

RECOVERY OF THE SOUND SPEED FOR THE ACOUSTIC WAVE EQUATION FROM PHASELESS MEASUREMENTS*

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Abstract. We recover the higher order terms for the acoustic wave equation from measurements of the modulus of the solution. The recovery of these coefficients is reduced to a question of stability for inverting a Hamiltonian flow transform, not the geodesic X-ray transform encountered in other inverse boundary problems like the determination of covector fields for the wave equation. Under some geometric assumptions, we reduce this to a question of boundary rigidity, which allows recovery of the sound speed for the acoustic wave equation. Previous techniques do not measure the full amplitude of the outgoing scattered wave, which is the main novelty in our approach.

Keywords. Phase less measurements; Helmholtz equation; acoustic wave equation; Gaussian beams; inverse problems; integral geometry.

AMS subject classifications. 35R30; 53C65; 35C07; 35Q99; 58J37.

1. Introduction and physical motivation

Scattering is a general physical process where some forms of radiation, such as light or sound, or moving particles are forced to deviate from a geodesic trajectory by a path due to localized non-uniformities in the medium through which they pass. Conventionally, this also includes the deviation of reflected radiation from the angle predicted by the law of reflection [43]. Scattering may also refer to particle-particle collisions between molecules, atoms, electrons, photons and other particles [12]. The types of non-uniformities which can cause scattering are sometimes known as scatterers. These include particles and surface roughness. The type of stability estimate we prove on the phaseless measurements of the solution insure that the energy of the waves uniquely determines the scatterer.

The theory of signal processing and inverse problems has seen a recent increase in a class of so-called phaseless measurements. Often in experiments, when a source wave is measured, the only part of the information available to experimenters is the modulus of the wave from the source. In signal processing, algorithms found in [13] and [11] are focused on the recovery of waves from a sequence of Fourier modes. We are interested in phaseless measurements to recover scattering terms for the acoustic wave equation and generalized Helmholtz equation. The modulus of the wave corresponds to the energy density of the wave at a given point. The inverse boundary value problem differs from the questions examined in signal processing where often times one is dealing with incomplete data sets. In particular, we show that having an idea of which partial differential equation the wave comes from is enough to give a full reconstruction of the coefficients modulo diffeomorphism. The scattered wave then completely determines the scatterer. These results are supported by the numerical work in [10] and are applicable to other operators which admit a Gaussian beam type solution. The problem differs from both of the author's previous work [46, 23] because the terms which we are recovering

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come from higher order terms. These control the bicharacteristic flow associated to the Hamiltonian governing the partial differential equation.

Practical applications to phaseless problems are varied — and one such application is multi-wave tomography. Usually, in multi-wave tomography some type of wave is sent to a portion of the body which is being imaged. In electromagnetic or optical radiation tomography the wave interaction with the tissues of the patient are measured to create the image, as explained in [4]. Naturally one cannot measure inside the patient, so some initial boundary value problem must be considered. Similarly, to image the Earth, one has to send waves of some kind through the planet and make measurements at the surface. One such mathematical model of the emitted ultrasound waves is the acoustic wave equation with a high-frequency source term.

Now, on the mathematical side, let $M \subset \mathbb{R}^d$ be a bounded and smooth manifold. Let g be a Riemannian metric on \mathbb{R}^d which agrees with the Euclidean one outside M and makes (\bar{M}, g) into a simple manifold. We recall that a simple manifold is one which is strictly geodesically convex with respect to the metric g .

Let the standard Laplace-Beltrami operator be denoted as

$$\Delta_g = \frac{1}{\sqrt{\det g(x)}} \frac{\partial}{\partial x^k} \left(g^{ki}(x) \sqrt{\det g(x)} \frac{\partial}{\partial x^i} \right) \quad (1.1)$$

in local coordinates with $g(x) = (g_{ik}(x))$, and $(g^{ik}(x)) = (g_{ki}(x))^{-1}$. We consider manifolds, M , which are smooth (C^∞). We write the local coordinates as (x^1, \dots, x^d) . We also assume the manifolds have a boundary.

The generalized Helmholtz equation may be written as

$$Lu = \Delta_g u + (i\alpha\lambda + \lambda^2)n^2(x)u = h(x, \lambda) \quad x \in \mathbb{R}^d. \quad (1.2)$$

The scalar λ is large and $n^2(x)$ represents sound speed. The number α helps describe the attenuation coefficient and $\alpha \in (0, 1)$. It is zero in the limit of zero absorption. The source $h(x, \lambda)$, which emits the waves, is assumed to depend on λ and be compactly supported in x with codimension 1. With these assumptions, we recall the *Sommerfeld radiation condition*:

DEFINITION 1.1 (Sommerfeld radiation condition). *Let g_n be the metric n^2g and $k = \sqrt{\lambda^2 + i\lambda\alpha}$. Provided that the compactly supported source $h(x, \lambda)$ satisfies the decay condition,*

$$|h(x, \lambda)| \leq (1 + |x|_{g_n})^{-d-\beta} \quad (1.3)$$

with $\beta > 0$, then we have that $u(x)$ in (1.2) satisfies the outgoing radiation condition:

$$\lim_{|x|_{g_n} \rightarrow \infty} |x|_{g_n}^{\frac{d-1}{2}} \left(\frac{\partial}{\partial |x|_{g_n}} - ik \right) u(x) = 0 \quad (1.4)$$

uniformly in all directions. Moreover, the Equation (1.2) has a unique solution in $L^2(\mathbb{R}^d)$, following [38] and [30].

Sources will be modelled after Gaussian beams following [30]. We may pick sources anywhere inside the domain M , however we chose a particular set of them in order to provide a complete reconstruction of $n^2(x)$. While this equation is known as the generalized Helmholtz equation, if we take multiplication by a prefactor, $\exp(\pm i\lambda t)$, then the solution u is turn into a solution

$$u(t, x) = \frac{1}{2} (\exp(i\lambda t)u(x) + \exp(-i\lambda t)u(x)) \quad (1.5)$$

to the acoustic wave equation when $n^2(x) \equiv 1$ on ∂M , and $\alpha \rightarrow 0$. In other words, for some fixed and finite T with this construction, $u(t, x)$ in (1.5) above solves

$$\partial_t^2 u(t, x) - n^{-2}(x) \Delta_g u(t, x) = \frac{1}{2} (\exp(i\lambda t)h(x, \lambda) + \exp(-i\lambda t)h(x, \lambda)) \quad \text{in } [0, T] \times \mathbb{R}^d. \tag{1.6}$$

$$\begin{aligned} \partial_t u(0, x) &= 0 \quad \text{in } \mathbb{R}^d \\ u(0, x) &= u(x) \quad \text{in } \mathbb{R}^d. \end{aligned}$$

The acoustic wave equation models the scattering of waves in the Earth’s core. Moreover the solution $u(t, x)$ in (1.5), is well posed with $u(t, x) \in C([0, T]; C^1(\mathbb{R}^d)) \cap C^1([0, T]; C(\mathbb{R}^d))$. This follows from the classical result Theorem 3.3 of [2], as by the Sommerfeld radiation condition, $u(x) \in L^2(\mathbb{R}^d)$ and by construction $\exp(i\lambda t)h(x, \lambda) \in L^2([0, T]; L^1(\mathbb{R}^d))$.

We chose to model our solutions to the wave equation with Gaussian beams. The existence of Gaussian beam solutions to the wave equation has been known since the 1960’s first in connection with lasers, see Babic and Buldyrev [8]. They were also used in the analysis of propagation of singularities in PDEs by Hörmander [22] and Ralston [35]. In the context of the Schrödinger equation, first order beams correspond to the so-called classical coherent states. The higher order versions of these equations have been introduced to approximate the Schrödinger equation solutions in quantum chemistry by Heller [19], Hagedorn [18], and Herman and Kluk [20].

Again, given a smooth, strictly convex, bounded domain M equipped with a metric g we assume that $n^2(x) \equiv 1$ on the boundary of M , ∂M . The measurements we consider give then data of the form

$$\{(x, |u(x)|) : x \in \partial M\} \tag{1.7}$$

with a collection of sources $h^{x_0, \omega_0}(x, \lambda)$ varying over all $x_0 \in \partial M$ and $\omega_0 \in S_{x_0} M$, which are inward pointing vectors into the manifold. That is, the data (1.7) for solutions u to (2.7) with source $h^{x_0, \omega_0}(x, \lambda)$ is known for all $x_0 \in \partial M$ and inward pointing $\omega_0 \in S_{x_0} M$. We need this collection of data in order to give a complete reconstruction of $n^2(x)$. We consider the metric g to be fixed and $n^2(x)$ to vary. This collection of measurements is a ‘true’ phaseless problem in contrast to phaseless backscattering measurements which were recently investigated in [26]. We see that the measurements for both Equations (1.6) and (1.2) coincide. Thus we have developed a robust model for the scattering of both waves traveling through the earth’s core (1.6) and quantum mechanical particles in (1.2).

Phaseless measurements are more physical, and the use of techniques in this article represents an improvement on existing literature. Indeed, the standard problem, see for example [31, 32], is to consider

$$\begin{aligned} (-n^{-2} \Delta_g - \lambda^2)u &= 0, \\ u|_{\partial M} &= \tilde{h}(x, \lambda). \end{aligned} \tag{1.8}$$

with $\tilde{h}(x, \lambda)$ a compactly supported function on ∂M . Here one would recover the factor n^{-2} from the Dirichlet-to-Neumann map with $\tilde{h}(x, \lambda)$ a high frequency source. With Dirichlet-to-Neumann map one considers a displaced field $u_1 - u_2$ corresponding to a displacement speed $n_1^{-2} - n_2^{-2}$. The measurements of the displacement field $u_1 - u_2$ are

then of the form

$$\int_{\partial M} |\partial_\nu(u_1 - u_2)|^2 d_g S. \tag{1.9}$$

Here $d_g S$ denotes the surface measure on M . For phaseless measurements with u_1 and u_2 corresponding to different factors in (1.8) above, the measurements are of the form (1.7) for each individual solution with codimension 1 sources. We then see that

$$\int_{\partial M} |u_1|^2 - |u_2|^2 d_g S \leq \int_{\partial M} |u_1 - u_2|^2 d_g S \tag{1.10}$$

with, again, u_1, u_2 corresponding to different n_i .

