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AGGREGATION EQUATIONS WITH FRACTIONAL DIFFUSION: PREVENTING CONCENTRATION BY MIXING*

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Abstract. We investigate a class of aggregation-diffusion equations with strongly singular kernels and weak (fractional) dissipation in the presence of an incompressible flow. Without the flow the equations are supercritical in the sense that the tendency to concentrate dominates the strength of diffusion and solutions emanating from sufficiently localised initial data may explode in finite time. The main purpose of this paper is to show that under suitable spectral conditions on the flow, which guarantee good mixing properties, for any regular initial datum the solution to the corresponding advection-aggregation-diffusion equation is global if the prescribed flow is sufficiently fast. This paper can be seen as a partial extension of [Kiselev & Xu, Arch. Rat. Mech. Anal., 222(2):1077-1112, 2016], and our arguments show in particular that the suppression mechanism for the classical 2D parabolic-elliptic Keller–Segel model devised by Kiselev and Xu also applies to the fractional Keller–Segel model (where $\triangle$ is replaced by $-(-\triangle)^{\frac{\gamma}{2}}$) requiring only that $\gamma>1$. In addition, we remove the restriction to dimension $d<4$. As a by-product, a characterisation of the class of relaxation enhancing flows on the d-torus is extended to the case of fractional dissipation.

Keywords. preventing blowup; Keller–Segel; transport-diffusion; mixing; fractional dissipation.

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1. Introduction

We are interested in the question of how the presence of a (prescribed, steady) incompressible flow may alter the long-time dynamics of solutions of a class of aggregation-diffusion equations with strongly singular kernels. More specifically, our starting point is the evolutionary problem

$$\partial_t\rho=-\Lambda^\gamma\rho+\nabla\cdot(\rho\nabla K*\rho) \text{ in } (0,\infty)\times\mathbb{T}^d \tag{1.1}$$

with the initial condition $\rho(0)=\rho_0$ for some sufficiently regular density $\rho_0\geq 0$. Here Λ denotes the half-Laplacian (see definition (1.7)), $\gamma>1$, $\mathbb{T}^d$ is the flat d-torus, henceforth identified with $[-\frac{1}{2},\frac{1}{2})^d$ subject to periodic boundary conditions, and we assume that the periodic convolution kernel K has the following properties:

- Smoothness away from the origin.
- $\nabla K(x)\sim\frac{x}{|x|^{2+a}}$ near $x=0$ for some $a\geq 0$. This is the case if $-K\sim|x|^{-a}$ in some neighbourhood of the origin (with the understanding $K\sim\log|x|$ if $a=0$). For simplicity of presentation, we will assume $\nabla K(x)=\frac{x}{|x|^{2+a}}$ on $B_\varepsilon(0)$ for some $0<\varepsilon\ll 1$.

We note that the behaviour of the kernel near its singularity at the origin (including its sign) determines to a large extent the interaction modelled by the nonlinear term in equation (1.1). Our choice of the sign guarantees a predominantly attractive interaction

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and is essential for the construction of exploding solutions. Next, notice that for $a=d-2$ the kernel K has the same singularity at the origin as the fundamental solution of the Laplacian on $\mathbb{T}^d$ so that, informally speaking, in this case equation (1.1) becomes a version of the fractional (or classical if $\gamma=2$) parabolic-elliptic Keller–Segel model, which is one of the fundamental models for aggregation in several physical and biological systems (in particular for chemotaxis), see e.g. [9, 19–21]. In this sense our model is a generalisation of Keller–Segel and, indeed, virtually the same analysis as in this paper can be used to give a direct derivation of the corresponding results for Keller–Segel. Let us also point out that for $a=0$ we essentially recover a version of the so-called *modified* Keller–Segel model [10, 12].

The motive to allow for fractional diffusion in our model is two-fold: besides experimental evidence suggesting that in certain applications the repulsive forces may be better described by fractional rather than standard diffusion (see e.g. [1, 4, 18] and references therein), another reason to consider the more general case of fractional diffusion is the quest for a better understanding of how the equation's dynamics depends on the strength of diffusion. The mathematical literature on models for aggregation with fractional dissipation is large, see [4, 7, 8, 16, 25–27] for a small selection.

The main reason for our choice of periodic boundary conditions lies in the fact that in this setting chaotic dynamics generated by a time-independent flow are possible already in the physically particularly relevant case of two spatial dimensions (see Section 4 and Appendix D for more details).

In order to describe our results, we first need to introduce some fundamental properties of equation (1.1).

Conservation of mean. First note that formally for any solution to equation (1.1) the mean value is conserved in time:

$$\int_{\mathbb{T}^d} \rho(t,x)dx = \int_{\mathbb{T}^d} \rho_0(x)dx.$$

In fact, all evolution equations which we shall consider here enjoy this property, and in this context we will abbreviate $\bar{\rho} = \int_{\mathbb{T}^d} \rho_0$. In applications ρ usually describes a density, and for the sake of exposition, we will henceforth assume $\rho_0 \geq 0$, a property, which by the maximum principle (see e.g. [27] for a proof in a related setting) is preserved in time for any sufficiently regular solution to equation (1.1). It will, however, be obvious that (apart from the blowup proof in Appendix A) our results remain valid without the assumption of positivity.

Scaling. Let us for the moment replace $\mathbb{T}^d$ by $\mathbb{R}^d$ and consider the scaling properties of the equation obtained by substituting in equation (1.1) the kernel ∇K for its homogeneous approximation near the origin, i.e. $\frac{x}{|x|^{2+a}}$. This equation is invariant under the scaling

$$\rho_\lambda(t,x) = \lambda^{\gamma-2+d-a}\rho(\lambda^\gamma t, \lambda x), \;\; \lambda > 0. \tag{1.2}$$

Moreover, by preservation of mean, non-negative solutions have conserved L^1_x-norm. Thus, the exponent $\gamma=\gamma_c$ which leaves the L^1_x-norm of the rescaled solutions ρ_λ invariant in the sense that $\|\rho_\lambda(t,\cdot)\|_{L^1} = \|\rho(\lambda^\gamma t,\cdot)\|_{L^1}$ plays a distinguished role and is generally referred to as the L^1-*critical* exponent. From the scaling (1.2) we obtain

$$\gamma_c = 2 + a.$$

In this degenerate case the conservation of mean property does not provide any control of the scaling parameter λ and, in principle, it would allow for self-similar blowup. For

$\gamma < \gamma_c$ (resp. $\gamma > \gamma_c$) equation (1.1) is called L^1-*supercritical* (resp. L^1-*subcritical*). In the case $a = 0$ and $\gamma \in (1,2]$ (which implies $\gamma \leq \gamma_c$) it is not difficult to produce[1] solutions exploding in finite time for suitably localized smooth initial data of large mass using a virial type argument similar to the strategy in [24, Appendix I]. This reflects the above scaling heuristics: in the L^1-supercritical (and critical) regime, diffusion is too weak to be generically able to compete with the aggregation effects induced by the quadratic drift term in equation (1.1) with velocity $-\nabla K * \rho$. Conceptually, one would therefore also expect the existence of blowup solutions for more singular kernels ($a > 0$) as long as diffusion is not too strong (at most critical). However, the standard moment method does not seem to be applicable directly in this case as for $a > 0$ the arising "perturbation" terms

$$\int\int \frac{x-y}{|x-y|^{a+2}} \cdot \Psi(x,y)\rho(x)\rho(y)\,dydx \tag{1.3}$$

(with Ψ being some smooth cut-off which in general does not vanish along the diagonal) can no longer be controlled only in terms of the (conserved) mass $\int \rho$. The emergence of terms of the form (1.3) in the moment method seems to be unavoidable on $\mathbb{T}^d$ even if restricting to more specific kernels K – the reason being that polynomials (such as $|x|^2$) are not periodic.

Background and results. One of our main goals (cf. Theorem 4.2) is to show that there exists an exponent $\gamma_0 < \gamma_c$ such that (the expected) blowup can be suppressed through the action of a suitable fast flow whenever $\gamma \in (\gamma_0, \gamma_c]$. This question is motivated by the work of Kiselev and Xu [24] where the authors prove a similar statement for the two- and three-dimensional parabolic-elliptic Keller–Segel model. We would like to stress that to the authors' knowledge in the arguably more realistic setting of a coupled chemotaxis-fluid system, where the flow is not explicitly prescribed but governed by the laws of fluid dynamics, there is no result in the literature proving the existence of global-in-time solutions for a model for which it is known that without the flow there exist solutions exploding in finite time. The class of flows we focus on is a generalisation of weakly mixing flows in the ergodic sense, and a natural adaptation of the class of *relaxation enhancing* flows considered in [24] to the case of fractional dissipation. The notion of relaxation enhancing flows was introduced in the work [14] by Constantin, Kiselev, Ryzhik and Zlatoš, which constitutes a core reference for our approach. For more background on fluid mixing and its possibly regularising effects in the context of reaction-diffusion equations, we refer to [24] and references therein. We conclude by pointing out another interesting work [3], which demonstrates that chemotactic singularity formation can also be prevented by mixing due to a fast shear flow. The underlying mechanism is, however, rather different from the one considered here and is not able to suppress more than one dimension (of the Keller–Segel model which is L^1-critical for $d = 2$ and L^1-supercritical in higher dimensions). Our second main result (Theorem 4.3) will show that the suppression mechanism by ergodic type mixing has a much weaker dimensional dependence in the sense that it applies to the Keller–Segel model in arbitrarily high dimension.

Let us remark that, heuristically, one would expect that for $d = 2$ the shear flow approach in [3] can be extended to the case of fractional dissipation. The same applies to the second type of flow considered in [24], a two-dimensional almost optimal mixing flow. In contrast to the previously discussed flows, the optimal mixing-type flow is

[1] A proof is given in Appendix A.

active in the sense that its construction depends on the solution of the partial differential equation itself. The interested reader may wish to investigate either of these problems.

We finish this section by introducing two technical assumptions on the kernel K needed in large parts of our analysis, commenting on local properties of solutions to equation (1.1), fixing basic notations and indicating the organisation of the rest of this text.

Further assumptions on K. For fixed $\varepsilon > 0$ and $p_0 > 1$ we note

$$\int_{B_\varepsilon(0)} \frac{1}{|x|^{(1+a)p_0}}\, dx = c_d \int_0^\varepsilon r^{d-1-p_0(1+a)}\, dr,$$

which shows that $\nabla K \in L^{p_0}(\mathbb{T}^d)$ if and only if

$$p_0 < \frac{d}{1+a}. \tag{1.4}$$

In the following we will therefore assume that the parameters $d \geq 2$ (integer) and $a \geq 0$ are such that $\frac{d}{1+a} > 1$, so that in particular there always exists $p_0 > 1$ satisfying inequality (1.4).

Moreover, since we focus on L^2-methods in our first main result (cf. Footnote 2), we will assume for this part that $2 + a - \frac{d}{2} < 2$, or equivalently,

$$\frac{d}{2a} > 1. \tag{1.5}$$

This condition ensures that the lower bound $\gamma_0 = 2 + a - \frac{d}{2}$ on γ, which makes the L^2-norm (heuristically) a subcritical quantity for equation (1.1), is less than 2.

LWP and smoothing. If $\gamma > 1$, problem (1.1) is locally well-posed in $H^s(\mathbb{T}^d)$ for sufficiently large $s \geq s_0(d)$. More specifically, if[2]

$$\gamma > \max\left\{2 + a - d\left(1 - \frac{1}{p}\right), 1\right\}, \tag{1.6}$$

then local existence and uniqueness already hold in $L^p(\mathbb{T}^d)$. This can be shown using semigroup estimates for $-\Lambda^\gamma$ and a fixed point argument similar to [23] and [7].

Throughout these notes, for simplicity of exposition, we will formulate auxiliary results under the assumption of a smooth initial datum ρ_0 (resp. a smooth solution). This assumption can be removed by standard arguments exploiting the fact that, as soon as condition (1.6) holds true, the smoothing effect induced by $-\Lambda^\gamma$ is strong enough to instantaneously regularise the (local) solution emanating from an L^p datum.

[2]Notice that for $\gamma = 2 + a - d\left(1 - \frac{1}{p}\right)$ the scaling (1.2) preserves the L^p_x-norm in the sense that $\|\rho_\lambda(t,\cdot)\|_{L^p} = \|\rho(\lambda^\gamma t,\cdot)\|_{L^p}$ so that the required strength of diffusion for making the L^p norm heuristically subcritical decreases with increasing p. Thus, one may expect to obtain improved lower bounds on γ by working in L^p spaces of higher integrability. In Theorem 4.3 we will illustrate that in some sense this is indeed the case using the example of the standard Keller–Segel model. In two spatial dimensions, for Keller–Segel type singularities ($a = d - 2$) L^2 methods work for any $\gamma > 1$, which is why we first focus on the case $p = 2$. See also the discussion in Section 4 (page 349) for difficulties arising in L^p.

