline{\eta}+\widetilde{\rho}\,\widetilde{u}\cdot n\partial_z\overline{\eta}-\mu\varepsilon\partial_{zz}\overline{\eta}$$
$$=\frac{1}{\overline{\gamma}}\left(\widetilde{F}+\mu\varepsilon\partial_{zz}\overline{\gamma}\cdot\overline{\eta}+\frac{\mu\varepsilon}{2}\partial_z(\ln|g|)\partial_z\overline{\gamma}\cdot\overline{\eta}-\widetilde{\rho}(\widetilde{u}\cdot\nabla\overline{\gamma})\overline{\eta}\right):=\mathcal{S}_1.$$

In the spirit of Wang et al. [7], we rewrite the equation as follows

$$\widetilde{\rho}(t,y,0)\left[\overline{\eta}_t+\widetilde{u}^1(t,y,0)\partial_{y^1}+\widetilde{u}^2(t,y,0)\partial_{y^2}\overline{\eta}+z\partial_z(\widetilde{u}\cdot n)(t,y,0)\partial_z\overline{\eta}\right]-\mu\varepsilon\partial_{zz}\overline{\eta}$$
$$=\mathcal{S}_1+\mathcal{S}_2, \tag{3.211}$$

where $\mathcal{S}_2$ is defined as

$$\begin{aligned}\mathcal{S}_2\triangleq&[\widetilde{\rho}(t,y,0)-\widetilde{\rho}(t,y,z)]\eta_t+\sum_{i=1,2}[(\widetilde{\rho}\,\widetilde{u}^i)(t,y,0)-(\widetilde{\rho}\,\widetilde{u}^i)(t,y,z)]\partial_{y_i}\overline{\eta}\\&-\widetilde{\rho}(t,y,z)[(\widetilde{u}\cdot n)(t,y,z)-z\partial_z(\widetilde{u}\cdot n)(t,y,0)]\partial_z\overline{\eta}\\&-[\widetilde{\rho}(t,y,z)-\widetilde{\rho}(t,y,0)]z\partial_z(\widetilde{u}\cdot n)(t,y,0)\partial_z\overline{\eta}.\end{aligned}$$

Consequently, by using Lemma 3.15, we get from equation (3.211) that for $m\geq 6$

$$\begin{aligned}\|\overline{\eta}\|_{\mathcal{H}^{1,\infty}}\lesssim&C\|\overline{\eta}_0\|_{\mathcal{H}^{1,\infty}}+C\int_0^t\|\widetilde{\rho}^{\;-1}\|_{L^\infty}\|(\mathcal{S}_1+\mathcal{S}_2)\|_{\mathcal{H}^{1,\infty}}d\tau\\&+C\int_0^t(1+\|\widetilde{\rho}^{\;-1}\|_{L^\infty})(1+\|(\rho,u,\nabla u)\|^2_{\mathcal{H}^{1,\infty}})\|\overline{\eta}\|_{\mathcal{H}^{1,\infty}}d\tau\\\lesssim&C\|\overline{\eta}_0\|_{\mathcal{H}^{1,\infty}}+C\int_0^t\|(\mathcal{S}_1+\mathcal{S}_2)\|_{\mathcal{H}^{1,\infty}}d\tau\\&+C\int_0^t(1+P(\Lambda_{1m})+\|\mathcal{Z}\nabla u\|^2_{L^\infty})\|\overline{\eta}\|_{\mathcal{H}^{1,\infty}}d\tau.\end{aligned}\tag{3.212}$$

On the other hand, following the same argument as [7], we have the following estimate

$$\begin{aligned}&\|(\mathcal{S}_1+\mathcal{S}_2)\|^2_{\mathcal{H}^{1,\infty}}\\&\leq C_4\left\{\varepsilon^2\|\nabla^2u\|^2_{\mathcal{H}^4}+\varepsilon^2[1+P(Q(t))]\|\Delta p\|^2_{\mathcal{H}^2}+P(Q(t))+P(\Lambda_m)\right\}.\end{aligned}\tag{3.213}$$

Then, we deduce that

$$\begin{aligned}\|\overline{\eta}(t)\|^2_{\mathcal{H}^{1,\infty}}\leq&\|\overline{\eta}_0\|^2_{\mathcal{H}^{1,\infty}}+C_4t\int_0^t(1+P(Q(\tau))+P(\Lambda_m))d\tau\\&+C_4t\varepsilon^2\int_0^t([1+P(Q(t))]\|\Delta p\|^2_{\mathcal{H}^2}+\|\nabla^2u\|^2_{\mathcal{H}^4})d\tau.\end{aligned}$$

which, together with inequalities (3.195), (3.210), (3.197) and (3.198), completes the proof of Lemma 3.16. □

3.5. Uniform estimate for Δp. In this subsection, we shall estimate Δp to complete the L^∞-estimates and prove that the boundary layers for the density is weaker that the one for the velocity. Taking divergence operator to equation (3.5), it is easy to deduce that

$$-\varepsilon\Delta\mathrm{div}u+\frac{1}{2\mu+\lambda}\Delta p=-\frac{1}{2\mu+\lambda}\mathrm{div}(\rho u_t+\rho u\cdot\nabla u)-\frac{1}{2\mu+\lambda}\mathrm{div}(\nabla d\cdot\Delta d).\tag{3.214}$$

On the other hand, it follows from equation (1.1) that

$$\mathrm{div} u = -(\ln\,\rho)_t - u\cdot\nabla\ln\,\rho = -\frac{p_t}{\gamma p} - \frac{u\cdot\nabla p}{\gamma p}. \tag{3.215}$$

Then, the combination of equations (3.214) and (3.215) yields directly

$$\begin{aligned}&\varepsilon\Delta(\ln\rho)_t + \varepsilon u\cdot\nabla\Delta\ln\rho + \varepsilon\Delta u\cdot\nabla\ln\rho + 2\varepsilon\sum_{k=1}^{3}\partial_k u\cdot\nabla\partial_k\ln\rho + \frac{1}{2\mu+\lambda}\Delta p\\ &= -\frac{1}{2\mu+\lambda}\mathrm{div}(\rho u_t + \rho u\cdot\nabla u) - \frac{1}{2\mu+\lambda}\mathrm{div}(\nabla d\cdot\Delta d).\end{aligned} \tag{3.216}$$

LEMMA 3.17. *For $m\geq 6$, it holds that*

$$\begin{aligned}&\sup_{0\leq\tau\leq t}\left(\|\Delta p(\tau)\|_{\mathcal{H}^1}^2 + \varepsilon\|\Delta p(\tau)\|_{\mathcal{H}^2}^2\right) + \int_0^t \|\Delta p(\tau)\|_{\mathcal{H}^2}^2 d\tau\\ &\leq CC_{m+2}\left\{P(N_m(0)) + [1+P(Q(t))]\int_0^t P(N_m(\tau))d\tau\right\}.\end{aligned} \tag{3.217}$$

Proof. Applying $\mathcal{Z}^\alpha(|\alpha|\leq 2)$ to equation (3.216) and multiplying by $\mathcal{Z}^\alpha\Delta\ln\rho$, one attains

$$\begin{aligned}&\varepsilon\|\mathcal{Z}^\alpha\Delta\ln\rho\|^2 - \varepsilon\|\mathcal{Z}^\alpha\Delta\ln\rho_0\|^2 + \underbrace{\frac{2}{2\mu+\lambda}\int_0^t\int\mathcal{Z}^\alpha\Delta p\cdot\mathcal{Z}^\alpha\Delta\ln\rho dx d\tau}_{VII_1}\\ &= -2\varepsilon\int_0^t\int\mathcal{Z}^\alpha(\Delta u\cdot\nabla\ln\rho)\mathcal{Z}^\alpha\Delta\ln\rho\; dx d\tau\\ &\quad + (-4\varepsilon)\sum_{k=1}^{3}\int_0^t\int\mathcal{Z}^\alpha(\partial_k u\cdot\nabla\partial_k\ln\rho)\mathcal{Z}^\alpha\Delta\ln\rho\; dx d\tau\\ &\quad + (-2\varepsilon)\int_0^t\int\mathcal{Z}^\alpha(u\cdot\nabla\Delta\ln\rho)\mathcal{Z}^\alpha\Delta\ln\rho dx d\tau\\ &\quad + \frac{-2}{2\mu+\lambda}\int_0^t\int\mathcal{Z}^\alpha\mathrm{div}(\rho u_t + \rho u\cdot\nabla u)\mathcal{Z}^\alpha\Delta\ln\rho dx d\tau\\ &\quad + \frac{-2}{2\mu+\lambda}\int_0^t\int\mathcal{Z}^\alpha\mathrm{div}(\nabla d\cdot\Delta d)\mathcal{Z}^\alpha\Delta\ln\rho\; dx d\tau\\ &:= VII_2 + VII_3 + VII_4 + VII_5 + VII_6.\end{aligned} \tag{3.218}$$

Using the same arguments as Lemma 3.16 of [7], one can obtain the following estimates

$$VII_5 \leq C_{m+2}[1+P(Q(t))]\int_0^t P(\Lambda_m)\tau, \tag{3.219}$$

$$VII_2 \leq \delta\int_0^t \|\mathcal{Z}^\alpha\Delta\ln\rho\|_{L^2}^2 d\tau + C_\delta[1+P(Q(t))]\int_0^t P(\Lambda_m)d\tau, \tag{3.220}$$

$$\begin{aligned}VII_3 \leq\ &\delta\int_0^t \|\mathcal{Z}^\alpha\Delta\ln\rho\|_{L^2}^2 d\tau + \varepsilon^2\int_0^t \|\nabla^2 u\|_{\mathcal{H}^3}^2 d\tau\\ &+ C_\delta\varepsilon^2[1+P(Q(t))]\int_0^t (P(\Lambda_m) + \|\Delta\ln\rho\|_{\mathcal{H}^2}^2)d\tau\end{aligned}$$

$$+C_\delta\varepsilon^2\int_0^t\|\nabla u\|_{\mathcal{H}^4}^2(\|\Delta\ln\rho\|_{L^2}^4+P(\Lambda_m))d\tau, \tag{3.221}$$

$$\begin{aligned}VII_1\geq&\frac{\gamma}{2}p(c_0)\int_0^t\|\mathcal{Z}^\alpha\Delta\ln\rho\|_{L^2}^2d\tau\\&-C[1+P(Q(t))]\int_0^t(P(\Lambda_m)+\|\Delta\ln\rho\|_{\mathcal{H}^1}^2)d\tau,\end{aligned} \tag{3.222}$$

$$\begin{aligned}VII_4\leq&\varepsilon^2\int_0^t\|\nabla^2u\|_{\mathcal{H}^3}^2d\tau+\delta\int_0^t\|\mathcal{Z}^\alpha\Delta\ln\rho\|_{L^2}^2d\tau\\&+C_\delta C_2[1+P(Q(t))]\int_0^t(\varepsilon^2\|\Delta\ln\rho\|_{\mathcal{H}^2}^2+P(\Lambda_m))d\tau.\end{aligned} \tag{3.223}$$