Our problem is more physical because it corresponds to measuring the amplitudes of two different waves as on the left hand side of (1.10), rather than super-imposing them and measuring their difference. We are also able to include an attenuation coefficient which is the $\alpha \rightarrow 0$ limit of (1.8). The introduction of these measurements for the coefficient recovery problem is one of the major contributions of this article.

2. Statement of the main theorem

Our main theorem can be stated as follows. We consider sources $h^{x_0, \omega_0}(x, \lambda)$ with support in \mathbb{R}^{d-1} . Here we view M as an embedded submanifold of M' , a larger manifold. Sources are indexed by x_0 and ω_0 . We let $\nu(x)$ denote the outward unit normal to the boundary at x . Let the set

$$\partial_+ S M = \{(x_0, \omega_0) : x_0 \in \partial M, \langle \nu(x_0), \omega_0 \rangle > 0, \omega_0 \in S_x M\}, \tag{2.1}$$

be the range of x_0, ω_0 where $S_x M$ denotes the unit sphere bundle of the manifold at x . The sources are defined for $x \in \mathbb{R}^{d-1}$ as:

$$h^{x_0, \omega_0}(x, \lambda) = 2i\lambda \chi_\lambda(x) (\omega_0 \cdot \nu + \mathcal{M}(0)(x - x_0) \cdot \nu + \nabla_g \chi_\lambda \cdot \nu) \exp(i\lambda \tilde{\psi}(x)) \tag{2.2}$$

with

$$\tilde{\psi}(x) = (x - x_0) \cdot \omega_0 + \frac{1}{2}(x - x_0) \cdot \mathcal{M}(0)(x - x_0). \tag{2.3}$$

Here $\chi_\lambda(x)$ is a smooth compactly supported function of codimension 1 in a neighborhood of the set

$$\{x \in M', |x - x_0| < \lambda^{-1/2d}\} \tag{2.4}$$

which is perpendicular to ω_0 . This set also contains the support of $\tilde{\psi}(x)$.

The flow for the ray path of H is defined by the set of ODEs in local coordinates on the manifold:

$$\frac{dx_i}{ds} = 2g^{ij}(x(s))p_j, \quad \frac{dp_i}{ds} = -\frac{\partial g^{jk}(x(s))p_k p_j}{\partial x_i} + \frac{\partial n^2(x(s))}{\partial x_i} \tag{2.5}$$

with initial data (x_0, ω_0) . As g, n are differentiable, the Hamiltonian system has a unique solution for short times T depending on the norm of g and n , c.f. Picard iteration (Theorem 1.7 in [1]). The matrix $\mathcal{M}(0)$ is a complex matrix satisfying

$$\mathcal{M}(0) = \mathcal{M}(0)^T, \quad \mathcal{M}(0)x(0) = \dot{p}(0), \quad \Im \mathcal{M}(0) \text{ positive definite on } \dot{x}(0)^\perp. \tag{2.6}$$

We know there is a solution u^{x_0, ω_0} to the following equation for each x_0, ω_0 :

$$(\Delta_g + (\lambda^2 + i\alpha\lambda)n^2(x))u^{x_0, \omega_0} = h^{x_0, \omega_0}(x, \lambda), \tag{2.7}$$

We let $\epsilon_0 \in (0, 1)$ and $\tilde{n}^2 = n_1^2 - n_2^2$ (notice that \tilde{n}^2 is not a square, just the difference of two squares). We have the following stability result on \tilde{n}^2 where the measurements are the amplitudes of the collection of u^{x_0, ω_0} corresponding to different n^2 , ranging over sources indexed by $\partial_+ SM$. Let $\text{diam}_H(M)$ denote the maximal radius of the manifold with respect to the flow H . We let g' denote the extended metric which is g on M and Euclidean on the exterior. We impose the condition the manifold M' contains M and be such that M' is simple with respect to the metric $n_1^2 g'$. We show in Section 7 that this is sufficient in our case to guarantee that M is strictly geodesically convex with respect to the Hamiltonian flow H .

THEOREM 2.1. *Let $N > \max\{(1-d)/2 + 2 + 2s, 0\} + 1$, $s > d/2$. Then there exists a constant C_1 , which depends on $\text{diam}_H(M)$, the $C^2(M)$ norm of \tilde{n}^2 , and a constant C_2 , which depends on $\text{diam}_H(M)$ and the $C^{N+s}(M)$ norm of $n_i^2(x), i=1, 2$, such that if $u_1^{x_0, \omega_0}$ and $u_2^{x_0, \omega_0}$ solve the radiation problem (2.7) with coefficients n_1^2 and n_2^2 , respectively, then it follows that there exists small $\epsilon, \epsilon_0 > 0$ such that if*

$$\|n_1^2 - n_2^2\|_{C^3(M)} < \epsilon; \lambda^{-1} < \epsilon_0, \quad \delta = \sup_{x \in \partial M; (x_0, \omega_0) \in \partial_+ SM} \|u_1^{x_0, \omega_0} - u_2^{x_0, \omega_0}\| < \epsilon_0, \tag{2.8}$$

then this implies

$$\|n_2 - n_1\|_{C^2(M)} \leq C_1 \left(\frac{C_2}{\lambda^{\beta'}} + \delta \right)^\mu. \tag{2.9}$$

for some $\mu, \beta' \in (0, 1/2)$, with $\lambda\epsilon > 1$.

The uniqueness corollary follows immediately.

COROLLARY 2.1. *Assume $\delta = 0$ and that the assumptions of Theorem 2.1 are satisfied for all large λ . Then $n_1^2 = n_2^2$.*

Our results indicate that there is very little stability to be expected from such a problem.

COROLLARY 2.2. *The stability result also holds for the acoustic wave Equation (1.6) under the hypothesis of Theorem 2.1.*

A partial data result is also possible for non-trapping manifolds with strictly geodesically convex subsets, but this is discussed in the Appendix.

3. Comparison with previous results

In this paper, we derive a stability result for the higher order coefficients of the acoustic wave Equation (1.2) for fully phaseless measurements. Stable reconstructions have been made from Robin conditions for lower order terms than the ones considered here [5]. In the related case, for the generalized Helmholtz equation for Dirichlet boundary conditions in [24, 33] and Robin conditions in [9, 44] the potential can also be recovered. However stability estimates from phaseless measurements have not been previously given.

In [28, 27], uniqueness results in dimension 3 for lower order terms than the ones considered are derived from phaseless measurements. These papers are predicated on analyticity arguments, which require data in a small neighborhood of the source. The

question of phaseless stability from internal measurements for Schrödinger was also examined in [3]. These measurements are in contrast to the boundary data we require. We are not able to prove uniqueness results unless $\lambda \rightarrow \infty$. However, physically this is equivalent to setting the $n^2(x)$ term equal to zero.

The inverse problem of recovering the source for wave equations from Dirichlet boundary conditions is also examined in [42, 40, 21, 34, 41]. The major difference is that we are able to recover higher order terms which make the bicharacteristic flow more complicated.

Generically, we are looking for an asymptotic model to (1.2) of the form

$$U(x) = A(x, \lambda) \exp(i\lambda\psi(x)) = \sum_j \frac{a_j(x)}{\lambda^j} \exp(i\lambda\psi(x)), \tag{3.1}$$

which we show in the high frequency limit solves the Equation (1.2) up to suitable error terms. The variable $x \in M$ and $s = s(x)$ is a parameter which helps describe the bicharacteristic flow in terms of the coordinates on M . We use a Gaussian beam Ansatz which involves the construction of a phase function $\psi(x)$ and an amplitude a for the first order beams in local coordinates as

$$\begin{aligned} \psi(x) &= S(s) + (x - x(s)) \cdot p(s) + \frac{1}{2}(x - x(s)) \cdot \mathcal{M}(s)(x - x(s)) \\ a(s, x) &= a_0(s) + \mathcal{O}(d_{g'}(x, x(s))) \end{aligned} \tag{3.2}$$

where $x(s)$ is a curve which describes a Hamiltonian flow, given by (3.9) below, and $S(s), \mathcal{M}(s)$ will be specified functions of s determined by the Gaussian beam Ansatz. For the construction of the Ansatz, we follow the work of [30] closely.

In the previous work [46] the main author considered the operator

$$\tilde{L} = \Delta_{\mathbb{R}^d} + \lambda^2 + i\lambda\alpha n^2(x), \tag{3.3}$$

while here we use

$$L = \Delta_g + \lambda^2 n^2(x) + i\lambda\alpha n^2(x). \tag{3.4}$$

This corresponds to the acoustic wave equation. These operators have corresponding Hamiltonian flows given by the Hamiltonian functions which are $\tilde{H} = |p|^2 - 1$ and $H = |p|_g^2 - n^2$. We recall the main theorem of [46] where we considered the problem

$$(-\Delta - \lambda^2 - i\lambda n^2(x))u^{x_0, \omega_0} = h^{x_0, \omega_0}(x, \lambda), \tag{3.5}$$

associated to \tilde{L} where we let x_0 denote the position of the center of the plane wave source. Sources are indexed by x_0 and ω_0 .