Notations. For smooth periodic functions $f(x)=\sum_{k\in\mathbb{Z}^d}\hat{f}(k)e^{2\pi i x\cdot k}$ and $\sigma\in\mathbb{R}$ we define

$$\|f\|_{\dot{H}^\sigma}^2=\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|k|^{2\sigma}|\hat{f}(k)|^2$$

and

$$\|f\|_{H^\sigma}^2=\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{2\sigma}|\hat{f}(k)|^2,$$

where $\langle k\rangle=(1+|k|^2)^{\frac{1}{2}}$. The space $H^\sigma(\mathbb{T}^d)$ is defined as the completion of $C^\infty(\mathbb{T}^d)$ under the norm $\|\cdot\|_{H^\sigma}$. We next define the fractional derivative Λ^σ via

$$\Lambda^\sigma f(x)=\sum_{k\in\mathbb{Z}^d\setminus\{0\}}|k|^\sigma\hat{f}(k)e^{2\pi i k\cdot x}. \tag{1.7}$$

For sufficiently regular periodic functions f,g the following identities are immediate

$$\|f\|_{\dot{H}^\sigma}=\|\Lambda^\sigma f\|_{L^2},$$
$$\Lambda^\sigma(f*g)=f*\Lambda^\sigma g.$$

Moreover,

$$\int_{\mathbb{T}^d}f\Lambda^\sigma g=\int_{\mathbb{T}^d}(\Lambda^\sigma f)g,$$
$$\Lambda^{\sigma_1}\Lambda^{\sigma_2}f=\Lambda^{\sigma_1+\sigma_2}f.$$

Constants C or $C(\dots)$ may change from line to line and unless explicitly indicated otherwise they are continuous and non-decreasing functions of their (non-negative) arguments. Their possible dependence on the parameters γ,a and d will usually not be indicated explicitly. For quantities $A,B\geq 0$ the notation $A\lesssim B$ means that there exists a constant $0<C<\infty$ (which may depend on fixed parameters) such that $A\leq CB$. Furthermore, $A\sim B$ stands for $A\lesssim B$ and $B\lesssim A$. If it is appropriate to indicate the dependence of the hidden constant in " $\lesssim$ " on certain parameters $p_1,\dots$, this will be done through $\lesssim_{p_1,\dots}$.

Outline. This text is structured as follows. In the next section we recall several well-known estimates needed for the subsequent analysis. Section 3 is devoted to the derivation of a priori estimates required for our L^2-based suppression result. In Section 4 we first introduce further concepts in order to determine the flows leading to the specific prevention of concentration mechanism which we here focus on. Then we turn to the proof of our main results.
In Appendix A the existence of exploding solutions to equation (1.1) is proved in the case $a=0$, $\gamma\in(1,2]$. The appendix to this text further contains two extensions of results in the literature which we require for our main argument in Section 4 (see Appendices B and C). Finally, in Appendix D we construct examples of incompressible flows, which provide a justification for our Definition 4.1 of γ-relaxation enhancing flows.

2. Auxiliary tools

Here we collect some standard inequalities, which will be used throughout the text.

LEMMA 2.1 (Interpolation). *Let $\sigma,\mu>0$. Then for all $f\in C^\infty(\mathbb{T}^d)$*

$$\|f\|_{\dot{H}^\sigma}\lesssim\|f\|_{L^2}^{1-b}\|f\|_{\dot{H}^{\sigma+\mu}}^b, \tag{2.1}$$

where $b = \frac{\sigma}{\sigma+\mu}$.

Proof. We compute using Plancherel's identity and Hölder inequality with $p = \frac{1}{1-b}$

$$\begin{aligned}\|f\|_{\dot{H}^\sigma}^2 &= \int |\Lambda^\sigma f|^2\,dx \approx \sum_k |k|^{2\sigma} |\hat{f}(k)|^2 = \sum_k |\hat{f}(k)|^{2(1-b)} |k|^{2\sigma} |\hat{f}(k)|^{2b} \\ &\le \left(\sum_k |\hat{f}(k)|^2\right)^{(1-b)} \left(\sum_k |k|^{2(\sigma+\mu)} |\hat{f}(k)|^2\right)^b,\end{aligned}$$

where in the last step we used $\frac{\sigma}{b} = \sigma + \mu$. ∎

The following result is an immediate consequence of Plancherel's identity and Cauchy–Schwarz.

Lemma 2.2 (Duality). *Let* $f, g \in C^\infty(\mathbb{T}^d)$ *satisfy* $\hat{f}(0)\hat{g}(0) = 0$. *Then for* $\sigma \in \mathbb{R}$

$$\int_{\mathbb{T}^d} f(x) g(x)\,dx \le \|f\|_{\dot{H}^\sigma} \|g\|_{\dot{H}^{-\sigma}}.$$

In our analysis we will frequently use the following product rule estimate (also known as Kato–Ponce inequality) combined with the subsequently stated Sobolev embedding for fractional derivatives.

Lemma 2.3 (Fractional product rule estimate). *Let* $\sigma \ge 0$ *be given. Then for all* $p_i, q_i \in (2,\infty)$ *with* $\frac{1}{2} = \frac{1}{p_i} + \frac{1}{q_i}$, $i = 1,2$ *the bound*

$$\|\Lambda^\sigma (fg)\|_{L^2} \lesssim \|\Lambda^\sigma f\|_{L^{p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{p_2}} \|\Lambda^\sigma g\|_{L^{q_2}} \tag{2.2}$$

holds true.

Proof. For the whole space this is a special case of e.g. [17]. In the case of the torus, we refer to [13] and references therein. ∎

Lemma 2.4 (Homogeneous Sobolev embedding). *Assume* $0 < \frac{\sigma}{d} < \frac{1}{p} < 1$ *and define* $q \in (p,\infty)$ *via*

$$\frac{\sigma}{d} = \frac{1}{p} - \frac{1}{q}.$$

Then for all $f \in C^\infty(\mathbb{T}^d)$ *with zero mean*

$$\|f\|_{L^q(\mathbb{T}^d)} \lesssim \|\Lambda^\sigma f\|_{L^p(\mathbb{T}^d)}. \tag{2.3}$$

Proof. See [5] for a direct Fourier analytic proof on the torus. ∎

3. L^2 a priori estimates

In this section we will establish a priori estimates for the evolution equation

$$\begin{aligned}\partial_t \rho + u \cdot \nabla \rho &= -\Lambda^\gamma \rho + \nabla \cdot (\rho \nabla K * \rho) \ \text{ in } \ (0,\infty) \times \mathbb{T}^d, \\ \rho(0) &= \rho_0,\end{aligned} \tag{3.1}$$

where $u = u(x)$ is a given smooth divergence-free vector field and ρ_0 a non-negative initial datum. Clearly, the conservation of mean property, preservation of positivity, LWP and the smoothing effects for the local solution mentioned in the introduction remain valid for problem (3.1). Strictly speaking, a thorough elaboration of the local well-posedness theory in L^2 and the smoothing properties would imply some (albeit possibly weaker) version of the results derived in this section. We nevertheless include this part, mainly because the derived lemmata will be used explicitly in and will facilitate the presentation of the proof of our first "blowup prevention theorem" (Theorem 4.2).

To simplify the exposition, we will prove the following results only in the (more interesting) cases $\gamma \le 2$ and $2+a-\frac{d}{2} \ge 1$. At the end of the proofs we sketch the modifications necessary to treat the remaining cases.

3.1. A blowup criterion. Here we illustrate by a formal derivation that a form of the standard blowup/continuation criteria for several classical aggregation equations (including the Keller–Segel model[3]) is also valid for our problem.

THEOREM 3.1 (L^2-control suffices). *Assume that $\gamma > \max\{2+a-\frac{d}{2},1\}$ and let[4] $\rho_0 \in C^\infty(\mathbb{T}^d)$. Then the following criterion holds: either the local solution ρ to problem (3.1) extends to a global smooth solution or there exists $T^* \in (0,\infty)$ and $1 \le r < \infty$ such that*

$$\int_0^t \|\rho(\tau)-\bar\rho\|_{L^2}^r \, d\tau \to \infty \ \text{ as } t \nearrow T^*.$$

Proof. **(Proof of Theorem 3.1 for $\gamma \le 2$, $2+a-\frac{d}{2} \ge 1$.)** It suffices to derive a priori bounds on higher order derivatives in terms of L^2, the rest of the argument then follows as in [23, Appendix I]. Let $s \ge s_0(d)$ be a sufficiently large integer. Then we estimate as in the proof of [24, Theorem 2.1]

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{\dot H^s}^2 \le -\|\rho\|_{\dot H^{s+\frac{\gamma}{2}}}^2 + C\|u\|_{C^s}\|\rho\|_{\dot H^s}^2 + \left|\int \nabla\cdot(\rho\nabla K*\rho)(-\triangle)^s\rho\right|. \tag{3.2}$$

The last term on the RHS is estimated using Lemmata 2.2 and 2.3

$$\begin{aligned}\left|\int \Lambda^s(\rho\nabla K*\rho)\cdot\nabla\Lambda^s\rho\,dx\right| &\lesssim \|\Lambda^s(\rho\nabla K*\rho)\|_{\dot H^{1-\frac{\gamma}{2}}}\|\nabla\Lambda^s\rho\|_{\dot H^{-1+\frac{\gamma}{2}}}\\ &\lesssim \left(\|\Lambda^{s+1-\frac{\gamma}{2}}\rho\|_{L^{p_1}}\|\nabla K*\rho\|_{L^{q_1}} + \|\rho\|_{L^{p_2}}\|\nabla K*\Lambda^{s+1-\frac{\gamma}{2}}\rho\|_{L^{q_2}}\right)\|\rho\|_{\dot H^{s+\frac{\gamma}{2}}}.\end{aligned} \tag{3.3}$$

This is valid for $p_i,q_i \in (2,\infty)$ whenever $\frac{1}{p_i}+\frac{1}{q_i}=\frac{1}{2}$ for $i=1,2$. In the following we estimate the terms on the RHS of estimate (3.3). We first choose $p_1 = 2+\varepsilon$ for $\varepsilon>0$ sufficiently small such that for $\sigma_1 = \left(\frac{1}{2}-\frac{1}{p_1}\right)d$ we have $b_1 := \frac{\sigma_1+s+1-\frac{\gamma}{2}}{s+\frac{\gamma}{2}} < 1$. This is possible since $\gamma>1$. Thus, using Lemmata 2.4 and 2.1, we find

$$\|\Lambda^{s+1-\frac{\gamma}{2}}\rho\|_{L^{p_1}} \le C\|\rho\|_{\dot H^{\sigma_1+s+1-\frac{\gamma}{2}}} \le C\|\rho-\bar\rho\|_{L^2}^{1-b_1}\|\rho\|_{\dot H^{s+\frac{\gamma}{2}}}^{b_1}.$$

Next, we apply Young's convolution inequality with suitable exponents $p_0,q_3 \in (1,\infty)$ satisfying $1+\frac{1}{q_1} = \frac{1}{p_0}+\frac{1}{q_3}$. More precisely, we choose $p_0 = \frac{d}{1+a}(1-\delta)$ for $\delta>0$ small and note that if $2+a-\frac{d}{2} \ge 1$, then $\frac{d}{1+a} \le 2$, thus implying $p_0<2$. Hence, for $\varepsilon>0$ sufficiently

[3]Concrete counterparts of Theorem 3.1 and Lemma 3.1 in the case of the parabolic-elliptic Keller–Segel model can be found in [24, Theorem 2.1 & Proposition 3.1].

[4]Recall that thanks to the assumed lower bound on γ, by the smoothing properties of equation (3.1), the assumption of smooth initial data can be removed, and the statement, mutatis mutandis, is valid for L^2 data.

small (which enforces q_1 to be sufficiently large) we have $q_3 \geq 2$. And clearly, for $s \geq s_0(d)$ sufficiently large we have $b_2 := \frac{(\frac{1}{2}-\frac{1}{q_3})d}{s+\frac{\gamma}{2}} < 1$. Thus,

$$\begin{aligned}\|\nabla K * \rho\|_{L^{q_1}} = \|\nabla K * (\rho - \bar\rho)\|_{L^{q_1}} &\leq \|\nabla K\|_{L^{p_0}} \|\rho - \bar\rho\|_{L^{q_3}} \\ &\leq C \|\nabla K\|_{L^{p_0}} \|\rho - \bar\rho\|_{L^2}^{1-b_2} \|\rho\|_{\dot H^{s+\frac{\gamma}{2}}}^{b_2},\end{aligned}$$

where the first identity holds since $\partial_{x_i} K$ has zero mean for all i. We note that

$$\begin{aligned} b_1 + b_2 &= \frac{\left(\frac{1}{2} - \frac{1}{p_1}\right) d + s + 1 - \frac{\gamma}{2} + (\frac{1}{2} - \frac{1}{q_3}) d}{s + \frac{\gamma}{2}} \\ &= \frac{\left(\frac{1}{p_0} - \frac{1}{2}\right) d + s + 1 - \frac{\gamma}{2}}{s + \frac{\gamma}{2}} \\ &= \frac{s + \frac{\gamma}{2} - \gamma - \frac{d}{2} + \frac{d}{p_0} + 1}{s + \frac{\gamma}{2}}. \end{aligned} \tag{3.4}$$

Since $\gamma > 2 + a - \frac{d}{2}$, the term $-\gamma - \frac{d}{2} + 1 + \frac{d}{p_0}$ is strictly negative if $\delta > 0$ is chosen sufficiently small. Then the strict inequality $b_1 + b_2 < 1$ holds.