On the other hand, by using the Proposition 2.2, it is easy to check that

$$\begin{aligned}VII_6\leq&\delta\int_0^t\|\Delta\ln\rho\|_{\mathcal{H}^2}^2\tau+C_\delta\|\Delta d\|_{L^\infty}^2\int_0^t\|\Delta d\|_{\mathcal{H}^2}^2d\tau\\&+C_\delta\|\nabla d\|_{L^\infty}^2\int_0^t\|\nabla\Delta d\|_{\mathcal{H}^2}^2d\tau+C_\delta\|\nabla\Delta d\|_{L^\infty}^2\int_0^t\|\nabla d\|_{\mathcal{H}^2}^2d\tau\\\leq&\delta\int_0^t\|\Delta\ln\rho\|_{\mathcal{H}^2}^2d\tau+C_\delta[1+P(Q(t))]\int_0^t\Lambda_m(\tau)d\tau.\end{aligned} \tag{3.224}$$

Hence, the combination of estimates (3.219)-(3.224) gives directly

$$\begin{aligned}&\varepsilon\|\Delta\ln\rho\|_{\mathcal{H}^2}^2+\int_0^t\|\Delta\ln\rho\|_{\mathcal{H}^2}^2d\tau\\\leq&C\varepsilon\|\Delta\ln\rho_0\|_{\mathcal{H}^2}^2+C\varepsilon^2\int_0^t\|\nabla^2u\|_{\mathcal{H}^3}^2d\tau\\&+C_\delta C_{m+2}[1+P(Q(t))]\int_0^t(\varepsilon\|\Delta\ln\rho\|_{\mathcal{H}^2}^2+\|\Delta\ln\rho\|_{\mathcal{H}^1}^4+P(\Lambda_m))d\tau.\end{aligned} \tag{3.225}$$

It is easy to check that

$$\begin{aligned}\|\Delta\ln\rho(t)\|_{\mathcal{H}^1}^2\leq&\|\Delta\ln\rho_0\|_{\mathcal{H}^1}^2+\int_0^t\|\partial_t\Delta\ln\rho(t)\|_{\mathcal{H}^1}^2d\tau\\\leq&\|\Delta\ln\rho_0\|_{\mathcal{H}^1}^2+\int_0^t\|\Delta\ln\rho(t)\|_{\mathcal{H}^2}^2d\tau,\end{aligned}$$

which, together with estimate (3.225), yields directly

$$\begin{aligned}&\sup_{0\leq\tau\leq t}(\|\Delta\ln\rho(\tau)\|_{\mathcal{H}^1}^2+\varepsilon\|\Delta\ln\rho(\tau)\|_{\mathcal{H}^2}^2)+\int_0^t\|\Delta\ln\rho(\tau)\|_{\mathcal{H}^2}^2d\tau\\\leq&CC_{m+2}\left\{N_m(0)+[1+P(Q(t))]\int_0^t(1+\varepsilon\|\Delta\ln\rho(\tau)\|_{\mathcal{H}^2}^2+\|\Delta\ln\rho(\tau)\|_{\mathcal{H}^1}^4+P(\Lambda_m))d\tau\right\}.\end{aligned}$$

Therefore, we complete the proof of Lemma 3.17. □

3.6. Proof of Theorem 3.1. By virtue of inequalities (3.164), (3.165) and (3.169), it is easy to deduce that

$$Q(t)\leq C_3\sup_{0\leq\tau\leq t}\left\{\|\nabla u(\tau)\|_{\mathcal{H}^{1,\infty}}^2+P(\Lambda_m(\tau))+P(\|\Delta p(\tau)\|_{\mathcal{H}^1}^2)\right\}$$

$$\leq CC_{m+2}\left\{P(N_m(0))+P(N_m(t))\int_0^t P(N_m(\tau))d\tau\right\}. \tag{3.226}$$

In order to close the a priori estimates, one still need to get the uniform estimate for $\|\nabla\partial_t^{m-1}u\|_{L^2}^2$. To this end, we combine estimates (3.163), (3.226) and Lemma 3.10 to deduce that

$$\begin{aligned}\|\nabla\partial_t^{m-1}u\|_{L^2}^2 &\leq CC_{m+2}\left\{\|u(t)\|_{\mathcal{H}^m}^2+\|\eta(t)\|_{\mathcal{H}^{m-1}}^2+\|\partial_t^{m-1}\mathrm{div}u\|_{L^2}^2\right\}\\ &\leq CC_{m+2}\left\{P(\Lambda_{1m})+\|\eta(t)\|_{\mathcal{H}^{m-1}}^2+P(Q(t))\right\}\\ &\leq CC_{m+2}\left\{P(N_m(0))+P(N_m(t))\int_0^t P(N_m(\tau))d\tau\right\}. \end{aligned}\tag{3.227}$$

Hence, the combination of inequalities (3.163), (3.217), (3.226) and (3.227) yields for $m\geq 6$ that

$$\begin{aligned}&N_m(t)+\int_0^t(\|\nabla\partial_t^{m-1}p(\tau)\|_{L^2}^2+\|\Delta p(\tau)\|_{\mathcal{H}^2}^2)d\tau+\varepsilon\int_0^t\|\nabla u(\tau)\|_{\mathcal{H}^m}^2d\tau\\ &\quad+\varepsilon\sum_{k=0}^{m-2}\int_0^t\|\nabla^2\partial_t^k u(\tau)\|_{m-1-k}^2d\tau+\varepsilon^2\int_0^t\|\nabla^2\partial_t^{m-1}u(\tau)\|_{L^2}^2d\tau\\ &\quad+\int_0^t\|\Delta d(\tau)\|_{\mathcal{H}^m}^2d\tau+\int_0^t\|\nabla\Delta d(\tau)\|_{\mathcal{H}^{m-1}}^2d\tau\\ &\leq\widetilde{C}_2C_{m+2}\left\{P(N_m(0))+P(N_m(t))\int_0^t P(N_m(\tau))d\tau\right\},\quad\forall t\in[0,T],\end{aligned}$$

which completes the proof of inequality (3.3). Furthermore, equation (1.1) implies that

$$|\rho(x,0)|\exp\left(-\int_0^t\|\mathrm{div}u(\tau)\|_{L^\infty}d\tau\right)\leq\rho(x,t)\leq|\rho(x,0)|\exp\left(\int_0^t\|\mathrm{div}u(\tau)\|_{L^\infty}d\tau\right),$$

which proves inequality (3.1). Therefore, we complete the proof of Theorem 3.1.

4. Proof of Theorem 1.1 (Uniform Regularity)

In this section, we will give the proof for the Theorem 1.1. Indeed, we shall indicate how to combine the a priori estimates obtained so far to prove the uniform existence result. Fixing $m\geq 6$, we consider the initial data $(p_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)\in X_{n,m}^\varepsilon$ such that

$$\mathcal{I}_m(0)=\sup_{0<\varepsilon\leq 1}\|(p_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)\|_{X_{n,m}^\varepsilon}\leq\widetilde{C}_0\quad\text{and}\quad 0<\widehat{C}_0^{-1}\leq\rho_0^\varepsilon\leq\widehat{C}_0. \tag{4.1}$$

For such initial data, since we are not aware of a local existence result for equations (1.1)-(1.3) and (1.4)(or (1.5)), we first establish the local existence of solution for equations (1.1)-(1.3) and (1.4) with initial data $(p_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)\in X_{n,m}^\varepsilon$. For such initial data $(p_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)$, it is easy to see that there exists a sequence of smooth approximate initial data $(p_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta},d_0^{\varepsilon,\delta})\in X_{n,m}^{\varepsilon,ap}$ (δ being a regularity parameter), which have enough space regularity so that the time derivatives at the initial time can be defined by equations (1.1)-(1.3) and the boundary compatibility conditions are satisfied. Fixed $\varepsilon\in(0,1]$, one constructs the approximate solutions as follows:

(1)Define $u^0=u_0^{\varepsilon,\delta}$ and $d^0=d_0^{\varepsilon,\delta}$.

(2)Assume that (u^{k-1},d^{k-1}) has been defined for $k\geq 1$. Let (ρ^k,u^k,d^k) be the unique solution to the following linearized initial data boundary value problem in $\Omega\times(0,T)$

$$\rho_t^k+\mathrm{div}(\rho^k u^{k-1})=0, \tag{4.2}$$
$$\rho^k u_t^k+\rho^k u^{k-1}\cdot\nabla u^k+\nabla p^k=\varepsilon\mu\Delta u^k+\varepsilon(\mu+\lambda)\nabla\mathrm{div}u^k-\nabla d^{k-1}\cdot\Delta d^{k-1}, \tag{4.3}$$
$$d_t^k-\Delta d^k=|\nabla d^{k-1}|^2 d^{k-1}-u^{k-1}\cdot\nabla d^{k-1}, \tag{4.4}$$

with initial data

$$(\rho^k,u^k,d^k)|_{t=0}=(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta},d_0^{\varepsilon,\delta}), \tag{4.5}$$

and the boundary condition

$$u^k\cdot n=0,\quad n\times(\nabla\times u^k)=[Bu^k]_\tau,\quad \text{and}\quad \frac{\partial d^k}{\partial n}=0,\quad \text{on } \partial\Omega. \tag{4.6}$$

Since ρ^k,u^k and d^k are decoupled, the existence of global unique smooth solution $(\rho^k,u^k,d^k)(t)$ of equations (4.2)-(4.6) can be obtained by using classical methods, for example, the similar argument in Cho et al. [34] (or the recent work in [35]). By virtue of $(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta},d_0^{\varepsilon,\delta})\in H^{4m}\times H^{4m}\times H^{4m+1}$, one proves that there exists a positive time $\widetilde{T}_1=\widetilde{T}_1(\varepsilon)$ (depending on ε, $\|(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta})\|_{H^{4m}}$ and $\|d_0^{\varepsilon,\delta}\|_{H^{4m+1}}$) such that

$$\|(\rho^k,u^k)(t)\|_{H^{4m}}^2+\|d^k(t)\|_{H^{4m+1}}^2\leq\widetilde{C}_1\quad\text{and}\quad\frac{\widehat{C}_0}{2}\leq\rho^k(t)\leq 2\widehat{C}_0\quad\text{for } 0\leq t\leq\widetilde{T}_1, \tag{4.7}$$

where the constant $\widetilde{C}_1$ depends on $\widetilde{C}_0,\varepsilon^{-1}$, $\|(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta})\|_{H^{4m}}$ and $\|d_0^{\varepsilon,\delta}\|_{H^{4m+1}}$. On one hand, due to equations (4.2)-(4.4) and estimate (4.7), one can obtain the following estimates

$$\|\partial_t\rho^k\|_{L^\infty(0,\widetilde{T}_1;H^{4m-1})}^2+\|\partial_t u^k\|_{L^2(0,\widetilde{T}_1;H^{4m-1})}^2+\|\partial_t d^k\|_{L^2(0,\widetilde{T}_1;H^{4m})}^2\leq\widetilde{C}_2, \tag{4.8}$$

where the constant $\widetilde{C}_2$ depends on $\widetilde{C}_0,\varepsilon^{-1}$, $\|(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta})\|_{H^{4m}}$ and $\|d_0^{\varepsilon,\delta}\|_{H^{4m+1}}$. Then, the combination of estimates (4.7)- (4.8) and the classical Lions–Aubin lemma yields the compactness of sequences (ρ^k,u^k,d^k) in Sobolev space. On the other hand, by the similar arguments as Section 3 in [34] (or the recent work of Section 3.2.2 in [35]), there exists a time $\widehat{T}_1(\leq\widetilde{T}_1)$(independent of k) such that (ρ^k,u^k,d^k) converges to a limit $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta},d^{\varepsilon,\delta})$ as $k\to+\infty$ in the following strong sense:

$$(\rho^k,u^k)\to(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})\text{ in } L^\infty(0,\widehat{T}_1;L^2)\quad\text{and}\quad\nabla u^k\to\nabla u^{\varepsilon,\delta}\text{ in } L^2(0,\widehat{T}_1;L^2),$$

and

$$d^k\to d^{\varepsilon,\delta}\text{ in } L^\infty(0,\widehat{T}_1;H^1)\quad\text{and}\quad\Delta d^k\to\Delta d^{\varepsilon,\delta}\text{ in } L^2(0,\widehat{T}_1;L^2).$$