In [46], the domain $\Omega \subset \mathbb{R}^d$ had the Euclidean metric instead of a Riemannian one. We then define the subset of the co-sphere bundle as

$$\partial\mathcal{S}\Omega^+ = \{(x_0, \omega_0) : x_0 \in \partial\Omega, \langle \nu, \omega_0 \rangle > 0\}, \tag{3.6}$$

where ν denotes the outward unit normal to the boundary. We recall the main theorem from [46]:

THEOREM 3.1. *Let $N \geq (1+d)/2 + 4$ and $\epsilon_0 > 0$, then there exists a constant C_1 which depends on $\text{diam}(\Omega)$, the $C(\Omega)$ norm of $n_i^2(x), i=1,2$ and a constant C_2 which depends*

on the $\text{diam}(\Omega)$ and the $C^{N+1}(\Omega)$ norm of $n_i^2(x)$, $i = 1, 2$ such that if $u_1^{x_0, \omega_0}$ and $u_2^{x_0, \omega_0}$ solve the radiation problem (3.5) with attenuation coefficients n_1^2 and n_2^2 respectively then it follows that if

$$\lambda^{-1} < \epsilon_0, \quad \delta = \sup_{\partial S\Omega^+} ||u_1^{x_0, \omega_0}| - |u_2^{x_0, \omega_0}|| < \epsilon_0, \tag{3.7}$$

then this implies

$$||n_2^2 - n_1^2||_{H^{-1/2}(\Omega)} \leq C_1 \left(\frac{C_2}{\lambda^{\beta'}} + \delta \right) \tag{3.8}$$

for some $\beta' \in (0, (2d)^{-1})$.

By contrast, in the main theorem, Theorem 2.1, we see that the additional conditional hypothesis that n_1 and n^2 be close $||n_1^2 - n^2||_{C^3(M)} < \epsilon$ for sufficiently small ϵ . This condition is important because it indicates the full problem for the acoustic wave equation is generically unstable.

To see why this is the case, we recall that our Gaussian beam solutions, both in [46] and here, are concentrated over curves $x(s)$ known as null-bicharacteristics. The top order terms of the approximation, the Ansatz, contain the explicit form of a ray transform which we can read off from the measurements we acquire. The main problem of isolating the X-ray transform of the coefficients $n^2(x)$ is to control the null-bicharacteristics.

In local coordinates, the ordinary differential equations which govern the ray path $\{(s, x(s)) : 0 \leq s \leq T\}$, along which solutions are concentrated, are

$$\frac{d^2 x_i(s)}{ds^2} = - \frac{\partial g^{jk}(x(s)) p_k p_j}{\partial x_i} + \frac{\partial n^2(x(s))}{\partial x_i}, \tag{3.9}$$

which is associated to L , resp.

$$\frac{d^2 x(s)}{ds^2} = 0, \tag{3.10}$$

which is associated to \tilde{L} . The second equation gives that the ray paths in \mathbb{R}^d are straight lines while the first one does not. Naturally one would expect the results to be more interesting if the ray paths are not straight lines. Thus, the problem of recovering $n^2(x)$ for the operator L is more difficult, and we address it in this paper.

In the Calderón Problem in Conformally Transversal Geometries, [25] and also [15] the question of boundary distance rigidity is reduce to a question of invertibility of the geodesic X-ray transform. As a consequence of their work, they reduce the question of recovery of source terms for several operators from the Dirichlet-to-Neumann maps to a question of invertibility of the geodesic X-ray transform. In this paper, we choose phaseless data as our measurements and show that the question of recovery of sound speed amounts to a question of invertibility and stability of a so-called flow transform.

The results presented here have applications to other operators for which the Hamiltonian flow and the geodesic flow do not coincide. In order to prove our results, we introduce a condition which one can think of as a generalization of the condition of Bardos-Lebeau-Rauch [29] for Hamiltonian flows. Specifically, we require the Hamiltonian flow to be simple.

As a result we view the significant contributions of this article as the following:

- We extend the Gaussian beam Ansatz developed in [30] to non-trapping manifolds M with compact closure and background metric denoted by g . Previously in [30] this was done for open sets in \mathbb{R}^d .
- We do not use any analyticity arguments to recover the coefficients or establish the stability estimates. As mentioned in the introduction we use the full phaseless measurements. This is the first time this has been done except by one of the authors in [46], representing an improvement over aforementioned works on phaseless scattering measurements [28, 27] or the Dirichlet to Neumann map [42, 40, 21, 34, 41].
- There is only one previous study on the flow transform in [7]. We make the connection between the flow transform and the boundary distance function under some hypotheses on n^2 . In particular it is shown the question of Hamiltonian flow stability without these hypotheses is distinct from the question of boundary distance rigidity. This is the first time in the literature these questions are shown to be distinct, and any stability has been demonstrated for the flow transform.

In order to achieve these points, the rest of this article takes the following outline. Section 4 introduces the Ansatz for the operator L and extends it to simple manifolds, where we highlight the precise form of the leading order terms which are necessary for the main theorem. Section 5 introduces the sources for the generalised Helmholtz equation, and Section 6 and 7 prove the required error estimates. The observability estimates, Section 8, establishes the main theorem by reading off from the measurements of the solutions the flow transform from the main order terms in the Ansatz. The Appendix on flow transforms contains stand alone results, but provides the necessary stability estimates to finish the proof of the main theorem in Section 8.

Notation: For two functions f, g , we write $f \sim g$ if there exists a constant $C > 0$ such that $C^{-1}f \leq g \leq Cg$. We denote $d_g(x, y)$ the distance between the points x, y defined by the Riemannian metric g .

4. Construction of solutions

We make more precise the explicit nature of the solutions. We let λ be a high frequency asymptotic parameter, $\lambda \in \mathbb{R}^+$, as in the introduction. Let $A(x, \lambda)$ denote the amplitude and $\psi(x)$ the (λ independent) phase. We use a Gaussian beam Ansatz to build asymptotic solutions to (1.2) in the high frequency limit, and then give estimates on the difference between these approximate solutions and the true solutions. The Gaussian beam Ansatz is predicated on an asymptotic expansion in inverse powers of λ followed by a Taylor series expansion around a curve $x(s)$ dictated by the Hamiltonian flow.

We begin with the following definition:

DEFINITION 4.1 (Order of the Ansatz). *Let*

$$U(x) = A(x, \lambda) \exp(i\lambda\psi(x)) \quad (4.1)$$

denote the Gaussian beam Ansatz corresponding to $Lu = 0$. For $C^N(M)$ coefficients n^2 , we say that a Gaussian beam is of order $N \in \mathbb{N}$ if the amplitude can be expanded as

$$A(x, \lambda) = \sum_{j=0}^{N-1} a_j(x) \lambda^{-j} \quad (4.2)$$

in inverse powers of λ with $a_j(x) \in H^{N+2}(M)$ independent of λ .

An N^{th} order Gaussian beam Ansatz U^N with superscript denoting the order, must satisfy

$$\|u - U^N\|_{H^m(M)}^2 \leq C\lambda^{-N+2+(1-d)/2+2m} \tag{4.3}$$

where u is the solution to (1.2), with sources $h(x, \lambda) \in C^N(M)$, and C depends on the $C^{N+m}(M)$ norm of n^2 . This estimate is proved explicitly in Section 6.

In order for the inequality (4.3) to be true, the order, N , of the Gaussian beam Ansatz then dictates the degree of accuracy of the Taylor expansion around the curve $x(s)$ required to solve (1.2) to a high degree in inverse powers of λ . We characterise this relationship using the eikonal and transport equations which are introduced later. We extend the work in [30] for the Gaussian beam Ansatz in constructing the phase $\psi(x)$ and amplitude $A(x, \lambda)$. Let R be a large positive real number, such that $M' \subset \{x; |x| < 6R\}$.

We claim:

THEOREM 4.1. *There is an N^{th} order Gaussian beam which solves the Helmholtz equation $Lu = 0$ in the set $\{x; |x| < 6R\}$. In local coordinates, it takes the form*

$$U^N(x) = A(x, \lambda) \exp(i\lambda\psi(x))\phi(x) \tag{4.4}$$

$$A(x, \lambda) = (a_0(s) + \mathcal{O}(d_g(x, x(s))) + \mathcal{O}(\lambda^{-1})) \tag{4.5}$$

where $x(s)$ is a curve in space defined by (3.9) with initial condition $(x_0, \omega_0) \in \partial_+ \mathcal{SM}$ and $\phi(x)$ is a cutoff which will be defined in a neighborhood of the curve. The coefficient $a_0(s)$ is given by

$$a_0(s) = \chi_\lambda(x) \exp\left(\int_0^s -\alpha n^2(x(t)) - \text{tr}\mathcal{M}(t) dt\right). \tag{4.6}$$

and $\psi(x)$ is given by (3.2) with $\mathcal{M}(s)$ a positive definite complex matrix whose evolution is governed by a Riccati equation.

We demonstrate that the N^{th} order Gaussian beam Ansatz can be explicitly constructed according to a system of ODEs. However, for the purposes of this paper the first order terms in the expansion are the most important.