The terms $\|\rho\|_{L^{p_2}}$ and $\|\nabla K * \Lambda^{s+1-\frac{\gamma}{2}} \rho\|_{L^{q_2}}$ on the RHS of estimate (3.3) are treated similarly and yield bounds with only minor differences (see the proof of Lemma 3.1).

Inserting the derived bounds into estimate (3.2), applying Young's inequality twice – once with the exponent $\frac{2}{b_1+b_2+1}$ (>1) applied to the factor involving the highest power of $\|\rho\|_{\dot H^{s+\frac{\gamma}{2}}}$ – we obtain, after absorption, a bound of the form

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_{\dot H^s}^2 \leq -\frac{1}{2}\|\rho\|_{\dot H^{s+\frac{\gamma}{2}}}^2 + C\|u\|_{C^s}\|\rho\|_{\dot H^s}^2 + C\|\rho - \bar\rho\|_{L^2}^r + C(\bar\rho)$$

for some possibly large $r \in (1,\infty)$, $r = r(a,d,\gamma,s)$. From this estimate the conclusion can easily be deduced.

Let us briefly comment on how to adapt the proof in order to obtain the result in the remaining cases where $2 + a - \frac{d}{2} < 1$ or $\gamma > 2$. If $2 + a - \frac{d}{2} < 1$ and $\gamma \leq 2$ the main difference lies in the fact that $q_3 < 2$ (using the same notation as in the above proof), and hence the estimate of the term $\|\nabla K * \rho\|_{L^{q_1}}$ simplifies to

$$\|\nabla K * \rho\|_{L^{q_1}} \leq \|\nabla K\|_{L^{p_0}} \|\rho - \bar\rho\|_{L^{q_3}} \leq \|\nabla K\|_{L^{p_0}} \|\rho - \bar\rho\|_{L^2}.$$

In consequence, when estimating the RHS of estimate (3.3), the factor $\|\rho\|_{\dot H^{s+\frac{\gamma}{2}}}$ appears with a power of $1 + b_1$ (instead of $1 + b_1 + b_2$). Since $1 + b_1 < 2$, one then argues as before.

In the case $\gamma > 2$ first note that assumption (1.5) guarantees $2 > 2 + a - \frac{d}{2}$. Next note that

$$\|\rho\|_{\dot H^r} \leq \|\rho\|_{\dot H^{r'}}$$

whenever $r' \geq r$. Therefore the exponent γ can be replaced by 2 in all estimates, which reduces the problem to the previous cases. ∎

3.2. Local control. We now prove that solutions are locally controlled in $L^2(\mathbb{T}^d)$ for some time which only depends on the L^2-distance of the solution to the mean, the mean value and model parameters.

LEMMA 3.1 (Local L^2-control). *Suppose* $\gamma > \max\{2+a-\frac{d}{2},1\}$ *and let* $\rho \geq 0$ *be a smooth (local) solution to problem* (3.1). *Assume that* $\|\rho(t_0)-\bar{\rho}\|_{L^2} = B > 0$ *for some* $t_0 \geq 0$. *Then*

$$\|\rho(t_0+\tau)-\bar{\rho}\|_{L^2} \leq 2B \text{ for all } 0 \leq \tau \leq \tau_0,$$

where

$$\tau_0 = C_1(\|\nabla K\|_{L^{p_0}})^{-1} \min\left\{B^{-r_1}, \bar{\rho}^{-r_2}\right\} > 0 \tag{3.5}$$

for some[5] *sufficiently large* $1 < p_0 < \frac{d}{1+a}$, *a non-decreasing function* $C_1(\dots) > 0$ *and positive (possibly large) constants* $r_i > 0, i = 1,2$, *which only depend on* γ, d, a *and the choice of* p_0.

Proof. **(Proof of Lemma 3.1 for $\gamma \leq 2$, $2+a-\frac{d}{2} \geq 1$.)** By multiplying equation (3.1) with $\rho-\bar{\rho}$ and integrating in space, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho-\bar{\rho}\|_{L^2}^2 &= -\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^2 - \int \rho \nabla K * \rho \cdot \nabla(\rho-\bar{\rho}) dx \\ &\leq -\|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^2 + \|\Lambda^{1-\frac{\gamma}{2}}(\rho \nabla K * \rho)\|_{L^2} \|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}.\end{aligned} \tag{3.6}$$

Here we used the incompressibility of the flow. By Lemma 2.3, for $p_i, q_i \in (2,\infty)$ with

$$p_i^{-1} + q_i^{-1} = 2^{-1}, \ i = 1,2, \tag{3.7}$$

we have

$$\|\Lambda^{1-\frac{\gamma}{2}}(\rho \nabla K * \rho)\|_{L^2} \leq C\left(\|\Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{p_1}} \|\nabla K * (\rho-\bar{\rho})\|_{L^{q_1}} + \|\rho\|_{L^{p_2}} \|\nabla K * \Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{q_2}}\right), \tag{3.8}$$

which means that the last term on the RHS of inequality (3.6) can be bounded from above by

$$C\left(\|\Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{p_1}} \|\nabla K * (\rho-\bar{\rho})\|_{L^{q_1}} + \|\rho\|_{L^{p_2}} \|\nabla K * \Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{q_2}}\right) \|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}. \tag{3.9}$$

We now claim that thanks to Young's convolution inequality and Gagliardo–Nirenberg–Sobolev estimates (see Lemma 2.4 and 2.1), term (3.9) is controlled by

$$C_{\ddagger}\|\nabla K\|_{L^{p_0}} \|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}(I_1+I_2), \tag{3.10}$$

where $C_{\ddagger}$ is a fixed positive constant (depending only on γ, a and d) and

$$I_1 = \|\rho-\bar{\rho}\|_{L^2}^{2-(b_1+b_2)} \|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^{b_1+b_2}, \tag{3.11}$$

$$I_2 = (\bar{\rho} + \|\rho-\bar{\rho}\|_{L^2}^{1-b_3} \|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^{b_3}) \|\rho-\bar{\rho}\|_{L^2}^{1-b_4} \|\rho\|_{\dot{H}^{\frac{\gamma}{2}}}^{b_4}. \tag{3.12}$$

Here $b_1, b_2 \in [0,1)$ are obtained as in the proof of Theorem 3.1 and satisfy $b_1+b_2 < 1$ (we choose again $p_0 = \frac{d}{a+1}(1-\delta)$ with $\delta = \delta(a,d,\gamma) > 0$ (at least) as small as in Theorem 3.1). The value of b_1+b_2 is precisely given by setting $s=0$ in formula (3.4), i.e.

$$b_1+b_2 = \frac{-\frac{d}{2}+\frac{d}{p_0}+1}{\frac{\gamma}{2}} - 1. \tag{3.13}$$

[5] Recall that hypothesis (1.4) ensures $1 < \frac{d}{a+1}$.

To see how the expression for I_2 and the exponents $b_3, b_4 \in [0,1)$ arise, we proceed similarly to the proof of Theorem 3.1: since $2+a-\frac{d}{2} \geq 1$ (which implies $p_0 < \frac{d}{a+1} \leq 2$), we can choose $p_2 > 2$ sufficiently close to 2 (thus enforcing q_2 defined via equation (3.7) to be arbitrarily large) such that q_4 defined via

$$1+\frac{1}{q_2} = \frac{1}{p_0} + \frac{1}{q_4}$$

satisfies $q_4 \geq 2$. We now apply Young's convolution inequality to the second convolution term in expression (3.9) estimating ∇K in L^{p_0} and use in a subsequent step Lemma 2.4 (twice) for the arising ρ-terms $\|\rho\|_{L^{p_2}}$ and $\|\Lambda^{1-\frac{\gamma}{2}}\rho\|_{L^{q_4}}$ with

$$\sigma_3 = \left(\frac{1}{2} - \frac{1}{p_2}\right) d,$$
$$\sigma_4 = \left(\frac{1}{2} - \frac{1}{q_4}\right) d$$

and then Lemma 2.1 (twice) with

$$b_3 = \frac{\sigma_3}{\gamma/2},$$
$$b_4 = \frac{\sigma_4 + 1 - \gamma/2}{\gamma/2} \tag{3.14}$$

to obtain the I_2-part of expression (3.10). Notice that

$$\begin{aligned} b_3 + b_4 &= \frac{\sigma_3 + \sigma_4 + 1 - \gamma/2}{\gamma/2} \\ &= \frac{(1-(p_2^{-1} + q_4^{-1}))d + 1}{\gamma/2} - 1 \\ &= \frac{(p_0^{-1} - 2^{-1})d + 1}{\gamma/2} - 1 \end{aligned} \tag{3.15}$$

and that the assumption $\gamma > 2 + a - \frac{d}{2}$ implies that for $p_0 < \frac{d}{1+a}$ sufficiently large the strict bound $\frac{(p_0^{-1}-2^{-1})d+1}{\gamma/2} - 1 < 1$ holds true. Hence

$$b_3 + b_4 < 1.$$

(Since $b_i \geq 0$, this justifies in particular the application of Lemma 2.1 above.) Note that comparison of equations (3.13) and (3.15) shows $b_1 + b_2 = b_3 + b_4$.

Abbreviating $b := b_3 + b_4 + 1 < 2$, we thus obtain the bound

$$\frac{1}{2}\frac{d}{dt}\|\rho - \bar\rho\|_{L^2}^2 \leq -\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^2 + C_\ddagger \|\nabla K\|_{L^{p_0}} \left(\|\rho - \bar\rho\|_{L^2}^{3-b} \|\rho\|_{\dot H^{\frac{\gamma}{2}}}^{b} + \bar\rho \|\rho - \bar\rho\|_{L^2}^{1-b_4} \|\rho\|_{\dot H^{\frac{\gamma}{2}}}^{1+b_4} \right). \tag{3.16}$$

For later use, we remark that from estimate (3.8) and the subsequent estimates up to (3.16), we immediately deduce

$$\|\Lambda^{1-\frac{\gamma}{2}}(\rho \nabla K * \rho)\|_{L^2} \leq C_\ddagger \|\nabla K\|_{L^{p_0}} \left(\|\rho - \bar\rho\|_{L^2}^{3-b} \|\rho\|_{\dot H^{\frac{\gamma}{2}}}^{b-1} + \bar\rho \|\rho - \bar\rho\|_{L^2}^{1-b_4} \|\rho\|_{\dot H^{\frac{\gamma}{2}}}^{b_4} \right). \tag{3.17}$$

We now define

$$c_1 = \left(1 - \frac{b}{2}\right)^{-1}(3-b) = 2\left(1 + \frac{1}{2-b}\right)$$

and note that

$$\left(1 - \frac{1+b_4}{2}\right)^{-1}(1-b_4) = 2.$$

Applying a standard absorption argument to estimate (3.16), we then find

$$\frac{d}{dt}\|\rho - \bar\rho\|_{L^2}^2 \le -\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^2 + C_\star(\|\nabla K\|_{L^{p_0}})\left(\|\rho-\bar\rho\|_{L^2}^{c_1} + \bar\rho^{\frac{2}{1-b_4}}\|\rho - \bar\rho\|_{L^2}^2\right). \tag{3.18}$$

Once more for later use, we note that Young's multiplication inequality applied to the RHS of estimate (3.17) yields

$$\|\Lambda^{1-\frac{\gamma}{2}}(\rho\nabla K * \rho)\|_{L^2}^2 \le \frac{1}{2}\|\rho\|_{\dot H^{\frac{\gamma}{2}}}^2 + C_\star(\|\nabla K\|_{L^{p_0}})\left(\|\rho-\bar\rho\|_{L^2}^{c_1} + \bar\rho^{\frac{2}{1-b_4}}\|\rho - \bar\rho\|_{L^2}^2\right), \tag{3.19}$$

with the same constants c_1 and $C_\star$ as in estimate (3.18).