It is easy to check that $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta},d^{\varepsilon,\delta})$ is a classical solution to the problem (1.1)-(1.3) and (1.4) with initial data $(\rho_0^{\varepsilon,\delta},u_0^{\varepsilon,\delta},d_0^{\varepsilon,\delta})$. In view of the lower semicontinuity of norms, one can deduce from the uniform bounds (4.7) that

$$\|(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta})(t)\|_{H^{4m}}^2+\|d^{\varepsilon,\delta}(t)\|_{H^{4m+1}}^2\leq\widetilde{C}_1,\quad 0\leq t\leq\widetilde{T}_1. \tag{4.9}$$

and

$$\frac{\widehat{C}_0}{2} \le \rho^{\varepsilon,\delta}(t) \le 2C_0, \quad 0 \le t \le \widetilde{T}_1. \tag{4.10}$$

Applying a priori estimates given in Theorem 3.1 to the solution $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})$, one can obtain a uniform time T_0 and constant C_3 (independent of ε and δ) such that

$$\begin{aligned} &N_m(p^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})(t) + \int_0^t (\|\nabla \partial_t^{m-1} p^{\varepsilon,\delta}(\tau)\|_{L^2}^2 + \|\Delta p^{\varepsilon,\delta}(\tau)\|_{\mathcal{H}^2}^2) d\tau \\ &+ \varepsilon \int_0^t \|\nabla u^{\varepsilon,\delta}(\tau)\|_{\mathcal{H}^m}^2 d\tau + \varepsilon \sum_{k=0}^{m-2} \int_0^t \|\nabla^2 \partial_t^k u^{\varepsilon,\delta}(\tau)\|_{m-1-k}^2 d\tau \\ &+ \varepsilon^2 \int_0^t \|\nabla^2 \partial_t^{m-1} u^{\varepsilon,\delta}(\tau)\|_{L^2}^2 d\tau + \int_0^t \|\Delta d^{\varepsilon,\delta}(\tau)\|_{\mathcal{H}^m}^2 d\tau \\ &+ \int_0^t \|\nabla \Delta d^{\varepsilon,\delta}(\tau)\|_{\mathcal{H}^{m-1}}^2 d\tau \le \widetilde{C}_3, \ \forall t \in [0, \min\{T_0, \widehat{T}_1\}], \end{aligned} \tag{4.11}$$

and

$$\frac{1}{2\widehat{C}_0} \le \rho^{\varepsilon,\delta}(t) \le 2\widehat{C}_0, \quad t \in [0, \min\{T_0, \widehat{T}_1\}]. \tag{4.12}$$

where T_0 and $\widetilde{C}_3$ depend only on $\widehat{C}_0$ and $\mathcal{I}_m(0)$. Based on the uniform estimates (4.11) and (4.12) for $(\rho^{\varepsilon,\delta}, u^{\varepsilon,\delta}, d^{\varepsilon,\delta})$, one can pass the limit $\delta \to 0$ to get a strong solution $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)$ of equations (1.1)-(1.3) and (1.4) with initial data $(\rho_0^\varepsilon, u_0^\varepsilon, d_0^\varepsilon)$ satisfying equations (4.2)-(4.4) by using a strong compactness arguments(see [36]). Indeed, it follows from estimate (4.11) that $(p^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \nabla d^{\varepsilon,\delta})$ is bounded uniformly in $L^\infty(0, \widetilde{T}_2; H_{co}^m)$, where $\widetilde{T}_2 = \min\{T_0, \widetilde{T}_1\}$, while $(\nabla p^{\varepsilon,\delta}, \nabla u^{\varepsilon,\delta}, \Delta d^{\varepsilon,\delta})$ is bounded uniformly in $L^\infty(0, \widetilde{T}_2; H_{co}^{m-1})$, and $(\partial_t p^{\varepsilon,\delta}, \partial_t u^{\varepsilon,\delta}, \partial_t \nabla d^{\varepsilon,\delta})$ is bounded uniformly in $L^\infty(0, \widetilde{T}_2; H_{co}^{m-1})$. Then, the strong compactness argument implies that $(p^{\varepsilon,\delta}, u^{\varepsilon,\delta}, \nabla d^{\varepsilon,\delta})$ is compact in $\mathcal{C}([0, \widetilde{T}_2]; H_{co}^{m-1})$. In particular, there exists a sequence $\delta_n \to 0^+$ and $(p^\varepsilon, u^\varepsilon, \nabla d^\varepsilon) \in \mathcal{C}([0, \widetilde{T}_2]; H_{co}^{m-1})$ such that

$$(p^{\varepsilon,\delta_n}, u^{\varepsilon,\delta_n}, \nabla d^{\varepsilon,\delta_n}) \to (p^\varepsilon, u^\varepsilon, \nabla d^\varepsilon) \text{ in } \mathcal{C}([0, \widetilde{T}_2]; H_{co}^{m-1}) \text{ as } \delta_n \to 0^+.$$

Moreover, applying the lower semicontinuity of norms to the bounds (4.11), one obtains the bounds (4.11) and (4.12) for $(p^\varepsilon, u^\varepsilon, d^\varepsilon)$. It follows from the bounds of (4.11) and (4.12) for $(p^\varepsilon, u^\varepsilon, d^\varepsilon)$, and the anisotropic Sobolev inequality (2.5), that

$$\begin{aligned} &\sup_{0 \le t \le \widetilde{T}_2} \|(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, d^{\varepsilon,\delta_n} - d^\varepsilon)(t)\|_{L^\infty}^2 \\ &\le C \sup_{0 \le t \le \widetilde{T}_2} \{\|\nabla(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{H_{co}^1} \\ &\qquad \times \|(\rho^{\varepsilon,\delta_n} - \rho^\varepsilon, u^{\varepsilon,\delta_n} - u^\varepsilon, d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{H_{co}^2}\} \to 0, \end{aligned}$$

and

$$\sup_{0 \le t \le \widetilde{T}_2} \|\nabla(d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{L^\infty}^2 \le C \sup_{0 \le t \le \widetilde{T}_2} \|\Delta(d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{H_{co}^1} \|\nabla(d^{\varepsilon,\delta_n} - d^\varepsilon)\|_{H_{co}^2} \to 0.$$

Hence, it is easy to check that $(\rho^\varepsilon,u^\varepsilon,d^\varepsilon)$ is a strong solution of the nematic liquid crystal flows (1.1)-(1.3). The uniqueness of the solution $(\rho^\varepsilon,u^\varepsilon,d^\varepsilon)$ comes directly from the Lipschitz regularity of solution, which will be proved in detail below. Thus, the whole family $(\rho^{\varepsilon,\delta},u^{\varepsilon,\delta},d^{\varepsilon,\delta})$ converge to $(\rho^\varepsilon,u^\varepsilon,d^\varepsilon)$. Therefore, we have established the local solution of Equation (1.1)-(1.3) and (1.4) with initial data $(p_0^\varepsilon,u_0^\varepsilon,d_0^\varepsilon)\in X_{n,m}^\varepsilon$, $t\in[0,T_2]$.

We shall use the local existence results to prove Theorem 1.1. If $T_0\le\widetilde{T}$, then Theorem 1.1 follows from estimates (4.11) and (4.12) with $\widetilde{C}_1=\widetilde{C}_3$. On the other hand, for the case $\widetilde{T}\le T_0$, based on the uniform estimate (4.11) and (4.12), we can use the local existence results established above to extend our solution step by step to the uniform time interval $t\in[0,T_0]$. Therefore, we complete the proof of Theorem 1.1.

Finally, we give the proof in detail for the uniqueness of solution to the system (1.1)-(1.3) and (1.5). Indeed, let $(\rho_i,u_i,d_i)(i=1,2)$ be two solutions on $\Omega\times(0,T]$ of system (1.1)-(1.3) and (1.5). Set $\bar\rho=\rho_2-\rho_1,\bar p=p_2-p_1,\bar u=u_2-u_1$ and $\bar d=d_2-d_1$, then we have

$$\rho_2\partial_t\bar u+\rho_2u_2\cdot\nabla\bar u+\nabla\bar p=-\mu\varepsilon\nabla\times(\nabla\times\bar u)+(2\mu+\lambda)\varepsilon\nabla\mathrm{div}\bar u-K_1, \tag{4.13}$$

$$\partial_t\bar d-\Delta\bar d=K_2, \tag{4.14}$$

where

$$K_1=\bar\rho\partial_tu_1+\rho_2\bar u\cdot\nabla u_1+\bar\rho u_1\cdot\nabla u_1+\nabla d_2\cdot\Delta\bar d+\nabla\bar d\cdot\Delta d_1, \tag{4.15}$$

$$K_2=-u_2\cdot\nabla\bar d-\bar u\cdot\nabla d_1+|\nabla d_2|^2\bar d+\nabla\bar d:\nabla(d_2+d_1)\cdot d_1. \tag{4.16}$$

The boundary condition for the system (4.13)-(4.14) is given by

$$\bar u\cdot n=0,\quad n\times(\nabla\times\bar u)=[B\bar u]_\tau,\quad n\cdot\nabla\bar d=0,\quad \text{on }\partial\Omega. \tag{4.17}$$

Multiplying equation (4.13) by $\bar u$, integrating by Ω and applying the boundary condition (4.17), we find

$$\begin{aligned}&\frac{d}{dt}\frac12\int\rho_2|\bar u|^2dx+\mu\varepsilon\int|\nabla\times\bar u|^2dx+(2\mu+\lambda)\varepsilon\int|\mathrm{div}\bar u|^2dx\\&=-\varepsilon\int_{\partial\Omega}[B\bar u]_\tau\cdot\bar u_\tau\ d\sigma-\int K_1\cdot\bar u\ dx-\int\nabla\bar p\cdot\bar u\ dx.\end{aligned} \tag{4.18}$$