We need the following lemma to start off with:

LEMMA 4.1. *Every geometric optics solution concentrates on an open set around the ray path $\{(s, x(s)) : 0 \leq s \leq T\}$ for some finite T , with $T > 0$. The flow for the ray path of H is defined by the set of ODEs in local coordinates on the manifold:*

$$\frac{dx_i}{ds} = 2g^{ij}(x(s))p_j, \quad \frac{dp_i}{ds} = -\frac{\partial g^{jk}(x(s))p_k p_j}{\partial x_i} + \frac{\partial n^2(x(s))}{\partial x_i} \tag{4.7}$$

and also

$$\frac{d}{ds}\psi(x(s)) = \nabla_g \psi(x(s)) \cdot \nabla_p H(x(s), p(s)) = n^2(x(s)). \tag{4.8}$$

Proof. We have $H(x, p) = |p|_g^2 - n^2$ associated to $H(x, \nabla_g \psi(x))$, so we set $p = \nabla_g \psi(x)$. We are looking for a solution to $H(x, \nabla_g \psi(x)) = 0$ because we want to solve

(4.14) to high order. We differentiate $H(x, \nabla_g \psi(x))$ with respect to x . This gives the following relationship:

$$\nabla_x H(x, \nabla_g \psi(x)) + D^2 \psi(x) \nabla_p H(x, \nabla_g \psi(x)) = 0 \tag{4.9}$$

where D^2 represents the Hessian associated to ψ . For any curve $y(s)$ we have the following identity

$$\begin{aligned} \frac{d}{ds} \nabla \psi(y(s)) &= D^2(\psi(y(s)) \frac{dy(s)}{ds}) \\ &= D^2(\psi(y(s)) \left(\frac{dy(s)}{ds} - \nabla_p H(y(s), \nabla_g \psi(y(s))) \right)) - \nabla_x H(y, \nabla_g \psi(y(s))) \end{aligned} \tag{4.10}$$

by substitution of (4.9). It follows that if we are seeking a null-bicharacteristic curve that

$$\begin{aligned} \frac{dx(s)}{ds} &= \nabla_p H((x(s), \nabla_g \psi(x(s)))) \\ \frac{dp(s)}{ds} &= \frac{d}{ds} (\nabla_g \psi(x(s))) = -\nabla_x H(x(s), \nabla_g \psi(x(s))) \end{aligned} \tag{4.11}$$

so that (4.10) vanishes. Moreover we see for our particular H that

$$\frac{d}{ds} \psi(x(s)) = \nabla_g \psi(x(s)) \cdot \nabla_p H(x(s), p(s)) = n^2(x(s)). \tag{4.12}$$

□

Proof. (Proof of Theorem 4.1.) We have that

$$LU = \sum_{j=-2}^l \exp(i\lambda\psi(x)) c_j(x) \lambda^{-j}. \tag{4.13}$$

The coefficients c_j , $j = 0, 1, \dots, l$, are defined recursively as follows

$$c_{-2} = (n^2(x) - |\nabla_g \psi|^2) a_0(x) \equiv E(x) a_0, \tag{4.14}$$

$$\begin{aligned} c_{-1} &= i\alpha n^2(x) a_0 + \nabla_g \cdot (a_0 \nabla_g \psi) + \nabla_g a_0 \cdot \nabla_g \psi + E(x) a_1, \\ c_j &= i\alpha n^2(x) a_{j+1} + \nabla_g \cdot (a_{j+1} \nabla_g \psi) + \nabla_g a_{j+1} \cdot \nabla_g \psi + E(x) a_{j+1} + \Delta_g a_j. \end{aligned} \tag{4.15}$$

The first Equation (4.14) is the eikonal and it dictates the degree of the Taylor expansion of the phase $\psi(x)$. The second set of Equations (4.15) are the transport equations and solving them recursively gives a definite system of equations for the coefficients $a_j(x)$.

If we Taylor expand the coefficients $a_j(x)$ around the central ray $x(s)$, we are looking for a system of differential equations for the coefficients. If the Gaussian beam Ansatz is of first order, then to solve (1.2) we chose c_{-2} and c_{-1} to vanish on the ray to third and first order respectively and define S as in 3.2.

This leads to the following set of differential equations:

$$d_s S(s) = 2n^2(x(s)), \quad d_s a_0(s) = -\text{tr}(\mathcal{M}(s)) a_0 - \alpha n^2(x(s)) a_0. \tag{4.16}$$

Denoting $\mathcal{M}_{jk}(s) = \partial_{x_j x_k}^2 \psi(x(s))$, we also must have

$$-d_s \mathcal{M}_{jk}(s) = \sum_{\ell,r} (\partial_{p_\ell p_r}^2 \tilde{H}(x(s), p(s))) \mathcal{M}_{j\ell}(s) \mathcal{M}_{kr}(s) \tag{4.17}$$

$$+ \sum_{\ell} \left(\partial_{x_j p_\ell}^2 \tilde{H}(x(s), p(s)) \mathcal{M}_{\ell k}(s) + \partial_{x_k p_\ell}^2 \tilde{H}(x(s), p(s)) \mathcal{M}_{\ell j}(s) \right) \tag{4.18}$$

$$+ \partial_{x_j x_k}^2 \tilde{H}(x(s), p(s)) \tag{4.19}$$

with $\tilde{H}(x, p) = p \cdot \nabla_p H(x, p)$.

These equations correspond to differentiating the equation

$$\partial_s \psi(x) + \nabla_g \psi(x) \cdot \tilde{H}(x, \nabla_g \psi) = 0 \tag{4.20}$$

with respect to x and evaluating along the curve $x(s)$ with respect to the constraint $H(x, \nabla_g \psi) = 0$. This equation is the correct one to consider by the derivation of (4.8). These equations can be easily checked [37]. The matrix \mathcal{M} is known as the Hessian matrix. Higher order beams can be constructed by requiring c_{-2} vanishing to higher order on γ .

We see that

$$a_0(s) = \chi_\lambda(x) \exp \left(\int_0^s -\alpha n^2(x(t)) - \text{tr} \mathcal{M}(t) dt \right). \tag{4.21}$$

The phase ψ needs to verify the conditions

$$\psi(x(s)) = S(s), \quad \nabla \psi(x(s)) = p(s), \quad D^2 \psi(x(s)) = \mathcal{M}(s), \tag{4.22}$$

compatible with 3.2, and differentiation of (4.8) with respect to x . We use the initial data $S(0) = 0$ and $\mathcal{M}(0)$ satisfies (2.6) cf. [30, Section 2]. If the matrix $\mathcal{M}(s)$ satisfies (2.6), then for all s the matrix $\mathcal{M}(s)$ inherits the properties of $\mathcal{M}(0)$;

$$\mathcal{M}(s) \dot{x}(s) = \dot{p}(s) \quad \mathcal{M}(s) = \mathcal{M}(s)^\perp \tag{4.23}$$

and $\Im \mathcal{M}(s)$ is positive definite on the orthogonal complement of $\dot{x}(s)$, cf [35].

In order to write down such an $\psi(x)$ we need to be able to write s as a function of the coordinate variables $x \in M$. We know $x(s)$ traces out a smooth curve $\tilde{\gamma}$ in \mathbb{R}^d and if we assume $x(s)$ is non-trapping then this curve is a straight line when s is sufficiently large.

Remember, we consider R large enough so that the set $\{x : |x|_{g'} < 6R\}$ contains \overline{M} . We set

$$\Omega(\epsilon_0) = \{x : |x|_{g'} \leq 6R \quad d_{g'}(x, \tilde{\gamma}) \leq \epsilon_0\} \tag{4.24}$$

as a tubular neighborhood of $\tilde{\gamma}$ with radius ϵ_0 in the ball $\{|x|_{g'} \leq 6R\}$. Choosing ϵ_0 sufficiently small we can uniquely define $s = s(x)$ for all $x \in \Omega(\epsilon_0)$ such that $x(s)$ is the closest point on $\tilde{\gamma}$ to x provided $\tilde{\gamma}$ has no self-intersections. The variable s is the analogue of the time variable for time dependent problems.

We now define a cutoff function $\phi_\lambda(x) \in C^\infty(\mathbb{R}^d)$ for $\lambda > 0$ such that

$$\phi_\lambda(x) = \begin{cases} 0 & \text{if } x \in \Omega^c(2\lambda^{-1/2d}) \\ 1 & \text{if } x \in \Omega(\lambda^{-1/2d}). \end{cases} \tag{4.25}$$

One can arrange that there is a constant C such that

$$\sup_{x \in M} |\nabla_x^m \phi_\lambda| \leq C \lambda^{-m}. \tag{4.26}$$

We drop the subscript λ for the rest of this paper. We used the fact that $\lambda^{-1} < \epsilon_0$ so the definition of the parameter s makes sense as above.

We also recall the following result:

LEMMA 4.2 ([47, Corollary 5]). *Let $\psi(x)$ be the phase function of a first order beam. We have*

$$\exp(-2\lambda\Im\psi(x)) \sim \exp(-\lambda C d_{g'}(x, x(s))^2),$$

where C is independent of λ .

Let B denote the set

$$B = \{x : d_{g'}(x, x(s)) > \lambda^{-(\frac{1}{2}-\sigma)}, 0 \leq s \leq 6R\}, \quad \sigma > 0, \sigma \in \mathbb{R}.$$

We conclude that since $2\Im\psi(s, x) \sim d_{g'}(x, x(s))^2$, $\exp(-2\lambda\Im\psi(x))$ is exponentially decreasing in λ for all $x \in B$. Notice that we are taking more care to construct the cutoff functions than in [30], as they are crucial for the phaseless measurements.