Now note that $c_1 > 2$ and that, by estimate (3.18), the function $f(t) = \|\rho(t) - \bar\rho\|_{L^2}^2$ satisfies

$$f' \le C_0 f^{c_1/2} + C_0\bar\rho^{\frac{2}{1-b_4}} f, \qquad f(t_0) = B^2$$

where $C_0 = C_\star(\|\nabla K\|_{L^{p_0}})$. Comparison with the explicit solution $\tilde f$ to

$$\tilde f' = C_0 \tilde f^{c_1/2} + C_0\bar\rho^{\frac{2}{1-b_4}} \tilde f, \qquad \tilde f(t_0) = B^2,$$

which is given by

$$\tilde f(t_0 + t) = R^{\frac{1}{q}}\exp(C_0 R t) B^2\left(R - B^{2q}[\exp(C_0 R q t) - 1]\right)^{-\frac{1}{q}}$$

with $q = \frac{c_1 - 2}{2}$ and $R = \bar\rho^{\frac{2}{1-b_4}}$, shows that

$$f(t_0+\tau) \le 4B^2, \quad \text{whenever } 0 \le \tau \le \tau_0 := \delta_0 C_0^{-1}\min\left\{\frac{2}{c_1-2}B^{-(c_1-2)}, \bar\rho^{-\frac{2}{1-b_4}}\right\}. \tag{3.20}$$

Here $\delta_0 > 0$ is a universal constant. Thus, the assertion of Lemma 3.1 is obtained by choosing $r_1 = c_1 - 2$ and $r_2 = \frac{2}{1-b_4}$.

The case where $2 + a - \frac{d}{2} < 1$ or $\gamma > 2$ is treated similarly to the sketch at the end of the proof of Theorem 3.1. ∎

4. Prevention of blowup

In the following we assume that our vector field u is relaxation enhancing in a sense analogous to [24, Definition 5.1], but adapted to the fractional diffusion $-\Lambda^\gamma$. In order to give a precise definition, let us recall that a divergence-free Lipschitz vector field u on $\mathbb{T}^d$ gives rise to a flow map $\Phi : \mathbb{R} \times \mathbb{T}^d \to \mathbb{T}^d$, $(t,x) \mapsto \Phi_t(x)$ via

$$\frac{d}{dt}\Phi_t(x) = u(\Phi_t(x)),$$

$$\Phi_0 = \mathrm{Id}_{\mathbb{T}^d},$$

where the transformations Φ_t are measure-preserving bi-Lipschitz mappings. Thus we obtain a one-parameter group of unitary operators $U^t f(x) = f(\Phi_t^{-1}(x))$ on $L^2(\mathbb{T}^d)$.

Definition 4.1. *Let $\gamma \geq 1$. We call a divergence-free Lipschitz vector field $u = u(x)$ γ-relaxation enhancing (γ-RE) if the corresponding unitary operator U^1 does not have any non-constant eigenfunctions in $H^{\frac{\gamma}{2}}(\mathbb{T}^d)$.*

Remark 4.1.

(i) Theorem 4.1 and Remark 4.2 below will explain the term “relaxation enhancing”.

(ii) The notion “relaxation enhancing” was first introduced in [14] in a somewhat more general context. The notion used in [24] corresponds in our definition to 2-RE. Any flow which is weakly mixing in the ergodic sense (so that U^1 does not have any non-constant eigenfunctions in L^2) is also γ-RE for any γ as above. The existence of weakly mixing flows on $\mathbb{T}^d$ for any $d \geq 2$ is classical and can be shown by considering suitable time changes of appropriate irrational translations on $\mathbb{T}^d$ (see [14, Section 6] and references therein). A concrete example for a 2-RE flow which is not weakly mixing can also be found in [14, Section 6].

(iii) In Appendix D we show that for any given $1 \leq \gamma_1 < \gamma_2$ there exists a smooth, incompressible flow on $\mathbb{T}^2$ which is γ_2-RE but not γ_1-RE.

We now consider for a parameter $A \gg 1$ the initial value problem

$$\begin{aligned} \partial_t \rho^A + A u \cdot \nabla \rho^A &= -\Lambda^\gamma \rho^A + \nabla \cdot (\rho^A \nabla K * \rho^A) \ \text{ in } (0,\infty) \times \mathbb{T}^d, \\ \rho^A(0) &= \rho_0, \end{aligned} \tag{4.1}$$

where the kernel K satisfies the conditions described in the introduction (Section 1) and $d \geq 2$. The crucial ingredient in the proof of our first main theorem (Theorem 4.2) is the following result (cf. [14]):

Theorem 4.1 (Enhanced relaxation). *Let $\gamma \geq 1$ and let the divergence-free smooth vector field u be γ-relaxation enhancing. Then for every $\tau > 0$, $\varepsilon > 0$ there exists a positive constant $A_0 = A_0(\tau, \varepsilon)$ such that for any $A \geq A_0$ and for any $\mu_0 \in L^2(\mathbb{T}^d)$ with $\int_{\mathbb{T}^d} \mu_0 = 0$ the solution μ^A to*

$$\begin{aligned} \partial_t \mu^A + A u \cdot \nabla \mu^A &= -\Lambda^\gamma \mu^A \ \ \textit{in } (0,\infty) \times \mathbb{T}^d, \\ \mu^A(0) &= \mu_0 \end{aligned} \tag{4.2}$$

satisfies $\|\mu^A(t)\|_{L^2} \leq \varepsilon \|\mu_0\|_{L^2}$ for all $t \geq \tau$.

Remark 4.2.

(i) As in the proof of the first part of Theorem 1.4 in [14] one can show that the γ-RE property of u is also necessary for the statement in Theorem 4.1 to hold.

(ii) If restricting to initial data in $H^{\frac{1}{2}}$ (instead of general L^2 data), one is still able to obtain enhanced relaxation for $\gamma \in (0,1)$ if the unitary evolution (cf. U^1 in Definition 4.1) does not have any non-constant eigenfunctions in $H^{\frac{\gamma}{2}}$.

(iii) Theorem 4.1 (at least with $\gamma = 2$) remains true when L^2 is replaced by L^p for any $p \in [1,\infty]$, see [14, Theorem 5.5].

In the case $\gamma \geq 2$ Theorem 4.1 is a consequence of the abstract criterion in [14] (combined with Proposition 4.1). We will sketch the extension to arbitrary $\gamma \geq 1$ in Appendix B. In any case, an important ingredient in the proof is the boundedness of the linear transport evolution in $H^{\frac{\gamma}{2}}$ for sufficiently regular vector fields:

PROPOSITION 4.1 (Estimate for transport equation). *Let $v = v(x)$ be a smooth divergence-free*[6] *vector field and assume $\gamma > 0$. Then any sufficiently regular solution η to*

$$\partial_t \eta + v \cdot \nabla \eta = 0 \quad in \ (0,\infty) \times \mathbb{T}^d, \qquad \eta(0) = \eta_0 \tag{4.3}$$

satisfies the bound

$$\|\eta(t)\|_{\dot{H}^{\frac{\gamma}{2}}(\mathbb{T}^d)} \lesssim \exp(C(v)t) \|\eta_0\|_{\dot{H}^{\frac{\gamma}{2}}(\mathbb{T}^d)}, \tag{4.4}$$

where $C(v) \lesssim_{\gamma,d} \|\Lambda^{\gamma+\frac{d}{2}+1} v\|_{L^2}$.

REMARK 4.3. A version of this result for Besov spaces on $\mathbb{R}^d$ can be found in [2]. In Appendix C we will provide a proof of the above result, which is suboptimal in terms of the regularity required for the vector field v.

We are now in a position to turn to our first main result. From now on we let $p_0 = p_0(\gamma, a, d) \in (1, \frac{d}{a+1})$ be an exponent for which both Theorem 3.1 and Lemma 3.1 are valid. Also recall that by Assumption (1.4) we have $\|\nabla K\|_{L^{p_0}} < \infty$. For simplicity, any dependence of constants on γ, a and d will (as before) be omitted.

THEOREM 4.2 (Prevention of blowup for model with fractional dissipation). *Let $\gamma > \max\{2 + a - \frac{d}{2}, 1\}$. Suppose that the divergence-free smooth vector field $u(x)$ is γ-relaxation enhancing. Then for any $\rho_0 \in L^2(\mathbb{T}^d)$ there exists an amplitude $A_0(\|\rho_0 - \bar{\rho}\|_{L^2}, \bar{\rho}, u, \|\nabla K\|_{L^{p_0}})$ such that, whenever $A \geq A_0$, problem (4.1) has a global solution $\rho^A \in C_b([0,\infty), L^2) \cap C^\infty((0,\infty) \times \mathbb{T}^d)$.*

REMARK 4.4. Prevention of blowup in the sense of Theorem 4.2 cannot be expected to hold for a threshold amplitude A_0 independent of the initial datum. This is essentially due to a scaling obstruction: in the L^1 critical or supercritical regime the linear advection term is unable to generically compete with the quadratic drift, see also Appendix A.

The rough idea of the proof of Theorem 4.2 can be described as follows. Recall that aggregation-diffusion equations are generally characterised by two competing forces: the tendency to concentrate due to aggregation versus the tendency to uniformly distribute the initial mass over the spatial domain thanks to diffusion. As long as diffusion dominates, the solution cannot concentrate too much and thus will not blow up. In the delicate case of small diffusion (when the $H^{\frac{\gamma}{2}}$ norm is not large enough compared to L^2) the γ-RE flow – if sufficiently strong – takes care of the low frequencies by quickly stirring the density[7]. This increases spatial gradients, thus enhancing dissipation, and eventually prevents blowup.

Proof. **(Proof of Theorem 4.2 for $\gamma \leq 2$ and $2 + a - \frac{d}{2} \geq 1$.)** Without loss of generality we may assume that ρ_0 is not constant, i.e. $\rho_0 \not\equiv \bar{\rho}$ and $\rho \in C^\infty$ (cf. page 336

[6]The assumption $\nabla \cdot v = 0$ is not necessary for the boundedness of the evolution (4.3) with respect to $\|\cdot\|_{\dot{H}^{\frac{\gamma}{2}}}$, see [2].

[7]Strictly speaking, this mechanism of stirring only fully applies if $\rho^A(t)$ lies in the continuous spectral subspace corresponding to U^1. In the case of a non-trivial component in the L^2-closure of the subspace spanned by all (rough) eigenfunctions the mechanism by which gradients are increased is somewhat more technical. The interested reader is referred to [14, Lemma 3.3].

(LWP and Smoothing)). By Theorem 3.1, it suffices to prove global control in $L^2(\mathbb{T}^d)$. For this purpose we first introduce the following parameters:

- Denote $B := \|\rho_0 - \bar{\rho}\|_{L^2} > 0$.
- Let $p_0 \in \left(1, \frac{d}{a+1}\right)$, $c_1 > 2$, b_4 (defined in equation (3.14)) and $C_\star(\|\nabla K\|_{L^{p_0}})$ be the constants introduced in the proof of Lemma 3.1. We recall that these quantities only depend on γ, a and d. Furthermore denote by $\tau_0 = \tau_0(B, \bar{\rho}, \|\nabla K\|_{L^{p_0}})$ the (possibly small) positive time span (3.5) in Lemma 3.1.
- Define now $\tau_1 = \min\left\{ \frac{1}{16}\left\{4C_\star(\|\nabla K\|_{L^{p_0}})\left((2B)^{c_1-2} + \bar{\rho}^{\frac{2}{1-b_4}}\right)\right\}^{-1}, \tau_0 \right\}$.
- Let $A_0 = A_0(\tau_1)$ be such that for any $A \geq A_0$ and any mean-zero $\mu_0 \in L^2(\mathbb{T}^d)$ the solution $\tilde{\mu}^A$ to equation (4.2) with initial value $\tilde{\mu}^A(0) = \mu_0$ saturates the bound
$$\|\tilde{\mu}^A(\tau_1)\|_{L^2} \leq \frac{1}{8}\|\mu_0\|_{L^2}.$$
The existence of such an A_0 is guaranteed by Theorem 4.1. Obviously, A_0 can be chosen to be non-increasing on $\mathbb{R}^+$ and it will necessarily become unbounded near $\tau_1 = 0$.

Now define $t_0 = \inf\{t > 0 : \|\rho^A(t) - \bar{\rho}\|_{L^2} \geq B\}$. If $t_0 = \infty$, there is nothing to prove. We therefore assume $t_0 < \infty$ so that by continuity $\|\rho^A(t_0) - \bar{\rho}\|_{L^2} = B$. Since $\nabla \cdot (Au) = 0$ the statement of Lemma 3.1 applies to $\rho = \rho^A$, and recalling $\tau_1 \leq \tau_0$, we deduce the bound

$$\|\rho^A(t_0+\tau) - \bar{\rho}\|_{L^2} \leq 2B \qquad \text{for all } \tau \in [0, \tau_1]. \tag{4.5}$$

In the following we will show that the above choice of A_0 implies the bound $\|\rho^A(t_0 + \tau_1) - \bar{\rho}\|_{L^2} \leq B$. The claim then follows by iterating the argument: define $t_1 = \inf\{t > t_0 + \tau_1 : \|\rho^A(t) - \bar{\rho}\|_{L^2} \geq B\}$ and proceed as before with t_0 replaced by t_1 etc. This then results in the global bound $\|\rho^A(t) - \bar{\rho}\|_{L^2} \leq 2B$ for all $t > 0$.

Denote $R(\tau) = \int_{t_0}^{t_0+\tau} \|\rho^A\|^2_{\dot{H}^{\frac{\gamma}{2}}}$. We distinguish the following cases, which reflect the idea described above.