By virtue of the Hölder inequality, one arrives at

$$|\int K_1\cdot\bar udx|\le C(\|(u_1,\nabla u_1,\partial_tu_1,\nabla d_2,\Delta d_1)\|_{L^\infty}^2+1)\|(\bar\rho,\bar u,\nabla\bar d)\|_{L^2}^2+\delta\|\Delta\bar d\|_{L^2}^2. \tag{4.19}$$

Similar to inequalities (3.14) and (3.15), we can obtain

$$|\bar u|_{L^2(\partial\Omega)}^2\le\delta\|\nabla\bar u\|_{L^2}^2+C_\delta\|\bar u\|_{L^2}^2. \tag{4.20}$$

and

$$\mu\|\nabla\times\bar u\|_{L^2}^2+(2\mu+\lambda)\|\mathrm{div}\bar u\|_{L^2}^2\ge2c_1\|\nabla\bar u\|_{L^2}^2-C\|\bar u\|_{L^2}^2. \tag{4.21}$$

In order to deal with the difficult term $\int\nabla\bar p\cdot\bar u\ dx$, we apply the following relation

$$\mathrm{div}\bar u=\frac{\partial_tp_2}{\gamma p_1p_2}\bar p-\frac{1}{\gamma p_1}\partial_t\bar p+\frac{u_2\cdot\nabla p_2}{\gamma p_1p_2}\bar p-\frac{\bar u\cdot\nabla p_2}{\gamma p_1}-\frac{1}{\gamma p_1}u_1\cdot\nabla\bar p.$$

Integrating by parts and applying the boundary condition (4.17), we have

$$\begin{aligned}
&-\int \nabla\bar{p}\cdot\bar{u}dx=\int \bar{p}\cdot\mathrm{div}\bar{u}dx\\
&=\int \frac{\partial_t p_2}{\gamma p_1 p_2}|\bar{p}|^2dx+\int\frac{-1}{\gamma p_1}\partial_t\bar{p}\cdot\bar{p}dx+\int\frac{u_2\cdot\nabla p_2}{\gamma p_1 p_2}|\bar{p}|^2dx\\
&\quad+\int\frac{-\bar{u}\cdot\nabla p_2}{\gamma p_1}\cdot\bar{p}dx+\int\frac{-1}{\gamma p_1}u_1\cdot\nabla\bar{p}\cdot\bar{p}dx\\
&:=J_1+J_2+J_3+J_4+J_5.
\end{aligned}\tag{4.22}$$

By computation, it is easy to check that

$$\begin{aligned}
J_2&=-\frac{d}{dt}\int\frac{1}{2\gamma p_1}|\bar{p}|^2dx-\int\frac{\partial_t p_1}{2\gamma p_1^2}|\bar{p}|^2dx\\
&\le-\frac{d}{dt}\int\frac{1}{2\gamma p_1}|\bar{p}|^2dx+C\|\partial_t p_1\|_{L^\infty}\|\bar{p}\|_{L^2}^2,
\end{aligned}\tag{4.23}$$

and

$$J_5=\int(\frac{\mathrm{div}u_1}{2\gamma p_1}-\frac{u_1\cdot\nabla p_1}{2\gamma p_1^2})|\bar{p}|^2dx\le C(\|(u_1,\nabla u_1,\nabla p_1)\|_{L^\infty}^2+1)\|\bar{p}\|_{L^2}^2.\tag{4.24}$$

On the other hand, it is easy to deduce that

$$J_1+J_3+J_4\le C(\|(\partial_t p_2,\nabla p_2,u_2)\|_{L^\infty}^2+1)(\|\bar{u}\|_{L^2}^2+\|\bar{p}\|_{L^2}^2).\tag{4.25}$$

Substituting the estimates (4.23)-(4.25) into equation (4.22), we find

$$\begin{aligned}
-\int\nabla\bar{p}\cdot\bar{u}dx\le&-\frac{d}{dt}\int\frac{1}{2\gamma p_1}|\bar{p}|^2dx+C\|(\bar{u},\bar{p})\|_{L^2}^2\\
&+C\|(u_1,u_2,\partial_t p_1,\partial_t p_2,\nabla p_1,\nabla p_2,\nabla u_1)\|_{L^\infty}^2\|(\bar{u},\bar{p})\|_{L^2}^2.
\end{aligned}\tag{4.26}$$

Then, plugging the estimates (4.19)-(4.21) and (4.26) into equation (4.18), one arrives at

$$\begin{aligned}
&\frac{d}{dt}\int\frac{\rho_2}{2}|\bar{u}|^2dx+\frac{d}{dt}\int\frac{1}{2\gamma p_1}|\bar{p}|^2dx+\varepsilon c_1\int|\nabla\bar{u}|^2dx-\delta\int|\Delta\bar{d}|^2dx\\
\le&C(\|(u_1,u_2,\partial_t p_1,\partial_t p_2,\partial_t u_1,\nabla p_1,\nabla p_2,\nabla u_1,\nabla d_2,\Delta d_1)\|_{L^\infty}^2+1)\|(\bar{\rho},\bar{p},\bar{u},\nabla\bar{d})\|_{L^2}^2.
\end{aligned}\tag{4.27}$$

Multiplying equation (4.14) by $-\Delta\bar{d}$, integrating by Ω and applying the boundary condition (4.17), we find

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int|\nabla\bar{d}|^2dx+\frac{3}{4}\int|\Delta\bar{d}|^2dx\\
\le&C(\|(u_2,\nabla d_1,\nabla d_2)\|_{L^\infty}^2+\|\nabla d_2\|_{L^\infty}^4+1)\|(\bar{d},\nabla\bar{d},u)\|_{L^2}^2.
\end{aligned}\tag{4.28}$$

Multiplying equation (4.14) by $\bar{d}$, integrating by Ω and applying the boundary condition (4.17), we obtain

$$\frac{d}{dt}\frac{1}{2}\int|\bar{d}|^2dx+\int|\nabla\bar{d}|^2dx\le C(\|(u_2,\nabla d_1,\nabla d_2)\|_{L^\infty}^2+1)\|(\bar{u},\bar{d},\nabla\bar{d})\|_{L^2}^2.\tag{4.29}$$

By virtue of the estimate (3.2), one finds the relation

$$\|\bar{\rho}\|_{L^2}^2 \leq C\|\bar{p}\|_{L^2}^2. \tag{4.30}$$

The combination of inequalities (4.27)-(4.29) and (4.30) gives directly

$$\begin{aligned}
&\frac{d}{dt}\left\{\int \frac{\rho_2}{2}|\bar{u}|^2 dx + \int \frac{1}{2\gamma p_1}|\bar{p}|^2 dx + \int |\bar{d}|^2 dx + \int |\nabla \bar{d}|^2 dx\right\} \\
\leq & CG(t)\left\{\int \frac{\rho_2}{2}|\bar{u}|^2 dx + \int \frac{1}{2\gamma p_1}|\bar{p}|^2 dx + \int |\bar{d}|^2 dx + \int |\nabla \bar{d}|^2 dx\right\}.
\end{aligned} \tag{4.31}$$

where

$$G(t) := \|(u_1, u_2, \partial_t p_1, \partial_t p_2, \partial_t u_1, \nabla p_1, \nabla p_2, \nabla u_1, \nabla d_1, \nabla d_2, \Delta d_1)\|_{L^\infty}^2 + \|\nabla d_2\|_{L^\infty}^4 + 1.$$

Then, it follows from Grönwall's inequality and the inequality (4.31) that

$$(\bar{d}, \bar{u}, \bar{p}, \nabla \bar{d}) = 0,$$

which implies $(p_1, u_1, d_1) = (p_2, u_2, d_2)$. Therefore, we complete the proof of uniqueness of solution for the system (1.1)-(1.3) and (1.5).

5. Proof of Theorem 1.2 (Inviscid Limit)

In this section, we study the vanishing viscosity of solutions for equations (1.1)-(1.3) to the solution for equations (1.6)-(1.8) with a rate of convergence. It is easy to see that the solution $(\rho, u, d) \in H^4 \times H^4 \times H^5$ of equations (1.1)-(1.3) and (1.4) with initial data $(\rho_0, u_0, d_0) \in H^4 \times H^4 \times H^5$ satisfies

$$\sum_{k=0}^{4} \|(\rho, u)\|_{C^k([0,T_1];H^{4-k})} + \sum_{k=0}^{2} \|d\|_{C^k([0,T_1];H^{5-2k})} \leq \widetilde{C}_4$$

where $\widetilde{C}_4$ depends only on $\|(\rho_0, u_0, d_0)\|_{H^4 \times H^4 \times H^5}$. On the other hand, it follows from the Theorem 1.1 that the solution $(\rho^\varepsilon, u^\varepsilon, d^\varepsilon)$ of equations (1.1)-(1.3) and (1.4) with initial data (ρ_0, u_0, d_0) satisfies

$$\|(p(\rho^\varepsilon), u^\varepsilon, d^\varepsilon)\|_{X_m^\varepsilon} \leq \widetilde{C}_1, \quad \frac{1}{2\widehat{C}_0} \leq \rho^\varepsilon(t) \leq 2\widehat{C}_0 \quad \forall t \in [0, T_0],$$

where T_0 and $\widetilde{C}_1$ are defined in Theorem 1.1. In particular, this uniform regularity implies the bound

$$\|(\rho^\varepsilon, u^\varepsilon)\|_{W^{1,\infty}} + \|d^\varepsilon\|_{W^{2,\infty}} + \|\partial_t(\rho^\varepsilon, u^\varepsilon)\|_{L^\infty} + \|d_t^\varepsilon\|_{W^{1,\infty}} \leq \widetilde{C}_1,$$

which plays an important role in the proof of Theorem 1.2. Let us define

$$\phi^\varepsilon = \rho^\varepsilon - \rho, \quad v^\varepsilon = u^\varepsilon - u, \quad \varphi^\varepsilon = d^\varepsilon - d.$$

It then follows from equations (1.1)-(1.3) that

$$\partial_t \phi^\varepsilon + \rho \mathrm{div} v^\varepsilon + u \cdot \nabla \phi^\varepsilon = R_1^\varepsilon, \tag{5.1}$$

$$\begin{aligned}
&\rho \partial_t v^\varepsilon + \rho u \cdot \nabla v^\varepsilon + \nabla(p^\varepsilon - p) + \Phi^\varepsilon \\
=& -\mu\varepsilon \nabla \times (\nabla \times v^\varepsilon) + (2\mu + \lambda)\varepsilon \nabla \mathrm{div} v^\varepsilon + R_2^\varepsilon + R_3^\varepsilon,
\end{aligned} \tag{5.2}$$

$$\partial_t \varphi^\varepsilon - \Delta\varphi^\varepsilon = R_4^\varepsilon, \tag{5.3}$$

where

$$\begin{aligned}
&R_1^\varepsilon = -\phi^\varepsilon \mathrm{div} v^\varepsilon - v^\varepsilon\cdot\nabla\phi^\varepsilon - \phi^\varepsilon \mathrm{div} u - \nabla\rho\cdot v^\varepsilon,\\
&R_2^\varepsilon = -\phi^\varepsilon v_t^\varepsilon - \phi^\varepsilon u_t + \mu\varepsilon\Delta u + (\mu+\lambda)\varepsilon\nabla \mathrm{div} u,\\
&R_3^\varepsilon = -\nabla d^\varepsilon\cdot\Delta\varphi^\varepsilon - \nabla\varphi^\varepsilon\cdot\Delta d,\\
&R_4^\varepsilon = -u\cdot\nabla\varphi^\varepsilon - v^\varepsilon\cdot\nabla d^\varepsilon + (\nabla\varphi^\varepsilon : \nabla(d^\varepsilon + d))d^\varepsilon + |\nabla d|^2\varphi^\varepsilon,\\
&\Phi^\varepsilon = (\rho^\varepsilon u^\varepsilon - \rho u)\cdot\nabla u^\varepsilon.
\end{aligned}$$