Proof. We need only observe that $\mathcal{M}(s)$ is a bounded and positive definite matrix. From the form of the phase functions constructed the desired result follows. \square

One requires the c'_j s with $j > -2$ also vanish to higher order and obtain a recursive set of linear equations for the partial derivatives of a_0, a_1, \dots, a_l , which are independent of λ . More precisely, for an N^{th} order beam $l = \lceil N/2 \rceil - 1$, the $c_j(x)$ should vanish to order $N - 2j - 2$ whenever $-2 \leq j \leq l - 1$. From the definition of the cutoffs and a standard energy estimate, we will see in Section 6 that (4.3) is then satisfied for appropriate source terms. It is important to note that we could construct, explicitly, higher order terms of the beams using this recipe, but we are only interested in the precise form of the first order terms as they are sufficient for our asymptotics. The construction of the localized cutoff finishes the construction of U_λ . A standard argument gives that U_λ extends across each local coordinate chart to cover the ray $x(s)$ iteratively, c.f. [25, Section 7]. \square

5. Introduction of the source terms

We now introduce source functions. We summarize the results from [30] to show we can build such sources. We claim:

THEOREM 5.1. *There exists a Gaussian beam solution which solves the Equation (1.2) and is found by solving the Dirichlet problem on one side of hyper-planes which contain a source point.*

Proof. This argument is similar to [30, Section 2.1] and is repeated for completeness. We let ρ be a function such that $|\nabla_g \rho| = 1$ on the hypersurface $\Sigma = \{x : \rho(x) = 0\}$. Let x_0 be a point in Σ and we let $(x(s), p(s))$ be the solution path — in other words the null-bicharacteristics with $(x(0), p(0)) = (x_0, n(x_0) \nabla_g \rho(x_0))$. The hypersurface Σ is given by $s = \sigma(y)$ with $\sigma(0) = 0$ and $\nabla \sigma(0) = 0$ where $x = (s, y)$ and $y = (y_1, \dots, y_{d-1})$ is transversal. We let the optics Ansatz $U(x)$ have initial data $(x(0), p(0))$, and be defined in this tubular neighborhood. We let U^+ be the restriction of U to $\{x : \rho(x) \geq 0\}$. Because we need to have a source term which is a multiple of $\delta(\rho)$, we also need a second, ‘outgoing’

solution U^- , defined on $\{x:\rho(x)\leq 0\}$. It is then equal to U^+ on the hypersurface Σ . For example, we can write the ingoing and outgoing optics solutions as

$$U^+ = A^+(x, \lambda)\exp(i\lambda\psi^+(x)), \quad U^- = A^-(x, \lambda)\exp(i\lambda\psi^-(x)), \tag{5.1}$$

where [30] have set $\psi^+ = \psi^-$ and $A^+ = A^-$ on Σ . The requirement that their Taylor series coincide on the boundary is equivalent to setting $\partial_y^\alpha|_{y=0}\psi^+(\sigma(y)) = \partial_y^\alpha|_{y=0}\psi^-(\sigma(y))$. We extend U^+ to be 0 on $\{x:\rho(x)<0\}$ and U^- to be 0 on $\{x:\rho(x)>0\}$. We define our geometric optics Ansatz solution U to be $U = U^+ + U^-$. We set $A = A^+ = A^-$ on Σ . In order to add the source term, we notice that

$$LU = i\lambda \left(\left(\frac{\partial\psi^+}{\partial\nu} - \frac{\partial\psi^-}{\partial\nu} \right) A(x, \lambda) + \frac{\partial A^+}{\partial\nu}(x, \lambda) - \frac{\partial A^-}{\partial\nu}(x, \lambda) \right) \exp(i\lambda\phi^+)\delta(\rho) \tag{5.2}$$

$$+ f_{gb} = g_0\delta(\rho) + f_{gb},$$

where $\nu(x) = \nabla\rho(x)$ is the unit normal to Σ . We consider the singular part $g_0\delta(\rho) = h(x, \lambda)$ to be the source term and f_{gb} the error. Then we obtain

$$f = \exp(i\lambda\psi^+) \sum_{j=-2}^l c_j^+(x)\lambda^{-j} + \exp(i\lambda\psi^-) \sum_{j=-2}^l c_j^-(x)\lambda^{-j}, \tag{5.3}$$

where we extend c_j^+ to be zero for $\rho(x) < 0$ and c_j^- to be zero for $\rho(x) > 0$. We know by construction that $c_{-2}^\pm = \mathcal{O}(d_g(x, x(s))^3)$ and $c_{-1}^\pm = \mathcal{O}(d_g(x, x(s)))$, respectively for 1st order beams.

As a result, we see that $h(x, \lambda)$ as defined in (2.2) satisfies (5.2), with $\rho(x)$ a boundary defining function. The incoming and outgoing Gaussian beams are matched at the boundary of \overline{M} , which we can consider to be our hyper-surface Σ by construction. \square

6. Error estimates

We complete in this section the error estimates on the beams which are needed for the main theorem. We claim for N^{th} order Gaussian beams:

LEMMA 6.1. *We have the following estimate for the Gaussian beam error:*

$$\|f_{gb}(x)\|_{H^m(|x|_{g'} < R)}^2 \leq C\lambda^{-N+2+(1-d)/2+2m}. \tag{6.1}$$

with C independent of λ depending on the $C^{N+m}(M')$ norm of both the initial data and $\tilde{n}^2(x)$.

Proof. We have that the $c_j(x)^\pm$ are bounded and

$$c_j^\pm(x) = \sum_{|\beta|=N-2j-2} d_{\beta,j}^\pm d_{g'}(x, x(s))^\beta, \quad j = -2, \dots, l-1, \tag{6.2}$$

where the $d_{\beta,j}$ are bounded by Taylor’s theorem. We obtain

$$|c_j^\pm(x)|_{g'} \leq C_j d_{g'}(x, x(s))^{N-2j-2} |\phi(x)|_{g'}, \tag{6.3}$$

with the C_j uniformly bounded independent of λ . As $\mathcal{M}(s)$ is a matrix with positive definite imaginary part,

$$\text{Im}\psi^\pm(x) \geq C d_{g'}(x, x(s))^2 \tag{6.4}$$

for some positive constant C . From the elementary inequality [30] for $a, b > 0$,

$$b^p \exp(-ab^2) \leq C_p a^{-p/2} \exp(-ab^2/2), \quad C_p = \left(\frac{p}{e}\right)^{\frac{p}{2}}, \tag{6.5}$$

with $p = N - 2j - 2$, $a = \lambda c$ and $b = d_{g'}(x, x(s))$, $x \in \Omega(\lambda^{-1/2})$, we obtain

$$|f_{gb}(x)|_{g'} \leq \exp(-\lambda \Im \psi^\pm(x)) \sum_{j=-2}^l |c_j(x)|_{g'} \lambda^{-j}. \tag{6.6}$$

see [17]. It follows that

$$\begin{aligned} |f_{gb}(x)|_{g'} &\leq \exp(-\lambda c d_{g'}(x, x(s))^2) \sum_{j=-1}^l C_j d_{g'}(x, x(s))^{N-2j-2} \lambda^{-j} \\ &\leq C \exp(-\lambda c d_{g'}(x, x(s))^2/2) \sum_{j=-2}^l \lambda^{-N/2+j+1} \lambda^{-j} \\ &\leq C \exp(-\lambda c d_{g'}(x, x(s))^2/2) \lambda^{-N/2+1} \end{aligned}$$

with C a generic constant independent of λ depending on then $C^{N+m}(M')$ norm of both the initial data and $\tilde{n}^2(x)$. If we differentiate $c_j^\pm(x)$ and $\exp(i\lambda\psi(x))$, then we have for the N^{th} order beams

$$\begin{aligned} &\|f_{gb}(x)\|_{H^m(|x|_{g'} < R)}^2 \\ &\leq C \lambda^{-N+2+2m} \int_{x \in \Omega(\lambda^{-1/2})} \exp(-2\lambda C d_g(x, x(s))^2) dx \leq C \lambda^{-N+2+(1-d)/2+2m}, \end{aligned} \tag{6.7}$$

as claimed since $\Omega(\lambda^{-1/2d}) \subset \Omega(\lambda^{-1/2})$ and the integrand is positive. □

Notice that this error now includes a function which is localized in a neighborhood of $\mathcal{O}(1/\sqrt{\lambda})$, which is more precise than in [30]. The reason for the precise ϕ_λ is twofold: We need a localized solution for (8.23) to hold, and we are also considering a bounded domain.

7. Extension of the Ansatz

The extension of the Gaussian beam Ansatz to M' will provide us with the means to make observations further away from the source and still identify source terms. In the last section we built an Ansatz which is a subset of a slightly extended version of M . We view the manifold M as an embedded subset of \mathbb{R}^d , so we have the following inclusions:

$$\overline{M} \subset \overline{M'} \subset \{x : |x|_{g'} < 6R\}. \tag{7.1}$$

for R sufficiently large. We briefly review the results of [30]. We set

$$\tilde{f}_u = LU - g_0 \delta(\rho). \tag{7.2}$$

We would like this function to be supported in $|x|_{g'} < 6R$ and be $\mathcal{O}(\lambda^{-1})$. We let $G_\lambda(x)$ be the Green's function for the Helmholtz operator L .

In order to extend our approximate solution U , we introduce a smooth cutoff $\chi_a(x)$ such that

$$\begin{aligned} \chi_a(x) &= 1 \text{ for } |x|_{g'} < (a-1)R, \\ \chi_a(x) &= 0 \text{ for } |x|_{g'} > aR. \end{aligned}$$

Now we set

$$\tilde{U}(x) = \chi_3(x)U(x) + \int G_\lambda(x-y)\chi_5(y)L[(1-\chi_3(y))U(y)]dy. \tag{7.3}$$

In [30], they prove

$$\left\| U - \tilde{U} \right\|_{H^m(|x|_{g'} < 6R)} = \mathcal{O}(\lambda^{-n}). \tag{7.4}$$

Therefore the size of \tilde{U} and the extension depends on the size of f_{gb} . Furthermore, using ideas in Vainberg [45], they also prove

$$\|U - u\|_{H^m(|x|_{g'} < 6R)} \leq C\lambda^{-1} \|f_{gb}\|_{H^m(|x|_{g'} < R)}. \tag{7.5}$$

The triangle inequality allows us to conclude

$$\left\| u - \tilde{U} \right\|_{H^m(|x|_{g'} < 6R)} \leq C\lambda^{-1} \|f_{gb}\|_{H^m(|x|_{g'} < R)}. \tag{7.6}$$

The proofs done in [30] are for when (M, g) has the Euclidean metric, but can be easily extended as long as M is simple with respect to g .