Case I: $R(\tau_1) > B^2$.
Here we apply estimate (3.18) (with ρ replaced by ρ^A), which is possible since Au is divergence-free. Hence on the time interval $[t_0, t_0 + \tau_1]$, we have

$$\begin{aligned}\frac{d}{dt}\|\rho^A - \bar{\rho}\|^2_{L^2} &\leq -\|\rho^A\|^2_{\dot{H}^{\frac{\gamma}{2}}} + C_\star(\|\nabla K\|_{L^{p_0}})\left(\|\rho^A - \bar{\rho}\|^{c_1}_{L^2} + \bar{\rho}^{\frac{2}{1-b_4}}\|\rho^A - \bar{\rho}\|^2_{L^2}\right)\\ &\leq -\|\rho^A\|^2_{\dot{H}^{\frac{\gamma}{2}}} + 4C_\star(\|\nabla K\|_{L^{p_0}})\left((2B)^{c_1-2} + \bar{\rho}^{\frac{2}{1-b_4}}\right)B^2,\end{aligned}$$

where we used the bound (4.5) in the second step. We now integrate in time from t_0 to $t_0 + \tau_1$ to obtain

$$\begin{aligned}\|\rho^A - \bar{\rho}\|^2_{L^2}(t_0+\tau_1) &\leq B^2 - B^2 + \tau_1 \cdot 4C_\star(\|\nabla K\|_{L^{p_0}})\left((2B)^{c_1-2} + \bar{\rho}^{\frac{2}{1-b_4}}\right)B^2\\ &\leq \frac{1}{16}B^2.\end{aligned}$$

Here we used the hypothesis (of Case I) and, in the second step, the choice of τ_1.

Case II: $R(\tau_1) \leq B^2$.
In this case we need to approximate $\rho^A(t_0+t)$ by the solution $\mu^A(t_0+t)$ to equation (4.2)

with datum $\mu^A(t_0)=\rho^A(t_0)$. We estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\rho^A-\mu^A\|_{L^2}^2+\|\rho^A-\mu^A\|_{\dot H^{\frac{\gamma}{2}}}^2&=-\int\rho^A\nabla K*\rho^A\cdot\nabla(\rho^A-\mu^A)\\&\le\frac{1}{2}\|\rho^A\nabla K*\rho^A\|_{\dot H^{1-\frac{\gamma}{2}}}^2+\frac{1}{2}\|\rho^A-\mu^A\|_{\dot H^{\frac{\gamma}{2}}}^2.\end{aligned}$$

Absorption yields

$$\frac{1}{2}\frac{d}{dt}\|\rho^A-\mu^A\|_{L^2}^2+\frac{1}{2}\|\rho^A-\mu^A\|_{\dot H^{\frac{\gamma}{2}}}^2\le\frac{1}{2}\|\rho^A\nabla K*\rho^A\|_{\dot H^{1-\frac{\gamma}{2}}}^2. \tag{4.6}$$

Thanks to estimate (3.19), the RHS of estimate (4.6) is bounded from above by

$$\frac{1}{2}\left\{\frac{1}{2}\|\rho^A\|_{\dot H^{\frac{\gamma}{2}}}^2+C_\star(\|\nabla K\|_{L^{p_0}})\left(\|\rho^A-\bar\rho\|_{L^2}^{c_1}+\bar\rho^{\frac{2}{1-b_4}}\|\rho^A-\bar\rho\|_{L^2}^2\right)\right\}.$$

Combination with the bound (4.5) implies on the time interval $[t_0,t_0+\tau_1]$

$$\frac{d}{dt}\|\rho^A-\mu^A\|_{L^2}^2+\|\rho^A-\mu^A\|_{\dot H^{\frac{\gamma}{2}}}^2\le\frac{1}{2}\|\rho^A\|_{\dot H^{\frac{\gamma}{2}}}^2+4C_\star(\|\nabla K\|_{L^{p_0}})\left((2B)^{c_1-2}+\bar\rho^{\frac{2}{1-b_4}}\right)B^2.$$

We now integrate from t_0 to $t_0+\tau_1$ to conclude using also the hypothesis (of Case II)

$$\begin{aligned}\|\rho^A-\mu^A\|_{L^2}^2(t_0+\tau_1)&\le\frac{1}{2}B^2+\tau_1\cdot 4C_\star(\|\nabla K\|_{L^{p_0}})\left((2B)^{c_1-2}+\bar\rho^{\frac{2}{1-b_4}}\right)B^2\\&\le\frac{1}{2}B^2+\frac{1}{16}B^2\\&=\frac{9}{16}B^2.\end{aligned}$$

In the second step of the last estimate, we used the choice of τ_1.

Note that as $\mu^A(t_0)-\bar\rho=\rho^A(t_0)-\bar\rho$ (whose L^2-norm equals B), by choice of A_0 and since $A\ge A_0$, the bound

$$\|\mu^A-\bar\rho\|_{L^2}(t_0+\tau_1)\le\frac{1}{8}B$$

holds true. We therefore obtain

$$\begin{aligned}\|\rho^A-\bar\rho\|_{L^2}(t_0+\tau_1)&\le\|\rho^A-\mu^A\|_{L^2}(t_0+\tau_1)+\|\mu^A-\bar\rho\|_{L^2}(t_0+\tau_1)\\&\le\frac{7}{8}B.\end{aligned}$$

In any case we have

$$\|\rho^A-\bar\rho\|_{L^2}(t_0+\tau_1)\le\frac{7}{8}B\le B,$$

which completes the proof in the case $\gamma\le 2$ and $2+a-\frac{d}{2}\ge 1$.

To ensure the validity of the assertion in the remaining cases, one needs to make sure that estimates analogous to estimates (3.18) and (3.19) hold true in these cases. This can be verified by following the ideas explained at the end of the proof of Lemma 3.1 and Theorem 3.1. ∎

REMARK 4.5.

(i) Theorem 4.2 can be refined in such a way as to obtain exponential convergence of the solution to the mean as $t\to\infty$. In fact, under the assumptions of Theorem 4.2, it follows that for any $\rho_0\in L^2(\mathbb{T}^d)$ and any $\kappa\in(0,\infty)$ there exists $A_0(\|\rho_0-\bar\rho\|_{L^2},\bar\rho,u,\|\nabla K\|_{L^{p_0}},\kappa)$ such that, whenever $A\geq A_0$, problem (4.1) has a global, regular solution ρ^A which satisfies

$$\|\rho^A(t)-\bar\rho\|_{L^2}\leq C\exp(-\kappa t)\|\rho_0-\bar\rho\|_{L^2}, \tag{4.7}$$

where C is a universal constant (in particular independent of κ).

Let us briefly sketch how this result is obtained by adapting the proof of Theorem 4.2. Given $\kappa\in(0,\infty)$ define $\tau(\kappa)=\frac{-\ln\theta}{\kappa}$, where $\theta=\frac{7}{8}$. Then define $\tau:=\min\{\tau_1,\tau(\kappa)\}$, where τ_1 and the quantities introduced before its definition are the same as in the proof of Theorem 4.2. As threshold amplitude choose $A_0=A_0(\tau)$ satisfying the same identity as $A_0(\tau_1)$ but with the possibly smaller time τ. Now start the iteration at time $t=0$ instead of t_0. By Lemma 3.1 the bound $\|\rho^A(t)-\bar\rho\|_{L^2}\leq 2B$ holds for all $t\in[0,\tau]$. Then, repeating the arguments in the two cases of the proof of Theorem 4.2, we can conclude

$$\|\rho^A(\tau)-\bar\rho\|_{L^2}\leq\theta B.$$

Let us now define $\rho_n=\rho^A(n\tau)$ for $n\in\mathbb{N}$ and $B_n=\|\rho_n-\bar\rho\|_{L^2}$. Then in the n-th iteration step one distinguishes the cases where $R_n:=\int_{n\tau}^{(n+1)\tau}\|\rho^A(t)\|^2_{\dot H^{\frac{\gamma}{2}}}\,dt$ is less than B_n^2, resp. greater than or equal to B_n^2. Since, by definition, τ_0,τ_1 are non-increasing in their argument "B", and since $\theta\in(0,1)$, we can again argue as in the proof of Theorem 4.2 (with B replaced by B_n) and inductively obtain

$$\|\rho_n-\bar\rho\|_{L^2}\leq\theta^nB\leq\exp(-\kappa(n\tau))\|\rho_0-\bar\rho\|_{L^2}.$$

The decay (4.7) is now easily obtained.

(ii) Note that for $d=2$ and $a=0$ the kernel ∇K has the same singularity at the origin as ∇N, where N denotes the two-dimensional Newton kernel. Although on the torus N is not a proper convolution kernel, an analysis almost completely analogous to the one established here, shows that the statement of Theorem 4.2 also applies to the two-dimensional parabolic-elliptic Keller–Segel model with fractional diffusion $-\Lambda^\gamma$ whenever $\gamma>1$. Similarly, for the three-dimensional parabolic-elliptic Keller–Segel model with fractional diffusion, we have blowup prevention whenever $\gamma>\frac{3}{2}$.

Note that for dimension $d\geq 4$ Theorem 4.2 no longer includes the Keller–Segel case as the lower bound $\gamma_0=d/2$ would enforce diffusion to be stronger than classical (more concretely, it is the fact that the assumption $\frac{d}{2(d-2)}>1$ (cf. condition (1.5)) is violated what makes our arguments break down). As alluded to in the introduction, the reason for this failure is the fact that the L^2-norm is no longer subcritical for Keller–Segel in $d\geq 4$.

Scaling suggests that by working in L^p spaces of higher integrability ($p>2$) smaller lower bounds on γ may be achieved, namely

$$\gamma>2+a-d\left(1-\frac{1}{p}\right) \tag{4.8}$$

(as long as γ is large enough so that the nonlinear equation is locally well-posed in a suitable Lebesgue (or Sobolev) space and for data of the corresponding regularity

Theorem 4.1 is valid for this γ. The additional condition $\gamma > 1$, for instance, would ensure these last two properties). For the Keller–Segel type (Newton kernel) singularity inequality (4.8) becomes $\gamma > \frac{d}{p}$. This may lead to the expectation that also in the higher-dimensional Keller–Segel model the mixing mechanism is able to prevent blowup for any $\gamma > 1$ when confining to e.g. $L^\infty(\mathbb{T}^d)$ initial data. However, when trying to prove suppression using L^p- instead of L^2-estimates the following issue arises: following the notation in the proof of Theorem 4.2, it appears that in L^p, $p > 2$, the approximation of ρ^A by μ^A requires an estimate of the form

$$\|\Lambda^{1-\frac{\gamma}{2}} f\|_{L^{p_1}} \lesssim \|\Lambda^{\frac{\gamma}{2}}(|f|^{\frac{p}{2}})\|_{L^2}^{2/p} \tag{4.9}$$

for some $p_1 > 2$. Certainly such an estimate cannot hold unless $\gamma > \left(\frac{1}{p} + \frac{1}{2}\right)^{-1}$, a lower bound which is strictly larger than 1 if $p > 2$. Despite the above scaling heuristics and although the weakly mixing/relaxation enhancing condition on the flow appears to be a very strong hypothesis, it is not obvious to the authors how to extend the approach in such a way that it includes the Keller–Segel model with fractional diffusion of any strength $\gamma > 1$ and in any dimension $d \geq 2$.

In the case $\gamma = 2$, however, estimate (4.9) becomes trivial, and indeed, in this case by working in L^p instead of L^2 the suppression mechanism can be extended as to include in particular the classical Keller–Segel model ($\gamma = 2$) in any dimension $d \geq 2$, which we would like to illustrate in the following. Let us consider the Keller–Segel model – in its precise form for clarity's sake – under the influence of a strong incompressible flow

$$\partial_t \rho^A + Au \cdot \nabla \rho^A = \triangle \rho^A + \nabla \cdot (\rho^A \nabla \triangle^{-1}(\rho^A - \bar{\rho})) \text{ in } (0,\infty) \times \mathbb{T}^d \tag{4.10}$$

with $d \geq 4$. The higher-dimensional Keller–Segel model with standard diffusion (i.e. equation (4.10) with $A = 0$) is $L^{\frac{d}{2}}$-critical and L^1-supercritical (choose $\gamma = 2$, $a = d-2$ in the scaling (1.2)). For $p > \frac{d}{2}$ local well-posedness in L^p and regularity for positive times are well-established in the community (see e.g. [6] for results on bounded domains and [11] for results on the whole space assuming sufficient decay at infinity), and at any (positive) level of mass (= L^1-norm for non-negative solutions) there exist smooth solutions which blow up in finite time [6,7,11]. Moreover, for global regularity it suffices to globally control the L^p-norm of the solution, and statements analogous to those established in Section 3 hold true whenever $p > \frac{d}{2}$. We will therefore directly proceed to the proof of global regularity for equation (4.10) whenever A is sufficiently large.