The boundary conditions to equations (5.1)-(5.3) are given as follows

$$v^\varepsilon\cdot n = 0, \quad n\times(\nabla\times v^\varepsilon) = [Bv^\varepsilon]_\tau + [Bu]_\tau - n\times w, \quad \frac{\partial\varphi^\varepsilon}{\partial n} = 0, \quad x\in\partial\Omega. \tag{5.4}$$

LEMMA 5.1. *For $t\in[0,\min\{T_0,T_1\}]$, it holds that*

$$\sup_{0\le\tau\le t}(\|(\phi^\varepsilon,v^\varepsilon)(\tau)\|_{L^2}^2 + \|\varphi^\varepsilon(\tau)\|_{H^1}^2) + \mu\varepsilon\int_0^t \|v^\varepsilon\|_{H^1}^2 d\tau + \int_0^t (\|\nabla\varphi^\varepsilon\|_{L^2}^2 + \|\Delta\varphi^\varepsilon\|_{L^2}^2)d\tau \le C\varepsilon^{\frac{3}{2}}. \tag{5.5}$$

where $C>0$ depend only on $\widetilde{C}_0, \widetilde{C}_1$ and $\widetilde{C}_4$.

Proof. Multiplying equation (5.2) by v^ε and integrating over Ω, one arrives at

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int \rho|v^\varepsilon|^2dx + \int \Phi^\varepsilon\cdot v^\varepsilon dx + \int\nabla(p^\varepsilon - p)\cdot v^\varepsilon dx\\
&= -\mu\varepsilon\int\nabla\times(\nabla\times v^\varepsilon)\cdot v^\varepsilon dx + (2\mu+\lambda)\varepsilon\int\nabla \mathrm{div} v^\varepsilon\cdot v^\varepsilon dx + \int R_2^\varepsilon\cdot v^\varepsilon dx + \int R_3^\varepsilon\cdot v^\varepsilon dx.
\end{aligned} \tag{5.6}$$

It is easy to check that

$$\int\Phi^\varepsilon\cdot v^\varepsilon dx \le C\|(\rho,u^\varepsilon,\nabla u^\varepsilon)\|_{L^\infty}(\|\phi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2). \tag{5.7}$$

Integrating by parts and applying equation (5.1), we find

$$\begin{aligned}
&\int\nabla(p^\varepsilon - p)\cdot v^\varepsilon dx = -\int(p^\varepsilon - p)\mathrm{div} v^\varepsilon dx\\
&\ge \int\frac{p'(\rho)}{\rho}\phi^\varepsilon(\phi_t^\varepsilon + u\cdot\nabla\phi^\varepsilon - R_1^\varepsilon)dx - C(1+\|\nabla u^\varepsilon\|_{L^\infty})\|\phi^\varepsilon\|_{L^2}^2\\
&\ge \frac{d}{dt}\int\frac{p'(\rho)}{2\rho}|\phi^\varepsilon|^2dx - C(1+\|(\rho,u,\rho^\varepsilon,u^\varepsilon)\|_{W^{1,\infty}})(\|\phi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2)\\
&\ge \frac{d}{dt}\int\frac{p'(\rho)}{2\rho}|\phi^\varepsilon|^2dx - C(\|\phi^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2).
\end{aligned} \tag{5.8}$$

Integrating by parts and applying the boundary condition (5.4), one attains directly

$$-\mu\varepsilon\int\nabla\times(\nabla\times v^\varepsilon)\cdot v^\varepsilon dx$$

$$
\begin{aligned}
&= -\mu\varepsilon\int_{\partial\Omega} n\times(\nabla\times v^\varepsilon)\cdot v^\varepsilon dx - \mu\varepsilon\int|\nabla\times v^\varepsilon|^2dx\\
&= -\mu\varepsilon\int_{\partial\Omega}([Bv^\varepsilon]_\tau+[Bu]_\tau-n\times w)\cdot v^\varepsilon d\sigma - \mu\varepsilon\int|\nabla\times v^\varepsilon|^2dx\\
&\le -\mu\varepsilon\|\nabla\times v^\varepsilon\|_{L^2}^2 + C\varepsilon(|v^\varepsilon|_{L^2(\partial\Omega)}^2+|v^\varepsilon|_{L^2(\partial\Omega)}),
\end{aligned}
\tag{5.9}
$$

and

$$
(2\mu+\lambda)\varepsilon\int\nabla\mathrm{div}v^\varepsilon\cdot v^\varepsilon dx = (2\mu+\lambda)\varepsilon\int|\mathrm{div}v^\varepsilon|^2dx. \tag{5.10}
$$

On the other hand, by virtue of the Hölder and Cauchy inequalities, one attains

$$
\int R_2^\varepsilon\cdot v^\varepsilon dx \le C\|(\phi^\varepsilon,v^\varepsilon)\|_{L^2}^2 + C\varepsilon^2, \tag{5.11}
$$

and

$$
\int R_3^\varepsilon\cdot v^\varepsilon dx \le \delta\|\Delta\varphi^\varepsilon\|_{L^2}^2 + C_\delta(\|v^\varepsilon\|_{L^2}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2). \tag{5.12}
$$

Substituting the estimates (5.7)-(5.12) into the identity (5.6), we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int\left(\frac{p'(\rho)}{\rho}|\phi^\varepsilon|^2+\frac{\rho}{2}|v^\varepsilon|^2\right)dx + \mu\varepsilon\|\nabla\times v^\varepsilon\|_{L^2}^2 + (2\mu+\lambda)\varepsilon\|\mathrm{div}v^\varepsilon\|_{L^2}^2\\
\le& C_\delta\|(\phi^\varepsilon,v^\varepsilon,\nabla\varphi^\varepsilon)\|_{L^2}^2 + C\varepsilon(|v^\varepsilon|_{L^2(\partial\Omega)}^2+|v^\varepsilon|_{L^2(\partial\Omega)}) + C\varepsilon^2 + \delta\|\Delta\varphi^\varepsilon\|_{L^2}^2.
\end{aligned}
\tag{5.13}
$$

The application of Proposition 2.1 gives directly

$$
\|\nabla v^\varepsilon\|_{H^1}^2 \le C(\|\nabla\times v^\varepsilon\|_{L^2}^2 + \|\mathrm{div}v^\varepsilon\|_{L^2}^2 + \|v^\varepsilon\|_{L^2}^2). \tag{5.14}
$$

By virtue of the trace theorem in Proposition 2.3 and Cauchy's inequality, one finds

$$
|v^\varepsilon|_{L^2(\partial\Omega)}^2 \le \delta\|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta\|v^\varepsilon\|_{L^2}^2, \tag{5.15}
$$

and

$$
\begin{aligned}
\varepsilon|v^\varepsilon|_{L^2(\partial\Omega)} &\le \varepsilon\|v^\varepsilon\|_{L^2}^{\frac12}\|\nabla v^\varepsilon\|_{L^2}^{\frac12} \le \delta\varepsilon\|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta\varepsilon\|v^\varepsilon\|_{L^2}^{\frac23}\\
&\le \delta\varepsilon\|\nabla v^\varepsilon\|_{L^2}^2 + C_\delta\|v^\varepsilon\|_{L^2}^2 + \varepsilon^{\frac32}.
\end{aligned}
\tag{5.16}
$$

Then, the combination of estimates (5.13)-(5.16) yields that

$$
\begin{aligned}
&\frac{d}{dt}\int\left(\frac{p'(\rho)}{\rho}|\phi^\varepsilon|^2+\frac{\rho}{2}|v^\varepsilon|^2\right)dx + \mu\varepsilon\|v^\varepsilon\|_{H^1}^2\\
\le& C\|(\phi^\varepsilon,v^\varepsilon,\nabla\varphi^\varepsilon)\|_{L^2}^2 + C\varepsilon^{\frac32} + \delta\|\Delta\varphi^\varepsilon\|_{L^2}^2.
\end{aligned}
\tag{5.17}
$$

Multiplying equation (5.3) by $-\Delta\varphi^\varepsilon$ and integrating over Ω, we find

$$
-\int\partial_t\varphi^\varepsilon\cdot\Delta\varphi^\varepsilon dx + \int|\Delta\varphi^\varepsilon|^2dx = -\int R_3^\varepsilon\cdot\Delta\varphi^\varepsilon dx. \tag{5.18}
$$

Integrating by parts and applying the boundary condition (5.4), it holds that

$$
-\int\partial_t\varphi^\varepsilon\cdot\Delta\varphi^\varepsilon dx = -\int_{\partial\Omega}\partial_t\varphi^\varepsilon\cdot(n\cdot\nabla\varphi^\varepsilon)d\sigma + \frac12\frac{d}{dt}\int|\nabla\varphi^\varepsilon|^2dx
$$

$$=\frac{1}{2}\frac{d}{dt}\int|\nabla\varphi^\varepsilon|^2dx. \tag{5.19}$$

Applying the Cauchy inequality, it is easy to deduce that

$$\begin{aligned}-\int R_2^\varepsilon\cdot\Delta\varphi^\varepsilon dx\le&\delta\|\Delta\varphi^\varepsilon\|_{L^2}^2+C_\delta\|u\|_{L^\infty}^2\|\nabla\varphi^\varepsilon\|_{L^2}^2\\&+C_\delta\|\nabla d^\varepsilon\|_{L^\infty}^2(\|\nabla\varphi^\varepsilon\|_{L^2}^2+\|v^\varepsilon\|_{L^2}^2)\\&+C_\delta\|\nabla d\|_{L^\infty}^2(\|\nabla\varphi^\varepsilon\|_{L^2}^2+\|\varphi^\varepsilon\|_{L^2}^2)\\\le&\delta\|\Delta\varphi^\varepsilon\|_{L^2}^2+C_\delta(\|v^\varepsilon\|_{L^2}^2+\|\varphi^\varepsilon\|_{L^2}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2).\end{aligned} \tag{5.20}$$

Substituting the estimates (5.19)-(5.20) into the identity (5.18), we obtain

$$\frac{1}{2}\frac{d}{dt}\int|\nabla\varphi^\varepsilon|^2dx+\frac{3}{4}\int|\Delta\varphi^\varepsilon|^2dx\le C(\|v^\varepsilon\|_{L^2}^2+\|\varphi^\varepsilon\|_{L^2}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2). \tag{5.21}$$