In conclusion, we have that:

LEMMA 7.1. *There exists an extension of the Gaussian beam solution of the problem (1.2) to the whole space such that*

$$\left\| u - \tilde{U} \right\|_{H^m(M')} \leq C\lambda^{-1} \|f_{gb}\|_{H^m(|x|_{g'} < R)}. \tag{7.7}$$

Here we know \tilde{U} is supported in $|x| < 6R$ and we know u is in $H^m(\mathbb{R}^d)$ with g' metric so that for R sufficiently large, $\left\| u - \tilde{U} \right\|_{H^m(M')}$ is essentially the same as $\left\| u - \tilde{U} \right\|_{H^m(|x|_{g'} < 6R)}$.

8. Observability inequalities

We prove the main theorem, Theorem 2.1 in this section. Recall that we are considering two different coefficients n_1^2 and n_2^2 and their associated solutions u_1 and u_2 . We consider our globally defined complex optics solutions \tilde{U}_1 and \tilde{U}_2 , which were constructed previously. We drop the superscripts x_0, ω_0 where it is understood. This section is rather technical, so we outline it as follows. We start with a sequence of Lemmas that allow us to read off from the phasesless measurements an amplitude from the Ansatz. The explicit form of the amplitude contains the flow transform and from the construction of from the approximate solutions we show it is the leading order term with a high degree of accuracy. Then we use the mean value theorem to recover the exact form of the flow transform, and finish the proof of the main theorem by applying the results in the appendix.

We start by finding the leading order solution to a high degree of accuracy. In local coordinates, for $j=1,2$, we let $\mathcal{A}_j(x)$ be defined as

$$\mathcal{A}_j(x) = a_{j0}^1(0) \exp \left(- \int_0^s \alpha(n_j^2(x_j(t))) dt - \int_0^s \text{tr} \mathcal{M}_j(t) dt \right) \exp(i\lambda\psi_j(x)) \phi_j(x). \tag{8.1}$$

We claim $\mathcal{A}_j(x)$ are the most important terms in the Ansatz construction.

LEMMA 8.1. *For $i=1,2$, and solutions u_j to (2.7) with respective attenuation coefficients n_j^2 , we have*

$$u_j(x) = \mathcal{A}_j(x) + E_j(x) \tag{8.2}$$

with $\|E_j\|_{C(M')} \leq \frac{C}{\sqrt{\lambda}}$. Here C is a generic constant, which depends on the $C^{N+s}(M)$ norm of $n_j^2(x)$ with N such that $\max\{(1-d)/2 + 2 + 2s, 0\} + 1$.

Proof. This estimate is a result of building an approximate solution with N sufficiently large. We conclude from (7.6) and (6.7)

$$\left\| \tilde{U}_j - u_j \right\|_{H^s(M')} \leq \frac{C}{\sqrt{\lambda}}. \tag{8.3}$$

Here C is a generic constant, which depends on the $C^{N+s}(M)$ norm of $n_j^2(x)$ with N such that $\max\{(1-d)/2 + 2 + 2s, 0\} + 1$. We use the fact that \tilde{U}_j , and u_j are bounded in $C^{N+s}(M)$ norm to conclude

$$\left\| \tilde{U}_j - u_j \right\|_{C^0(M')} \leq \frac{C}{\sqrt{\lambda}}. \tag{8.4}$$

This estimate is a consequence of Sobolev embedding, as $H^s(M) \subseteq L^\infty(M)$ provided $s > d/2$. We use the estimate on the first order terms

$$\left\| U_j - a_j^1 \exp(i\lambda\psi_j) \right\|_{C^0(M')} \leq \frac{C}{\lambda} \tag{8.5}$$

where a_j^1 is the first order term, found to leading order by (4.4) and (4.21). Combining the estimates (8.5) and (8.4), gives the estimate (8.2). Now we need to combine the estimates to recover the X-ray transform. We would like to use the first order coefficients to reconstruct the coefficient $n^2(x)$. Using (4.4), we know that

$$\left| a_j^1(x) \exp(i\lambda\psi_j(x)) - a_{j0}^1(0) \exp \left(- \int_0^s \alpha(n_j^2(x_j(t))) dt - \int_0^s \text{tr} \mathcal{M}_j(t) dt \right) \exp(i\lambda\psi_j(x)) \phi(x) \right| = \mathcal{O}(\lambda^{-1/2}), \tag{8.6}$$

by choice of the cutoff function in (4.25). This completes the proof of the Lemma. \square

Examining (8.2), we are interested in the right hand side. Using (8.6), we would like to approximate its modulus for $j=2$. We claim

LEMMA 8.2. *The function $\mathcal{A}_2(x)$ can be approximated in local coordinates as:*

$$|\mathcal{A}_2(x)| \tag{8.7}$$

$$= \left| a_0(0) \left(-\exp \left(-\int_0^s \alpha(n_2^2(x_2(t)) dt \right) \right) \exp \left(-\int_0^s \text{tr} \mathcal{M}_1(t) dt \right) \exp(\lambda \underline{\psi}_1(x)) \phi_1(x) \right| \times (\mathcal{E}_2(x)), \tag{8.8}$$

where

$$\underline{\psi}_1(x) = -(x - x_1(s)) \Im \mathcal{M}_1(s) \cdot (x - x_1(s)). \tag{8.9}$$

and

$$\mathcal{E}_2(x) = \mathcal{O}(\exp(-\lambda \epsilon)). \tag{8.10}$$

Before we start, we need the following lemma to control the error, \mathcal{E}_2 . The Lemma below works for level sets of general Hamiltonians H , however we chose to focus on the case $H(x, p) = |p|_g^2 - n^2$ which is relevant for our solution curve. Again, we remind the reader the Equation (4.20) is the correct one that we want to solve to high order by the derivation of Lemma 4.1. We let $\epsilon \in (0, 1)$ be a fixed positive constant. We set $\tilde{H}(x, p) = p \cdot \nabla_p H(x, p)$.

LEMMA 8.3 (Uniqueness). *Assume that $\psi_1, \psi_2 \in C^\infty(M; \mathbb{C})$, $s_0, s \in (a, b)$, satisfy*

$$\begin{aligned} \psi_1(x) &= \psi_2(x) + \mathcal{O}((d_{g'}(x, x_1(s_0)))^{m+1}), \text{ for } s_0 \quad (\partial_x \psi_j)(x_j(s)) = p_j(s), \\ \partial_s \psi_j(x) + \tilde{H}(x, \nabla_g \psi_j(x)) &= \mathcal{O}(d_{g'}(x, x_j(s))^{m+1}) \end{aligned}$$

for some $m \in \mathbb{N}$. Then the following holds

$$|x_1(s) - x_2(s)| = \mathcal{O}(\epsilon) \quad |\mathcal{M}_1(s) - \mathcal{M}_2(s)| = \mathcal{O}(\epsilon) \tag{8.11}$$

the order terms depend uniformly on $\text{diam}(M)$ and the $C^1(M)$ and $C^3(M)$ norm of n_2^2 , respectively.

Proof. We can map the neighborhood of the flow on the manifold to the Euclidean plane and use the metric $g_{ij} = \delta_{ij}$ the standard Euclidean one. In this case we have by definition of the Hamiltonian flow.

$$\left| \frac{d|\tilde{x}|}{ds} \right| \leq \left| \frac{d\tilde{x}}{ds} \right| \leq |(n_1 - n_2)(x_1(s))| + C|\tilde{x}| \tag{8.12}$$

where $C = \sup_x |\nabla n_2(x)|$ by the mean value theorem and $\tilde{x}(s) = x_1(s) - x_2(s)$. Given initial conditions $\tilde{x}(0) = 0$ we can use Gronwall's inequality with $w = |\tilde{x}|$ to conclude

$$|\tilde{x}(s)| = \mathcal{O}(\epsilon) \tag{8.13}$$

since $\|n_1^2 - n_2^2\|_{C^3(M')} < \epsilon$. The \mathcal{O} terms depend on the length of the geodesic.

We also have by definition of the matrix Ricatti equations:

$$\frac{d|\tilde{\mathcal{M}}(s)|}{ds} \leq \left| \frac{d}{ds} \tilde{\mathcal{M}}(s) \right| = -D^2 n_1^2(x_1(s)) + D^2 n_2^2(x_2(s)) + \mathcal{M}_1^2(s) - \mathcal{M}_2^2(s) \tag{8.14}$$

which then implies

$$\left| \frac{d}{ds} \tilde{\mathcal{M}}(s) \right| \leq |D^2(n_1^2 - n_2^2)(x_1(s))| + C_1|\tilde{x}| + C_2|\tilde{\mathcal{M}}| \tag{8.15}$$

where $C_1 = \sup(D^3 n_2(x))$ by the mean value theorem and $\tilde{M}(s) = \mathcal{M}_1(s) - \mathcal{M}_2(s)$, $C_2 = \sup_{j=1,2,s} \{\mathcal{M}_j(s)\}$. Applying Gronwall's theorem again gives the second desired inequality. □

Proof. The expansion implies

$$\begin{aligned} (x - x_1(s)) &= (x - x_2(s)) + \mathcal{O}(\epsilon) + \mathcal{O}(d_{g'}(x, x_1(s))^2); \\ (x - x_1(s))\Im \mathcal{M}_1(s)(x - x_1(s)) & \\ = (x - x_2(s))\Im \mathcal{M}_2(s)(x - x_2(s)) + \mathcal{O}(\epsilon) + \mathcal{O}(d_{g'}(x, x_1(s))^3) & \end{aligned} \tag{8.16}$$

and also

$$\Im \mathcal{M}_1(s) = \Im \mathcal{M}_2(s) + \mathcal{O}(\epsilon) \tag{8.17}$$

We now proceed to the proof of Lemma 8.2. It follows Lemma 8.3 in local coordinates:

$$|\mathcal{A}_2(x)| = |a_0(0) \exp\left(-\int_0^s \alpha(n_2^2(x_2(t)) dt)\right) \exp\left(-\int_0^s \text{tr} \mathcal{M}_1(t) dt\right) \exp(\lambda \psi_1(x)) \phi_1(x)| \times (\mathcal{E}_2) \tag{8.18}$$

with $\psi_1(x)$ as above. Since Lemma 8.3 is local, the order terms depend on $\text{diam}(M)$ and $C^3(M)$ norm of $n_1^2(x)$ and $n_2^2(x)$, made precise by (8.16), (8.17). This allows us to conclude (8.7), with appropriate loss of error. □

We need one more auxiliary Lemma which is taken from [46] before proving the main theorem.