Theorem 4.3 (Prevention of blowup for Keller–Segel model in higher dimensions). *Assume $d \geq 6$ and let $p > \frac{d}{2}$. Suppose that the divergence-free smooth vector field $u(x)$ is 2-relaxation enhancing. Then for any initial datum $\rho_0 \in L^p(\mathbb{T}^d)$ there exists an amplitude $A_0(\|\rho_0 - \bar{\rho}\|_{L^p}, \bar{\rho}, u, p)$ such that, whenever $A \geq A_0$, Equation (4.10) has a global solution $\rho^A \in C_b([0,\infty), L^p) \cap C^\infty((0,\infty) \times \mathbb{T}^d)$ with initial value $\rho^A(0) = \rho_0$. For $d = 4,5$ the statement holds true under the stronger condition $p > \frac{4d}{d+2}$.*

Remark 4.6. For $d \geq 6$ Theorem 4.3 is optimal in terms of the regularity required for the initial data in the sense that equation (4.10) with $A = 0$ is $L^{\frac{d}{2}}$-critical.

Proof. **(Proof of Theorem 4.3.)** The result follows from arguments similar to Theorem 4.2 with L^2 replaced by L^p. In contrast to the proof of Theorem 4.2, here we do not (need to) distinguish the cases of small and large diffusion: for any time $t_0 \geq 0$ – even if diffusion is large – the local solution $\rho^A(t_0 + \tau)$ to equation (4.10) can be approximated sufficiently well by the solution $\mu^A(t_0 + \tau)$ to equation (4.2) with datum $\mu^A(t_0) = \rho^A(t_0)$ for small enough times $\tau > 0$, as will be shown in the following.

We first prove the case $d \geq 6$. Without loss of generality we may assume $p < d$. Note that since $p < d$ we can define $q \in (p,\infty)$ via

$$\left(\frac{1}{p} - \frac{1}{q}\right) d = 1. \tag{4.11}$$

Since $d \geq 6$ and $p > \frac{d}{2}$, we have

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{p} - \frac{1}{d} < \frac{3}{d} \leq \frac{1}{2}$$

so that there exists $r \in (2,\infty)$ satisfying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1}{2}.$$

We now let $h = |\rho^A - \mu^A|^{p/2}$ and estimate using equation (4.10) and $\nabla \cdot u = 0$

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\rho^A - \mu^A\|_{L^p}^p + \frac{4(p-1)}{p^2}\|\nabla h\|_{L^2}^2 \\
\leq& -\int \rho^A \nabla \triangle^{-1}(\rho^A - \bar{\rho}) \cdot \nabla\left((\rho^A - \mu^A)|\rho^A - \mu^A|^{p-2}\right) \\
\leq& C\|\rho^A\|_{L^p}\|\nabla \triangle^{-1}(\rho^A - \bar{\rho})\|_{L^q}\|\rho^A - \mu^A\|_{L^{r(p/2-1)}}^{p/2-1}\|\nabla h\|_{L^2} \\
\leq& C\|\rho^A\|_{L^p}\|\rho^A - \bar{\rho}\|_{L^p}\|h\|_{L^{r_1}}^{(p-2)/p}\|\nabla h\|_{L^2},
\end{aligned}$$

where r_1 is defined via

$$r_1 \cdot p/2 = r(p/2 - 1).$$

In the last estimate we used Lemma 2.4 (exploiting our choice of q) and the boundedness of the Riesz transform on $L^p, p \in (1,\infty)$. For $p \in (\frac{d}{2}, d)$ and $d \geq 6$ an elementary check yields $r_1 > 2$. (Of course, $r_1 \in [1,2]$ would be even easier.) Now note that by Lemmata 2.4 and 2.1 for $\sigma = \left(\frac{1}{2} - \frac{1}{r_1}\right) d$ we have

$$\|h\|_{L^{r_1}} \lesssim \|\Lambda^\sigma h\|_{L^2} \lesssim \|\nabla h\|_{L^2}^\sigma \|h\|_{L^2}^{1-\sigma}.$$

Hence we obain

$$\begin{aligned}
&\frac{d}{dt}\|\rho^A - \mu^A\|_{L^p}^p + \|\nabla h\|_{L^2}^2 \\
\leq& C(\|\rho^A - \bar{\rho}\|_{L^p} + \bar{\rho})\|\rho^A - \bar{\rho}\|_{L^p}\|h\|_{L^2}^{(1-\sigma)(p-2)/p}\|\nabla h\|_{L^2}^{1+\sigma(p-2)/p}.
\end{aligned} \tag{4.12}$$

It is elementary to verify that $p > \frac{d}{2}$ guarantees $\sigma(p-2)/p < 1$. Thus an absorption argument yields

$$\frac{1}{p}\frac{d}{dt}\|\rho^A - \mu^A\|_{L^p}^p \leq C(\|\rho^A - \bar{\rho}\|_{L^p} + \bar{\rho})^{c_3}\|\rho^A - \bar{\rho}\|_{L^p}^{c_3}\|h\|_{L^2}^{c_4} \tag{4.13}$$

with $c_i = c_i(\sigma,p), i = 3,4$, suitable positive exponents. Similarly to Lemma 3.1, for $B := \max\{\|\rho^A(t_0) - \bar{\rho}\|_{L^p}, 1\}$ one can show[8] that $\|\rho^A - \bar{\rho}\|_{L^p} \leq 2B$ on some small time interval

[8]Since for the Keller–Segel model this is a well-known result, its proof is omitted here. Of course, the condition $p > \frac{d}{2}$ is crucial for its validity.

$[t_0,t_0+\tau_0]$ where $\tau_0>0$ only depends on $B,\bar\rho$ and fixed parameters. Also notice that on $[t_0,t_0+\tau_0]$ we then have $\|h\|_{L^2}=\|\rho^A-\mu^A\|_{L^p}^{p/2}$ and $\|\rho^A-\mu^A\|_{L^p}\le\|\rho^A-\bar\rho\|_{L^p}+\|\mu^A-\bar\rho\|_{L^p}\le 3B$, where in the last bound we used the fact that $\|\mu^A-\bar\rho\|_{L^p}$ is non-increasing on $[t_0,\infty)$. The rest of the argument is similar to the reasoning in Case II of the proof of Theorem 4.2 except that here we need to use Remark 4.2 (iii) instead of Theorem 4.1.

If $d=4,5$ we assume again without loss of generality $p<d$ and define q via equation (4.11). The condition $p>\frac{4d}{d+2}$ ensures that $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$. The rest of the proof then follows as before. ∎

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Appendix A. Blowup in the absence of advection. In this section we aim to show that in the case $a=0$ and in the absence of strong advection there exist smooth initial data which lead to blowup in finite time. We stress that blowup can also be produced in the presence of the advective term if one *first* fixes the flow Au (including its amplitude) and choses appropriate data *afterwards*.

We consider the equation

$$\partial_t\rho=-\Lambda^\gamma\rho+\nabla\cdot(\rho\nabla K*\rho)\ \text{ in }\ (0,\infty)\times\mathbb{T}^d, \tag{A.1}$$

where $\nabla K(x)\sim\frac{x}{|x|^2}$ near $x=0$, $d\ge 2$ and $\gamma\in(1,2]$. In this case, blowup can be produced by a construction very similar to the one in [24]. We therefore confine ourselves to sketching the main argument and indicating the steps which deviate from [24]. Let us introduce the following parameters and auxiliary functions:

- $0<2a<b<\frac{1}{8}$ (sufficiently small).
- $\rho_0\in C^\infty(\mathbb{T}^d)$ nonnegative with $\operatorname{supp}\rho_0\subset B_a(0)$ and mass $M\ge 1$ (sufficiently large).
- ϕ a smooth cut-off at scale b: Fix $\phi_0\in C^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi_0\subset B_1$, $\phi_0\equiv 1$ on $B_{\frac{1}{2}}$, $0\le\phi_0\le 1$. Then $\phi(x):=\phi_0(\frac{x}{b})$ can be considered as a function on the periodic box $\mathbb{T}^d$.

For simplicity we assume equality $\nabla K(x)=\frac{x}{|x|^2}$ on $B_{\frac{1}{4}}$. The parameters a,b,M will be fixed later. As long as the solution ρ stays regular, it preserves positivity and mass.

The main ingredient in the blowup proof is a virial argument, which can be exploited when considering the evolution of the second moment. This is a standard technique for proving blowup of the two- and higher-dimensional Keller–Segel model in bounded domains and the whole space.

Lemma A.1 (Decrease of 2nd moment). *Let $T>0$ and assume that problem* (A.1) *subject to initial condition $\rho(0)=\rho_0$ has a regular solution ρ on $[0,T]$. Then for all $t\in[0,T]$*

$$\frac{d}{dt}\int_{\mathbb{T}^d}|x|^2\rho(t,x)\phi(x)\,dx\le-\left(\int\rho(t,x)\phi(x)\,dx\right)^2+C_2M\|\rho(t,\cdot)\|_{L^1(\mathbb{T}^d\setminus B_b)}$$
$$+C_3bM^2+C_4M.$$

Remark A.1. Note that since $\operatorname{supp}\phi\subset(-\frac{1}{2},\frac{1}{2})^d$ the integrand on the LHS is well-defined and smooth on the periodic box $\mathbb{T}^d$.

Proof. **(Proof of Lemma A.1.)** We compute

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^d}|x|^2\rho(t,x)\phi(x)\,dx &= -\int_{\mathbb{T}^d}\rho(t,x)\Lambda^\gamma(|x|^2\phi(x))\,dx\\ &\quad -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\nabla(|x|^2\phi(x))\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,dydx\\ &=: (i)+(ii).\end{aligned}$$

In order to estimate the first term on the RHS, let us recall that for $\gamma\in(0,2)$ the fractional Laplacian has the following representation (see e.g. [15] or [28]):

$$\Lambda^\gamma f(x) = \text{p.v.}\int_{\mathbb{T}^d}(f(x)-f(y))G_{\gamma,d}(x-y)dy,$$

where

$$G_{\gamma,d}(z) = c_{\gamma,d}\sum_{\alpha\in\mathbb{Z}^d}\frac{1}{|z-\alpha|^{d+\gamma}},\ z\neq 0,$$

and $c_{\gamma,d}$ is a normalisation constant. Using the above formula and the smoothness of ϕ_0, it is easy to see that there exists a positive constant $C_{\phi_0}<\infty$ such that for all $b\in(0,1]$

$$\left\|\Lambda^\gamma\left(|x|^2\phi_0\left(\frac{x}{b}\right)\right)\right\|_{L^\infty(\mathbb{T}^d)} \le C_{\phi_0}b^{2-\gamma}.$$

Recalling $\phi(x)=\phi_0(\frac{x}{b})$, we conclude $(i)\le CMb^{2-\gamma}$.
To estimate the second term, we introduce the splitting

$$\begin{aligned}(ii) &= -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,dydx\\ &\quad -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|x|^2\nabla\phi(x)\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,dydx\\ &= -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\phi(y)\,dydx\\ &\quad -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,dydx\\ &\quad -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|x|^2\nabla\phi(x)\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,dydx\\ &=: (iii)+(iv)+(v).\end{aligned}$$

On $\{x-y : x,y\in\operatorname{supp}\phi\}$ we have $\nabla K(z)=\frac{z}{|z|^2}$. Thus, upon symmetrisation,

$$\begin{aligned}(iii) &= -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|x|^2-2x\cdot y+|y|^2}{|x-y|^2}\phi(x)\rho(t,y)\rho(t,x)\phi(y)\,dydx\\ &= -\left(\int_{\mathbb{T}^d}\rho(t,x)\phi(x)\,dx\right)^2.\end{aligned}$$

Next, we note

$$(iv) = -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\phi(x)x\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,dydx$$

$$
\begin{aligned}
&= -2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \phi(x)x\cdot \frac{x-y}{|x-y|^2}\chi_{B_{\frac14}}(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,dydx\\
&\quad +2\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \phi(x)x\cdot \nabla K(x-y)\chi_{\mathbb{T}^d\setminus B_{\frac14}}(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,dydx\\
&= -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} [\phi(x)(1-\phi(y))x-\phi(y)(1-\phi(x))y]\cdot \frac{x-y}{|x-y|^2}\chi_{B_{\frac14}}(x-y)\rho(t,y)\rho(t,x)\,dydx\\
&\quad +\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \phi(x)x\cdot \nabla K(x-y)\chi_{\mathbb{T}^d\setminus B_{\frac14}}(x-y)\rho(t,y)\rho(t,x)(1-\phi(y))\,dydx\\
&\le CM\|\rho(t)\|_{L^1(\mathbb{T}^d\setminus B_{\frac b2})} + CbM^2.
\end{aligned}
$$

In the last step we used

$$
|[\phi(x)(1-\phi(y))x-\phi(y)(1-\phi(x))y]| \le C\chi_{\mathbb{T}^d\times\mathbb{T}^d\setminus B_{\frac b2}\times B_{\frac b2}}(x,y).
$$

Similar arguments yield

$$
\begin{aligned}
(v) &= -\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} |x|^2\nabla\phi(x)\cdot\nabla K(x-y)\rho(t,y)\rho(t,x)\,dydx\\
&\le CM\|\rho(t)\|_{L^1(\mathbb{T}^d\setminus B_{\frac b2})} + CbM^2.
\end{aligned}
$$

(In both estimates (and thus also in claimed estimate) the term CbM^2 can actually be dropped.)