In order to control the term $\int|\varphi^\varepsilon|^2dx$ on the right-hand side of inequality (5.21), we multiply equation (5.3) by φ^ε and integrate by parts to get that

$$\frac{1}{2}\frac{d}{dt}\int|\varphi^\varepsilon|^2dx+\int|\nabla\varphi^\varepsilon|^2dx=\int R_4^\varepsilon\cdot\varphi^\varepsilon dx. \tag{5.22}$$

In view of the Hölder inequality, one arrives at

$$\begin{aligned}\int R_4^\varepsilon\cdot\varphi^\varepsilon dx\le&\|u\|_{L^\infty}\|\varphi^\varepsilon\|_{L^2}\|\nabla\varphi^\varepsilon\|_{L^2}+\|\nabla d\|_{L^\infty}^2\|\varphi^\varepsilon\|_{L^2}^2\\&+\|\nabla d^\varepsilon\|_{L^\infty}(\|v^\varepsilon\|_{L^2}\|\varphi^\varepsilon\|_{L^2}+\|\varphi^\varepsilon\|_{L^2}\|\nabla\varphi^\varepsilon\|_{L^2})\\&+\|\nabla d\|_{L^\infty}\|\nabla\varphi^\varepsilon\|_{L^2}\|\varphi^\varepsilon\|_{L^2}\\\le&C(\|v^\varepsilon\|_{L^2}^2+\|\varphi^\varepsilon\|_{L^2}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2),\end{aligned}$$

which, together with equation (5.22), yields directly

$$\frac{1}{2}\frac{d}{dt}\int|\varphi^\varepsilon|^2dx+\int|\nabla\varphi^\varepsilon|^2dx\le C(\|v^\varepsilon\|_{L^2}^2+\|\varphi^\varepsilon\|_{L^2}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2). \tag{5.23}$$

Then the combination of estimates (5.17), (5.21) and (5.23) yields immediately

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{p'(\rho)}{\rho}|\phi^\varepsilon|^2+\frac{\rho}{2}|v^\varepsilon|^2+\frac{1}{2}|\varphi^\varepsilon|^2+\frac{1}{2}|\nabla\varphi^\varepsilon|^2\right)dx\\&+\mu\varepsilon\|v^\varepsilon\|_{H^1}^2+\frac{3}{4}\int(|\nabla\varphi^\varepsilon|^2+|\Delta\varphi^\varepsilon|^2)dx\\\le&C(\|\phi^\varepsilon\|_{L^2}^2+\|v^\varepsilon\|_{L^2}^2+\|\varphi^\varepsilon\|_{L^2}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2)+C\varepsilon^{\frac{3}{2}},\end{aligned}$$

which, together with the Grönwall inequality, completes the proof of Lemma 5.1. □

LEMMA 5.2. *For $t\in[0,\min\{T_0,T_1\}]$, it holds that*

$$\sup_{0\le\tau\le t}\|\Delta\varphi^\varepsilon(\tau)\|_{L^2}^2+\int_0^t\|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2d\tau\le C\varepsilon^{\frac{1}{2}}. \tag{5.24}$$

Proof. Taking ∇ operator to equation (5.3), we find

$$\nabla\varphi^\varepsilon-\nabla\Delta\varphi^\varepsilon=\nabla R_4^\varepsilon,$$

which, multiplying by $-\nabla\varphi^\varepsilon$, reads

$$-\int \partial_t \nabla\varphi^\varepsilon \cdot \nabla\Delta\varphi^\varepsilon dx + \int |\nabla\Delta\varphi^\varepsilon|^2 dx = -\int \nabla R_4^\varepsilon \cdot \nabla\Delta\varphi^\varepsilon dx. \tag{5.25}$$

Integrating by parts and applying the boundary condition (5.4), it is easy to deduce

$$\begin{aligned}
&-\int \partial_t \nabla\varphi^\varepsilon \cdot \nabla\Delta\varphi^\varepsilon dx \\
=&-\int_{\partial\Omega} n\cdot\nabla\varphi^\varepsilon \cdot \nabla\Delta\varphi^\varepsilon d\sigma + \frac{1}{2}\frac{d}{dt}\int |\Delta\varphi^\varepsilon|^2 dx \\
=&\frac{1}{2}\frac{d}{dt}\int |\Delta\varphi^\varepsilon|^2 dx.
\end{aligned} \tag{5.26}$$

On the other hand, it is easy to check that

$$\begin{aligned}
\|\nabla R_4^\varepsilon\|_{L^2}^2 \le & C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{H^1}^2) + C(\|\nabla^2\varphi^\varepsilon\|_{L^2}^2 + \|\nabla v^\varepsilon\|_{L^2}^2) \\
\le & C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{H^1}^2) + C(\|\Delta\varphi^\varepsilon\|_{L^2}^2 + \|\nabla v^\varepsilon\|_{L^2}^2),
\end{aligned} \tag{5.27}$$

where we have used the standard elliptic estimates in the last inequality. Hence, by virtue of the Cauchy inequality, equation (5.26) and inequality (5.27), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int |\Delta\varphi^\varepsilon|^2 dx + \int |\nabla\Delta\varphi^\varepsilon|^2 dx \\
\le& \delta\|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2 + C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{H^1}^2) + C(\|\Delta\varphi^\varepsilon\|_{L^2}^2 + \|\nabla v^\varepsilon\|_{L^2}^2).
\end{aligned} \tag{5.28}$$

Choosing δ small enough in inequality (5.28) and integrating over $[0,t]$, one attains

$$\begin{aligned}
&\int |\Delta\varphi^\varepsilon(t)|^2 dx + \int_0^t \|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2 d\tau \\
\le& C(\|v^\varepsilon\|_{L^2}^2 + \|\varphi^\varepsilon\|_{H^1}^2) + C\int_0^t (\|\Delta\varphi^\varepsilon\|_{L^2}^2 + \|\nabla v^\varepsilon\|_{L^2}^2) d\tau \le C\varepsilon^{\frac{1}{2}},
\end{aligned}$$

where we have used the estimate (5.5) in the last inequality. Therefore, we complete the proof of Lemma 5.2. □

Lemma 5.3. *For $t \in [0, \min\{T_0, T_1\}]$, it holds that*

$$\begin{aligned}
&\|(\mathrm{div} v^\varepsilon, \nabla(p^\varepsilon - p))\|_{L^2}^2 + (2\mu+\lambda)\varepsilon \int_0^t \|\nabla \mathrm{div} v^\varepsilon(\tau)\|_{L^2}^2 d\tau \\
\le& \delta\int_0^t \|v_t^\varepsilon\|_{L^2}^2 d\tau + C_\delta \int_0^t \|(\varphi^\varepsilon, v^\varepsilon)\|_{H^1}^2 d\tau + C_\delta \varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{5.29}$$

Proof. Multiplying (5.2) by $\nabla \mathrm{div} v^\varepsilon$ and integrating over Ω, one attains

$$\begin{aligned}
&\underbrace{\int (\rho v_t^\varepsilon + \rho u\cdot\nabla v^\varepsilon) dx}_{VIII_1} + \underbrace{\int \nabla(p^\varepsilon - p)\cdot\nabla \mathrm{div} v^\varepsilon dx}_{VIII_2} + \underbrace{\int \Phi^\varepsilon \cdot \nabla \mathrm{div} v^\varepsilon dx}_{VIII_3} \\
=& \underbrace{-\mu\varepsilon \int \nabla\times(\nabla\times v^\varepsilon)\cdot\nabla \mathrm{div} v^\varepsilon dx}_{VIII_4} + \underbrace{(2\mu+\lambda)\varepsilon\int |\nabla \mathrm{div} v^\varepsilon|^2 dx}_{VIII_5}
\end{aligned}$$

$$+\underbrace{\int R_2^\varepsilon\cdot\nabla\mathrm{div}v^\varepsilon dx}_{VIII_6}+\underbrace{\int R_3^\varepsilon\cdot\nabla\mathrm{div}v^\varepsilon dx}_{VIII_7}. \tag{5.30}$$

Following the same argument as Lemma 6.2 of [7], it is easy to obtain that

$$VIII_1\le -\frac{d}{dt}\int\frac{\rho}{2}|\mathrm{div}v^\varepsilon|^2dx+\delta\|v_t^\varepsilon\|_{L^2}^2+C_\delta\|\nabla v^\varepsilon\|_{L^2}^2+C|v^\varepsilon|_{L^2(\partial\Omega)}, \tag{5.31}$$

$$VIII_2\le -\frac{d}{dt}\int\frac{1}{2\gamma p^\varepsilon}|\nabla(p^\varepsilon-p)|^2dx+C(1+\|(u^\varepsilon,p^\varepsilon)\|_{W^{1,\infty}})\|(p^\varepsilon-p,v^\varepsilon)\|_{H^1}^2, \tag{5.32}$$

$$VIII_3\le C(1+\|(\rho^\varepsilon,u^\varepsilon)\|_{W^{1,\infty}})(\|(\varphi^\varepsilon,v^\varepsilon)\|_{H^1}^2+|(\varphi^\varepsilon,v^\varepsilon)|_{L^2(\partial\Omega)}), \tag{5.33}$$

$$VIII_4\le \delta\varepsilon\|\nabla\mathrm{div}v^\varepsilon\|_{L^2}^2+C_\delta\varepsilon(1+\|v^\varepsilon\|_{H^1}^2), \tag{5.34}$$

$$VIII_6\le \frac{(2\mu+\lambda)\varepsilon}{8}\|\nabla\mathrm{div}v^\varepsilon\|_{L^2}^2+\delta\|v_t^\varepsilon\|_{L^2}^2+C_\delta(\|(\varphi^\varepsilon,v^\varepsilon)\|_{H^1}^2+\varepsilon^{\frac32}). \tag{5.35}$$

On the other hand, integrating by parts and applying the Cauchy inequality, we find

$$\begin{aligned}VIII_7=&-\int_{\partial\Omega}n\cdot(\nabla d^\varepsilon\cdot\Delta\varphi^\varepsilon+\nabla\varphi^\varepsilon\cdot\Delta d)\mathrm{div}v^\varepsilon d\sigma\\&+\int\mathrm{div}(\nabla d^\varepsilon\cdot\Delta\varphi^\varepsilon+\nabla\varphi^\varepsilon\cdot\Delta d)\mathrm{div}v^\varepsilon dx\\=&\int\mathrm{div}(\nabla d^\varepsilon\cdot\Delta\varphi^\varepsilon+\nabla\varphi^\varepsilon\cdot\Delta d)\mathrm{div}v^\varepsilon dx\\\le& C(1+\|(\nabla d^\varepsilon,\Delta d^\varepsilon)\|_{L^\infty})(\|\nabla(\varphi^\varepsilon,v^\varepsilon)\|_{L^2}^2+\|\Delta\varphi^\varepsilon\|_{L^2}^2+\|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2).\end{aligned} \tag{5.36}$$