LEMMA 8.4. *Let $A(x)$ and $B(x)$ be positive functions in $C^0(\mathbb{R})$ and $\epsilon \in (0, 1)$ such that*

$$\|\exp(-A(x)) - \exp(-B(x))\|_{C^0(\mathbb{R})} < \epsilon. \tag{8.19}$$

Then there is a constant C depending on the $C^0(\mathbb{R})$ norms of A and B , such that

$$\|A(x) - B(x)\|_{C^0(\mathbb{R})} < C\epsilon. \tag{8.20}$$

Proof. By the mean value theorem, there exists an r_* between $B(x)$ and $A(x)$ for each fixed x such that

$$|(\exp(-A) - \exp(-B))| = \left| \left(-\int_B^A \exp(-r) dr \right) \right| = |B - A| \exp(-r_*). \tag{8.21}$$

The desired result follows by taking the supremum over x and applying (8.19). □

We can now begin the proof of Theorem 2.1.

Proof. It follows from Lemma 8.1 and 8.2:

$$|\mathcal{A}_1(x)| - |\mathcal{A}_2(x)| \leq |u_1| - |u_2| + E(x) \tag{8.22}$$

with $\|E(x)\|_{C(M)} \leq \frac{C}{\sqrt{\lambda}}$ and C a generic constant depending on the $C^{N+s}(M)$ norm of n_j^2 for $j = 1, 2$.

We use (8.7) to look at the supremum over $x \in \partial M$ of (8.22). Because $\Im \mathcal{M}_1(s)$ is a positive definite matrix, by Corollary 4.2 we have

$$\sup_{x \in \partial M} \left| \exp(\lambda \underline{\psi}_1(x)) \phi(x) \right| = 1. \tag{8.23}$$

Indeed, $x_1(s)$ reaches the boundary and is contained in $\Omega(\lambda^{-1/2})$. Using (8.7), we obtain that the supremum of the left hand side of (8.22) over $x \in \partial M$ equals

$$C(s) \left| a_0^1(0) \left(\exp \left(- \int_0^{\tau(x_0, \omega_0)} \alpha(n_1^2(x_1(t))) dt \right) - \exp \left(- \int_0^{\tau(x_0, \omega_0)} \alpha(n_2^2(x_2(t))) dt \right) \right) \times E_0(x) \right|. \tag{8.24}$$

Here

$$C(s) = \exp \left(- \int_0^{\tau(x_0, \omega_0)} \text{tr} \mathcal{M}_1(t) dt \right). \tag{8.25}$$

with $\tau(x_0, \omega_0)$ the time it takes for $x(s)$ defined by (3.9) with initial conditions (x_0, ω_0) to exit M . We also have

$$\|E_0(x)\|_{C^0(M')} = \exp(\mathcal{O}(-\lambda\epsilon)) \tag{8.26}$$

by (8.18) from Lemma (8.3). Note that $\lambda\epsilon > 1$ implies $E_0(x)$ is very nearly 1. Combining (8.22) and (8.24), we see that:

$$\begin{aligned} & \left| C(s) \left| a_0^1(0) \left(\exp \left(- \int_0^{\tau(x_0, \omega_0)} \alpha(n_1^2(x_1(t))) dt \right) - \exp \left(- \int_0^{\tau(x_0, \omega_0)} \alpha(n_2^2(x_2(t))) dt \right) \right) \right| \right| \\ & \leq \left(\delta + \frac{C_2}{\lambda^{\beta'}} \right). \end{aligned} \tag{8.27}$$

where we have used the fact $\lambda\epsilon > 1$ and the triangle inequality to fold the error from $1 - \mathcal{E}_2(x)$ into the term $\frac{C_2}{\lambda^{\beta'}}$.

Now we can see by assumption that δ, λ^{-1} are less than ϵ_0 . Applying Lemma 8.4, to (8.27) with $\epsilon = \epsilon_0$ and

$$A = \int_0^{\tau(x_0, \omega_0)} n_1^2(x_1(t)) dt \quad B = \int_0^{\tau(x_0, \omega_0)} n_2^2(x_2(t)) dt \tag{8.28}$$

we obtain:

$$\|A - B\|_{C^0(\partial SM^+)} \leq \tilde{C}_1 \left(\delta + \frac{C_2}{\lambda^{\beta'}} \right). \tag{8.29}$$

\tilde{C}_1 denotes a generic constant depending on the $C^0(M)$ norm of n_1^2, n_2^2 , while C_2 depends on the $C^{N+s}(M)$ norm. Applying the result, Corollary A.2, directly from the appendix allows us to conclude the desired result. \square

Concluding Remarks: In this article we were able to use a Gaussian beam Ansatz to solve the inverse source problem for the acoustic wave and generalised Helmholtz equation. Open challenges are the inclusion of trapped rays, and invertability and stability of the flow transform for more general geometries.

Appendix A. Flow transform and boundary distance rigidity. We now consider the problem of recovering a function stably from its integrals over the integral curves of a Hamiltonian flow. A related nonlinear problem was considered in [7], and generalizations of the geodesic X-ray transform problem have been studied before (see e.g. [6, 16, 36]). However, we are not aware of any discussion of the present problem in the literature. Although all key ingredients are already known, we present the statement and proof for completeness.

Let (M, g) be a compact Riemannian manifold with boundary and $q: M \rightarrow \mathbb{R}$ a smooth function. We let Ω denote a magnetic field which is a closed 2-form. We consider the law of motion governed by Newton’s equation:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = Y(\dot{\gamma}) - \nabla q(\gamma). \tag{A.1}$$

We have $Y: TM \rightarrow TM$ is the Lorentz force which is associated to Ω , which is the map uniquely determined by

$$\Omega_x(\xi, \eta) = \langle Y_x(\xi), \eta \rangle \tag{A.2}$$

$\forall x \in M, \xi, \eta \in T_x M$. The curve which satisfies (A.1) is called an \mathcal{MP} geodesic as per [7]. The \mathcal{MP} stands for Maupertius principle. The Equation (A.1) defines a flow ϕ_t on TM . which we call an \mathcal{MP} flow. Whenever $\Omega=0$ the flow is the potential flow and this corresponds to the Hamiltonian flow, corresponding to the Hamiltonian function $H: T^*M \rightarrow \mathbb{R}$ given by $H(x, p) = \frac{1}{2}g_{ij}(x)p^i p^j + q(x)$ in local coordinates. For our purposes it is sufficient to examine potential flows, but our results could extend to situations with a magnetic term included, in which case the Hamiltonian would be $H(x, p) = |p + \alpha|_g^2 + q(x)$, $\Omega = d\alpha$ for Ω exact.

We refer to the curves generated by the Hamiltonian flow as H -geodesics. For an \mathcal{MP} flow, the energy $\mathcal{E}(x, \xi) = \frac{1}{2}|\xi|_g^2 + q(x)$ is an integral of motion. The law of conservation of energy says for all \mathcal{MP} geodesics the energy is constant along the geodesic.

The \mathcal{MP} flow depends on the choice of energy level. Consider an H -geodesic $\gamma: [0, T] \rightarrow M$. The value of $H(\gamma(t), \dot{\gamma}(t))$ is independent of t , and we write H_0 for this constant. We assume that $H_0 > \max_{x \in M} q$ so we can set $S_{H_0}M = \mathcal{E}^{-1}(H_0)$, the bundle with energy H_0 over M . It is then necessary for H_0 to be larger than $\max_{x \in M} q(x)$. Otherwise some fibers of the bundle are empty. We let $\nu(x)$ the inward normal to ∂M at x . We denote

$$\partial_+ S_{H_0}M = \{(x, \xi) \in S_{H_0}M : \langle \xi, \nu \rangle \geq 0\}. \tag{A.3}$$

For $(x, \xi) \in S_{H_0}M$ let $\tau(x, \xi)$ be the time such that the \mathcal{MP} geodesic with $\gamma(0) = x, \dot{\gamma}(0) = \xi$ reaches ∂M . The function $\tau(x, \xi): S_{H_0}M \rightarrow \mathbb{R}$ is smooth by Lemma A.5 of [7].

Let $\Lambda(\cdot, \cdot)$ denote the second fundamental form of ∂M . The boundary of ∂M is strictly \mathcal{MP} convex if

$$\Lambda(x, \xi) > \langle Y_x(\xi), \nu(x) \rangle - d_x q(\nu(x)) \tag{A.4}$$

for all $(x, \xi) \in \partial_+ S_{H_0} M$.