Using all these estimates, we conclude

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^d}|x|^2\rho(t,x)\phi(x)\,dx \le -\left(\int_{\mathbb{T}^d}\rho(t,x)\phi(x)\,dx\right)^2 + CM\|\rho(t)\|_{L^1(\mathbb{T}^d\setminus B_{\frac b2})}\\
+CM^2b + CMb^{2-\gamma}.
\end{aligned}
$$

Since $\gamma \le 2$, the claimed bound follows. □

Next, we need to ensure that the mass – initially localised near the origin – cannot escape too fast. The statement and proof are analogous to [24, Lemma 8.3], where the extension to $\gamma\in(1,2]$ follows as in the previous lemma.

The existence of exploding solutions is shown completely analogously to [24, Proof of Theorem 8.1].

Appendix B. Transport-diffusion equation. In this section we will prove Theorem 4.1 in the remaining case $\gamma\in[1,2)$. The proof of this theorem follows along the lines of the proof of [14, Theorem 1.4], and we therefore only point out the differences. First of all, if $\gamma<2$, condition (2.1) in [14] is no longer satisfied. We have the following replacement for [14, Theorem 2.1].

Theorem B.1 (Well-posedness). *Assume $\gamma\in(1,2)$ and let $v=v(x)$ be a smooth divergence-free vector field. For any $T>0$ and $\mu_0\in H^{\frac{\gamma}{2}}(\mathbb{T}^d)$ there exists a unique solution*

$$
\mu\in L^2(0,T;H^\gamma)\cap C([0,T];H^{\frac{\gamma}{2}}) \text{ with } \partial_t\mu\in L^2(0,T;L^2)
$$

of the Cauchy problem

$$\begin{aligned}\partial_t\mu + v\cdot\nabla\mu &= -\Lambda^\gamma\mu \quad \textit{in } (0,T)\times\mathbb{T}^d, \\ \mu(0) &= \mu_0.\end{aligned} \tag{B.1}$$

Proof. The existence of weak solutions

$$\mu \in L^2(0,T;H^{\frac{\gamma}{2}})\cap C([0,T];L^2) \text{ with } \partial_t\mu \in L^2(0,T;H^{-(1-\frac{\gamma}{2})}) \tag{B.2}$$

to initial datum $\mu_0 \in L^2(\mathbb{T}^d)$ can be shown via a simple Galerkin scheme. Since $\gamma > 1$, regularity and uniqueness are straightforward as well. ∎

Remark B.1. If $\gamma \in (0,1]$, local existence and uniqueness of a weak solution $\mu \in C([0,T];H^{\frac{1}{2}})$ with $\partial_t\mu \in C([0,T];H^{-\frac{1}{2}})$ to the Cauchy problem (B.1) with initial datum in $H^{\frac{1}{2}}$ can still be established: the existence of rough solutions is again obtained via a Galerkin method. To prove the claimed regularity and uniqueness one first notes that the constructed weak solution μ satisfies the pointwise equality

$$\partial_t S_k\mu + \nabla\cdot S_k(v\mu) = -\Lambda^\gamma S_k\mu,$$

where S_k are the LP-projections introduced in Appendix C, and then proceeds as in the proof of Proposition C.1.

Owing to the worse regularity, more care has to be taken when approximating the advection-diffusion equation by the pure transport equation. Our replacement for [14, Lemma 2.4] is the following.

Lemma B.1 (Approximation by pure transport). *Let $v = v(x)$ be a smooth divergence-free vector field. Assume $\gamma \in [1,2)$ and let $\eta_0 \in H^{\frac{\gamma}{2}}(\mathbb{T}^d)$. Let $\eta^0 \in C([0,\infty);H^{\frac{\gamma}{2}})$ be a weak solution of the transport problem* (4.3) *and let $\eta^\varepsilon = \mu$ solve* (B.1) *with $-\Lambda^\gamma$ replaced*[9] *by $-\varepsilon\Lambda^\gamma$ and initial datum η_0. Then*

$$\frac{d}{dt}\|\eta^\varepsilon(t) - \eta^0(t)\|_{L^2}^2 \le \frac{\varepsilon}{2}\|\eta^0(t)\|_{\dot H^{\gamma/2}}^2 \le \frac{\varepsilon}{2}\exp(C(v)t)\|\eta_0\|_{\dot H^{\gamma/2}}^2, \tag{B.3}$$

where $C(v)$ is the constant from Proposition 4.1.

Proof. The difference $\eta^\varepsilon - \eta^0$ satisfies

$$\partial_t(\eta^\varepsilon - \eta^0) + u\cdot\nabla(\eta^\varepsilon - \eta^0) = -\varepsilon\Lambda^\gamma\eta^\varepsilon, \tag{B.4}$$

where for fixed time t the equality is to be understood in $H^{\frac{\gamma}{2}-1} \subseteq H^{-\frac{\gamma}{2}}$. We can therefore take the dual pairing $\dot H^{-\frac{\gamma}{2}} \times \dot H^{\frac{\gamma}{2}}$ of the equation with $(\eta^\varepsilon - \eta^0)(t) \in H^{\frac{\gamma}{2}}$ to obtain after an absorption argument the first inequality in (B.3). (Here we also used the incompressibility and the smoothness of the flow which guarantee that $B(f,g) := \langle u\cdot\nabla f, g\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}}$ satisfies $B(f,f) = 0$ for all $f \in C^\infty$ and extends uniquely to a bounded bilinear form on $H^{\frac{1}{2}} \times H^{\frac{1}{2}}$.) The second inequality in (B.3) is just the boundedness of the transport evolution with respect to $\|\cdot\|_{\dot H^{\frac{\gamma}{2}}}$ (cf. the bound (4.4) or Appendix C). ∎

Remark B.2. The statement of Lemma B.1 remains true for $\gamma \in (0,1)$ if restricting to initial data in $H^{\frac{1}{2}}$. Indeed, in this case one only needs to notice that (for fixed time) the equation (B.4) holds in $H^{-\frac{1}{2}}$ and that $(\eta^\varepsilon - \eta^0)(t) \in H^{\frac{1}{2}}$.

[9] In order to facilitate the comparison with [14], we adopt the rescaling to "small diffusion on long time scales" as introduced in [14].

The remaining lemmata used in the proof of [14, Theorem 1.4] can either be shown by similar arguments as in Lemma B.1 (where for mere L^2 data the regularity (B.2) has to be used) or require only a formal adaptation (such as replacing the "diffusion operator" $-\Gamma$ by $-\Lambda^\gamma$).

Appendix C. Transport equation in $H^\sigma(\mathbb{T}^d)$. Here we are concerned with the linear transport equation with a (prescribed) divergence-free smooth velocity field $v=v(x)$:

$$\begin{aligned} \partial_t\eta + v\cdot\nabla\eta = 0 \ \text{ in } \ (0,\infty)\times\mathbb{T}^d, \\ \eta(0)=\eta_0. \end{aligned} \tag{C.1}$$

Our aim is to prove Proposition 4.1, i.e. the boundedness of the associated evolution in fractional Hilbert spaces $H^\sigma(\mathbb{T}^d), \sigma>0$, where we do not aim for optimal regularity with respect to v. In the whole space case fairly general a priori estimates in Besov spaces can be found in [2]. As in [2] we will make use of a standard tool from harmonic analysis, which we shall introduce in the following.

C.1. Preliminaries. We consider a Littlewood–Paley decomposition: let $\phi_0\in C_c^\infty(\mathbb{R}^d)$ be a radial bump function with $\operatorname{supp}\phi_0\subset B_{11/10}(0)$ which is equal to 1 on $B_1(0)$ and satisfies $0\le\phi_0\le 1$. Denoting $\phi(\xi):=\phi_0(\xi)-\phi_0(2\xi)$, we then have

$$\phi_0(2\xi)+\sum_{k\ge 0}\phi(2^{-k}\xi)=1, \qquad \xi\in\mathbb{R}^d.$$

For smooth functions η on $\mathbb{T}^d$ we then define the operators

$$S_{-1}\eta(x)=\sum_{\alpha\in\mathbb{Z}^d}\phi_0(2\alpha)\hat\eta(\alpha)\mathrm{e}^{2\pi\mathrm{i}x\cdot\alpha}=\phi_0(0)\hat\eta(0)$$

and for $k\ge 0$

$$S_k\eta(x)=\sum_{\alpha\in\mathbb{Z}^d}\phi(2^{-k}\alpha)\hat\eta(\alpha)\mathrm{e}^{2\pi\mathrm{i}x\cdot\alpha}.$$

Note that S_k localises to frequency $\sim 2^k$, i.e. $\operatorname{supp}\widehat{S_k\eta}\subset\{\alpha\in\mathbb{Z}^d:|\alpha|\approx 2^k\}$ and we have equivalence of (semi-) norms

$$\|\eta\|_{\dot H^\sigma}^2\sim\sum_{k\ge 0}2^{2\sigma k}\|S_k\eta\|_{L^2}^2. \tag{C.2}$$

We will at times also use the notation $S_{\le N}, S_{M<\dots<N}$ and $S_{\ge N}$ to denote the sums of operators corresponding to $\sum_{-1\le k\le N}S_k, \sum_{M<k<N}S_k$ and $\sum_{k\ge N}S_k$.

C.2. Boundedness of evolution. We will now provide a proof of the transport estimate:

Proposition C.1. *Assume $\sigma>0$. Any sufficiently regular solution η of equation* (C.1) *satisfies*

$$\|\eta(t)\|_{\dot H^\sigma}^2\le\exp(C(v)t)\|\eta_0\|_{\dot H^\sigma}^2, \qquad t\ge 0, \tag{C.3}$$

where the positive constant $C(v)$ saturates the bound

$$C(v)\lesssim_{\sigma,d}\|\Lambda^{\sigma+\frac{d}{2}+1}v\|_{L^2}.$$

The proof exploits the following gain at level k for the commutator involving an LP projection S_k for $k \gg 1$.

LEMMA C.1. *For smooth functions f, g on the torus the following commutator estimate holds*

$$\|[S_k, g]f\|_{L^2(\mathbb{T}^d)} \leq 2^{-k} \|\nabla\phi\|_{L^\infty} \|\hat{g}(\beta)\beta\|_{l^1_\beta} \|f\|_{L^2(\mathbb{T}^d)}. \tag{C.4}$$

Proof. **(Proof of Lemma C.1.)** We first note

$$\|[S_k, g]f\|_{L^2(\mathbb{T}^d)} = \|\widehat{[S_k, g]}f\|_{l^2(\mathbb{Z}^d)} \tag{C.5}$$

and therefore consider

$$\begin{aligned}\widehat{[S_k, g]}f(\alpha) &= \widehat{S_k(gf)}(\alpha) - \widehat{g} * \widehat{S_k f}(\alpha) \\ &= \sum_{\beta \in \mathbb{Z}^d} [\phi(2^{-k}\alpha) - \phi(2^{-k}(\alpha - \beta))]\hat{g}(\beta)\hat{f}(\alpha - \beta) \\ &= \sum_{\beta} 2^{-k} \int_0^1 \nabla\phi(2^{-k}(\alpha - (1-s)\beta))ds \cdot \beta\hat{g}(\beta)\hat{f}(\alpha - \beta).\end{aligned}$$

Hence

$$|\widehat{[S_k, g]}f(\alpha)| \leq 2^{-k} \|\nabla\phi\|_{L^\infty(\mathbb{R}^d)} \sum_{\beta} |\beta\hat{g}(\beta)||\hat{f}(\alpha - \beta)|.$$

Young's convolution inequality then yields the claim

$$\|\widehat{[S_k, g]}f\|_{l^2(\mathbb{Z}^d)} \leq 2^{-k} \|\nabla\phi\|_{L^\infty(\mathbb{R}^d)} \|\beta\hat{g}(\beta)\|_{l^1_\beta} \|f\|_{L^2},$$

where we used $\|\hat{f}\|_{l^2} = \|f\|_{L^2}$. □

We are now in a position to show the boundedness of the evolution (C.1) in $\dot{H}^\sigma(\mathbb{T}^d)$.