Substituting the estimates (5.31)-(5.35) and (5.36) into the identity (5.30), we find

$$\begin{aligned}&\frac{d}{dt}\int\left(\frac{\rho}{2}|\mathrm{div}v^\varepsilon|^2+\frac{1}{2\gamma p^\varepsilon}|\nabla(p^\varepsilon-p)|^2\right)+(2\mu+\lambda)\varepsilon\int|\nabla\mathrm{div}v^\varepsilon|^2dx\\\le&\delta\|v_t^\varepsilon\|_{L^2}^2+C\|(\varphi^\varepsilon,v^\varepsilon)\|_{H^1}^2+C|(\varphi^\varepsilon,v^\varepsilon)|_{L^2(\partial\Omega)}+C\varepsilon^{\frac32}\\&+C(\|\nabla\varphi^\varepsilon\|_{L^2}^2+\|\Delta\varphi^\varepsilon\|_{L^2}^2+\|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2).\end{aligned} \tag{5.37}$$

By virtue of the trace theorem in Proposition 2.3, we obtain

$$|(\varphi^\varepsilon,v^\varepsilon)|_{L^2}\le C(\|(\varphi^\varepsilon,v^\varepsilon)\|_{H^1}+\|(\varphi^\varepsilon,v^\varepsilon)\|_{L^2}^{\frac23})\le(\|(\varphi^\varepsilon,v^\varepsilon)\|_{H^1}+\varepsilon^{\frac12}). \tag{5.38}$$

Integrating inequality (5.37) over $[0,t]$ and substituting inequality (5.38) into the resulting inequality, we find

$$\begin{aligned}&\|(\mathrm{div}v^\varepsilon,\nabla(p^\varepsilon-p))\|_{L^2}^2+(2\mu+\lambda)\varepsilon\int_0^t\|\nabla\mathrm{div}v^\varepsilon(\tau)\|_{L^2}^2d\tau\\\le&\delta\int_0^t\|v_t^\varepsilon\|_{L^2}^2d\tau+C_\delta\int_0^t\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}^2d\tau+C_\delta\varepsilon^{\frac12}.\end{aligned}$$

Therefore, we complete the proof of Lemma 5.3. ∎

LEMMA 5.4. *For $t\in[0,\min\{T_0,T_1\}]$, it holds that*

$$\|\nabla\times v^\varepsilon\|_{L^2}^2+\varepsilon\int_0^t\|(\nabla\times v^\varepsilon)(\tau)\|_{H^1}^2d\tau$$

$$
\le \delta\|\nabla(\phi^\varepsilon,v^\varepsilon)\|_{L^2}^2+C\delta\int_0^t(\|v_t^\varepsilon\|_{L^2}^2+\varepsilon\|\nabla^2v^\varepsilon\|_{L^2}^2)d\tau
+C_\delta\int_0^t\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}^2d\tau+C_\delta\varepsilon^{\frac16}. \tag{5.39}
$$

Proof. Multiplying equation (5.2) by $\nabla\times(\nabla\times v^\varepsilon)$ and integrating over Ω, we find

$$
\begin{aligned}
&\underbrace{\int\rho^\varepsilon v_t^\varepsilon\cdot\nabla\times(\nabla\times v^\varepsilon)dx}_{IX_1}+\underbrace{\int\nabla(p^\varepsilon-p)\cdot\nabla\times(\nabla\times v^\varepsilon)dx}_{IX_2}\\
=&-\mu\varepsilon\|\nabla\times(\nabla\times v^\varepsilon)\|_{L^2}^2+(2\mu+\lambda)\varepsilon\int\nabla\mathrm{div}v^\varepsilon\cdot\nabla\times(\nabla\times v^\varepsilon)dx\\
&\underbrace{-\int\widetilde{\Phi}^\varepsilon\cdot\nabla\times(\nabla\times v^\varepsilon)dx}_{IX_3}+\underbrace{\int\widetilde{R}_2^\varepsilon\cdot\nabla\times(\nabla\times v^\varepsilon)dx}_{IX_4}+\underbrace{\int R_3^\varepsilon\cdot\nabla\times(\nabla\times v^\varepsilon)dx}_{IX_5},
\end{aligned}\tag{5.40}
$$

where

$$
\widetilde{\Phi}^\varepsilon=\rho^\varepsilon u^\varepsilon\cdot\nabla v^\varepsilon+(\rho^\varepsilon u^\varepsilon-\rho u)\cdot\nabla u,
$$
$$
\widetilde{R}_2^\varepsilon=-\phi^\varepsilon u_t+\mu\varepsilon\Delta u+(\mu+\lambda)\varepsilon\nabla\mathrm{div}u.
$$

Following the same argument as Lemma 6.3 of [7], it is easy to obtain that

$$
IX_1\ge\frac{d}{dt}\left\{\int\frac{\rho^\varepsilon}{2}|\nabla\times v^\varepsilon|^2dx+\int_{\partial\Omega}\left(\frac{\rho^\varepsilon}{2}v^\varepsilon(Bv^\varepsilon)+\rho^\varepsilon v^\varepsilon\cdot(Bu-n\times w)\right)d\sigma\right\}
-\delta\|v_t^\varepsilon\|_{L^2}^2-C_\delta(\|v^\varepsilon\|_{H^1}^2+|v^\varepsilon|_{L^2}), \tag{5.41}
$$
$$
|IX_2|\le C(\|p^\varepsilon-p\|_{H^1}^2+\|v^\varepsilon\|_{H^1}^2+|p^\varepsilon-p|_{L^2}), \tag{5.42}
$$
$$
|IX_3|\le C(\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}^2+|(\phi^\varepsilon,v^\varepsilon)|_{L^2}), \tag{5.43}
$$
$$
|IX_4|\le\delta\varepsilon\|\nabla\times(\nabla\times v^\varepsilon)\|_{L^2}^2+C_\delta(\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}^2+|(\phi^\varepsilon,v^\varepsilon)|_{L^2}+\varepsilon^{\frac32}). \tag{5.44}
$$

On the other hand, integrating by parts and applying boundary condition (5.4), we find

$$
\begin{aligned}
IX_5&=\int_{\partial\Omega}R_3^\varepsilon\cdot(n\times(\nabla\times v^\varepsilon))d\sigma+\int\nabla\times R_3^\varepsilon\cdot\nabla\times v^\varepsilon dx\\
&=\int_{\partial\Omega}R_3^\varepsilon\cdot[Bv^\varepsilon]_\tau d\sigma+\int_{\partial\Omega}R_3^\varepsilon\cdot([Bu]_\tau-n\times w)d\sigma\\
&\quad+\int\nabla\times R_3^\varepsilon\cdot\nabla\times v^\varepsilon dx\\
&:=IX_{51}+IX_{52}+IX_{53}.
\end{aligned}\tag{5.45}
$$

Integrating by parts and applying the Hölder inequality, we obtain

$$
\begin{aligned}
IX_{51}&=\int_{\partial\Omega}(n\times R_3^\varepsilon)\cdot(n\times[Bv^\varepsilon]_\tau)d\sigma\\
&=\int_{\partial\Omega}(n\times R_3^\varepsilon)\cdot(n\times(Bv^\varepsilon))d\sigma\\
&=\int(\nabla\times R_3^\varepsilon)\cdot(n\times(Bv^\varepsilon))dx+\int R_3^\varepsilon\cdot\nabla\times(n\times(Bv^\varepsilon))d\sigma
\end{aligned}
$$

$$\le C(\|v^\varepsilon\|_{H^1}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2+\|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2). \tag{5.46}$$

It is easy to check that

$$|IX_{52}|\le C(|\nabla\varphi^\varepsilon|_{L^2(\partial\Omega)}+|\Delta\varphi^\varepsilon|_{L^2(\partial\Omega)}), \tag{5.47}$$

and

$$|IX_{53}|\le C(\|\nabla\times v^\varepsilon\|_{L^2}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2+\|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2). \tag{5.48}$$

Then, substituting the estimates (5.46)-(5.48) into equation (5.45) yields

$$IX_5\le C(\|v^\varepsilon\|_{H^1}^2+\|\nabla\varphi^\varepsilon\|_{L^2}^2+\|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2)+C(|\nabla\varphi^\varepsilon|_{L^2(\partial\Omega)}+|\Delta\varphi^\varepsilon|_{L^2(\partial\Omega)}). \tag{5.49}$$

The application of the trace theorem in Proposition 2.3 yields that

$$|(\phi^\varepsilon,v^\varepsilon)|_{L^2}\le C\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}^{\frac12}\|(\phi^\varepsilon,v^\varepsilon)\|_{L^2}^{\frac12}\le C\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}^2+C\varepsilon^{\frac12}, \tag{5.50}$$

$$|\nabla\varphi^\varepsilon|_{L^2(\partial\Omega)}\le C\|\nabla\varphi^\varepsilon\|_{H^1}\le C\varepsilon^{\frac12}, \tag{5.51}$$

$$|\Delta\varphi^\varepsilon|_{L^2(\partial\Omega)}\le\|\Delta\varphi^\varepsilon\|_{H^1}^{\frac12}\|\Delta\varphi^\varepsilon\|_{L^2}^{\frac12}\le C\|\nabla\Delta\varphi^\varepsilon\|_{L^2}^2+C\varepsilon^{\frac16}. \tag{5.52}$$

Plugging inequalities (5.41)-(5.44), (5.50)-(5.52) and (5.49), into equation (5.40) reads immediately

$$\frac{d}{dt}\left\{\int\frac{\rho^\varepsilon}{2}|\nabla\times v^\varepsilon|^2dx+\int_{\partial\Omega}\left(\frac{\rho^\varepsilon}{2}v^\varepsilon(Bv^\varepsilon)+\rho^\varepsilon v^\varepsilon\cdot(Bu-n\times w)\right)d\sigma\right\}+\frac{\mu\varepsilon}{2}\int|\nabla\times(\nabla\times v^\varepsilon)|^2dx$$
$$\le C\delta\|v_t^\varepsilon\|_{L^2}^2+C\delta\varepsilon\|\nabla^2v^\varepsilon\|_{L^2}^2+C_\delta(\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}^2+\varepsilon^{\frac16}). \tag{5.53}$$

In view of the Proposition 2.1, one arrives at

$$\begin{aligned}\|\nabla\times v^\varepsilon\|_{H^1}^2\le& C_1(\|\nabla\times(\nabla\times v^\varepsilon)\|_{L^2}^2+\|\mathrm{div}(\nabla\times v^\varepsilon)\|_{L^2}^2)\\&+C_1(\|\nabla\times v^\varepsilon\|_{L^2}^2+|n\times(\nabla\times v^\varepsilon)|_{H^{\frac12}}^2)\\\le& C_1(\|\nabla\times(\nabla\times v^\varepsilon)\|_{L^2}^2+\|\nabla\times v^\varepsilon\|_{L^2}^2)\\&+C_1(|Bv^\varepsilon|_{H^{\frac12}}^2+|(Bu)_\tau-n\times w|_{H^{\frac12}}^2).\end{aligned} \tag{5.54}$$

By virtue of the trace inequality in Proposition 2.3, we have

$$\|(\phi^\varepsilon,v^\varepsilon)\|_{L^2}^2\le C\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}\|(\phi^\varepsilon,v^\varepsilon)\|_{L^2}\le\delta\|\nabla(\phi^\varepsilon,v^\varepsilon)\|_{L^2}^2+C_\delta\varepsilon^{\frac32}. \tag{5.55}$$