For all $x \in M$, the \mathcal{MP} exponential map at x is the partial map

$$\exp_x^{\mathcal{MP}} : T_x M \rightarrow M \tag{A.5}$$

which is given by

$$\exp_x^{\mathcal{MP}}(t\xi) = \pi \circ \phi_t(\xi) \quad \text{for } t \geq 0, (x, \xi) \in S_{H_0} M. \tag{A.6}$$

For all $x \in M$, $\exp_x^{\mathcal{MP}}$ is a C^1 smooth partial map and $T_x M$ is C^∞ smooth on $T_x M \setminus \{0\}$. We let \exp_x^H denote the exponential map associated to the potential Hamiltonian ($\Omega = 0$). Notice that the exponential map $\exp_x^{\mathcal{MP}}$ depends on the energy level.

Since our manifold has a boundary, this is only a partial map. We say that the manifold is simple at energy H_0 if the map $\exp_x^H : (\exp_x^H)^{-1}(M) \rightarrow M$ is a diffeomorphism for each $x \in M$, and also that ∂M is strictly \mathcal{MP} convex. We remark that simplicity implies $H_0 > \max_{x \in M} q$. It is more stringent to assume a system to be simple than to assume it to be non-trapping.

We define the Hamiltonian flow transform of a function $f : M \rightarrow \mathbb{R}$ at energy H_0 for $(x, \xi) \in \partial_+ S_{H_0} M$ as

$$I_{H_0} f(x, \xi) = \int_0^{\tau(x, \xi)} f(\gamma_{x, \xi}(t)) dt, \tag{A.7}$$

$$I_{H_0} : C^\infty(M) \rightarrow C^\infty(\partial_+ S_{H_0} M), \tag{A.8}$$

where $\gamma_{x, \xi}$ is an H -geodesic of energy H_0 starting at x in the direction ξ . We denote the corresponding normal operator by $N_{H_0} = I_{H_0}^* I_{H_0}$, with the adjoint operator

$$I_{H_0}^* f : C^\infty(\partial_+ S_{H_0} M) \rightarrow C^\infty(M). \tag{A.9}$$

Let $\gamma(t)$ be an \mathcal{MP} geodesic of energy level H_0 . We consider the change of variables

$$s(t) = \int_0^t 2(H_0 - q(\gamma(t))) dt. \tag{A.10}$$

It follows that s is the arc-length of $\sigma(s) = \gamma(t(s))$ under the metric $\tilde{g} = 2(H_0 - q)g$. A version of Maupertuis principle says that $\sigma(s) = \gamma(t(s))$ is a unit speed magnetic geodesic of the system $(\tilde{g}, 0)$. We quote these results from [7] without proof.

THEOREM A.1 (Theorem 2.1 in [7]). *Let (M, g, q) denote an \mathcal{MP} potential system on M and H_0 a constant such that $H_0 > \max_{x \in M} q(x)$. We suppose $\gamma(t)$ is an \mathcal{MP} geodesic of H_0 , then $\sigma(s) = \gamma(t(s))$ is a unit speed geodesic of the system $(M, \tilde{g}, 0)$ on M .*

PROPOSITION A.1 (Prop 2.2 in [7]). *The \mathcal{MP} potential system (M, g, q) on M of energy H_0 is simple if and only if so is system $(M, \tilde{g}, 0)$.*

We use these two results to prove the following:

LEMMA A.1. *Suppose (M, g, q) is simple at energy H_0 w.r.t. the Hamiltonian H . Define a new metric \tilde{g} conformal to g by $\tilde{g} = 2(H_0 - q)g$. Then (M, \tilde{g}) is a simple Riemannian manifold and a reparametrization turns H -geodesics of energy H_0 to unit speed geodesics w.r.t. \tilde{g} .*

Moreover, if \tilde{I} denotes the X-ray transform w.r.t. the metric \tilde{g} , then $I_{H_0}f(\gamma) = \tilde{I}[f/2(H_0 - q)](\gamma \circ r)$, where r is the reparametrization.

Proof. The first part follows from Theorem A.1 and the second part is a straightforward calculation. \square

Let us remark that the boundary ∂M is strictly convex with respect to \tilde{g} if and only if $\Lambda(v, v) > -\partial_\nu q(x)$ for all $(x, v) \in S_{H_0}^* M$ with $x \in \partial M$ and $v \perp \partial M$. Here ∂_ν is the inward normal derivative and $\Lambda(\cdot, \cdot)$ the second fundamental form. This calculation can be found in [7, Lemma A.4].

The theorem below is not strictly necessary for this article, but could be used to find the lower order terms for the operator $L + q_0$ with q_0 a potential.

THEOREM A.2. *Suppose $\dim M \geq 2$. If the manifold is simple for H_0 , then I_{H_0} is injective on $L^2(M)$. Moreover, we have the stability estimate*

$$\|f\|_{L^2(M)} \leq C \|N_{H_0} f\|_{H^1(M')}, \tag{A.11}$$

where $M' \supset M$ is a slightly extended manifold and C is a constant depending on the manifold, q and H_0 .

Proof. By Lemma A.1 it suffices to show that the X-ray transform on the manifold (M, \tilde{g}) is injective and stable. Injectivity [14, Theorem 7.1] and stability [39, Theorem 3] are known for simple manifolds. \square

In fact, it suffices to assume that the underlying manifold (M, g) is simple if H_0 can be taken arbitrarily large. The following corollary follows immediately from Theorem A.2 above and Lemma A.2 below.

COROLLARY A.1. *Let (M, g) be a simple manifold of dimension two or higher. Then for sufficiently large H_0*

$$\|f\|_{L^2(M)} \leq C \|N_{H_0} f\|_{H^1(M')}, \tag{A.12}$$

where $M' \supset M$ is a slightly extended manifold and C is a constant depending on the manifold, q and H_0 .

The corollary applies, in particular, to the closure of any strictly convex and bounded Euclidean domain.

LEMMA A.2. *If (M, g) is simple, then the manifold M is simple at energy H_0 for large enough H_0 .*

Proof. It is well known that simplicity is an open condition: small perturbations of simple metrics are still simple. Therefore for H_0 large enough the metric $(1 - q/H_0)g$ is simple. Rescaling the metric does not alter simplicity, so also the metric $\tilde{g} = 2(H_0 - q)g$ is simple. The claim then follows from Proposition A.1. \square

We now relate our results to known theorems on boundary distance rigidity. Let $d = \dim M \geq 3$ and let $c > 0, \tilde{c} > 0$ be smooth functions. M is a Riemannian manifold with metric g_0 . Let ρ_c (resp. $\rho_{\tilde{c}}$) be the Riemannian distance function with respect to the metric $c^{-2}g_0$ (resp. $\tilde{c}^{-2}g_0$). Let ∂M be strictly convex with respect to both $g = c^{-2}g_0$ and $\tilde{g} = \tilde{c}^{-2}g_0$ near a fixed $p \in \partial M$.

THEOREM A.3. *There exists $k > 0$ and $0 < \mu < 1$ with the following property. For any $0 < c_0 \in C^k(M)$, $p \in \partial M$ and $A > 0$ there exists $\epsilon_0 > 0$ and $C > 0$ with the property that*

for any two positive c, \tilde{c} with

$$\|c - c_0\|_{C^2} + \|\tilde{c} - c_0\|_{C^2} \leq \epsilon_0 \quad \|c\|_{C^k} \leq A \tag{A.13}$$

and for any neighbourhood Γ of p on ∂M , we have the stability estimate

$$\|c - \tilde{c}\|_{C^2(U)} \leq C \|\rho - \tilde{\rho}\|_{C(\Gamma \times \Gamma)}^\mu \tag{A.14}$$

for some neighbourhood U of p in M .

We have the following corollary

COROLLARY A.2. *Let c and \tilde{c} satisfy the hypothesis of the Theorem A.3 above. Then for any neighbourhood Γ of p on ∂M , we have the stability estimate*

$$\|c - \tilde{c}\|_{C^2(U)} \leq C \|I_{H_0}^c(c^{-1}) - I_{H_0}^{\tilde{c}}(\tilde{c}^{-1})\|_{C(\partial SM^+)_U}^\mu \tag{A.15}$$

where $I_{H_0}^c$ and $I_{H_0}^{\tilde{c}}$ denote the Hamiltonian flow transform at energy H_0 with respect to the potentials c^{-2} and \tilde{c}^{-2} respectively. Here we denote the space $C(\partial SM^+)_U$ as the restriction of the vectors (x_0, ω_0) whose endpoints under the geodesic flow are on $\partial M \cap U$.

Proof. Let U be a geodesically convex subset of (M, g) . Chose a point $p \in U$, then for each $p_0 \in U$ there exist a unique minimising geodesic starting at p and ending at p_0 . In other words there is a unique tangent vector $\omega \in T_p M$ such that

$$p_0 = \exp_p \omega \tag{A.16}$$

and a diffeomorphism

$$\hat{U} = \{\omega \in T_p M | p_0 \in U\} \cong U \tag{A.17}$$

We note ρ_c is the geodesic ray transform of c with respect to the metric $c^{-2}g_0$, so the transformation $\tilde{g} = 2(H_0 - q)g$ with $\tilde{g} = c^{-2}g_0$, $H_0 = 1$ and $-q = c^{-2}$ in Lemma A.1 applies to turn it into the flow transform. In other words we have

$$I_{H_0}^c(c^{-1}) = \int_0^{\tau(x_0, \omega_0)} c^{-1}(x_1(s)) ds = \rho_c(x, \tau(x_0, \omega_0)) \tag{A.18}$$

by Lemma A.1. Here $\tau(x_0, \omega_0)$ is a point on the boundary which is the flow out of the Hamiltonian flow with potential c^{-2} which had initial conditions x_0, ω_0 . By strict geodesic convexity, the norms $C(\partial SM^+)_U$ and $C(\Gamma \times \Gamma)$ are equivalent. \square

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