Proof. **(Sketch proof of Proposition C.1.)** Without loss of generality we can assume $\hat{\eta}(0) = 0$. In the following we will omit any possible dependence of constants on σ and d. Now let $k \geq 0$ be a fixed but arbitrary integer. The equation implies

$$\partial_t S_k \eta = -\nabla \cdot S_k(v\eta)$$

and hence

$$\frac{1}{2}\frac{d}{dt}\|S_k\eta\|^2_{L^2(\mathbb{T}^d)} = \int -\nabla \cdot S_k(v\eta)\, S_k\eta.$$

Since by incompressibility

$$\int \nabla \cdot (v\, S_k\eta)\, S_k\eta = -\frac{1}{2}\int v \cdot \nabla |S_k\eta|^2 = 0,$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\|S_k\eta\|^2_{L^2(\mathbb{T}^d)} = \int \nabla \cdot [v, S_k]\eta\, S_k\eta$$

$$
\begin{aligned}
&= \int \nabla\cdot \tilde S_k[v,S_k]\eta\, S_k\eta \qquad\qquad (\tilde S_k S_k = S_k)\\
&\le \|\nabla\cdot \tilde S_k[v,S_k]\eta\|_{L^2}\|S_k\eta\|_{L^2}\\
&\lesssim 2^k\|\tilde S_k[v,S_k]\eta\|_{L^2}\|S_k\eta\|_{L^2}, \qquad (C.6)
\end{aligned}
$$

where $\tilde S_k$ denotes a suitable Fourier multiplier localising to frequency $\sim 2^k$ whose symbol is equal to 1 on $\operatorname{supp}\phi(2^{-k}\cdot)$. We now assume $k\gg 1$ and split

$$v = S_{\le k-4}v + S_{>k-4}v$$

and consider

$$\tilde S_k[v,S_k]\eta = \tilde S_k[S_{\le k-4}v,S_k]\eta + \tilde S_k[S_{>k-4}v,S_k]\eta. \qquad (C.7)$$

With regard to the regularity of η, the first term is the delicate one. It can be estimated using Lemma C.1, as we will show now. Note that there exists a multiplier S_k' localising to frequency $\sim 2^k$ such that

$$\tilde S_k[S_{\le k-4}v,S_k]\eta = \tilde S_k[S_{\le k-4}v,S_k]S_k'\eta.$$

Now Lemma C.1 applied to $g=S_{\le k-4}v, f=S_k'\eta$ yields

$$
\begin{aligned}
\|\tilde S_k[S_{\le k-4}v,S_k]S_k'\eta\|_{L^2} &\le \|[S_{\le k-4}v,S_k]S_k'\eta\|_{L^2}\\
&\le C2^{-k}\|\widehat{S_{\le k-4}v}(\alpha)\alpha\|_{l^1_\alpha}\|S_k'\eta\|_{L^2}\\
&\le C2^{-k}\|\hat v(\alpha)\alpha\|_{l^1_\alpha}\|S_k'\eta\|_{L^2},
\end{aligned}
$$

where in the last step we used

$$\|\widehat{S_{\le k-4}v}(\alpha)\alpha\|_{l^1_\alpha} = \sum_\alpha \Big|\sum_{j\le k-4}\phi(2^{-j}\alpha)\hat v(\alpha)\alpha\Big| \le \sum_\alpha |\hat v(\alpha)\alpha|.$$

Finally notice that by the equivalence of norms (C.2)

$$\sum_{k\gg 1} 2^{2k\sigma}2^k\left(2^{-k}\|\hat v(\alpha)\alpha\|_{l^1_\alpha}\|S_k'\eta\|_{L^2}\right)\|S_k\eta\|_{L^2} \lesssim \|\hat v(\alpha)\alpha\|_{l^1_\alpha}\|\eta(t)\|^2_{\dot H^\sigma(\mathbb{T}^d)}.$$

Estimating the second term in equation (C.7) is straightforward if one is not interested in optimal regularity results for v. For a rough estimate, we note that the part of this term which requires the highest regularity of v is

$$S_k(S_{k-4<\cdots<k+4}v\eta)$$

as it may involve low frequencies of η. We first estimate using a Bernstein inequality (see e.g. [2, Lemma 2.1])

$$
\begin{aligned}
\|S_k(S_{k-4<\cdots<k+4}v\eta)\|_{L^2} &\lesssim 2^{\frac{kd}{2}}\|S_k(S_{k-4<\cdots<k+4}v\eta)\|_{L^1}\\
&\lesssim 2^{\frac{kd}{2}}\|S_{k-4<\cdots<k+4}v\|_{L^2}\|\eta\|_{L^2}
\end{aligned}
$$

and note that thanks to Cauchy–Schwarz and $\hat\eta(0)=0$

$$\sum_{k\gg 1} 2^{2k\sigma}2^k\left(2^{\frac{kd}{2}}\|S_{k-4<\cdots<k+4}v\|_{L^2}\|\eta\|_{L^2}\right)\|S_k\eta\|_{L^2}$$

$$\lesssim \sum_{k\gg 1} \|S_{k-4<\cdots<k+4}(\Lambda^{\sigma+\frac{d}{2}+1}v)\|_{L^2} 2^{k\sigma}\|S_k\eta\|_{L^2}\|\eta\|_{L^2}$$
$$\lesssim \|\Lambda^{\sigma+\frac{d}{2}+1}v\|_{L^2}\|\eta\|^2_{\dot{H}^\sigma(\mathbb{T}^d)}.$$

For the low frequencies $k\leq k_0$ (k_0 being a suitable fixed positive integer), we estimate using estimate (C.6) and omitting the k_0 dependence

$$\frac{d}{dt}\sum_{0\leq k\leq k_0} 2^{2\sigma k}\|S_k\eta\|^2_{L^2(\mathbb{T}^d)} \lesssim \sum_{0\leq k\leq k_0} \|[v,S_k]\eta\|_{L^2}\|S_k\eta\|_{L^2}$$
$$\lesssim \|\hat{v}(\alpha)\alpha\|_{l^1_\alpha}\|\eta(t)\|^2_{\dot{H}^\sigma(\mathbb{T}^d)}.$$

In the second step, we used Lemma C.1 (mainly in order to illustrate that the estimate is independent of $\hat{v}(0)$).

We now recall estimate (C.6) and combine our estimates for high and low frequencies to conclude

$$\frac{d}{dt}\|\eta(t)\|^2_{\dot{H}^\sigma(\mathbb{T}^d)} \lesssim \left(\|\Lambda^{\sigma+\frac{d}{2}+1}v\|_{L^2} + \|\hat{v}(\alpha)\alpha\|_{l^1_\alpha}\right)\|\eta(t)\|^2_{\dot{H}^\sigma(\mathbb{T}^d)}. \tag{C.8}$$

Finally note that since $\sigma>0$

$$\sum_{\alpha\neq 0}|\hat{v}(\alpha)\alpha| \leq \left(\sum_{\alpha\neq 0}|\hat{v}(\alpha)|^2|\alpha|^{2(1+\frac{d}{2}+\sigma)}\right)^{\frac{1}{2}}\left(\sum_{\alpha\neq 0}|\alpha|^{-d-2\sigma}\right)^{\frac{1}{2}} \lesssim \|\Lambda^{\sigma+\frac{d}{2}+1}v\|_{L^2}.$$

Hence, Gronwall's inequality applied to estimate (C.8) yields the claim. ☐

REMARK C.1. The reader interested in optimising the regularity required for v may consult the comprehensive analysis in [2, Theorem 3.14 & Lemma 2.100].

Appendix D. Examples of γ-RE flows. In this section we provide examples which show that in general the classes of γ-relaxation enhancing flows introduced in Definition 4.1 are different for different γ. Our construction is an adaptation of [14, Proposition 6.2].

PROPOSITION D.1. *For any $\gamma>\frac{1}{2}$ and any (small) $\varepsilon>0$ there exists a smooth, divergence-free vector field $u(x)$ on $\mathbb{T}^2$ such that the induced unitary evolution U on $L^2(\mathbb{T}^2)$ has discrete spectrum and all non-constant eigenfunctions lie in $H^{\gamma-\varepsilon}\setminus H^{\gamma+\varepsilon}$. In particular, u is $2(\gamma+\varepsilon)$-RE but not $2(\gamma-\varepsilon)$-RE.*

Proof. (Sketch.) The proof adapts the construction in [14, Proposition 6.2]. We therefore only point out the necessary modifications. Let $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ be a positive Liouvillean number. Then, by [14, Proposition 6.3] (see also the original statement [22, Theorem 4.5]), there exists a smooth function $h\in C^\infty(\mathbb{T}^1)$ and a nowhere continuous, integrable function $\tilde{R}$ on $\mathbb{T}^1$ such that

$$\tilde{R}(\xi+\alpha)-\tilde{R}(\xi)=h(\xi) \text{ for all } \xi\in\mathbb{T}^1. \tag{D.1}$$

Since $\tilde{R}\in L^1(\mathbb{T}^1)$, it can naturally be identified with an element in $H^\sigma(\mathbb{T}^1)$ for sufficiently small $\sigma\in\mathbb{R}$. Thus, we can define

$$r:=\inf\{s\in\mathbb{R}:\Lambda^{-s}\tilde{R}\in H^\gamma\}.$$

The discontinuity of $\tilde{R}$ and $\gamma > \frac{1}{2}$ imply that $r \in (0,\infty)$. We now set $R := \Lambda^{-r}\tilde{R}$ and $Q := \Lambda^{-r}h + 1$. Let further $\varepsilon > 0$ be small enough such that $\gamma - \varepsilon > \frac{1}{2}$. Clearly

$$R \in H^{\gamma-\varepsilon}(\mathbb{T}^1) \setminus H^{\gamma+\varepsilon}(\mathbb{T}^1), \tag{D.2}$$

and thanks to the Sobolev embedding into Hölder spaces we may henceforth identify R with its Hölder continuous representative. Furthermore,

$$Q \in C^\infty(\mathbb{T}^1) \text{ with } \int_{\mathbb{T}^1} Q = 1,$$

and from property (D.1) we deduce

$$R(\xi + \alpha) - R(\xi) = Q(\xi) - 1 \text{ for all } \xi \in \mathbb{T}^1. \tag{D.3}$$

Thanks to equation (D.3) and the smoothness of Q, we may now proceed as in the proof of [14, Proposition 6.2]. Our arguments only deviate when it comes to determining the regularity of the eigenfunctions $\psi^w_{nl} \in L^2(\mathbb{T}^2)$, where we use the same notation as in [14]. For this part, let us recall (cf. [14, equation (6.2)]) that the eigenfunctions have the form

$$\psi(x,y) := \psi^w_{nl}(x,y) = \zeta(x,y)\mathrm{e}^{2\pi\mathrm{i}(n\alpha+l)R(x-\alpha y)},$$

where $n,l \in \mathbb{Z}$. Here $\zeta(x,y)$ is a smooth complex-valued function with $|\zeta| = 1$, which is not periodic in y. To complete the proof, it remains to show that the regularity of R implies the asserted regularity of ψ. The remaining steps are then exactly the same as in [14].

Regarding the regularity of ψ, we may henceforth assume $(n,l) \neq (0,0)$ since otherwise the explicit form of ζ in [14, equation (6.2)] implies that ψ is constant. Since R is Hölder continuous and bounded, the regularity (D.2) implies that for any $\lambda \in \mathbb{R}^*$

$$R_\lambda(\xi) := \mathrm{e}^{\mathrm{i}\lambda R(\xi)} \in H^{\gamma-\varepsilon}(\mathbb{T}^1) \setminus H^{\gamma+\varepsilon}(\mathbb{T}^1). \tag{D.4}$$

This can easily be seen by noting that $\mathrm{e}^{\mathrm{i}\lambda\cdot} : \mathbb{R} \to \mathbb{S}^1$ is a local C^∞ diffeomorphism and by using standard fractional chain rule/Moser type estimates (see e.g. [29, Chapter 3]).

Let us next fix $\lambda = 2\pi(n\alpha + l)$, which is different from 0, and consider the function

$$\Theta_\lambda(x,y) := R_\lambda(x - \alpha y) : \mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}} \to \mathbb{S}^1,$$

where $\mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}}$ denotes the periodic box $[0,1) \times [0,\alpha^{-1})$. By using the explicit definition of $\|\cdot\|_{\dot{H}^s}$ (in terms of Fourier coefficients) one quickly finds

$$\|\Theta_\lambda\|_{\dot{H}^s(\mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}})} = C_{s,\alpha}\|R_\lambda\|_{\dot{H}^s(\mathbb{T}^1)}$$

for some positive constant $C_{s,\alpha} > 0$. Thus, property (D.4) yields

$$\Theta_\lambda \in (H^{\gamma-\varepsilon} \setminus H^{\gamma+\varepsilon})(\mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}}). \tag{D.5}$$

To conclude the regularity

$$\psi \in (H^{\gamma-\varepsilon} \setminus H^{\gamma+\varepsilon})(\mathbb{T}^1 \times \mathbb{T}^1)$$

one can use a smooth partition of unity of $\mathbb{T}^2$ in y-direction corresponding to a finite number of overlapping cylinders of height $\frac{1}{2}\alpha^{-1}$ (if $\alpha > 1$). This allows us to split ψ into a finite sum of functions, which may be considered (by first (smoothly) extending by zero to $\mathbb{T}^1 \times \mathbb{R}^1$ and then suitably periodising) as being defined on $\mathbb{T}^1 \times \mathbb{T}^1_{\alpha^{-1}}$. Each of these summands is the product of a smooth function with Θ_λ so that property (D.5) implies $\psi \in H^{\gamma-\varepsilon}$. In order to see $\psi \notin H^{\gamma+\varepsilon}$ one can use similar arguments together with the fact that $|\zeta| = 1$ everywhere. ∎

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