Substituting inequalities (5.54) and (5.55) into inequality (5.53) and integrating the resulting inequality over $[0,t]$ yield the estimate (5.39). Therefore, we complete the proof of Lemma 5.4. □

Proof. **(Proof of Theorem 1.2.)** By virtue of Proposition 2.1, we have

$$\begin{aligned}\|v^\varepsilon\|_{H^1}^2\le& C(\|\nabla\times v^\varepsilon\|_{L^2}^2+\|\mathrm{div}v^\varepsilon\|_{L^2}^2+\|v^\varepsilon\|_{L^2}^2+\|v^\varepsilon\cdot n\|_{H^{\frac12}}^2)\\\le& C(\|\nabla\times v^\varepsilon\|_{L^2}^2+\|\mathrm{div}v^\varepsilon\|_{L^2}^2+\|v^\varepsilon\|_{L^2}^2),\end{aligned} \tag{5.56}$$

and

$$\|v^\varepsilon\|_{H^2}^2\le C(\|\nabla\times v^\varepsilon\|_{L^2}^2+\|\mathrm{div}v^\varepsilon\|_{H^1}^2+\|v^\varepsilon\|_{H^1}^2+\|v^\varepsilon\cdot n\|_{H^{\frac32}}^2)$$

$$\leq C(\|\nabla\times v^\varepsilon\|_{H^2}^2+\|\mathrm{div} v^\varepsilon\|_{H^1}^2+\|v^\varepsilon\|_{H^1}^2). \tag{5.57}$$

On the other hand, it follows from the Equation (5.2) that

$$\|v_t^\varepsilon\|_{L^2}^2\leq C(\|(\phi^\varepsilon,v^\varepsilon)\|_{H^1}^2+\varepsilon^2\|\nabla^2 v^\varepsilon\|_{L^2}^2+\varepsilon^{\frac{1}{2}}). \tag{5.58}$$

The combination of inequalities (5.29), (5.39), (5.56)-(5.58) and choosing δ small enough, it is easy to check that

$$\|\nabla(v^\varepsilon,p^\varepsilon-p)\|_{L^2}^2+\varepsilon\int_0^t\|v^\varepsilon(\tau)\|_{H^2}^2d\tau\leq C\int_0^t\|\nabla(v^\varepsilon,p^\varepsilon-p)\|_{L^2}^2d\tau+C\varepsilon^{\frac{1}{6}},$$

which, together with the Grönwall inequality, gives

$$\|\nabla(v^\varepsilon,p^\varepsilon-p)\|_{L^2}^2+\varepsilon\int_0^t\|v^\varepsilon(\tau)\|_{H^2}^2d\tau\leq C\varepsilon^{\frac{1}{6}}. \tag{5.59}$$

On the other hand, by virtue of Sobolev's inequality, uniform estimate (1.19) and convergence rate (5.5), it is easy to deduce

$$\begin{aligned}&\|(\rho^\varepsilon-\rho,u^\varepsilon-u)\|_{L^\infty(0,T_2;L^\infty(\Omega))}\\ \leq& C\|(\rho^\varepsilon-\rho,u^\varepsilon-u)\|_{L^2}^{\frac{2}{5}}\|(\rho^\varepsilon-\rho,u^\varepsilon-u)\|_{W^{1,\infty}}^{\frac{3}{5}}\leq C\varepsilon^{\frac{3}{10}},\end{aligned} \tag{5.60}$$

and

$$\|d^\varepsilon-d\|_{L^\infty(0,T^2;W^{1,\infty}(\Omega))}\leq C\|d^\varepsilon-d\|_{H^1}^{\frac{2}{5}}\|d^\varepsilon-d\|_{W^{2,\infty}}^{\frac{3}{5}}\leq C\varepsilon^{\frac{3}{10}}, \tag{5.61}$$

The combination of inequalities (5.5), (5.24) and (5.59)-(5.61) completes the proof of Theorem 1.2

□

Acknowledgements. The authors express their sincere thanks to the referee for their comments and suggestions, which improved the presentation of the paper. The author Jincheng Gao would like to thank Yong Wang for fruitful discussion and suggestion. Jincheng Gao's research was partially supported by Guangdong Natural Science Foundation (Grant No.2014A030313161), China Postdoctoral Science Foundation Project (Grant No.2016M600064 and No.2017T100053), and NNSF of China(Grant No.11571380). Boling Guo's research was partially supported by the NNSF of China(Grants No.11731014).

REFERENCES

[1] J.L. Ericksen, *Hydrostatic theory of liquid crystals*, Arch. Ration. Mech. Anal., 9(1):371–378, 1962.
[2] F.M. Leslie, *Some constitutive equations for liquid crystals*, Arch. Ration. Mech. Anal., 28(4):265–283, 1968.
[3] P.G. de Gennes, *The Physics of Liquid Crystals*, Oxford, 1974.
[4] G.M. Gie and J.P. Kelliher, *Boundary layer analysis of the Navier-Stokes equations with generalized Navier boundary conditions*, J. Diff. Eqs., 253(6):1862–1892, 2012.
[5] Y.L. Xiao and Z.P. Xin, *On 3D Lagrangian Navier-Stokes α model with a class of vorticityslip boundary conditions*, J. Math. Fluid Mech., 15(2):215–247, 2013.
[6] Y.L. Xiao and Z.P. Xin, *On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition*, Commun. Pure Appl. Math., 60(7):1027–1055, 2007.
[7] Y. Wang, Z.P. Xin, and Y. Yong, *Uniform regularity and vanishing viscosity limit for the compressible Navier–Stokes with general Navier-Slip boundary conditions in 3-dimensional domains*, SIAM J. Math. Anal., 47(6):4123–4191, 2015.

[8] F. Jiang, S. Jiang, and D.H. Wang, *Global weak solutions to the equations of compressible flow of nematic liquid crystals in two dimensions*, Arch. Ration. Mech. Anal., 214(2):403–451, 2014.
[9] J.Y. Lin, B.S. Lai, and C.Y. Wang, *Global finite energy weak solutions to the compressible nematic liquid crystal flow in dimension three*, SIAM J. Math. Anal., 47(4):2952–2983, 2015.
[10] T. Huang, C.Y. Wang, and H.Y. Wen, *Strong solutions of the compressible nematic liquid crystal flow*, J. Diff. Eqs., 252(3):2222–2265, 2012.
[11] X.P. Hu and H. Wu, *Global solution to the three-dimensional compressible flow of liquid crystals*, SIAM J. Math. Anal., 45(5):2678–2699, 2013.
[12] J.C. Gao, Q. Tao, and Z.A. Yao, *Long-time behavior of solution for the compressible nematic liquid crystal flows in* $\mathbb{R}^3$, J. Diff. Eqs., 261(4):2334–2383, 2016.
[13] J.C. Gao, Q. Tao, and Z.A. Yao, *Optimal decay rates of classical solutions for the full compressible MHD equations*, Z.Angew. Math. Phys., 67(2):23, 2016.
[14] T. Huang, C.Y. Wang, and H.Y. Wen, *Blow up criterion for compressible nematic liquid crystal flows in dimension three*, Arch. Ration. Mech. Anal., 204(1):285–311, 2012.
[15] S.J. Ding, J.R. Huang, H.Y. Wen, and R.Z. Zi, *Incompressible limit of the compressible nematic liquid crystal flow*, J. Funct. Anal., 264(7):1711–1756, 2013.
[16] K. Schade and Y. Shibata, *On strong dynamics of compressible nematic liquid crystals*, SIAM J. Math. Anal., 47(5):3963–3992, 2015.
[17] Q. Tao, J.C. Gao, and Z.A. Yao, *Global strong solutions of the compressible nematic liquid crystal flow with the cylinder symmetry*, Commun. Math. Sci., 13(8):2065–2096, 2015.
[18] Q. Tao and Y. Yang, *Exponential stability for the compressible nematic liquid crystal flow with large initial data*, Commun. Pure Appl. Anal., 15(5):1661–1669, 2016.
[19] Q. Tao and Y. Yang, *Time periodic solution to the compressible nematic liquid crystal flows in periodic domain*, DOI:10.1002/mma.4592, 14 September 2017.
[20] P. Constantin, *Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations*, Comm. Math. Phys., 104(2):311–326, 1986.
[21] P. Constantin and C. Foias, *Navier-Stokes Equation*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, 1988.
[22] T. Kato, *Nonstationary flows of viscous and ideal fluids in* $\mathbb{R}^3$, J. Funct. Anal., 9(3):296–305, 1972.
[23] N. Masmoudi, *Remarks about the inviscid limit of the Navier-Stokes system*, Comm. Math. Phys., 270(3):777–788, 2007.
[24] H. Beirao da Veiga, *Vorticity and regularity for flows under the Navier boundary condition*, Commun. Pure Appl. Anal., 5(4):907–918, 2006.
[25] H. Beirao da Veiga and F. Crispo, *Concerning the* $W^{k,p}$*-inviscid limit for 3-d flows under a slip boundary condition*, J. Math. Fluid Mech., 13(1):117–135, 2011.
[26] N. Masmoudi and F. Rousset, *Uniform Regularity for the Navier-Stokes equation with Navier boundary condition*, Arch. Ration. Mech. Anal., 203(2):529–575, 2012.
[27] Y.L. Xiao and Z.P. Xin, *On the inviscid limit of the 3D Navier-Stokes equations with generalized Navier-slip boundary conditions*, Comm. Math. Stat., 1(3):259–279, 2013.
[28] J.C. Gao, B.L. Guo, and X.Y. Xi, *Uniform Regularity and Vanishing Viscosity Limit for the Nematic Liquid Crystal Flows in Three Dimensional Domain*, preprint, arXiv:1606.03914, 2016.
[29] M. Paddick, *The strong inviscid limit of the isentropic compressible NavierCStokes equations with Navier boundary conditions*, preprint, arXiv:1410.2811v1.
[30] Z.P. Xin and T. Yanagisawa, *Zero-viscosity limit of the linearized NavierCStokes equations for a compressible viscous fluid in the half-plane*, Comm. Pure Appl. Math., 52(4):479–541, 1999.
[31] Y.G. Wang and M. Williams, *The inviscid limit and stability of characteristic boundary layers for the compressible Navier-Stokes equations with Navier-friction boundary conditions*, Ann. Inst. Fourier, 62(6):2257–2314, 2013.
[32] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, AMS, Providence, RI, 343, 1984.
[33] O. Gues, *Probleme mixte hyperbolique quasi-lineaire caracteristique*, Comm. Part. Diff. Eqs., 15(5):595–645, 1990.
[34] Y.G. Cho, H.J. Choe, and H. Kim, *Unique solvability of the initial boundary value problems for compressible viscous fluids*, J. Math. Pures Appl., 83(2):243–275, 2004.
[35] J.C. Gao, Q. Tao, and Z.A. Yao, *Strong solutions to the density-dependent incompressible nematic liquid crystal flows*, J. Diff. Eqs., 260(4):3691–3748, 2016.
[36] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl., 146(1):65–96, 1986.